# List Decodable Learning via Sum of Squares 

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#### Abstract

In the list-decodable learning setup, an overwhelming majority (say a $1-\beta$-fraction) of the input data consists of outliers and the goal of an algorithm is to output a small list $\mathcal{L}$ of hypotheses such that one of them agrees with inliers. We develop a framework for listdecodable learning via the Sum-of-Squares SDP hierarchy and demonstrate it on two basic statistical estimation problems - Linear regression: Suppose we are given labelled examples $\left\{\left(X_{i}, y_{i}\right)\right\}_{i \in[N]}$ containing a subset $S$ of $\beta N$ inliers $\left\{X_{i}\right\}_{i \in S}$ that are drawn i.i.d. from standard Gaussian distribution $N(0, I)$ in $\mathbb{R}^{d}$, where the corresponding labels $y_{i}$ are well-approximated by a linear function $\ell$. We devise an algorithm that outputs a list $\mathcal{L}$ of linear functions such that there exists some $\hat{\ell} \in \mathcal{L}$ that is close to $\ell$. This yields the first algorithm for linear regression in a list-decodable setting. Our results hold for any distribution of examples whose concentration and anticoncentration can be certified by Sum-of-Squares proofs. - Mean Estimation: Given data points $\left\{X_{i}\right\}_{i \in[N]}$ containing a subset $S$ of $\beta N$ inliers $\left\{X_{i}\right\}_{i \in S}$ that are drawn i.i.d. from a Gaussian distribution $N(\mu, I)$ in $\mathbb{R}^{d}$, we devise an algorithm that generates a list $\mathcal{L}$ of means such that there exists $\hat{\mu} \in \mathcal{L}$ close to $\mu$. The recovery guarantees of the algorithm are analogous to the existing algorithms for the problem by Diakonikolas et al. [DKS18] and Kothari et al. [KS17a]. In an independent and concurrent work, Karmalkar et al. [KKK19] also obtain an algorithm for list-decodable linear regression using the Sum-of-Squares SDP hierarchy.


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## 1 Introduction

The presence of outliers in data poses a fundamental challenge to algorithms for high-dimensional statistical estimation. While robust statistics have been explored extensively for several decades now [Hub11], a flurry of recent work starting with [KLS09, ABL14, LRV16, DKK ${ }^{+}$16] have led to new robust algorithms for high-dimensional statistical tasks such as mean estimation, covariance estimation, linear regression and learning linear separators.

More recently, a promising line of work [HL18, KS17b, KS17a, KKM18] has brought to bear the sum-of-squares SDP hierarchy on problems from robust statistics, resulting in new algorithms under fairly minimal assumptions. Continuing this line of inquiry, we further develop the SoS SDP based approach to robust statistics. Specifically, we develop a framework for list-decodable learning via the SoS SDP hierarchy. We demonstrate the framework by devising the first polynomial-time algorithm for linear regression that can extract an underlying linear function even in the presence of an overwhelming majority of outliers.

Linear regression is a corner-stone problem statistics and the underlying optimization problem is perhaps the central example of convex optimization. In the classical setup for linear regression, the input data consists of labelled examples $\left\{\left(X_{i}, y_{i}\right)\right\}_{i \in[N]}$ where $\left\{X_{i}\right\}_{i \in[N]}$ are drawn i.i.d. from a distribution $\mathcal{D}$ over $\mathbb{R}^{d}$, and the labels $\left\{y_{i}\right\}_{i \in[N]}$ are noisy evaluations of a linear function. Specifically, the labels $y_{i}$ are given by $y_{i}=\left\langle\hat{\ell}, X_{i}\right\rangle+\gamma_{i}$ where $\gamma_{i}$ denotes the noise. The goal is to recover an estimate $\ell$ to the linear function $\hat{e} l l$.

In its simplest form, the distribution $\mathcal{D}=N(0$, Id $)$ is the standard Gaussian measure, the noise $\gamma_{i}$ is mean zero and independent of the example $X_{i}$. The linear function eill can be recovered (up to statistical deviations) by minimizing the squared loss namely,

$$
\ell=\operatorname{argmin} \underset{(X, y) \sim \mathcal{D}}{\mathbb{E}}\left[(\langle\ell, X\rangle-y)^{2}\right] .
$$

From an algorithmic standpoint, the realizable setting of linear regression is fairly well understood.
The focus of this work is on algorithms for linear regression that are robust to the presence of outliers. While there is an extensive literature on robust linear regression (see [RL05, BJK15, BJKK17] and the references therein), there are no algorithms that are robust to an overwhelming majority of outliers. Concretely, consider the following problem setup: we are given labelled examples $\left\{\left(X_{i}, y_{i}\right)\right\}_{i \in[N]}$ such that a $\beta$-fraction of these examples are drawn from the underlying distribution, while the remaining ( $1-\beta$ )-fraction of examples are adversarially chosen. Formally, let us suppose $\beta N$ examples are drawn from the distribution with $X \sim N(0$, Id $)$ and $y=\hat{\ell}(X)+\gamma$, while the rest of the examples are arbitrary.

For $\beta<\frac{1}{2}$, it is information theoretically impossible to estimate the linear function $\hat{\ell}$, since the input data can potentially be consistent with $\frac{1}{\beta}$-different linear functions $\ell$. It is natural to ask if an efficient algorithm can recover a small list of candidate linear functions $\mathcal{L}=\left\{\ell_{1}, \ldots, \ell_{t}\right\}$ such that one of them is close to $\ell$. The learning model such as the one above where the goal of the algorithm is to find a small list of candidate hypotheses is referred to as list-decodable learning.

This model was introduced by Balcan et al. [BBV08] in the context of clustering, and has been the subject of a line of work [CSV17, DKS18, SVC16, SKL17, KS17b] in the recent past. The problem of linear regression in the setup of list-decodable learning had remained open.

Our Results. In this work, we use the sum-of-squares SDP hierarchy to devise an efficient algorithm for the list-decodable linear regression problem. Formally, we show the following result.

Theorem 1.1. There is an algorithm $\mathcal{A}$ such that the following holds for every $\beta>0$.
Suppose for a sequence of labelled examples $\left\{\left(X_{i}, y_{i}\right)\right\}_{i \in[N]}$, there exists a linear function $\hat{\ell}(X)=\langle\hat{\ell}, X\rangle$ and a subset $S \subset[N]$ of $\beta N$ examples such that,

1. For any $\varepsilon>0$, the $k^{\text {th }}$ emprical moments of $\left\{X_{i}\right\}_{i \in S}$ are close to that of the underlying distribution of examples for each $k=1 \ldots$ K i.e.,

$$
\left\|\underset{i \in S}{\mathbb{E}} X_{i}^{\otimes k}-\underset{X \sim \mathcal{D}}{\mathbb{E}} X^{\otimes k}\right\|_{2} \leq \varepsilon
$$

for some $K=O\left(1 / \beta^{4}\right)$.
2. The injective tensor norm of the covariates is bounded. That is to say for all degree $D \geq 4$ pseudoexpectations $\tilde{\mathbb{E}}$ over indeterminate $v_{1}, \ldots, v_{d}$ the fourth injective tensor norm

$$
\left.\right|_{4} \underset{X \sim \mathcal{D}}{\mathbb{E}}\langle X, v\rangle^{4} \leq B\|v\|^{4}
$$

is certifiably upper bounded by a constant B. For standard gaussians $B=3$. More generally, any distribution satisfying a poincare inequality has certifiably upper bounded injective tensor norms, see [KS17a].
3. The empirical loss of $\hat{\ell}$ is $(2,4)$-hypercontractive, i.e.,

$$
\underset{i \in S}{\mathbb{E}}\left(y_{i}-\left\langle\hat{\ell}, X_{i}\right\rangle\right)^{4} \leq g \cdot\left(\underset{i \in S}{\mathbb{E}}\left(y_{i}-\left\langle\hat{\ell}, X_{i}\right\rangle\right)^{2}\right)^{2}
$$

for some constant $g$
Then the algorithm $\mathcal{A}$ running on the set of examples $\left\{\left(X_{i}, y_{i}\right)\right\}_{i \in[N]}$ outputs a list of $O\left(\left(\frac{\|\hat{\ell}\|}{\sigma}\right)^{\log 1 / \beta}\right)$ candidate linear functions $\mathcal{L}$ such that there exists $\ell \in \mathcal{L}$ satisfying

$$
\|\ell-\hat{\ell}\|_{2} \leq O\left(\frac{\sigma}{\beta^{3 / 2}}\right)
$$

where $\sigma^{2} \stackrel{\text { def }}{=} \mathbb{E}_{i \in S}\left(y_{i}-\left\langle\hat{\ell}, X_{i}\right\rangle\right)^{2}$. The runtime of the algorithm $\mathcal{A}$ is $\left(\frac{\|\hat{\ell}\|}{\sigma}\right)^{\log 1 / \beta} \cdot N^{O\left(1 / \beta^{4}\right)}$ for $N=d^{O\left(\frac{1}{\beta^{4}}\right)}$.
Even in the absence of outliers, the information theoretic limit on the accuracy $\|\ell-\hat{\ell}\|=\Omega(\sigma)$. To interpret the list size and runtime bounds, consider the setting $\beta=1 / 4,\|\hat{\ell}\|=1$ and noise rate $\sigma=10^{-7}$. In this case, the linear function $\hat{\ell}$ is specified by an arbitrary point in the unit ball, and the algorithm finds a constant-sized list such that one of the points $\ell$ in the ball satisfies $\|\ell-\hat{\ell}\|<0.01$. More generally, $\left(\frac{\|\hat{\ell}\|}{\sigma}\right)^{d}$ would be the size of a $\sigma$-net for the ball of radius $\|\hat{\ell}\|$, but the list size is a fixed polynomial in $\frac{\|\hat{\ell}\|}{\sigma}$.

Our results on linear regression apply to a broader class of probability distributions on examples we term "SoS certifiably anti-concentrated" (see Definition 8.2). Informally, these are probability distributions $\mathcal{D}$ that admit an sum-of-squares proof of their anti-concentration along every direction.

Sum-of-Squares SDPs yield a unified framework for statistical estimation tasks [HL18, KS17b, KS17a, KKM18] through the notion of identifiability proofs. Roughly speaking, if there exists a sum-of-squares proof that the statistical parameter of interest is identifiable from the data, then the sum-of-squares SDP can be utilized to estimate the statistic. In the setting of list decodable learning, the parameter of interest say, the underlying linear function is not uniquely determined by the data, thus breaking the paradigm of SoS proof of identifiability. Alternately, the SoS SDP solution is potentially a convex combination of the different hypotheses on the list. Therefore, a
list-decodable learning algorithm via SoS SDP will have to involve some randomized rounding to isolate one hypotheses from the mixture.

We use the technique of conditioning [BRS11, RT12] to randomly isolate one hypothesis from the SoS SDP solution. More precisely, our algorithm iteratively conditions the SoS SDP solution on specific data points being inliers. The analysis of the algorithm argues that after conditioning on a small number of appropriately chosen data points being inliers, the SoS SDP solution is more or less supported on a unique hypothesis, that we can output.

The framework of rounding by iterative conditioning can also be applied to list-decodable mean estimation problem. In the mean-estimation problem, a $\beta$-fraction of inliers in a set of $N$ data points $\left\{X_{1}, \ldots, X_{N}\right\}$ are sampled from a distribution $\mathcal{D}$. The goal is to recover a list of points $\left\{\hat{\mu}_{1}, \ldots, \hat{\mu}_{t}\right\}$ such that one of them is close to mean of the inliers. Diakonikolas et al. [DKS18] devise an algorithm for the problem when $\mathcal{D}$ is a spherical Gaussian, while Kothari and Steinhardt [KS17b] solve it on a broader class of distributions referred to as SoS-certifiable distributions.

A probability distribution $\mathcal{D}$ is $(2 k, B)$-SoS certifiable if the polynomial inequality $\mathbb{E}_{X \sim \mathcal{D}}[\langle v, X-$ $\left.\mathbb{E}[X]\rangle^{2 k}\right] \leq B^{2 k}\|v\|^{2 k}$ admits a sum-of-squares proof. Similarly, an empirical distribution $\left\{X_{i}\right\}_{i \in S}$ is said to be $(2 k, B)$-SoS certifiable if $\mathbb{E}_{i \in S}\left[\left\langle v, X_{i}-\mathbb{E}\left[X_{i}\right]\right\rangle^{2 k}\right] \leq B^{2 k}\|v\|^{2 k}$ admits a sum-of-squares proof. By applying our framework, we recover an algorithm for list-decodable mean estimation for SoS-certifiable distributions analogous to the work of [KS17b]. Formally, we show the following.

Theorem 1.2. There is an algorithm $\mathcal{A}$ such that the following holds for every $\beta>0$. Suppose for a sequence of labelled examples $\left\{X_{i}\right\}_{i \in[N]}$, there exists a subset $S \subset[N]$ of $\beta N$ examples such that the empirical distribution $\left\{X_{i}\right\}_{i \in S}$ is a $(2 k, B)$-SoS certifiable then the algorithm $\mathcal{A}$ returns a list of $L$ points of length poly $\left(\frac{1}{\beta}\right)$ such that there exists a point $\mu \in \mathcal{L}$ with $\left\|\mu-\mathbb{E}_{i \in S} X_{i}\right\| \leq O\left(\frac{B}{\beta^{1 / k}}\right)$. The runtime of the algorithm is $d^{\operatorname{poly}\left(\frac{1}{\beta}\right)}$

## Related Work.

List Decodable Learning. Balcan et al. [BBV08] introduced the notion of list-decodable learning, specifically, the notion of list-clustering. Charikar et al. [CSV17] formally defined the notions of list-decodable learning and semi-verified learning, and showed that learning problems in the two models reduce to one another. Charikar et al. [CSV17] obtained algorithms for list-decodable learning in the general setting of stochastic convex optimization, and applied the algorithm to a variety of settings including mean estimation, density estimation and planted partition problems (also see [SVC16, SKL17]). The same model of list-decodable learning has been studied for the case of mean estimation [KS17b] and Gaussian mixture learning [KS17a, DKS18].

Linear Regression. Several heuristics have been developed for modifying the ordinary least squares objective with the intent of minimizing the effect of outliers (see [RL05]).

Often, the terminology of âĂIJrobust regressionâĂI is used to refer to a more restricted noise model where only the labels are allowed to be corrupted adversarially (see [NT13a, NT13b, BJK15, BJKK17]). The work of Bhatia et al. [BJKK17] yields an algorithm for regression when the noise introduced is oblivious to the examples, but with a desirable property called consistency, in that the error rate approaches zero with increasing sample size.

There are several works on regression in the more stringent noise models. Balakrishnan et al. [BDLS17] devise algorithms for sparse linear regression in Huber's contamination model. Diakonikolas et al. [DKS19] and Klivans et al. [KKM18] yield algorithms in the most stringent noise models where both the examples and the labels can be arbitrarily corrupted. The latter appeals to SoS SDP relaxations and is applicable to a broad class of distributions under very minimal assumptions. All of the work described above apply at small noise rates, where the total fraction of corruptions are bounded by a small constant.

In a setting where the outliers are an overwhelming majority of the samples, linear regression algorithms have been studied for recovering a sparse vector $x$ [WM10, NT13a, NT13b].

Finally, Hardt and Moitra [HM13] consider a related problem of robust subspace recovery where a fraction of the samples lie within a $d$-dimensional subspace of $\mathbb{R}^{n}$. They devise an algorithm when there are at least $d / n$-fraction of inliers (which corresponds to ( $1-1 / n$ )-fraction of inliers for linear regression). Furthermore, they show that if we make no additional distributional assumptions on the points, then it is computationally intractable to solve robust subspaec recovery with less than $d / n$-fraction of inliers under the Small-Set Expansion Hypothesis.

In an independent and concurrent work, Karmalkar et al. [KKK19] also devise algorithms for list-decodable linear regression using the sum-of-squares SDP hierarchy. The runtime and recovery guarantees of the algorithm are very similar to this work.

## 2 Preliminaries

### 2.1 Proofs to Algorithms: Identifiability and Why it Fails

At a high level, The proofs to algorithms method sets up a system of polynomial equalities and inequalities $\mathcal{P}=\left\{f_{1}(x)=0, f_{2}(x)=0, \ldots, f_{m}(x)=0, g_{1}(x) \geq 0, g_{2}(x) \geq 0, \ldots, g_{n}(x) \geq 0\right\}$ and aims to output a solution $\theta$ to $\mathcal{P}$. Here we think of $\theta$ as a statistical parameter which in our case is either a mean estimate or a hyperplane. In general, this is too much to ask for as the solution set of $\mathcal{P}$ may be nonconvex and admit no discernible structure. The SoS hierarchy is a powerful tool in convex optimization, designed to approximately solve polynomial systems. The hierarchy is parameterized by its 'degree' $k$. The degree corresponds to the size of a Semidefinite Program (SDP) used to solve for solutions to $\mathcal{P}$. The hope is that with higher degree, larger SDP's can obtain sharper approximations to $\theta$. Thus, an immediate hurdle in designing efficient algorithms is to control ' $\mathrm{k}^{\prime}$ with respect to the desired approximation guarantee.

In general, outputting a solution $\theta$ even approximately is still too much to ask for. Instead the SoS algorithm aims to output a fake distribution or 'pseudodistribution' over solutions to $\mathcal{P}$. Furthermore the SoS algorithm returns only the degree up to $k$ moments of the pseudodistribution $\zeta$. That the pseudodistribution is not a true distribution lies at the heart of obtaining computationally efficient algorithms from SoS. Thus, it can be said that pseudodistributions are relaxations of actual probability distributions over the solution set of $\mathcal{P}$. We will defer discussion of pseudodistributions and their dual objects pseudoexpectations to section 2.2.

In the context of unsupervised learning the goal is to estimate a parameter $\theta^{\prime}$ from samples $x_{1}, \ldots, x_{n}$. Identifiability refers to the property that any solution $\theta$ to $\mathcal{P}$ is close to the true parameter $\theta^{\prime}$, i.e $\left\|\theta-\theta^{\prime}\right\| \ll$ small. Furthermore, if this proof of identifiability is captured by a sufficiently simple proof (a low degree SoS) then up to rounding issues $\theta$ can be found efficiently. This paradigm has been immensely successful in designing SoS algorithms in machine learning settings.

A key challenge for the list decoding problem is that even if it were possible to output a true distribution $v$ over solutions to $\mathcal{P}$, another $\beta$ fraction of the dataset can imitate a solution to $\mathcal{P}$. Thus, direct identifiability fails. A natural fix would be to sample from the distribution $v$ in the hopes of finding a region of substantial probability mass around $\theta^{\prime}$. The analogue of sampling on pseudodistributions is 'rounding'. The core technical contribution of this work is developing new techniques for rounding pseudodistributions for high dimensional parameter estimation tasks. Our method 'concentration rounding' has its roots in conditioning SoS SDP's, see [BRS11, RT12]. Next we present some standard tools when working with SoS and some properties of conditional pseudoexpectation.

### 2.2 Sum-of-Squares Toolkit

Sum-of-Squares Proofs. Fix a set of polynomial inequalities $\mathcal{A}=\left\{p_{i}(x) \geq 0\right\}_{i \in[m]}$ in variables $x_{1}, \ldots, x_{n}$. Starting with these "axioms $\mathcal{A}$, a sum-of-squares proof of $q(x) \geq 0$ is given by an
identity of the form,

$$
\left(1+\sum_{k \in\left[m^{\prime}\right]} b_{k}^{2}(x)\right) \cdot q(x)=\sum_{j \in\left[m^{\prime \prime}\right]} s_{j}^{2}(x)+\sum_{i \in[m]} a_{i}^{2}(x) \cdot p_{i}(x),
$$

where $\left\{s_{j}(x)\right\}_{j \in\left[m^{\prime \prime}\right]},\left\{a_{i}(x)\right\}_{i \in[m]},\left\{b_{k}(x)\right\}_{i \in\left[m^{\prime}\right]}$ are real polynomials. It is clear that any identity of the above form manifestly certifies that the polynomial $q(x) \geq 0$, whenever each $p_{i}(x) \geq 0$ for real $x$. The degree of the sum-of-squares proof is the maximum degree of all the summands, i.e., $\max \left\{\operatorname{deg}\left(s_{j}^{2}\right), \operatorname{deg}\left(a_{i}^{2} p_{i}\right)\right\}_{i, j}$.

Sum-of-squares proofs extend naturally to polynomial systems that involve a set of equalities $\left\{r_{i}(x)=0\right\}$ along with a set of inequalities $\left\{p_{i}(x) \geq 0\right\}$. We can extend the definition syntactically by replacing each equality $r_{i}(x)=0$ by a pair of inequalities $r_{i}(x) \geq 0$ and $-r_{i}(x) \geq 0$.

We will the use the notation $\mathcal{A} \left\lvert\, \frac{x}{d}\{q(x) \geq 0\}\right.$ to denote that the assertion that, there exists a degree- $d$ sum-of-squares proof of $q(x) \geq 0$ from the set of axioms $\mathcal{A}$. The superscript $x$ in the notation $\mathcal{A} \left\lvert\, \frac{x}{d}\{q(x) \geq 0\}\right.$ indicates that the sum-of-squares proof is an identity of polynomials where $x$ is the formal variable.

A useful quality of SoS proofs is that they can be composed in the following sense.
Fact 2.1. For polynomial systems $\mathcal{A}$ and $\mathcal{B}$, if $\left.\mathcal{A} \left\lvert\, \frac{\{ }{d} p(x) \geq 0\right.\right\}$ and $\left.\mathcal{B} \left\lvert\, \frac{\{ }{d^{\prime}} q(x) \geq 0\right.\right\}$ then $\mathcal{A} \cup \mathcal{B} \left\lvert\, \frac{\{ }{\max \left(d, d^{\prime}\right)}\right.$ $p(x)+q(x) \geq 0\}$. Also $\left.\left.\mathcal{A} \cup \mathcal{B}\right|_{d d^{\prime}} ^{\{ } p(x) q(x) \geq 0\right\}$

We now turn to pseudoexpectations, the dual object to SoS proofs.

