Efficiently list-edge coloring multigraphs asymptotically optimally

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Abstract

We give polynomial time algorithms for the seminal results of Kahn [22, 23], who showed that the Goldberg-Seymour and List-Coloring conjectures for (list-)edge coloring multigraphs hold asymptotically. Kahn's arguments are based on the probabilistic method and are non-constructive. Our key insight is that we can combine sophisticated techniques due to Achlioptas, Iliopoulos and Kolmogorov [2] for the analysis of local search algorithms with correlation decay properties of the probability spaces on matchings used by Kahn in order to construct efficient edge-coloring algorithms.

Keywords— edge-coloring, mulitgraphs, Goldberg-Seymour conjecture, list-edge-coloring conjecture

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1 Introduction

In graph edge coloring one is given a (multi)graph G=(V,E) and the goal is to find an assignment of one of q colors to each edge $e \in E$ so that no pair of adjacent edges share the same color. The *chromatic index*, $\chi_e(G)$, of G is the smallest integer q for which this is possible. In the more general *list-edge coloring* problem, a list of q allowed colors is specified for each edge. A graph is q-list-edge colorable if it has a list-coloring no matter how the lists are assigned to each edge. The *list chromatic index*, $\chi_e^{\ell}(G)$, is the smallest q for which G is q-list-edge colorable.

Edge coloring is one of the most fundamental and well-studied coloring problems with various applications in computer science (e.g., [7, 12, 21, 22, 23, 34, 36, 38, 39, 40, 42]). To give just one representative example, if edges represent data packets then an edge coloring with q colors specifies a schedule for exchanging the packets directly and without node contention. In this paper we are interested in designing algorithms for efficiently edge coloring and list-edge coloring multigraphs. To formally describe our results, we need some notation.

For a multigraph G, let $\mathcal{M}(G)$ denote the set of matchings of G. A fractional edge coloring is a set $\{M_1,\ldots,M_\ell\}$ of matchings and corresponding positive real weights $\{w_1,\ldots,w_\ell\}$, such that the sum of the weights of the matchings containing each edge is one, i.e., $\forall e \in E, \sum_{M_i:e\in M_i} w_i = 1$. A fractional edge coloring is a fractional edge c-coloring if $\sum_{M\in\mathcal{M}(G)} w_M = c$. The fractional chromatic index of G, denoted by $\chi_e^*(G)$, is the minimum c such that G has a fractional edge c-coloring.

Let $\Delta = \Delta(G)$ be the maximum degree of G and define $\Gamma := \max_{H \subseteq V, |H| \ge 2} \frac{|E(H)|}{|H|/2|}$, where E(H) is the set of edges of the induced subgraph H. Both of these quantities are obvious lower bounds for the chromatic index and it is known [9] that $\chi_e^*(G) = \max(\Delta, \Gamma)$. Furthermore, Padberg and Rao [35] show that the fractional chromatic index of a multigraph, and indeed an optimal fractional edge coloring, can be computed in polynomial time.

Goldberg and Seymour independently stated the now famous conjecture that every multigraph G satisfies $\chi_e(G) \le \max(\Delta+1, \lceil \chi_e^*(G) \rceil)$. In a seminal paper [22], Kahn showed that the Goldberg-Seymour conjecture holds asymptotically:

Theorem 1.1 ([22]). The chromatic index of a multigraph G satisfies $\chi_e(G) \leq (1 + o(1))\chi_e^*(G)$.

(Here o(1) denotes a term that tends to zero as $\chi_e(G) \to \infty$.) Later Kahn proved the analogous result for list-edge coloring [23], establishing that the List Coloring Conjecture, which asserts that $\chi_e^{\ell}(G) = \chi_e(G)$ for any multigraph G, also holds asymptotically:

Theorem 1.2 ([23]). The list chromatic index of a multigraph G satisfies $\chi_e^{\ell}(G) \leq (1+o(1))\chi_e^*(G)$.

The proofs of Kahn use the probabilistic method and are not constructive. The main contribution of this paper is to provide polynomial time algorithms for the above results, as follows:

Theorem 1.3. There exists a randomized algorithm that, given a multigraph G on n vertices, constructs a $(1 + o(1))\chi_e^*(G)$ -edge coloring of G in expected polynomial time.

Theorem 1.4. There exists a randomized algorithm that, given a multigraph G on n vertices and an arbitrary list of $(1+o(1))\chi_e^*(G)$ colors for each edge, constructs a valid list-edge coloring of G in expected polynomial time.

Clearly, Theorem 1.4 subsumes Theorem 1.3. Moreover, in a very recent breakthrough [6], Chen, Jing and Zang proved the (non-asymptotic) Goldberg-Seymour conjecture without exploiting the arguments of Kahn. Even before this work, the results of Sanders and Steurer [38] and Scheide [40] already give deterministic polynomial time algorithms for edge coloring multigraphs asymptotically optimally, again without exploiting the arguments of Kahn. Nonetheless, we choose to present the proof of Theorem 1.3 for three reasons. First and most importantly, its proof is significantly easier than that of Theorem 1.4, while it contains many of the key ideas required for proving Theorem 1.4. Second, our algorithms and techniques are very different from those of [6, 38, 40]. Finally, we show that the algorithm of Theorem 1.3 is *commutative*, a notion introduced by Kolmogorov [27]. This fact may be of independent interest as we discuss in Remark 2.1 in Section 2.2.

To the best of our knowledge, Theorem 1.4 is the first result to give an asymptotically optimal polynomial time algorithm for list-edge coloring multigraphs.

1.1 Technical Overview

The proofs of Theorems 1.1 and 1.2 are based on a very sophisticated variation of what is known as the *semi-random method* (also known as the "naive coloring procedure"), which is the main technical tool behind some of the strongest graph coloring results, e.g., [20, 21, 25, 30]. The idea is to gradually color the graph in iterations, until we reach a point where we can finish the coloring using a greedy algorithm. In its most basic form, each iteration consists of the following simple procedure: assign to each edge a color chosen uniformly at random; then uncolor any edge which receives the same color as one of its neighbors. Using the Lovász Local Lemma (LLL) [10] and concentration inequalities, one typically shows that, with positive probability, the resulting partial proper coloring has useful properties that allow for the continuation of the argument in the next iteration. For a nice exposition of both the method and the proofs of Theorems 1.1 and 1.2, the reader is referred to [31].

The key new ingredient in Kahn's arguments is the method of assigning colors to edges. For each color c, we choose a matching M_c from some *hard-core* distribution on $\mathcal{M}(G)$ and assign the color c to the edges in M_c . The idea is that, by assigning each color exclusively to the edges of one matching, we avoid conflicting color assignments and the resulting uncolorings.

The existence of such hard-core distributions is guaranteed by the characterization of the matching polytope due to Edmonds [9] and a result by Lee [28] (also shown independently by Rabinovich et al. [37]). The crucial fact about them is that they are endowed with very useful approximate stochastic independence properties, as was shown by Kahn and Kayll in [24]. In particular, for every edge e, conditioning on events that are determined by edges far enough from e in the graph does not effectively alter the probability of e being in the matching.

The reason why this property is important is because it enables the application of a sophisticated version of what is known as the *Lopsided* Lovász Local Lemma. Recall that the original statement of the LLL asserts, roughly, that, given a family of "bad" events in a probability space, if each bad event individually is not very likely and, in addition, is independent of all but a small number of other bad events, then the probability of avoiding all bad events is strictly positive. The Lopsided LLL used by Kahn generalizes this criterion as follows. For each bad event B, we fix a positive real number μ_B and require that conditioning on all but a small number of other bad events doesn't make the probability of B larger than μ_B . Then, provided the μ_B are small enough, the conclusion of the LLL still holds. In other words, one replaces the "probability of a bad event" in the original LLL statement with the "boosted" probability of the event, and the notion of "independence" by the notion of "sufficiently mild negative correlation".

Notably, the breakthrough result of Moser and Tardos [32, 33] that made the LLL constructive for the vast majority of its applications does not apply in this case, mainly for two reasons. First, the algorithm of Moser and Tardos applies only when the underlying probability measure of the LLL application is a *product* over explicitly presented variables. Second, it relies on a particular type of dependency (defined by shared variables). The lack of an efficient algorithm for Lopsided LLL applications is the primary obstacle to making the arguments of Kahn constructive.

Our main technical contribution is the design and analysis of such algorithms. Towards this goal, we use the flaws-actions framework introduced in [1] and further developed in [2, 3, 4, 16, 18]. In particular, we use the algorithmic LLL criterion for the analysis of stochastic local search algorithms developed by Achlioptas, Iliopoulos and Kolmogorov in [2]. We start by showing that there is a connection between this criterion and the version of the Lopsided LLL used by Kahn, in the sense that the former can be seen as the constructive counterpart of the latter. However, this observation by itself is not sufficient, since the result of [2] is a tool for analyzing a *given* stochastic local search algorithm. Thus, we are still left with the task of designing the algorithm before using it. Nonetheless, this connection provides valuable intuition on how to realize this task. Moreover, we believe it is of independent interest as it provides an explanation for the success of various algorithms (such as [29]) inspired by the techniques of Moser and Tardos, which were not tied to a known form of the LLL.

To get a feeling for the nature of our algorithms, it is helpful to have some intuition for the criterion of [2]. There, the input is the algorithm to be analyzed and a probability measure μ over the state space of the algorithm. The goal of the algorithm is to reach a state that avoids a family of bad subsets of the space which we call *flaws*. It does this by focusing on a flaw that is currently present at each step, and taking a (possibly randomized) action to address it. At a high level, the role of the measure is to gauge how efficiently the algorithm rids the state of flaws, by quantifying the trade-off between the probability that a flaw is present at some inner state of the execution of the algorithm and the number of other flaws each flaw can possibly introduce when the algorithm addresses it. In particular, the quality of the convergence criterion is affected by the *compatibility* between the measure and the algorithm.

Roughly, the states of our algorithm will be matchings in a multigraph (corresponding to color classes) and the goal will be to construct matchings that avoid certain flaws. To that end, our algorithm will locally modify each

flawed matching by (re)sampling matchings in subgraphs of G according to distributions induced by the hard-core distributions used in Kahn's proof. The fact that correlations decay with distance in these distributions allows us to prove that, while the changes are local, and hence not many new flaws are introduced at each step, the compatibility of our algorithms with these hard-core distributions is high enough to allow us to successfully apply the criterion of [2].

1.2 Organization of the Paper

In Section 2 we present the necessary background. In Section 3 we show a useful connection between the version of the Lopsided LLL used by Kahn and the algorithmic LLL criterion of [2]. In Section 4 we present the proof of Theorem 1.3. In Section 5, we sketch the proof of Theorem 1.2 and then prove Theorem 1.4.

2 Background and Preliminaries

2.1 The Lopsided Lovász Local Lemma

Erdős and Spencer [11] noted that independence in the LLL can be replaced by positive correlation, yielding the original version of what is known as the *Lopsided* LLL, more sophisticated versions of which have also been established in [5, 8]. Below we state the Lopsided LLL in one of its most powerful forms.

Theorem 2.1 (General Lopsided LLL). Let (Ω, μ) be a probability space and $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ be a set of m (bad) events. For each $i \in [m]$, let $L(i) \subseteq [m] \setminus \{i\}$ be such that $\mu(B_i \mid \bigcap_{j \in S} \overline{B_j}) \leq b_i$ for every $S \subseteq [m] \setminus (L(i) \cup \{i\})$. If there exist positive real numbers $\{x_i\}_{i=1}^m$ such that

$$b_i \le x_i \prod_{j \in L(i)} (1 - x_j) \quad \text{for all } i \in [m], \tag{1}$$

then the probability that none of the events in $\mathcal B$ occurs is at least $\prod_{i=1}^m (1-x_i)>0$.

Note that in most applications of the Lopsided LLL the definition of sets $\{L(i)\}_{i\in[m]}$ is "symmetric", in the sense that if $j\in L(i)$ then $i\in L(j)$ for every $i,j\in[m]$. With that in mind, any (undirected) graph on [m] that includes every edge (i,j) such that $j\in L(i)$ or $i\in L(j)$ is called a *lopsidependency* graph.

2.2 An Algorithmic LLL Criterion.

Let Ω be a discrete state space, and let $F = \{f_1, f_2, \dots, f_m\}$ be a collection of subsets (which we call *flaws*) of Ω . We define $\bigcup_{i \in [m]} f_i = \Omega^*$. Our goal is to find a state $\sigma \in \Omega \setminus \Omega^*$; we refer to such states as *flawless*.

For a state σ , we denote by $U(\sigma)=\{f_j\in F \text{ s.t. } f_j\ni\sigma\}$ the set of flaws present in σ . We consider local search algorithms working on Ω which, in each flawed state $\sigma\in\Omega^*$, choose a flaw f_i in $U(\sigma)$ and randomly move to a nearby state in an effort to fix f_i . We will assume that, for every flaw f_i and every state $\sigma\in f_i$, there is a non-empty set of $actions\ a(i,\sigma)\subseteq\Omega$ such that $addressing\ flaw\ f_i$ in state σ amounts to selecting the next state τ from $a(i,\sigma)$ according to some probability distribution $\rho_i(\sigma,\tau)$. Note that potentially $a(i,\sigma)\cap f_i\neq\emptyset$, i.e., addressing a flaw does not necessarily imply removing it. We write $\sigma\xrightarrow{i}\tau$ to denote the fact that the algorithm addresses flaw f_i at σ and moves to τ .

