

COUNTING THIN SUBGRAPHS VIA PACKINGS FASTER THAN MEET-IN-THE-MIDDLE TIME

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ABSTRACT. Vassilevska and Williams (STOC 2009) showed how to count simple paths on k vertices and matchings on $k/2$ edges in an n -vertex graph in time $n^{k/2+O(1)}$. In the same year, two different algorithms with the same runtime were given by Koutis and Williams (ICALP 2009), and Björklund *et al.* (ESA 2009), via $n^{st/2+O(1)}$ -time algorithms for counting t -tuples of pairwise disjoint sets drawn from a given family of s -sized subsets of an n -element universe. Shortly afterwards, Alon and Gutner (TALG 2010) showed that these problems have $\Omega(n^{\lfloor st/2 \rfloor})$ and $\Omega(n^{\lfloor k/2 \rfloor})$ lower bounds when counting by color coding.

Here we show that one can do better, namely, we show that the “meet-in-the-middle” exponent $st/2$ can be beaten and give an algorithm that counts in time $n^{0.45470382st+O(1)}$ for t a multiple of three. This implies algorithms for counting occurrences of a fixed subgraph on k vertices and pathwidth $p \ll k$ in an n -vertex graph in $n^{0.45470382k+2p+O(1)}$ time, improving on the three mentioned algorithms for paths and matchings, and circumventing the color-coding lower bound. We also give improved bounds for counting t -tuples of disjoint s -sets for $s = 2, 3, 4$.

Our algorithms use fast matrix multiplication. We show an argument that this is necessary to go below the meet-in-the-middle barrier.

1. INTRODUCTION

Suppose we want to count the number of occurrences of a k -element pattern in an n -element universe. This setting is encountered, for example, when P is a k -vertex pattern graph, H is an n -vertex host graph, and we want to count the number of subgraphs that are isomorphic to P in H . If k is a constant independent of n , enumerating all the k -element subsets or tuples of the n -element universe can be done in time $O(n^k)$, which presents a trivial upper bound for counting small patterns.

In this paper we are interested in patterns that are *thin*, such as pattern graphs that are paths or cycles, or more generally pattern graphs with *bounded pathwidth*. Characteristic to such patterns is that they can be split into two or more parts, such that the interface between the parts is easy to control. For example, a simple path on k vertices can be split into two paths of half the length that have exactly one vertex in common; alternatively, one may split the path into two independent sets of vertices.

The possibility to split into two controllable parts immediately suggests that one should pursue an algorithm that runs in no worse time than $n^{k/2+O(1)}$; such an algorithm was indeed discovered in 2009 by Vassilevska and Williams [26] for counting k -vertex subgraphs that admit an independent set of size $k/2$. This result was accompanied, within the same year, of two publications presenting the same

runtime restricted to counting paths and matchings. Koutis and Williams [19] and Björklund *et al.* [6] describe different algorithms for the related problem of counting the number of t -tuples of disjoint sets that can be formed from a given family of s -subsets of an n -element universe in $n^{st/2+O(1)}$ time. Fomin *et al.* [13] generalized the latter result into an algorithm that counts occurrences of a k -vertex pattern graph with pathwidth p in $n^{k/2+2p+O(1)}$ time.

Splitting into three parts enables faster listing of the parts in $n^{k/3+O(1)}$ time, but requires more elaborate control at the interface between parts. This strategy enables one to count also dense subgraphs such as k -cliques via an algorithm of Nešetřil and Poljak [24] (see also [11, 18]) that uses fast matrix multiplication to achieve a pairwise join of the three parts, resulting in running time $n^{\omega k/3+O(1)}$, where $2 \leq \omega < 2.3728639$ is the limiting exponent of square matrix multiplication [22, 27]. Even in the case $\omega = 2$ this running time is, however, $n^{2k/3+O(1)}$, which is inferior to “meeting in the middle” by splitting into two parts.

But is meet-in-the-middle really the best one can do? For many problems it appears indeed that the worst-case running time given by meet-in-the-middle is difficult to beat. Among the most notorious examples in this regard is the Subset Sum problem, for which the 1974 meet-in-the-middle algorithm of Horowitz and Sahni [15] remains to date the uncontested champion. Related problems such as the k -Sum problem have an equally frustrating status, in fact to such an extent that the case $k = 3$ is regularly used as a source of hardness reductions in computational geometry [14].

Against this background one could perhaps expect a barrier at the meet-in-the-middle time $n^{k/2+O(1)}$ for counting thin subgraphs, and such a position would not be without some supporting evidence. Indeed, not only are the algorithms of Vassilevska and Williams [26], Koutis and Williams [19], and Björklund *et al.* [6] fairly recent discoveries, but they all employ rather different techniques. Common to all three algorithms is however the need to consider the $k/2$ -element subsets of the n -element vertex set, resulting in time $n^{k/2+O(1)}$. Yet further evidence towards a barrier was obtained by Alon and Gutner [1] who showed that color-coding based counting approaches relying on a perfectly k -balanced family of hash functions face an unconditional $c(k)n^{\lfloor k/2 \rfloor}$ lower bound for the size of such a family. From a structural complexity perspective Flum and Grohe [12] have shown that counting k -paths is $\#W[1]$ -hard with respect to the parameter k , and a very recent breakthrough of Curticapean [9] establishes a similar barrier to counting k -matchings. This means that parameterized counting algorithms with running time $f(k)n^{O(1)}$ for a function $f(k)$ independent of n are unlikely for these problems, even if such a structural complexity approach does not pinpoint precise lower bounds of the form $n^{\Omega(g(k))}$ for some function $g(k)$.

Contrary to the partial evidence above, however, our objective in this paper is to show that *there is a crack in the meet-in-the-middle barrier*, albeit a modest one. In particular, we show that it is possible to count subgraphs on k vertices such as paths and matchings—and more generally any k -vertex subgraphs with pathwidth p —within time $n^{0.45470382k+2p+O(1)}$ for $p \ll k$.

Our strategy is to reduce the counting problem to the task of evaluating a particular trilinear form on weighted hypergraphs, and then show that this trilinear form admits an evaluation algorithm that breaks the meet-in-the-middle barrier.

This latter algorithm is our main contribution, which we now proceed to present in more detail.

1.1. Weighted disjoint triples. Let U be an n -element set. For a nonnegative integer q , let us write $\binom{U}{q}$ for the set of all q -element subsets of U . Let $f, g, h : \binom{U}{q} \rightarrow \mathbb{Z}$ be three functions given as input. We are interested in computing the trilinear form

$$(1) \quad \Delta(f, g, h) = \sum_{\substack{A, B, C \in \binom{U}{q} \\ A \cap B = A \cap C = B \cap C = \emptyset}} f(A)g(B)h(C).$$

To ease the running time analysis, we make two assumptions. First, q is a constant independent of n . Second, we assume that the values of the functions f, g, h are bounded in bit-length by a polynomial in n , which will be the setup in our applications (Theorems 3 and 4).

Let us write ω for the limiting exponent of square matrix multiplication, $2 \leq \omega < 2.3728639$ [22, 27]. Similarly, let us write α for the limiting exponent such that multiplying an $N \times N^\alpha$ matrix with an $N^\alpha \times N$ matrix takes $N^{2+o(1)}$ arithmetic operations, $0.3 < \alpha \leq 3 - \omega$ [21].

The next theorem is our main result; here the intuition is that we take $k = 3q$ in our applications, implying that we break the meet-in-the-middle exponent $k/2$.

Theorem 1 (Fast weighted disjoint triples). *There exists an algorithm that evaluates $\Delta(f, g, h)$ in time $O(n^{3q(\frac{1}{2}-\tau)+c})$ for constants c and τ independent of the constant q , with $c \geq 0$ and*

$$(2) \quad \tau = \begin{cases} \frac{(3-\omega)(1-\alpha)}{36-6(1+\omega)(1+\alpha)} & \text{if } \alpha \leq 1/2; \\ \frac{1}{18} & \text{if } \alpha \geq 1/2. \end{cases}$$

Remark 1. For $\omega = 2.3728639$ and $\alpha = 0.30$ we obtain $\tau = 0.045296182$ and hence $O(n^{3q \cdot 0.45470382+c})$ time. For $\alpha \geq 1/2$ we obtain $\tau = 0.055555556$ and hence $O(n^{3q \cdot 0.44444445+c})$ time. Note that the latter case occurs in the case $\omega = 2$ because then $\alpha = 1$.

Remark 2. We observe that the trilinear form (1) admits an evaluation algorithm analogous to the algorithm of Nešetřil and Poljak [24] discussed above. Indeed, (1) can be split into a multiplication of two $n^q \times n^q$ square matrices, which gives running time $O(n^{\omega q+c})$. Even in the case $\omega = 2$ the running time $O(n^{2q+c})$ is however inferior to Theorem 1.

Remark 3. Theorem 1 can be stated in an alternative form that counts the number of arithmetic operations (addition, subtraction, multiplication, and exact division of integers) performed by the algorithm on the inputs f, g, h to obtain $\Delta(f, g, h)$. This form is obtained by simply removing the constant c from the bound in Theorem 1.

Finally, we show that one can improve upon Theorem 1 via case by case analysis. Here our intent is to pursue only the cases $q = 2, 3, 4$ and leave the task of generalizing from here to further work.

When considering specific values of q , it is convenient to measure efficiency using the number of arithmetic operations (addition, subtraction, multiplication, and exact division of integers) performed by an algorithm.

Theorem 2. *There exist algorithms that solve the weighted disjoint triples problem*

- (1) for $q = 2$ in $O(n^\omega)$ arithmetic operations,
- (2) for $q = 3$ in $O(n^{\omega+1})$ arithmetic operations, and
- (3) for $q = 4$ in $O(n^{2\omega})$ arithmetic operations.

Remark. In the case $\omega = 2$ we observe that the three algorithms in Theorem 2 all run in $O(n^q)$ arithmetic operations, which is linear in the size of the input.

1.2. Counting thin subgraphs and packings. Once Theorem 1 is available, the following theorem is an almost immediate corollary of techniques for counting injective homomorphisms of bounded-pathwidth graphs developed by Fomin *et al.* [13] (see also §3 in Amini *et al.* [2]). In what follows τ is the constant in (2).

Theorem 3 (Fast counting of thin subgraphs). *Let P be a fixed pattern graph with k vertices and pathwidth p . Then, there exists an algorithm that takes as input an n -vertex host graph H and counts the number of subgraphs of H that are isomorphic to P in time $O(n^{(\frac{1}{2}-\tau)k+2p+c} + n^{k/3+3p+c})$ where $c \geq 0$ is a constant independent of the constants k, p, τ .*

Remark. The running time in Theorem 3 simplifies to $O(n^{(\frac{1}{2}-\tau)k+2p+c})$ if $p \leq k/9$.

Theorem 1 gives also an immediate speedup for counting set packings. In this case we use standard dynamic programming to count, for each q -subset A with $q = st/3$, the number of $t/3$ -tuples of pairwise disjoint s -subsets whose union is A . We then use Theorem 1 to assemble the number of t -tuples of pairwise disjoint s -subsets from triples of such q -subsets. This results in the following corollary.