## Pseudoexpectations.

Definition 2.2. Fix a polynomial system $\mathcal{P}$ in $n$ variables $x \in \mathbb{R}^{n}$ consisting of inequalities $\left\{p_{i}(x) \geq\right.$ $0\}_{i \in[m]}$. A degree- $d$ pseudoexpectation $\tilde{\mathbb{E}}: \mathbb{R}[x]_{\leq d} \rightarrow \mathbb{R}$ satisfying $\mathcal{P}$ is a linear functional over polynomials of degree at most $d$ with the properties that $\tilde{\mathbb{E}}[1]=1, \tilde{\mathbb{E}}\left[p(x) a^{2}(x)\right] \geq 0$ for all $p \in \mathcal{P}$ and polynomials $a$ such that $\operatorname{deg}\left(a^{2} \cdot p\right) \leq d$, and $\tilde{\mathbb{E}}\left[q(x)^{2}\right] \geq 0$ whenever $\operatorname{deg}\left(q^{2}\right) \leq d$.

The properties above imply that when $\mathcal{A} \left\lvert\, \frac{x}{d}\{q(x) \geq 0\}\right.$, then if $\tilde{\mathbb{E}}$ is a degree- $d$ pseudoexpectation operator for the polynomial system defined by $\mathcal{A}, \tilde{\mathbb{E}}[q(x)] \geq 0$ as well. This implies that $\tilde{\mathbb{E}}$ satisfies several useful inequalities; for example, the Cauchy-Schwarz inequality. (See e.g. [ $\mathrm{BBH}^{+} 12$ ] for details.)

SoS Algorithm. The degree $D$ moment tensor of a pseudoexpectation $\tilde{\mathbb{E}}_{\zeta}$ is the tensor $\tilde{\mathbb{E}}_{\zeta}\left(1, x_{1}, x_{2}, \ldots, x_{n}\right)^{\otimes D}$. Each entry corresponds to the pseudo-expectation of all monomials of degree at most $D$ in $x$. The set of all degree- $D$ moment tensors of degree $D$ pseudoexpectations is convex, and there's a separation oracle that runs in time $n^{O(D)}$.

Fact 2.3. ([Nes00], [Par00], [Las01], [Sho87]). For any $n, D \in \mathbb{Z}^{+}$, let $\tilde{\mathbb{E}}_{\zeta}$ be degree $D$ pseudoexpectation satisfying a polynomial system $\mathcal{P}$. Then the following set has a $n^{O(D)}$-time weak separation oracle (in the sense of [GLS81]):

$$
\left\{\underset{\zeta}{\tilde{\mathbb{E}}}\left(1, x_{1}, x_{2}, \ldots, x_{n}\right)^{\otimes D} \mid \text { degree } D \text { pseudoexpectations } \underset{\zeta}{\tilde{\mathbb{E}}} \text { satisfying } \mathcal{P}\right\}
$$

Armed with a separation oracle, the ellipsoid algorithm finds a degree $D$ pseudoexpectation in time $n^{O(D)}$, which we call the degree $D$ sum-of-squares algorithm.

Next we present some useful inequalities for working with SoS proofs and pseudoexpectations.

## Useful Inequalities.

Fact 2.4. (Cauchy Schwarz) Let $x_{1}, . ., x_{n}, y_{1}, \ldots, y_{n}$ be indeterminates, than

$$
\overline{4}_{4}\left(\sum_{i \leq n} x_{i} y_{i}\right)^{2} \leq\left(\sum_{i \leq n} x_{i}^{2}\right)\left(\sum_{i \leq n} y_{i}^{2}\right)
$$

Fact 2.5. (Triangle Inequality) Let $x, y$ be $n$-length vectors of indeterminates, then

$$
\sqrt{2}\|x+y\|^{2} \leq 2\|x\|^{2}+2\|y\|^{2}
$$

Fact 2.6. (Moment Bounds) Let $u=\left(u_{1}, \ldots, u_{k}\right)$ be a vector of indeterminants. Let $\mathcal{D}$ be Gaussian with variance proxy 1. Let $t \geq 0$ be an integer. Then we have

$$
\overline{2 t} \underset{x \sim \mathcal{D}}{\mathbb{E}}\langle X, u\rangle^{2 t} \leq(2 t)!\|u\|^{2 t}
$$

Fact 2.7. (Pseudoexpectation Cauchy Schwarz). Let $f(x)$ and $g(x)$ be degree at most $\ell \leq \frac{D}{2}$ polynomial in indeterminate $x$, then

$$
\tilde{\mathbb{E}}[f(x) g(x)]^{2} \leq \tilde{\mathbb{E}}\left[f(x)^{2}\right] \tilde{\mathbb{E}}\left[g(x)^{2}\right]
$$

Fact 2.8. (Pseudoexpectation Holder's) Let p be a degree $\ell$ sum of squares polynomial, $t \in \mathbb{N}$, and $\tilde{\mathbb{E}}$ a degree $O(t \ell)$ pseudoexpectation. Then

$$
\tilde{\mathbb{E}} p(x)^{t-2} \leq\left(\tilde{\mathbb{E}} p(x)^{t}\right)^{\frac{t-2}{t}}
$$

Fact 2.9. (SoS Holder) Let $X_{1}, . ., X_{n}$ and $w_{1}, \ldots, w_{n}$ be indeterminates. Let $q \in \mathbb{N}$ be a power of 2 , then

$$
\left.\left\{w_{i}^{2}=w_{i} \forall i \in[n]\right\}\right|_{O(q)}\left(\sum_{i \leq n} w_{i} X_{i}\right)^{q}\left(\sum_{i \leq n} X_{i}^{q}\right)
$$

and

$$
\left.\left\{w_{i}^{2}=w_{i} \forall i \in[n]\right\}\right|_{O(q)}\left(\sum_{i \leq n} w_{i} X_{i}\right)^{q}\left(\sum_{i \leq n} w_{i} X_{i}^{q}\right)
$$

Fact 2.10. (Spectral Bounds) Let $A \in \mathbb{R}^{d \times d}$ be a positive semidefinite matrix with $\lambda_{\text {max }}$ and $\lambda_{\text {min }}$ being the largest and smallest eigenvalues of $A$ respectively. Let $\tilde{\mathbb{E}}$ be a pseudoexpectation with degree greater than or equal to 2 over indeterminates $v=\left(v_{1}, \ldots, v_{d}\right)$. Then we have

$$
\left.\right|_{2}\left\langle A, v v^{T}\right\rangle \leq \lambda_{\max }\|v\|^{2}
$$

and

$$
\bar{I}_{2}\left\langle A, v v^{T}\right\rangle \geq \lambda_{\text {min }}\|v\|^{2}
$$

Remark 2.11. We will make use of the following notation for $\|\cdot\|_{o p}$ and $\|\cdot\|_{n u c}$ for operator and nuclear norm.

Next we will discuss some useful properties of conditional pseudoexpectation.

## Conditional Pseudoexpectation.

Definition 2.12. (Conditioning) Given a degree $D$ pseudoexpectation operator
$\tilde{\mathbb{E}}: \mathbb{R}\left[x_{1}, \ldots, x_{n}\right]^{\leq D} \rightarrow \mathbb{R}$ and a polynomial $e(x)$ of degree $d<D / 2$, the conditioned pseudoexpectation operator $\left.\tilde{\mathbb{E}}\right|_{e}$ is given by,

$$
\tilde{\mathbb{E}}[s(x) \mid e(x)]=\frac{\tilde{\mathbb{E}}\left[s(x) e^{2}(x)\right]}{\tilde{\mathbb{E}}\left[e^{2}(x)\right]} .
$$

$\left.\tilde{\mathbb{E}}\right|_{e}$ is a degree $D-2 d$ pseudoexpectation functional that satisfies the same polynomial constraints as $\tilde{\mathbb{E}}$.

For a indeterminate $w$ satisfying the boolean constraint $w^{2}=w$, we will use $\tilde{\mathbb{E}}[\cdot \mid w]$ to denote the conditioned functional $\tilde{\mathbb{E}}_{\mid w^{2}}$. Given a degree $D$ pseudo-expectation operator $\tilde{\mathbb{E}}$ satisfies a polynomial system $\mathcal{P}$ all of whose polynomials are of degree at most $d$, for every polynomial $e$ of degree $\leq \frac{D-d}{2}$, the conditioned pseudoexpectation functional $\tilde{\mathbb{E}}[\cdot \mid \ell]$ also satisfies the system $\mathcal{P}$,

For any two polynomials $p, q$ we define the pseudovariance as

$$
\widetilde{\mathbb{V}}[p] \stackrel{\text { def }}{=} \tilde{\mathbb{E}}\left[p^{2}\right]-\tilde{\mathbb{E}}[p]^{2}
$$

and pseudo-covariance as

$$
\widetilde{\operatorname{Cov}}[p, q] \stackrel{\text { def }}{=} \tilde{\mathbb{E}}[p, q]-\tilde{\mathbb{E}}[p] \tilde{\mathbb{E}}[q]
$$

We will also be making extensive use of the conditional pseudoexpectation toolkit.
Fact 2.13. (Conditional Pseudoexpectation Toolkit)
Let $w$ satisfy the boolean constraint $w^{2}=w$. For a random variable $b$ taking values in $\{0,1\}$ such that $\mathbb{P}[b=1]=\tilde{\mathbb{E}}[w]$ and $\mathbb{P}[b=0]=\tilde{\mathbb{E}}[1-w]$ we have the following useful facts

1. (Law of Total Pseudoexpectation) $\mathbb{E}_{b} \tilde{\mathbb{E}}[p(x, w) \mid w=b]=\tilde{\mathbb{E}}[p(x, w)]$
2. (Law of Total Pseudovariance) $\widetilde{\mathbb{V}}[p(x, w)]-\mathbb{E}_{b} \widetilde{\mathbb{V}}[p(x, w) \mid w=b]=\mathbb{V}_{b}[\tilde{\mathbb{E}}[p(x, w) \mid w=b]]$
3. (Correlation Identity) $\tilde{\mathbb{E}}[p(x, w) \mid w=b]=\frac{\widetilde{\operatorname{Cov}}[p(x, w), w]}{\tilde{\mathbf{V}}[w]} b+\left(\tilde{\mathbb{E}}[p(x, w)]-\frac{\widetilde{\operatorname{Cov}}[p(x, w), w]}{\tilde{\mathbf{V}}[w]} \tilde{\mathbb{E}}[w]\right)$

Remark 2.14. (Numerical accuracy). To make our error guarantees precise, we have to discuss issues of bit complexity. The SoS algorithm obtains a degree $D$ pseudoexpectation $\tilde{\mathbb{E}}_{\zeta}$ satisfying polynomial system $\mathcal{P}$ approximately. That is to say, for every $g$ a sum of squares and $p_{1}, \ldots, p_{\ell} \in \mathcal{P}$ with $\operatorname{deg}\left[g \prod p_{i} \leq D\right]$, one has $\tilde{\mathbb{E}} g \prod_{i \in \mathcal{P}} p_{i} \geq-2^{-\Omega(n)}\|g\|$, where $\|g\|$ is $\ell_{2}$ norm of the coefficients of $g$. So long as all of the polynomials involved in our SoS proofs have coefficients bounded by $n^{B}$ for some large constant $B$, then for any polynomial system $Q$ such that $\mathcal{P} \vdash Q$, we have $Q$ satisfied up to error $2^{-\Omega(n)}$.

### 2.3 Organization

In Section 3 we go over the main ideas of list decoding robust regression for the covariate distribution $\mathcal{N}(0, I)$. Then in section Section 4 we wrap our algorithms in an iterative framework for sharp error guarantees. In Section 5 we prove the lemmas relevant to conditioning SoS SDP's. In Section 7 we present the exhibit the proof of concentration rounding. In Section 8 we define anticoncentration and prove that the Gaussian is certifiably anticoncentrated. Subsequently in Section 9 we prove that certifiable anticoncentration is closed under linear transformation, and that spherically symmetric strongly log concave distributions are certifiably anticoncentrated. We defer remaining regression lemmas to the appendix. In particular, we present our proof of list decoding mean estimation to Section C.

## 3 Technique Overview: Robust Regression

In this section we introduce many of the ideas involved in designing our rounding algorithm. We leave sharper error/runtime guarantees and general distributional assumptions to later sections.

Let $N$ be the size of the data set. Let $\beta<\frac{1}{2}$, and let $M=\beta N$. We receive a data set $\mathcal{D}=\left\{\left(X_{i}, y_{i}\right)\right\}_{i=1}^{N}$ where the $X_{i} \in \mathbb{R}^{d}$ are the covariates and the $y_{i} \in \mathbb{R}$ are the labels. Of the covariates, $M$ points are drawn $X_{i} \sim \mathcal{N}(0, I)$. We will refer to these points as "inliers". Furthermore, we introduce boolean variables $w_{1}^{\prime}, \ldots, w_{N}^{\prime} \in\{0,1\}$ indicating if a data point is an inlier, equal to 1 ; outlier, equal to 0 . Let $\ell^{\prime} \in \mathbb{R}^{d}$ be the $\ell_{2}$ minimizer of the error over the inliers i.e

$$
\ell^{\prime} \stackrel{\text { def }}{=} \underset{\ell \in \mathbb{R}^{d}}{\operatorname{argmin}} \frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime}\left(y_{i}-\left\langle\ell, X_{i}\right\rangle\right)^{2}
$$

Let $\sigma^{2}$ be a constant upper bounding the $\ell_{2}$ error

$$
\underset{\ell \in \mathbb{R}^{d}}{\operatorname{argmin}} \frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime}\left(y_{i}-\left\langle\ell^{\prime}, X_{i}\right\rangle\right)^{2} \leq \sigma^{2}
$$

Likewise, let $g$ be a constant such that the $\ell_{4}$ error of $\ell^{\prime}$ is

$$
\underset{\ell \in \mathbb{R}^{d}}{\operatorname{argmin}} \frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime}\left(y_{i}-\left\langle\ell^{\prime}, X_{i}\right\rangle\right)^{4} \leq g \sigma^{4}
$$

In the special case of Gaussian noise $\mathcal{N}\left(0, \sigma^{2}\right)$ we have $g=3$. Then for any $d \in \mathbb{Z}^{+}$sufficiently large, our algorithm recovers a list of estimates $L=\left\{\ell_{1}, \ldots, \ell_{A}\right\}$ for $|L|=O\left(\left(\frac{\rho}{\sigma}\right)^{\log \left(\frac{1}{\beta}\right)}\right)$ such that for some $\ell_{i} \in L$ we have

$$
\left\|\ell_{i}-\ell^{\prime}\right\|_{2} \leq O\left(\frac{\sigma}{\beta^{3 / 2}}\right)
$$

with high probability over the data. If we regard $\frac{\rho}{\sigma}$ as a fixed constant then the list is of length poly $\left(\frac{1}{\beta}\right)$. Furthermore, our algorithm is efficient, running in polynomial time $\operatorname{poly}(d, N)$ where we take $N=d^{O\left(\frac{1}{\beta^{4}}\right)}$. Here we take $N$ to be large enough to certify arbitrary closeness of the $K^{\prime}$ th empirical moments to the distributional moments of the covariates in $\ell_{2}$ norm for $K=O\left(\frac{1}{\beta^{4}}\right)$. That is to say, for any constant $\varepsilon>0$, and for $X_{1}, \ldots, X_{N} \sim \mathcal{N}\left(0, I_{d}\right)$ we have with high probability

$$
\left\|\frac{1}{N} \sum_{i=1}^{N} X_{i}^{\otimes \frac{K}{2}}\left(X_{i}^{\otimes \frac{K}{2}}\right)^{T}-M_{K}\right\|_{F}^{2} \leq \varepsilon
$$

where $M_{K}$ is the empirical $K^{\prime}$ th moment tensor of the spherical Gaussian. For our analyses we take $d>\rho$ and $\varepsilon=O\left(\sigma d^{-K}\right)$. For these settings of $N$ and $d$, and fixing $\rho, \sigma, \beta$ to be constants, we often write $o_{d}(1)$ without ambiguity.

Our approach is to run an SoS SDP, and then round out the list. We begin by describing the Robust Regression SoS SDP.

$$
\begin{align*}
& \underset{\text { degree } \mathrm{D} \text { pseudoexpectations } \tilde{\mathrm{E}}}{\text { minimize }} \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[w_{i}\right]^{2} \\
& \begin{array}{l}
\text { Such that } \tilde{\mathbb{E}} \\
\text { sthe polynomial system }
\end{array}\left(w_{i}^{2}-w_{i}\right)=0, i \in[N], \\
& \sum_{i=1}^{N} w_{i}-M=0, i \in[N], \\
& \left\|\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i}^{\otimes \frac{t}{2}}\left(X_{i}^{\otimes \frac{t}{2}}\right)^{T}-M_{t}\right\|_{F}^{2}-\varepsilon \leq 0, t \in[K]  \tag{3.1}\\
& \left.\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(\left\langle\ell, X_{i}\right\rangle-y_{i}\right)^{4}-g \sigma^{4}\right) \leq 0, \\
& \frac{1}{M} \sum_{i=1}^{N} w_{i}\left(\left\langle\ell, X_{i}\right\rangle-y_{i}\right)^{2}-\sigma^{2} \leq 0, \\
& \left\|\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\ell, X_{i}\right\rangle\right) X_{i}\right\|_{2}^{2}=0, \\
& \|\ell\|^{2}-\rho^{2} \leq 0
\end{align*}
$$

return: $\tilde{\mathbb{E}}_{\zeta}$
Let $\mathcal{P}$ be the set of polynomial constraints of Robust Regression SDP. We elaborate on the meaning of each constraint below, and we will often refer to them in our analyses according to the numbering below.

1. The first constraint $\left\{w_{i}^{2}=w_{i}\right\}$ enforces $w_{i} \in\{0,1\}$ and we refer to it as the booleaness constraint.
2. The next constraint $\left\{\sum_{i=1}^{N} w_{i}-M\right\}$ ensures we select a $\beta$ fraction of the data set.
3. The third constraint ensures that the pseudodistribution is over subsets with moments that match the distribution of the covariates. We refer to them as the moment constraints.
4. The next constraints ensures the error incurred by $\ell$ is small, and we refer to them as the $\ell_{2}$ noise constraint.
5. Similarly, we have a $\ell_{4}$ noise constraint.
6. We have the $\ell_{2}$ minimization constraint, which sets $\ell$ equal to the $\ell_{2}$ minimizer of the selected $w_{i}$.
7. Finally, the scaling constraint restricts the length of $\ell,\|\ell\| \leq \rho^{2}$.

The RobustRegression SDP minimizes a convex objective which we refer to as Frobenius Minimization. This technique first used in the work of Hopkins and Steurer [HS17], ensures that the SDP
solution is a convex combination over every possible solution to the system. This turns out to be crucial. To see why, consider an actual solution $W_{\text {fake }}$ consisting of variables $w_{1}, \ldots, w_{N} \in\{0,1\}$ and $\ell \in \mathbb{R}^{d}$ satisfying $\mathcal{P}$. The distribution that places mass 1 on $W_{\text {fake }}$ and no mass on the clean data is a valid distribution over the solutions to $\mathcal{P}$ and therefore also a valid pseudodistribution. Since we only have assumptions on less than half the data, a malicious $W_{\text {fake }}$ can be planted anywhere confounding our efforts to recover $\ell^{\prime}$. What we need is a way to produce a distribution over solutions to $\mathcal{P}$ that is a convex combination over all the possible solutions. The objective function $\sum_{i=1}^{N} \tilde{\mathbb{E}}\left[w_{i}\right]^{2}$ is a strictly convex function, minimizing which ensures that SDP solution is spread over all solutions to $\mathcal{P}$. More precisely, we have the following guarantee.

Lemma 3.1. (Frobenius minimization $\Longrightarrow$ Correlation) Let $\mathcal{P}$ be a polynomial system in variables $\left\{w_{i}\right\}_{i \in[N]}$ and a set of indeterminates $\left\{\ell_{i}\right\}_{i \in \mathbb{N}}$, that contains the set of inequalities:

$$
w_{i}^{2}=w_{i} \forall i \in[N] \quad \sum_{i} w_{i}=\beta N
$$

Let $\tilde{\mathbb{E}}_{\zeta}: \mathbb{R}\left[\left\{w_{i}\right\}_{i \in[N]},\{\ell\}\right]^{\leq D} \rightarrow \mathbb{R}$ denote a degree $D$ pseudoexpectation that satisfies $\mathcal{P}$ and minimizes the norm $\left\|\tilde{\mathbb{E}}_{\zeta}[w]\right\|$. If $w_{i}^{\prime} \in\{0,1\}$ and $\ell^{\prime}$ is a satisfying assignment to $\mathcal{P}$ then there is correlation with the inliers,

$$
\begin{equation*}
\underset{\zeta}{\tilde{\mathbb{E}}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right] \geq \beta \tag{3.2}
\end{equation*}
$$

We defer the proof of this statement to Section 6.1
Remark: The lemma does not guarantee that $\tilde{\mathbb{E}}_{\zeta}\left[\left(\left\langle w, w^{\prime}\right\rangle-\beta M\right) p(w, \ell)\right] \geq 0$ for all SoS polynomials $p(w, \ell)$ of $\operatorname{deg}(p(w, \ell)) \leq D-1$. That is to say, the guarantees of Lemma 3.1 are only in pseudoexpectation.

At this point we have found a pseudodistribution $\zeta$ satisfying $\mathcal{P}$ that in pseudoexpectation is correlated with the inliers. Pursuing a line of wishful thinking, we would like to sample from this pseudodistribution armed with access to its degree $D$ moments. This is the algorithmic task of rounding the SDP solution, and it is chief intellectual thrust of this work.