Throughout the paper we consider algorithms that start from a state $\sigma \in \Omega$ picked from an initial distribution θ , and then repeatedly pick a flaw that is present in the current state and address it. The algorithm always terminates when it encounters a flawless state.

Definition 2.2 (Causality). We say that flaw f_i causes f_j if there exists a transition $\sigma \xrightarrow{i} \tau$ such that (i) $f_j \ni \tau$; (ii) either $f_i = f_j$ or $f_j \not\ni \sigma$.

Definition 2.3 (Causality Graph). Any (undirected) graph $C = C(\Omega, F)$ on [m] that includes every edge (i, j) such that either f_i causes f_j or f_j causes f_i is called a causality graph. We write $\Gamma(i)$ for the set of neighbors of i in this graph. We also write $i \sim j$ to denote that $j \in \Gamma(i)$ (or equivalently, $i \in \Gamma(j)$).

For a given probability measure μ supported on the state space Ω , and for each flaw f_i , we define the *charge*

$$\gamma_i = \max_{\tau \in \Omega} \sum_{\sigma \in f_i} \frac{\mu(\sigma)}{\mu(\tau)} \rho_i(\sigma, \tau). \tag{2}$$

In Section 3 we give the intuition behind the definition of charges and also draw a connection with the parameters b_i in Theorem 2.1. We are now ready to state the main result of [2].

Theorem 2.4 ([2]). Assume that, at each step, the algorithm chooses to address the lowest indexed flaw according to an arbitrary, but fixed, permutation of [m]. If there exist positive real numbers $x_i \in (0,1)$ for $1 \le i \le m$ such that

$$\gamma_i \le (1 - \epsilon)x_i \prod_{j \in \Gamma(i)} (1 - x_j) \text{ for every } i \in [m]$$
 (3)

for some $\epsilon \in (0,1)$, then the algorithm reaches a flawless object within $(T_0 + s)/\epsilon$ steps with probability at least $1 - 2^{-s}$, where

$$T_0 = \log_2 \left(\max_{\sigma \in \Omega} \frac{\theta(\sigma)}{\mu(\sigma)} \right) + \sum_{j \in [m]} \log_2 \left(\frac{1}{1 - x_j} \right).$$

We also describe another theorem that can be used to show convergence in a polynomial number of steps, even when the number of flaws is super-polynomial, assuming that the algorithm has a nice "commutativity" property which we describe next.

Definition 2.5. For $i \in [m]$, let A_i denote the $|\Omega| \times |\Omega|$ matrix defined by $A_i[\sigma, \sigma'] = \rho_i(\sigma, \sigma')$ if $\sigma \in f_i$, and $A_i[\sigma, \sigma'] = 0$ otherwise. An algorithm defined by matrices A_i , $i \in [m]$, is commutative with respect to a causality relation \sim if for every $i, j \in [m]$ such that $i \nsim j$ we have $A_i A_j = A_j A_i$.

Remark 2.1. As shown in [27, 18, 16], commutative algorithms have several additional nice properties: they are often parallelizable, their output distribution approximates the so-called "LLL-distribution", etc. Here we use the fact that commutative algorithms converge in polynomial time even in the presence of superpolynomially many flaws, assuming that the causality graph can be covered by a polynomial number of cliques (see Theorem 2.6 below). It is also worth noting that, if there were an efficient parallel algorithm for sampling matchings in multigraphs, namely a parallel version of the MCMC algorithm of Theorem 2.10 which we discuss in the next section and which we use in our algorithm for Theorem 1.3, then our proof directly implies a parallel algorithm for Theorem 1.3. The study of parallel versions of MCMC sampling algorithms has been initiated recently in [13, 14].

We note that Definition 2.5 was introduced in [16], as a generalization of the combinatorial definition of commutativity introduced in [27]. While the latter would suffice for our purposes, we choose to work with Definition 2.5 due to its compactness.

Theorem 2.6. Let \mathcal{A} be a commutative algorithm with respect to a causality relation \sim . Assume there exist positive real numbers $\{x_i\}_{i\in[m]}$ in (0,1) such that condition (3) holds. Assume further that the causality graph induced by \sim can be covered by n cliques with potentially further edges between them. Setting $\delta := \min_{i\in[m]} x_i \prod_{j\in\Gamma(i)} (1-x_j)$, the expected number of steps performed by \mathcal{A} is at most $t = O\left(\max_{\sigma\in\Omega}\frac{\theta(\sigma)}{\mu(\sigma)}\cdot\frac{n}{\epsilon}\log\frac{n\log(1/\delta)}{\epsilon}\right)$, and for any parameter $\lambda \geq 1$, \mathcal{A} terminates within λt resamplings with probability $1-\mathrm{e}^{-\lambda}$.

As shown in [18, Theorem 3.2], the proof of Theorem 2.6 reduces to that of the analogous result of Hauepler, Saha and Srinivasan [15] for the Moser-Tardos algorithm, and hence we omit it.

2.3 Hard-Core Distributions on Matchings

A probability distribution ν on the matchings of a multigraph G is *hard-core* if it is obtained by associating to each edge e a positive real $\lambda(e)$ (called the *activity* of e) so that the probability of any matching M is proportional to

 $\prod_{e \in M} \lambda(e)$. Thus, recalling that $\mathcal{M}(G)$ denotes the set of matchings of G, and setting $\lambda(M) = \prod_{e \in M} \lambda(e)$ for each $M \in \mathcal{M}(G)$, we have

$$\nu(M) = \frac{\lambda(M)}{\sum_{M' \in \mathcal{M}(G)} \lambda(M')}.$$

The characterization of the matching polytope due to Edmonds [9] and a result of Lee [28] (which was also shown independently by Rabinovich et al. [37]) imply the following connection between fractional edge colorings and hard-core probability distributions on matchings. Before describing it, we need a definition.

For any probability distribution ν on the matchings of a multigraph G, we refer to the probability that a particular edge e is in the random matching as the *marginal* of ν at e. We write $(\nu_{e_1}, \ldots, \nu_{e_{|E(G)|}})$ for the collection of marginals of ν at all the edges $e_i \in E(G)$.

Theorem 2.7 ([28, 37]). There is a hard-core probability distribution ν with marginals $(\frac{1}{c}, \dots, \frac{1}{c})$ if and only if there is a fractional c'-edge coloring of G with c' < c, i.e., if and only if $\chi_e^* < c$.

Kahn and Kayll [24] proved that the probability distribution promised by Theorem 2.7 is endowed with very useful approximate stochastic independence properties.

Definition 2.8. Suppose we choose a random matching M from some probability distribution. We say that an event Q is t-distant from a vertex v if Q is completely determined by the choice of all matching edges at distance at least t from v. We say that Q is t-distant from an edge e if it is t-distant from both endpoints of e.

Theorem 2.9 ([24]). For any $\delta > 0$, there exists a $K = K(\delta)$ such that for any multigraph G with fractional chromatic index c there is a hard-core distribution ν with marginals $(\frac{1-\delta}{c},\ldots,\frac{1-\delta}{c})$ such that:

- (a) for every $e \in E(G)$, $\lambda(e) \leq \frac{K}{c}$ and hence $\forall v \in V(G)$, $\sum_{e \ni v} \lambda(e) \leq K$;
- (b) for every $\epsilon \in (0,1)$, if we choose a matching M according to ν then, for any edge e and event Q which is t-distant from e,

$$\Pr(e \in M \mid Q) \in (1 \pm \epsilon) \Pr(e \in M),$$

where
$$t = t(\epsilon) = 8(K+1)^2 \epsilon^{-1} + 2$$
.

We conclude this subsection by stating the result of Jerrum and Sinclair [19] for sampling from hard-core distributions on matchings. We also describe a few of its applications that will be helpful in our proofs. The algorithm of [19] works by simulating a rapidly mixing Markov chain on matchings, whose stationary distribution is the desired hard-core distribution ν , and outputting the final state.

Theorem 2.10 ([19], Corollary 4.3). Let G be a multigraph, $\{\lambda(e)\}_{e \in E(G)}$ a vector of activities associated with the edges of G, and ν the corresponding hard-core distribution. Let n = |V(G)| be the number of vertices of G and define $\lambda' = \max\{\max_{u,v \in V(G)} \sum_{e \ni \{u,v\}} \lambda(e), 1\}$. There exists an algorithm that, for any $\epsilon > 0$, runs in time poly $(n, \lambda', \log \epsilon^{-1})$ and outputs a matching in G drawn from a distribution ν' such that $\|\nu - \nu'\|_{\text{TV}} \le \epsilon$.

Remark 2.2. [19] establishes this result for matchings in (simple) graphs. However, the extension to multigraphs is immediate: make the graph simple by replacing each set of multiple edges e_1, \ldots, e_ℓ between a pair of vertices u, v by a single edge e of activity $\lambda(e) = \sum_i \lambda(e_i)$; then use the algorithm to sample a matching from the hard-core distribution in the resulting simple graph; finally, for each edge $e = \{u, v\}$ in this matching, select one of the corresponding multiple edges $e_i \ni \{u, v\}$ with probability $\lambda(e_i) / \sum_i \lambda(e_i)$. Note that the running time will depend polynomially on the maximum activity λ' in the simple graph, as claimed.

Note that, via a standard argument, the algorithm of Theorem 2.10 can be used to design a *fully-polynomial* randomized approximation scheme (f.p.r.a.s.) for the partition function of a hard-core probability distribution on the matchings of a multigraph G — namely, for the quantity $Z_{\lambda}(G) = \sum_{M \in \mathcal{M}(G)} \lambda(M)$.

Theorem 2.11 ([19], Corollary 4.4). Let G be a multigraph, $\{\lambda(e)\}_{e \in E(G)}$ a vector of activities associated with the edges of G, and $Z_{\lambda}(G)$ the corresponding partition function. Let n = |V(G)| be the number of vertices of G and define $\lambda' = \max\{\max_{u,v \in V(G)} \sum_{e \ni \{u,v\}} \lambda(e), 1\}$. There exists an algorithm that, for any $\epsilon > 0$, runs in time $\operatorname{poly}(n,\lambda',1/\epsilon)$ and outputs a quantity $\widetilde{Z}_G(\lambda)$ such that $\operatorname{Pr}\left((1-\epsilon)Z_G(\lambda) \le \widetilde{Z}_G(\lambda) \le (1+\epsilon)Z_G(\lambda)\right) \ge 3/4$.

Remark 2.3. The estimate in Theorem 2.11 could be arbitrarily bad with probability 1/4. However, this probability can be reduced to any desired $\delta > 0$ by performing $O(\log \delta^{-1})$ trials and taking the median.

Theorem 2.11 allows us to design a f.p.r.a.s. for the edge-marginals of a hard-core probability distribution on the matchings of a multigraph G.

Corollary 2.12. Let G be a multigraph, $\{\lambda(e)\}_{e\in E(G)}$ a vector of activities associated with the edges of G, and ν the corresponding hard-core distribution. Let n=|V(G)| be the number of vertices of G and define $\lambda'=\max\{\max_{u,v\in V(G)}\sum_{e\ni\{u,v\}}\lambda(e),1\}$. There exists an algorithm that, for any edge e, $\epsilon>0$ and $\delta>0$, runs in time $\operatorname{poly}(n,\lambda',1/\epsilon,\log\delta^{-1})$ and outputs a quantity $\widetilde{\nu}_e$ such that $\Pr\left((1-\epsilon)\nu_e\leq\widetilde{\nu}_e\leq(1+\epsilon)\nu_e\right)\geq 1-\delta$, where ν_e is the marginal of ν at e.

Proof. Let G_e be the mutligraph obtained by removing e along with every other edge of G adjacent to it. Let $Z_{\lambda}(G)$, $Z_{\lambda}(G_e)$ denote the partition functions corresponding to multigraphs G, G_e with respect to $\{\lambda(e)\}_{e \in E(G)}$, respectively. Observe now that $\nu_e = \lambda(e) \cdot Z_{\lambda}(G_e)/Z_{\lambda}(G)$. Using the f.p.r.a.s. promised by Theorem 2.11 (and Remark 2.3) to get appropriately accurate estimates for $Z_{\lambda}(G)$, $Z_{\lambda}(G_e)$, we directly obtain an estimate for ν_e that satisfies the guarantees of Corollary 2.12.

Finally, one can use Theorem 2.11 as a subroutine in the algorithm of Singh and Vishnoi [41] to obtain the following result.

Corollary 2.13. Let G be a multigraph on n vertices and let $\delta \in (0,1)$ be a parameter. Let $\nu = \nu_{\delta}$ be the hard-core probability distribution over the matchings of G promised by Theorem 2.9. For every $\eta > 0$, there exists a poly $(n, \log \eta^{-1}, \log \delta^{-1})$ -time algorithm that computes a set of edge activities $\{\lambda'(e)\}_{e \in E(G)}$ such that the corresponding hard-core distribution ν' satisfies $\|\nu - \nu'\|_{\text{TV}} \leq \eta$.

Proof. Corollary 2.13 follows in a straightforward way from the main results of Singh and Vishnoi [41] and Jerrum and Sinclair [19]. Briefly, the main result of [41] states that finding a distribution that approximates ν can be seen as the solution of a max-entropy distribution estimation problem which can be efficiently solved given a "generalized counting oracle" for ν . The latter oracle is provided by Theorem 2.11.