Theorem 4 (Fast counting of set packings). *There exists an algorithm that takes as input a family \mathcal{F} of s -element subsets of an n -element set and an integer t that is divisible by 3, and counts the number of t -tuples of pairwise disjoint subsets from \mathcal{F} in time $O(n^{(\frac{1}{2}-\tau)st+c})$ where $c \geq 0$ is a constant independent of the constants s, t, τ .*

1.3. On the hardness of counting in disjoint parts. We present two results that provide partial justification why there was an apparent barrier at “meet-in-the-middle time” for counting in disjoint parts.

First, in the case of two disjoint parts, the problem appears to contain no algebraic dependency that one could exploit towards faster algorithms beyond those already presented in Björklund *et al.* [5, 6]. Indeed, we can provide some support towards this intuition by showing that the associated 2-tensor has full rank over the rationals, see Lemma 10. This observation is most likely not new but we were unable to find the right reference.

Second, recall that our algorithms mentioned in the previous section use fast matrix multiplication. We show an argument that this is necessary to go below the meet-in-the-middle barrier. More precisely, we show that any *trilinear algorithm* (cf. [25, §9]) for $\Delta(f, g, h)$ whose rank over the integers is below the meet-in-the-middle barrier implies a sub-cubic algorithm for matrix multiplication:

Theorem 5. *Suppose that for all constants q there exists a trilinear algorithm for $\Delta(f, g, h)$ with rank $r = O(n^{3q(1/2-\tau)+c})$ over the integers, where $\tau > 0$ and $c \geq 0$ are constants independent of n and q . Then, $\omega \leq 3 - \tau$.*

1.4. Overview of techniques and discussion. The main idea underlying Theorem 1 is to design a system of linear equations whose solution contains the weighted disjoint triples (1) as one indeterminate. The main obstacle to such a design is of course that we must be able to construct and solve the system within the allocated time budget.

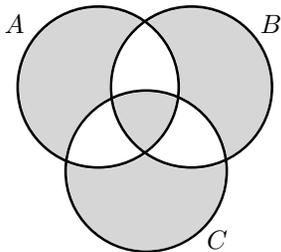
In our case the design will essentially be a balance between two families of linear equations, the *basic* (first) family and the *cheap* (second) family, for the same indeterminates. The basic equations alone suffice to solve the system in meet-in-the-middle time $O(n^{3q/2+c})$, whereas the cheap equations solve directly for selected indeterminates *other than* (1). The virtue of the cheap equations is that their right-hand sides can be evaluated efficiently using fast (rectangular) matrix multiplication, which enables us to throw away the most expensive of the basic equations and still have sufficient equations to solve for (1), thereby breaking the meet-in-the-middle barrier. Alternatively one can view the extra indeterminates and linear equations as a tool to expand the scope of our techniques beyond the extent of the apparent barrier so that it can be circumvented.

Before we proceed to outline the design in more detail, let us observe that the general ingredients outlined above, namely fast matrix multiplication and linear equations, are well-known techniques employed in a number of earlier studies. In particular in the context of subgraph counting such techniques can be traced back at least to the triangle- and cycle-counting algorithms of Itai and Rodeh [16], with more recent uses including the algorithms of Kowaluk, Lingas, and Lundell [20] that improve upon algorithms of Nešetřil and Poljak [24] and Vassilevska and Williams [26] for counting small dense subgraphs ($k < 10$) with a maximum independent set of size 2. Also the counting-in-halves technique of Björklund *et al.* [6] can be seen to solve an (implicit) system of linear equations to recover weighted disjoint packings.

Let us now proceed to more detailed design considerations. Here the main task is to relax (1) into a collection of trilinear forms related by linear constraints. A natural framework for relaxation is to parameterize the triples (A, B, C) so that the pairwise disjoint triples required by (1) become an extremal case.

A first attempt at such parameterization is to parameterize the triples (A, B, C) by the size of the union $|A \cup B \cup C| = j$. In particular, the triple (A, B, C) is pairwise disjoint if and only if $j = 3q$. With this parameterization we obtain $2q + 1$ indeterminates, one for each value of $q \leq j \leq 3q$. In this case *inclusion-sieving* (trimmed Möbius inversion [6, 7]) on the subset lattice $(2^{[n]}, \cup)$ enables a system of linear equations on the indeterminates. This is in fact the approach underlying the counting-in-halves technique of Björklund *et al.* [6], which generalizes also to Möbius algebras of lattices with the set union (set intersection) replaced by the join (meet) operation of the lattice [8]. Unfortunately, it appears difficult to break the meet-in-the-middle barrier via this parameterization, in particular due to an apparent difficulty of arriving at a cheap system of equations to complement the basic equations arising from the inclusion sieve.

A second attempt at parameterization is to replace the set union $X \cup Y$ with the *symmetric difference* $X \oplus Y = (X \setminus Y) \cup (Y \setminus X)$ and parameterize the triples (A, B, C) by the size of the symmetric difference $|A \oplus B \oplus C| = j$. The set $A \oplus B \oplus C$ is illustrated in Fig. 1.

FIGURE 1. The set $A \oplus B \oplus C$.

Recalling that $X \cup Y$ and $X \oplus Y$ coincide if and only if X and Y are disjoint, we again recover the pairwise disjoint triples as the extremal case $j = 3q$. With this parameterization we obtain $\lfloor 3q/2 \rfloor + 1$ indeterminates, one for each $0 \leq j \leq 3q$ such that $j \equiv q \pmod{2}$. In this case *parity-sieving* (trimmed “parity-of-intersection transforms”, see §3.2) on the group algebra of the elementary Abelian group $(2^{\lfloor n \rfloor}, \oplus)$ enables a system of linear equations on the indeterminates. While this second parameterization via symmetric difference is *a priori* less natural than the first parameterization via set union, it turns out to be more successful in breaking the meet-in-the-middle barrier. In particular the basic equations (Lemma 1) on the $\lfloor 3q/2 \rfloor + 1$ indeterminates alone suffice to obtain an algorithm with running time $O(n^{3q/2+c})$, which is precisely at the meet-in-the-middle barrier. The key insight then to break the barrier is that the indeterminates with *small* values of j can be solved directly (Lemma 2) via fast rectangular matrix multiplication. In particular this is because small j implies large overlap between the sets A, B, C and a “triangle-like” structure that is amenable to matrix multiplication techniques. That is, from the perspective of the symmetric difference $D = A \oplus B$, it suffices to control the differences $A \setminus B$ and $B \setminus A$ (outer dimensions in matrix multiplication), whereas the overlap $A \cap B$ (inner dimension) is free to range across sets disjoint from D (see §3.4).

A further design constraint is that the basic equations (Lemma 1) must be mutually independent of the cheap equations (Lemma 2) to enable balancing the running time (see §3.6) while retaining invertibility of the system; here we have opted for an analytically convenient design where the basic equations are in general position (the coefficient matrix is a Vandermonde matrix) that enables easy combination with the cheap equations, even though this design may not be the most efficient possible from a computational perspective.

From an efficiency perspective we can in fact do better than Theorem 1 for small values of q by proceeding via case by case analysis (see §3.7). We show that faster algorithms exist for at least $q = 2, 3, 4$ (Theorem 2).

An open problem that thus remains is whether the upper bound $n^{3q(\frac{1}{2}-\tau)+O(1)}$ in Theorem 1 can be improved to the asymptotic form $\binom{n}{3q(\frac{1}{2}-\delta)}n^{O(1)}$ for some constant $\delta > 0$ independent of $0 \leq q \leq n/3$. In particular, such an improvement would parallel the asymptotic running time $\binom{n}{k/2}n^{O(1)}$ of the counting-in-halves technique [6]. Furthermore, such an improvement would be of considerable interest since it would, for example, lead to faster algorithms for computing the permanent of an integer matrix. Unfortunately, this also suggests that such an improvement is unlikely, or at least difficult to obtain given the relatively modest progress in

improved algorithms for the permanent problem [4]. Some further evidence towards the subtlety of counting in disjoint parts is that we can show (Theorem 5) that to break the “meet-in-the-middle” barrier for the weighted disjoint triples problem with a trilinear algorithm, it is in fact *necessary* to use fast matrix multiplication (see §6). Put otherwise, the proofs of Theorems 1 and 5 reveal that for constant q the structural tensors for weighted disjoint triples and matrix multiplication are loosely rank-equivalent in terms of existence of low-rank decompositions.

1.5. Organization. The proof of Theorem 1 is split into two parts. First, in §2 we derive a linear system whose solution contains $\Delta(f, g, h)$. Then, in §3 we derive an algorithm that constructs and solves the system within the claimed running time bound. We then proceed with the two highlighted applications of Theorem 1: in §4 we give a proof of Theorem 3 by relying on techniques of Fomin *et al.* [13], and in §5 we prove Theorem 4. We conclude the paper in §6 by connecting fast trilinear algorithms for $\Delta(f, g, h)$ to fast matrix multiplication.

2. THE LINEAR SYSTEM

We now proceed to derive a linear system whose solution contains $\Delta(f, g, h)$. Towards this end it is convenient to start by recalling some elementary properties of the symmetric difference operator on sets.

For sets $X, Y \subseteq U$, let us write $X \oplus Y = (X \setminus Y) \cup (Y \setminus X)$ for the symmetric difference of X and Y . We immediately observe that

$$(3) \quad |X \oplus Y| = |X| + |Y| - 2|X \cap Y|$$

and hence

$$(4) \quad |X \oplus Y| \equiv |X| + |Y| \pmod{2}.$$

In particular, for any $A, B, C \in \binom{U}{q}$ we have

$$|A \oplus B \oplus C| \equiv |A| + |B| + |C| = 3q \equiv q \pmod{2}.$$

Thus, the size $|A \oplus B \oplus C|$ is always even if q is even and always odd if q is odd. In both cases $j = |A \oplus B \oplus C|$ may assume exactly $e = \lfloor 3q/2 \rfloor + 1$ values in

$$(5) \quad J_q = \{j \in \{0, 1, \dots, 3q\} : j \equiv q \pmod{2}\}.$$

We are now ready to define the $e \times e$ linear system. We start with the indeterminates of the system.

2.1. The indeterminates. For each $j \in J_q$, let

$$(6) \quad x_j = x_j(f, g, h) = \sum_{\substack{A, B, C \in \binom{U}{q} \\ |A \oplus B \oplus C| = j}} f(A)g(B)h(C).$$

In particular, since $A, B, C \in \binom{U}{q}$ are pairwise disjoint if and only if $|A \oplus B \oplus C| = 3q$, we observe that $\Delta(f, g, h) = x_{3q}$. Thus, it suffices to solve for the indeterminate x_{3q} to recover (1). We proceed to formulate a linear system towards this end. The system is based on two families of equations. The first family will contribute d equations, and the second family will contribute $e - d$ equations.

2.2. A first family of equations. Our first family of equations is based on a parity construction. For now we will be content in simply defining the equations and providing an illustration in Fig. 2. (The eventual algorithmic serendipity of this construction will be revealed only later in (20) and (21).) Let $i = 0, 1, \dots, d-1$ be an index for the equations, let $p \in \{0, 1\}$ denote parity, and let $s = 0, 1, \dots, i$. For all $Z \in \binom{U}{s}$ let

$$(7) \quad T_p(Z) = \sum_{\substack{A, B, C \in \binom{U}{q} \\ |(A \oplus B \oplus C) \cap Z| \equiv p \pmod{2}}} f(A)g(B)h(C).$$

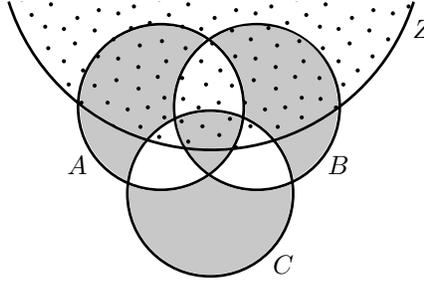


FIGURE 2. The set $(A \oplus B \oplus C) \cap Z$ (grey and dotted) from the definition of $T_p(Z)$.