For the sake of exposition, let us say that $\tilde{\mathbb{E}}$ corresponds an actual distribution over solutions to the polynomial system. Recall that the goal of the rounding algorithm is to recover the linear function $\ell^{\prime}$. Suppose the variance of $\ell^{\prime}$ as a random variable is sufficiently small, then its expectation $\tilde{\mathbb{E}}[\ell]$ is a good approximation to an actual solution. Formally,
Lemma 3.2. Let $\eta \in\left[0, \frac{1}{2}\right]$ be a small constant. Let $\tilde{\mathbb{E}}_{\zeta}$ be a degree $O\left(\frac{1}{\eta^{4}}\right)$ pseudoexpectation satisfying the constraints of RobustRegression $\operatorname{SDP}(\mathcal{D}, \rho)$. Then if the pseudovariance of the estimator $\tilde{\mathbb{E}}_{\zeta}[\ell]$ is small in every direction

$$
\begin{equation*}
\max _{u} \widetilde{\mathbb{V}}_{\zeta}[\langle\ell, u\rangle] \leq \eta \rho^{2} \tag{3.3}
\end{equation*}
$$

and there is correlation with the inliers

$$
\begin{equation*}
\underset{\zeta}{\underset{\mathrm{E}}{\tilde{E}}}\left[\frac{1}{M} \sum_{i} w_{i} w_{i}^{\prime}\right] \geq \beta \tag{3.4}
\end{equation*}
$$

then our estimator satisfies,

$$
\begin{equation*}
\left\|\tilde{\mathbb{E}}[\ell]-\ell^{\prime}\right\| \leq \sqrt{\frac{\eta \rho^{2}+O\left(\frac{\sigma^{2}}{\eta^{2}}\right)}{\beta}} \tag{3.5}
\end{equation*}
$$

In particular, for $\eta=\frac{\beta}{8}$ and $\rho^{2}>\Omega\left(\frac{\sigma^{2}}{\beta^{3}}\right)$ the degree $O\left(\frac{1}{\beta^{4}}\right)$ pseudoexpectation satisfies

$$
\left\|\underset{\zeta}{\tilde{E}}[\ell]-\ell^{\prime}\right\| \leq \frac{\rho}{2}
$$

Provided we can take the pseudovariance down in every direction, the error guarantee 'contracts' from the trivial $\rho$ to $\frac{\rho}{2}$. It is then possible to iterate such a contraction procedure to achieve optimal error guarantees which is the subject of Section 4 . Without going into details, the proof of Lemma 3.2 critically relies on both the concentration and anticoncentration of the covariates. For instance, if the covariates were drawn from a degenerate point distribution at the origin, then nothing can be inferred about $\ell^{\prime}$. In this sense, concentration is insufficient to recover $\ell^{\prime}$ meaningfully. To overcome this hurdle, we formalize what it means for a distribution to be SoS certifiably anticoncentrated.

Certifiable Anticoncentration As will become clear in Section 8, the smaller $\eta$ is, the harder it is for SoS to certify the bounds in the above lemma. For purposes of anticoncentration, $\eta$ is a parameter representing an interval about the origin. For any distribution $\mathcal{D}$, we think of $\mathcal{D}$ as being anticoncentrated if the mass of $\mathcal{D}$ falling within the $\eta$ interval is small. For example, in the case of $\mathcal{D}=\mathcal{N}(0,1)$, the mass within the $\eta$ interval is upper bounded by $\frac{\eta}{\sqrt{2 \pi}}$. Characterizing this "anticoncentration" of $\mathcal{D}$ about the origin becomes increasingly difficult (higher degree) as $\eta$ falls, intuitively, because it requires a finer grained picture of the distribution $\mathcal{D}$. It turns out the $\mathcal{N}\left(0, I_{d}\right)$ is $O\left(\frac{1}{\eta^{4}}\right)$ SoS certifiably anticoncentrated the proof of which is detailed in Section ??. That this proof is independent of the dimension $d$, along with the rounding algorithm, is what enables the list decoding to run in polynomial time.

Now we move on to the actual statement of Lemma 3.2. In general, the variance of the SDP solution will not be small. Thus, we will iteratively reduce the variance by conditioning on the $w_{i}$ variables. Intuitively, we are conditioning on specific data points $\left(X_{i}, y_{i}\right)$ being part of the inliers ( $w_{i}=1$ ) or being part of the outliers $\left(w_{i}=0\right)$.

Towards these ends, let $\tilde{\mathbb{E}}_{1}, \tilde{\mathbb{E}}_{2}, \ldots, \tilde{\mathbb{E}}_{R}$ be a sequence of pseudoexpectations where $\tilde{\mathbb{E}}_{1}$ is the output of RobustRegressionSDP $(\mathcal{D}, \rho)$. We want to define an algorithm to update $\tilde{\mathbb{E}}_{t}$ to $\tilde{\mathbb{E}}_{t+1}$ where $\max _{u} \widetilde{\mathbb{V}}_{t+1}[\langle\ell, u\rangle]<\max _{u} \widetilde{\mathbb{V}}_{t}[\langle\ell, u\rangle]$ so that eventually $\tilde{\mathbb{E}}_{R}<\eta \rho^{2}$.
let $Q_{t}$ be the pseudocovariance matrix defined

$$
Q_{t}={\underset{\mathbb{E}}{t}}_{\tilde{E}}\left[(\ell-\underset{t}{\tilde{\mathbb{E}}}[\ell])(\ell-\underset{t}{\tilde{\mathbb{E}}}[\ell])^{T}\right]
$$

We have $\left\|Q_{t}\right\|_{o p}=\max _{u} \widetilde{\mathbb{V}}_{t}[\langle\ell, u\rangle]$. Let's say we have a strategy $\mathcal{S}$ for selecting a $w_{j} \in w_{1}, \ldots, w_{N}$, and then apply the following update

$$
\underset{t+1}{\underset{\mathbb{E}}{ }}= \begin{cases}\left.\tilde{\mathbb{E}}_{t}\right|_{w_{j}=1} & \text { with probability } \tilde{\mathbb{E}}_{t}\left[w_{j}\right] \\ \left.\tilde{\mathbb{E}}_{t}\right|_{w_{j}=0} & \text { with probability } \tilde{\mathbb{E}}_{t}\left[1-w_{j}\right]\end{cases}
$$

Let $b_{j}$ be a $\{0,1\}$ random variable satisfying $\mathbb{P}\left[b_{j}=1\right]=\tilde{\mathbb{E}}_{t}\left[w_{j}\right]$. We wish to argue that there is a large expected decrease in the direction of largest variance.

$$
\left\|Q_{t}\right\|_{\text {op }}-\underset{\mathcal{S}}{\mathbb{E}} \mathbb{E}\left\|\left.Q_{b_{j}}\right|_{w_{j}=b_{j}}\right\|_{\text {op }}>\text { large }
$$

Unfortunately, controlling $\|Q\|_{\text {op }}$ is difficult. We will instead control $\|Q\|_{\text {nuc }}$, i.e trace norm, and prove

$$
\|Q\|_{\text {nuc }}-\underset{\mathcal{S}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}}\left\|\left.Q\right|_{w_{j}=b_{j}}\right\|_{\text {nuc }} \geq \Omega\left(\beta^{3} \rho^{2}\right)
$$

For the strategy $\mathcal{S}$ defined below
Lemma 3.3. (Variance Decrease Strategy) Let $\tilde{\mathbb{E}}_{\zeta}$ satisfy the pseudoexpectation constraints of RobustRegression $\operatorname{SDP}(\mathcal{D}, \rho)$. Let $Q$ be the associated pseudocovariance matrix. Let $v$ be any direction in the unit ball $S^{d-1}$. Let $\mathcal{S}_{v}$ be a probability distribution over $[N]$ where for any $j \in N$ we have

$$
\boldsymbol{S}_{v}(j) \stackrel{\text { def }}{=} \frac{\widetilde{\mathbb{V}}_{\zeta}\left[w_{j}\left(y_{j}-\tilde{\mathbb{E}}_{\zeta}\left[\left\langle\ell, X_{i}\right\rangle\right]\right)\left\langle X_{j}, v\right\rangle\right]}{\sum_{i=1}^{N} \widetilde{\mathbb{V}}_{\zeta}\left[w_{j}\left(y_{j}-\tilde{\mathbb{E}}_{\zeta}\left[\left\langle\ell, X_{i}\right\rangle\right]\right)\left\langle X_{j}, v\right\rangle\right]}
$$

Then for $M_{4}$ being the fourth moment matrix of the Gaussian defined in RobustRegressionSDP, and for $\|Q\|_{\text {nuc }}>\sigma^{2} \sqrt{g}$ we have

$$
\widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]-\underset{j \sim S_{v}}{\mathbb{E}} \mathbb{E} \widetilde{\mathbb{V}}_{\zeta}\left[\langle\ell, v\rangle \mid w_{j}=b_{j}\right] \geq \Omega\left(\frac{\beta \widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]^{2}}{\|Q\|_{n u c}}\right)
$$

The above lemma allows the rounding algorithm to decrease the variance along a single direction, thereby decreasing the nuclear norm as follows.

Corollary 3.4. (Connecting variance decrease strategy to nuclear norm rounding) For $(\lambda, v)$ being the largest eigenvalue/vector pair of $Q$, and $\mathcal{S}_{v}$ defined in Lemma 3.3. Let $\gamma>0$ be a constant. If $\|Q\|_{o p} \geq \gamma$ and $\|Q\|_{n u c}>\sigma^{2} \sqrt{g}$, then

$$
\|Q\|_{n u c}-\underset{j \sim \sim_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}}\left\|\left.Q\right|_{w_{j}=b_{j}}\right\|_{n u c} \geq \Omega\left(\frac{\beta \gamma^{2}}{\rho^{2}}\right)
$$

In particular, for $\gamma=\eta \rho^{2}$, we have

$$
\|Q\|_{n u c}-\underset{j \sim \mathcal{S}_{v}}{\mathbb{E}} \mathbb{E}\left\|\left.Q\right|_{w_{j}=b_{j}}\right\|_{n u c} \geq \Omega\left(\beta \eta^{2} \rho^{2}\right)
$$

The corollary establishes a win-win. Either $\|Q\|_{o p}<\gamma$ in which case the variance of our estimator is small in every direction, or we can round and decrease an upper bound on $\|Q\|_{\text {op }}$. We defer the proof of Lemma 3.3 to Section 7 and the proof of Corollary 3.4 to section Section 5.

Taken together, the conditioning strategy iteratively chases the variance down by selecting the direction of largest variance in the pseudocovariance of our estimator, and conditions on the $w_{j}$ exhibiting the largest scaled variance.

We are now ready to state our main algorithm and prove the main result of this section

```
Algorithm 2: Regression Rounding Algorithm
    Result: a \(d\) dimensional hyperplane
    Inputs: \(\left(\tilde{\mathbb{E}}_{1}, \rho\right)\) The output of RobustRegressionSDP and the scaling parameter
    for \(t=1: O\left(\frac{1}{\eta^{2} \beta^{2}}\right)\) do
        Let \(Q_{t}=\tilde{\mathbb{E}}_{t}\left[\left(\ell-\tilde{\mathbb{E}}_{t}[\ell]\right)\left(\ell-\tilde{\mathbb{E}}_{t}[\ell]\right)^{T}\right]\) be the pseudocovariance matrix of the estimator \(\ell\)
        Let \((\lambda, v)\) be top eigenvalue/vector of \(Q_{t}\)
        if \(\lambda>\eta \rho^{2}\) then
            Let \(\mathcal{S}_{v}\) be a probability distribution over \([N]\);
            Where for any \(j \in N\) we have \(\mathcal{S}_{v}(j) \stackrel{\text { def }}{=} \frac{\widetilde{\mathbb{V}}_{t}\left[w_{j}\left(y_{j}-\tilde{\mathbb{E}}_{t}\left[\left\langle\ell, X_{i}\right\rangle\right]\right)\left\langle X_{j}, v\right\rangle\right]}{\sum_{i=1}^{N} \widetilde{\mathbb{V}}_{t}\left[w_{j}\left(y_{j}-\tilde{\mathbb{E}}_{h}\left[\left\langle\ell, X_{i}\right\rangle\right]\right)\left\langle X_{j}, v\right\rangle\right]}\)
            Sample \(j \sim \mathcal{S}_{v}\)
            Sample \(b_{j} \in \operatorname{Bern}\left(\tilde{\mathbb{E}}_{t}\left[w_{j}\right]\right)\)
            Let \(\tilde{\mathbb{E}}_{t+1}=\left.\tilde{\mathbb{E}}_{t}\right|_{w_{j}=b_{j}}\)
        else
            return: \(\tilde{\mathbb{E}}_{t}[\ell]\)
        end
    end
```

Theorem 3.5. Let $\ell^{\prime}$ be a solution to the constraints of RobustRegressionSDP. Let $\eta$ be a constant greater than 0 . Let $\tilde{\mathbb{E}}_{\zeta}$ be the output of RobustRegressionSDP $(\mathcal{D}, \rho)$ for degree $D=O\left(\max \left(\frac{1}{\beta^{2} \eta^{2}}, \frac{1}{\eta^{4}}\right)\right)$. Then after $R=O\left(\frac{1}{\beta^{2} \eta^{2}}\right)$ rounds of updates according to the strategy $\mathcal{S}$ in Lemma 3.3, the resulting pseudoexpectation, which we denote $\tilde{\mathbb{E}}_{R}$, satisfies

$$
\left\|\underset{R}{\tilde{\mathbb{E}}}[\ell]-\ell^{\prime}\right\| \leq \sqrt{\frac{\eta \rho^{2}+O\left(\frac{\sigma^{2}}{\eta^{2}}\right)}{\beta}}
$$

with probability greater than $\Omega(\beta)$ over the randomness in the algorithm. In particular for $\eta=\Omega(\beta)$ and for $\rho^{2} \geq \Omega\left(\frac{\sigma^{2}}{\beta^{3}}\right)$, the degree $D=O\left(\frac{1}{\beta^{4}}\right)$ pseudoexpectation satisfies

$$
\left\|\underset{R}{\tilde{\mathbb{E}}}[\ell]-\ell^{\prime}\right\| \leq \frac{\rho}{2}
$$

## Remark:

As stated, Theorem 3.5 takes down the error guarantee to $\frac{\rho}{2}$ and is not yet an iterative algorithm that obtains the optimal error guarantees, yet it contains most of the elements of the full algorithm. Issues concerning iteration are the subject of the next section on algorithms.
Proof. We now have all the tools to prove Theorem 3.5. By frobenius minimization Lemma 3.1 we have,

$$
\underset{\zeta}{\underset{\zeta}{\mathrm{E}}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right] \geq \beta
$$

Now we show that after $R$ rounds of conditioning,

$$
\max _{u \in \mathcal{S}^{d-1}} \widetilde{\mathbb{V}}_{\zeta, R}[\langle\ell, u\rangle] \leq \eta \rho^{2}
$$

To apply Lemma 3.2 we iteratively round according to $\mathcal{S}_{v}$ in Lemma 3.3 to decrease $\|Q\|_{n u c}$. For $\|Q\|_{o p}>\eta \rho^{2}$, Corollary 3.4 gives us

$$
\|Q\|_{\text {nuc }}-\underset{j \sim s_{v}}{\mathbb{E}} \mathbb{E}\left\|\left.Q\right|_{w_{j}=b_{j}}\right\|_{\text {nuc }} \geq \Omega\left(\beta \eta^{2} \rho^{2}\right)
$$

We aim to show that after $R=O\left(\frac{1}{\beta^{2} \eta^{2}}\right)$ iterations, the algorithm outputs $\left\|Q_{R}\right\|_{\text {nuc }} \leq \eta \rho^{2}$ with probability greater than $1-\frac{\beta}{4}$ over the randomness in the selection strategy and 0,1 conditionings. We denote the probability and expectation over the randomness in the algorithm $\mathbb{P}_{\mathcal{A}}[\cdot]$ and $\mathbb{E}_{\mathcal{A}}[\cdot]$ respectively. Thus, to prove the following

$$
\underset{\mathcal{A}}{\mathbb{P}}\left[\left\|Q_{R}\right\|_{o p} \leq \eta \rho^{2}\right] \geq 1-\frac{\beta}{4}
$$

we proceed by contradiction. Suppose that at each iteration of $t=1,2, \ldots, R$, that $\left\|Q_{t}\right\|_{\text {op }}>\eta \rho^{2}$ with probability greater than $\frac{\beta}{4}$. Then in expectation over $\mathcal{S}$ we have that each round of conditioning decreases $\|Q\|_{\text {nuc }}$ by $\frac{\beta}{4}$ (the probability that the assumption in 3.6 holds) times $\Omega\left(\beta \eta^{2} \rho^{2}\right)$ (the expected decrease in 3.6). Thus,

$$
\begin{equation*}
\underset{\mathcal{A}}{\mathbb{E}}\left[\left\|Q_{t}\right\|_{\text {nuc }}\right]-\underset{\mathcal{A}}{\mathbb{E}}\left[\left\|Q_{t+1}\right\|_{\text {nuc }}\right] \geq \underset{\mathcal{A}}{\mathbb{P}}\left[\left\|Q_{t}\right\|_{\text {op }} \geq \eta \rho^{2}\right] \cdot \Omega\left(\beta \eta^{2} \rho^{2}\right) \geq \Omega\left(\beta^{2} \eta^{2} \rho^{2}\right) \tag{3.6}
\end{equation*}
$$

We also know that the initial pseudocovariance is upper bounded in nuclear norm i.e

$$
\begin{equation*}
\left\|Q_{\zeta}\right\|_{\text {nuc }}=\underset{\zeta}{\tilde{E}}\left[\|\ell-\underset{\zeta}{\tilde{\mathbb{E}}}[\ell]\|^{2}\right]=\underset{\zeta}{\tilde{\mathbb{E}}}\left[\|\ell\|^{2}\right]-\underset{\zeta}{\tilde{\mathbb{E}}}[\|\ell\|]^{2} \leq \underset{\zeta}{\tilde{\mathbb{E}}}\left[\|\ell\|^{2}\right] \leq \rho^{2} \tag{3.7}
\end{equation*}
$$

Where the last inequality is an application of the scaling constraint (7). Thus, putting together 3.6 and 3.7 after $R=O\left(\frac{\rho^{2}}{\beta^{2} \eta^{2} \rho^{2}}\right)=O\left(\frac{1}{\beta^{2} \eta^{2}}\right)$ iterations, $\mathbb{E}_{\mathcal{A}}\left[\left\|Q_{R}\right\|_{\text {nuc }}\right] \leq 0$ which is impossible because $\left\|Q_{R}\right\|_{\text {nuc }}=\tilde{\mathbb{E}}_{R}\left[\left\|\ell-\tilde{\mathbb{E}}_{R}[\ell]\right\|^{2}\right] \geq 0$. Thus our assumption is false, and $\mathbb{P}_{\mathcal{A}}\left[\left\|Q_{R}\right\|_{o p} \leq \eta \rho^{2}\right] \geq 1-\frac{\beta}{4}$ as desired.

We also know by the law of total pseudoexpectation that in expectation over the selection strategy and 0,1 conditionings,

$$
\underset{\mathcal{A}}{\mathbb{E}}\left[\underset{R}{\tilde{\mathbb{E}}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]\right]=\underset{\zeta}{\tilde{\mathbb{E}}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right] \geq \beta
$$

Note that this is a generic fact that is true regardless of which conditioning strategy we choose. Thus by Markov for random variables taking values in $[0,1]$ we have

$$
\underset{\mathcal{A}}{\mathbb{P}}\left[\underset{R}{\tilde{\mathbb{E}}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right] \geq \frac{\beta}{2}\right] \geq \frac{\beta}{2}
$$

Now that we know $\mathbb{P}_{\mathcal{A}}\left[\left\|Q_{R}\right\|_{o p} \leq \eta \rho^{2}\right] \geq 1-\frac{\beta}{4}$ and $\mathbb{P}_{\mathcal{A}}\left[\tilde{\mathbb{E}}_{R}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right] \geq \frac{\beta}{2}\right] \geq \frac{\beta}{2}$, we conclude via union bound that the failure probability of both events is upper bounded by $\frac{\beta}{4}+1-\frac{\beta}{2}=1-\frac{\beta}{4}$. Thus the conditions of Lemma 3.2 are satisfied with probability greater than $\frac{\beta}{4}$ in which case

$$
\left\|\frac{\tilde{\mathbb{E}}}{R}[\ell]-\ell^{\prime}\right\| \leq \sqrt{\frac{\eta \rho^{2}+O\left(\frac{\sigma^{2}}{\eta^{2}}\right)}{\beta}}
$$

In particular for $\eta=\Omega(\beta)$ and for $\rho^{2} \geq \Omega\left(\frac{\sigma^{2}}{\beta^{3}}\right)$ we have

$$
\left\|\underset{R}{\tilde{E}}[\ell]-\ell^{\prime}\right\| \leq \frac{\rho}{2}
$$

Lemma 3.6. Running Algorithm Algorithm 2 a total of $O\left(\frac{1}{\beta}\right)$ times produces a list $L=\left\{\ell_{1}, \ldots, \ell_{O\left(\frac{1}{\beta}\right)}\right\}$ such that with probability $1-c$, there exists a list element $\ell_{i} \in L$ satisfying $\left\|\ell_{i}-\ell^{\prime}\right\| \leq \frac{\rho}{2}$ where $c$ is a small constant. Minor modifications enable the algorithm to succeed with high probability.

We defer the modifications required to succeed with high probability to the appendix. We proceed under the assumption that Algorithm 2 outputs a list $L$ satisfying the guarantees in Lemma 3.6 with high probability. For variety, we present the mean estimation algorithms with these modifications in place.

## 4 Iterative Contraction for Sharp Rates

The Regression Rounding Algorithm 2 generates a list $L$ which contracts the error guarantee from $\rho$ to $\frac{\rho}{2}$ with high probability. In this section we wrap the algorithm in an iterative framework to obtain sharp error guarantees.