3 Causality, Lopsidependency and Approximate Resampling Oracles

In this section we show a connection between Theorem 2.1 and Theorem 2.4. While this section is not essential to the proof of our main results, it does provide useful intuition since it implies the following natural approach to making applications of the Lopsided LLL algorithmic: we start designing a local search algorithm for addressing the flaws that correspond to bad events by considering the family of probability distributions $\{\rho_i(\sigma,\cdot)\}_{i\in[m],\sigma\in f_i}$ whose supports induce a causality graph that coincides with the lopsidependency graph of the Lopsided LLL application of interest. This is typically a straightforward task. The key to successful implementation is our ability to make the way in which the algorithm addresses flaws sufficiently *compatible* with the underlying probability measure μ . To make this precise, we first recall an algorithmic interpretation of the notion of *charges* defined in (2).

As shown in [2], the charge γ_i captures the compatibility between the actions of the algorithm for addressing flaw f_i and the measure μ . To see this, consider the probability, $\nu_i(\tau)$, of ending up in state τ after (i) sampling a state $\sigma \in f_i$ according to μ , and then (ii) addressing f_i at σ . Define the *distortion* associated with f_i as

$$d_i := \max_{\tau \in \Omega} \frac{\nu_i(\tau)}{\mu(\tau)} \ge 1,\tag{4}$$

i.e., the maximum possible inflation of a state probability incurred by addressing f_i (relative to its probability under μ , and averaged over the initiating state $\sigma \in f_i$ according to μ). Now observe from (2) that

$$\gamma_i = \max_{\tau \in \Omega} \frac{1}{\mu(\tau)} \sum_{\sigma \in f_i} \mu(\sigma) \rho_i(\sigma, \tau) = d_i \cdot \mu(f_i).$$
 (5)

An algorithm for which $d_i = 1$ is called a *resampling oracle* [17] for f_i , and notice that it perfectly removes the conditional of the addressed flaw. However, designing resampling oracles for sophisticated measures can be impossible by local search. This is because small, but non-vanishing, correlations can travel arbitrarily far in Ω . Thus, allowing for some distortion can be very helpful, especially in cases where correlations decay with distance.

Theorem 3.1 below shows that Theorem 2.4 is the algorithmic counterpart of Theorem 2.1.

Theorem 3.1. Given a family of flaws $F = \{f_1, \dots, f_m\}$ over a state space Ω , an algorithm A with causality graph C with neighborhoods $\Gamma(\cdot)$, and a measure μ over Ω , then for each $S \subseteq F \setminus \Gamma(i)$ we have

$$\mu\left(f_i \mid \bigcap_{j \in S} \overline{f_j}\right) \le \gamma_i,\tag{6}$$

where the γ_i are the charges of the algorithm as defined in (2).

Proof. Let $F_S := \bigcap_{i \in S} \overline{f_i}$. Observe that

$$\mu(f_i \mid F_S) = \frac{\mu(f_i \cap F_S)}{\mu(F_S)}$$

$$= \frac{\sum_{\sigma \in f_i \cap F_S} \mu(\sigma) \sum_{\tau \in a(i,\sigma)} \rho_i(\sigma,\tau)}{\mu(F_S)}$$

$$= \frac{\sum_{\sigma \in f_i \cap F_S} \mu(\sigma) \sum_{\tau \in F_S} \rho_i(\sigma,\tau)}{\mu(F_S)}, \tag{7}$$

where the second equality holds because each $\rho_i(\sigma, \cdot)$ is a probability distribution, and the third by the definition of causality and the fact that $S \subseteq F \setminus \Gamma(i)$. Now notice that changing the order of summation in (7) gives

$$\frac{\sum_{\tau \in F_S} \sum_{\sigma \in f_i \cap F_S} \mu(\sigma) \rho_i(\sigma, \tau)}{\mu(F_S)}$$

$$= \frac{\sum_{\tau \in F_S} \mu(\tau) \sum_{\sigma \in f_i \cap F_S} \frac{\mu(\sigma)}{\mu(\tau)} \rho_i(\sigma, \tau)}{\mu(F_S)}$$

$$\leq \frac{\sum_{\tau \in F_S} \mu(\tau) \left(\max_{\tau' \in \Omega} \sum_{\sigma \in f_i} \frac{\mu(\sigma)}{\mu(\tau')} \rho_i(\sigma, \tau') \right)}{\mu(F_S)}$$

$$= \gamma_i.$$

In words, Theorem 3.1 shows that causality graph C is a lopsidependency graph with respect to measure μ with $b_i = \gamma_i$ for all $i \in [m]$. Thus, when designing an algorithm for an application of Theorem 2.1 using Theorem 3.1, we have to make sure that the induced causality graph coincides with the lopsidependency graph, and that the measure distortion induced when addressing flaw f_i is sufficiently small so that the resulting charge γ_i is bounded above by b_i .

4 Edge Coloring Multigraphs: Proof of Theorem 1.3

We follow the exposition of the proof of Kahn in [31]. Note that throughout the proof we assume that the maximum degree Δ of the input multigraph G satisfies $\Delta \geq \Delta_0$ for some appropriately large constant Δ_0 .

The key to the proof of Theorem 1.3 is the following lemma.

Lemma 4.1. For all $\epsilon > 0$, there exists $\chi_0 = \chi_0(\epsilon)$ such that if $\chi_e^*(G) \ge \chi_0$ then we can find $N = \lfloor \chi_e^*(G)^{\frac{3}{4}} \rfloor$ matchings in G whose deletion leaves a multigraph G' with $\chi_e^*(G') \le \chi_e^*(G) - (1+\epsilon)^{-1}N$ in expected poly $(n, \ln \frac{1}{\epsilon})$ time.

Remark 4.1. Since $\chi_e^*(G) = \text{poly}(n)$, we may assume that $\epsilon \geq \frac{1}{\text{poly}(n)}$ without loss of generality. Therefore, the expected running time of the algorithm promised by Lemma 4.1 is poly(n).

Using the algorithm of Lemma 4.1 recursively, for every $\epsilon>0$ we can efficiently find an edge coloring of G using at most $(1+\epsilon)\chi_e^*+\chi_0$ colors as follows. First, we compute $\chi_e^*(G)$ using the algorithm of Padberg and Rao [35]. If $\chi_e^*\geq\chi_0$, then we apply Lemma 4.1 to get a multigraph G' with $\chi_e^*(G')\leq\chi_e^*(G)-(1+\epsilon)^{-1}N$. We can now color G' recursively using at most $(1+\epsilon)\chi_e^*(G')+\chi_0\leq (1+\epsilon)\chi_e^*(G)-N+\chi_0$ colors. Using one extra color for each of the N matchings promised by Lemma 4.1, we can then complete the coloring of G, proving the claim. In the base case where $\chi_e^*(G)<\chi_0$, we color G greedily using $2\Delta-1$ colors. The fact that $2\Delta-1\leq 2\chi_e^*-1<\chi_e^*+\chi_0$ concludes the proof of Theorem 1.3 as the number of recursive calls is at most n.

In the following sections, we prove Lemma 4.1. In Section 4.1 we describe the local search algorithm behind Lemma 4.1, and in Section 4.2 we prove its convergence. In Sections 4.3, 4.4 we prove two important auxiliary lemmas that are used in our convergence proof.

4.1 The Algorithm

Observe that we only need to prove Lemma 4.1 for $\epsilon < \frac{1}{10}$ since, clearly, if it holds for ϵ then it holds for all $\epsilon' > \epsilon$. So we fix $\epsilon \in (0,0.1)$ and let $c^* = \chi_e^*(G) - (1+\epsilon)^{-1}N$. Our goal will be to delete N matchings from G to get a multigraph G' which has fractional chromatic index at most c^* .

The flaws. Let $\Omega = \mathcal{M}(G)^N$ be the set of possible N-tuples of matchings of G. For a state $\sigma = (M_1, \dots, M_N) \in \Omega$ let G_σ denote the multigraph obtained by deleting the N matchings M_1, \dots, M_N from G. For a vertex $v \in V(G)$ we define $d_{G_\sigma}(v)$ to be the degree of v in G_σ . We now define the following flaws. For every vertex $v \in V(G)$ let

$$f_v = \left\{ \sigma \in \Omega : d_{G_\sigma}(v) > c^* - \frac{\epsilon}{4} N \right\}.$$

For every connected subgraph H of G with an odd number of vertices and such that (i) $|V(H)| \leq \frac{8\Delta}{\epsilon N}$, and (ii) $|E(H)| > \left(\frac{|V(H)|-1}{2}\right)c^*$, let

$$f_H = \{ \sigma \in \Omega : H \subseteq G_\sigma \}.$$

The following lemma implies that it suffices to find a flawless state. (This lemma was proved in [22], but we include a proof here for completeness.)

Lemma 4.2 ([22]). Any flawless state σ satisfies $\chi_e^*(G_\sigma) \leq c^*$.

Proof. Edmonds' characterization [9] of the matching polytope implies that the chromatic index of G_{σ} is at most c^* if

- 1. $\forall v: d_{G_{\sigma}}(v) \leq c^*$; and
- 2. $\forall H \subseteq G_{\sigma}$ with an odd number of vertices:

$$E(H) \le \frac{|V(H)| - 1}{2} c^*.$$

Now clearly, addressing every flaw of the form f_v establishes condition 1. By summing degrees it also implies that, for every subgraph F, $|E(F)| \leq \frac{|V(F)|(c^* - \epsilon N/4)}{2} \leq \frac{|V(F)|}{2}c^*$.

Moreover, any odd subgraph H can be decomposed into a connected component H' with an odd number of vertices, and a (possibly empty) subgraph F with an even number of vertices. Since there are no edges between F and H', in the absence of f_v flaws we obtain

$$|E(H)| = |E(F)| + |E(H')| \le \frac{|V(F)|c^*}{2} + |E(H')|.$$

Thus it suffices to prove condition 2 for the connected odd subgraph H', for if $|E(H')| \le (|V(H')| - 1)c^*/2$ then we have

$$|E(H)| \le (|V(F)| + |V(H')| - 1)c^*/2 = (|V(H)| - 1)c^*/2.$$

Now, again by summing degrees, we see that if no f_v flaw is present then condition 2 can fail only for H with fewer than $\frac{8\Delta}{\epsilon N}$ vertices, concluding the proof. Indeed, in the absence of f_v flaws, we have $|E(H)| \leq |V(H)|(c^* - \epsilon N/4)/2$ and, since $c^* \leq \chi_e^*(G) \leq 2\Delta$, if $|V(H)|(c^* - \epsilon N/4)/2 \geq (|V(H)| - 1)c^*/2$ then $|V(H)| \leq c^*/((\epsilon/4)N) \leq 8\Delta/\epsilon N$. \square

To describe an efficient algorithm for finding flawless states we need to (i) determine the initial distribution of the algorithm and show that it is efficiently samplable; (ii) show how to address each flaw efficiently; (iii) show that the expected number of steps of the algorithm is polynomial; and finally (iv) show that we can search for flaws in polynomial time, so that each step is efficiently implementable.

The initial distribution. Apply Theorem 2.9 with $\delta = \frac{\epsilon}{4}$. Let ν be the promised hard-core probability distribution, $\lambda = \{\lambda(e)\}$ the vector of activities associated with it, and K the corresponding constant. Note that the activities $\lambda(e)$ defining ν are not readily available. However, recalling Corollary 2.13 we see that we can efficiently compute a set of activities that gives an arbitrarily good approximation to the desired distribution ν .

For a parameter $\eta>0$ and a distribution p, we say that we η -approximately sample from p to express the fact that we sample from a distribution \tilde{p} such that $\|p-\tilde{p}\|_{\mathrm{TV}}\leq\eta$. Set $\eta=\frac{1}{n^{\beta}}$, where β is a sufficiently large constant to be specified later, and let ν' be the distribution promised by Corollary 2.13. The initial distribution of our algorithm, θ , is obtained by η -approximately sampling N random matchings (independently) from ν' . Observe that $\|\theta-\mu\|_{\mathrm{TV}}\leq 2\eta N$, where μ denotes the probability distribution over Ω induced by taking N independent samples from ν .

Addressing flaws. For an integer d>0 and a connected subgraph H, let $S_{< d}(H)$ be the set of vertices within distance strictly less than d of a vertex $u\in V(H)$. Given a state $\sigma=(M_1,\ldots,M_N)$, a subgraph H, and d>0 let

$$Q_H(d, \sigma) = (M_1 - S_{< d}(H), \dots, M_N - S_{< d}(H)),$$

where we define $M-X=M\cap E(G-X)$. Moreover, let $Q_H^i(d,\sigma)=M_i-S_{< d}(H)$ denote the i-th entry of $Q_H(d,\sigma)$. (In words, $Q_H^i(\sigma,d)$ is the set of edges of M_i with the property that both their endpoints are at distance at least d from H.) Finally, let $G_{< d+1}(H)$ be the multigraph induced by $S_{< d+1}(H)$ and $\mathcal{M}_{d+1}^i=\mathcal{M}_{d+1}^i(H,\sigma)$ be the set of matchings of $G_{< d+1}(H)$ that are "compatible" with $Q_H^i(d,\sigma)$. That is, for any matching M in \mathcal{M}_{d+1}^i we have that $M \cup Q_H^i(d,\sigma)$ is also a matching of G. More specifically, note that $\mathcal{M}_{d+1}^i(H,\sigma)$ corresponds to the set of matchings of the following multigraph $G_{i,< d+1}(H)$. Let $V_{i,d}$ denote the set of vertices of $S_{< d+1}(H)$ that belong to edges in $Q_H^i(\sigma,d)$. Multigraph $G_{i,< d+1}(H)$ is induced by $S_{< d+1}(H) \setminus V_{i,d}$.

