# Inference from Auction Prices* 

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#### Abstract

Econometric inference allows an analyst to back out the values of agents in a mechanism from the rules of the mechanism and bids of the agents. This paper gives an algorithm to solve the problem of inferring the values of agents in a dominant-strategy mechanism from the social choice function implemented by the mechanism and the per-unit prices paid by the agents (the agent bids are not observed). For single-dimensional agents, this inference problem is a multi-dimensional inversion of the payment identity and is feasible only if the payment identity is uniquely invertible. The inversion is unique for single-unit proportional weights social choice functions (common, for example, in bandwidth allocation); and its inverse can be found efficiently. This inversion is not unique for social choice functions that exhibit complementarities. Of independent interest, we extend a result of Rosen (1965), that the Nash equilbria of "concave games" are unique and pure, to an alternative notion of concavity based on Gale and Nikaido (1965).


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## 1 Introduction

Traditional econometric inference allows an analyst to determine the values of agents from their equilibrium actions and the rules of a mechanism (Guerre et al., 2000; Haile and Tamer, 2003). This paper studies an inference problem when only the profile of the agents' per-unit prices is available to the analyst. Such an inference may be applicable when bids are kept private but prices are published; moreover, it is of interest even for incentive compatible mechanisms (where agents truthfully report their preferences). As a motivating example, with the per-unit prices from the incentive compatible mechanism for allocating a divisible item proportionally to agent values (cf. Johari and Tsitsiklis, 2004), we prove that agents' values are uniquely determined and can be computed efficiently.

Econometric inference is a fundamental topic in a data-driven approach to mechanism design and a number of recent papers have been developing its algorithmic foundations. The following are prominent examples. Chawla et al. $(2014,2016)$ show that the revenue and welfare of a counter factual auction can be estimated directly from Bayes-Nash equilibrium bids in an incumbent auction. Nekipelov et al. (2015) develop methods for identifying the rationalizable set of agent values and regret parameters in repeated auctions with learning agents. Hoy et al. (2017) show that the quantities that govern price-of-anarchy analyses can be determined directly from bid data and, thus, empirical price-of-anarchy bounds can be established that improve on the theoretical worst case.

There are two important questions in algorithmic econometrics. First, when are the values uniquely identified? Second, can the values be efficiently computed when the values are identifiable? The first question is studied in depth by the econometrics literature (for inference from actions); the second question is an opportunity for algorithms design and analysis.

We consider inference in single-dimensional environments where a stochastic social choice function maps profiles of agent values to profiles of allocation probabilities. The characterization of incentive compatibility (Myerson, 1981) requires the allocation probability of an agent be monotonically non-decreasing in that agent's value and that an agent's expected payments satisfy a payment identity. Per-unit prices - the expected payments conditioned on winning - are easily determined from the expected payments in the payment identity by normalizing by the allocation probability 1 Consequentially, given any social choice function and valuation profile, the allocation probabilities and prices of an incentive compatible mechanism that implements the social choice function are uniquely and easily determined. Our inference problem is the opposite. Given the profile of the agents' prices, determine the valuation profile that leads to these prices. The social choice function and, thus, the function mapping values to prices is known. The resulting inversion problem is multi-dimensional and this multi-dimensionality leads to a possibility of non-uniqueness (and consequentially, non-identifiability) and computational challenges.

The first goal of this paper is to understand what social choice functions admit inference from prices and which do not. Fundamentally, social choice functions with induced allocation rules that are not strictly increasing do not admit inference. For example, the only inference possible from the outcome of a second-price auction is that the winner has value above the winner's price and the

[^1]losers have value below the winner's price. On the other hand, a "soft max" social choice function like proportional values, where an agent receives a fraction of the item proportional to her value, is strictly continuous and, as we will show, the valuation profile can be uniquely inferred from the winner-pays prices. We will show sufficiency for social choice functions to admit inference from prices as ones where the Jacobian of the payment identity has all minors positive on (almost) all inputs and, as a class, proportional weights social choice functions (with general strictly monotonic weight functions) satisfy this property. In contrast we show that this property does not generally hold for social choice functions that exhibit complementarities.

These identification and non-identification results are complemented by an algorithm for efficiently computing the valuation profile from the prices that corresponds to any proportional weights social choice function for single-item environments.

Our focus is on proportional weights allocation rules for (probabilistically) sharing a unit resource. Such mechanisms have been previously considered in the literature on bandwidth allocation (e.g., Johari and Tsitsiklis, 2004). Another point of contact with the literature is the special case of exponential weights. The mechanism that implements the exponential weights allocation rule is known as the exponential mechanism (Huang and Kannan, 2012). The exponential mechanism is often considered because its realized allocation has good privacy properties. Huang and Kannan (2012) recommend additionally adding Laplacian noise to the payments of the exponential mechanism so that its realized outcome (allocation and payments) is differentially private. Our main result shows that, in fact, without such noise added to the payments the exponential mechanism is not private.

Organization. The rest of this paper is organized as follows. Section 2 gives notation for discussing social choice functions, mechanisms, and agents; reviews the characterization of incentivecompatible single-dimensional mechanisms; and reviews proportional weights allocations. Section 3, then, gives an algorithmic framework for robustly identifying values from prices. It shows that values are identified from payments corresponding to social choice functions given by proportional weights in single-item and multi-unit environments. Section 3.3 shows that values are not identifiable from prices for proportional weights allocations that correspond to environments with complementarities. Section 4 gives an efficient algorithm for inferring values from prices for proportional weights social choice functions in single-item environments.

## 2 Preliminaries

This paper considers general environments for single-dimensional linear agents. Each agent $i$ has value $v_{i} \in[0, h]$. For allocation probability $x_{i}$ and expected payment $p_{i}$, the agent's utility is $v_{i} x_{i}-p_{i}$. A profile of $n$ agent values is denoted $\boldsymbol{v}=\left(v_{1}, \ldots, v_{n}\right)$; the profile with agent $i$ 's value replaced with $z$ is $\left(z, \boldsymbol{v}_{-i}\right)=\left(v_{1}, \ldots, v_{i-1}, z, v_{i+1}, \ldots, v_{n}\right)$.

A stochastic social choice function $\boldsymbol{x}$ maps a profile of values $\boldsymbol{v}$ to a profile of allocation probabilities. A dominant strategy incentive compatible (DSIC) mechanism ( $\boldsymbol{x}, \boldsymbol{p}$ ) maps a profile of values $\boldsymbol{v}$ to profiles of allocations $\boldsymbol{x}(\boldsymbol{v})$ and payments $\boldsymbol{p}(\boldsymbol{v})$ so that: for all agents $i$, values $v_{i}$, and other agent values $\boldsymbol{v}_{-i}$, it is optimal for agent $i$ to bid her value $v_{i}$. The following theorem of Myerson (1981) characterizes social choice functions that can be implemented by DSIC mechanisms.

Theorem 1 (Myerson, 1981). Allocation and payment rules ( $\boldsymbol{x}, \boldsymbol{p}$ ) are induced by a dominant strategy incentive compatible mechanism if and only if for each agent $i$,

1. (monotonicity) allocation rule $x_{i}\left(v_{i}, \boldsymbol{v}_{-i}\right)$ is monotone non-decreasing in $v_{i}$, and
2. (payment identity) payment rule $p_{i}(\boldsymbol{v})$ satisfies

$$
\begin{equation*}
p_{i}(\boldsymbol{v})=v_{i} x_{i}(\boldsymbol{v})-\int_{0}^{v_{i}} x_{i}\left(z, \boldsymbol{v}_{-i}\right) \mathrm{d} z+p_{i}\left(0, \boldsymbol{v}_{-i}\right) \tag{1}
\end{equation*}
$$

where the payment of an agent with value zero is often zero, i.e., $p_{i}\left(0, \boldsymbol{v}_{-i}\right)=0$.
Most DSIC mechanisms are implemented to satisfy an ex post individual rationality constraint; specifically, an agent pays nothing when not allocated. The payment when allocated, i.e., the perunit price, is thus the expected payment normalized by the probability of winning. Throughout this work, we assume $p_{i}\left(0, \boldsymbol{v}_{-i}\right)=0$. Denote the price function by $\boldsymbol{\pi}: \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}_{+}^{n}$, as

$$
\begin{align*}
\pi_{i}(\boldsymbol{v}) & =p_{i}(\boldsymbol{v}) / x_{i}(\boldsymbol{v}) \\
& =v_{i}-\frac{\int_{0}^{v_{i}} x_{i}\left(z, \boldsymbol{v}_{-i}\right) \mathrm{d} z}{x_{i}(\boldsymbol{v})} \tag{2}
\end{align*}
$$

for all agents $i$.
The main objective of this paper is to infer the agents' values from observations of the per-unit prices of the mechanism. A price profile $\boldsymbol{\rho}$ is observed, and it is desired to infer the valuation profile $\boldsymbol{v}$ that generated this price profile by $\boldsymbol{\rho}=\boldsymbol{\pi}(\boldsymbol{v})$. The key question of this paper is to identify sufficient conditions on the social choice function $\boldsymbol{x}$ such that the price function $\boldsymbol{\pi}$ is invertible.

An important special case is the case where there is $n=1$ agent and the price function $\pi(\cdot)$ is single-dimensional. When the social choice function $x(\cdot)$ is strictly increasing, the price function $\pi(\cdot)$ is strictly increasing (apply Lemma 1 with only one agent), and is uniquely invertible. Thus, the agent's value can be identified from her observed price $\rho$, e.g., by binary search.

Lemma 1. Assume $\partial x_{i} / \partial v_{i}(\boldsymbol{v})>0$ everywhere. Then $\partial \pi_{i} / \partial v_{i}(\boldsymbol{v})>0$ for all values except 0 .
Proof. The partial of the price function $\pi_{i}^{\prime}\left(v_{i}, \boldsymbol{v}_{-i}\right)=\frac{x_{i}^{\prime}\left(v_{i}, \boldsymbol{v}_{-i}\right) \int_{0}^{v_{i}} x_{i}\left(z, \boldsymbol{v}_{-i}\right) d z}{\left(x_{i}\left(v_{i}, \boldsymbol{v}_{-i}\right)\right)^{2}}$ is positive if $x_{i}^{\prime}\left(v_{i}, \boldsymbol{v}_{-i}\right)$ is positive, unless the numerator is 0 because $v_{i}=0$ and the integral endpoints are the same.