The right-hand sides of the first system are now defined by

$$(8) \quad y_i = \sum_{(u_1, u_2, \dots, u_i) \in U^i} T_0(\oplus_{\ell=1}^i \{u_\ell\}) - T_1(\oplus_{\ell=1}^i \{u_\ell\}).$$

Let us recall that the universe U has n elements. For nonnegative integers i and j , let us define the Vandermonde matrix $V = (v_{ij})$ by setting

$$(9) \quad v_{ij} = (n - 2j)^i.$$

Remark. We recall from basic linear algebra that any $d \times d$ submatrix of a $d \times e$ Vandermonde matrix with entries z_j^i for $i = 0, 1, \dots, d-1$ and $j = 0, 1, \dots, e-1$ has nonzero determinant if the values z_j are pairwise distinct. This makes a Vandermonde matrix particularly well-suited for building systems of independent equations from multiple families of equations.

Lemma 1 (First family). *For all $i = 0, 1, \dots, d-1$ it holds that*

$$(10) \quad \sum_{j \in J_q} v_{ij} x_j = y_i.$$

Proof. Let us fix a triple $A, B, C \in \binom{U}{q}$ with $|A \oplus B \oplus C| = j$. From (5) we have $j \in J_q$. Let us write $m_p(i)$ for the number of i -tuples $(u_1, u_2, \dots, u_i) \in U^i$ such that

$$(11) \quad |(A \oplus B \oplus C) \cap (\oplus_{\ell=1}^i \{u_\ell\})| \equiv p \pmod{2}.$$

From (6), (7), and (8) we observe that the lemma is implied by

$$(12) \quad v_{ij} = m_0(i) - m_1(i).$$

Indeed, v_{ij} and $m_0(i) - m_1(i)$ are the coefficients before $f(A)g(B)h(C)$ in the LHS and RHS of (10), respectively. To prove (12), we proceed by induction on i . The base case $i = 0$ is set up by observing

$$(13) \quad \begin{aligned} m_0(0) &= 1, \\ m_1(0) &= 0. \end{aligned}$$

For $i \geq 1$, let us study what happens if we extend an arbitrary $(i-1)$ -tuple $(u_1, u_2, \dots, u_{i-1}) \in U^{i-1}$ by a new element $u_i \in U$. We observe that we have exactly $n-j$ choices for the value u_i among the elements of U outside $A \oplus B \oplus C$ and j choices inside $A \oplus B \oplus C$. The parity (11) changes if and only if we choose an element inside $A \oplus B \oplus C$. Thus, for $i \geq 1$ we have

$$(14) \quad \begin{aligned} m_0(i) &= (n-j)m_0(i-1) + jm_1(i-1), \\ m_1(i) &= jm_0(i-1) + (n-j)m_1(i-1). \end{aligned}$$

From (13) and (14) we thus have

$$\begin{aligned} m_0(0) - m_1(0) &= 1, \\ m_0(i) - m_1(i) &= (n-2j)(m_0(i-1) - m_1(i-1)). \end{aligned}$$

Hence, $m_0(i) - m_1(i) = (n-2j)^i$ and from (9) we conclude that the lemma holds. \square

2.3. A second family of equations. Our second family of equations is based on solving for the indeterminates (6) directly. We state the following lemma in a general form, but for performance reasons we will in fact later use only the equations indexed by the $e-d$ smallest values $j \in J_q$ in our linear system.

Lemma 2 (Second family). *For all $j \in J_q$ it holds that*

$$(15) \quad x_j = \sum_{\ell=q-j}^{q+j} \sum_{D \in \binom{U}{\ell}} \sum_{\substack{A, B \in \binom{U}{q} \\ A \oplus B = D}} f(A)g(B) \sum_{\substack{C \in \binom{U}{q} \\ |C \cap D| = (q+\ell-j)/2}} h(C).$$

Proof. We must show that the right-hand side of (15) equals (6). Let us study a triple $A, B, C \in \binom{U}{q}$ with $|A \oplus B \oplus C| = j$. We observe that $q-j \leq |A \oplus B| \leq q+j$ because otherwise taking the symmetric difference with C will either leave too many elements uncanceled or it cannot cancel enough of the elements in $D = A \oplus B$. Since $|A| = |B| = q$ from (3) it follows that $|D|$ is in fact always even. Furthermore, when $|D| = \ell$ we observe that

$$\begin{aligned} j &= |A \oplus B \oplus C| \\ &= |C \oplus D| \\ &= |C| + |D| - 2|C \cap D| \\ &= q + \ell - 2|C \cap D|. \end{aligned}$$

The lemma follows by solving for $|C \cap D|$ and observing that each triple A, B, C uniquely determines $D = A \oplus B$. \square

2.4. The linear system. We are now ready to combine equations from the two families to a system

$$(16) \quad M\vec{x} = \vec{y}$$

of independent linear equations for the indeterminates $\vec{x} = (x_j : j \in J_q)$. Recalling (5), there are exactly $e = \lfloor 3q/2 \rfloor + 1$ indeterminates, and hence exactly e independent equations are required.

Let us use a parameter $0 \leq \gamma \leq 1/2$ in building the system. (The precise value of γ will be determined later in §3.6.)

We now select $d = \lfloor (3/2 - \gamma)q \rfloor + 1$ equations from the first family (Lemma 6), and $e - d$ equations from the second family (Lemma 8). More precisely, we access the first family for d equations indexed by $i = 0, 1, \dots, d - 1$, and the second family for the $e - d$ equations indexed by the smallest $e - d$ values $j \in J_q$. That is, if q is even, we use equations indexed by $j \in \{0, 2, \dots, 2(e - d - 1)\}$, and if q is odd, we use equations indexed by $j \in \{1, 3, \dots, 2(e - d) - 1\}$. Thus, for all q we conclude that

$$i \leq \lfloor (3/2 - \gamma)q \rfloor$$

and that

$$\begin{aligned} j &\leq 2(e - d) - 1 \\ &= 2(\lfloor 3q/2 \rfloor - \lfloor (3/2 - \gamma)q \rfloor) - 1 \\ &\leq 2(\lfloor \gamma q \rfloor + 1) - 1 \\ &\leq \lfloor 2\gamma q \rfloor + 1. \end{aligned}$$

Let us now verify that the selected system consists of independent equations. To verify this it suffices to solve the system. The equations from the second family (Lemma 8) by construction solve directly for $e - d$ indeterminates. We are thus left with d equations from the first family (Lemma 6). Now observe that since we know the values of $e - d$ indeterminates, we can subtract their contribution from both sides of the remaining equations, leaving us d equations over d indeterminates. In fact (see the remark before Lemma 6), the coefficient matrix of the remaining system is a $d \times d$ submatrix of the original Vandermonde matrix, and hence invertible. We conclude that the equations are independent.

It remains to argue that the system (16) can be constructed and solved within the claimed running time.

3. EFFICIENT CONSTRUCTION AND SOLUTION

This section proves Theorem 1 by constructing and solving the system derived in §2 within the claimed running time. We start with some useful subroutines that enable us to efficiently construct the right-hand sides for (8) and (15).

3.1. The intersection transform. Let s and $0 \leq t \leq s$ be nonnegative integers. For a function $f : \binom{U}{q} \rightarrow R$, define the *intersection transform* $f_{\iota_t} : \binom{U}{s} \rightarrow R$ of f for all $Z \in \binom{U}{s}$ by

$$(17) \quad f_{\iota_t}(Z) = \sum_{\substack{A \in \binom{U}{q} \\ |A \cap Z| = t}} f(A).$$

The following lemma is an immediate corollary of a theorem of Björklund *et al.* [5, Theorem 1].

Lemma 3. *There exists an algorithm that evaluates all the $\binom{|U|}{s}$ values of the intersection transform for all $0 \leq t \leq s$ in time $O(n^{\max(s,q)+c})$ for a constant $c \geq 0$ independent of constants s and q .*

Remark. Lemma 3 can be stated in an alternative form that counts the number of arithmetic operations (addition, subtraction, multiplication, and exact division of integers) performed by the algorithm on the input f to obtain $f\iota_t$ for all $0 \leq t \leq s$. This form is obtained by simply removing the constant c from the bound in Lemma 3. (Indeed, we can use Bareiss's algorithm [3] to solve the underlying linear system with exact divisions.)

3.2. The parity transform. Let s be a nonnegative integer and let $p \in \{0, 1\}$. For a function $f : \binom{U}{q} \rightarrow R$, define the *parity transform* $f\pi_p : \binom{U}{s} \rightarrow R$ of f for all $Z \in \binom{U}{s}$ by

$$(18) \quad f\pi_p(Z) = \sum_{\substack{A \in \binom{U}{q} \\ |A \cap Z| \equiv p \pmod{2}}} f(A).$$

Lemma 4. *There exists an algorithm that evaluates the parity transform for $p \in \{0, 1\}$ in time $O(n^{\max(s,q)+c})$ for a constant $c \geq 0$ independent of constants s and q .*

Proof. We observe that

$$f\pi_p = \sum_{\substack{t \in \{0, 1, \dots, s\} \\ t \equiv p \pmod{2}}} f\iota_t$$

and apply Lemma 3. □

3.3. Evaluating the right-hand side of the first family. Let i be a nonnegative integer. Our objective is to evaluate the right-hand side of (8). Let us start by observing that it suffices to compute the values (7) for all $Z \subseteq U$ with $|Z| \leq i$.

The following lemma will be useful towards this end. Denote by $L_n(i, s)$ the number of tuples $(u_1, u_2, \dots, u_i) \in U^i$ with $s = |\oplus_{\ell=1}^i \{u_\ell\}|$ and $n = |U|$.

Lemma 5. *We have*

$$L_n(i, s) = \begin{cases} 1 & \text{if } i = s = 0; \\ 0 & \text{if } i < s; \\ L_n(i-1, 1) & \text{if } i \geq 1 \text{ and } s = 0; \\ (n-s+1)L_n(i-1, s-1) \\ \quad + (s+1)L_n(i-1, s+1) & \text{if } i \geq s \geq 1. \end{cases}$$

In particular, the values $L_n(i, s)$ can be computed for all $0 \leq s \leq i \leq 3q/2 \leq n$ in time $O(q^2)$.