Our iterative framework, ListDecodeRegression Algorithm 3, iterates over the list $L$ generated by RoundingRobustRegression Algorithm 2, and uses the list elements to shift the data so as to obtain sharper estimates. This will involve rerunning both RobustRegressionSDP Algorithm 4 and RoundingRobustRegression Algorithm 2. Formally, for each $\ell_{i} \in L$ create a new dataset $\left\{\left(X_{j}, y_{j}^{\prime}\right)\right\}_{j=1}^{N}$ with the same covariates with shifted labels $y_{1}^{\prime}, \ldots, y_{N}^{\prime}$. The labels are shifted according to the hyperplane $\ell_{i}$ as follows, $y_{j}^{\prime}:=y_{j}-\left\langle\ell_{i}, X_{j}\right\rangle$ for all $j \in[N]$. Then the scaling constraint $\left\{\|\ell\|^{2} \leq \frac{\rho^{2}}{2}\right\}$ is added to the RobustRegressionSDP, and we resolve the SDP and rerun the rounding. Iterating this procedure, the error guarantee contracts each iteration from $\frac{\rho}{2}, \frac{\rho}{4}, \ldots$ so on and so forth, whilst the list length increases multiplicatively by factors of $O\left(\frac{1}{\beta}\right)$ until the ubiquitous assumption $\rho^{2} \geq \Omega\left(\frac{\sigma^{2}}{\beta^{3}}\right)$ no longer holds and we are left with the sharp error guarantee $\left\|\ell_{i}-\ell^{\prime}\right\| \leq O\left(\frac{\sigma}{\beta^{3 / 2}}\right)$ for
some list element $\ell_{i} \in L$. The following theorem formalizes the discussion above.

```
Algorithm 3: ListDecodeRegression
    Result: A list of hyperplanes \(L=\left\{\ell_{1}, \ldots, \ell_{A}\right\}\)
    inputs: \((\mathcal{D}, \rho)\)
    \(L=\{0\}\)
    for \(t \in \log _{2}\left(\frac{\rho \beta^{3 / 2}}{\sigma}\right)\) do
        \(\rho_{t}=\frac{\rho}{2^{t}}\)
        \(\%\) Let \(Y\) to be a list of pseudoexpectations
        \(Y=\emptyset\)
        for \(\ell_{i} \in L\) do
            \((X, y)=\mathcal{D}\)
            for \(y_{j} \in \mathcal{Y}\) do
                \(y_{j}=y_{j}-\left\langle\ell_{i}, X_{i}\right\rangle\)
            end
            Let \(Y=Y \cup\) RobustRegressionSDP \(\left(\mathcal{D}, \rho_{t}\right)\)
        end
        \(L=\emptyset\)
        for \(\tilde{\mathbb{E}}_{\zeta} \in Y\) do
            \(L^{\prime}=\) RegressionRounding \(\left(\tilde{\mathbb{E}}_{\zeta}, p_{t}\right)\)
            \(L=L \cup L^{\prime}\)
        end
    end
    return: L
```

Theorem 4.1. ListDecodeRegression $(\mathcal{D}, \rho)$ outputs a list of hyperplanes $L=\left\{\ell_{1}, \ldots, \ell_{A}\right\}$ where $A=$ $O\left(\left(\frac{1}{\beta}\right)^{\log \left(\frac{\beta^{3} / 2 \rho}{\sigma}\right)}\right)$ such that for some $\ell_{i} \in L$

$$
\left\|\ell_{i}-\ell^{\prime}\right\| \leq O\left(\frac{\sigma}{\beta^{3 / 2}}\right)
$$

with high probability in time $\left(\frac{\rho}{\sigma}\right)^{\log (1 / \beta)} N^{O\left(\frac{1}{\beta^{4}}\right)}$ for $N=d^{O\left(\frac{1}{\beta^{4}}\right)}$. Here we are running solving RobustRegressionSDP Algorithm 4 for degree $D=O\left(\frac{1}{\beta^{4}}\right)$, and running $R=O\left(\frac{1}{\beta^{4}}\right)$ rounds of the RegressionRounding Algorithm 2

Proof. For any call to RegressionRounding Algorithm 2, we have by Corollary ?? a list $L$ and a list element $\ell_{i} \in L$ satisfying $\left\|\ell_{i}-\ell^{\prime}\right\| \leq \frac{\rho}{2}$. After each iteration we construct a new data set $\left\{\left(X_{j}, y_{j}^{\prime}\right)\right\}_{j=1}^{N}$ by shifting the labels according to the rule $y_{j}^{\prime}:=y_{j}-\left\langle\ell_{i}, X_{j}\right\rangle$ and enforce the scaling constraint $\left\{\|\ell\|^{2} \leq \frac{\rho^{2}}{4}\right\}$. The key point is that this new constraint is feasible for at least one iterate $\ell_{i} \in L$. This is all we need to iterate RobustRegressionSDP Algorithm 4 and subsequently the RegressionRounding Algorithm Algorithm 2.

The list length grows by a factor of $O\left(\frac{1}{\beta}\right)$ per iteration for $O\left(\log \left(\frac{\beta^{3 / 2} \rho}{\sigma}\right)\right)$ iterations. Thus, we run RobustRegressionSDP Algorithm 4 no more than $O\left(\left(\frac{1}{\beta}\right)^{\log \left(\frac{\beta^{3 / 2} \rho}{\sigma}\right)}\right)$ times. From Lemma 3.6 we solve RobustRegressionSDP for degree $D=O\left(\frac{1}{\beta^{4}}\right)$. This concludes our treatment of list decoding robust regression.

Thus far we have assumed the covariates are distributed $\mathcal{N}(0, I)$ with a fourth injective tensor norm of $B=3$. In addition, we regarded the fourth moment of the noise model upper bounded
by $g \sigma^{4}$ for a constant $g$. We conclude this section by stating a general theorem relevant for large values of $B$ and $g$.

Theorem 4.2. Let a $\beta$ fraction of $X_{1}, \ldots, X_{N} \in \mathbb{R}^{d}$ be drawn from a distribution $\mathcal{D}$ with identity covariance and a fourth injective tensor norm upper bounded by a constant B. Let $N, d, \ell^{\prime}, g, \sigma, \rho$ be defined as they were previously. Then ListDecodeRegression $(\mathcal{D}, \rho)$ outputs a list of hyperplanes $L=\left\{\ell_{1}, \ldots, \ell_{A}\right\}$ where $A=O\left(\left(\frac{1}{\beta}\right)^{\log \left(\frac{\beta^{3 / 2} \rho}{\sigma}\right)}\right)$ such that for some $\ell_{i} \in L$

$$
\left\|\ell_{i}-\ell^{\prime}\right\| \leq O\left(\max \left(\frac{\sigma}{\beta^{3 / 2}}, \sigma^{2} \sqrt{g}\right)\right)
$$

with high probability in time $\left(\frac{\rho}{\sigma}\right)^{\log (1 / \beta)} N^{O\left(\frac{B}{\beta^{4}}\right)}$ for $N=d^{O\left(\frac{1}{\beta^{4}}\right)}$. Here we are running solving RobustRegressionSDP Algorithm 4 for degree $D=O\left(\frac{B}{\beta^{4}}\right)$, and running $R=O\left(\frac{B}{\beta^{4}}\right)$ rounds of the RegressionRounding Algorithm 2

The proof follows by direct inspection of the proof of Theorem 4.1.

## 5 On Conditioning SoS SDP Solutions

In this section we prove facts about concentration rounding.

### 5.1 Concentration Rounding: One Dimensional Case

Fact 5.1. (Conditional Pseudoexpectation Toolkit)
For any two polynomials $p, q$ we define $\widetilde{\mathbb{V}}[p] \stackrel{\text { def }}{=} \tilde{\mathbb{E}}\left[p^{2}\right]-\tilde{\mathbb{E}}[p]^{2}, \widetilde{\operatorname{Cov}}[p, q] \stackrel{\text { def }}{=} \tilde{\mathbb{E}}[p, q]-\tilde{\mathbb{E}}[p] \tilde{\mathbb{E}}[q]$.
Let $w$ satisfy the boolean constraint $w^{2}=w$. For a random variable $b$ taking values in $\{0,1\}$ such that $\mathbb{P}[b=1]=\tilde{\mathbb{E}}[w]$ and $\mathbb{P}[b=0]=\tilde{\mathbb{E}}[1-w]$ we have the following useful facts

1. (Law of Total Pseudoexpectation) $\mathbb{E}_{b} \tilde{\mathbb{E}}[p(x, w) \mid w=b]=\tilde{\mathbb{E}}[p(x, w)]$
2. (Law of Total Pseudovariance) $\widetilde{\mathbb{V}}[p(x)]-\mathbb{E}_{b} \widetilde{\mathbb{V}}[p(x, w) \mid w=b]=\mathbb{V}_{b}[\tilde{\mathbb{E}}[p(x, w) \mid w=b]]$
3. (Correlation Identity) $\tilde{\mathbb{E}}[p(x, w) \mid w=b]=\frac{\widetilde{\operatorname{Cov}}[p(x, w), w]}{\widetilde{\mathbf{V}}[w]} b+\left(\tilde{\mathbb{E}}[p(x, w)]-\frac{\widetilde{\operatorname{Cov}}[p(x, w), w]}{\widetilde{\mathbf{V}}[w]} \tilde{\mathbb{E}}[w]\right)$

Proof. (facts) It is easy to check that $\mathbb{E}[b]=\tilde{\mathbb{E}}[w]$ and $\mathbb{V}[b]=\widetilde{\mathbb{V}}[w]$ and $\operatorname{Cov}[b]=\widetilde{\operatorname{Cov}}[w]$. The law of total pseudoexpectation is an application of definitions. The law of total pseudovariance is an application of the law of total pseudoexpectation. The proof is as follows.

$$
\begin{aligned}
& \widetilde{\mathbb{V}}[p(x, w)]-\underset{b}{\mathbb{E}} \tilde{\mathbb{V}}[p(x, w) \mid w=b]=\tilde{\mathbb{E}}\left[p(x, w)^{2}\right]-\tilde{\mathbb{E}}[p(x, w)]^{2}-\left(\underset{b}{\mathbb{E}} \tilde{\mathbb{E}}\left[p(x, w)^{2}\right]-\underset{b}{\mathbb{E}} \tilde{\mathbb{E}}[p(x, w)]^{2}\right) \\
& \quad=\underset{b}{\mathbb{E}}\left[\tilde{\mathbb{E}}[p(x, w)]^{2}\right]-\tilde{\mathbb{E}}[p(x, w)]^{2}=\underset{b}{\mathbb{E}}\left[\tilde{\mathbb{E}}[p(x, w)]^{2}\right]-\underset{b}{\mathbb{E}}[\tilde{\mathbb{E}}[p(x, w) \mid b]]^{2}=\underset{b}{\mathbb{V}}[\tilde{\mathbb{E}}[p(x, w) \mid w=b]]
\end{aligned}
$$

Lastly, we prove the correlation identity. We know $\tilde{\mathbb{E}}[p(x, w) \mid w=b]$ is a function of $b$. Therefore there exists constants $c$ and $d$ such that $\tilde{\mathbb{E}}[p(x, w) \mid w=b]=c b+d$. First we determine $c$. We know

$$
\begin{gathered}
c \tilde{\mathbb{V}}[w]=c \mathbb{V}[b]=\operatorname{Cov}(c b+d, b)=\operatorname{Cov}(\tilde{\mathbb{E}}[p(x, w) \mid w=b], b) \\
=\mathbb{E}[b \tilde{\mathbb{E}}[p(x, w) \mid w=b]]-\mathbb{E}[\tilde{\mathbb{E}}[p(x, w) \mid w=b]] \mathbb{E}[b]
\end{gathered}
$$

$$
\begin{gathered}
=\mathbb{P}[b=1] \tilde{\mathbb{E}}[p(x, w) \mid w=1]-\tilde{\mathbb{E}}[p(x, w)] \mathbb{E}[b] \\
=\tilde{\mathbb{E}}[w] \frac{\tilde{\mathbb{E}}[p(x, w) w]}{\tilde{\mathbb{E}}[w]}-\tilde{\mathbb{E}}[p(x, w)] \tilde{\mathbb{E}}[w] \\
=\widetilde{\operatorname{Cov}}[p(x, w), w]
\end{gathered}
$$

Thus

$$
c=\frac{\widetilde{\operatorname{Cov}}[p(x, w), w]}{\widetilde{\mathbb{V}}[w]}
$$

Then to obtain $d$ we apply expectation on both sides of $\tilde{\mathbb{E}}[p(x, w) \mid w=b]=c b+d$.
Let $w_{1}, \ldots, w_{N}$ be variables that satisfy the boolean constraint for all $i \in[N]$. Let $Z_{1}, \ldots, Z_{N} \in \mathbb{R}$ be numbers and let $\hat{\mu} \stackrel{\text { def }}{=} \frac{1}{N} \sum_{i=1}^{N} w_{i} Z_{i}$. We show that pseudo-variance $\widetilde{\mathbb{V}}[\hat{\mu}]$ decreases in expectation when we condition on the variables $w_{i}$ according to a carefully chosen strategy.

Theorem 5.2. Let $w_{1}, \ldots, w_{N}$ denote variables satisfying $\left\{w_{i}^{2}=w_{i} \mid i \in[N]\right\}$ and let $\hat{\mu}=\frac{1}{N} \sum_{i \in[N]} w_{i} Z_{i}$ for some sequence of real numbers $\left\{Z_{i}\right\}_{i \in[N]}$. Define a probability distribution $\mathcal{S}:[N] \rightarrow \mathbb{R}^{+}$as

$$
\mathcal{S}(j) \stackrel{\text { def }}{=} \frac{\widetilde{\mathbb{V}}\left[w_{j} Z_{j}\right]}{\sum_{i=1}^{N} \widetilde{\mathbb{V}}\left[w_{j} Z_{j}\right]}
$$

If we condition on the value of $w_{j}$ where $j$ is drawn from $\mathcal{S}$, then the pseudovariance decreases by

$$
\widetilde{\mathbb{V}}[\hat{\mu}]-\underset{j \sim \mathcal{S}}{\mathbb{E}} \mathbb{E} \mathbb{b _ { j }} \mathbb{\mathbb { V }}\left[\hat{\mu} \mid w_{j}=b_{j}\right] \geq \frac{(\widetilde{\mathbb{V}}(\hat{\mu}))^{2}}{\frac{1}{N} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left(z_{i}\right)}
$$

Where $b_{j}$ is $[0,1]$ random variable with $\mathbb{P}\left[b_{j}=1\right]=\tilde{\mathbb{E}}\left[w_{j}\right]$ and $\mathbb{P}\left[b_{j}=0\right]=\tilde{\mathbb{E}}\left[1-w_{j}\right]$
This also immediately yields for $\hat{\mu}=\frac{1}{M} \sum_{i=1}^{N} w_{i} Z_{i}$

$$
\widetilde{\mathbb{V}}[\hat{\mu}]-\underset{j \sim \mathcal{S}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}\left[\hat{\mu} \mid w_{j}=b_{j}\right] \geq \beta \frac{(\widetilde{\mathbb{V}}(\hat{\mu}))^{2}}{\frac{1}{M} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left(z_{i}\right)}
$$

Proof. Let $z_{i}=w_{i} Z_{i}$ for all $i \in[N]$. Since $z_{i}$ is a constant multiple of $w_{i}$, conditioning on $z_{i}$ is equivalent to conditioning on $w_{i}$. We begin with the law of total variance

$$
\widetilde{\mathbb{V}}(\hat{\mu})-\underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}\left(\hat{\mu} \mid w_{j}=b_{j}\right)=\underset{b_{j}}{\mathbb{V}}\left(\tilde{\mathbb{E}}\left[\hat{\mu} \mid w_{j}=b_{j}\right]\right)
$$

Then we apply the expectation over the strategy $\mathcal{S}$ to both sides to obtain

$$
\widetilde{\mathbb{V}}[\hat{\mu}]-\underset{\mathcal{S}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}\left[\hat{\mu} \mid w_{j}=b_{j}\right]=\underset{\mathcal{S}}{\mathbb{E}} \underset{b_{j}}{\mathbb{V}}\left[\tilde{\mathbb{E}}\left[\hat{\mu} \mid w_{j}=b_{j}\right]\right]
$$

$$
=\underset{\mathcal{S}}{\tilde{\mathbb{E}}} \widetilde{\mathbb{V}}_{b_{j}}\left[\frac{\widetilde{\operatorname{Cov}}\left[\hat{\mu}, w_{j}\right]}{\widetilde{\mathbb{V}}\left[w_{j}\right]} b_{j}\right]=\underset{\mathcal{S}}{\tilde{\mathbb{E}}} \frac{\widetilde{\operatorname{Cov}}\left[\hat{\mu}, w_{j}\right]^{2}}{\widetilde{\mathbb{V}}\left[w_{j}\right]^{2}} \mathbb{V}\left[b_{j}\right]=\underset{\mathcal{E}}{\tilde{\mathbb{E}}} \frac{\widetilde{\operatorname{Cov}}\left[\hat{\mu}, z_{j}\right]^{2}}{\widetilde{\mathbb{V}}\left[z_{j}\right]}
$$

Writing out the distribution of $\mathcal{S}$ we obtain

$$
=\sum_{j=1}^{N} \frac{\widetilde{\mathbb{V}}\left[z_{j}\right]}{\sum_{i=1}^{N} \widetilde{\mathbb{V}}\left[z_{j}\right]} \frac{\widetilde{\operatorname{Cov}}\left[\hat{\mu}, z_{j}\right]^{2}}{\widetilde{\mathbb{V}}\left[z_{j}\right]}=\frac{\sum_{i=1}^{N} \widetilde{\operatorname{Cov}}\left[\hat{\mu}, z_{j}\right]^{2}}{\sum_{i=1}^{N} \widetilde{\mathbb{V}}\left[z_{j}\right]}=\frac{\frac{1}{N} \sum_{i=1}^{N} \widetilde{\operatorname{Cov}}\left[\hat{\mu}, z_{j}\right]^{2}}{\frac{1}{N} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left[z_{j}\right]}
$$

by Jensen's inequality

$$
\geq \frac{\left(\frac{1}{N} \sum_{i=1}^{N} \widetilde{\operatorname{Cov}}\left[\hat{\mu}, z_{j}\right]\right)^{2}}{\frac{1}{N} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left[z_{j}\right]}=\frac{\widetilde{\mathbb{V}}[\hat{\mu}]^{2}}{\frac{1}{N} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left[z_{i}\right]}
$$

Corollary 5.3. (Connecting variance decrease strategy to nuclear norm rounding) For $\mathcal{S}_{v}$ and $Q$ defined in Lemma 3.3. Let $\gamma>0$ be a constant. If $\|Q\|_{o p} \geq \gamma$ and $\|Q\|_{n u c}>\sigma^{2} \sqrt{g}$, then

$$
\|Q\|_{n u c}-\underset{j \sim S_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}}\left\|\left.Q\right|_{w_{j}=b_{j}}\right\|_{n u c} \geq \Omega\left(\frac{\beta \gamma^{2}}{\rho^{2}}\right)
$$

In particular for $\gamma=\eta \rho^{2}$, we have

$$
\|Q\|_{n u c}-\underset{j \sim \mathcal{S}_{v}}{\mathbb{E}} \mathbb{E}\left\|\left.Q\right|_{w_{j}=b_{j}}\right\|_{n u c} \geq \Omega\left(\beta \eta^{2} \rho^{2}\right)
$$

Proof. Let $v, e_{1}, \ldots, e_{d-1} \in R^{d}$ be an orthonormal basis. First we write the nuclear norm as a decomposition along an orthonormal basis i.e

$$
\|Q\|_{\text {nuc }}=\widetilde{\mathbb{V}}[\langle\ell, v\rangle]+\sum_{j=1}^{d-1} \widetilde{\mathbb{V}}\left[\left\langle\ell, e_{j}\right\rangle\right]
$$

Now we write down the expected decrease in $\|Q\|_{\text {nuc }}$ for a single conditioning to obtain

$$
\begin{array}{r}
\|Q\|_{\text {nuc }}-\left.\underset{j \sim \mathcal{S}_{v}}{\mathbb{E}} \mathbb{E}\| \|\right|_{w_{j}=z_{i}} \|_{\text {nuc }}=\left(\widetilde{\mathbb{V}}[\langle\ell, v\rangle]-\underset{j \sim S_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}\left[\langle\ell, v\rangle \mid w_{j}=b_{j}\right]\right) \\
+\left(\sum_{j=1}^{d-1} \widetilde{\mathbb{V}}\left[\left\langle\ell, e_{j}\right\rangle\right]-\underset{j \sim \mathcal{S}_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}\left[\left\langle\ell, e_{j}\right\rangle \mid w_{j}=b_{j}\right]\right)
\end{array}
$$

Then we apply Lemma 3.3 to the first term, and we apply the fact that pseudovariance is monotonically decreasing after conditioning (law of total pseudovariance) to the second term to obtain.

$$
\geq \Omega\left(\frac{\beta \widetilde{\mathbb{V}}_{\zeta}(\langle\ell, v\rangle)^{2}}{\|Q\|_{n u c}}\right)-o_{d}(1)
$$

Using the fact that $\widetilde{\mathbb{V}}_{\zeta}(\langle\ell, v\rangle)=\|Q\|_{\text {op }} \geq \gamma$ and $\|Q\|_{\text {nuc }}=\tilde{\mathbb{E}}_{\zeta}\left[\left\|\ell-\tilde{\mathbb{E}}_{\zeta}[\ell]\right\|^{2}\right] \leq \tilde{\mathbb{E}}_{\zeta}\left[\|\ell\|^{2}\right] \leq \rho^{2}$ we further lower bound by

$$
\geq \Omega\left(\frac{\beta \gamma^{2}}{\rho^{2}}\right)-o_{d}(1) \geq \Omega\left(\frac{\beta \gamma^{2}}{\rho^{2}}\right)-o_{d}(1)
$$

for $\gamma=\eta \rho^{2}$, we conclude

$$
\|Q\|_{\text {nuc }}-\underset{j \sim \mathcal{S}_{v}}{\mathbb{E}} \mathbb{E}\left\|\left.Q\right|_{w_{j}=z_{i}}\right\|_{\text {nuc }} \geq \Omega\left(\beta \eta^{2} \rho^{2}\right)
$$

as desired.