Our goal is to understand families of (multi-agent) social choice functions $\boldsymbol{x}$ that allow values to be inferred from prices. Clearly, as in the single-agent case, if the allocation rule is not strictly increasing in each agent's value, then the values of the agents cannot be inferred. We assume that the social choice function $\boldsymbol{x}$ is such that it has strictly-increasing allocation functions $x_{i}$ for any given $\boldsymbol{v}_{-i}$, for all $v_{i}>0$.

Mechanisms in the literature for welfare and revenue maximization are based on social choice functions that map agents' values to weights and allocate to maximize the sum of the weights of the agents allocated. In order to satisfy the required strict monotonicity property, our focus is on smoothed versions of these social choice functions under feasibility constraints that correspond to single-item auctions (or single-minded combinatorial auctions admitting only one winner).

In single-item environments a natural "soft max" is given by proportional weights allocations. A weight function is given for each agent $i$ as a strictly monotone and continuously differentiable function $w_{i}: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$and the proportional weights social choice function maps each agent's value
to a weight and then allocates to agents with probabilities proportional to weights. 2 A canonical example of proportional weights is exponential weights: $w_{i}\left(v_{i}\right)=e^{v_{i}}$ for each agent $i$.

Given the assumptions on functions $\boldsymbol{w}$, they are invertible. Where appropriate we will overload $v_{i}$ to allow it to be the functional inverse of $w_{i}$ mapping a weight back to its value. We also overload the notations $\boldsymbol{x}, \boldsymbol{\pi}$ to take weights $\boldsymbol{w}$ as an input, with $\boldsymbol{x}(\boldsymbol{w}):=\boldsymbol{x}(\boldsymbol{v}(\boldsymbol{w}))$ and $\boldsymbol{\pi}(\boldsymbol{w}):=\boldsymbol{\pi}(\boldsymbol{v}(\boldsymbol{w}))$.

## 3 Identification and Non-identification

This section considers sufficient conditions under which values can be inferred from the observed prices $\boldsymbol{\rho}$ of a DSIC mechanism ( $\boldsymbol{x}, \boldsymbol{p}$ ). The critical challenge to identification arises from the observation that values can only possibly be identified from prices if the price function $\boldsymbol{\pi}$ is invertible. We solve this challenge both in theory here in Section 3, and algorithmically in Section 4.

Our theoretical and algorithmic results are simpler to prove as inversions from prices to intermediate weights, and then from weights to values. Describing the inversion via weights is without loss because weights functions $w_{i}(\cdot)$ are continuously differentiable, positive, strictly increasing functions mapping an agent's value to weight. The weights can be inverted as $v_{i}\left(w_{i}\right):=w_{i}^{-1}\left(v_{i}\right)$.

Our approach is to write the problem of inverting the price function $\boldsymbol{\pi}$ at prices $\boldsymbol{\rho}$ as a proxy game between proxy players where the actions are weights. The proxy game is a tool for computing the inverse: with proxy actions corresponding to weights, its unique Nash will be the desired inversion point. Each proxy player represents an agent of the mechanism ( $\boldsymbol{x}, \boldsymbol{p}$ ). A proxy player $i$ is responsible for identifying its agent's weight $w_{i}$, in the proxy game parameterized by $\rho$.

Towards designing the proxy game to have a specific (and unique) Nash equilibrium, we design the proxy game's payoff function $\Phi_{i}^{\rho}$ of a proxy player $i$ for (action) $\tilde{w}_{i}-$ given the profile of weightactions from the other proxy players $\tilde{\boldsymbol{w}}_{-i}-$ to be optimized where $\pi_{i}(\tilde{\boldsymbol{w}})$ on the proxy action profile is closest to the observed price $\rho_{i} \cdot \frac{3}{3}$ The first goal here is to give the technical description a pricefunction inversion algorithm using a proxy game, and reduce the question of its correctness to the uniqueness of a pure Nash equilibrium in the proxy game (Proposition (1).

Recalling equation (2), we transform the price function $\pi$ to weights-space using calculus-change-of-variables as

$$
\begin{equation*}
\pi_{i}(\boldsymbol{w})=v_{i}\left(w_{i}\right)-\frac{\int_{w_{i}(0)}^{w_{i}} x_{i}\left(z, \boldsymbol{w}_{-i}\right) v_{i}^{\prime}(z) d z}{x_{i}(\boldsymbol{w})} . \tag{3}
\end{equation*}
$$

For fixed observed prices $\boldsymbol{\rho}$, define the price-imbalance function $\phi_{i}^{\boldsymbol{\rho}}(\cdot)$ and the cumulative priceimbalance $\Phi_{i}^{\rho}(\cdot)$ respectively as follows, and we set $\Phi^{\rho}$ as the proxy game utility function:

$$
\begin{align*}
& \phi_{i}^{\boldsymbol{\rho}}\left(\tilde{w}_{i}, \tilde{\boldsymbol{w}}_{-i}\right)=\rho_{i}-\pi_{i}\left(\tilde{w}_{i}, \tilde{\boldsymbol{w}}_{-i}\right)=\rho_{i}-v_{i}\left(\tilde{w}_{i}\right)+\frac{\int_{w_{i}(0)}^{\tilde{w}_{i}} x_{i}\left(z, \tilde{\boldsymbol{w}}_{-i}\right) v_{i}^{\prime}(z) \mathrm{d} z}{x_{i}(\tilde{\boldsymbol{w}})},  \tag{4}\\
& \Phi_{i}^{\boldsymbol{\rho}}\left(\tilde{w}_{i}, \tilde{\boldsymbol{w}}_{-i}\right)=\int_{w_{i}(0)}^{\tilde{w}_{i}} \phi_{i}^{\boldsymbol{\rho}}\left(z, \tilde{\boldsymbol{w}}_{-i}\right) \mathrm{d} z . \tag{5}
\end{align*}
$$

The proxy game is defined with weights $\tilde{\boldsymbol{w}}$ as proxy actions, and with utilities for the proxy agents given by the cumulative price-imbalance functions $\boldsymbol{\Phi}^{\rho}$. Each function $\Phi_{i}^{\rho}$ is strictly concave in

[^2]dimension $i$, except at the lower end point of its domain where it is weakly concave (see Lemma 10 in Appendix (A.2). From concavity of $\Phi_{i}^{\rho}$ in (5), a "zero" of $\phi_{i}^{\rho}$ in (4) is optimal. As desired, when other players select proxy weights $\tilde{\boldsymbol{w}}_{-i}$, proxy player $i$ would select proxy weight $\tilde{w}_{i}$ so that agent $i$ 's price according to $\boldsymbol{\pi}$ on $\tilde{\boldsymbol{w}}$ is closest to agent $i$ 's observed payment $\rho_{i}$ (and $\rho_{i}=\pi_{i}(\tilde{\boldsymbol{w}})$ if possible). Based on this proxy game, we define the following inference algorithm.
Definition 1. The price-inversion algorithm $\mathcal{A}$ on price space $[0, \infty)^{n}$ for social choice function $\boldsymbol{x}$ on value space $[0, h]^{n}$ is

1. Observe price profile $\boldsymbol{\rho}$.
2. Select a Nash equilibrium $\tilde{\boldsymbol{w}}$ in the proxy game defined in weight space with utility functions given by the cumulative price-imbalance $\boldsymbol{\Phi}^{\boldsymbol{\rho}}$ for $\boldsymbol{\rho}$.
3. Return inferred values based on inferred weights $\tilde{\boldsymbol{w}}$ as $\left(v_{1}\left(\tilde{w}_{1}\right), \ldots, v_{n}\left(\tilde{w}_{n}\right)\right)$.

A key property for the proper working of the price-inversion algorithm is whether the proxy game admits a unique pure Nash equilibrium. For example, if there are multiple distinct valuation profiles that map to the same prices via $\boldsymbol{\pi}$ (values of agents in the original auction), then each of these valuation profiles will have a corresponding equilibrium in the proxy game (in proxy game action-weights space). Proposition $\square$ formalizes the correctness of the price-inversion algorithm, subject to the proxy game having unique pure Nash equilibrium.
Proposition 1. Any weights profile $\boldsymbol{w} \in\left[w_{i}(0), w_{i}(h)\right]^{n}$ such that observed price profile $\boldsymbol{\rho}$ satisfies $\boldsymbol{\rho}=\boldsymbol{\pi}(\boldsymbol{w})$ is a Nash equilibrium of the proxy game on the social choice function $\boldsymbol{x}$ and prices $\boldsymbol{\rho}$; if this Nash equilibrium $\boldsymbol{w}$ of the proxy game is unique then the inverse $\boldsymbol{\pi}^{-1}(\boldsymbol{\rho})$ is unique and given by the price inversion algorithm $\mathcal{A}$.

Proof. The second part follows from the first part. For the first part, assume $\boldsymbol{\rho}=\boldsymbol{\pi}(\boldsymbol{w})$ for some $\boldsymbol{w}$ in weights space domain. Action profile $\boldsymbol{w}$ in the proxy game is a Nash equilibrium as follows. Each proxy agent's first-order condition is satisfied. Specifically, with utilities given by the cumulative imbalances $\boldsymbol{\Phi}^{\rho}$, the first-order condition is given by $\phi_{i}^{\boldsymbol{\rho}}\left(w_{i}, \boldsymbol{w}_{-i}\right)=\rho_{i}-\pi_{i}\left(w_{i}, \boldsymbol{w}_{-i}\right)$ and is zero by the choice of $\boldsymbol{w}$. Further, checking first-order conditions is sufficient because $\boldsymbol{\Phi}^{\boldsymbol{\rho}}$ is strictly concave by Lemma 10, i.e., $\frac{\partial \phi_{i}^{\rho}\left(w_{i}, \boldsymbol{w}_{-i}\right)}{\partial w_{i}}=-\pi_{i}^{\prime}(\boldsymbol{w})<0$ (except at the lower bound where the partial is 0 , but this can not affect player $i$ 's strict preference over actions).