Proof. When we insert an element u_i into a tuple we may obtain a tuple with exactly $s \geq 1$ elements that occur an odd number of times in two different ways: either we had $s-1$ such elements and insert a new element ($n-s+1$ choices), or we had $s+1$ such elements and insert one of them. The running time follows

by tabulating the values $L_n(i, s)$ in increasing lexicographic order in (i, s) . This completes the lemma. \square

Now let us reduce (8) to (7). In particular, we have

$$(19) \quad y_i = \sum_{s=0}^i \binom{n}{s}^{-1} L_n(i, s) \sum_{Z \in \binom{U}{s}} T_0(Z) - T_1(Z).$$

Indeed, by symmetry each $Z \in \binom{U}{s}$ has the same number $l_{i,s}$ of tuples (u_1, u_2, \dots, u_i) such that $|\oplus_{\ell=1}^i \{u_\ell\}| = Z$. Hence, $L_n(i, s) = \binom{n}{s} l_{i,s}$ and (19) follows. Thus it remains to compute the values (7). At this point it is convenient to recall Fig. 2. We have

$$(20) \quad \begin{aligned} |(A \oplus B \oplus C) \cap Z| &= |(A \cap Z) \oplus (B \cap Z) \oplus (C \cap Z)| \\ &\equiv |A \cap Z| + |B \cap Z| + |C \cap Z| \pmod{2}, \end{aligned}$$

where the congruence follows from (4). Let us use the shorthand $\bar{p} = 1 - p$ for the complement of $p \in \{0, 1\}$. Denoting pointwise multiplication of functions by “ \cdot ”, from (20), (7) and (18) it immediately follows that

$$(21) \quad T_p = f\pi_p \cdot g\pi_p \cdot h\pi_p + f\pi_{\bar{p}} \cdot g\pi_{\bar{p}} \cdot h\pi_p + f\pi_{\bar{p}} \cdot g\pi_p \cdot h\pi_{\bar{p}} + f\pi_p \cdot g\pi_{\bar{p}} \cdot h\pi_{\bar{p}}.$$

Lemma 6. *There exists an algorithm that for $0 \leq \gamma \leq 1/2$ evaluates the right-hand side (8) for all $0 \leq i \leq (3/2 - \gamma)q$ in time $O(n^{(3/2 - \gamma)q + c})$ for a constant $c \geq 0$ independent of constants γ and q .*

Proof. First evaluate the parity transforms $f\pi_p, g\pi_p, h\pi_p$ for all $p \in \{0, 1\}$ and $s \leq (3/2 - \gamma)q$ using Lemma 4. Then use (21) to evaluate T_p for all $p \in \{0, 1\}$ and $s \leq (3/2 - \gamma)q$. Finally compute the coefficients $L_n(i, s)$ using Lemma 5 and evaluate the right-hand sides (8) via (19). \square

3.4. The symmetric difference product. Let ℓ be a nonnegative even integer. For $f, g : \binom{U}{q} \rightarrow R$ define the symmetric difference product $f \oplus g : \binom{U}{\ell} \rightarrow R$ for all $D \in \binom{U}{\ell}$ by

$$(22) \quad (f \oplus g)(D) = \sum_{\substack{A, B \in \binom{U}{q} \\ A \oplus B = D}} f(A)g(B).$$

From (3) we observe that if $|A \oplus B| = \ell$ with $|A| = |B| = q$ then $|A \cap B| = q - \ell/2$ and $|A \setminus B| = |B \setminus A| = \ell/2$. Define the matrix F with rows indexed by $I \in \binom{U}{\ell/2}$ and columns indexed by $K \in \binom{U}{q - \ell/2}$ with the (I, K) -entry defined by

$$(23) \quad F(I, K) = \begin{cases} f(I \cup K) & \text{if } I \cap K = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Define the matrix G with rows indexed by $K \in \binom{U}{q - \ell/2}$ and columns indexed by $J \in \binom{U}{\ell/2}$ with the (K, J) -entry defined by

$$(24) \quad G(K, J) = \begin{cases} g(K \cup J) & \text{if } K \cap J = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

From (23) and (24) we observe that the product matrix FG enables us to recover the symmetric difference product (22) for all $D \in \binom{U}{\ell}$ by

$$(25) \quad (f \oplus g)(D) = \sum_{I \in \binom{D}{\ell/2}} FG(I, D \setminus I).$$

Recall that we write ω for the limiting exponent of square matrix multiplication, $2 \leq \omega < 2.3728639$, and α for the limiting exponent such that multiplying an $N \times N^\alpha$ matrix with an $N^\alpha \times N$ matrix takes $N^{2+o(1)}$ arithmetic operations, $0.30 < \alpha \leq 3 - \omega$. For generic rectangular matrices it is known (cf. [23]) that the product of an $N \times M$ matrix and an $M \times N$ matrix can be computed using

- (M1) $O(N^{2-\alpha\beta}M^\beta + N^2)$ arithmetic operations, with $\beta = (\omega - 2)/(1 - \alpha)$ when $M \leq N$, and
- (M2) $O(N^{\omega-1}M)$ arithmetic operations by decomposing the product into a sum of $\lceil M/N \rceil$ products of $N \times N$ square matrices when $M \geq N$.

Remark. The bounds above are not the best possible [21]; however, to provide a clean exposition we will work with these somewhat sub-state-of-the-art bounds.

Lemma 7. *There exists an algorithm that for $0 \leq \gamma \leq 1/2$ evaluates the symmetric difference product (22) for all even $(1 - 2\gamma)q - 1 \leq \ell \leq (1 + 2\gamma)q + 1$ in time*

$$O(n^{\omega q/2 + (2-\alpha\beta-\beta)\gamma q + c} + n^{(1+2\gamma)q + c})$$

for a constant $c \geq 0$ independent of constants γ and q .

Proof. For convenience we may pad F and G with all-zero rows and columns so that F becomes an $n^{\ell/2} \times n^{q-\ell/2}$ matrix and G becomes an $n^{q-\ell/2} \times n^{\ell/2}$ matrix.

For $\ell \leq q$ by (M2) we can thus multiply F and G using $O(n^{\omega\ell/2 + q - \ell}) \subseteq O(n^{\omega q/2})$ arithmetic operations and hence in time $O(n^{\omega q/2 + c})$.

For $\ell \geq q$ by (M1) we can thus multiply F and G using

$$O(n^{(2-\alpha\beta)\ell/2 + \beta(q-\ell/2)} + n^\ell)$$

arithmetic operations. For $q \leq \ell \leq (1 + 2\gamma)q + 1$ the linear function

$$u(\ell) = (2 - \alpha\beta)\ell/2 + \beta(q - \ell/2)$$

has its maximum at $\ell = q$ or at $\ell = (1 + 2\gamma)q + 1$. Noting that $2 - \alpha\beta + \beta = \omega \geq 2$, at $\ell = q$ we obtain the bound $O(n^{\omega q/2 + c})$ for the running time. At $\ell = (1 + 2\gamma)q + 1$ we obtain the running time bound

$$O(n^{(2-\alpha\beta)(1/2+\gamma)q + \beta(1/2-\gamma)q + c} + n^{(1+2\gamma)q + c}).$$

Again noting that $2 - \alpha\beta + \beta = \omega$, this bound simplifies to

$$O(n^{\omega q/2 + (2-\alpha\beta-\beta)\gamma q + c} + n^{(1+2\gamma)q + c}).$$

It remains to analyze the quantity $2 - \alpha\beta - \beta$. Towards this end, recall that $0.3 < \alpha \leq 3 - \omega$ and $2 \leq \omega < 2.3728639$, with $\alpha = 1$ if and only if $\omega = 2$. The

lemma now follows by observing that

$$\begin{aligned}
2 - \alpha\beta - \beta &= \omega - 2\beta \\
&= \omega - \frac{2(\omega - 2)}{1 - \alpha} \\
&= \frac{4 - (1 + \alpha)\omega}{1 - \alpha} \\
&\geq \frac{4 - (4 - \omega)\omega}{1 - \alpha} \\
&\geq 0.
\end{aligned}$$

□

3.5. Evaluating the right-hand side of the second family. Let j be a non-negative integer. Our objective is to evaluate the right-hand side of (15). Let us start by observing that (15) can be stated using the symmetric difference product (22) and the intersection transform (17) in equivalent form

$$(26) \quad x_j = \sum_{\ell=q-j}^{q+j} \sum_{D \in \binom{U}{\ell}} (f \oplus g)(D) \cdot h_{\ell(q+\ell-j)/2}(D).$$

Lemma 8. *There exists an algorithm that for $0 \leq \gamma \leq 1/2$ evaluates the right-hand-side of (15) for all $0 \leq j \leq 2\gamma q + 1$ in time*

$$O(n^{\omega q/2 + (2 - \alpha\beta - \beta)\gamma q + c} + n^{(1+2\gamma)q + c})$$

for a constant $c \geq 0$ independent of constants γ and q .

Proof. Because $0 \leq j \leq 2\gamma q + 1$ we observe that $(1 - 2\gamma)q - 1 \leq \ell \leq (1 + 2\gamma)q + 1$ in (26). Using Lemma 7 we can evaluate $f \oplus g$ for all required ℓ within the claimed time bound. Using Lemma 3 with $s = \lfloor (1 + 2\gamma)q + 1 \rfloor$ we can evaluate h_{ℓ} for all $t \leq s$ in $O(n^{(1+2\gamma)q + c})$ time which is within the claimed time bound. Finally, the sums in (26) can be computed in the claimed time bound by using the evaluations $f \oplus g$ and h_{ℓ} . □

3.6. Running-time analysis. We now balance the running times from Lemma 6 and Lemma 8 by selecting the value of $0 \leq \gamma \leq 1/2$. Disregarding the constant c which is independent of q and γ , the contribution of Lemma 6 is $O(n^{(3/2 - \gamma)q})$ and the contribution of Lemma 8 is

$$O(n^{\omega q/2 + (2 - \alpha\beta - \beta)\gamma q} + n^{(1+2\gamma)q}).$$

In particular, we must minimize the maximum of the three contributions

$$\begin{aligned}
&O(n^{(3/2 - \gamma)q}), \\
&O(n^{\omega q/2 + (2 - \alpha\beta - \beta)\gamma q}), \quad \text{and} \\
&O(n^{(1+2\gamma)q}).
\end{aligned}$$

We claim that if $\alpha \leq 1/2$ then the maximum is controlled by

$$(27) \quad \frac{3}{2} - \gamma = \frac{\omega}{2} + (2 - \alpha\beta - \beta)\gamma.$$

Let us select the value of γ given by (27). Recalling that $\beta = (\omega - 2)/(1 - \alpha)$, we have

$$(28) \quad \gamma = \frac{3 - \omega}{2(3 - \alpha\beta - \beta)} = \frac{(3 - \omega)(1 - \alpha)}{12 - 2(1 + \omega)(1 + \alpha)}.$$

In (28) we have $\gamma = 1/6$ if and only if $\alpha = 1/2$. In particular, we have $\gamma \leq 1/6$ if $\alpha \leq 1/2$, implying

$$\frac{3}{2} - \gamma \geq 1 - 2\gamma$$

and thus (28) and (27) determine the maximum as claimed. In this case we can achieve running time

$$\begin{aligned} & O\left(n^{\left(\frac{3}{2} - \frac{(3-\omega)(1-\alpha)}{12-2(1+\omega)(1+\alpha)}\right)q+c}\right) \\ &= O\left(n^{3q\left(\frac{1}{2} - \frac{(3-\omega)(1-\alpha)}{36-6(1+\omega)(1+\alpha)}\right)+c}\right). \end{aligned}$$

Conversely, if $\alpha \geq 1/2$ then the maximum is controlled by

$$\frac{3}{2} - \gamma = 1 - 2\gamma,$$

in which case we select $\gamma = 1/6$ and achieve running time

$$O\left(n^{\left(\frac{3}{2} - \frac{1}{6}\right)q+c}\right) = O\left(n^{3q\left(\frac{1}{2} - \frac{1}{18}\right)+c}\right).$$

Since the system (16) and its solution (6) are integer-valued and have bit-length bounded by a polynomial in n that is independent of the constant q , for example Bareiss's algorithm [3] solves the constructed system in the claimed running time. This completes the proof of Theorem 1.