We consider the procedure RESAMPLE below which takes as input a connected subgraph H, a state σ and a positive integer $d \leq n$, and which will be used to address flaws.

```
1: procedure RESAMPLE(H, \sigma, d)

2: Let \sigma = (M_1, M_2, \dots, M_N)

3: for i = 1 to N do

4: Let p be the hard-core distribution over matchings in \mathcal{M}_{d+1}^i induced by \{\lambda'(e)\}_{e \in E(G_{< d+1})}.

5: \eta-approximately sample a matching M from distribution p

6: Let M_i' = M \cup Q_H^i(d, \sigma) \triangleright By definition, M_i' is a matching

7: Output \sigma' = (M_1', M_2', \dots, M_N')
```

Throughout the proof, we fix the parameter

$$t = 8(K+1)^2 \delta^{-1} + 2.$$

To address f_v, f_H in state σ , we invoke procedures RESAMPLE $(\{v\}, \sigma, t)$ and RESAMPLE (H, σ, t) , respectively.

Searching for flaws. Notice that we can compute c^* in polynomial time using the algorithm of Padberg and Rao [35]. Therefore, given a state $\sigma \in \Omega$, we can search for flaws of the form f_v in polynomial time. However, the flaws of the form f_H are potentially exponentially many, so a brute-force search does not suffice for our purposes.

Fortunately, the result of Padberg and Rao provides a polynomial time oracle for this problem as well. Recall Edmonds' characterization used in the proof of Lemma 4.2. The constraints over odd subgraphs H are called *matching constraints*. Recall further that in the proof of Lemma 4.2 we showed that, in the absence of f_v -flaws, the only matching constraints that could possibly be violated correspond to f_H flaws. On the other hand, the oracle of Padberg and Rao can decide in polynomial time whether G has a fractional c-coloring or return a violated matching constraint, for every $c \geq 0$. Hence, if our algorithm prioritizes f_v flaws over f_H flaws, this oracle can be used to detect the latter in polynomial time.

4.2 Proof of Lemma 4.1

We are left to show that the expected number of steps of the algorithm is polynomial and that each step can be executed in polynomial time. To that end, we will show that both of these statements are true assuming that the initial distribution θ is μ instead of approximately μ , and that in Lines 4, 5 of the procedure RESAMPLE (H, σ, d) we perfectly sample from the hard-core probability distribution induced by activities $\{\lambda(e)\}_{e \in E(G_{i,< d}(H))}$ instead of η -approximately sampling from p. We can maximally couple the approximate and ideal distributions, and then take the constant β in the definition of the approximation parameter η to be sufficiently large. The latter implies that the probability that the coupling will fail during the execution of the algorithm is negligible (i.e., at most $\frac{1}{n^c}$). Since the fractional chromatic index of a multigraph can be computed in polynomial time, we can absorb the probability that the coupling fails into the polynomial expected running time by executing our algorithm sufficiently many times. That is, we execute our algorithm for a number of steps that is twice its expected running time, and if the edge coloring it produces is not a desirable one, we repeat the process.

For an integer d>0 and a vertex v, let $F_d(v)$ be the set of flaws indexed by a vertex of $S_{< d}(v)$ or a subgraph H intersecting $S_{< d}(v)$. For each set H for which we have defined f_H we let $F_d(H) = \bigcup_{v \in V(H)} F_d(v)$. For each flaw f_v we define the causality neighborhood $\Gamma(f_v) = F_{t+2}(v)$, and for each flaw f_H we define $\Gamma(f_H) = F_{t+2}(H)$, where t is as defined in the previous subsection. Notice that this is a valid choice because flaw f_v can only cause flaws in $F_{t+1}(v)$ and flaw f_H can only cause flaws in $F_{t+1}(H)$. The reason why we choose these neighborhoods to be larger than seemingly necessary is because, as we will see, with respect to this causality graph our algorithm is commutative, allowing us to apply Theorem 2.6.

Lemma 4.3. Let $f \in \{f_v, f_H\}$ for a vertex v and a connected subgraph H of G with an odd number of vertices and let $D = \Lambda^{t+2\Delta^{\frac{1}{3}}+4}$. We have:

- (a) $\gamma_f \leq \frac{1}{2eD}$;
- (b) $|\Gamma(f)| \leq D$,

where the charges are computed with respect to the measure μ and the algorithm that samples from the ideal distributions.

Lemma 4.4. For each pair of flaws $f \nsim g$, the matrices A_f , A_g commute.

The proof of Lemma 4.3 can be found in Section 4.3. Lemma 4.4 establishes that our algorithm is commutative with respect to the causality relation \sim induced by neighborhoods $\Gamma(\cdot)$. Its proof can be found in Section 4.4.

Now, setting $x_f = \frac{1}{1 + \max_{f' \in F} |\Gamma(f')|}$ for each flaw f, we see that condition (3) with $\epsilon = 1/4$ is implied by

$$\gamma_f \cdot \left(1 + \max_{f' \in F} (|\Gamma(f')|\right) \cdot e \le 3/4 \text{ for every flaw } f,$$
 (8)

which is true for large enough Δ according to Lemma 4.3. Notice further that the causality graph induced by \sim can be covered by n cliques, one for each vertex of G, with potentially further edges between them. Indeed, flaws indexed by subgraphs that contain a certain vertex of G form a clique in the causality graph. Combining Lemma 4.4 with the latter observation, we are able to apply Theorem 2.6, which implies that our algorithm terminates after an expected number of at most $O\left(\max_{\sigma \in \Omega} \frac{\theta(\sigma)}{\mu(\sigma)} \cdot n \log(n \log(1/\delta))\right) = O(n \log n)$ steps. (This is because we assume that $\theta = \mu$ per our discussion above.)

This completes the proof of Lemma 4.1 and hence, as explained at the beginning of Section 4, Theorem 1.3 follows. It remains, however, to go back and prove Lemmas 4.3 and 4.4, which we do in the next two subsections.

4.3 Proof of Lemma 4.3

Proof of part (a). We will need the following key lemma, which was essentially proved in [22]. Its proof can be found in Appendix A. Recall that μ is the distribution over Ω induced by taking N independent samples from ν .

Lemma 4.5. For any random state σ distributed according to μ :

- (i) for every flaw f_v and state $\tau \in \Omega$: $\mu(\sigma \in f_v \mid Q_v(t,\sigma) = Q_v(t,\tau)) \leq \frac{1}{2eD}$; and
- (ii) for every flaw f_H and state $\tau \in \Omega$: $\mu(\sigma \in f_H \mid Q_H(t,\sigma) = Q_H(t,\tau)) \leq \frac{1}{2eD}$.

We show the proof of part (a) of Lemma 4.3 only for the case of f_v - flaws, as the proof for f_H - flaws is very similar. Specifically, our goal will be to prove that

$$\gamma_{f_v} = \max_{\tau \in \Omega} \mu(\sigma \in f_v \mid Q_v(t, \sigma) = Q_v(t, \tau)). \tag{9}$$

Lemma 4.5 then concludes the proof.

Recalling the definition of γ_f from (2) we see that, in order to prove (9), it suffices to show that, for σ distributed according to μ and any state $\tau \in \Omega$,

$$\sum_{\omega \in f_v} \frac{\mu(\omega)}{\mu(\tau)} \rho_{f_v}(\omega, \tau) = \mu(\sigma \in f_v \mid Q_v(t, \sigma) = Q_v(t, \tau)). \tag{10}$$

Indeed, maximizing (10) over $\tau \in \Omega$ yields (9) and completes the proof.

Fix $\tau=(M_1,M_2,\ldots,M_N)\in\Omega$. To compute the sum on the left-hand side of (10) we need to determine the set of states $\operatorname{In}_v(\tau)\subseteq f_v$ for which $\rho_{f_v}(\omega,\tau)>0$. To do this, recall that given as input a state $\omega=(M_1^\omega,M_2^\omega,\ldots,M_N^\omega)\in f_v$, procedure RESAMPLE (v,ω,t) modifies one by one each matching $M_i, i\in[N]$, "locally" around v. In particular, observe that the support of the distribution for updating M_i is exactly the set $\mathcal{M}_{t+1}^i(v,\omega)$, and hence it must be the case that $Q_v^i(t,\omega)=Q_v^i(t,\tau)$ for every $i\in[N]$ and state $\omega\in\operatorname{In}_v(\tau)$. This also implies that, for every such ω ,

$$\frac{\mu(\omega)}{\mu(\tau)} = \prod_{i=1}^{N} \frac{\nu(M_i^{\omega})}{\nu(M_i)} = \prod_{i=1}^{N} \frac{\lambda(M_i^{\omega} \cap E(G_{< t+1}(v)))}{\lambda(M_i \cap E(G_{< t+1}(v)))}.$$
(11)

Recall now that we have assumed that the hard-core distribution in Lines 4, 5 of RESAMPLE (v, ω, t) is induced by the ideal vector of activities λ . In particular, we have

$$\rho_{f_v}(\omega, \tau) = \prod_{i=1}^N \frac{\lambda(M_i \cap E(G_{< t+1}(v)))}{\sum_{M \in \mathcal{M}_{t+1}^i(v,\omega)} \lambda(M)}$$

$$= \prod_{i=1}^N \frac{\lambda(M_i \cap E(G_{< t+1}(v)))}{\sum_{M \in \mathcal{M}_{t+1}^i(v,\tau)} \lambda(M)}$$
(12)

since $Q_v^i(t,\omega) = Q_v^i(t,\tau)$, which combined with (11) yields

$$\frac{\mu(\omega)}{\mu(\tau)}\rho_{f_v}(\omega,\tau) = \prod_{i=1}^N \frac{\lambda(M_i^\omega \cap E(G_{< t+1}(v)))}{\sum_{M \in \mathcal{M}_{t+1}^i(v,\tau)} \lambda(M)}.$$
(13)

Letting $\sigma = (M_1^{\sigma}, \dots, M_N^{\sigma})$ be a random state distributed from μ we see that, by definition, the right-hand side of (13) equals:

$$\prod_{i=1}^{N} \frac{\lambda(M_{i}^{\omega} \cap E(G_{< t+1}(v)))}{\sum_{M \in \mathcal{M}_{t+1}^{i}(v,\tau)} \lambda(M)} = \prod_{i=1}^{N} \nu(M_{i}^{\sigma} = M_{i}^{\omega} \mid Q_{v}^{i}(t,\sigma) = Q_{v}^{i}(t,\tau)) = \mu(\sigma = \omega \mid Q_{v}(t,\sigma) = Q_{v}(t,\tau)).$$
(14)

Finally, combining (13) and (14), we obtain that

$$\sum_{\omega \in f_v} \frac{\mu(\omega)}{\mu(\tau)} \rho_{f_v}(\omega, \tau) = \sum_{\omega \in f_v} \mu(\sigma = \omega \mid Q_v(t, \sigma) = Q_v(t, \tau)) = \mu(\sigma \in f_v \mid Q_v(t, \sigma) = Q_v(t, \tau)),$$

concluding the proof of the first part of Lemma 4.3.

Proof of part (b). For this proof we will use the following well-known proposition, which we also prove here for completeness.

Proposition 4.6. For every vertex v there are at most $(e\Delta)^{s-1}$ sets of vertices S such that (i) $v \in S$; (ii) |S| = s; and (iii) G[S] is connected.

Proof. The number of such sets is bounded by the number of distinct s-vertex trees which are rooted at v. The latter quantity is bounded by the number of distinct Δ -ary rooted trees with s vertices, which is

$$T_{\Delta}(s) := \frac{\binom{\Delta s}{s}}{(\Delta - 1)s + 1},\tag{15}$$

see e.g. [26]. It is not hard to see that $T_{\Delta}(s) \leq (e\Delta)^{s-1}$ for $s \in \{1,2\}$ and $\Delta \geq 1$. For $s \geq 3$, we obtain

$$T_{\Delta}(s) \le \frac{\left(\frac{\Delta s \cdot e}{s}\right)^s}{(\Delta - 1)s + 1} = \frac{(\Delta \cdot e)^s}{(\Delta - 1)s + 1} \le (e\Delta)^{s - 1}$$

for sufficiently large Δ , concluding the proof. Note that in deriving the first inequality we used that $\binom{a}{b} \leq (a \cdot e/b)^b$ for positive integers $b \leq a$.

To prove part (b) of Lemma 4.3 it suffices to show that

$$|F_{t+2}(v)| \le \Delta^{t+2\Delta^{1/3}+3} \tag{16}$$

for every vertex v. Indeed, (16) clearly suffices if $f = f_v$. If $f = f_H$, notice that every H for which we define $F_{t+2}(H)$ has fewer than Δ vertices (assuming Δ is sufficiently large) and, therefore, every $F_{t+2}(H)$ has less than $D = \Delta^{t+2}\Delta^{1/3}+4$ elements.

Towards proving (16), at first notice that every set $S_{< t+2}(v)$ has at most Δ^{t+2} elements. Moreover, using Proposition 4.6 we obtain that, for sufficiently large Δ , every vertex u is in at most

$$H_u := \sum_{s=1}^{\frac{8\Delta}{\epsilon N}} (e\Delta)^{s-1} \le \frac{1}{e\Delta} \cdot \frac{(e\Delta)^{\frac{8\Delta}{\epsilon N} + 1} - 1}{e\Delta - 1} \le \Delta^{2\Delta^{1/3}}$$

sets H corresponding to a flaw f_H . Note that in deriving the second inequality above we used the fact that $N=\lfloor\chi_e^*(G)^{3/4}\rfloor=\Theta(\Delta^{3/4})$, which in turn implies that $\frac{8\Delta}{\epsilon N}\leq 2\Delta^{1/3}$ for sufficiently large Δ . Overall:

$$|F_{t+2}(v)| \le |S_{< t+2}(v)| \cdot \left(\max_{u \in S_{< t+2}(v)} H_u + 1\right) \le \Delta^{t+2} \cdot \left(\Delta^{2\Delta^{1/3}} + 1\right) \le \Delta^{t+2\Delta^{1/3} + 3}$$

for sufficiently large Δ , concluding the proof.