Motivated by Proposition [1, the remainder of this section identifies proportional weights as a large natural class of social choice functions for which the proxy game has a unique pure Nash equilibrium for all price profiles, which we will state in Theorem 5. The computational question of finding the Nash equilibrium of the proxy game is deferred to Section 4.

We outline the rest of the section. As mentioned previously, a necessary condition for the uniqueness of pure Nash in the proxy game is that the price function $\boldsymbol{\pi}$ is one-to-one. In Section 3.1, we show that $\boldsymbol{\pi}$ being one-to-one is implied by a slightly weaker condition than the following: for all inputs the Jacobian of $\boldsymbol{\pi}$ - denoted $J_{\boldsymbol{\pi}}$ - has all positive principal minors (i.e. it is a $P$-matrix, see Definition 2below). In Section 3.2 we show that all proportional weights social choice functions (for single-unit environments) induce price functions that satisfy this condition. In contrast, Section 3.3 describes a natural variant of proportional weights social choice functions for environments which resemble single-minded combinatorial auctions, and shows that the price functions for these social choice functions are not generally invertible, and therefore the proxy game does not have a unique pure Nash equilibrium in this extended setting.

### 3.1 Sufficiency of "Interior P-Matrix Functions"

This section shows that a sufficient condition for the uniqueness of a pure Nash equilibrium in the proxy game defined in algorithm $\mathcal{A}$ (Definition (1) - necessary for its correctness - is that the price function $\boldsymbol{\pi}$ (for the social choice rule $\boldsymbol{x}$ ) is an "interior $P$-matrix function," a property on its Jacobian $J_{\boldsymbol{\pi}}$ (to be defined shortly in Definition (3). An intuitive outline of the technique is:

- existence is by algorithm design, as the vector of true weights exists as a pure Nash point, in particular one with all first-order conditions equal to 0 ;
- uniqueness results because the mapping between proxy game action vectors and proxy agent utility gradients is a bijection with "high-dimensional monotonicity," for which interior $P$ matrix functions are sufficient; so the proxy game has "high-dimensional concavity."

We will address existence in Theorem 4 and its proof. First we set up the structure towards uniqueness (also Theorem (4). The next definition for $P$-matrix ("positive matrix") comes from Gale and Nikaido (1965), and so does Theorem 2 (below) connecting $P$-matrices to bijection and invertibility $4^{4}$ We give their definition and extend it to include "weak" and "negative" cases, and list facts about $P$-matrices to be used in this and subsequent sections:

Definition 2. $A K \times K$ matrix is a $P$-matrix if all of its principal minors are positive (i.e., have strictly positive determinant). Such a matrix is a $P_{0}$-matrix if all of its principal minors are non-negative. Further, the terms $N$-matrix and $N_{0}$-matrix are used to describe matrices that when negated (all entries multiplied by -1) are, respectively, a $P$-matrix and a $P_{0}$-matrix.

Fact 1. The following are true about P-matrices:

1. a P-matrix is downward-closed, i.e., each of its principal minors is a $P$-matrix too;
2. the class of $P$-matrices contains the class of all positive definite matrices as a special case (where for our purposes, the definition of a positive definite matrix $M$ is $\boldsymbol{z}^{\top} M \boldsymbol{z}>0 \forall \boldsymbol{z} \neq \mathbf{0}$ with $M$ not necessarily symmetric);
3. the product of a strictly positive, diagonal matrix and a $P$-matrix is also a $P$-matrix.

Theorem 2 (Gale and Nikaido, 1965). A continuously differentiable function $f: \Omega \rightarrow \mathbb{R}^{n}$ with compact and convex product domain $\Omega \subset \mathbb{R}^{n}$ is one-to-one if its Jacobian is everywhere a $P$-matrix.

We will need a generalization of Theorem 2 that relaxes the strict $P$-matrix condition on the Jacobian, on the axis-aligned boundaries. The problem for our price-function setting is that the pseudogradient of the utility function is only a $P_{0}$-matrix on the lower boundaries (from equation (6) below).

Define a function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ to be a $P$-matrix function if its Jacobian is a $P$-matrix at all points of the function's domain. We need to extend this definition. Note, Definition 3 for interior $P$-matrix functions depends on Definition 4 for identified boundaries (next).

Definition 3. $5^{5}$ For product space $\Omega=\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right] \times \cdots \times\left[a_{n}, b_{n}\right]$ and function $f: \Omega \rightarrow \mathbb{R}^{n}$, a function $f: \Omega \rightarrow \mathbb{R}^{n}$ is an interior $P$-matrix function (respectively interior $N$-matrix function) if for every point $\boldsymbol{\omega} \in \Omega$ :

[^3]- the Jacobian of $f$ evaluated at $\boldsymbol{\omega}$ as $J_{f}(\boldsymbol{\omega})$ is a $P_{0}$-matrix (respectively $N_{0}$-matrix);
- and choosing the minor of $J_{f}(\boldsymbol{\omega})$ that removes row/column pairs corresponding to the dimensions in which $\boldsymbol{\omega}$ is in identified boundaries of $\Omega$, this principal minor of $J_{f}(\boldsymbol{\omega})$ is strictly a $P$-matrix (respectively $N$-matrix).
Before giving the definition of identified boundaries, we give their intuition and justification. They describe conditions which address the problem of Jacobians having determinant 0 at the boundaries. By Theorem2, Jacobian as $P$-matrix everywhere is sufficient for inversion. An identified boundary (in input space dimension $i$ ) must first allow unilateral inversion of its coordinate, by mapping to a unique, constant output in dimension $i$ for all inputs in this identified boundary of $i$ (i.e., bijectively); and second, after fixing the input in all such identified-boundary dimensions $i$ as parameters, the reduced function in the remaining dimensions must have Jacobian as a strict $P$-matrix, which will imply that it can be inverted; hence entire output vectors can be inverted.
Definition 4. For compact and convex product space $\Omega \subset \mathbb{R}^{n}$, and function $f: \Omega \rightarrow \mathbb{R}^{n}$, a boundary (described by $c_{i} \in\left\{a_{i}, b_{i}\right\}$ ) is identified if both of the following hold for all $\boldsymbol{\omega}_{-i} \in \Omega_{-i}$ :
- fixing $\boldsymbol{\omega}_{i}=c_{i}$, function $f_{i}\left(c_{i}, \boldsymbol{\omega}_{-i}\right)$ is constant for all $\boldsymbol{\omega}_{-i}$; or equivalently, all cross-partials on the $c_{i}$ boundary are 0 : $\partial f_{i} / \partial \omega_{j}\left(c_{i}, \boldsymbol{\omega}_{-i}\right)=0$ for all $j \neq i$;
- the output is unique to the boundary: $f_{i}\left(c_{i}, \boldsymbol{\omega}_{-i}\right) \neq f_{i}\left(d_{i}, \boldsymbol{\omega}_{-i}\right)$ for all $d_{i} \in\left[a_{i}, b_{i}\right], d_{i} \neq c_{i}$.

As previously suggested, the implication of an identified boundary is that, (e.g.) the low point of the domain in dimension $i$ maps identically to the low point of the function's range in dimension $i$ as a unilateral bijection. Further, note that a sufficient condition for the second point of the definition is having partial $\partial f_{i} / \partial \omega_{i}>0$ for all inputs $\omega_{i}$ off the boundary.

Our Theorem 3 generalizes Theorem 2 of Gale and Nikaido. We use it as an interim result towards our more pertinent result in Theorem [4, which connects interior $P$-matrix functions to proxy games. Theorem $\mathbb{4}$ lets us reduce the correctness of price-inversion algorithm $\mathcal{A}$ to the condition that $\boldsymbol{\pi}$ is an interior $P$-matrix function, stated formally in Corollary [1. Proofs for the next two theorems are given in Appendix A.1.
Theorem 3. If function $f: \Omega \rightarrow \mathbb{R}^{n}$ on compact and convex product domain $\Omega \subset \mathbb{R}^{n}$ is an interior $P$-matrix function (Definition (3), then it is one-to-one, and therefore invertible on its image.
Theorem 4. A game with $n$ players and

- a compact and convex product action space $\Omega_{1} \times \ldots \times \Omega_{n}=\Omega \subset \mathbb{R}^{n}$;
- a continuous and twice-differentiable utility function $\boldsymbol{U}: \Omega \rightarrow \mathbb{R}^{n}$ such that:
- the pseudogradient $\left[\frac{\partial U_{i}}{\partial \omega_{i}}\right]_{i}$ of the utility function $\boldsymbol{U}$ is an interior $N$-matrix function;
- and there exists $\boldsymbol{\omega}^{0} \in \Omega$ such that the pseudogradient evaluated at $\boldsymbol{\omega}^{0}$ is $\mathbf{0}$ (the 0-vector);
has a unique Nash equilibrium, which is $\boldsymbol{\omega}^{0}$, and this equilibrium is pure.
Corollary 1. Given agents with (unknown) values $\boldsymbol{v} \in[0, h]^{n}$. Consider price function $\boldsymbol{\pi}$ resulting from a dominant-strategy incentive-compatible mechanism implementing $\boldsymbol{x}$, with Jacobian $J_{\boldsymbol{\pi}}$.

If $\boldsymbol{\pi}$ is an interior $P$-matrix function, then on observed prices from restricted domain $\boldsymbol{\rho} \in$ $\operatorname{Image}(\boldsymbol{\pi})$, the price-inversion algorithm $\mathcal{A}$ (Definition 1) infers successively the true weights $\boldsymbol{w}$ and the true values $\boldsymbol{v}$ from the mechanism's outcome (as summarized by the prices $\boldsymbol{\rho}=\boldsymbol{\pi}(\boldsymbol{v})$ ).