3.7. Speedup for $q = 2, 3, 4$. In this section we prove Theorem 2. We split the proof into three parts.

Proof ($q = 2$). Let us study the first family of equations (Lemma 1). For $q = 2$ we have indeterminates x_0, x_2, x_4, x_6 and equations indexed by $i = 0, 1, 2, 3$, where equation i can be constructed in $O(n^{\max(i, q)})$ arithmetic operations; cf. Lemma 3 to Lemma 6. Thus, it suffices to replace the equation for $i = 3$ with an equation independent of the equations $i = 0, 1, 2$ to solve for all the indeterminates, and in particular for x_6 , which gives the weighted disjoint triples. Our strategy is to solve directly for the indeterminate x_0 . We observe that x_0 requires to sum over all triples (A, B, C) of q -subsets such that the q -uniform hypergraph $\{A, B, C\}$ has no vertices of odd degree. Up to isomorphism the only such hypergraph for $q = 2$ is the triangle. From now on we abuse the notation slightly and extend the domains of the functions f, g and h to sets of size *at most* q so that they evaluate to 0 for sets of size strictly smaller than q . Accordingly, we have

$$(29) \quad x_0 = \sum_{1 \leq p, r \leq n} f(\{p, r\}) \sum_{s=1}^n g(\{p, s\}) h(\{r, s\}),$$

where we can evaluate the inner sum simultaneously for all p, r by multiplying two $n \times n$ matrices using $O(n^\omega)$ arithmetic operations. \square

$$\text{I : } \begin{array}{|c|c|c|} \hline p & rst & \\ \hline 1 & 110 & \\ \hline 1 & 101 & \\ \hline 1 & 011 & \\ \hline \end{array}, \quad \text{II : } \begin{array}{|c|c|c|} \hline p & rs & tu \\ \hline 1 & 11 & 00 \\ \hline 0 & 10 & 11 \\ \hline 0 & 01 & 11 \\ \hline \end{array}.$$

FIGURE 3. Two nonisomorphic types of 3-uniform hypergraphs with one vertex of odd degree. We display these hypergraphs below as incidence matrices where the rows correspond to hyperedges and the columns correspond to vertices, with a 1-entry indicating incidence and a 0-entry indicating non-incidence between a hyperedge and a vertex. Vertical and horizontal lines to partition the vertices and the hyperedges to orbits with respect to the action of the automorphism group of the hypergraph.

Proof ($q = 3$). Let us imitate the proof for $q = 2$. For $q = 3$ our indeterminates are x_1, x_3, x_5, x_7, x_9 , and the equations are indexed by $i = 0, 1, 2, 3, 4$. Again it suffices to replace the $i = 4$ equation. We will do this by solving directly for the indeterminate x_1 . For $q = 3$ there are, up to isomorphism, exactly two q -uniform hypergraphs $\{A, B, C\}$ with a unique vertex p of odd degree. In Type I hypergraphs p is of degree 3 and in Type II hypergraphs p is of degree 1. Let $x_{1,\text{I}}$ and $x_{1,\text{II}}$ denote the contribution to x_1 of triples (A, B, C) corresponding to Type I and Type II hypergraphs, respectively. Then $x_1 = x_{1,\text{I}} + x_{1,\text{II}}$.

Note that for every Type I hypergraph the hypergraph $\{A \setminus \{p\}, B \setminus \{p\}, C \setminus \{p\}\}$ is 2-uniform and has no odd vertices. Hence the contribution $x_{1,\text{I},p}$ of Type I triples such that $p \in A \cap B \cap C$ can be computed in time $O(n^\omega)$ by applying the formula (29) to functions $f_p, g_p, h_p : \binom{U}{2} \rightarrow \mathbb{Z}$, where $f_p(X) = f(\{p\} \cup X)$, $g_p(X) = g(\{p\} \cup X)$ and $h_p(X) = h(\{p\} \cup X)$. (Note that we use here the fact that f, g and h evaluate to 0 for sets with less than 3 elements.) Since

$$x_{1,\text{I}} = \sum_{p=1}^n x_{1,\text{I},p},$$

the value $x_{1,\text{I}}$ can be computed in $O(n^{\omega+1})$ time.

Now consider Type II hypergraphs. Let us further partition Type II triples (A, B, C) according to which of the sets A, B, C contains p . Let $z_{1,\text{II}}^{f|g,h}$ denote the contribution to x_1 of Type II triples where $p \in A$. Note that then $z_{1,\text{II}}^{g|f,h}$ and $z_{1,\text{II}}^{h|f,g}$ are the contributions of Type II triples where $p \in B$ and $p \in C$, respectively, and hence

$$(30) \quad x_{1,\text{II}} = z_{1,\text{II}}^{f|g,h} + z_{1,\text{II}}^{g|f,h} + z_{1,\text{II}}^{h|f,g}.$$

Let us focus on $z_{1,\text{II}}^{f|g,h}$, i.e. assume that $A = \{p, r, s\}$ for some $r, s \in U$. Since r and s are of degree 2, either both are in one of the remaining sets, say B , or each of r, s is in exactly one of B and C . However we can assume the latter, because in the former C has at least two degree 1 vertices. So let r be the vertex of $A \cap B$ and let s be the vertex of $A \cap C$. Since the remaining vertices in $B \cup C$ are of degree 2, there are exactly two of them, say, t and u , and $\{t, u\} \in B \cap C$, see Fig 3. It follows that

$$\begin{aligned}
z_{1,\text{II}}^{f|g,h} &= \sum_{1 \leq p,r,s \leq n} \sum_{\substack{1 \leq t < u \leq n \\ p \notin \{t,u\}}} f(\{p,r,s\})g(\{r,t,u\})h(\{s,t,u\}) \\
&= \sum_{1 \leq p,r,s \leq n} f(\{p,r,s\}) \left(\sum_{1 \leq t < u \leq n} g(\{r,t,u\})h(\{s,t,u\}) \right) - \\
&\quad \underbrace{\sum_{1 \leq p,r,s \leq n} \sum_{1 \leq t \leq n} f(\{p,r,s\})g(\{p,r,t\})h(\{p,s,t\})}_{x_{1,\text{I}}}.
\end{aligned}$$

Note that it is sufficient to assume only $p \notin \{t, u\}$; indeed, since f, g and h evaluate to 0 for sets with less than 3 elements any choice of p, r, s, t, u which satisfies this assumption but $|\{p, r, s, t, u\}| < 5$ produces a zero term in the sum. Here the sum $\sum_{1 \leq p,r,s \leq n} f(\{p,r,s\}) \left(\sum_{1 \leq t < u \leq n} g(\{r,t,u\})h(\{s,t,u\}) \right)$ can be evaluated with an $n \times n^2$ by $n^2 \times n$ rectangular matrix multiplication in $O(n^{1+\omega})$ arithmetic operations; cf. (M2). Hence it takes $O(n^{1+\omega})$ time to compute $z_{1,\text{II}}^{f|g,h}$, since we have shown that $x_{1,\text{I}}$ can also be computed within this time bound. \square

Proof ($q = 4$). Let us imitate the proof for $q = 3$. For $q = 4$ our indeterminates are $x_0, x_2, x_4, x_6, x_8, x_{10}, x_{12}$, and the equations are indexed by $i = 0, 1, 2, 3, 4, 5, 6$. It suffices to replace the $i = 5$ and $i = 6$ equations. We will do this by solving directly for the indeterminates x_0 and x_2 , that is, the cases $j = 0$ and $j = 2$ for $j = |A \oplus B \oplus C|$.

The case $j = 0$. For $q = 4$ there is, up to isomorphism, a unique q -uniform hypergraph $\{A, B, C\}$ with no vertex of odd degree:

$$\begin{array}{|c|}
\hline
111100 \\
110011 \\
001111 \\
\hline
\end{array}.$$

Accordingly, we have

$$x_0 = \sum_{1 \leq p < q \leq n} \sum_{1 \leq r < s \leq n} f(\{p,q,r,s\}) \sum_{1 \leq t < u \leq n} g(\{p,q,t,u\})h(\{r,s,t,u\}),$$

where we can evaluate the inner sum simultaneously for all p, q, r, s by multiplying two $n^2 \times n^2$ matrices. This takes $O(n^{2\omega})$ arithmetic operations.

The case $j = 2$. For $q = 4$ we will show that there are, up to isomorphism, exactly four q -uniform hypergraphs $\{A, B, C\}$ with exactly two vertices p, r of odd degree (see Fig 4). For $t \in \{\text{I, II, III, IV}\}$ let $x_{2,t}$ denote the contribution to x_2 of triples (A, B, C) such that the corresponding hypergraph is of type t .

In hypergraphs of Type I both odd degree vertices p and r are of degree 3. Then $\{A \setminus \{p, r\}, B \setminus \{p, r\}, C \setminus \{p, r\}\}$ is 2-uniform and has no odd vertices. Hence the contribution $x_{2,\text{I},p,r}$ of Type I triples such that both p and r are of degree 3 can be computed in time $O(n^\omega)$ by applying the formula (29) to functions $f_{pr}, g_{pr}, h_{pr} : \binom{U}{2} \rightarrow \mathbb{Z}$, where $f_{pr}(X) = f(\{p, r\} \cup X)$, $g_{pr}(X) = g(\{p, r\} \cup X)$ and $h_{pr}(X) = h(\{p, r\} \cup X)$. Since

$$x_{2,\text{I}} = \sum_{1 \leq p < r \leq n} x_{2,\text{I},p,r},$$

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10	1100	1											
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FIGURE 4. Four nonisomorphic 4-uniform hypergraphs with exactly two vertices of odd degree.

the value $x_{2,I}$ can be computed in $O(n^{\omega+2})$ time.

In hypergraphs of Type II there is one vertex p of degree 3 and one vertex r of degree 1. Then $\{A \setminus \{p\}, B \setminus \{p\}, C \setminus \{p\}\}$ is 3-uniform and has exactly one odd degree vertex, in fact a degree 1 vertex. Hence the contribution $x_{2,II,p}$ of Type II triples such that p is of degree 3 can be computed in time $O(n^{\omega+1})$ by applying the formula (30) to functions $f_p, g_p, h_p : \binom{U}{3} \rightarrow \mathbb{Z}$, where $f_p(X) = f(\{p\} \cup X)$, $g_p(X) = g(\{p\} \cup X)$ and $h_p(X) = h(\{p\} \cup X)$. Since

$$x_{2,II} = \sum_{1 \leq p \leq n} x_{2,II,p},$$

the value $x_{2,II}$ can be computed in $O(n^{\omega+2})$ time. Note also that by the same reasoning the contribution $z_{2,II}^{f|g,h}$ of Type II triples (A, B, C) such that the degree 1 vertex belongs to A is equal to $\sum_{1 \leq p \leq n} z_{1,II}^{f_p|g_p,h_p}$, hence it also can be computed in $O(n^{\omega+2})$ time. (We will use this quantity while computing $x_{2,III}$ and $x_{2,IV}$)

In the remaining hypergraphs both p and r are of degree 1, but we have two nonisomorphic types of such hypergraphs. In Type III hypergraphs both p and r are in the same hyperedge. Let $z_{2,III}^{f|g,h}$ denote the contribution to x_2 of Type III triples where $p, r \in A$. Note that then $z_{2,III}^{g|f,h}$ and $z_{2,III}^{h|f,g}$ are the contributions of Type III triples where $p, r \in B$ and $p, r \in C$, respectively, and hence

$$(31) \quad x_{2,III} = z_{2,III}^{f|g,h} + z_{2,III}^{g|f,h} + z_{2,III}^{h|f,g}.$$