## 6 Frobenius Minimization

### 6.1 Frobenius Norm Minimization

Lemma (Restatement of Lemma 3.1). (Frobenius minimization $\Longrightarrow$ Correlation) Let $\mathcal{P}$ be a polynomial system in variables $\left\{w_{i}\right\}_{i \in[N]}$ and a set of indeterminates $\left\{\ell_{i}\right\}_{i \in \mathbb{N}}$, that contains the set of inequalities:

$$
w_{i}^{2}=w_{i} \forall i \in[N] \quad \sum_{i} w_{i}=\beta N
$$

Let $\tilde{\mathbb{E}}_{\zeta}: \mathbb{R}\left[\left\{w_{i}\right\}_{i \in[N]},\{\ell\}\right]^{\leq D} \rightarrow \mathbb{R}$ denote a degree $D$ pseudoexpectation that satisfies $\mathcal{P}$ and minimizes the norm $\left\|\tilde{\mathbb{E}}_{\zeta}[w]\right\|$. If $w_{i}^{\prime} \in\{0,1\}$ and $\ell^{\prime}$ is a satisfying assignment to $\mathcal{P}$ then there is correlation with the inliers,

$$
\begin{equation*}
\underset{\zeta}{\tilde{\mathbb{E}}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right] \geq \beta \tag{6.1}
\end{equation*}
$$

Proof. Let $\tilde{\mathbb{E}}_{P}$ denote the pseudo-expectation operator corresponding to the actual assignment $\left\{w_{i}^{\prime}\right\}_{i \in[N]}$ and $\left\{\ell^{\prime}\right\}$. Note that $\tilde{\mathbb{E}}_{P}$ is an actual expectation over an assignment satisfying the polynomial constraints. For a constant $\mathcal{K} \in[0,1]$, let us consider the pseudoexpectation operator $\tilde{\mathbb{E}}_{R}$ defined as follows for a polynomial $p(w)$,

$$
\underset{R}{\tilde{\mathbb{E}}} \stackrel{\text { def }}{=} \mathcal{\kappa} \underset{P}{\tilde{\mathbb{E}}}+(1-\kappa) \underset{D}{\tilde{\mathbb{E}}}
$$

Since $\tilde{\mathbb{E}}_{D}$ is the pseudoexpecation operator that minimizes $\left\|\tilde{\mathbb{E}}_{D}[w]\right\|$, we get that

$$
\begin{equation*}
\langle\underset{R}{\tilde{\mathrm{E}}}[w], \underset{R}{\tilde{\mathrm{E}}}[w]\rangle \geq\langle\underset{D}{\tilde{\mathrm{E}}}[w], \underset{D}{\tilde{\mathrm{E}}}[w]\rangle \tag{6.2}
\end{equation*}
$$

Expanding the LHS with the definition of $R$ we have

$$
(1-\kappa)^{2} \cdot\left\langle\underset{D}{\tilde{\mathbb{E}}}[w], \frac{\tilde{\mathrm{E}}}{D}[w]\right\rangle+2 \kappa(1-\kappa)\langle\underset{D}{\tilde{\mathbb{E}}}[w], \underset{P}{\tilde{\mathrm{E}}}[w]\rangle+\kappa^{2}\langle\underset{P}{\tilde{\mathbb{E}}}[w], \underset{P}{\tilde{\mathrm{E}}}[w]\rangle \geq\langle\underset{D}{\tilde{\mathbb{E}}}[w], \underset{D}{\tilde{\mathbb{E}}}[w]\rangle
$$

Rearranging the terms we get

$$
\left\langle\underset{D}{\tilde{\mathbb{E}}}[w], \frac{\underset{P}{\mathbb{E}}}{P}[w]\right\rangle \geq \frac{1}{2 \kappa(1-\kappa)}\left(\left(2 \kappa-\kappa^{2}\right)\left\langle\underset{D}{\tilde{E}}[w], \frac{\tilde{D}}{\mathbb{E}}[w]\right\rangle-\kappa^{2}\left\langle\underset{P}{\tilde{E}}[w], \frac{\tilde{E}}{\underset{E}{E}}[w]\right\rangle\right)
$$

By definition, we have that $\left\langle\tilde{\mathbb{E}}_{P}[w], \tilde{\mathbb{E}}_{P}[w]\right\rangle=\sum_{i} w_{i}^{\prime 2}=\beta N$. By Cauchy-Schwartz inequality, $\left\langle\tilde{\mathbb{E}}_{D}[w], \tilde{\mathbb{E}}_{D}[w]\right\rangle \geq \frac{1}{N}\left(\sum_{i} \tilde{\mathbb{E}}_{D}\left[w_{i}\right]\right)^{2}=\frac{1}{N}(\beta N)^{2}=\beta^{2} N$. Substituting these bounds back we get that,

$$
\langle\underset{D}{\tilde{\mathrm{E}}}[w], \underset{P}{\tilde{\mathrm{E}}}[w]\rangle \geq \frac{\left(\left(2 \kappa-\kappa^{2}\right) \beta^{2}-\kappa^{2} \beta\right)}{2 \kappa(1-\kappa)} \cdot N
$$

Taking limits as $\kappa \rightarrow 0$, we get the desired result.

## 7 Regression Rounding

In this section we prove that concentration rounding decreases $\|Q\|_{n u c}$. First we closely approximate $\widetilde{\mathbb{V}}[\langle\ell, u\rangle]$ by $\widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]$ for any unit vector $u \in S^{d-1}$. Then we apply Theorem 5.2 with the strategy $\mathcal{S}_{v}$ to analyze a single iteration of concentration rounding. We begin with the following useful lemma for working with pseudovariance.
Lemma 7.1 (Pseudovariance Triangle Inequality). Let $f(x)$ and $g(x)$ be polynomials. Then for any $\psi>0$ there is a degree 2 SoS proof of the following.

$$
\widetilde{\mathbb{V}}[f(x)+g(x)] \leq(1+\psi) \widetilde{\mathbb{V}}[f(x)]+\left(\frac{1+\psi}{\psi}\right) \widetilde{\mathbb{V}}[g(x)]
$$

Proof.

$$
\begin{equation*}
\widetilde{\mathbb{V}}[f(x)+g(x)]=\tilde{\mathbb{E}}\left[(f(x)+g(x)-\tilde{\mathbb{E}}[f(x+g(x))])^{2}\right]=\tilde{\mathbb{E}}\left[((f(x)-\tilde{\mathbb{E}}[f(x)])+(g(x)-\tilde{\mathbb{E}}[g(x)]))^{2}\right] \tag{7.1}
\end{equation*}
$$

Then we observe that there is a degree 2 SoS proof of the fact

$$
(f(x)+g(x))^{2}=f(x)^{2}+g(x)^{2}+2 \psi f(x) \frac{g(x)}{\psi} \leq\left(1+\psi^{2}\right) f(x)^{2}+\left(\frac{1+\psi^{2}}{\psi^{2}}\right) g(x)^{2}
$$

Plugging this into 7.1
$\leq\left(1+\psi^{2}\right) \tilde{\mathbb{E}}\left[(f(x)-\tilde{\mathbb{E}}[f(x)])^{2}\right]+\left(\frac{1+\psi^{2}}{\psi^{2}}\right) \tilde{\mathbb{E}}\left[(g(x)-\tilde{\mathbb{E}}[g(x)])^{2}\right]=\left(1+\psi^{2}\right) \widetilde{\mathbb{V}}[f(x)]+\frac{1+\psi^{2}}{\psi^{2}} \widetilde{\mathbb{V}}[g(x)]$
Substituting any variable $\psi^{\prime}=\psi^{2}>0$ we obtain the desired result.
Lemma (Restatement of Lemma 3.3). Let $\tilde{\mathbb{E}}_{\zeta}$ satisfy the pseudoexpectation constraints of RobustRegression $\operatorname{SDP}(\mathcal{D}, \rho)$. Let $Q$ be the associated pseudocovariance matrix. Let $v$ be any direction in the unit ball $S^{d-1}$. Let $\mathcal{S}_{v}$, be a probability distribution over $[N]$ where for any $j \in N$ we have

$$
\mathcal{S}_{v}(j) \stackrel{\text { def }}{=} \frac{\widetilde{\mathbb{V}}_{\zeta}\left[w_{j}\left(y_{j}-\tilde{\mathbb{E}}_{\zeta}\left[\left\langle\ell, X_{i}\right\rangle\right]\right)\left\langle X_{j}, v\right\rangle\right]}{\sum_{i=1}^{N} \widetilde{\mathbb{V}}_{\zeta}\left[w_{j}\left(y_{j}-\tilde{\mathbb{E}}_{\zeta}\left[\left\langle\ell, X_{i}\right\rangle\right]\right)\left\langle X_{j}, v\right\rangle\right]}
$$

Then for $M_{4}$ being the fourth moment matrix of the Gaussian defined in RobustRegressionSDP, and for $\|Q\|_{n u c}>\sigma^{2} \sqrt{g}$ we have

$$
\widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]-\underset{j \sim S_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}_{\zeta}\left[\langle\ell, v\rangle \mid w_{j}=b_{j}\right] \geq \Omega\left(\frac{\beta \widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]^{2}}{\|Q\|_{n u c}}\right)
$$

To prove Lemma 3.3 we will need the following lemma
Lemma 7.2. Let $\tilde{\mathbb{E}}$ be a pseudodistribution satisfying $\mathcal{P}$. The following holds.

$$
\begin{align*}
& \widetilde{\mathbb{V}}[\langle\ell, u\rangle] \leq\left(1+o_{d}(1)\right) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]+o_{d}(1)  \tag{7.2}\\
& \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right] \leq\left(1+o_{d}(1)\right) \widetilde{\mathbb{V}}[\langle\ell, u\rangle]+o_{d}(1) \tag{7.3}
\end{align*}
$$

Informally, 7.2 gives us an arbitrarily good approximation (up to a negligible additive error term) to $\widetilde{\mathbb{V}}[\langle\ell, u\rangle]$ by the variance of an estimator that is amenable to rounding via Theorem 5.2. We defer the proof to the end of the section. Now we're ready to prove Lemma 3.3

Proof. (Proof of Lemma 3.3) First we apply Lemma 7.2 to obtain an arbitrarily good constant factor approximation of the variance decrease.

$$
\begin{array}{r}
\widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]-\underset{j \sim S_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}_{\zeta}\left[\langle\ell, v\rangle \mid w_{j}=b_{j}\right] \geq\left(1-o_{d}(1)\right) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right] \\
-\left(1+o_{d}(1)\right) \underset{j \sim S_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}\left[\left.\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle \right\rvert\, w_{j}=b_{j}\right]-o_{d}(1)
\end{array}
$$

Using the setting $d>\rho^{2}$ we have $\rho^{2} o_{d}(1)=o_{d}(1)$ and we simplify the above expression to obtain

$$
\begin{equation*}
=\left(\widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]-\underset{j \sim S_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}\left[\left.\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle \right\rvert\, w_{j}=b_{j}\right]\right)-o_{d}(1) \tag{7.4}
\end{equation*}
$$

To lower bound the first term above, we apply Theorem 5.2 with $Z_{i}=w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle$. This immediately gives us,

$$
\begin{gathered}
\widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]-\underset{j \sim S_{u}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}\left[\left.\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle \right\rvert\, w_{j}=b_{j}\right] \\
\geq \frac{\beta \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]^{2}}{\frac{1}{M} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left[w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]}
\end{gathered}
$$

Applying Lemma 7.2 to the numerator we obtain

$$
\geq \frac{\left(1-o_{d}(1)\right) \beta \widetilde{\mathbb{V}}[\langle\ell, u\rangle]^{2}-o_{d}(1)}{\frac{1}{M} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left[w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]}
$$

Now we upper bound the denominator by

$$
\frac{1}{M} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left[w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right] \leq 2\left(g \sigma^{4} B\right)^{\frac{1}{2}}+2 B\|Q\|_{n u c}
$$

The proof is as follows. First we use $\widetilde{\mathbb{V}}(X) \leq \tilde{\mathbb{E}}\left[X^{2}\right]$ to obtain

$$
\begin{gathered}
\left.\frac{1}{M} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left(w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right) \leq \frac{1}{M} \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)^{2}\left\langle X_{i}, u\right\rangle^{2}\right)\right] \\
\left.=\frac{1}{M} \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[w_{i}\left(y_{i}-\left\langle\ell, X_{i}\right\rangle+\left\langle\ell, X_{i}\right\rangle-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)^{2}\left\langle X_{i}, u\right\rangle^{2}\right)\right]
\end{gathered}
$$

Then we use degree 2 SoS triangle inequality to obtain

$$
\begin{equation*}
\left.\leq 2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\ell, X_{i}\right\rangle\right)^{2}\left\langle X_{i}, u\right\rangle^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\ell-\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle^{2}\left\langle X_{i}, u\right\rangle^{2}\right)\right] \tag{7.5}
\end{equation*}
$$

The first term is upper bounded by pseudoexpectation Cauchy-Schwarz

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\ell, X_{i}\right\rangle\right)^{2}\left\langle X_{i}, u\right\rangle^{2}\right] & =\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}^{2}\left(y_{i}-\left\langle\ell, X_{i}\right\rangle\right)^{2}\left\langle X_{i}, u\right\rangle^{2}\right] \\
& =\left(\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}^{2}\left(y_{i}-\left\langle\ell, X_{i}\right\rangle\right)^{4}\right]\right)^{\frac{1}{2}}\left(\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}^{2}\left\langle X_{i}, u\right\rangle^{4}\right]\right)^{\frac{1}{2}}
\end{aligned}
$$

Then by degree 2 SoS Cauchy-Schwarz, followed by applying the fourth moment constraints on noise (4) we obtain

$$
\begin{gathered}
\leq\left(g \sigma^{4}\right)^{1 / 2} \tilde{\mathbb{E}}\left[\left\langle\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i}^{\otimes 2}\left(X_{i}^{\otimes 2}\right)^{T}, u^{\otimes 2}\left(u^{\otimes 2}\right)^{T}\right\rangle\right]^{\frac{1}{2}} \\
=\left(g \sigma^{4}\right)^{\frac{1}{2}} \tilde{\mathbb{E}}\left[\left\langle\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i}^{\otimes 2}\left(X_{i}^{\otimes 2}\right)^{T}-M_{4}, u^{\otimes 2}\left(u^{\otimes 2}\right)^{T}\right\rangle+\left\langle M_{4}, u^{\otimes 2}\left(u^{\otimes 2}\right)^{T}\right\rangle\right]^{\frac{1}{2}}
\end{gathered}
$$

Then applying Cauchy-Schwarz, followed by applying the fourth moment constraints on the covariates (3) we obtain

$$
\leq\left(g \sigma^{4}\right)^{\frac{1}{2}} \tilde{\mathbb{E}}\left[\left\|\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i}^{\otimes 2}\left(X_{i}^{\otimes 2}\right)^{T}-M_{4}\right\|_{F}^{2}+B\right]^{\frac{1}{2}} \leq\left(2 g \sigma^{4} B\right)^{\frac{1}{2}}
$$

Next we upper bound the second term in 7.5 by SoS Cauchy Schwarz

$$
\left.\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\ell-\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle^{2}\left\langle X_{i}, u\right\rangle^{2}\right)\right]=\tilde{\mathbb{E}}\left[\left\langle(\ell-\tilde{\mathbb{E}}[\ell])^{\otimes 2}\left(u^{\otimes 2}\right)^{T}, \frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i}^{\otimes 2}\left(X_{i}^{\otimes 2}\right)^{T}\right\rangle\right]
$$

$$
=\tilde{\mathbb{E}}\left[\left\langle(\ell-\tilde{\mathbb{E}}[\ell])^{\otimes 2}\left(u^{\otimes 2}\right)^{T}, \frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i}^{\otimes 2}\left(X_{i}^{\otimes 2}\right)^{T}-M_{4}\right\rangle\right]+\tilde{\mathbb{E}}\left[\left\langle(\ell-\tilde{\mathbb{E}}[\ell])^{\otimes 2}\left(u^{\otimes 2}\right)^{T}, M_{4}\right\rangle\right]
$$

Applying SoS Cauchy-Schwarz to the first term we obtain

$$
=\tilde{\mathbb{E}}\left[\|\ell-\tilde{\mathbb{E}}[\ell]\|^{2}\left\|\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i}^{\otimes 2}\left(X_{i}^{\otimes 2}\right)^{T}-M_{4}\right\|_{F}^{2}\right]+\tilde{\mathbb{E}}\left[\left\langle(\ell-\tilde{\mathbb{E}}[\ell])^{\otimes 2}\left(u^{\otimes 2}\right)^{T}, M_{4}\right\rangle\right]
$$

Then applying the fourth moment constraints on the covariates (3) and applying the definition of $\|Q\|_{\text {nuc }}$ we obtain

$$
=\varepsilon\|Q\|_{n u c}+\tilde{\mathbb{E}}\left[\left\langle(\ell-\tilde{\mathbb{E}}[\ell])^{\otimes 2}\left(u^{\otimes 2}\right)^{T}, M_{4}\right\rangle\right]
$$

We upper bound the second term above using the assumption upper bounding the fourth injective norm of the covariates.

$$
\tilde{\mathbb{E}}\left[\left\langle(\ell-\tilde{\mathbb{E}}[\ell])^{\otimes 2}\left(u^{\otimes 2}\right)^{T}, M_{4}\right\rangle\right] \leq B\|Q\|_{n u c}
$$

For $\mathcal{N}(0, I)$, we have $B=3$. Plugging both terms back into 7.5 , we obtain

$$
\begin{gathered}
\widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]-\underset{j \sim S_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}_{\zeta}\left[\langle\ell, v\rangle \mid w_{j}=b_{j}\right] \\
\geq \frac{\left(1-o_{d}(1)\right) \beta \widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]^{2}-o_{d}(1)}{2 \sigma^{2}(g B)^{\frac{1}{2}}+2 B\|Q\|_{n u c}}-o_{d}(1)=\frac{\beta \widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]^{2}}{2 \sigma^{2}(g B)^{\frac{1}{2}}+2 B\|Q\|_{n u c}}-o_{d}(1)
\end{gathered}
$$

Using the assumption $\|Q\|_{n u c}>\sigma^{2} \sqrt{g}$ and setting $B=3$ we have

$$
\widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]-\underset{j \sim s_{v}}{\mathbb{E}} \underset{b_{j}}{\mathbb{E}} \widetilde{\mathbb{V}}_{\zeta}\left[\langle\ell, v\rangle \mid w_{j}=b_{j}\right] \geq \Omega\left(\frac{\beta \widetilde{\mathbb{V}}_{\zeta}[\langle\ell, v\rangle]^{2}}{\|Q\|_{n u c}}\right)
$$

### 7.1 Snapping for Regression

Lemma (Restatement of Lemma 3.2). Let $\eta \in\left[0, \frac{1}{2}\right]$ be a small constant. Let $\tilde{\mathbb{E}}_{\zeta}$ be a degree $O\left(\frac{1}{\eta^{4}}\right)$ pseudoexpectation satisfying the constraints of RobustRegressionSDP( $\mathcal{D}, \rho)$. Then if the pseudovariance of the estimator $\tilde{\mathbb{E}}_{\zeta}[\ell]$ is small in every direction

$$
\begin{equation*}
\max _{u} \widetilde{\mathbb{V}}_{\zeta}[\langle\ell, u\rangle] \leq \eta \rho^{2} \tag{7.6}
\end{equation*}
$$

and there is correlation with the inliers

$$
\begin{equation*}
\underset{\zeta}{\tilde{\mathbb{E}}}\left[\frac{1}{M} \sum_{i} w_{i} w_{i}^{\prime}\right] \geq \beta \tag{7.7}
\end{equation*}
$$

then our estimator satisfies,

$$
\begin{equation*}
\left\|\tilde{\mathbb{E}}[\ell]-\ell^{\prime}\right\| \leq \sqrt{\frac{\eta \rho^{2}+O\left(\frac{\sigma^{2}}{\eta^{2}}\right)}{\beta}} \tag{7.8}
\end{equation*}
$$

In particular, for $\eta=\frac{\beta}{8}$ and $\rho^{2}>\Omega\left(\frac{\sigma^{2}}{\beta^{3}}\right)$ the degree $O\left(\frac{1}{\beta^{4}}\right)$ pseudoexpectation satisfies
Proof. Let $u \in S^{d-1}$, we have by linearity

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]\left\langle\tilde{\mathbb{E}}[\ell]-\ell^{\prime}, u\right\rangle^{2}=\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\left\langle\tilde{\mathbb{E}}[\ell]-\ell^{\prime}, u\right\rangle^{2}\right]
$$

And by degree 2 SoS triangle inequality

$$
=\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\left\langle\tilde{\mathbb{E}}[\ell]-\ell+\ell-\ell^{\prime}, u\right\rangle^{2}\right] \leq 2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\left\langle\ell-\ell^{\prime}, u\right\rangle^{2}\right]
$$

The following expression is a sum of squares $\mathcal{R} \leftarrow_{2}\left\{\frac{1}{M} \sum_{i=1}^{N}\left(1-w_{i}\right) w_{i}^{\prime}\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2} \geq 0\right\}$ so we add it to the right hand side to obtain

$$
\leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\left\langle\ell-\ell^{\prime}, u\right\rangle^{2}\right]
$$

applying degree 2 SoS Cauchy-Schwarz to the second term we obtain,

$$
\leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\left\|\ell-\ell^{\prime}\right\|^{2}\right]
$$

Consider the second term above. By the properties of ( $c, D(\eta)$ )-SoS-anticoncentration (see Definition 8.2) we upper bound by,

$$
\begin{aligned}
& \leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+2\left(c \rho \eta+\frac{1}{\eta^{2}} \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\left\langle\ell-\ell^{\prime}, X_{i}\right\rangle^{2}\right]\right) \\
& \leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+2\left(\rho^{2} \eta+\eta^{2} \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\left(\left\langle\ell, X_{i}\right\rangle-y_{i}\right)^{2}+\left(\left\langle\ell^{\prime}, X_{i}\right\rangle-y_{i}\right)^{2}\right]\right)
\end{aligned}
$$

By SoS triangle inequality
$\leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+2\left(\rho^{2} \eta+\eta^{2} 2\left(\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\left(\left\langle\ell, X_{i}\right\rangle-y_{i}\right)^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\left(\left\langle\ell^{\prime}, X_{i}\right\rangle-y_{i}\right)^{2}\right]\right)\right)$

Using the fact that $\left.\mathcal{P}\right|_{2}\left\{\frac{1}{M} \sum_{i=1}^{N}\left(1-w_{i}\right) w_{i}^{\prime}\left(\left\langle\ell, X_{i}\right\rangle-y_{i}\right)^{2} \geq 0, \frac{1}{M} \sum_{i=1}^{N}\left(1-w_{i}^{\prime}\right) w_{i}\left(\left\langle\ell, X_{i}\right\rangle-y_{i}\right)^{2} \geq 0\right\}$ we add in both polynomials to obtain

$$
\leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+2\left(\rho^{2} \eta+\eta^{2} 2\left(\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(\left\langle\ell, X_{i}\right\rangle-y_{i}\right)^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime}\left(\left\langle\ell^{\prime}, X_{i}\right\rangle-y_{i}\right)^{2}\right]\right)\right)
$$

Applying the SDP noise constraint (4) we obtain

$$
\leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+\left(2 c \rho^{2} \eta+\frac{8 \sigma^{2}}{\eta^{2}}\right)
$$

Thus far we've shown in degree $D(\eta)$ the following inequality

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]\left\langle\tilde{\mathbb{E}}[\ell]-\ell^{\prime}, u\right\rangle^{2} \leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+\left(2 c \rho^{2} \eta+\frac{8 \sigma^{2}}{\eta^{2}}\right)
$$

This inequality holds for all $u \in S^{d-1}$, in particular for the unit vector $u$ along $\tilde{\mathbb{E}}[\ell]-\ell^{\prime}$ we have

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]\left\|\tilde{\mathbb{E}}[\ell]-\ell^{\prime}\right\|^{2} \leq 2 \max _{u \in S^{d-1}} \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+\left(2 c \rho^{2} \eta+\frac{8 \sigma^{2}}{\eta^{2}}\right)
$$

Dividing both sides by $\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]$ and taking a square root we obtain

$$
\left\|\tilde{\mathbb{E}}[\ell]-\ell^{\prime}\right\| \leq \sqrt{\frac{2 \max _{u \in S^{d-1}} \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\ell]-\ell, u\rangle^{2}\right]+\left(2 c \rho^{2} \eta+\frac{8 \sigma^{2}}{\eta^{2}}\right)}{\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]}}
$$

Plugging in the assumptions on frobenius minimization 7.7 and variance reduction 7.6 we obtain

$$
\left\|\tilde{\mathbb{E}}[\ell]-\ell^{\prime}\right\| \leq \sqrt{\frac{4 c \rho^{2} \eta+\frac{8 \sigma^{2}}{\eta^{2}}}{\beta}}
$$

Since $\eta$ is any constant in $\left[0, \frac{1}{2}\right]$ we conclude by writing

$$
\left\|\tilde{\mathbb{E}}[\ell]-\ell^{\prime}\right\| \leq \sqrt{\frac{\rho^{2} \eta+O\left(\frac{\sigma^{2}}{\eta^{2}}\right)}{\beta}}
$$

## 8 Certifying Anticoncentration

Anticoncentration is a measure of the "spread" of a distribution. For any distribution $\mathcal{D}$, let $\eta$ be a parameter $0<\eta<\frac{1}{2}$. If the probability mass of $\mathcal{D}$ contained in the $\eta$ interval around the origin is small, than $\mathcal{D}$ is anticoncentrated. For example, in the case of $\mathcal{D}=\mathcal{N}(0,1)$, the mass of $\mathcal{D}$ in any $\eta$ interval about the origin is upper bounded by $\frac{2}{\sqrt{2 \pi}} \eta$. This property of the probability mass decaying linearly with $\eta$ as $\eta$ goes to zero is what motivates the following definition.