Proof. We show that under the given assumptions, the proxy game meets the conditions of Theorem 4. The action space of the proxy game is equal to the agents' weights space which is a compact and convex product space. The proxy game has payoffs given by $\boldsymbol{\Phi}^{\boldsymbol{\rho}}$ such that utility functions are continuous and twice-differentiable.

The pseudogradient of the payoffs is given by $\phi^{\rho}$, and the Jacobian of the pseudogradient is the negation of the matrix $J_{\boldsymbol{\pi}}$. Given $\boldsymbol{\pi}$ as an interior $P$-matrix function, its negation $-\boldsymbol{\pi}$ is an interior $N$-matrix function. The true values $\boldsymbol{w}$ as input-actions to the proxy game will result in evaluation of the pseudogradient as $\boldsymbol{\phi}^{\boldsymbol{\rho}}(\boldsymbol{w})=\mathbf{0}$ by design of the game, so $\boldsymbol{\omega}^{0}=\boldsymbol{w}$ exists.

In conclusion, the proxy game indeed satisfies the conditions of Theorem 4 and admits $\boldsymbol{w}$ as a unique Nash equilibrium which is pure. Defining the inverse function $\boldsymbol{\pi}^{-1}$ to output the unique Nash of the proxy game is sufficient for its output to be unique and correct.

### 3.2 Single Item Proportional Weights Social Choice Functions

The goal of this section is to show that every proportional weights social choice function awarding a single item has a price function $\boldsymbol{\pi}$ meeting the conditions of Corollary (1. We state this now as the main theoretical result of the paper.

Theorem 5. A price function $\boldsymbol{\pi}$ (of equation (2)) - corresponding to a strictly monotone, continuous, differentiable proportional weights social choice rule - is an interior P-matrix function, and it is uniquely invertible.

Proof. We only need to show that $J_{\pi}$ is an interior $P$-matrix function. In Lemma 11 in Appendix A.2, we show that under $\boldsymbol{\pi}$, the lower boundaries of the weights space domain are identified boundaries. Lemma 2 (next) shows that when an input is in the lower boundary for any dimension $i, J_{\pi}$ has all-zero elements in row $i$, such that its determinant is trivially 0 , meeting the (weakened) identified-boundary condition of a $P_{0}$-matrix. Otherwise at all points of the weights space domain, Theorem 7 (at the end of this section) shows that the critical minor of $J_{\boldsymbol{\pi}}$-i.e., the minor which removes row/ column indexes corresponding to the dimensions in which its input exists in identified (lower) boundaries - is strictly a $P$-matrix.

The rest of this section builds towards Theorem[7. We start with the straightforward calculation of the partial derivatives of $\boldsymbol{\pi}$, which in particular give the entries of the Jacobian $J_{\boldsymbol{\pi}}$. The steps of the calculations and the proof of Lemma 2 are given in Appendix A. 2 ,

$$
\begin{align*}
& \frac{\partial \pi_{i}}{\partial w_{i}}(\boldsymbol{w})=\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\left(\sum_{k} w_{k}\right)-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{w_{i}}-1\right] d z  \tag{6}\\
& \frac{\partial \pi_{i}}{\partial w_{j}}(\boldsymbol{w})=\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\left(\sum_{k} w_{k}\right)-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{\left(\sum_{k} w_{k}\right)-w_{i}+z}-1\right] d z \tag{7}
\end{align*}
$$

Lemma 2. Given the price function $\boldsymbol{\pi}$ for proportional weights, for $j, k \neq i$, the cross derivatives are the same: $\frac{\partial \pi_{i}}{\partial w_{j}}=\frac{\partial \pi_{i}}{\partial w_{k}}$. Evaluating the Jacobian at $\boldsymbol{w}$, further, all elements of the Jacobian matrix $J_{\boldsymbol{\pi}}$ are positive, i.e., $\frac{\partial \pi_{i}}{\partial w_{i}}>0, \frac{\partial \pi_{i}}{\partial w_{j}}>0$, except at the $w_{i}(0)$ lower boundary in dimension $i$ where the elements of row $i$ are $\frac{\partial \pi_{i}}{\partial w_{i}}=\frac{\partial \pi_{i}}{\partial w_{j}}=0$.

We need to prove that $\boldsymbol{\pi}$ is an interior $P$-matrix function. Consider weights input $\boldsymbol{w}$. Let $K$ be the count of dimensions $i$ such that coordinate $w_{i}$ is "off" the lower identified boundary in
dimensioni，i．e．，$w_{i}>w_{i}(0)$ ．Without loss of generality we can assume the dimensions of identified boundaries have the largest indexes（if any）．

We critically consider only the principal minor of $J_{\pi}$ which results from keeping the first $K$ interior dimensions，as is sufficient to check an interior $P$－matrix function．We explicitly define the ratio of an agent＇s＂self－partial＂to its＂cross－partial＂for any $j \neq i$ by $h_{i}$ ，which will be needed for analysis throughout the rest of the paper 6

$$
\begin{equation*}
h_{i}=\frac{\partial \pi_{i}}{\partial w_{i}} / \frac{\partial \pi_{i}}{\partial w_{j}} \tag{8}
\end{equation*}
$$

The derivatives that appear are positive（Lemma（2）．We write the principal minor＇s Jacobian as

$$
J_{\pi, K}=D \cdot H=\left[\begin{array}{ccccc}
\partial \pi_{1} / \partial w_{2} & 0 & 0 & \ldots & 0  \tag{9}\\
0 & \partial \pi_{2} / \partial w_{1} & 0 & \ldots & 0 \\
0 & 0 & \partial \pi_{3} / \partial w_{1} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \partial \pi_{K} / \partial w_{1}
\end{array}\right] \cdot\left[\begin{array}{ccccc}
h_{1} & 1 & 1 & \ldots & 1 \\
1 & h_{2} & 1 & \ldots & 1 \\
1 & 1 & h_{3} & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \ldots & h_{K}
\end{array}\right]
$$

Multiplying by a positive diagonal matrix $D$ is a benign operation with respect to the deter－ mination of a matrix as a $P$－matrix（Fact $⿴ 囗 十 ⺝ 丶(3)$ ）．We will define $H$ to be the rightmost matrix of equation（9）which is composed of $h_{i}$ elements in the diagonal and all ones elsewhere．By reduction， we need only show that $H$ is a $P$－matrix，for which it is sufficient to show $H$ is positive definite （Fact［（2））．

We claim the following results，starting with a complete characterization of when an arbitrary matrix $G$（with structure of $H$ ）is positive definite，a result which could be of independent interest．

Theorem 6．Consider a $K \times K$ matrix $G$ with diagonal $g_{1}, g_{2}, \ldots, g_{K}$ and all other entries equal to 1 （and without loss of generality $g_{1} \leq g_{2} \leq \ldots \leq g_{K}$ ）．The following is a complete characterization describing when $G$ is positive definite．

1．if $g_{1} \leq 0$ ，then the matrix $G$ is not positive definite；
2．if $g_{1} \geq 1$ and $g_{2}>1$ ，then $G$ is positive definite；
3．if $0<g_{1}, g_{2} \leq 1$ ，then $G$ is not positive definite；
4．if $0<g_{1}<1$ and $g_{2}>1$ ，then $G$ is positive definite if and only if $\sum_{k} \frac{1}{1-g_{k}}>1$ ．
The proof of Theorem 6 is given in Appendix A．3 where the main difficulty is part（4）．Theo－ rem 6 is for arbitrary $G$ ．We now return to the specific consideration of $H$ resulting from $\pi$ and $J_{\boldsymbol{\pi}, K}$ ，showing in Theorem 7 that it must be covered by cases（2）or（4）from Theorem 6，The proofs of Lemma 3 and Lemma 4 are given in Appendix A．4．

Lemma 3．If $h_{i} \leq 1$ ，then $w_{i}>0.5 \sum_{k} w_{k}$ ，and all other weights must have $w_{j}<0.5 \sum_{k} w_{k}$（and $h_{j}>1$ ）．

Lemma 4．When $h_{1}<1$ and $h_{j}>1 \forall j \neq 1$ ，we have $\sum_{k} \frac{1}{1-h_{k}}>1$ ．

[^4]Theorem 7. Let matrix $J_{\boldsymbol{\pi}}$ be the Jacobian of $\boldsymbol{\pi}$ at weights $\boldsymbol{w}$ of a positive, strictly increasing, and differentiable proportional weights social choice functions. $\boldsymbol{\pi}$ is an interior $P$-matrix function.

Proof. By the definition of an interior $P$-matrix function (Definition (3), we consider the restriction to the minor $J_{\pi, K}=D \cdot H$ at $\left(w_{1}, \ldots, w_{K}\right)$, where coordinates in identified (lower) boundaries of weights space have been discarded (see equation (9)). Because weights $\left(w_{1}, \ldots, w_{K}\right)$ are definitively off their respective lower boundaries, Lemma 2 implies that all $h_{i} \in\left\{h_{1}, \ldots, h_{K}\right\}$ are strictly positive. By Lemma 3, at most one agent $i$ has $h_{i} \leq 1$. Without loss of generality, we can set this $i=1$. So there are just two cases:

1. $h_{1} \geq 1$ and $h_{j}>1 \forall j \neq 1$, and
2. $0<h_{1}<1$ and $h_{j}>1 \forall j \neq 1$.

These are respectively cases (2) and (4) of Theorem 6. To satisfy the condition within case (4) of Theorem 6, Lemma 4 is sufficient. Thus, the factor $H$ of the Jacobian minor $J_{\pi, K}$ is positive definite. Finally, using Fact 1 , $H$ is a $P$-matrix and the product $J_{\pi, K}=D \cdot H$ is also a $P$-matrix.