We focus on $z_{2,III}^{f|g,h}$, i.e. we assume $A = \{p, r, s, t\}$ for some vertices s and t such that $|A| = 4$. Then since the remaining vertices are all of degree 2, the remaining sets in the triple are $B = \{s, u, v, w\}$ and $C = \{t, u, v, w\}$ for some three different vertices u, v, w outside A . Then,

$$\begin{aligned} z_{2,III}^{f|g,h} &= \sum_{1 \leq p < r \leq n} \sum_{1 \leq s, t \leq n} \sum_{\substack{1 \leq u < v < w \leq n \\ \{p, r\} \cap \{u, v, w\} = \emptyset}} f(\{p, r, s, t\}) g(\{s, u, v, w\}) h(\{t, u, v, w\}) \\ &= \underbrace{\sum_{1 \leq p < r \leq n} \sum_{1 \leq s, t \leq n} f(\{p, r, s, t\}) \left(\sum_{1 \leq u < v < w \leq n} g(\{s, u, v, w\}) h(\{t, u, v, w\}) \right)}_{(*)} - \\ &\quad \underbrace{\sum_{1 \leq p < r \leq n} \sum_{1 \leq s, t \leq n} \sum_{\substack{1 \leq u < v < w \leq n \\ \{p, r\} \cap \{u, v, w\} \neq \emptyset}} f(\{p, r, s, t\}) g(\{s, u, v, w\}) h(\{t, u, v, w\})}_{o^{f|g,h}}, \end{aligned}$$

where $o^{f|g,h}$ is an overcount which we specify in a moment. Note that it is sufficient to assume only $\{p, r\} \cap \{u, v, w\} = \emptyset$; indeed, since f, g and h evaluate to 0 for sets with less than 4 elements any choice of p, r, s, t, u, v, w which satisfies this assumption but $|\{p, r, s, t, u, v, w\}| < 7$ produces a zero term in the sum. Observe also that the sum $\sum_{1 \leq u < v < w \leq n} g(\{s, u, v, w\})h(\{t, u, v, w\})$ can be evaluated for each s, t with an $n \times n^3$ by $n^3 \times n$ rectangular matrix multiplication in $O(n^{2+\omega})$ arithmetic operations; cf. (M2). Once these values are tabulated for every s, t , the sum (*) can be evaluated in $O(n^4) \subseteq O(n^{2+\omega})$ time. It remains to compute the value of overcount $o^{f|g,h}$ efficiently. This can be split as $o^{f|g,h} = o_1^{f|g,h} + o_2^{f|g,h}$, where $o_1^{f|g,h}$ and $o_2^{f|g,h}$ denote the contributions of the terms where $|\{p, r\} \cap \{u, v, w\}| = 1$ and $|\{p, r\} \cap \{u, v, w\}| = 2$, respectively.

Let us focus on $o_1^{f|g,h}$. In these triples either $p \in \{u, v, w\}$ or $r \in \{u, v, w\}$, but not both (recall that $p < r$). Since u, v , and w are symmetric, we can assume that either $p = w$ or $r = w$. In other words either $A = \{p, r, s, t\}$, $B = \{p, s, u, v\}$ and $C = \{p, t, u, v\}$ or $A = \{p, r, s, t\}$, $B = \{r, s, u, v\}$ and $C = \{r, t, u, v\}$. Equivalently, we can drop the assumption $p < r$ and just consider all triples $A = \{p, r, s, t\}$, $B = \{p, s, u, v\}$ and $C = \{p, t, u, v\}$. It follows that

$$o_1^{f|g,h} = \sum_{1 \leq p, r, s, t \leq n} \sum_{\substack{1 \leq u < v \leq n \\ \{p, r\} \cap \{u, v\} = \emptyset}} f(\{p, r, s, t\})g(\{p, s, u, v\})h(\{p, t, u, v\}).$$

Observe that $o_1^{f|g,h}$ is equal to the contribution of Type II triples (A, B, C) such that the unique degree 1 vertex is in A , i.e. $o_1^{f|g,h} = z_{2, \text{II}}^{f|g,h}$, and we know how to compute this value in time $O(n^{\omega+2})$.

Now consider $o_2^{f|g,h}$. In these triples $\{p, r\} \subseteq \{u, v, w\}$. Since u, v and w are symmetric, we can assume $\{p, r\} = \{v, w\}$. It means that we count triples of the form $A = \{p, r, s, t\}$, $B = \{p, r, s, u\}$, and $C = \{p, r, t, u\}$, for every five different vertices $p < r$ and s, t, u . It follows that $o_2^{f|g,h} = x_{2, \text{I}}$, which is computable in $O(n^{\omega+2})$ time.

Finally we focus on Type IV triples, where there are two degree 1 vertices p and r , the remaining vertices are of degree 2 and p and r are in different hyperedges. Let $z_{2, \text{IV}}^{f, g|h}$ denote the contribution to x_2 of Type IV triples where $p, r \in A \cup B$. Note that then $z_{2, \text{IV}}^{f, h|g}$ and $z_{2, \text{IV}}^{g, h|f}$ are the contributions of Type IV triples where $p, r \in A \cup C$ and $p, r \in B \cup C$, respectively, and hence

$$(32) \quad x_{2, \text{IV}} = z_{2, \text{IV}}^{f, g|h} + z_{2, \text{IV}}^{f, h|g} + z_{2, \text{IV}}^{g, h|f}.$$

We focus on $z_{2, \text{IV}}^{f, g|h}$, i.e. we can assume $p \in A$ and $r \in B$. Then $C = \{s, t, u, v\}$ for some four different vertices s, t, u, v different from p and r . Since all vertices except for p and r are of degree 2 from the handshaking lemma the number of vertices k in the hypergraph satisfies $1 \cdot 2 + 2 \cdot (k - 2) = 4 \cdot 3$, and hence $k = 7$, i.e. there is exactly one vertex more, call it w . Since w is of degree 2, $w \in A \cap B$. Since all vertices in C are of degree 2, two of them, say s, t are in A , and the other two, say u, v in B . Then,

$$\begin{aligned}
z_{2,IV}^{f,g|h} &= \sum_{1 \leq p \neq r \leq n} \sum_{1 \leq w \leq n} \sum_{\substack{1 \leq s < t \leq n \\ r \notin \{s,t\}}} \sum_{\substack{1 \leq u < v \leq n \\ p \notin \{u,v\}}} f(\{p, w, s, t\}) g(\{r, w, u, v\}) h(\{s, t, u, v\}) \\
&= \underbrace{\sum_{\substack{1 \leq p, w \leq n \\ p \notin \{u,v\}}} \sum_{\substack{1 \leq u < v \leq n \\ p \notin \{u,v\}}} \left(\sum_{1 \leq s < t \leq n} f(\{p, w, s, t\}) h(\{s, t, u, v\}) \right)}_{(**)} \sum_{\substack{1 \leq r \leq n \\ r \neq p}} g(r, w, u, v) - \\
&\quad \underbrace{\sum_{1 \leq p \neq r \leq n} \sum_{1 \leq w \leq n} \sum_{\substack{1 \leq s < t \leq n \\ r \in \{s,t\}}} \sum_{\substack{1 \leq u < v \leq n \\ p \notin \{u,v\}}} f(\{p, w, s, t\}) g(\{r, w, u, v\}) h(\{s, t, u, v\})}_{o^{f,g|h}},
\end{aligned}$$

where $o^{f,g|h}$ is an overcount which we specify in a moment. Note that by similar arguments as before, it is sufficient to assume only $\{p, r\} \cap \{u, v, w\} = \emptyset$. Observe also that the sum $s_{p,w,u,v} = \sum_{1 \leq s < t \leq n} f(\{p, w, s, t\}) h(\{s, t, u, v\})$ can be evaluated for each p, w, u, v by multiplying two $n^2 \times n^2$ matrices in $O(n^{2\omega})$ arithmetic operations. Moreover, for every u, v, w we compute and store the sum $s_{u,v,w} = \sum_{1 \leq r \leq n} g(r, u, v, w)$; this takes time $O(n^4) \subseteq O(n^{2\omega})$. By noting that $\sum_{\substack{1 \leq r \leq n \\ r \neq p}} g(r, u, v, w) = s_{u,v,w} - g(p, u, v, w)$ it follows that once all the values of $s_{p,w,u,v}$ and $s_{u,v,w}$ are computed the sum $(**)$ can be evaluated in $O(n^4) \subseteq O(n^{2+\omega})$ time. It remains to compute the value of overcount $o^{f,g|h}$ efficiently.

Observe that $o^{f,g|h}$ is equal to the contribution of Type IV triples considered above where $r \in \{s, t\}$, i.e. the triples (A, B, C) such that $A = \{p, r, s, w\}$, $B = \{r, w, u, v\}$ and $C = \{r, s, u, v\}$. These are exactly the Type II triples such that the degree 1 vertex is in A . Hence, $o^{f,g|h} = z_{2,II}^{f|g,h}$, and we know how to compute this value in time $O(n^{\omega+2})$. This finishes the proof of Theorem 2. \square

4. COUNTING THIN SUBGRAPHS IN THREE PARTS

This section proves Theorem 3 by relying on the techniques in §4 of Fomin *et al.* [13] and invoking our Theorem 1 as a subroutine that enables fast counting of injective homomorphisms in three parts. Whereas Fomin *et al.* [13] use the path decomposition to split P into two halves of size roughly $k/2$ joined by a separator of size at most p , we split P into a sequence of three parts of size roughly $k/3$ joined by two separators of size at most p . Accordingly, the following lemma is an immediate analog of Proposition 2 in Fomin *et al.* [13] (cf. [17]).

Lemma 9. *Let P be a graph with k vertices and pathwidth p . Then, we can partition the vertices of P into five pairwise disjoint sets L, S, M, T, R such that (i) $|L|, |M|, |R| \leq k/3$, (ii) $|S|, |T| \leq p$, and (iii) every edge of P joins vertices in one or two consecutive sets in the sequence L, S, M, T, R .*

Imitating the design in §4 of Fomin *et al.* [13], we now iterate over all possible $O(n^{2p})$ guesses φ how an injective homomorphism from P to H can map the elements of the disjoint sets S and T to $V(H)$. For each such guess φ , we use the algorithm in Lemma 2 of Fomin *et al.* [13] to compute for each $A, B, C \subseteq V(H) \setminus (\varphi(S) \cup \varphi(T))$ with $|A|, |B|, |C| \leq k/3$ the following three quantities: (a) the number $f_\varphi(A)$ of injective homomorphisms from $P[L \cup S]$ to $H[A \cup \varphi(S)]$ that

extend φ , (b) the number $g_\varphi(B)$ of injective homomorphisms from $P[S \cup M \cup T]$ to $H[\varphi(S) \cup B \cup \varphi(T)]$ that extend φ , and (c) the number $h_\varphi(C)$ of injective homomorphisms from $P[T \cup R]$ to $H[\varphi(T) \cup C]$ that extend φ . This takes $O(n^{k/3+3p+c})$ time for a constant $c \geq 0$ independent of the constants k and p ; in particular the running-time bottleneck occurs with the functions g_φ where we run an $O(n^{p+c})$ -time algorithm of Díaz *et al.* [10] to compute the number of homomorphisms from $P[S \cup M \cup T]$ to $H[\varphi(S) \cup B \cup \varphi(T)]$ that extend φ for each of the $O(n^{2p+k/3})$ possibilities for $\varphi(S) \cup B \cup \varphi(T)$.