4.4 Proof of Lemma 4.4

Fix states $\sigma_1=(M_1,M_2,\ldots,M_N)\in f$ and $\sigma_2=(M_1',M_2',\ldots,M_N')\in g$ such that $f\not\sim g$. To prove that the matrices A_f,A_g commute, we need to show that for every such pair

$$\sum_{\tau} \rho_f(\sigma_1, \tau) \rho_g(\tau, \sigma_2) = \sum_{\tau} \rho_g(\sigma_1, \tau) \rho_f(\tau, \sigma_2). \tag{17}$$

To that end, let H_f , H_g be the subgraphs (which may consist only of a single vertex) associated with flaws f and g, respectively. Since $f \sim g$ we have $\min_{u \in V(H_f), v \in V(H_g)} \operatorname{dist}(u, v) \geq t + 2$, where $\operatorname{dist}(u, v)$ denotes the length of the shortest path between u and v. Notice that this implies $S_{< t+2}(H_f) \cap S_{< t+2}(H_g) = \emptyset$.

Consider a pair of transitions $\sigma_1 \xrightarrow{f} \tau$, $\tau \xrightarrow{g} \sigma_2$, where $\tau = (M_1'', \dots, M_N'')$, and so that $\rho_f(\sigma_1, \tau) > 0$, $\rho_g(\tau, \sigma_2) > 0$. The facts that procedure RESAMPLE (σ, f, t) only modifies the input set of matchings locally within $S_{< t+1}(H_f)$, that $\rho_g(\tau, \sigma_2) > 0$, and that $S_{< t+2}(H_f) \cap S_{< t+2}(H_g) = \emptyset$ imply that (i) $\sigma_1 \in g$; and (ii) for every $i \in [N]$, $M_i \cap (S_{< t+2}(H_g)) = M_i'' \cap (S_{< t+2}(H_g))$. Notice now that the probability distribution $\rho_g(\tau, \cdot)$ depends only on $(M_1'' \cap S_{< t+2}(H_g), \dots, M_N'' \cap S_{t+2}(H_g))$. Hence, (i) and (ii) imply that the probability distribution $\rho_g(\sigma_1, \cdot)$ is well defined and, in addition, there exists a natural bijection b_g between the action set $a(g, \tau)$ and the action set $a(g, \sigma_1)$ so that $\rho_g(\tau, \tau') = \rho_g(\sigma_1, b_g(\tau'))$ for every $\tau' \in a(g, \tau)$. This is because both distributions are implemented by sampling from the set of matchings of the same multigraph according to the same probability distribution.

Now let $\tau'=b_g(\sigma_2)$. A symmetric argument implies that $\tau'\in f$ and that there exists a natural bijection b_f between $a(f,\sigma_1)$ and $a(f,\tau')$ so that $\rho_f(\sigma_1,\sigma)=\rho_f(\tau',b_f(\sigma))$ for every $\sigma\in a(f,\sigma_1)$. In particular, notice that $\sigma_2=b_f(\tau)$ and that

$$\rho_f(\sigma_1, \tau)\rho_g(\tau, \sigma_2) = \rho_g(\sigma_1, \tau')\rho_f(\tau', b_f(\tau))$$

$$= \rho_g(\sigma_1, \tau')\rho_f(\tau', \sigma_2). \tag{18}$$

Overall, what we have shown is a bijective mapping that sends any pair of transitions $\sigma_1 \xrightarrow{f} \tau, \tau \xrightarrow{g} \sigma_2$ to a pair of transitions $\sigma_1 \xrightarrow{g} \tau', \tau' \xrightarrow{f} \sigma_2$ and which satisfies (18). This establishes (17), concluding the proof.

5 List-Edge Coloring Multigraphs: Proof of Theorem 1.4

In this section we review the proof of Theorem 1.2 and then prove its constructive version, Theorem 1.4. Again, throughout the proof we assume that the maximum degree Δ of the input multigraph G satisfies $\Delta \geq \Delta_0$ for some appropriately large constant Δ_0 .

In Section 5.1 we give a high-level sketch of the existential proof of Kahn, and we state the key technical results from that paper (Theorems 5.1, 5.2, and Lemma 5.3). As we will see, our main contribution is to make Theorem 5.1 constructive. Towards this end, we describe our local search algorithm in Section 5.2, where we also prove its correctness assuming it converges (Lemma 5.4), as well as an important property of the flaws we consider (Lemma 5.5). Finally, in Section 5.3 we prove that our search algorithm has expected polynomial running time, concluding the proof of Theorem 1.4.

5.1 A High Level Sketch of the Existential Proof

As we explained in the introduction, the non-constructive proof of Theorem 1.2 is a sophisticated version of the semi-random method and proceeds by partially coloring the edges of the multigraph in iterations, until at some point the coloring can be completed greedily. (More accurately, the method establishes the *existence* of such a sequence of desirable partial colorings.)

We will follow the exposition in [31]. In each iteration, we have a list L_e of acceptable colors for each edge e. We assume that each L_e originally has C colors for some $C \geq (1+\epsilon)\chi_e^*(G)$, where $\epsilon > 0$ is an arbitrarily small constant. For each color i, we let G_i be the subgraph of G formed by the edges for which i is acceptable. Since $G_i \subseteq G, \chi_e^*(G_i) \leq \chi_e^*(G)$. Thus, Theorem 2.9 implies that we can find a hard-core distribution on the matchings of G_i with marginals $(\frac{1}{C}, \dots, \frac{1}{C})$ whose activity vector λ_i satisfies $\lambda_i(e) \leq \frac{K}{C}$ for all e, where $K = K(\epsilon)$ is a constant.

In each iteration, we will use the *same* activity vector λ_i to generate the random matchings assigned to color i. Of course, in each iteration we restrict our attention to the subgraph of G_i obtained by deleting the set E^* of edges colored (with any color) in previous iterations, and the endpoints of the set of edges E_i^* colored i in previous iterations. (Thus, although we use the same activity vector for each color in each iteration, the induced hard-core distributions may vary significantly.) Further, we will make sure that our distributions have the property that for each edge e, the expected number of matchings containing e is very close to 1. (In other words, the sum over e0 of the probabilities that edge e1 is a part of the matching corresponding to color e1 is close to 1.)

We apply the Lopsided LLL in the following probability space. For each color i, independently, we choose a matching $M_i \in G_i$ from the corresponding distribution. Next, we *activate* each edge in M_i independently with probability $\alpha := \frac{1}{\log \Delta(G)}$; we assign colors only to activated edges in order to ensure that very few edges are assigned more than one color. We then update the multigraph by deleting the colored edges, and update the lists L_e by deleting any color assigned to an edge incident on e. We give a more detailed description below.

Notice that our argument needs to ensure that (i) at the beginning of each iteration the induced hard-core distributions are such that, for each uncolored edge e, the expected number of random matchings containing e is very close to 1; and (ii) after some number of iterations, we can complete the coloring greedily.

As far as the latter condition is concerned, notice that if (i) holds throughout then, in each iteration, the probability that an edge retains a color remains close to the activation probability α . This allows us to prove that the maximum degree in the uncolored multigraph drops by a factor of about $1-\alpha$ in each iteration. Hence, after $\log_{\frac{1}{1-\alpha}} 3K$ iterations, the maximum degree in the uncolored multigraph will be less than $\frac{\Delta}{2K}$. Furthermore, for each e and e, the

probability that e is in the random matching of color i is at most $\lambda_i(e) \leq \frac{K}{C}$. Since (i) continues to hold, this implies there are at least $\frac{C}{K} > \frac{\Delta}{K}$ colors available for each edge, and so the coloring can be completed greedily. (Recall that the $C > \chi_e^*(G) \geq \Delta$.)

An Iteration.

- 1. For each color i, pick a matching M_i according to a hard-core probability distribution μ_i on $\mathcal{M}(G_i)$ with activities λ_i such that for some constant K:
 - (a) $\forall e \in E(G), \sum_i \mu_i (e \in M_i) \approx 1$; and
 - (b) $\forall i, \forall e \in E(G), \lambda_i(e) \leq \frac{K}{C}$ and hence $\forall v \in V(G), \sum_{L_e \ni i} \lambda_i(e) \leq K$.
- 2. For each i, activate each edge of M_i independently with probability $\alpha = \frac{1}{\log \Delta(G)}$, to obtain a new matching $F_i \subseteq M_i$. We color the edges of F_i with color i and delete $V(F_i)$ from G_i . We also delete from G_i every edge not in M_i which is in F_j for some $j \neq i$. We do not delete edges of $(M_i F_i) \cap F_j$ from G_i . (Note that this may result in edges receiving more than one color, which is not a problem since we can always pick one of them arbitrarily at the end of the iterative procedure.)
- 3. Perform an equalizing coin flip for each edge e of G_i so that the probability that e is both colored and removed from G_i in either Step 2 or Step 3 is exactly α . (See also Remark 5.1 below.)

Remark 5.1. Note that the expected number of edges that are both colored and removed from G_i in Step 2 is less than $\alpha |E(G_i)|$ because, although the expected number of colors retained by an edge is very close to α , some edges may be assigned more than one color. Performing "equalizing coin flips" in Step 3 is a standard idea that helps in avoiding several technical difficulties that stem from the latter fact.

The *outcome* of an iteration is defined to be the choices of matchings, activations, and equalizing coin flips. Let $Out = Out_{\ell}$ denote the random variable that equals the outcome of the ℓ -th iteration. (In what follows, we will focus on a specific iteration ℓ and so we will omit the subscript.)

For each edge e=(u,v), we define a bad event A_e as follows. Let G_i' be the multigraph obtained after carrying out the modifications to G_i in Steps 2 and 3 of the above iteration. Let $t=8(K+1)^2(\log\Delta)^{20}+2$, recall the definition of $S_{< t}(H)$ for subgraph H, and let $G_{< t}(H)$ denote the corresponding induced subgraph. Let Z_i be a random matching in $G_i' \cap G_{< t}(\{u,v\})$ sampled from the hard-core probability distribution induced by activity vector λ_i . Let A_e be the event that

$$\left| \sum_{i:G'\ni e} \Pr(e \in Z_i \mid \text{Out}) - \sum_{i:G_i\ni e} \Pr(e \in M_i) \right| > \frac{1}{2(\log \Delta)^4}.$$
 (19)

To get some intuition behind the definition of event A_e , let M_i' be a random matching in G_i' chosen according to the hard-core distribution with activities λ_i . Since correlations decay with distance, one can show that $\Pr(e \in M_i' \mid \text{Out})$ is within a factor $1 + \frac{1}{(\log \Delta)^{20}}$ of $\Pr(e \in Z_i \mid \text{Out})$. Thus, according to (19), avoiding bad event A_e implies that $\sum_i \Pr(e \in M_i') \approx \sum_i \Pr(e \in M_i) \approx 1$, which is what is required in order to maintain property (i) at the beginning of the next iteration. In particular, it is straightforward to see that avoiding all bad events $\{A_e\}_{e \in E(G)}$ guarantees that

$$\left| \sum_{i:G'\ni e} \Pr(e \in M_i' \mid \text{Out}) - \sum_{i:G_i\ni e} \Pr(e \in M_i) \right| \le \frac{1}{(\log \Delta)^4},\tag{20}$$

for sufficiently large Δ , which is what we really need. The reason we consider Z_i and not M'_i is that events defined with respect to the former are mildly negatively correlated with most other bad events, making it possible to apply the Lopsided LLL.

Further, for each vertex v we define A_v to be the event that the proportion of edges incident on v which are colored in the iteration is less than $\alpha - \frac{1}{(\log \Delta)^4}$.

It can be formally shown that, if we avoid all bad events, then (i) holds, i.e., at the beginning of the next iteration we can choose new probability distributions so that for each uncolored edge e we maintain the property that the expected number of random matchings containing e is very close to 1, and, moreover, after $\log_{\frac{1}{1-\alpha}} 3K$ iterations we can complete the coloring greedily.

Theorem 5.1 ([23]). Assume that (20) holds for the edge marginals of the matching distributions of iteration ℓ . Then, with positive probability, the same is true for the matching distributions of iteration $\ell + 1$.

Theorem 5.2 ([23]). If we can avoid the bad events of the first $\log_{\frac{1}{1-\alpha}} 3K$ iterations, then we can complete the coloring greedily.

Proving Theorems 5.1 and 5.2 is the heart of the proof of Theorem 1.2. The most difficult part is proving that, for any $x \in V \cup E$, the probability of event A_x is very close to 0 conditioned on any choice of outcomes for distant events. (This is needed in order to apply the Lopsided LLL.) Given Theorem 5.1, the proof of Theorem 5.2 follows, as we have already explained, from the fact that in each iteration the expected number of random matchings containing each uncolored edge e is very close to 1 and, therefore, the probability that e retains a color remains close to α .