### 3.3 Impossibility Results for Complementarities

In this section we show that for a natural generalization of the proportional weights social choice function to an environment with complementarities between agents, the values of the agents cannot necessarily be identified from the prices output by the mechanism.

The impossibility result we present will consider a generalization of exponential weights to environments with complementarities. We will consider the special case where the agents are partitioned and the mechanism can allocate to all agents in any one part, but agents from multiple parts may not be simultaneously allocated. We prove that a natural extension of exponential weights to partition set systems results in a price function $\boldsymbol{\pi}$ that is not one-to-one, by counterexample. Thus, the price function is generally not invertible: no algorithm can distinguish between two (or more) valuation profiles which give the same prices.

Definition 5. The exponential weights social choice function for an n-agent partition set system with parts $\mathcal{S}=\left(S_{1}, \ldots, S_{r}\right)$ is given by:

- $v_{S}=\sum_{i \in S} v_{i}$ for $S \in \mathcal{S}$;
- $x_{S}(\boldsymbol{v})=\frac{e^{v_{S}}}{\sum_{T \in \mathcal{S}} e^{v_{T}}}$ for $S \in \mathcal{S}$;
- $x_{i}(\boldsymbol{v})=x_{S}(\boldsymbol{v})$ for $i \in S$

The resulting price function corresponding to the exponential weights social choice function for partition set systems is

$$
\begin{aligned}
\pi_{i}(\boldsymbol{v}) & =v_{i}-\frac{\int_{0}^{v_{i}} x_{i}\left(z, \boldsymbol{v}_{-i}\right) d z}{x_{i}(\boldsymbol{v})} \\
& =v_{i}-\frac{\sum_{T} e^{v_{T}}}{e^{v_{S}}} \int_{0}^{v_{i}} \frac{e^{z} e^{v_{S \backslash\{i\}}}}{e^{z} e^{v_{S \backslash\{i\}}}+\sum_{T \neq S} e^{v_{T}}} d z \\
& =v_{i}-\frac{\sum_{T} e^{v_{T}}}{e^{v_{S}}}\left(\ln \left(\sum_{T} e^{v_{T}}\right)-\ln \left(e^{v_{S \backslash\{i\}}}+\sum_{T \neq S} e^{v_{T}}\right)\right)
\end{aligned}
$$

The completion of the counterexample is in the following lemma.


Figure 1: Graphing the function $\left[\pi_{1}(\alpha)-\pi_{5}(\alpha)\right]$ from the proof of Lemma 5. The zeroes of the function parameterize values for agents in $S_{1}$ and $S_{2}$ such that all agents across both parts have identical prices, despite the agents of each group having strictly distinct values from each other. (Note, by design, the curve is rotationally symmetric around the point $(5,0)$.)

Lemma 5. The price function $\boldsymbol{\pi}$ corresponding to the exponential weights social choice function for partition set systems (with at least one partition containing two or more agents) is not one-to-one.

Proof. We prove that the price function is not one-to-one (and consequentially by the contrapositive of Corollary $\mathbb{1}$ its Jacobian is not positive definite). We first set up a parameterized analysis and then choose the parameters later.

Let there be $k$ agents in set $S_{1}$ who all have the same valuation $\alpha / k$, and another $k$ agents in set $S_{2}$ who all have the same valuation $(\beta-\alpha) / k$. Note $\beta=v_{S_{1}}+v_{S_{2}}$. Players in all other sets $S_{r}$ for $r>2$ have a constant value $v_{\text {others }}$ and can be summarized by a single parameter $\delta$ by letting $e^{\delta}=\sum_{r>2} e^{v_{S_{r}}}$. Parameters $k, \alpha, \beta$ and $v_{\text {others }}$ will be determined later.

The price for agent 1 in part $S_{1}$ is

$$
\pi_{1}=\frac{\alpha}{k}-\frac{e^{\alpha}+e^{\beta-\alpha}+e^{\delta}}{e^{\alpha}}\left[\ln \left(e^{\alpha}+e^{\beta-\alpha}+e^{\delta}\right)-\ln \left(e^{(1-1 / k) \alpha}+e^{\beta-\alpha}+e^{\delta}\right)\right]
$$

The price for agent $k+1$ in part $S_{2}$ is

$$
\pi_{k+1}=\frac{\beta-\alpha}{k}-\frac{e^{\alpha}+e^{\beta-\alpha}+e^{\delta}}{e^{\beta-\alpha}}\left[\ln \left(e^{\alpha}+e^{\beta-\alpha}+e^{\delta}\right)-\ln \left(e^{(1-1 / k)(\beta-\alpha)}+e^{\alpha}+e^{\delta}\right)\right]
$$

We now show that it is possible that player 1 and player $k+1$ have different valuations but are charged the same prices. Consider the case $k=4, \beta=10$, and there is one additional part $S_{3}$ with a single agent 9 with $v_{9}=v_{\text {others }}=4$ inducing $\delta=4$. Then we can consider the quantity ( $\pi_{1}-\pi_{5}$ ) as a function of parameter $\alpha$ with $v_{1}=\ldots=v_{4}=\alpha / 4$ and $v_{5}=\ldots=v_{8}=(10-\alpha) / 4$.

This function $\left.\left.\left[\pi_{1}\right) \alpha\right)-\pi_{5}(\alpha)\right]$ is graphed in Figure [1, where we can see that there are three solutions for $\pi_{1}=\pi_{5}$. Without showing the explicit calculation, $\pi_{1}=\pi_{5}$ holds for a value profile where $v_{1}=\ldots=v_{4}=\alpha / 4 \approx 0.375$ and $v_{5}=\ldots=v_{8}=(\beta-\alpha) / 4 \approx 2.125$, and $v_{9}=4$. In this
case, the seller cannot distinguish between $S_{1}$ and $S_{2}$ which part has agents with identical values $\approx 0.375$ versus the other part whose agents all have values $\approx 2.125$.

This lemma can be generalized as follows. A set system is downward-closed if all subsets of feasible sets are feasible. Agents are substitutes if the set system satisfies the matroid augmentation property, i.e., for any pair of feasible sets with distinct cardinalities, there exists an element from the larger set that is not in the smaller set that can be added to the smaller set and the resulting set remains feasible. A set system exhibits complementarities if agents are substitutes (i.e., there exist sets that fail the augmentation property). Exponential weights can be generalized to any set system by choosing a maximal set with probability proportional to its exponentiated weight. The impossibility result above can then be easily generalized to any set system that exhibits complementarities by identifying the sets and taking $S_{1}$ and $S_{2}$ to be the agents uniquely in each set (i.e., not in their intersection), and setting all other agent values to zero.

## 4 Computational Methods for Inverting the Price Function

In Section 3 we gave the price-inversion algorithm (Definition (1), which is a well-defined, continuous function that inverts the payment identity $\boldsymbol{\pi}$ to map prices $\boldsymbol{\rho}$ back to values $\boldsymbol{v}$ (Theorem 5). The algorithm is straightforward except for Step 2 which requires the computation of a Nash equilibrium in the defined proxy game. In this section we give a simple algorithm for identifying an equilibrium of the proxy game and thus show that the inverse function can be efficiently computed.

The algorithm for solving the proxy game is enabled by two observations. First, for player $i$, the sum of weights $s=\sum_{k} w_{k}$ summarizes everything that needs to be known about the other players and this observation leads to a many-to-one reduction in the dimension of search space. Consequently, the price function can be rewritten as a function $\bar{\pi}_{i}\left(s, w_{i}\right) .7$ Second, because the price function $\boldsymbol{\pi}$ is invertible, the sum $s$ is uniquely determined from the prices.

Obviously at most one agent can have strictly more than half the total weight $s$. For the rest of this section, without loss of generality we fix agent $i^{*}$ to mean that $w_{i^{*}}$ is not restricted and $w_{i} \leq s / 2$ for all $i \neq i^{*}$.

Fix observed input prices $\boldsymbol{\rho}$. For any agent $i \neq i^{*}$, consider the set of points $\left(s, w_{i}\right)$ for which $\bar{\pi}_{i}$ outputs $\rho_{i}$. Our first key Lemma 6 (below) will show that, restricting to the space $w_{i} \leq s / 2$, this set of points can be interpreted as a real-valued, monotone decreasing function of $s$, denoted $w_{i}^{\rho}(\cdot)$. With this property holding for all agents other than $i^{*}$, we can express the price function for agent $i^{*}$ with dependence on prices $\boldsymbol{\rho}_{-i^{*}}$ and sum $s 8^{8}$

$$
\begin{equation*}
\pi_{i^{*}}^{\boldsymbol{\rho}}(s):=\pi_{i^{*}}\left(\max \left\{s-\sum_{i \neq i^{*}} w_{i}^{\boldsymbol{\rho}}(s), w_{i^{*}}(0)\right\}, \boldsymbol{w}_{-i^{*}}^{\boldsymbol{\rho}}(s)\right) \tag{10}
\end{equation*}
$$

where, for guess of total weight $s$, the quantity $s-\sum_{i \neq i^{*}} w_{i}^{\boldsymbol{\rho}}(s)$ assigns an intermediate guess of $w_{i^{*}}$ as the "balance" of the quantity $s$ having subtracted the implied weights of the "small" agents for guess $s \frac{9}{9}$ Our second key Lemma 7 (below) will show that, on the range of $s$ for which it is

[^5]well-defined, the function $\pi_{i^{*}}^{\rho}$ is strictly monotonically increasing.
This setup suggests a natural binary search procedure. For some agent $i^{*}$ and small initial guess of $s$, the implied price for $i^{*}$ is smaller than the observed input, i.e. $\pi_{i^{*}}^{\rho}(s)<\rho_{i^{*}}$. A large guess of $s$ implies too big of a price for $i^{*}$, and monotonicity will then guarantee a crossing. The algorithm has the following steps:

1. Find an agent $i^{*}$ by iteratively running the following for each fixed assignment of agent $i \in\{1, \ldots, n\}$ :
(a) temporarily set $i^{*}=i$;
(b) determine the range of $s$ on which $\pi_{i^{*}}^{\rho}$ is well-defined and searching is appropriate 10
(c) if this range of $s$ is non-empty, permanently fix $i^{*}=i$ and break the for-loop;
2. use the monotonicity of $\pi_{i^{*}}^{\boldsymbol{\rho}}$ to binary search on $s$ for the true $s^{*}$, converging $\pi_{i^{*}}^{\boldsymbol{\rho}}(s)$ to $\rho_{i^{*}}$;
3. when the binary search has been run to satisfactory precision and reached a final estimate $\tilde{s}$, output weights $\tilde{\boldsymbol{w}}=\left(\tilde{s}-\sum_{i \neq i^{*}} w_{i}^{\boldsymbol{\rho}}(\tilde{s}), \boldsymbol{w}_{-i^{*}}^{\boldsymbol{\rho}}(\tilde{s})\right)$ which invert to values $\tilde{\boldsymbol{v}}$ via respective $v_{i}(\cdot)$ functions.