Definition 8.1. A probability distribution $\mathcal{D}$ over $\mathbb{R}^{d}$ is said to be $c$-anticoncentrated if for any $0<\eta<\frac{1}{2}$ there exists $\tau \leq c \eta$ such that for any measurable subset $\mathcal{E} \in \mathbb{R}^{n}$, and for all $v \in \mathbb{R}^{d}$ with $\|v\| \leq 1$, we have that

$$
\mathbb{E}\left[\langle X, v\rangle^{2} \cdot \mathbb{I}[\mathcal{E}]\right] \geq \eta^{2} \cdot \mathbb{P}[\mathcal{E}] \cdot\|v\|^{2}-\eta^{2} \tau
$$

We now state the SoS version of anticoncentration
Definition 8.2. Let $D:[0,1 / 2] \rightarrow \mathbb{N}$. A probability distribution $\mathcal{D}$ over $\mathbb{R}^{d}$ is said to $(c, D(\eta))$-SoSanticoncentrated, If for any $0<\eta<\frac{1}{2}$ there exists $\tau \leq c \eta$ and there exists a constant $k \in \mathbb{N}$ such that for all $N>d^{k}$, with probability $1-d^{-k}$, over samples $x_{1}, \ldots, x_{N} \sim \mathcal{D}$ the following polynomial system

$$
\mathcal{P}= \begin{cases}w_{i}^{2}=w_{i} & i \in[N] \\ \|v\|^{2} \leq \rho^{2} & \\ \left\|\frac{1}{N} \sum_{i=1}^{N} X_{i}^{\otimes \frac{t}{2}}\left(X_{i}^{\otimes \frac{t}{2}}\right)^{T}-M_{t}\right\|<\varepsilon & t \in[k]\end{cases}
$$

yields a degree $D(\eta)$ SoS proof of the following inequality

$$
\left.\mathcal{P}\right|_{D(\eta)}\left\{\frac{1}{N} \sum_{i=1}^{N} w_{i}\left\langle X_{i}, v\right\rangle^{2} \geq \eta^{2}\left(\frac{1}{N} \sum_{i} w_{i}\right)\|v\|^{2}-\eta^{2} \tau \rho^{2}\right\}
$$

Theorem 8.3. (Sufficient conditions for SoS anti-concentration) If the degree $D(\eta)$ empirical moments of $\mathcal{D}$ converge to the corresponding true moments $M_{t}$ of $\mathcal{D}$, that is for all $t \leq D(\eta)$

$$
\lim _{N \rightarrow \infty}\left\|\frac{1}{N} \sum_{i=1}^{N} X_{i}^{\otimes \frac{t}{2}}\left(X_{i}^{\otimes \frac{t}{2}}\right)^{T}-M_{t}\right\|=0
$$

And if there exists a uni-variate polynomial $I_{\eta}(z) \in \mathbb{R}[z]$ of degree at most $D(\eta)$ such that

1. $I_{\eta}(z) \geq 1-\frac{z^{2}}{\eta^{2} \rho^{2}}$ for all $z \in \mathbb{R}$.
2. $\left.\mathcal{P}\right|_{D(\eta)}\left\{\|v\|^{2} \cdot \mathbb{E}_{x \in \mathcal{D}}\left[I_{\eta}(\langle v, x\rangle)\right] \leq c \eta \rho^{2}\right\}$.

Then $\mathcal{D}$ is $(c, D(\eta))$ certifiably anticoncentrated.
Lemma 8.4. For every $d \in \mathbb{N}$, the standard Gaussian distribution $\mathcal{N}\left(0, I_{d}\right)$ is $\left(c, O\left(\frac{1}{\eta^{4}}\right)\right)$-SoSanticoncentrated. In particular there exists a construction for $c \leq 2 \sqrt{e}$

First we will prove Theorem 8.3
Proof. (Theorem 8.3) First, it is a standard fact that every uni-variate polynomial inequality has a sum of squares proof. More precisely, for any $p(x) \in \mathbb{R}[x]$ satisfying $p(x) \geq 0$, then it is true that $p(x) \geq_{\operatorname{deg}(p(x))} 0$. Furthermore, this is also true over any interval [a, b]
Fact 8.5. Let $a<b$. Then, a degree $2 d$ polynomial $p(x)$ is non-negative on $[a, b]$, if and only if it can be written as

$$
\left\{\begin{array}{l}
p(x)=s(x)+(x-a)(b-x) t(x), \text { if deg }(p) \text { is even } \\
p(x)=(x-a) s(x)+(b-x) t(x), \text { if deg }(p) \text { is odd }
\end{array}\right.
$$

where $s(x), t(x)$ are SoS. In the first case, we have $\operatorname{deg}(p)=2 d$, and $\operatorname{deg}(s) \leq 2 d, \operatorname{deg}(t) \leq 2 d-2$. In the second, $\operatorname{deg}(p)=2 d+1$, and $\operatorname{deg}(s) \leq 2 d, \operatorname{deg}(t) \leq 2 d$.

In light of this fact, we use Theorem 8.3 condition 1 to lower bound $\left\langle X_{i}, v\right\rangle^{2}$ by

$$
\left.\mathcal{P}\right|_{D(\eta)}\langle X, v\rangle^{2} \geq \eta^{2} \rho^{2}\left(1-I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)\right)
$$

Therefore,

$$
\left.\mathcal{P}\right|_{D(\eta)} \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle X, v\rangle^{2} \geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2} \rho^{2}\left(1-I_{\delta}\left(\left\langle X_{i}, v\right\rangle\right)\right)
$$

Then using the certificate that $\left\{\|v\|^{2}<\rho^{2}\right\}$ we obtain

$$
\begin{gathered}
\mathcal{P} \overleftarrow{\overline{D(\eta)}} \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2} \rho^{2}\left(1-I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)\right) \geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2}\|v\|^{2}\left(1-I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)\right) \\
=\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2}\|v\|^{2}-\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2}\|v\|^{2} I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)
\end{gathered}
$$

Then using the fact that $I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)$ is SoS and $\left\{w_{i}^{2}=w_{i}\right\}-\left(1-w_{i}\right) \geq_{2} 0$, we subtract $\frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime}(1-$ $\left.w_{i}\right) I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)$ to obtain

$$
\geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2}\|v\|^{2}-\eta^{2}\|v\|^{2} \frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime} I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)
$$

Expanding out $I_{\delta}\left(\left\langle X_{i}, v\right\rangle\right)$ as a degree $D(\eta)$ polynomial with coefficients $\alpha_{1}, \ldots, \alpha_{D(\eta)}$ we have

$$
I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)=\sum_{t=1}^{T} \alpha_{t}\left\langle X_{i}, v\right\rangle^{t}
$$

We want replace the empirical average $\frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime} I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)$ with the expectation $\mathbb{E}_{X \sim N(0, I)} I_{\eta}(\langle X, v\rangle)$ and bound the error term. Indeed, we know that

$$
\begin{aligned}
& \frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime} I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)=\sum_{t=1}^{T} \alpha_{t} \frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime}\left\langle X_{i}, v\right\rangle^{t}=\sum_{t=1}^{D(\eta)} \alpha_{t}\left\langle\frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime} X_{i}^{\otimes \frac{t}{2}}\left(X_{i}^{\otimes \frac{t}{2}}\right)^{T}, v^{\otimes \frac{t}{2}}\left(v^{\otimes \frac{t}{2}}\right)^{T}\right\rangle \\
= & \sum_{t=1}^{T} \alpha_{t}\left\langle\frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime} X_{i}^{\otimes \frac{t}{2}}\left(X_{i}^{\otimes \frac{t}{2}}\right)^{T}-\underset{X \sim N(0, I)}{\mathbb{E}} X^{\otimes \frac{t}{2}}\left(X^{\otimes \frac{t}{2}}\right)^{T}, v^{\otimes \frac{t}{2}}\left(v^{\otimes \frac{t}{2}}\right)^{T}\right\rangle+\sum_{t=1}^{D(\eta)} \alpha_{t}\left\langle\underset{X \sim N(0, I)}{\mathbb{E}} X^{\otimes \frac{t}{2}}\left(X^{\otimes \frac{t}{2}}\right)^{T}, v^{\otimes \frac{t}{2}}\left(v^{\otimes \frac{t}{2}}\right)^{T}\right\rangle
\end{aligned}
$$

Then by degree $D(\eta)$ SoS Cauchy Schwarz we obtain

$$
\leq \sum_{t=1}^{D(\eta)} \alpha_{t}\left\|\frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime} X_{i}^{\otimes \frac{t}{2}}\left(X_{i}^{\otimes \frac{t}{2}}\right)^{T}-M_{t}\right\|_{F}^{2}\|v\|^{t}+\sum_{t=1}^{D(\eta)} \alpha_{t} \underset{X \sim N(0, I)}{\mathbb{E}}\langle X, v\rangle^{t}
$$

Thus for our setting of $N$ and $d$ we obtain,

$$
\begin{equation*}
=\underset{X \sim N(0, I)}{\mathbb{E}} I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)+o_{d}(1) \tag{8.1}
\end{equation*}
$$

Note that it is important that the coefficients of $I_{\eta}(z)$ are chosen independently of $d$ or at the very least don't grow too fast with respect to $d$. Our final bound is,

$$
\left.\mathcal{P}\right|_{D(\eta)} \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle X, v\rangle^{2} \geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2}\|v\|^{2}-\eta^{2}\|v\|^{2} \underset{z \sim N\left(0,\|v\|^{2}\right)}{\mathbb{E}} I_{\eta}(z)+o_{d}(1)
$$

Applying sufficient condition 2 we obtain

$$
\left.\mathcal{P}\right|_{D(\eta)}\left\{\frac{1}{N} \sum_{i=1}^{N} w_{i}\left\langle X_{i}, v\right\rangle^{2} \geq \eta^{2}\left(\frac{1}{N} \sum_{i} w_{i}\right)\|v\|^{2}-\eta^{2} \tau \rho^{2}\right\}
$$

as desired.

## 9 Certifiably Anticoncentrated Distributions

Lemma (Restatement of Lemma 8.4). For every $d \in \mathbb{N}$, the standard Gaussian distribution $\mathcal{N}\left(0, I_{d}\right)$ is (c, $O\left(\frac{1}{\eta^{4}}\right)$ )-SoS-anticoncentrated. In particular there exists a construction for $c \leq 2 \sqrt{e}$
Proof. By Theorem 8.3 it suffices to exhibit a polynomial $I_{\eta}(x)$ satisfying

1. $I_{\eta}(x) \geq 1-\frac{x^{2}}{\eta^{2} \rho^{2}}$
2. $\left.\mathcal{P}\right|_{O\left(\frac{1}{\eta^{4}}\right.}\left\{\|v\|^{2} \mathbb{E}_{x \sim N(0, I)} I_{\eta}(\langle X, v\rangle) \leq c \eta \rho^{2}\right\}$

Firstly, without loss of generality the scaling $\rho$ can be set to 1 so that $\rho=1$ and $\|v\| \leq 1$. This is because any polynomial $I_{\eta}(x)$ satisfying conditions 1 and 2 for $\rho=1$ and $\|v\| \leq 1$ can be reparameterized as $I\left(x^{\prime}\right)=I_{\eta}\left(\frac{x^{\prime}}{\rho}\right)$ and satisfy conditions 1 and 2 for $\|v\| \leq \rho$ for general $\rho$.

Next we observe that owing to the spherical symmetry of the standard Gaussian we have $\|v\|^{2} \mathbb{E}_{x \sim N(0, I)} I_{\eta}(\langle X, v\rangle)$ is a spherically symmetric polynomial in $X$ which implies it is a polynomial in $\|v\|$. Thus define

$$
H(\|v\|):=\|v\|^{2} \underset{x \sim N(0, I)}{\mathbb{E}} I_{\eta}(\langle X, v\rangle)=\|v\|^{2} \underset{x \sim N\left(0,\|v\|^{2}\right)}{\mathbb{E}} I_{\eta}(x)
$$

Furthermore we have $\left\{\|v\|^{2} \leq 1\right\} \in \mathcal{P}$ and $\|v\| \geq 0$ is SoS. Therefore, it suffices to prove the inequality $H(\|v\|) \leq \eta$ and Fact 8.5 implies condition 2. Now we construct $I_{\eta}(x)$, which we refer to as the anticoncentration polynomial. Note that the indicator function of the $[-\eta, \eta]$ interval satisfies both anticoncentration conditions. The idea is to approximate the indicator function with a polynomial. It is difficult to directly approximate the indicator function as it is not continuous. Thus we dominate the indicator by a scaled Gaussian denoted $f(x)$ which satisfies the anticoncentration conditions. We then interpolate an explicit sum of squares polynomial through $f(x)$ denoted $I_{\eta}(x)$. The key here is that any uni variate positive polynomial blows up at its tails. Thus, we must prove the approximation error of $\left|f(x)-I_{\eta}(x)\right|$ is small for some interval around the origin, and far away from the origin that the decay of the Gaussian tail dominates the growth of the approximation error.

We note that there are many different strategies to construct polynomials satisfying the above criterion, and we will satisfy ourselves with proving the Gaussian is $\left(2 \sqrt{e}, O\left(\frac{1}{\eta^{4}}\right)\right.$ )-certifiably anticoncentrated.

First let $f(x)=\sqrt{e} \exp \left(-\frac{x^{2}}{2 \eta^{2}}\right)$. For simplicity we will design $f(x)$ such that $f( \pm \eta)=1$ to satisfy the first anticoncentration condition. Checking the second condition we find that

$$
\begin{gathered}
\|v\|^{2} \underset{x \sim N\left(0,\|v\|^{2}\right)}{\mathbb{E}} f(x)=\|v\|^{2} \int \frac{\sqrt{e}}{\sqrt{2 \pi}\|v\|} \exp \left(-\frac{x^{2}}{2 \eta^{2}}-\frac{x^{2}}{2\|v\|^{2}}\right) d x \\
=\|v\|^{2} \frac{\sqrt{e}}{\sqrt{2 \pi}\|v\|} \int \exp \left(-\frac{x^{2}}{2\left(\frac{\eta^{2}\|v\|^{2}}{\eta+\|v\|^{2}}\right)}\right) d x=\|v\|^{2} \frac{\eta \sqrt{e}}{\sqrt{\eta^{2}+\|v\|^{2}}} \\
\leq\|v\|^{2} \frac{\eta \sqrt{e}}{\|v\|} \leq \eta \sqrt{e}
\end{gathered}
$$

Where in the last inequality we used $0 \leq\|v\|^{2} \leq 1$.
Intuitively, if we interpolate a sum of squares polynomial $I_{\eta}(x)$ that closely approximates $f(x)$ in an interval around the origin, then $\mathbb{E}_{x \sim N\left(0,\|v\|^{2}\right)} I_{\eta}(x) \approx \mathbb{E}_{x \sim N\left(0,\|v\|^{2}\right)} f(x)$. Let $\left(x_{0}, x_{1}, \ldots, x_{n}\right)$ be evenly spaced points at intervals of length $v \eta^{r}$ ranging from $\left[-\frac{v \eta \eta^{r}}{2}, \frac{v \eta \eta^{r}}{2}\right]$ where we eventually set $v$ to be a constant and $r=4$. Let $\left(y_{0}, \ldots, y_{n}\right)$ be the set of evaluations $y_{i}=f\left(x_{i}\right)$. Let $I_{\eta}(x)$ be the following degree $2 n$ polynomial.

$$
\begin{aligned}
& I_{\eta}(x)=\frac{\left(x-x_{1}\right)^{2}\left(x-x_{2}\right)^{2} \ldots\left(x-x_{n}\right)^{2}}{\left(x_{0}-x_{1}\right)^{2}\left(x_{0}-x_{2}\right)^{2} \ldots\left(x_{0}-x_{n}\right)^{2}} y_{0}+ \frac{\left(x-x_{0}\right)^{2}\left(x-x_{1}\right)^{2} \ldots\left(x-x_{n}\right)^{2}}{\left(x_{1}-x_{0}\right)^{2}\left(x_{1}-x_{2}\right)^{2} \ldots\left(x_{1}-x_{n}\right)^{2}} y_{1}+\ldots \\
&+\frac{\left(x-x_{0}\right)^{2}\left(x-x_{1}\right)^{2} \ldots\left(x-x_{n-1}\right)^{2}}{\left(x_{n}-x_{0}\right)^{2}\left(x_{n}-x_{1}\right)^{2} \ldots\left(x_{n}-x_{n-1}\right)^{2}} y_{n} \\
&=\sum_{i=1}^{n}\left(\prod_{\substack{0 \leq j \leq n \\
i \neq j}} \frac{\left(x-x_{i}\right)^{2}}{\left(x_{i}-x_{j}\right)^{2}}\right) y_{j}
\end{aligned}
$$

$I_{\eta}(x)$ is the standard interpolation polynomial where each term is squared so as to be a sum of squares. Let $R_{2 n}(y)$ be the error term over the interval $[-y, y]$ be $R_{2 n}(y)=\max _{x \in[-y, y]}\left|f(x)-I_{\eta}(x)\right|$. It is easy to show the interpolation error is

$$
R_{2 n}(y)=\frac{1}{(2 n+1)!} \max _{x \in[-y, y]}\left|f^{2 n+1}(x)\right| \prod_{i=0}^{n}\left(x-x_{i}\right)^{2}
$$

One way to prove the above equality is to think of the construction of $I_{\eta}(x)$ as follows. Let $\tilde{I}(x)$ be the unique degree $2 n$ interpolation of points $\left\{\left(x_{i}, f\left(x_{i}\right)\right)\right\}_{i \in[n]}$ and $\left\{\left(x_{i}^{\prime}, f\left(x_{i}^{\prime}\right)\right)\right\}_{i \in[n]}$, which is not necessarily a sum of squares. . It is a standard fact in polynomial approximation theory, [Sau97], that the error $\tilde{I}_{2 n}(y)=\max _{x \in[-y, y]}|f(x)-\tilde{I}(x)|$ has the form.

$$
\tilde{R}_{2 n}(y)=\frac{1}{(2 n+1)!} \max _{x \in[-y, y]}\left|f^{2 n+1}(x)\right| \prod_{i=0}^{n}\left(x-x_{i}\right) \prod_{i=0}^{n}\left(x-x_{i}^{\prime}\right)
$$

It is easy to check that $I_{\eta}(x)=\lim _{\left(x_{0}^{\prime}, \ldots, x_{n}^{\prime}\right) \rightarrow\left(x_{0}, \ldots, x_{n}\right)} \tilde{I}(x)$. Thus

$$
R_{2 n}(y)=\lim _{\left(x_{0}^{\prime}, \ldots, x_{n}^{\prime}\right) \rightarrow\left(x_{0}, \ldots, x_{n}\right)} \tilde{R}(y)=\frac{1}{(2 n+1)!} \max _{x \in[-y, y]}\left|f^{2 n+1}(x)\right| \prod_{i=0}^{n}\left(x-x_{i}\right)^{2}
$$

as desired.
Now we verify anticoncentration condition 2

$$
\begin{align*}
& \|v\|^{2} \underset{x \sim N\left(0,\|v\|^{2}\right)}{\mathbb{E}} I_{\eta}(x) \leq\|v\|^{2} \underset{x \sim N\left(0,\|v\|^{2}\right)}{\mathbb{E}}\left|f(x)-I_{\eta}(x)\right|+\|v\|^{2} \underset{x \sim N\left(0,\|v\|^{2}\right)}{\mathbb{E}} f(x) \\
& \leq\|v\|^{2} \underset{x \sim N\left(0,\|v\|^{2}\right)}{\mathbb{E}}\left|f(x)-I_{\eta}(x)\right|+\sqrt{e} \eta  \tag{9.1}\\
& =\|v\|^{2}\left(\int_{-\infty}^{-\frac{v \eta^{r} n}{2}} R_{2 n}(y) \frac{1}{\sqrt{2 \pi}\|v\|} \exp \left(-\frac{y^{2}}{2\|v\|^{2}}\right) d y+\int_{-\frac{v \eta^{r} n}{2}}^{\frac{v \eta^{r} n}{2}} R_{2 n}(y) \frac{1}{\sqrt{2 \pi}\|v\|} \exp \left(-\frac{y^{2}}{2\|v\|^{2}}\right) d y\right. \\
& \left.+\int_{\frac{v \eta^{2} n}{2}}^{\infty} R_{2 n}(y) \frac{1}{\sqrt{2 \pi}\|v\|} \exp \left(-\frac{y^{2}}{2\|v\|^{2}}\right) d y\right)+\sqrt{e} \eta
\end{align*}
$$

Since we defined $R_{2 n}(y)$ to be the maximum error in the $[-y, y]$ interval, it is monotonic, and we upper bound it by its evaluation at its rightmost endpoint $R_{2 n}\left(\frac{v \eta^{r} n}{2}\right)$.