Below we state the key lemma that is proven in [23], and which we will also use in the analysis of our algorithm. Recall the definition of t. For a subgraph H, we let R_H be the random outcome of our iteration in $G - S_{< t^2}(H)$, i.e., R_H consists of $\bigcup_i (M_i - S_{< t^2}(H))$ together with the choices of the activated edges in $G - S_{< t^2}(H)$ which determine the $\bigcup_i (F_i - S_{< t^2}(H))$, and the outcomes of the equalizing coin flips for edges in this subgraph.

Lemma 5.3 ([23]). For every $x \in E \cup V$ and possible choice R_x^* for R_x , we have $\Pr(A_x \mid R_x = R_x^*) \leq \frac{1}{\Lambda^{3(t^2+t+2)}}$.

In the remaining sections we will focus on providing an efficient algorithm for Theorem 5.1 which, combined with Theorem 5.2, will imply the proof of Theorem 1.4.

As a final remark, we note that detecting whether bad events $\{A_e\}_{e\in E(G)}$ are present in a state is not a tractable task since it entails the exact computation of edge marginals of hardcore distributions over matchings. In order to overcome this obstacle, we will define flaws $\{f_e\}_{e\in E(G)}$ whose absence provides somewhat weaker guarantees than the removal of bad events $\{A_e\}_{e\in E(G)}$, but nonetheless implies (20) for every edge. To decide whether a flaw f_e is present in a state, we will use the results of [19] to estimate the corresponding edge marginals of random variables M_i and Z_i for every color i. Note that since we will only perform an approximation, we will not be able to check for (19) directly. However, our approximation will be tight enough so that, even in this case, (20) will still hold for every edge. We give the details below.

5.2 The Algorithm

Let $\mathcal U$ denote the set of uncolored edges and $N=|\bigcup_{e\in\mathcal U}L_e|$, the cardinality of the set of colors that appear in the list of available colors of some uncolored edge. For a color $i\in[N]$, recall that G_i denotes the subgraph of uncolored edges that contain i in their list of available colors. Finally, let $E_i=|E(G_i)|$ and $\mathcal S=\mathcal S(T)$ be the set of all binary strings of length T, where T is a parameter to be defined later. An element of $\mathcal S$ should be thought of as the input "randomness" to a subroutine of our algorithm whose purpose will be to estimate edge-marginals of distributions $\{\mu_i\}_{i\in[N]}$.

Define $\Omega = \prod_{i \in [N]} (\mathcal{M}(G_i) \times \{0,1\}^{E_i} \times \{0,1\}^{E_i} \times \mathcal{S}^{E_i})$. We consider an arbitrary but fixed ordering over \mathcal{U} , so that each state $\sigma \in \Omega$ can be represented as $\sigma = ((M_1, a_1, h_1, s_1), \dots, (M_N, a_N, h_N, s_N))$, where M_i, a_i, h_i are the matching, activation and equalizing coin flip vectors, respectively, that correspond to color i, so that edge e is activated in G_i if $a_i(e) = 1$ and is marked to be removed if $h_i(e) = 1$. Additionally, s_i is the tuple of strings corresponding to the particular element of \mathcal{S}^{E_i} at state σ . As we will see, tuples $\{s_i\}_{i \in [N]}$ are defined for purely technical reasons, and specifically for properly bypassing the issue of detecting the presence of f_e -flaws that we mentioned earlier.

Recalling Corollary 2.12, we see that we are able to obtain a $1 \pm 1/n^{\beta}$ approximation for the marginal $\mu_i(e)$, $i \in [N]$, of an edge e with probability at least $1 - 1/n^{\beta}$ in polynomial time, where β is a fixed and sufficiently large positive constant. This fact will be useful to us in two ways.

First, recall that for color i we choose a matching according to probability distribution μ_i , and we define $\operatorname{Eq}_i(e)$ to be the probability of success of the equalizing coin flip that corresponds to edge e and color i. Note that, given access to the marginals of μ_i , the value of $\operatorname{Eq}_i(e)$ can be computed efficiently. Of course, and as we just explained, we will have only (arbitrarily good) estimates of the marginals of μ_i , but as in the proof of Theorem 1.3, this suffices for our purposes. Indeed, through sampling we can efficiently get an estimate $\operatorname{Eq}_i'(e)$ that is within a $1 \pm 1/n^c$ factor of the correct value $\operatorname{Eq}_i(e)$ with probability at least $1 - 1/n^c$, where $c = c(\beta)$ is a sufficiently large constant, and hence guarantee that the total variation distance between the resampling probability distributions used by the algorithm and the ideal ones is negligible, i.e., at most $1/n^c$. (Later we will argue that we can maximally couple the approximate and ideal distributions and proceed with an argument identical to the one we used in the proof of Theorem 1.3, where we

absorb the probability that the coupling fails into the expected polynomial running time of the algorithm — recall our discussion in the beginning of Section 4.2)

Second, we let $T_1 = T_1(\beta) = \operatorname{poly}(n)$ be a fixed polynomial upper bound on the number of random bits required by the sampling algorithm (whose existence is guaranteed by Theorem 2.10) for approximating $\Pr(e \in M_i)$, for an arbitrary color $i \in [N]$ and an arbitrary edge e, within a factor $1 \pm 1/n^{\beta}$ with probability at least $1 - 1/n^{\beta}$. We let T_2 be an analogous fixed polynomial upper bound for estimating $\Pr(e \in Z_i \mid \operatorname{Out})$ for arbitrary Out , and define $T = T_1 + T_2$.

We let p be the probability distribution over Ω that is induced by the product of the μ_i 's, activation flips, equalizing coin flips, and the uniform distribution over \mathcal{S}^{E_i} , for each color $i \in [N]$. In other words, p is the probability distribution over Ω induced by the iteration along with some extra randomness that is used for sampling from $\mathcal{S}(T)^{E_i}$.

The initial distribution. Recall that each edge e initially has a list L_e of size at least $(1+\epsilon)\chi_e^*(G)$. As we have already seen in Corollary 2.13, the results of [19, 41] imply that for every color i and parameter $\eta=1/n^\beta$, where $\beta>0$ is a sufficiently large constant, there exists a $\operatorname{poly}(n,\ln\frac{1}{\epsilon})$ -algorithm that computes a vector λ_i' such that the induced hard-core distribution η -approximates in variation distance the hard-core distribution induced by vector λ_i . Let p' be the distribution obtained in an identical way to p but using vectors λ_i' instead of vectors λ_i . The initial distribution θ of our algorithm is obtained by η -approximately sampling from p'. Theorem 2.10 implies that this can be done in polynomial time.

Finding and addressing flaws. We define a flaw f_v for each bad event A_v . To define flaw f_e corresponding to an edge e, we first recall the definitions of T_1, T_2 . In particular, recall that the description of a state σ determines a binary string $s = s(\sigma) \in \mathcal{S}$ of length $T_1 + T_2$ for each color i and edge $e \in E(G_i)$. We will think of s as a concatenation of two strings of length T_1 and T_2 , respectively, that can and will be used as the "input randomness" to a sampling algorithm that estimates $\Pr(e \in M_i)$ and $\Pr(e \in Z_i \mid \operatorname{Out}(\sigma))$, respectively. (Here $\operatorname{Out}(\sigma)$ is the evaluation of random variable Out at σ .) Indeed, let $\Pr_{\sigma}(e \in M_i)$ be the resulting, $\operatorname{deterministic}$ (given $s(\sigma)$) estimation of $\Pr(e \in M_i)$ and, similarly, let $\Pr_{\sigma}(e \in Z_i \mid \operatorname{Out}(\sigma))$ be the resulting estimation of $\Pr(e \in Z_i \mid \operatorname{Out}(\sigma))$. Finally, we define flaw f_e to be the set of states $\sigma \in \Omega$ such that

$$\left| \sum_{i:G_i'\ni e} \widetilde{\Pr}(e \in Z_i \mid \operatorname{Out}(\sigma)) - \sum_{i:G_i\ni e} \widetilde{\Pr}(e \in M_i) \right| > \frac{2}{3(\log \Delta)^4}.$$
 (21)

We fix an arbitrary ordering π over $V \cup E$. In each step, the algorithm finds the lowest indexed flaw according to π that is present in the current state and addresses it.

Clearly, checking if vertex-flaws f_v are present in the current state can be done efficiently. The same is true for edge-flaws f_e given Theorem 2.10. What is perhaps not so clear, however, is whether the definition of f_e -flaws is sufficient for our purposes, and how it relates to the definition of bad events A_e .

To address these questions, recall first that we can use the results of [19] to approximate the edge marginals of the corresponding distributions within a $(1\pm\eta)$ -factor with probability at least $1-\eta$, in time $\operatorname{poly}(n,\frac{1}{\eta})$, where $\eta=1/n^{\beta}$. Our approach will be to first argue that, assuming our edge marginal estimates were *always* within a $(1\pm\eta)$ -factor of the true values, then our algorithm would terminate in expected polynomial time, and then use a coupling argument similar to the one described in the beginning of Section 4.2 to show that we can make this assumption in our analysis at a negligible price.

More formally, given a state $\sigma=((M_1,a_1,h_1,s_1),\ldots,(M_N,a_N,h_N,s_N)),$ let $M(\sigma)=(M_1,\ldots,M_N),$ $a(\sigma)=(a_1,\ldots,a_N),$ and $h(\sigma)=(h_1,\ldots,h_N),$ and define $\xi(\sigma)=(M(\sigma),a(\sigma),h(\sigma)).$ For each edge e, color i, and state σ , let $\mathcal{S}_i'(e)=\mathcal{S}_i'(e,\xi(\sigma))\subseteq\mathcal{S}$ be the set of strings with the property that, if our marginal estimators use them as input randomness in state σ for edge e, then they are guaranteed to provide a $(1\pm\eta)$ -factor approximation of the true marginals of e. Crucially, observe that $|\mathcal{S}_i'(e)|/|\mathcal{S}|\geq 1-1/n^c$ for a constant $c=c(\beta)$ which can be made arbitrarily large by increasing β . Let $\Omega'\subseteq\Omega$ be the subspace of Ω induced by removing every state $\sigma=((M_1,a_1,h_1,s_1),\ldots,(M_N,a_N,h_N,s_N))$ such that there exists an $i\in[N]$ for which $s_i\notin\prod_{e\in E(G_i)}\mathcal{S}_i'(e)$. That is, Ω' is the subspace of Ω in which our edge-marginal approximations are guaranteed to be within a $(1\pm\eta)$ -factor of the true values. Finally, let μ be the distribution induced by conditioning on the event that a sample from p belongs to Ω' . Equivalently, to take a sample from μ we first sample from the product of the μ_i 's, activation flips, and equalizing coin flips to obtain a tuple $\xi=(M,a,h)$, and then sample uniformly an element from $\prod_{i=1}^N\prod_{e\in E(G_i)}\mathcal{S}_i'(e,\xi).$

The following two lemmas justify our definition of f_e -flaws. Specifically, Lemma 5.4 shows that avoiding all f_e -flaws is sufficient for the purposes of our analysis (recall Theorem 5.1), while Lemma 5.5 bounds the probability of each flaw (with respect to μ).

Lemma 5.4. Condition (20) holds for every edge e and every state $\sigma \in \Omega'$ such that $\sigma \notin f_e$.

Proof. Since for every state $\sigma \in \Omega'$ we have that $\widetilde{\Pr}_{\sigma}(e \in Z_i \mid \operatorname{Out}(\sigma)), \widetilde{\Pr}_{\sigma}(e \in M_i)$ are within a $(1 \pm \eta)$ factor of the respective true marginals, we have that for every state $\sigma \in \Omega' \setminus f_e$:

$$\left| \sum_{i:G_i'\ni e} \Pr(e \in Z_i \mid \operatorname{Out}(\sigma)) - \sum_{i:G_i\ni e} \Pr(e \in M_i) \right| \le \frac{2}{3(\log \Delta)^4} + 2\eta.$$

Recalling that $\Pr(e \in M_i' \mid \sigma)$ is within a $(1 + \frac{1}{(\log \Delta)^{20}})$ -factor of $\Pr(e \in Z_i \mid \sigma)$, we can deduce that if flaw f_e is not present in a state $\sigma \in \Omega'$, then (20) holds for sufficiently large β, Δ , as claimed.

Lemma 5.5. For every $x \in E \cup V$ and possible choice R_x^* for R_x we have $\mu(f_x \mid R_x = R_x^*) \leq \frac{1}{\Delta^{3(t^2+t+2)}}$.

Proof. For f_v flaws the claim follows almost immediately from Lemma 5.3, so we focus on proving it for the case of f_e -flaws. In particular, we show that

$$\mu(f_e \mid R_e = R_e^*) \le \Pr(A_e \mid R_e = R_e^*)$$
 (22)

as this implies our claim per Lemma 5.3.

Recall that f_e is a subset of Ω , i.e., the "augmented" space where each state is associated with a tuple of strings from $\prod_{i \in [N]} \mathcal{S}^{E_i}$, while event A_e is a subset of the original probability space that is induced by the family of random matchings, activations, and equalizing coin flips for each edge. Recall also that, by definition, μ assigns zero probability mass to $f_e \setminus \Omega'$, i.e., the part of f_e where we have no guarantees about the quality of approximation of our edge-marginal estimators. In order to establish (22) we "project" $f_e \cap \Omega'$ to the original probability space to get an event \widetilde{A}_e . That is, the elements of \widetilde{A}_e are induced by the elements of f_e by ignoring the coordinate that corresponds to the tuple of strings from $\prod_{i \in [N]} \mathcal{S}^{E_i}$. By definition, $\mu(f_e \mid R_e = R_e^*) = \Pr(\widetilde{A}_e \mid R_e = R_e^*)$.