The rest of this section formalizes our key results.

### 4.1 Computation through Total Sum Weights

The following theorem claims correctness of the algorithm, and is the main result of this section.
Theorem 8. Given weights $\boldsymbol{w}$ and payments $\boldsymbol{\rho}=\boldsymbol{\pi}(\boldsymbol{w})$ according to a proportional weights social choice function, the algorithm identifies weights $\tilde{\boldsymbol{w}}$ to within $\epsilon$ of the true weights $\boldsymbol{w}$ in time polynomial in the number of agents $n$, the logarithm of the ratio of high to low weights $\max _{i} \ln \left(w_{i}(h) / w_{i}(0)\right)$, and the logarithm of the desired precision $\ln 1 / \epsilon$.

A major object of interest for this sequence of results is the price level set defined by $\mathcal{Q}_{i}^{\rho}=$ $\left\{\left(s, w_{i}\right) \mid \bar{\pi}_{i}\left(s, w_{i}\right)=\rho_{i}\right\}$, i.e., all of the $\left(s, w_{i}\right)$ pairs mapping to the price $\rho_{i}$ under $\bar{\pi}_{i}$, and also in particular its subset $\mathcal{P}_{i}^{\rho}=\left\{\left(s, w_{i}\right) \mid \bar{\pi}_{i}\left(s, w_{i}\right)=\rho_{i}\right.$ and $\left.w_{i} \leq s / 2\right\} \subseteq \mathcal{Q}_{i}^{\rho}$ which restricts the set to the region where $w_{i}$ is at most half the total weight $s$. Define $r_{i}^{\boldsymbol{\rho}}=\min \left\{s:\left(s, w_{i}\right) \in \mathcal{P}_{i}^{\boldsymbol{\rho}}\right\}$ as the lower bound on the sum $s$ on which the set $\mathcal{P}_{i}^{\rho}$ is supported. These quantities are depicted in Figure 2,

We give the formal statements of the two most critical lemmas supporting Theorem 8 ,
Lemma 6. The price level set $\mathcal{Q}_{i}^{\rho}$ is a curve; further, restricting $\mathcal{Q}_{i}^{\rho}$ to the region $w_{i} \leq s / 2$, the resulting subset $\mathcal{P}_{i}^{\rho}$ can be written as $\left\{\left(s, w_{i}^{\rho}(s)\right): s \in\left[r_{i}^{\rho}, \infty\right)\right\}$ for a real-valued decreasing function $w_{i}^{\rho}$ mapping sum $s$ to a weight $w_{i}$ that is parameterized by the observed price $\rho_{i}$.

Lemma 7. For any agent $i^{*}$ and $s \in\left[\max _{j \neq i^{*}} j_{j}^{\rho}, \infty\right)$, function $\pi_{i^{*}}^{\rho}$ is weakly increasing; specifically, $\pi_{i^{*}}^{\rho}$ is constant when $s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s) \leq w_{i^{*}}(0)$ and strictly increasing otherwise.

[^6]

Figure 2: The price level set curve $\mathcal{Q}_{i}^{\boldsymbol{\rho}}=\left\{\left(s, w_{i}\right): \bar{\pi}_{i}\left(s, w_{i}\right)=\rho_{i}\right\}$ (thick, gray, dashed), is decreasing below the $w_{i}=s / 2$ line (Lemma (6) where it is defined by its subset $\mathcal{P}_{i}^{\rho}$ (thin, black, solid). It is bounded above by the $w_{i}=s$ line (trivially as $s$ sums over all weights) and the $w_{i}=w_{i}(h)$ line (the maximum weight in the support of the values), and below by the $w_{i}=w_{i}(0)$ line which we have assumed to be strictly positive. $r_{i}^{\rho}$ is the minimum weight-sum consistent with observed price $\rho_{i}$ and weights $w_{i} \leq s / 2$.

A key step in the proof of Lemma 7 will depend on Lemma 4. The $\frac{1}{1-h_{k}}$ terms in the statement of Lemma 4 are realized to correspond to derivatives of $w_{k}^{\rho}$ functions. Consequently, the correctness of the algorithm critically relies on the proof of a unique inverse to the price function.

We give the proofs of Theorem 8, Lemma 6, and Lemma 7 in Appendix B.3, Preceding these proofs is supporting material: Appendix B.1 gives a detailed analysis of the structure of the search space, and Appendix B. 2 gives the description of the binary search algorithm with full details.

## References

Chawla, S., Hartline, J., and Nekipelov, D. (2014). Mechanism design for data science. In Proceedings of the fifteenth ACM conference on Economics and computation, pages 711-712. ACM.

Chawla, S., Hartline, J., and Nekipelov, D. (2016). A/b testing of auctions. In Proceedings of the 2016 ACM Conference on Economics and Computation, pages 19-20. ACM.

Gale, D. and Nikaido, H. (1965). The Jacobian matrix and global univalence of mappings. Mathematische Annalen, 159(2):81-93.

Guerre, E., Perrigne, I., and Vuong, Q. (2000). Optimal nonparametric estimation of first-price auctions. Econometrica, 68(3):525-574.

Haile, P. A. and Tamer, E. (2003). Inference with an incomplete model of english auctions. Journal of Political Economy, 111(1):1-51.

Hoy, D., Nekipelov, D., and Syrgkanis, V. (2017). Welfare guarantees from data. In Advances in Neural Information Processing Systems, pages 3768-3777.

Huang, Z. and Kannan, S. (2012). The exponential mechanism for social welfare: Private, truthful, and nearly optimal. In Proceedings of the 2012 IEEE 53rd Annual Symposium on Foundations of Computer Science, FOCS '12, pages 140-149, Washington, DC, USA. IEEE Computer Society.

Johari, R. and Tsitsiklis, J. N. (2004). Efficiency loss in a network resource allocation game. Mathematics of Operations Research, 29(3):407-435.

Myerson, R. B. (1981). Optimal auction design. Mathematics of Operations Research, 6(1):58-73.
Nekipelov, D., Syrgkanis, V., and Tardos, E. (2015). Econometrics for learning agents. In Proceedings of the Sixteenth ACM Conference on Economics and Computation, pages 1-18. ACM.

Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games. Econometrica, 33(3):520-534.

## A Supporting Material for Section 3

## A. 1 Proofs of Theorem 3 and Theorem 4

Before getting to results, we define a dimensionally-reduced function by a parameterized procedure. This procedure will be useful as a sub-routine in multiple proofs.
Definition 6. Given a function $f: \Omega \rightarrow \mathbb{R}^{n}$, two points $\boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2}$ in compact and convex product space $\Omega \subset \mathbb{R}^{n}$, and a set $K$ of dimensions with identified lower boundaries with cardinality $k=|K|$. Define $a$ dimensionally-reduced function $D: \Omega_{-K} \rightarrow \mathbb{R}^{n-k}$ where

- $\Omega_{-K} \subset \Omega$ is the projection of product space $\Omega$ into dimensions not in $K$, and further the lower bounds of each remaining dimension $i$ is (weakly) increased to $\min \left\{\omega_{i}^{1}, \omega_{i}^{2}\right\}$ respectively, and analogously each upper bound decreased to $\max \left\{\omega_{i}^{1}, \omega_{i}^{2}\right\}$;
- $D\left(\boldsymbol{\omega}_{-K}\right)=f\left(\boldsymbol{\omega}_{-K}, \boldsymbol{\omega}_{K}=\mathbf{c}_{K}\right)$ for (vector) $\mathbf{c}_{K}$ the fixed inputs of (removed dimension) identified boundaries, input to $f$ as constant parameters.

We restate and prove Theorem 3 here. Recall it is an extension of Theorem 2 (Gale and Nikaidd, 1965). Its proof depends on Lemma 8 given immediately following.

Theorem 3. If function $f: \Omega \rightarrow \mathbb{R}^{n}$ on compact and convex product domain $\Omega \subset \mathbb{R}^{n}$ is an interior $P$-matrix function (Definition圂), then it is one-to-one, and therefore invertible on its image.
Proof. By contradiction, assume there exist two distinct inputs $\boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2}$ such that $f\left(\boldsymbol{\omega}^{1}\right)=f\left(\boldsymbol{\omega}^{2}\right)$. With equal outputs under $f$ it must be that $\boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2}$ exist in the same set of identified boundaries because by Lemma 8 (given next), an input in each of these identified boundaries outputs a unique constant in its respective dimension. Let the common set of dimensions in identified boundaries be $K$. We consider dimensionally-reduced function $D$ applied to $f, \boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2}$ and set $K$ (of Definition (6)). $D$ now meets all of the conditions of Theorem 2 (in particular $D$ has Jacobian as a $P$-matrix everywhere because no coordinate of $\boldsymbol{\omega}_{-K}^{1}$ or $\boldsymbol{\omega}_{-K}^{2}$ is in the original identified boundaries, and $f$ is an interior $P$-matrix function which must be a strict $P$-matrix function when excluding identified boundaries by definition). Therefore $D$ is one-to-one on its restricted domain, which includes $\boldsymbol{\omega}_{-K}^{1}, \boldsymbol{\omega}_{-K}^{2}$, a result which extends to analysis under $f$ such that $f$ must also be one-to-one. I.e., it must be that $f\left(\boldsymbol{\omega}^{1}\right) \neq f\left(\boldsymbol{\omega}^{2}\right)$ in some coordinate outside the set $K$, giving the contradiction.