Using the algorithm in Theorem 1 for each guess φ , we obtain the number of injective homomorphisms from P to H as $\sum_\varphi \Delta(f_\varphi, g_\varphi, h_\varphi)$ in time $O(n^{(\frac{1}{2}-\tau)k+2p+c})$. Dividing by the number of automorphisms of P , we obtain the number of subgraphs isomorphic to P in H (cf. [13, Theorem 2]). This completes the proof of Theorem 3.

5. COUNTING SET PACKINGS IN THREE PARTS

This section proves Theorem 4. Let U be the n -element universe and let $\mathcal{F} \subseteq \binom{U}{s}$ be a set of s -element subsets of U given as input. A further input is the integer $t \equiv 0 \pmod{3}$. Our task is to count the number of t -tuples $(S_1, S_2, \dots, S_t) \in \mathcal{F}^t$ that are pairwise disjoint, that is, $S_i \cap S_j = \emptyset$ holds for all $1 \leq i < j \leq t$.

The structure of the proof is to rely on standard dynamic programming techniques to execute the count for pairwise disjoint $t/3$ -tuples, and then invoke the weighted disjoint triples algorithm (Theorem 1) to arrive at the desired count. Let us now proceed with the details.

We start by defining a sequence of functions $f_\ell : \binom{U}{\ell s} \rightarrow \mathbb{Z}$ that we will then compute using dynamic programming. For $\ell = 1, 2, \dots, t/3$ and all $X \in \binom{U}{\ell s}$, let $f_\ell(X)$ be the number of ℓ -tuples $(S_1, S_2, \dots, S_\ell) \in \mathcal{F}^\ell$ that (a) are pairwise disjoint, and (b) satisfy $X = S_1 \cup S_2 \cup \dots \cup S_\ell$.

To set up a base case for the dynamic programming, we observe that $f_1 : \binom{U}{s} \rightarrow \mathbb{Z}$ is the indicator function for the subsets in \mathcal{F} . That is, $f_1(X) = 1$ if and only if $X \in \mathcal{F}$, and $f_1(X) = 0$ otherwise. Since $|\mathcal{F}| \leq \binom{n}{s}$ and s is a constant independent of n , we have that f_1 can be computed in time $O(n^{s+c})$. Next, suppose that we have computed $f_{\ell-1}$ and want to compute f_ℓ . For each $X \in \binom{U}{\ell s}$, we use the following recurrence:

$$f_\ell(X) = \sum_{Y \in \binom{X}{s} \cap \mathcal{F}} f_{\ell-1}(X \setminus Y).$$

To see that the recurrence is correct, observe that for every $(S_1, S_2, \dots, S_\ell) \in \mathcal{F}^\ell$ that is pairwise disjoint with $X = S_1 \cup S_2 \cup \dots \cup S_\ell$ there is a unique $Y \in \mathcal{F} \cap \binom{X}{s}$ such that $(S_1, S_2, \dots, S_{\ell-1}) \in \mathcal{F}^{\ell-1}$ is pairwise disjoint with $X \setminus Y = S_1 \cup S_2 \cup \dots \cup S_{\ell-1}$, namely $Y = S_\ell$. In particular, the left-hand side and the right-hand side of the recurrence count the same ℓ -tuples.

To obtain the running time of the recurrence, observe that we iterate over all $X \in \binom{U}{\ell s}$ and then over all $Y \in \binom{X}{s}$, checking for each Y whether $Y \in \mathcal{F}$. Since both s and t are constants independent of n , also ℓ is a constant independent of n , and the running time bound becomes $O(n^{s\ell+c}(s\ell)^s) = O(n^{s\ell+c})$. In particular, to compute the function $f_{t/3}$ using the recurrence thus takes $O(n^{st/3+c})$ time.

We will now apply Theorem 1. Let us take $q = st/3$ and compute $\Delta(f_{t/3}, f_{t/3}, f_{t/3})$ using the algorithm in Theorem 1. This will take $O(n^{3q(\frac{1}{2}-\tau)+c}) = O(n^{(\frac{1}{2}-\tau)st+c})$, which is exactly the claimed running time.

To complete the proof of Theorem 4, we observe that $\Delta(f_{t/3}, f_{t/3}, f_{t/3})$ is exactly the number of t -tuples of pairwise disjoint subsets from \mathcal{F} , multiplied by the multinomial coefficient $\binom{t}{t/3, t/3, t/3}$. Indeed, each t -tuple $(S_1, S_2, \dots, S_t) \in \mathcal{F}^t$ of pairwise disjoint sets is counted in $\Delta(f_{t/3}, f_{t/3}, f_{t/3})$ exactly $\binom{t}{t/3, t/3, t/3}$ times, once for each possible way of partitioning the index set $\{1, 2, \dots, t\}$ into a three-tuple (I, J, K) with $|I| = |J| = |K| = t/3$ such that $A = \cup_{\ell \in I} S_\ell$, $B = \cup_{\ell \in J} S_\ell$, and $C = \cup_{\ell \in K} S_\ell$; cf. (1). Thus, $\binom{t}{t/3, t/3, t/3}^{-1} \Delta(f_{t/3}, f_{t/3}, f_{t/3})$ is the count we want. This completes the proof of Theorem 4.

6. ON THE HARDNESS OF COUNTING IN DISJOINT PARTS

This section presents two results that provide partial justification why there was an apparent barrier at “meet-in-the-middle time” for counting in disjoint parts.

First, in the case of two disjoint parts, the problem appears to contain no algebraic dependency that one could exploit towards faster algorithms beyond those already presented in Björklund *et al.* [5, 6]. Indeed, we can provide some support towards this intuition by recalling that the associated 2-tensor has full rank over the rationals (Lemma 10).

Second, in the case of three disjoint parts, we have already witnessed (in the proof of Theorem 1) that the associated 3-tensor does not have full rank, in essence because the 3-tensor for matrix multiplication does not have full rank. This prompts the question whether it was *necessary* to rely on fast matrix multiplication to break the barrier. We can provide some support towards an affirmative answer by showing that any *trilinear* algorithm for Δ that breaks the barrier implies a nontrivial algorithm for matrix multiplication (Theorem 5).

6.1. Disjoint pairs. It will be convenient to use Iverson’s bracket notation; for a logical proposition P we have $[P] = 1$ if P is true and $[P] = 0$ if P is false. The (n, k) -*disjointness matrix* is the $\binom{n}{k} \times \binom{n}{k}$ matrix with entries $[A \cap B = \emptyset]$ for all $A, B \in \binom{U}{k}$, $0 \leq k \leq n$. The following lemma is well known.

Lemma 10. *The (n, k) -disjointness matrix has full rank over the rationals.*

Proof. It suffices to show that the matrix is invertible. Observe that

$$\sum_{B \in \binom{U}{k}} [A \cap B = \emptyset] z_{|B \cap C|} = [A = C]$$

holds for all $A, C \in \binom{U}{k}$ when the values z_j for $j = 0, 1, \dots, k$ are the solutions to the $(k+1) \times (k+1)$ linear system with equations

$$\sum_{j=0}^k \binom{i}{j} \binom{n-k-i}{k-j} z_j = [i=0], \quad i = 0, 1, \dots, k.$$

The coefficient matrix of this system is a lower triangular matrix with nonzero entries on the diagonal. Thus, the system is invertible. \square

6.2. Fast disjoint triples implies fast matrix multiplication. Let us prove Theorem 5. For every q we have by assumption a trilinear algorithm of rank r for

computing $\Delta(f, g, h)$ for inputs $f, g, h : \binom{U}{q} \rightarrow \mathbb{Z}$ over an n -element universe U . That is, for every q and n there exist coefficients $\lambda_s, \alpha_A, \beta_B, \gamma_C \in \mathbb{Z}$ such that

$$\Delta(f, g, h) = \sum_{s=1}^r \lambda_s F_s G_s H_s$$

with

$$F_s = \sum_{A \in \binom{U}{q}} \alpha_A f(A), \quad G_s = \sum_{B \in \binom{U}{q}} \beta_B g(B), \quad H_s = \sum_{C \in \binom{U}{q}} \gamma_C h(C).$$

Let us now derive from these trilinear algorithms bilinear algorithms for matrix multiplication. Let us recall from (6) the indeterminates x_j with $j \equiv q \pmod{2}$. In particular, a trilinear algorithm for $\Delta = \Delta(f, g, h)$ is precisely a trilinear algorithm for x_{3q} .

Our proof strategy is to derive a bilinear algorithm for matrix multiplication from a trilinear algorithm for x_0 , and then derive a trilinear algorithm for x_0 from the first family of equations (§2) and the low-rank algorithm for Δ . Finally, we use recursion on the bilinear algorithm to conclude that $\omega \leq 3 - \tau$.

Let P and Q be matrices of size $N \times N$. Without loss of generality we may assume that q is even and that n is divisible by 3, with $N = \binom{n/3}{q/2} = \Omega(n^{q/2})$.

Partition U into three disjoint sets U_1, U_2, U_3 of size $n/3$. Define the function $f_P : \binom{U}{q} \rightarrow R$ for $K_1 \in \binom{U_1}{q/2}$ and $K_2 \in \binom{U_2}{q/2}$ by setting $f_P(K_1 \cup K_2) = P(K_1, K_2)$, and let f_P vanish elsewhere. Similarly, define the function $g_Q : \binom{U}{q} \rightarrow R$ for $K_2 \in \binom{U_2}{q/2}$ and $K_3 \in \binom{U_3}{q/2}$ by setting $g_Q(K_2 \cup K_3) = Q(K_2, K_3)$, and let g_Q vanish elsewhere.

Assume that we have a trilinear algorithm of rank r over the integers that computes x_0 . We claim that we can transform this trilinear algorithm into a bilinear algorithm of rank r that multiplies P and Q . Indeed (cf. [25, §9]), we can fix $f = f_P$ and $g = g_Q$ in the trilinear equations, and for each $K_1 \in \binom{U_1}{q/2}$ and $K_3 \in \binom{U_3}{q/2}$ solve for the indeterminate $h(C)$ with $C = K_1 \cup K_3$ to determine the (K_1, K_3) -entry of the product matrix PQ .

Recalling the first family of linear equations from §2, from the proof of Lemma 6 and the structure of equations (19) and (21) we can observe that the right-hand side y_i of the first family has trilinear rank at most

$$(33) \quad O(n^{(3/2-\gamma)q+c})$$

whenever $0 \leq i \leq (3/2 - \gamma)q$. Thus, using the first family we can show that the trilinear rank of x_0 is small by showing that the indeterminates x_j for large values of j have low trilinear rank.