$$
\begin{aligned}
& \leq\|v\|^{2}\left(\int_{-\infty}^{-\frac{v \eta^{r} n}{2}} R_{2 n}(y) \frac{1}{\sqrt{2 \pi}\|v\|} \exp \left(-\frac{y^{2}}{2\|v\|^{2}}\right) d y\right.+R_{2 n}\left(\frac{v \eta^{r} n}{2}\right) \int_{-\frac{v \eta^{r} n}{2}}^{\frac{v \eta^{r} n}{2}} \frac{1}{\sqrt{2 \pi}\|v\|} \exp \left(-\frac{y^{2}}{2\|v\|^{2}}\right) d y \\
&\left.+\int_{\frac{v \eta^{r} n}{2}}^{\infty} R_{2 n}(y) \frac{1}{\sqrt{2 \pi}\|v\|} \exp \left(-\frac{y^{2}}{2\|v\|^{2}}\right) d y\right)+\sqrt{e} \eta \\
&=\|v\|^{2}\left(R_{2 n}\left(\frac{v \eta^{r} n}{2}\right)+2 \int_{\frac{v \eta^{r} n}{2}}^{\infty} R_{2 n}(y) \exp \left(-\frac{y^{2}}{2}\right) d y\right)+\sqrt{e} \eta
\end{aligned}
$$

Thus it suffices to show

$$
R_{2 n}\left(\frac{v \eta^{r} n}{2}\right)+2 \int_{\frac{v \eta^{r} n}{2}}^{\infty} R_{2 n}(y) \exp \left(-\frac{y^{2}}{2}\right) d y \leq \sqrt{e} \eta
$$

Without loss of generality let $\rho=1$. Let's start with $R_{2 n}\left(\frac{v \eta^{r} n}{2}\right)$

$$
\begin{equation*}
R_{2 n}\left(\frac{v \eta^{r} n}{2}\right) \leq \frac{1}{(2 n+1)!} \max _{x \in[-y, y]}\left|f^{2 n+1}(x)\right| \prod_{i=0}^{n}\left(x-x_{i}\right)^{2} \tag{9.2}
\end{equation*}
$$

Since $f(x)$ is a scaled Gaussian, we have directly from its Taylor expansion

$$
\max _{x \in \mathbb{R}}\left|f^{2 n+1}(x)\right|<\max _{x \in \mathbb{R}}\left|f^{2 n+2}(x)\right|=\left|f^{2 n+2}(0)\right|=\frac{(2 n+2)!}{(n+1)!}\left(\frac{1}{2 \eta^{2}}\right)^{2 n+2}
$$

Plugging the above bound into 9.2 we obtain

$$
\begin{aligned}
& R_{2 n}\left(\frac{v \eta^{r} n}{2}\right)<\frac{v^{2} \eta^{2 r}\left(2^{2} v^{2} \eta^{2 r}\right) \ldots\left(n^{2} v^{2} \eta^{2 r}\right)}{(2 n+1)!}\left|f^{2 n+2}(0)\right|=\frac{v^{2 n} \eta^{2 r n}(n!)^{2}}{(2 n+1)!}\left|f^{2 n+2}(0)\right|=\frac{v^{2 n} \eta^{2 r n}(n!)^{2}}{(2 n+1)!} \frac{(2 n+2)!}{(n+1)!}\left(\frac{1}{2 \eta^{2}}\right)^{2 n+2} \\
& =2 v^{2 n} \eta^{2 r n}(n!)\left(\frac{1}{2 \eta^{2}}\right)^{2 n+2}=2\left(\frac{1}{4 \eta^{4}}\right)\left(\frac{v \eta^{r-2}}{2}\right)^{2 n} n!=2\left(\frac{1}{4 \eta^{4}}\right) \sqrt{2 \pi n}\left(\frac{n}{e}\right)^{n}\left(\frac{v \eta^{r-2}}{2}\right)^{2 n}=2\left(\frac{1}{4 \eta^{4}}\right) \frac{\sqrt{2 \pi n}}{2^{2 n} e^{n}}\left(v \eta^{r-2} \sqrt{n}\right)^{2 n}
\end{aligned}
$$

Where the factorial approximation is Stirling's. Thus a sufficient condition for error decay is $v \eta^{r-2} \sqrt{n} \leq 1$. Then for the benefit of tail error decay, we will set $n$ to saturate the center interval error $n:=\frac{1}{\sqrt{e} \nu^{2} \eta^{2(r-2)}}$ where the $\sqrt{e}$ will be to accommodate for some discrepancy in error in the tail bound. Intuitively, the larger the value of $n$ the further our the interpolation points, and the better the Gaussian tail dominates the polynomial growth in error.

Next we show the tail error is small.

$$
\begin{gathered}
\int_{\frac{v \eta^{r} n}{2}}^{\infty} R_{2 n}(y) \exp \left(-\frac{y^{2}}{2}\right) d y \leq \int_{\frac{v \eta^{r} n}{2}}^{\infty} \frac{\left(y+\frac{v \eta^{r} n}{2}\right)^{2 n}}{(2 n+1)!}\left|f^{2 n+1}(0)\right| \exp \left(-\frac{y^{2}}{2}\right) d y \\
\quad \leq \int_{\frac{v \eta^{r} n}{2}}^{\infty} \frac{\left(y+\frac{v \eta^{r} n}{2}\right)^{2 n}}{(2 n+1)!} \frac{(2 n+2)!}{(n+1)!}\left(\frac{1}{2 \eta^{2}}\right)^{2 n+2} \exp \left(-\frac{y^{2}}{2}\right) d y \\
\quad \leq 2 \int_{\frac{v \eta^{r} n}{2}}^{\infty} \frac{\left(y+\frac{v \eta^{r} n}{2}\right)^{2 n}}{n!}\left(\frac{1}{2 \eta^{2}}\right)^{2 n+2} \exp \left(-\frac{y^{2}}{2}\right) d y
\end{gathered}
$$

The integrand evaluated at $y=\frac{v \eta^{r} n}{2}$ is

$$
\leq \frac{1}{\sqrt{2 \pi n}}\left(\frac{1}{4 \eta^{4}}\right)\left(\frac{v \eta^{r-2} \sqrt{e n}}{2}\right)^{2 n} \exp \left(-\frac{\left(v \eta^{r} n\right)^{2}}{8}\right)
$$

By our choice of $n$ we have both the exponential and the error term falling to zero rapidly. For $r=4$ and $v=1 / 100$ we have $n=O\left(\frac{1}{\eta^{4}}\right)$ for a degree $2 n=O\left(\frac{1}{\eta^{4}}\right)$ polynomial.

The following lemma establishes that the sufficient conditions of Theorem 8.3 are naturally extended under linear transformations of the data set.

Lemma 9.1. (Anticoncentration under Linear Transformation) Let $\mathcal{D}$ be a (c, $D(\eta))$ certifiably anticoncentrated distribution. Let $I_{\eta}(z) \in \mathbb{R}[z]$ be a uni-variate polynomial satifying the conditions of Theorem 8.3. Let $x \sim \mathcal{D}$ be a random variable drawn from $\mathcal{D}$. Then for any invertible linear transformation $A \in \mathbb{R}^{d \times d}$, we denote the distribution of $A x$ as $A(\mathcal{D})$. Let $\Sigma=A A^{T}$ be the covariance of $A(\mathcal{D})$ with eigenvalues $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{d}$. Then $A(\mathcal{D})$ is $\left(c \frac{\lambda_{1}^{3 / 2}}{\lambda_{d}^{3 / 2}}, D(\eta)\right)$ certifiably anticoncentrated.

Proof. In light of this fact, we use Theorem 8.3 condition 1 to lower bound $\left\langle X_{i}, v\right\rangle^{2}$ by

$$
\left.\mathcal{P}\right|_{\overline{D(\eta)}}\langle X, v\rangle^{2} \geq \eta^{2}\left(1-I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)\right)
$$

Therefore,

$$
\left.\mathcal{P}\right|_{D(\eta)} \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle X, v\rangle^{2} \geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2}\left(1-I_{\delta}\left(\left\langle X_{i}, v\right\rangle\right)\right)
$$

Then using the certificate that $\left\{\|v\|^{2}<1\right\}$ we obtain

$$
\begin{gathered}
\left.\mathcal{P}\right|_{D(\eta)} \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2}\left(1-I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)\right) \geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{\eta} \frac{\left\|\Sigma^{1 / 2} v\right\|^{2}}{\left\|\Sigma^{1 / 2}\right\|_{o p}^{2}}\left(1-I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)\right) \\
\quad=\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2} \frac{\left\|\Sigma^{1 / 2} v\right\|^{2}}{\left\|\Sigma^{1 / 2}\right\|_{o p}^{2}}-\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2} \frac{\left\|\Sigma^{1 / 2} v\right\|^{2}}{\left\|\Sigma^{1 / 2}\right\|_{o p}^{2}} I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)
\end{gathered}
$$

Then using the fact that $I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)$ is SoS and $\left\{w_{i}^{2}=w_{i}\right\}-\left(1-w_{i}\right) \geq_{2} 0$, we subtract $\eta^{2} \frac{\left\|\Sigma^{1 / 2} v\right\|^{2}}{\left\|\Sigma^{1 / 2}\right\|_{o p}^{2}} \frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime}\left(1-w_{i}\right) I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)$ to obtain

$$
\geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2} \frac{\left\|\Sigma^{1 / 2} v\right\|^{2}}{\left\|\Sigma^{1 / 2}\right\|_{o p}^{2}}-\eta^{2} \frac{\left\|\Sigma^{1 / 2} v\right\|^{2}}{\left\|\Sigma^{1 / 2}\right\|_{o p}^{2}} \frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime} I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)
$$

We know from the moment certificates, 9.2 , that

$$
\frac{1}{M} \sum_{i=1}^{N} w_{i}^{\prime} I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)=\underset{X \sim N(0, \Sigma)}{\mathbb{E}} I_{\eta}\left(\left\langle X_{i}, v\right\rangle\right)+O(\varepsilon)=\underset{X \sim N(0, I)}{\mathbb{E}} I_{\eta}\left(\left\langle X, \Sigma^{1 / 2} v\right\rangle\right)+O(\varepsilon)
$$

so thus far we have shown,

$$
\left.\mathcal{P}\right|_{D(\eta)} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle X, v\rangle^{2} \geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2} \frac{\left\|\Sigma^{1 / 2} v\right\|^{2}}{\left\|\Sigma^{1 / 2}\right\|_{o p}^{2}}-\eta^{2} \frac{\left\|\Sigma^{1 / 2} v\right\|^{2}}{\left\|\Sigma^{1 / 2}\right\|_{o p}^{2}} \underset{x \sim N(0, I)}{\mathbb{E}} I_{\eta}\left(\left\langle X, \Sigma^{1 / 2} v\right\rangle\right)+O(\varepsilon)
$$

For the first term on the right hand side, lower bound $\left\|\Sigma^{1 / 2} v\right\| \geq \lambda_{d}^{2}\|v\|^{2}$. This follows by the PSD'ness of $\Sigma^{1 / 2}$ via degree 2 SoS. Then change the variable $\omega=\Sigma^{1 / 2} v$ to obtain

$$
\geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2} \frac{\lambda_{d}\|v\|^{2}}{\lambda_{1}}-\eta^{2} \frac{\|w\|^{2}}{\lambda_{1}} \underset{x \sim N(0, I)}{\mathbb{E}} I_{\eta}(\langle X, w\rangle)+O(\varepsilon)
$$

Consider the second term. Observing that $0 \leq\|w\|^{2} \leq \lambda_{1}$ and scaling Theorem 8.3 condition 2 by $\lambda_{1}$ we obtain

$$
\geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{2} \frac{\lambda_{d}\|v\|^{2}}{\lambda_{1}}-\eta^{2}(c \eta)+O(\varepsilon)
$$

Let $\eta^{\prime}=\eta \sqrt{\frac{\lambda_{d}}{\lambda_{1}}}$, then we conclude

$$
\geq \frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \eta^{\prime 2}\|v\|^{2}-\eta^{\prime 2}\left(c \frac{\lambda_{1}^{3 / 2}}{\lambda_{d}^{3 / 2}} \eta^{\prime}\right)+O(\varepsilon)
$$

as desired.
Corollary 9.2. (Anticoncentration of Spherically Symmetric Strongly Log Concave Distributions) Let $p\left(x_{1}, \ldots, x_{d}\right)$ be a distribution of the form

$$
p\left(x_{1}, \ldots, x_{d}\right) \propto \exp (-h(\|x\|))
$$

For $h(x) m$-strongly convex. Then $p(x)$ is $\left(\sqrt{2 e m}, O\left(\frac{1}{\eta^{4}}\right)\right)$-certifiably anticoncentrated.
Proof. The proof follows exactly as that of the Gaussian. We begin with

$$
H(\|v\|):=\|v\|^{2} \underset{x \sim p(x)}{\mathbb{E}} I_{\eta}(\langle X, v\rangle)=\|v\|^{2} \underset{x \sim p\left(\frac{x}{\|v\|}\right) \frac{1}{\|v\|}}{\mathbb{E}} I_{\eta}(x)
$$

Applying $m$-strong concavity we obtain

$$
\|v\|^{2} \underset{x \sim p\left(\frac{x}{\|v\|}\right) \frac{1}{\|v\|}}{\mathbb{E}} f(x) \leq\|v\|^{2} \int \frac{\sqrt{e m / 2}}{\sqrt{2 \pi}\|v\|} \exp \left(-\frac{x^{2}}{2 \eta^{2}}-\frac{m x^{2}}{\|v\|^{2}}\right) d x \leq\|v\|^{2} \frac{\eta \rho \sqrt{e m / 2}}{\|v\|} \leq \eta \sqrt{e m / 2}
$$

With the polynomial approximation calculations following the exact same template.

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## A Regression Missing Proofs

Lemma (Restatement of Lemma 3.6). Running Algorithm Algorithm 2 a total of $O\left(\frac{1}{\beta}\right)$ times produces a list $L=\left\{\ell_{1}, \ldots, \ell_{O\left(\frac{1}{\beta}\right)}\right\}$ such that with probability $1-c$, there exists a list element $\ell_{i} \in L$ satisfying $\left\|\ell_{i}-\ell^{\prime}\right\| \leq \frac{\rho}{2}$ where $c$ is a small constant. Minor modifications enable the algorithm to succeed with high probability.

Proof. There are a variety of techniques for boosting the success probability to $1-\frac{1}{\text { poly(d) }}$. One such technique is to make the rounding algorithm deterministic. Instead of using selection strategy $j \in \mathcal{S}_{v}$, simply condition on a variable $j \in[N]$ satisfying

$$
\|Q\|_{\text {nuc }}-\underset{b_{j}}{\mathbb{E}}\left\|\left.Q\right|_{w_{j}=b_{j}}\right\|_{\text {nuc }} \geq \Omega\left(\beta \eta^{2} \rho^{2}\right)
$$

Such a variable necessarily exists, because we found a distribution over $j$ where the above inequality holds in expectation. Furthermore, enumerate every $\{0,1\}$ conditioning up to a depth of $R=O\left(\frac{1}{\beta^{4}}\right)$. This implicitly defines a tree of pseudoexpectations. We can compute the probability of reaching each leaf via its $\{0,1\}$ conditioning sequence. In effect, we can compute a probability distribution over a list $L$ of $2^{R}$ estimates to $\ell^{\prime}$. Then applying the same analysis in Theorem 3.5 the probability over this distribution that $\ell_{i} \in L$ is close to $\ell^{\prime}$ is greater than $\frac{\beta}{4}$. A simple clustering algorithm which groups vectors endowed with high probability mass generates a list of length $O\left(\frac{1}{\beta}\right)$. This procedure is deterministic, and can be run changing neither the runtime nor error gaurantees.

Lemma (Restatement of Lemma 7.2). Let $\tilde{\mathbb{E}}$ be a pseudodistribution satisfying $\mathcal{P}$. The following holds.

$$
\begin{align*}
& \widetilde{\mathbb{V}}[\langle\ell, u\rangle] \leq\left(1+o_{d}(1)\right) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]+o_{d}(1)  \tag{A.1}\\
& \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right] \leq\left(1+o_{d}(1)\right) \widetilde{\mathbb{V}}[\langle\ell, u\rangle]+o_{d}(1) \tag{A.2}
\end{align*}
$$

The proofs of A. 9 and A. 10 are nearly identical. We include both below.
Proof. (Proof of A.9)
Let $u$ be a unit direction $u \in \mathcal{S}^{d-1}$. We know

$$
\widetilde{\mathbb{V}}[\langle\ell, u\rangle]=\widetilde{\mathbb{V}}\left[\ell^{T}\left(I-\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}\right)+\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\ell, X_{i}\right\rangle X_{i}^{T} u\right]
$$

Using pseudovariance triangle inequality

$$
\begin{equation*}
\left.\leq(1+\psi) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\ell, X_{i}\right\rangle\left\langle X_{i}, u\right\rangle\right]+\left(\frac{1+\psi}{\psi}\right) \widetilde{\mathbb{V}}\left[\left\langle\ell u^{T}, I-\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}\right)\right\rangle\right] \tag{A.3}
\end{equation*}
$$

Applying $\ell_{2}$ minimization constraint 6 to the first term we obtain

$$
\begin{equation*}
\left.=(1+\psi) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} y_{i}\left\langle X_{i}, u\right\rangle\right]+\left(\frac{1+\psi}{\psi}\right) \widetilde{\mathbb{V}}\left[\left\langle\ell u^{T}, I-\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}\right)\right\rangle\right] \tag{A.4}
\end{equation*}
$$

Consider the second term on the right hand side, which we upper bound as follows.

$$
\left.\left.\widetilde{\mathbb{V}}\left[\left\langle\ell u^{T}, I-\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}\right)\right\rangle\right] \leq \tilde{\mathbb{E}}\left[\left\langle\ell u^{T}, I-\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}\right)\right\rangle^{2}\right]
$$

First using deg 2 SOS Cauchy-Schwarz, and then both the scaling constraint and the moment bound constraint we obtain,

$$
\begin{equation*}
\leq 2 \tilde{\mathbb{E}}\left[\|\ell\|^{2}\left\|I-\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}\right\|^{2}\right] \leq \rho^{2} \varepsilon \tag{A.5}
\end{equation*}
$$

Plugging back into equation A. 4 we obtain

$$
\begin{gather*}
\widetilde{\mathbb{V}}[\langle\ell, u\rangle] \leq(1+\psi) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} y_{i}\left\langle X_{i}, u\right\rangle\right]+\left(\frac{1+\psi}{\psi}\right) \rho^{2} \varepsilon  \tag{A.6}\\
=(1+\psi) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle+\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\left\langle X_{i}, u\right\rangle\right]+\frac{1+\psi}{\psi} \rho^{2} \varepsilon
\end{gather*}
$$

Using the pseudovariance triangle inequality we obtain

$$
\begin{array}{r}
\leq(1+\psi)^{2} \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]+\frac{(1+\psi)^{2}}{\psi} \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\left\langle X_{i}, u\right\rangle\right]  \tag{A.7}\\
+\frac{1+\psi}{\psi} \rho^{2} \varepsilon
\end{array}
$$

Consider the second term. Subtracting the identity and adding it back we obtain

$$
\widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\left\langle X_{i}, u\right\rangle\right] \leq \widetilde{\mathbb{V}}\left[\left\langle\tilde{\mathbb{E}}[\ell] u^{T}, \frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}-I\right\rangle+\langle\tilde{\mathbb{E}}[\ell], u\rangle\right]
$$

Noting that pseudovariance is invariant under constant shifts we obtain.

$$
\leq \widetilde{\mathbb{V}}\left[\left\langle\tilde{\mathbb{E}}[\ell] u^{T}, \frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}-I\right\rangle\right] \leq \tilde{\mathbb{E}}\left[\left\langle\tilde{\mathbb{E}}[\ell] u^{T}, \frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}-I\right\rangle^{2}\right]
$$

Applying Cauchy-Schwarz, then the moment constraints, then pseudoexpectation CauchySchwarz, then the scaling constraints we obtain

$$
\begin{equation*}
\leq \tilde{\mathbb{E}}\left[\|\tilde{\mathbb{E}}[\ell]\|^{2}\left\|\frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}-I\right\|_{F}^{2}\right] \leq\|\tilde{\mathbb{E}}[\ell]\|^{2} \varepsilon \leq \tilde{\mathbb{E}}\left[\|\ell\|^{2}\right] \varepsilon \leq \rho^{2} \varepsilon \tag{A.8}
\end{equation*}
$$

Plugging this bound back into A. 7 we obtain

$$
\widetilde{\mathbb{V}}[\langle\ell, u\rangle] \leq(1+\psi)^{2} \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]+\frac{(1+\psi)^{2}}{\psi} \rho^{2} \varepsilon+\frac{1+\psi}{\psi} \rho^{2} \varepsilon
$$

For $\psi=\sqrt{\varepsilon}$ we obtain

$$
\begin{equation*}
\widetilde{\mathbb{V}}[\langle\ell, u\rangle] \leq\left(1+o_{d}(1)\right) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]+o_{d}(1) \tag{A.9}
\end{equation*}
$$

Proof. (Proof of A.10)

$$
\widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right]=\widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} y_{i}\left\langle X_{i}, u\right\rangle-\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\left\langle X_{i}, u\right\rangle\right]
$$

by pseudovariance triangle inequality

$$
\leq(1+\psi) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} y_{i}\left\langle X_{i}, u\right\rangle\right]+\frac{1+\psi}{\psi} \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\left\langle X_{i}, u\right\rangle\right]
$$

applying the bound in A. 8

$$
\leq(1+\psi) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} y_{i}\left\langle X_{i}, u\right\rangle\right]+\frac{1+\psi}{\psi} \rho^{2} \varepsilon
$$

Applying the $\ell_{2}$ minimization constraint we obtain

$$
\begin{gathered}
=(1+\psi) \widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle\ell, X_{i}\right\rangle\left\langle X_{i}, u\right\rangle\right]+\frac{1+\psi}{\psi} \rho^{2} \varepsilon \\
=(1+\psi) \widetilde{\mathbb{V}}\left[\left\langle\ell u^{T}, \frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}-I\right\rangle+\langle\ell, u\rangle\right]+\frac{1+\psi}{\psi} \rho^{2} \varepsilon
\end{gathered}
$$

then by pseudovariance triangle inequality

$$
\leq(1+\psi)^{2} \widetilde{\mathbb{V}}[\langle\ell, u\rangle]+\frac{(1+\psi)^{2}}{\psi} \widetilde{\mathbb{V}}\left[\left\langle\ell u^{T}, \frac{1}{M} \sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}-I\right\rangle\right]+\frac{1+\psi}{\psi} \rho^{2} \varepsilon
$$

Applying the bound in A. 5 to the second term we conclude

$$
\widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right] \leq(1+\psi)^{2} \widetilde{\mathbb{V}}[\langle\ell, u\rangle]+\frac{(1+\psi)^{2}}{\psi} \rho^{2} \varepsilon+\frac{1+\psi}{\psi} \rho^{2} \varepsilon
$$

For $\psi=\sqrt{\varepsilon}$ we conclude

$$
\begin{equation*}
\widetilde{\mathbb{V}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(y_{i}-\left\langle\tilde{\mathbb{E}}[\ell], X_{i}\right\rangle\right)\left\langle X_{i}, u\right\rangle\right] \leq\left(1+o_{d}(1)\right) \widetilde{\mathbb{V}}[\langle\ell, u\rangle]+o_{d}(1) \tag{A.10}
\end{equation*}
$$

as desired.