Lemma 8. If a function $f: \Omega \rightarrow \mathbb{R}^{n}$ on domain $\Omega \subset \mathbb{R}^{n}$ is an interior P-matrix function (of Definition (3) , then function $f_{i}$ evaluates to constant $f_{i}\left(c_{i}, \cdot\right)$ on an identified boundary in dimension $i$ with coordinate $c_{i}$ if and only if the input $\boldsymbol{\omega}$ to $f_{i}$ has $\omega_{i}=c_{i}$.

Proof. Without loss of generality, assume $c_{i}=a_{i}$ the lower boundary in dimension $i$, with the upper boundary argument by symmetry. For sufficiency, note that by definition of an identified boundary (Definition (4), all cross-partials on the function $f_{i}$ (evaluated at the identified boundary) are identically 0 . I.e., $\partial f_{i} / \partial \omega_{j}\left(a_{i}, \boldsymbol{\omega}_{-i}\right)=0$ for all $j \neq i$ and for all $\boldsymbol{\omega}_{-i}$. Therefore $f_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}\right)$ is a constant.

For necessity, consider an input $\left(d_{i}, \boldsymbol{\omega}_{-i}\right)$ with $d_{i}>a_{i}$ "off" the identified boundary. Evaluated at all inputs $\omega_{i} \in\left(a_{i}, d_{i}\right]$, the self-partial $\partial f_{i} / \partial \omega_{i}>0$ is necessary by the assumption that $f$ is an internal $P$-matrix function, because an implication of its definition is that, when $\omega_{i}$ is not in the lower boundary, the diagonal element of the Jacobian $\partial f_{i} / \partial \omega_{i}$ at index $(i, i)$ must be strictly positive (because diagonal elements are principal minors with dimension $1 \times 1$ ).

Therefore $f_{i}\left(d_{i}, \boldsymbol{\omega}_{-i}\right)=f_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}\right)+\int_{a_{i}}^{d_{i}}\left[\partial f_{i} / \partial \omega_{i}\left(z, \boldsymbol{\omega}_{-i}\right)\right] d z>f_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}\right)$ because the integral of a strictly positive function is strictly positive when $d_{i}>a_{i}$.

Note, an implication of Lemma 8 in the context of our price functions is that we observe agent $i$ to have price $\rho_{i}=0$ if and only if agent $i$ had minimal weight $w_{i}(0)$, trivially implying value $v_{i}=0$.

The rest of this appendix section is devoted to proving Theorem 4. Additionally, we develop the following corollary, which should be of independent interest to the game theory community.

Corollary 2. A game with a compact and convex product action space and pseudogradient that is an $N$-matrix function has a unique Nash equilibrium, which is pure.

A significance of Corollary 2 is that it extends a classic result by Rosen (1965).
Theorem 9 (Rosen, 1965). A game with a compact and convex product action space $\Omega$ and pseudogradient $\left[\frac{\partial U_{i}}{\partial \omega_{i}}\right]_{i}$ such that for all inputs $\boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2} \in \Omega$ :

$$
\left(\left[\frac{\partial U_{i}}{\partial \omega_{i}}\right]_{i}\left(\boldsymbol{\omega}^{2}\right)-\left[\frac{\partial U_{i}}{\partial \omega_{i}}\right]_{i}\left(\boldsymbol{\omega}^{1}\right)\right) \cdot\left(\boldsymbol{\omega}^{2}-\boldsymbol{\omega}^{1}\right)<0
$$

has a unique Nash equilibrium, which is pure.
We continue by listing three results from Gale and Nikaido (1965). The first, Theorem 10, is a result which appears in their paper. The third restates their result which we have already given as Theorem 2 in this paper. The second, Theorem [11, is a new intermediate sub-result statement, which summarizes the preliminary analysis within Gale and Nikaido's proof of Theorem 2. Theorem 11 is a generalization of Theorem $10{ }^{11}$

Theorem 11 is indispensable for our Theorem 4 and Corollary 2 results, yet a proof for this statement explicitly does not exist in continuous, cohesive form. To spare the reader the task of personally piecing it together, we give the proof here, adapted from Gale and Nikaido. For completeness, we will then finish the proof of Theorem 2 which basically becomes a corollary.

[^7]Theorem 10 (Gale and Nikaido, 1965). If function $f: \Omega \rightarrow \mathbb{R}^{n}$ on compact and convex product domain $\Omega \subset \mathbb{R}^{n}$ has Jacobian $J_{f}$ which is a P-matrix at every $\boldsymbol{\omega} \in \Omega$, then for any fixed input $\boldsymbol{\omega}^{1} \in \Omega$, and variable $\boldsymbol{\omega}^{2}$ from the domain $\Omega$, the inequalities

$$
f\left(\boldsymbol{\omega}^{1}\right) \leq f\left(\boldsymbol{\omega}^{2}\right), \boldsymbol{\omega}^{1} \geq \boldsymbol{\omega}^{2}
$$

have only the solution $\boldsymbol{\omega}^{1}=\boldsymbol{\omega}^{2}$.
Theorem 10 has an interpretation in the context of our proxy games, with $\boldsymbol{\omega}$ as a vector of actions, and $f$ as the pseudogradient function on utilities. For games maximizing utility we would use the equivalent analogous statement for Jacobian as $N$-matrix everywhere (and flip the sign of the first vector inequality). What the N-matrix version of Theorem 10 says when it holds for a game is: given $\boldsymbol{\omega}^{2}$, there can not exist distinct pointwise "weakly larger" actions $\boldsymbol{\omega}^{1}$ such that all local preference gradients (with respect to own action) are also weakly larger at $\boldsymbol{\omega}^{1}$ compared to $\boldsymbol{\omega}^{2}$.

However there is nothing special about the "weakly larger" direction - i.e., the "all-positives" orthant. The pure-math interpretation of Theorem 10 (still for $N$-matrix) is that "moves" from $\boldsymbol{\omega}^{2}$ in the direction of the all-positive orthant to $\boldsymbol{\omega}^{1}$ can not also move the output in the direction of the all-positive orthant. The generalization says, given an $N$-matrix Jacobian everywhere, moving the input in the direction of any orthant can not also move the output in the direction of the same orthant, i.e., by the contrapositive, there must exist a dimension in which the change in the input and the change of the corresponding output have opposite signs. This idea is immediately pertinent in game theory with actions as inputs and utility gradients as outputs, as the basis of a technique to contradict two action profiles supposedly both being in equilibrium.

We state this intermediate result formally here with Theorem (1) (but for continuity of language in result statements, we write it as the $P$-matrix version). To repeat, the proof here mirrors the first steps of Gale and Nikaidd's proof of Theorem 2 , with slight re-working to be explicitly restated as a generalization of Theorem 10. Note the following definition for use in Theorem 11.

Definition 7. Define the operators $\mathbb{1},-\mathbb{1}$ applied to inequalities by: multiplying an inequality by $\mathbb{1}$ leaves it unchanged, and multiplying it by $-\mathbb{1}$ reverses the sign of the inequality.

Theorem 11 (Gale and Nikaido, 1965). If function $f: \Omega \rightarrow \mathbb{R}^{n}$ on compact and convex product domain $\Omega \subset \mathbb{R}^{n}$ has Jacobian $J_{f}$ which is a P-matrix at every $\boldsymbol{\omega} \in \Omega$, then for any fixed input $\boldsymbol{\omega}^{1} \in \Omega$, and variable $\boldsymbol{\omega}^{2}$ from the domain $\Omega$, for every binary vector $\mathbb{B} \in\{\mathbb{1},-\mathbb{1}\}^{n}$ the inequalities

$$
\begin{gathered}
\mathbb{B}_{1}\left(f_{1}\left(\boldsymbol{\omega}^{1}\right) \leq f_{1}\left(\boldsymbol{\omega}^{2}\right)\right), \mathbb{B}_{1}\left(\omega_{1}^{1} \geq \omega_{1}^{2}\right) \\
\vdots \\
\mathbb{B}_{n}\left(f_{n}\left(\boldsymbol{\omega}^{1}\right) \leq f_{n}\left(\boldsymbol{\omega}^{2}\right)\right), \mathbb{B}_{n}\left(\omega_{n}^{1} \geq \omega_{n}^{2}\right)
\end{gathered}
$$

have only the solution $\boldsymbol{\omega}^{1}=\boldsymbol{\omega}^{2}$. Equivalently (the contrapostive), given inputs $\boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2} \neq \boldsymbol{\omega}^{1}$, there must exist a dimension $i$ such that $\left(\omega_{i}^{1}-\omega_{i}^{2}\right) \cdot\left(f_{i}\left(\boldsymbol{\omega}^{1}\right)-f_{i}\left(\boldsymbol{\omega}^{2}\right)\right)>0$.

Proof. Note that we will write the proof to parallel the argument as given by Gale and Nikaidd, and connect it back to the binary vector $\mathbb{B}$ as appropriate.

Inputs $\boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2} \in \Omega$ are explicitly indexed by $\left(\omega_{1}^{1}, \ldots, \omega_{n}^{1}\right)$ and $\left(\omega_{1}^{2}, \ldots, \omega_{n}^{2}\right)$. By contradiction, assume $\boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2}$ are distinct but there exists vector $\mathbb{B}^{*}$ such that all of the inequalities listed in the theorem statement are satisfied.