Towards this end, let us solve for x_j with $j \geq 3q - 2d$ using the trilinear algorithm for Δ as a subroutine. That is, we iterate over all possible choices for the intersecting part of a triple (A, B, C) with $|A \oplus B \oplus C| = j$, and for each such choice use Δ to sum over the disjoint parts to accumulate x_j . Let us now make this more precise. For sets $X, Y \subseteq U$ let us abbreviate $XY = X \cap Y$ and $\bar{X} = U \setminus X$. For $A, B, C \in \binom{U}{q}$, the *intersecting part* of the triple (A, B, C) is the tuple of disjoint sets

$$(34) \quad I(A, B, C) = (A\bar{B}\bar{C}, \bar{A}BC, \bar{A}\bar{B}C, ABC).$$

The *size* of $I(A, B, C)$ is

$$(35) \quad \begin{aligned} |I(A, B, C)| &= |ABC\bar{C}| + |A\bar{B}C| + |\bar{A}BC| + |ABC| \\ &= |AB| + |AC| + |BC| - 2|ABC|. \end{aligned}$$

We have that $j = |A \oplus B \oplus C|$ is large if and only if $|I(A, B, C)|$ is small. In more precise terms, from (35) we have

$$(36) \quad \begin{aligned} j &= |A \oplus B \oplus C| \\ &= |A| + |B| + |C| - 2|AB| - 2|AC| - 2|BC| + 4|ABC| \\ &= 3q - 2|I(A, B, C)|. \end{aligned}$$

The *disjoint part* of (A, B, C) is the tuple of disjoint sets

$$(37) \quad D(A, B, C) = (A\bar{B}\bar{C}, \bar{A}B\bar{C}, \bar{A}\bar{B}C).$$

It is immediate that $I(A, B, C)$ and $D(A, B, C)$ together uniquely determine the triple (A, B, C) .

Now consider an arbitrary triple (A, B, C) with $j = |A \oplus B \oplus C| \geq 3q - 2d$. We know that there is a unique quadruple (I_1, I_2, I_3, I_4) of disjoint sets with $I(A, B, C) = (I_1, I_2, I_3, I_4)$. Furthermore, from (36) and $j \geq 3q - 2d$ it follows that $|I_1| + |I_2| + |I_3| + |I_4| \leq d$. Thus, it suffices to iterate over at most $4^d(d+1)n^d$ quadruples (I_1, I_2, I_3, I_4) to match the intersecting part of (A, B, C) .

So suppose we have fixed (I_1, I_2, I_3, I_4) with $|I_1| + |I_2| + |I_3| + |I_4| \leq d$, and let $I = I_1 \cup I_2 \cup I_3 \cup I_4$. We can now capture each (A, B, C) with $I(A, B, C) = (I_1, I_2, I_3, I_4)$ by means of the disjoint part $D(A, B, C)$. That is, there exists a unique disjoint triple (D_1, D_2, D_3) of subsets $D_1, D_2, D_3 \subseteq \bar{I}$ such that $D(A, B, C) = (D_1, D_2, D_3)$. Furthermore, from (34) and (37) it is immediate that

$$\begin{aligned} |D_1| &= q - |I_1| - |I_2| - |I_4|, \\ |D_2| &= q - |I_1| - |I_3| - |I_4|, \\ |D_3| &= q - |I_2| - |I_3| - |I_4|. \end{aligned}$$

Since the sets D_1, D_2, D_3 in general have size different from q , let us introduce the following padding to obtain a valid input for Δ . Let E_1, E_2, E_3 be three sets that are disjoint from each other and U , with $|E_1| = |I_1| + |I_2| + |I_4|$, $|E_2| = |I_1| + |I_3| + |I_4|$, and $|E_3| = |I_2| + |I_3| + |I_4|$. Let $U' = U \cup E_1 \cup E_2 \cup E_3$ and $n' = n + |E_1| + |E_2| + |E_3| \leq n + 3q$.

We are now ready to construct an input for Δ . Define three functions $f', g', h' : \binom{U'}{q} \rightarrow R$ for all $A', B', C' \in \binom{U'}{q}$ by

$$\begin{aligned} f'(A') &= \begin{cases} f((A' \cap U) \cup I_1 \cup I_2 \cup I_4) & \text{if } A' \cap U \subseteq \bar{I} \text{ and } A' \cap (U' \setminus U) = E_1; \\ 0 & \text{otherwise.} \end{cases} \\ g'(B') &= \begin{cases} g((B' \cap U) \cup I_1 \cup I_3 \cup I_4) & \text{if } B' \cap U \subseteq \bar{I} \text{ and } B' \cap (U' \setminus U) = E_2; \\ 0 & \text{otherwise.} \end{cases} \\ h'(C') &= \begin{cases} h((C' \cap U) \cup I_2 \cup I_3 \cup I_4) & \text{if } C' \cap U \subseteq \bar{I} \text{ and } C' \cap (U' \setminus U) = E_3; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By construction, we now have that to every triple (A, B, C) with $I(A, B, C) = (I_1, I_2, I_3, I_4)$ there corresponds a unique disjoint triple (A', B', C') such that

$$f(A)g(B)h(C) = f'(A')g'(B')h'(C').$$

Taking the sum over all (I_1, I_2, I_3, I_4) , we have

$$x_j = \sum_{\substack{(I_1, I_2, I_3, I_4) \\ |I_1|+|I_2|+|I_3|+|I_4|=(3q-j)/2}} \Delta(f', g', h').$$

Thus, using the trilinear algorithm for Δ for each choice of (I_1, I_2, I_3, I_4) , we can compute x_j for all $3q - 2d \leq j \leq 3q$ with a trilinear algorithm of rank

$$(38) \quad O(4^d(d+1)n^d(n+3q)^{3q(1/2-\tau)+c}) = O(n^{d+3q(1/2-\tau)+c}).$$

Take $d = \lceil \gamma q \rceil$ so that together with equations from the first family (33) we have enough equations to solve for x_0 . It remains to select γ so that (33) and (38) are balanced. We have that (33) and (38) are balanced when

$$(3/2 - \gamma)q = \gamma q + 3q(1/2 - \tau).$$

That is, when $\gamma = 3\tau/2$. We thus have a trilinear algorithm for x_0 that has rank $r = O(n^{(3-\tau)q/2+c})$ for a constant c independent of n and q . That is, we have a bilinear algorithm of rank $r = O(n^{(3-\tau)q/2+c})$ to multiply two $N \times N$ matrices with $N = \Omega(n^{q/2})$. For any constant $\epsilon > 0$ we can now obtain, by selecting a large enough q , a bilinear algorithm of rank $r = O(N^{3-\tau+\epsilon})$ to multiply two $N \times N$ matrices. Taking a large enough N and using recursion (cf. [25, Theorem 2.1]), we conclude that $\omega \leq 3 - \tau + \epsilon$. Since ϵ was arbitrary, $\omega \leq 3 - \tau$.

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REFERENCES

- [1] N. Alon and S. Gutner. Balanced families of perfect hash functions and their applications. *ACM Transactions on Algorithms*, 6(3), 2010.
- [2] O. Amini, F. V. Fomin, and S. Saurabh. Counting subgraphs via homomorphisms. *SIAM J. Discrete Math.*, 26(2):695–717, 2012.
- [3] E. H. Bareiss. Sylvester’s identity and multistep integer-preserving Gaussian elimination. *Math. Comp.*, 22:565–578, 1968.
- [4] A. Björklund. Below all subsets for some permutational counting problems. *CoRR*, abs/1211.0391, 2012.
- [5] A. Björklund, T. Husfeldt, P. Kaski, and M. Koivisto. The fast intersection transform with applications to counting paths. *CoRR*, abs/0809.2489, 2008.
- [6] A. Björklund, T. Husfeldt, P. Kaski, and M. Koivisto. Counting paths and packings in halves. In A. Fiat and P. Sanders, editors, *ESA*, volume 5757 of *Lecture Notes in Computer Science*, pages 578–586. Springer, 2009.
- [7] A. Björklund, T. Husfeldt, P. Kaski, and M. Koivisto. Trimmed Moebius inversion and graphs of bounded degree. *Theory Comput. Syst.*, 47(3):637–654, 2010.
- [8] A. Björklund, T. Husfeldt, P. Kaski, M. Koivisto, J. Nederlof, and P. Parviainen. Fast zeta transforms for lattices with few irreducibles. In Y. Rabani, editor, *SODA*, pages 1436–1444. SIAM, 2012.

- [9] R. Curticapean. Counting matchings of size k is $\#W[1]$ -hard. In F. V. Fomin, R. Freivalds, M. Kwiatkowska, and D. Peleg, editors, *ICALP (1)*, volume 7965 of *Lecture Notes in Computer Science*, pages 352–363. Springer, 2013.
- [10] J. Díaz, M. Serna, and D. M. Thilikos. Counting H -colorings of partial k -trees. *Theoret. Comput. Sci.*, 281(1-2):291–309, 2002. Selected papers in honour of Maurice Nivat.
- [11] F. Eisenbrand and F. Grandoni. On the complexity of fixed parameter clique and dominating set. *Theoret. Comput. Sci.*, 326(1-3):57–67, 2004.
- [12] J. Flum and M. Grohe. The parameterized complexity of counting problems. *SIAM J. Comput.*, 33(4):892–922, 2004.
- [13] F. V. Fomin, D. Lokshtanov, V. Raman, S. Saurabh, and B. V. R. Rao. Faster algorithms for finding and counting subgraphs. *J. Comput. Syst. Sci.*, 78(3):698–706, 2012.
- [14] A. Gajentaan and M. H. Overmars. On a class of $O(n^2)$ problems in computational geometry. *Comput. Geom.*, 45(4):140–152, 2012.
- [15] E. Horowitz and S. Sahni. Computing partitions with applications to the knapsack problem. *J. Assoc. Comput. Mach.*, 21:277–292, 1974.
- [16] A. Itai and M. Rodeh. Finding a minimum circuit in a graph. *SIAM J. Comput.*, 7(4):413–423, 1978.
- [17] N. G. Kinnersley. The vertex separation number of a graph equals its path-width. *Inform. Process. Lett.*, 42(6):345–350, 1992.
- [18] T. Kloks, D. Kratsch, and H. Müller. Finding and counting small induced subgraphs efficiently. *Inform. Process. Lett.*, 74(3-4):115–121, 2000.
- [19] I. Koutis and R. Williams. Limits and applications of group algebras for parameterized problems. In S. Albers, A. Marchetti-Spaccamela, Y. Matias, S. E. Nikolettseas, and W. Thomas, editors, *ICALP (1)*, volume 5555 of *Lecture Notes in Computer Science*, pages 653–664. Springer, 2009.
- [20] M. Kowaluk, A. Lingas, and E.-M. Lundell. Counting and detecting small subgraphs via equations. *SIAM J. Discrete Math.*, 27(2):892–909, 2013.
- [21] F. Le Gall. Faster algorithms for rectangular matrix multiplication. In *FOCS*, pages 514–523. IEEE Computer Society, 2012.
- [22] F. Le Gall. Powers of tensors and fast matrix multiplication. arXiv:1401.7714, 2014.
- [23] G. Lotti and F. Romani. On the asymptotic complexity of rectangular matrix multiplication. *Theoret. Comput. Sci.*, 23(2):171–185, 1983.
- [24] J. Nešetřil and S. Poljak. On the complexity of the subgraph problem. *Comment. Math. Univ. Carolin.*, 26(2):415–419, 1985.
- [25] V. Pan. How can we speed up matrix multiplication? *SIAM Rev.*, 26(3):393–415, 1984.
- [26] V. Vassilevska and R. Williams. Finding, minimizing, and counting weighted subgraphs. In M. Mitzenmacher, editor, *STOC*, pages 455–464. ACM, 2009.
- [27] V. Vassilevska Williams. Multiplying matrices faster than Coppersmith-Winograd. In H. J. Karloff and T. Pitassi, editors, *STOC*, pages 887–898. ACM, 2012.