In addition, for every elementary event $\xi \in \widetilde{A}_e$ we have

$$\left| \sum_{i:G_i'\ni e} \Pr(e \in Z_i \mid \operatorname{Out}(\xi)) - \sum_{i:G_i\ni e} \Pr(e \in M_i) \right| \ge \frac{2}{3(\log \Delta)^4} - 2\eta > \frac{1}{2(\log \Delta)^4}, \tag{23}$$

for sufficiently large Δ . Note that the first inequality follows from (21) and the fact that we only consider elements in $f_e \cap \Omega'$, i.e., states in which our edge-marginal approximations are within a $(1 \pm \eta)$ -factor from the true values. Recalling (19), we see that inequality (23) implies that $\Pr(\widetilde{A}_e \mid R_e = R_e^*) \leq \Pr(A_e \mid R_e = R_e^*)$ (and, therefore, also (22)), concluding the proof.

Summarizing, we may assume without loss of generality that we are able to accurately and efficiently search for edge-flaws f_e , and that their probability with respect to measure μ is bounded above by $\Delta^{-3(t^2+t+2)}$ conditional on any instantiation of R_e .

Recall the procedure RESAMPLE described in Section 4.1. Below we describe procedure FIX that takes as input a subgraph H and a state σ . In the description of FIX below we invoke procedure RESAMPLE with an extra parameter, namely an activity vector λ_i' for each color i. By that we mean that in Lines 4, 5 of RESAMPLE we use the vector λ_i' to define p. Finally, recall that we defined $t = 8(K+1)^2(\log \Delta)^{20} + 2$.

Theorem 2.10 implies that procedure FIX runs in polynomial time for any input subgraph H and state σ . To address flaws $f_v, f_{\{u_1, u_2\}}$ in a state σ we invoke $\text{FIX}(\{v\}, \sigma)$ and $\text{FIX}(\{u_1, u_2\}, \sigma)$, respectively.

```
1: procedure FIX(H, \sigma)

2: Let \sigma = ((M_1, b_1, h_1, s_1), \dots, (M_N, b_N, h_N, s_N))

3: (M'_1, \dots, M'_N) \leftarrow \text{RESAMPLE}(H, (M_1, \dots, M_N), t^2, \lambda'_i)

4: for i = 1 to N do

5: Update a_i to a'_i by activating independently each edge in G_{i, < t^2 + 1}(H) with probability \alpha

6: Update h_i to h'_i by flipping the equalizing coin corresponding to each edge in G_{i, < t^2 + 1}(H)

7: Update s_i to s'_i by uniformly sampling from S^{E_i}

8: Output \sigma = ((M'_1, a'_1, h'_1, s'_1), \dots, (M'_N, a'_N, h'_N, s'_N))
```

5.3 Proof of Theorem 1.4

Similarly to the proof of Theorem 1.3, for our analysis we will assume that our algorithm samples from the "ideal" matchings distributions, i.e., the ones induced by the vectors λ_i rather than by the approximations λ_i' . We will also assume that each equalizing coin corresponding to a color $i \in [N]$ and an edge e is flipped with probability of success $\operatorname{Eq}_i(e)$ instead of $\operatorname{Eq}_i'(e)$, and that we update string $s_i(e)$ by sampling uniformly from $\mathcal{S}_i'(e)$ instead of \mathcal{S} . Under these assumptions, we will prove that our algorithm terminates in expected polynomial time. Recalling the proof of Theorem 1.3, the latter allows us to invoke an identical coupling argument and show that the price of making these assumptions is to increase the failure probability of our algorithm by an additive $1/n^{\gamma}$, where $\gamma = \gamma(\beta)$ can be made arbitrarily large by increasing β . This error probability can be subsumed by the expected running time of our algorithm.

For two flaws f_{x_1}, f_{x_2} , where $x_1, x_2 \in V \cup E$, we consider the causality relation $f_{x_1} \sim f_{x_2}$ iff $\operatorname{dist}(x_1, x_2) \leq t^2 + t + 2$. By inspecting procedure FIX it is not hard to verify that this is a valid choice for a causality graph in the sense that no flaw f can cause flaws outside $\Gamma(f)$. This is because, in order to determine whether a flaw f_x is present in a state σ , we only need information about σ in $G \cap S_{< t}(x)$, and procedure FIX locally modifies the state within a radius at most t^2 of the input subgraph H.

The algorithmic proof of Theorem 5.1, which as we explained earlier is the key ingredient in making Kahn's result constructive, follows almost immediately by combining Theorem 2.4 with Lemma 5.6 below, whose proof can be found in Section 5.3.1.

Lemma 5.6. Let $f \in \{f_e, f_v\}$ for an edge e and a vertex v. Then:

$$\gamma_f \le \frac{1}{\Delta^{3(t^2+t+2)}},$$

where the charges are computed with respect to measure μ and the algorithm that samples from the ideal distributions.

Constructive Proof of Theorem 5.1. Recall from (8) that, setting $x_f = \frac{1}{1 + \max_{f \in F} |\Gamma(f)|}$ for each flaw f, condition (3) with $\epsilon = 1/4$ is implied by

$$\max_{f \in F} \gamma_f \cdot \left(1 + \max_{f \in F} |\Gamma(f)| \right) \cdot e \le 3/4. \tag{24}$$

Clearly, for each flaw f, $|\Gamma(f)| = O(\Delta^{2(t^2+t+2)})$ so, by Lemma 5.6, condition (24) is satisfied for all sufficiently large Δ . Thus, Theorem 2.4 implies that, for every multigraph with large enough degree Δ_0 , the algorithm for each iteration terminates after an expected number of

$$O\left((m+n)\log_2\left(\frac{1}{1-1/\Delta^{2(t^2+t+2)}}\right)\right) = O(n^2)$$

steps.

Finally, the proof of Theorem 1.4 is concluded by combining the algorithm for Theorem 5.1 with the greedy algorithm of Theorem 5.2. It remains only for us to prove Lemma 5.6 stated above. This we do in the next subsection.

5.3.1 Proof of Lemma 5.6.

Let $\Omega_1 = \prod_{i=1}^N \mathcal{M}(G_i)$ and $\Omega_2 = \Omega_3 = \prod_{i=1}^N \{0,1\}^{E_i}$. For notational convenience, sometimes we write $\Omega_1^i = \mathcal{M}(G_i)$ and $\Omega_2^i = \Omega_3^i = \{0,1\}^{E_i}$, for $i \in [N]$.

Let ν_1 be the distribution over Ω_1 induced by the product of distributions $\mu_i, i \in [N]$. Let also ν_2, ν_3 be the distributions over Ω_2 and Ω_3 induced by the product of activation and equalizing coin flips of each color $i \in [N]$, respectively. Recall that we can take a sample from μ by sampling from $\nu_1 \times \nu_2 \times \nu_3$ to obtain a tuple $\xi = (M, a, h) \in \Omega_1 \times \Omega_2 \times \Omega_3$, and then sample uniformly from an element from $\prod_{i=1}^N \prod_{e \in E(G_i)} \mathcal{S}'_i(e, \xi)$. Moreover, note that each ν_j is the product of N distributions ν_j^i , one for each color $i \in [N]$. For example, notice that ν_1^i is another name for μ_i , while ν_2^i is the product measure over the edges of G_i induced by flipping a coin with probability α for each edge.

For $\sigma_1=(M_1,M_2,\ldots,M_N)\in\Omega_1$, a subgraph H, and an integer d>0, we define the quantities $Q_H(d,\sigma_1)=(M_1-S_{< d}(H),\ldots,M_N-S_{< d}(H))$ and $Q_H^i(d,\sigma_1)=M_i-S_{< d}(H)$, similarly to the proof of Lemma 4.3. Moreover, for $\sigma_2\in\Omega_2$ that represents the outcome of the activations, we let $A_H(d,\sigma_2)$ denote the restriction of σ_2 to $M_i-S_{< d}(H)$ for each color $i\in[N]$. In the same fashion, for $\sigma_3\in\Omega_3$ that represents the outcome of the equalizing coin flips, we let $C_H(d,\sigma_3)$ denote the restriction of σ_3 to $M_i-S_{< d}(H)$ for each color $i\in[N]$. For $\sigma_2\in\Omega_2,\sigma_3\in\Omega_3$, we also define $A_H^i(d,\sigma_2)$ and $C_H^i(d,\sigma_3), i\in[N]$, similarly to $Q_H^i(d,\sigma_1)$. Finally, for $\xi=(\sigma_1,\sigma_2,\sigma_3)\in\Omega_1\times\Omega_2\times\Omega_3$, define $R_H(d,\xi)=(Q_H(d,\sigma_1),A_H(d,\sigma_2),C_H(d,\sigma_3))$.

Our goal will be to show that, for every $x \in V \cup E$,

$$\gamma_{f_x} = \max_{\tau \in \Omega} \mu(\sigma \in f_x \mid R_x(t^2, \sigma) = R_x(t^2, \tau)), \tag{25}$$

where σ is a random state distributed according to μ . Note that combining (25) with Lemma 5.5 will conclude the proof of Lemma 5.6.

We only prove (25) for f_e -flaws, since the proof for f_v flaws is very similar. Observe that whether flaw f_e is present at a state σ is determined by $\bigcup_{i=1}^N (G_i \cap G_{< t}(e))$, the entries of the activation and equalizing flip vectors of each color $i \in [N]$ that correspond to edges in $G_i \cap G_{< t}(e)$, and the value of the "input randomness" strings $\{s_i(e)\}_{i=1}^N$. With that in mind, for each color i let $M_i(t,e) = M_i \cap E(G_i \cap G_{< t}(e))$ and $a_i(t,e), h_i(t,e)$ denote the (random) vectors constraining the entries of the activation and equalizing coin flip vectors for color i that correspond to the edges of $G_i \cap G_{< t}(e)$. Let also $\mathcal{D}_i(t,e)$ denote the domain of possible values of $(M_i(t,e), a_i(t,e), h_i(t,e), s_i(e))$.

The fact that we can determine whether f_e is present in a state by examining local information around e implies that there exists a set $X_e = X_e(t)$ of vectors of size N such that the i-th entry of a vector $x \in X_e$ is an element of $\mathcal{D}_i(t,e)$, and so that

$$f_e = \bigcup_{x \in X_e} \bigcap_{i \in [N]} ((M_i(t, e), a_i(t, e), h_i(t, e), s_i(e)) = x_i).$$
(26)

For a state $\sigma \in \Omega'$, let x_e^{σ} be the N-dimensional random vector whose i-th entry is $(M_i(t,e), a_i(t,e), h_i(t,e), s_i(e))$. According to (26), for $\tau \in \Omega'$ we have

$$\mu(\sigma \in f_e \mid R_e(t^2, \sigma) = R_e(t^2, \tau)) = \sum_{x \in X_e} \prod_{i=1}^N \mu(x_{e,i}^\sigma = x_i \mid R_e(t^2, \sigma) = R_e(t^2, \tau)), \tag{27}$$

since the random choices of matching, activation, and equalizing coin flips for each color are independent. For an N-dimensional vector x whose i-th entry is an element of $\mathcal{D}_i(t,e)$, we write $x_i(j)$ to denote the j-th element of tuple x_i . Thus, recalling the definition of the distributions ν_i^i , we have

$$\mu(x_{e,i}^{\sigma} = x_i \mid R_e(t^2, \sigma) = R_e(t^2, \tau)) = \prod_{j=1}^{3} \nu_j^i(x_{e,i}^{\sigma}(j) = x_i(j) \mid R_e(t^2, \sigma) = R_e(t^2, \tau)) \cdot \frac{1}{|\mathcal{S}_i'(e, \xi_i)|}, \tag{28}$$

where $\xi_i = (x_i(1), x_i(2), x_i(3))$.

Recall now that for a subgraph H, multigraph $G_{< d+1}(H)$ is induced by $S_{< d+1}(H)$, and $\mathcal{M}_{d+1}^i(H,\sigma)$ is the set

of matchings of $G_{< d+1}(H)$ that are compatible with $Q_H^i(d, \sigma_1)$. Hence,

$$\nu_{1}^{i}(x_{e,i}^{\sigma}(1) = x_{i}(1) \mid R_{e}(t^{2}, \sigma) = R_{e}(t^{2}, \tau))$$

$$= \nu_{1}^{i}(x_{e,i}^{\sigma}(1) = x_{i}(1) \mid Q_{e}^{i}(t^{2}, \sigma_{1}) = Q_{e}^{i}(t^{2}, \tau_{1}))$$

$$= \frac{\nu_{1}^{i}((x_{e,i}^{\sigma}(1) = x_{i}(1)) \cap (Q_{e}^{i}(t^{2}, \sigma_{1}) = Q_{e}^{i}(t^{2}, \tau_{1})))}{\nu_{1}^{i}(Q_{e}^{i}(t^{2}, \sigma_{1}) = Q_{e}^{i}(t^{2}, \tau_{1}))}$$

$$= \frac{\sum_{M \in \mathcal{M}_{t^{2}+1}^{i}(e, \tau_{1}), M \cap E(G_{
(29)$$

Moreover, we clearly have

$$\nu_2^i(x_{e,i}^{\sigma}(2) = x_i(2) \mid R_e(t^2, \sigma) = R_e(t^2, \tau))
= \nu_2^i(a_i(t, e) = x_i(2));
\nu_2^i(x_{e,i}^{\sigma}(3) = x_i(3) \mid R_e(t^2, \sigma) = R_e(t^2, \tau))$$
(30)

$$\nu_3^i(x_{e,i}^o(3) = x_i(3) \mid R_e(t^2, \sigma) = R_e(t^2, \tau))$$

$$= \nu_3^i(h_i(t, e) = x_i(3)). \tag{31}$$

We will use (27)-(31) to show that, for σ distributed according to μ , and any state $\tau \in \Omega'$,

$$\sum_{\omega \in f_e} \frac{\mu(\omega)}{\mu(\tau)} \rho_{f_e}(\omega, \tau) = \mu(\sigma \in f_e \mid R_e(t^2, \sigma) = R_e(t^2, \tau)). \tag{32}$$

According to the definition of γ_{f_e} , maximizing (32) over $\tau \in \Omega'$ yields (25).