## B Mean Algorithms

## SDP for Robust Mean Estimation

Here we write down the list decoding algorithm for mean estimation. Let RobustMeanSDP( $\mathcal{D}, \rho)$ take as input the dataset $\mathcal{D}$, and the parameter $\rho$.

## Algorithm 4: RobustMeanSDP

Result: A degree $D$ pseudoexpectation functional $\tilde{\mathbb{E}}_{\zeta}$
1 Inputs: $(\mathcal{D}, \rho)$ Dataset, and upper bound on $\|\mu\|$
2

$$
\begin{array}{ll}
\underset{\text { degree } \mathrm{D} \text { pseudoexpectations } \tilde{\mathbb{E}}}{\operatorname{minimize}} & \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[w_{i}\right]^{2} \\
\text { such that for all } & \tilde{\mathbb{E}}\left[\left(w_{i}^{2}-w_{i}\right)\right]=0, i \in[N], \\
& \tilde{\mathbb{E}}\left[\left(\sum_{i=1}^{N} w_{i}-M\right)\right]=0, \\
& \tilde{\mathbb{E}}\left[\|\left(\frac{1}{M} \sum_{i=1}^{N} w_{i}\left(X_{i}-\hat{\mu}\right)^{\otimes \frac{k}{2}}\left(\left(X_{i}-\hat{\mu}\right)^{\otimes \frac{k}{2}}\right)^{T}-M_{k} \|_{F}^{2}-\varepsilon_{k}\right)\right] \leq 0, \\
& \tilde{\mathbb{E}}\left[\left(\|\hat{\mu}\|^{2}-\rho^{2}\right)\right] \leq 0
\end{array}
$$

3 return: $\tilde{\mathbb{E}}_{\zeta}$

## Algorithms for Robust Mean Estimation

The algorithms are identical to those of robust regression up to parameter choices and the choice of rounding strategy $\mathcal{S}$.

```
Algorithm 5: Mean Preprocessing Algorithm
    Result: A list \(L\) of means of length \(O\left(\frac{1}{\beta}\right)\)
    inputs: \(\tilde{\mathbb{E}}_{\zeta}\)
    \(L=\emptyset\)
    for \(j=1\) to \(N\) do
        \(L=L \cup\left(\tilde{\mathbb{E}}\left[\hat{\mu} \mid w_{j}=1\right], \tilde{\mathbb{E}}\left[w_{j}\right]\right)\)
    end
    return: ExtractList(L)
```

```
Algorithm 6: Extract List
    Result: A list of vectors \(L=\left\{\mu_{1}, \ldots, \mu_{A}\right\}\)
    Inputs: A list \(L^{\prime}:=\left\{\left(\mu_{i}, p_{i}\right)\right\}\)
    \(L=\emptyset\)
    while \(L^{\prime} \neq \emptyset\) do
        Let \(\mu_{0}\) be any leaf in \(L^{\prime}\)
        for \((\mu, p) \in L^{\prime}\) do
            \(M=0\)
            if \(\left\|\mu-\mu_{0}\right\| \leq \frac{\rho}{4}\) then
                \(M=M+p\)
            end
    end
    if \(M \geq \frac{\beta}{4}\) then
            \(L=L \cup \mu_{0}\)
        else
            \(L=L^{\prime} \backslash \mu_{0}\)
        end
    end
    return: L
```

```
Algorithm 7: Mean RoundTree Algorithm
    Result: A set of leaves \(\mathcal{L}=\left\{T_{h}\right\}\) indexed by their position \(h\). Each leaf is a tuple
                                    \(T_{h}=\left(\mathbb{P}[h], \tilde{\mathbb{E}}_{\zeta, h}\right)\) consisting of the probability assigned to \(T_{h}\), and the corresponding
                    pseudoexpectation at \(T_{h}\)
    inputs: \(\left(\tilde{\mathbb{E}}_{\zeta, h}, \mathbb{P}[h], \rho\right)\)
    Let \(Q=\tilde{\mathbb{E}}_{\zeta, h}\left[\left(\hat{\mu}-\tilde{\mathbb{E}}_{\zeta, h}[\hat{\mu}]\right)\left(\hat{\mu}-\tilde{\mathbb{E}}_{\zeta, h}[\hat{\mu}]\right)^{T}\right]\) be the pseudocovariance matrix of the estimator \(\hat{\mu}\)
    Let \((\lambda, v)\) be top eigenvalue/vector of \(Q\)
    if \(\lambda>\frac{\beta \rho^{2}}{16}\) then
    Let \(\mathcal{S}_{v}\) be a probability distribution over \([N]\);
        Where for any \(j \in N\) we have \(\mathcal{S}_{v}(j) \stackrel{\text { def }}{=} \frac{\widetilde{\mathbf{V}}_{\zeta, h}\left[w_{j}\left\langle X_{j}-\tilde{\mathbb{E}}_{\zeta, h}[\hat{\mu}], v\right\rangle\right]}{\sum_{i=1}^{N} \tilde{\mathbf{V}}_{\zeta, h}\left[w_{j}\left\langle X_{j}-\tilde{E}_{\zeta, h}[\hat{\mu}], v\right\rangle\right]}\)
        Sample \(j \sim \mu\)
        Let \(\mathbb{P}[h \circ 1]=\mathbb{P}[h] \tilde{\mathbb{E}}_{\zeta, h}\left[w_{j}\right]\)
        Let \(\mathbb{P}[h \circ 0]=\mathbb{P}[h] \tilde{\mathbb{E}}_{\zeta, h}\left[1-w_{j}\right]\)
        Let \(\tilde{\mathbb{E}}_{\zeta, h \circ 0}=\left.\tilde{\mathbb{E}}_{\zeta, h}\right|_{w_{j}=0}\)
        Let \(\tilde{\mathbb{E}}_{\zeta, h \circ 1}=\left.\tilde{\mathbb{E}}_{\zeta, h}\right|_{w_{j}=1}\)
        return: \(\left\{\operatorname{RoundTree}\left(\tilde{\mathbb{E}}_{\zeta, h \circ 0}, \mathbb{P}[h \circ 0], \rho\right)\right\} \cup\left\{\operatorname{RoundTree}\left(\tilde{\mathbb{E}}_{\zeta, h \circ 1}, \mathbb{P}[h \circ 1], \rho\right)\right\}\);
    else
        return: \(\left\{\left(\tilde{\mathbb{E}}_{\zeta, h}[\hat{\mu}], \mathbb{P}[h], \rho\right)\right\}\)
    end
```

```
Algorithm 8: ListDecodeMean Algorithm
    Result: A list of mean \(L=\left\{\mu_{1}, \ldots, \mu_{A}\right\}\)
    inputs: ( \(\tilde{\mathbb{E}}_{\zeta}, \rho\) )
    \(L=\) MeanPreprocessing \(\left(\tilde{\mathbb{E}}_{\zeta}\right)\)
    for \(t \in \log \left(\frac{m_{2}}{\beta^{1-1 / k}}\right)\) do
    \(\rho_{t}=\frac{\rho}{2^{t}}\)
    \(\%\) Let \(Y\) to be a list of pseudoexpectations
    \(Y=\emptyset\)
    for \(\mu_{i} \in L\) do
        for \(X_{j} \in \mathcal{D}\) do
                \(X_{j}=X_{j}-\mu_{i}\)
        end
        Let \(Y=Y \cup\) RobustMeanSDP \(\left(\mathcal{D}, \mu_{i}\right)\)
    end
    \(L=\emptyset\)
        for \(\tilde{\mathbb{E}}_{\zeta} \in Y\) do
            \(L^{\prime}=\operatorname{RoundTree}\left(\tilde{\mathbb{E}}_{\zeta}, 1, \frac{\rho}{2}\right)\)
        \(L=L \cup \operatorname{ExtractList}\left(L^{\prime}\right)\)
    end
    end
    return: \(L\)
```


## C Mean Estimation Overview

A convenient feature of our list decoding framework is that any setting for which we can prove "variance reduction" and "snapping" gives us a list decoding algorithm. We prove the analogues Lemma 3.3 and Lemma 3.2 for the setting of mean estimation.

## C. 1 Mean Estimation

In this section, we will lay out the broad overview of the proof of our algorithm for mean estimation. Specifically, we will show the following.
Theorem C.1. ListDecodeMean $(\mathcal{D}, \rho)$ outputs a list $L=\left\{\mu_{1}, \ldots, \mu_{A}\right\}$ of length $|L|=O\left(1_{\beta}^{\log \left(\frac{m_{2}}{\left.\beta^{1-1 / k}\right)}\right)}\right.$ such that for some $\hat{\mu} \in L$ we have

$$
\left\|\mu^{\prime}-\hat{\mu}\right\| \leq O\left(\frac{1}{\beta^{1 / k}}\right)
$$

with high probability. The algorithm runs in time $\left(\frac{1}{\beta}\right)^{\log \left(\frac{m_{2}}{\beta^{1-1 / k}}\right)} d^{O\left(\max \left(k, \frac{1}{\beta^{4}}\right)\right)}$
Theorem C. 1 gives us recovery guarantees that are information theoretically optimal. A straightforward post-processing step takes the list length down to $O\left(\frac{1}{\beta}\right)$. See [DKS18] Appendix B proposition B.1.

To prove Theorem C. 1 we will need the following lemmas.
Lemma C.2. (Snapping) Suppose $\tilde{\mathbb{E}}_{\zeta}$ is a degree $k$-pseudoexpectation operator that satisfies the constraints of RobustMean SOS SDP. If the variance is small

$$
\max _{u} \widetilde{\mathbb{V}}[\langle\hat{\mu}, u\rangle] \leq \frac{\beta \rho^{2}}{16}
$$

and there is correlation with the plant

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right] \geq \beta
$$

then,

$$
\|{\left.\underset{\zeta}{\tilde{E}}[\hat{\mu}]-\mu \| \leq O\left(\frac{1}{\beta^{1 / k}}\right), ~\right) ~}_{\text {and }}
$$

Lemma C.3. Let $\tilde{\mathbb{E}}_{\zeta}$ satisfy the pseudoexpectation constraints of RobustMeanSDP( $\left.\mathcal{D}, \rho\right)$. $Q$ be the pseudocovariance matrix. Let $(\lambda, v)$ be the largest eigenpair of $Q$. Let $\mathcal{S}_{v}$ be a probability distribution over [ $N$ ] Where for any $j \in N$ we have

$$
\mathcal{S}_{v}(j) \stackrel{\text { def }}{=} \frac{\widetilde{\mathbb{V}}\left[w_{j}\left\langle x_{j}-\tilde{\mathbb{E}}[\hat{\mu}], v\right\rangle\right]}{\sum_{i=1}^{N} \tilde{\mathbb{V}}\left[w_{j}\left\langle x_{j}-\tilde{\mathbb{E}}[\hat{\mu}], v\right\rangle\right]}
$$

Then

$$
\widetilde{\mathbb{V}}[\langle\hat{\mu}, v\rangle]-\underset{j \sim S}{\mathbb{E}} \mathbb{E} \underset{b_{j}}{\mathbb{V}}\left[\langle\hat{\mu}, v\rangle \mid w_{j}=b_{j}\right] \geq \frac{\beta}{2} \frac{\widetilde{\mathbb{V}}[\langle\hat{\mu}, v\rangle]^{2}}{2\left(\tilde{\mathbb{E}}\left[\frac{1}{N} \sum_{i=1}^{N} w_{i}\left\langle X_{i}-\hat{\mu}, v\right\rangle^{2}\right]+\widetilde{\mathbb{V}}[\langle\hat{\mu}, v\rangle]\right)}
$$

Now we can prove Theorem C. 1

Proof. For $\|Q\|_{\text {op }} \geq \Omega\left(\beta \rho^{2}\right)$, we have by Corollary 3.4 a nuclear norm decrease $\phi=\Omega\left(\beta^{3} \rho^{2}\right)$. Then by Lemma ?? we have an algorithm that runs in time $d^{O\left(\max \left(k, \frac{1}{\left.\beta^{4}\right)}\right)\right.}$. Iterating the RoundTree algorithm as we do for regression gives us the final error gaurantee and list length.

## D Mean Lemmas

Proof. Lemma C. 3
We invoke Theorem 5.2 for $z_{i}:=w_{i}\left\langle X_{i}-\tilde{\mathbb{E}}[\hat{\mu}], v\right\rangle$ for $i \in[N]$. Now it suffices to upper bound $\frac{1}{\beta N} \sum_{i=1}^{N} \widetilde{\mathbb{V}}\left(z_{i}\right)$.

$$
\frac{1}{M} \sum_{i=1}^{N} \tilde{\mathbb{V}}\left(z_{i}\right) \leq \frac{1}{M} \sum_{i=1}^{N} \tilde{\mathbb{E}}\left[z_{i}^{2}\right]=\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle X_{i}-\mathbb{E}[\hat{\mu}], v\right\rangle^{2}\right]
$$

We add and subtract $\hat{\mu}$ and expand the expression with triangle inequality to obtain

$$
\begin{gathered}
=\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle X_{i}-\hat{\mu}+\hat{\mu}-\mathbb{E} \hat{\mu}, v\right\rangle^{2}\right]=\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle X_{i}-\hat{\mu}+\hat{\mu}-\mathbb{E} \hat{\mu}, v\right\rangle^{2}\right] \\
\leq 2\left(\tilde{\mathbb{E}}\left[\frac{1}{N} \frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle X_{i}-\hat{\mu}, v\right\rangle^{2}\right]+\tilde{\mathbb{E}}\left[\langle\hat{\mu}-\tilde{\mathbb{E}} \hat{\mu}, v\rangle^{2}\right]\right)=2\left(\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i}\left\langle X_{i}-\hat{\mu}, v\right\rangle^{2}\right]+\tilde{\mathbb{V}}[\langle\hat{\mu}, v\rangle]\right)
\end{gathered}
$$

as desired.
Proof. Lemma C. 2
First, we will make the following claim which we will prove later
Claim D.1.

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right](\langle\tilde{\mathbb{E}}[\hat{\mu}]-\mu, u\rangle)^{2} \leq \tilde{\mathbb{E}}\left[\langle\hat{\mu}-\tilde{\mathbb{E}} \hat{\mu}, u\rangle^{2}\right]+\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\|\hat{\mu}-\mu\|^{2}\right]
$$

Which implies

$$
\langle\tilde{\mathbb{E}}[\hat{\mu}]-\mu, u\rangle^{2} \leq \frac{\tilde{\mathbb{E}}\left[\langle\hat{\mu}-\mathbb{E}[\hat{\mu}], u\rangle^{2}\right]}{\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]}+\tilde{\mathbb{E}}\left[\frac{\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}}{\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]}\|\hat{\mu}-\mu\|^{2}\right]
$$

Now we bound the second term on the right hand side. Let $\boldsymbol{\mathcal { G }}(w)$ be the space of polynomials over $w_{1}, \ldots, w_{N}$. Consider the functional $\tilde{\mathbb{E}}^{\prime}: \mathcal{G}(w) \rightarrow \mathbb{R}$ which takes a polynomial $f(w)$ and maps it to $\tilde{\mathbb{E}}^{\prime} f(w)=\tilde{\mathbb{E}}\left[\frac{\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}}{\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]} f(w)\right]$. We observe that $\tilde{\mathbb{E}}^{\prime}$ is a valid pseudoexpectation. Therefore, applying cauchy-schwarz we obtain

$$
\tilde{\mathbb{E}}\left[\frac{\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}}{\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]}\|\hat{\mu}-\mu\|^{2}\right]=\tilde{\mathbb{E}}\left[\|\hat{\mu}-\mu\|^{2}\right] \leq \tilde{\mathbb{E}}\left[\|\hat{\mu}-\mu\|^{k}\right]^{\frac{2}{k}}=\tilde{\mathbb{E}}\left[\frac{\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}}{\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]}\|\hat{\mu}-\mu\|^{k}\right]^{2 / k}
$$

Rearranging we have

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\|\hat{\mu}-\mu\|^{2}\right] \leq \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\|\hat{\mu}-\mu\|^{k}\right]^{\frac{2}{k}} \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]^{1-\frac{2}{k}}
$$

Then by SOS Cauchy-Schwarz and then the SOS inequality [HL18] lemma 5.5 we have

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\|\hat{\mu}-\mu\|^{k}\right] \leq \tilde{\mathbb{E}}\left[\left(\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right)^{2}\|\hat{\mu}-\mu\|^{2 k}\right]^{\frac{1}{2}} \leq \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w, w_{i}\|\hat{\mu}-\mu\|^{k}\right]^{\frac{1}{2}}
$$

Any number that is smaller than its square root is less than one i.e

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\|\hat{\mu}-\mu\|^{k}\right] \leq 1
$$

So we conclude that

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\|\hat{\mu}-\mu\|^{2}\right] \leq \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]^{1-\frac{2}{k}}
$$

Plugging back into the second term above we obtain

$$
\leq \frac{\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]^{1-\frac{2}{k}}}{\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]}=\frac{1}{\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]^{\frac{2}{k}}} \leq \frac{1}{\beta^{\frac{2}{k}}}
$$

Where in the last line we used the result of frobenius norm minimization $\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right] \geq \beta$. Then by taking square root on both sides we obtain

$$
\|\mathbb{E}[\hat{\mu}]-\mu\| \leq 1+\frac{1}{\beta^{\frac{1}{k}}}=O\left(\frac{1}{\beta^{\frac{1}{k}}}\right)
$$

Now we prove the claim.

Proof. D. 1

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]\langle\tilde{\mathbb{E}}[\hat{\mu}]-\mu, u\rangle^{2}=\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle\tilde{\mathbb{E}}[\hat{\mu}]-\mu, u\rangle^{2}\right]
$$

Applying triangle inequality
$=\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle\tilde{\mathbb{E}}[\hat{\mu}]-\hat{\mu}+\hat{\mu}-\mu, u\rangle^{2}\right] \leq 2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle\tilde{\mathbb{E}}[\hat{\mu}]-\hat{\mu}, u\rangle^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle\hat{\mu}-\mu, u\rangle^{2}\right]$
Using $w_{i} w_{i}^{\prime} \leq 1$ in the first term, we have

$$
\leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\hat{\mu}]-\hat{\mu}, u\rangle^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\langle\hat{\mu}-\mu, u\rangle^{2}\right]
$$

by SOS Cauchy-Schwarz on the second term we obtain

$$
\tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\right]\langle\tilde{\mathbb{E}}[\hat{\mu}]-\mu, u\rangle^{2} \leq 2 \tilde{\mathbb{E}}\left[\langle\tilde{\mathbb{E}}[\hat{\mu}]-\hat{\mu}, u\rangle^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime}\|\hat{\mu}-\mu\|^{2}\right]
$$

## D. 1 Preprocessing via Conditioning

Theorem D.2. Consider the strategy $\mathcal{S}$ of conditioning on $w_{j}=1$ where $j$ is selected with probability $\mathcal{S}(j)=\frac{\tilde{\mathrm{E}}\left[w_{j}\right]}{M}$. Then in expectation over the selection of $j$, we have that $\|Q\|_{o p}$ is small in expectation.

$$
\underset{j \sim \mathcal{S}}{\mathbb{E}} \widetilde{\mathbb{V}}\left(\langle\hat{\mu}, u\rangle \mid w_{j}=1\right)<4 m_{2}
$$

and there is correlation with the plant.

$$
\underset{j \sim \mathcal{S}}{\mathbb{E}} \tilde{\mathbb{E}}\left[\left.\frac{1}{M} \sum_{i=1}^{N} w_{i} w_{i}^{\prime} \right\rvert\, w_{j}=1\right] \geq \beta
$$

Proof. The correlation with the plant follows by the definition of $\mathcal{S}$. We now upper bound $\|Q\|_{o p}$ explicitly.

$$
\underset{j \sim \mathcal{S}}{\mathbb{E}} \widetilde{\mathbb{V}}\left(\langle\hat{\mu}, u\rangle \mid w_{j}=1\right)=\underset{j \sim \mathcal{S}}{\mathbb{E}} \tilde{\mathbb{E}}\left[\left\langle\hat{\mu}-\tilde{\mathbb{E}}\left[\hat{\mu} \mid w_{j}=1\right], u\right\rangle^{2} \mid w_{j}=1\right]
$$

by triangle inequality

$$
\leq 2 \underset{j \sim \mathcal{S}}{\mathbb{E}} \tilde{\mathbb{E}}\left[\left\langle\hat{\mu}-X_{j}, u\right\rangle^{2} \mid w_{j}=1\right]+2 \underset{j \sim \mathcal{S}}{\tilde{\mathbb{E}}}\left\langle X_{j}-\tilde{\mathbb{E}}\left[\hat{\mu} \mid w_{j}=1\right], u\right\rangle^{2}
$$

Applying the definition of $\mathcal{S}$ to the first term we obtain

$$
=2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{j=1}^{N} w_{j}\left\langle X_{j}-\hat{\mu}, u\right\rangle^{2}\right]+2 \underset{j \sim \mathcal{S}}{\mathbb{E}}\left\langle X_{j}-\tilde{\mathbb{E}}\left[\hat{\mu} \mid w_{j}=1\right], u\right\rangle^{2}
$$

Then using pseudoexpectation cauchy-schwarz on the second term we obtain

$$
\leq 2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{j=1}^{N} w_{j}\left\langle X_{j}-\hat{\mu}, u\right\rangle^{2}\right]+2 \underset{j \sim \mathcal{S}}{\mathbb{E}} \tilde{\mathbb{E}}\left[\left\langle X_{j}-\hat{\mu}, u\right\rangle^{2} \mid w_{j}=1\right]
$$

Applying the definition of $\mathcal{S}$ to the second term we obtain

$$
\leq 2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{j=1}^{N} w_{j}\left\langle X_{j}-\hat{\mu}, u\right\rangle^{2}\right]+2 \tilde{\mathbb{E}}\left[\frac{1}{M} \sum_{j=1}^{N} w_{j}\left\langle X_{j}-\hat{\mu}, u\right\rangle^{2}\right] \leq 4 m_{2}
$$

as desired.


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