To compute the sum in (32) we need to determine the set of states $\operatorname{In}_e(\tau) = \{\omega : \rho_{f_e}(\omega, \tau) > 0\}$. We claim that for each $\omega \in \operatorname{In}_e(\tau)$ we have that $R_e(t^2, \omega) = R_e(t^2, \tau)$.

To see this, let

$$\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$$

$$= ((\omega_1^1, \dots, \omega_1^N), (\omega_2^1, \dots, \omega_2^N), (\omega_3^1, \dots \omega_3^N), \omega_4);$$

$$\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$$

$$= ((\tau_1^1, \dots, \tau_1^N), (\tau_2^1, \dots, \tau_2^N), (\tau_3^1, \dots, \tau_3^N), \tau_4),$$

where $\omega_j, \tau_j \in \Omega_j$ and $\omega_j^i, \tau_j^i \in \Omega_j^i$ for $j \in \{1,2,3\}$ and ω_4, τ_4 are tuples of input randomness strings. To express the probability distribution $\rho_{f_e}(\omega,\cdot)$ in a convenient way we consider the following 3N distributions. For each $i \in [N]$ we have a probability distribution $\rho_{f_e}^{i,1}(\omega_1^i,\cdot)$ corresponding to Line 3 of FIX and color i, and similarly, for ω_2^i, ω_3^i we have probability distributions $\rho_{f_e}^{i,2}(\omega_2^i,\cdot), \rho_{f_e}^{i,3}(\omega_3^i,\cdot)$, corresponding to Lines 5, 6 of FIX and color i, respectively. Recalling procedure RESAMPLE, we see that the support of $\rho_{f_e}^{i,1}(\omega_1^i,\cdot)$ is $\mathcal{M}_{t^2+1}^i(e,\omega_1)$, and hence it must be the case that $Q_e^i(t^2,\omega_1)=Q_e^i(t^2,\tau_1)$ for every $i\in[N]$ and state $\omega\in\operatorname{In}_e(\tau)$. Similarly, by inspecting procedure FIX one can verify that $A_e^i(t^2,\omega_2)=A_e^i(t^2,\tau_2)$ and that $C_e^i(t^2,\omega_3)=C_e^i(t^2,\tau_3)$ for each $i\in[N]$. Hence, $R_e(t^2,\omega)=R_e(t^2,\tau)$, as claimed.

For each $\omega \in f_e$,

$$\frac{\mu(\omega)}{\mu(\tau)}\rho_{f_e}(\omega,\tau) = \prod_{i=1}^{N} \frac{1}{|\mathcal{S}_i'(e,\xi(\omega))|} \prod_{j=1}^{3} \frac{\nu_j^i(\omega_j^i)}{\nu_j^i(\tau_j^i)} \rho_{f_e}^{i,j}(\omega_j^i,\tau_j^i) =: \prod_{i=1}^{N} \frac{1}{|\mathcal{S}_i'(e,\xi(\omega))|} \prod_{j=1}^{3} r_{i,j}(\omega), \tag{33}$$

since we have assumed that in Line 7 of FIX we update string $s_i(e)$ by sampling uniformly from $\mathcal{S}_i'(e)$ instead of \mathcal{S} . We will now give an alternative expression for each $r_{i,j}(\omega)$ in order to relate (33) to (32). We start with $r_{i,1}(\omega)$. The fact that $Q_e^i(t^2,\omega_1)=Q_e^i(t^2,\tau_1)$ for each $\omega\in\operatorname{In}_e(\tau)$ implies that

$$\frac{\nu_1^i(\omega_1^i)}{\nu_1^i(\tau_1^i)} = \frac{\lambda_i(\omega_1^i \cap E(G_{i, < t^2 + 1}(e)))}{\lambda_i(\tau_1^i \cap E(G_{i, < t^2 + 1}(e)))}.$$
(34)

To see this recall the definition of mutligraph $G_{i, < d+1}(H)$ in the text above the definition of procedure RESAMPLE.

Furthermore, since we have assumed that the hard-core distribution in Lines 4, 5 of RESAMPLE is induced by the ideal vector of activities λ_i , we have

$$\rho_{f_e}(\omega_1^i, \tau_1^i) = \frac{\lambda_i(\tau_1^i \cap E(G_{i, < t^2 + 1}(e)))}{\sum_{M \in \mathcal{M}_{t^2 + 1}^i(e, \omega_1)} \lambda_i(M)}.$$
(35)

Combining (34) with (35) and the fact that $Q_e^i(t^2, \omega_1) = Q_e^i(t^2, \tau_1)$ we obtain

$$r_{i,1}(\omega) = \frac{\lambda_i(\omega_1^i \cap E(G_{i, < t^2 + 1}(e)))}{\sum_{M \in \mathcal{M}_{t^2 + 1}^i(e, \tau_1)} \lambda_i(M)}.$$
(36)

Recall now the definitions of $a_i(t,e)$ and $h_i(t,e)$. The fact that $A_e^i(t^2,\omega_2)=A_e^i(t^2,\tau_2)$ for each $\omega\in\operatorname{In}_e(\tau)$ implies that

$$\frac{\nu_2^i(\omega_2^i)}{\nu_2^i(\tau_2^i)} = \frac{\nu_2^i(a_i(t,e) = x_{e,i}^{\omega}(2))}{\nu_2^i(a_i(t,e) = x_{e,i}^{\tau}(2))}.$$
(37)

Further, since in Line 5 of FIX we simply flip a coin independently with success probability α for each edge of $G_{i,< t^2+1}(e)$, we have

$$\rho_{f_e}(\omega_2^i, \tau_2^i) = \frac{\nu_2^i(a_i(t, e) = x_{e,i}^{\tau}(2))}{\sum_a \nu_2^i(a_i(t, e) = a)},\tag{38}$$

where the sum in the denominator ranges over all the possible values for $a_i(t, e)$. Thus, combining (37) with (38) we get

$$r_{i,2}(\omega) = \frac{\nu_2^i(a_i(t,e) = x_{e,i}^{\omega}(2))}{\sum_a \nu_2^i(a_i(t,e) = a)}.$$
(39)

Finally, an identical argument shows that

$$r_{i,3}(\omega) = \frac{\nu_3^i(h_i(t,e) = x_{e,i}^{\omega}(2))}{\sum_h \nu_3^i(h_i(t,e) = h)},\tag{40}$$

where the sum in the denominator ranges over all the possible values for $h_i(t, e)$.

For $x \in X_e$, let $\Omega_{e,x} = \{\omega : x_e^\omega = x\}$. For σ distributed according to μ , the left-hand side of (32) can be written as

$$\sum_{x \in X_{e}} \sum_{\omega \in \Omega_{e}} \frac{\mu(\omega)}{\mu(\tau)} \rho_{f_{e}}(\omega, \tau) = \sum_{x \in X_{e}} \sum_{\omega \in \Omega_{e,x}} \prod_{i=1}^{N} \frac{1}{|S'_{i}(e, \xi(\omega))|} \prod_{j=1}^{3} r_{i,j}(\omega)$$

$$= \sum_{x \in X_{e}} \prod_{i=1}^{N} \frac{1}{|S'_{i}(e, (x_{i}(1), x_{i}(2), x_{i}(3)))|} \prod_{j=1}^{3} \sum_{\substack{\omega \in \Omega_{e,x} \\ x_{e,i} = x_{i}(j)}} r_{i,j}(\omega) \tag{41}$$

$$= \sum_{x \in X_e} \prod_{i=1}^{N} \frac{1}{|S'_i(e, \xi_i)|} \prod_{j=1}^{3} \nu_j^i(x_{e,i}^{\sigma}(j) = x_i(j) \mid R_e(t^2, \sigma) = R_e(t^2, \tau))$$

$$= \mu(\sigma \in f_e \mid R_e(t^2, \sigma) = R_e(t^2, \tau)),$$
(42)

where $\xi_i = (x_i(1), x_i(2), x_i(3))$, concluding the proof of (32). Note that (42) follows by (29) and (36) for j = 1, (30) and (39) for j = 2, and (31) and (40) for j = 3. This concludes the proof of (25) and hence of Lemma 5.6

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A Proof of Lemma 4.5

We will need the following standard concentration bound (see, e.g., [31, Section 10.1]).

Lemma A.1. Let X be a random variable determined by n independent trials T_1, \ldots, T_n , and such that changing the outcome of any one trial can affect X by at most c. Then

$$\Pr(|X - \mathbb{E}[X]| > \lambda) \le 2e^{-\frac{\lambda^2}{2c^2n}}.$$

Proof of Part (a) of Lemma 4.5. Recall that $t=8(K+1)^2\delta^{-1}+2$ and that $\delta=\frac{\epsilon}{4}$. Consider a random state σ distributed according to μ and a fixed state $\tau\in\Omega$, and notice that applying Theorem 2.9 with the parameter ϵ instantiated to δ and our choice of t imply that

$$\mu(e \in M_i \mid Q_v^i(t,\sigma) = Q_v^i(t,\tau)) \ge (1-\delta) \frac{1-\delta}{\chi_e^*(G)} \ge \frac{1-\frac{\epsilon}{2}}{\chi_e^*(G)},$$

for any vertex v, any edge e incident on v and any $i \in [N]$. This implies

$$\mathbb{E}[d_{G_{\sigma}}(v) \mid Q_v^i(t,\sigma) = Q_v^i(t,\tau)] \leq \Delta \left(1 - \frac{1 - \frac{\epsilon}{2}}{\chi_e^*(G)}\right)^N \leq \chi_e^*(G) \left(1 - \frac{1 - \frac{\epsilon}{2}}{\chi_e^*(G)}\right)^N.$$

Now, recalling that $N = \lfloor \chi_e^*(G)^{\frac{3}{4}} \rfloor \sim \Delta^{3/4}$ and $\epsilon \leq \frac{1}{10}$, for sufficiently large Δ we have

$$\mathbb{E}[d_{G_{\sigma}}(v) \mid Q_{v}^{i}(t,\sigma) = Q_{v}^{i}(t,\tau)] \le \chi_{e}^{*}(G) \left(1 - (1 - \epsilon^{3}) \frac{(1 - \frac{\epsilon}{2})N}{\chi_{e}^{*}(G)}\right) \le \chi_{e}^{*}(G) - \left(1 - \frac{9\epsilon}{17}\right)N. \tag{43}$$

Further, since $c^* = \chi_e^*(G) - (1+\epsilon)^{-1}N$ and $\epsilon \leq \frac{1}{10}$, (43) yields

$$\mathbb{E}[d_{G_{\sigma}}(v) \mid Q_v^i(t,\sigma) = Q_v^i(t,\tau)] \le c^* - \left(1 - \frac{9\epsilon}{17} - (1+\epsilon)^{-1}\right)N \le c^* - \frac{\epsilon}{3}N.$$

As the choices of the M_i are independent and each affects the degree of v in G' by at most 1, we can apply Lemma A.1 with $\lambda = (\frac{\epsilon}{3} - \frac{\epsilon}{4})N = \frac{\epsilon}{12}N$ to prove part (a). In particular, since $N = \lfloor \chi_e^*(G)^{\frac{3}{4}} \rfloor \sim \Delta^{3/4}$ we have

$$\mu\left(d_{G_{\sigma}(v)} > c^* - \frac{\epsilon}{4}N \mid Q_v^i(t,\sigma) = Q_v^i(t,\tau)\right) \le 2e^{-\frac{\lambda^2}{2N}} \le \frac{1}{\Delta^{C+2\Delta^{\frac{1}{3}}}},$$

for any constant C for sufficiently large Δ .

Proof of Part (b) of Lemma 4.5. The proof of part (b) is similar. Consider again a random state σ distributed according to μ and fix a state $\tau \in \Omega$. Theorem 2.9 implies that for each $i \in [N]$, the probability that an edge e with both endpoints in H is in M_i , conditional on $Q_H^i(t,\sigma) = Q_H^i(t,\tau)$, is at least $(1-\delta)\frac{1-\delta}{\chi_e^*(G)} \geq \frac{1-\frac{\epsilon}{2}}{\chi_e^*(G)}$. Moreover, Edmonds' characterization of the matching polytope (which we have already seen in the the proof of Lemma 4.2) implies that the number of edges in G with both endpoints in H is at most $\chi_e^*(G)\lfloor \frac{V(H)-1}{2} \rfloor$. Similar calculations to those in part (a) reveal that

$$\mathbb{E}[|E_{\sigma}(H)| \mid Q_H^i(t,\sigma) = Q_H^i(t,\tau)] \le \left(\frac{V(H) - 1}{2}\right) (c^* - \frac{\epsilon}{3}N),$$

where $E_{\sigma}(H)$ is the set of edges of G_{σ} induced by H. Since the choices of matchings M_i are independent and each affects $|E_{\sigma}(H)|$ by at most $\frac{|V(H)|-1}{2}$, we can again apply Lemma A.1 to prove part (b).