Without loss of generality we may assume there exists index $k$ such that $\omega_{i}^{2} \leq \omega_{i}^{1}$ for $i \leq k$ and $\omega_{i}^{2} \geq \omega_{i}^{1}$ for $i>k$. If $k=n$ (or by symmetry $k=0$ ) then we are in the exact setting of Theorem 10 (here with the vector $\mathbb{B}^{*}=\{\mathbb{1}\}^{n}$ ), which requires $\boldsymbol{\omega}^{1}=\boldsymbol{\omega}^{2}$.

So from here on we assume $0<k<n$. To satisfy the second inequality in each line of the set of inequalities in the theorem statement, it must be that $\mathbb{B}^{*}$ is the vector of $k \mathbb{1}$ s followed by $(n-k)$ $-\mathbb{1}$ s. Define the analogous mapping $D: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ by

$$
D\left(\omega_{1}, \ldots, \omega_{n}\right)=\left(\omega_{1}, \ldots, \omega_{k},-\omega_{k+1}, \ldots,-\omega_{n}\right)
$$

Clearly $D$ is a bijection on $\mathbb{R}^{n}$ with inverse $D^{-1}=D$, and further $D(\Omega)$ is still a compact and convex product space. Let $E: D(\Omega) \rightarrow \mathbb{R}^{n}$ be the composite mapping $E=D \circ f \circ D$. (I.e., the function $E$ on the domain $D(\Omega)$ operates as follows: the first application of $D$ maps back to $\Omega$, to which $E$ can then properly apply $f$, and finally $D$ is applied again to this output.) At this point, we confirm that the following inequalities hold by inspection, because the application of $\mathbb{B}^{*}$ to the system of inequalities in the theorem statement dovetails with the use of the mapping $D$.

$$
\begin{equation*}
E\left(D\left(\boldsymbol{\omega}^{1}\right)\right) \leq E\left(D\left(\boldsymbol{\omega}^{2}\right)\right), D\left(\boldsymbol{\omega}^{1}\right) \geq D\left(\boldsymbol{\omega}^{2}\right) \tag{11}
\end{equation*}
$$

The Jacobian $J_{E}$ of $E$ is a $P$-matrix because it is obtained from the Jacobian $J_{f}$ by simple changes of row/column signs which preserve the classification as $P$-matrix. We use Lemma 9 to make this explicit (given immediately following this proof). In comparison to the Jacobian of $f$, the Jacobian of $E$ is obtained by multiplying each row and each column of $f$ with index at least $k+1$ by a factor of -1 . If we "transform" the Jacobian of $f$ into the Jacobian of $E$ by considering each $i>k$ in turn one step at a time, by multiplying the $i$ row and $i$ column each by -1 in each one step, we have that the resulting matrix is still a $P$-matrix as an invariant after each step (by Lemma (9), such that $J_{E}$ is a $P$-matrix when the transformation concludes.

With $J_{E}$ a $P$-matrix and equation (11), we can invoke Theorem 10 to conclude that $D\left(\boldsymbol{\omega}^{1}\right)=$ $D\left(\boldsymbol{\omega}^{2}\right)$, which immediately implies that $\boldsymbol{\omega}^{1}=\boldsymbol{\omega}^{2}$ by applying $D^{-1}$ to both sides. This gives the desired contradiction, as the analyzed contradiction also holds by analogy for $f$.

Lemma 9. Given $K \times K$ matrix $M$ as a $P$-matrix. For any index $i \in\{1, \ldots K\}$, the matrix $M^{\prime}$ resulting from multiplying row $i$ by -1 and successively column $i$ by -1 is also a $P$-matrix.

Proof. As a first note, the element of matrix $M^{\prime}$ at index $(i, i)$ gets multiplied by -1 in both the row-multiplication and column-multiplication operations, so its sign remains unchanged. All other elements of either the $i$ row or $i$ column have sign flipped from $M$.

Consider within matrix $M^{\prime}$, the determinant of any principal minor $M^{\prime \prime}$ of $M^{\prime}$, including possibly $M^{\prime}$ itself. Without loss of generality, the following argument holds for any $M^{\prime \prime}$, we don't need to explicitly consider any missing indexes from the original $M^{\prime}$. First in particular, if $M^{\prime \prime}$ excludes row/column $i$ then its determinant remains unchanged.

Otherwise we use the algebraic definition of a determinant. The determinant of $M^{\prime \prime}$ is a sum over product-terms with the following property: each product-term includes exactly one element from each row and each column of $M^{\prime \prime}$, and includes such exhaustively. Any such additive productterm (within the sum making up the determinant calculation) that includes the element of $M^{\prime}$ at index $(i, i)$ can not include any other element of $M^{\prime}$ from row $i$ or column $i$, therefore this term is exactly equal to the respective principal minor determinant term when calculated for the matrix $M$.

Any additive term that does not include the element of $M^{\prime}$ at index $(i, i)$ must use some term as $(i, x)$ and also some term as $(y, i)$ for $x \neq i$ and $y \neq i$, both of which are negated from the corresponding elements at the analogous indexes of $M^{\prime}$ such that again this determinant (additive) term is equal to the respective determinant term using $M$.

This shows that term by term within their summed computations, the determinants of minors of $M^{\prime}$ are everywhere equal to the respective determinants of minors of $M$. The conclusion is that $M^{\prime}$ is indeed a $P$-matrix, because $M$ is.

For completeness, before continuing we restate Theorem 2 and conclude its proof.
Theorem 2 (Gale and Nikaido, 1965). A continuously differentiable function $f: \Omega \rightarrow \mathbb{R}^{n}$ with compact and convex product domain $\Omega \subset \mathbb{R}^{n}$ is one-to-one if its Jacobian is everywhere a P-matrix.

Proof. The statement now follows as a corollary. By contradiction, assume there exist distinct $\boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2} \in \Omega$ with $f\left(\boldsymbol{\omega}^{1}\right)=f\left(\boldsymbol{\omega}^{2}\right)$. Let $\mathcal{P}$ be the program of constraints described in the statement of Theorem 11, Fix a binary vector $\mathbb{B}^{\text {RHS }} \in\{\mathbb{1},-\mathbb{1}\}$ to satisfy the right-hand side equations of $\mathcal{P}$ for these $\boldsymbol{\omega}^{1}, \boldsymbol{\omega}^{2}$. Assumptions in the theorem statement here meet the conditions of Theorem 11 , therefore $\boldsymbol{\omega}^{1} \neq \boldsymbol{\omega}^{2}$ implies that the left-hand side equations of $\mathcal{P}$ can not all be satisfied for $\mathbb{B}^{\text {RHS }}$. In particular it cannot be that $f\left(\boldsymbol{\omega}^{1}\right)=f\left(\boldsymbol{\omega}^{2}\right)$ (which would satisfy the left-hand side of $\mathcal{P}$ ).

Theorem 11has implications for our proxy games which become apparent in the proof of Theorem4. The intuition was previously described in the discussion immediately following Theorem 10 .

Theorem 4, A game with $n$ players and

- a compact and convex product action space $\Omega_{1} \times \ldots \times \Omega_{n}=\Omega \subset \mathbb{R}^{n}$;
- a continuous and twice-differentiable utility function $\boldsymbol{U}: \Omega \rightarrow \mathbb{R}^{n}$ such that:
- the pseudogradient $\left[\frac{\partial U_{i}}{\partial \omega_{i}}\right]_{i}$ of the utility function $\boldsymbol{U}$ is an interior $N$-matrix function;
- and there exists $\boldsymbol{\omega}^{0} \in \Omega$ such that the pseudogradient evaluated at $\boldsymbol{\omega}^{0}$ is $\mathbf{0}$ (the 0-vector);
has a unique Nash equilibrium, which is $\boldsymbol{\omega}^{0}$, and this equilibrium is pure.
Proof. For existence, the action vector $\omega^{0}$ is assumed to exist. It is a pure Nash equilibrium by the assumption that first-order conditions at $\boldsymbol{\omega}^{0}$ are all identically 0 , and utility functions $U_{i}$ are strictly concave with respect to their own unilateral changes (except possibly at the single points of the lower and upper boundaries where it can be weakly concave, but this exception at a single boundary point can not affect the uniqueness of a player's optimal action). This concavity follows from the pseudogradient as an interior N -matrix function, such that at all points (except boundaries), the diagonal elements of the pseudogradient's Jacobian $\left[\frac{\partial^{2} U_{i}}{\partial \omega_{i}^{2}}\right]_{i}$ must be strictly negative.

The Nash is unique first because the pseudogradient function is one-to-one by application of Theorem 3, so no other action vector $\boldsymbol{\omega}^{\prime} \neq \boldsymbol{\omega}^{0}$ can also map to $\mathbf{0}$ (under the pseudogradient function). Next, the rest of this proof is devoted to showing that a second equilibrium $\omega^{\prime}$ can not also exist in the boundaries by having non-zero first-order conditions (i.e., if an agent with action on the boundary has a gradient pointing outside the action space). An outline is given as follows.

- First we argue to ignore consideration of any coordinates of $\boldsymbol{\omega}^{\prime}$ which are in identified boundaries in their respective dimensions, with respect to the pseudogradient function as the output. Our goal here is to show that $\boldsymbol{\omega}^{\prime}$ and $\boldsymbol{\omega}^{0}$ are the same in these coordinates.
- Second, we consider a dimensionally-reduced function $D$ (Definition (6) applied to the pseudogradient function, $\boldsymbol{\omega}^{0}, \boldsymbol{\omega}^{\prime}$, and $K=K_{\boldsymbol{\omega}^{\prime}}=K_{\boldsymbol{\omega}^{0}}$ all the same set of dimensions, where $K_{\boldsymbol{\omega}^{\prime}}, K_{\boldsymbol{\omega}^{0}}$ are the sets of dimensions in which respectively $\boldsymbol{\omega}^{\prime}$ and $\boldsymbol{\omega}^{0}$ exist in identified boundaries. Our arguments under $D$ extend by analogy to our original pseudogradient function if and only if the parameters $\mathbf{c}_{K}$ used in the definition of $D$ represent the same assignment as the values of the respective coordinates in both $\boldsymbol{\omega}^{\prime}$ and $\boldsymbol{\omega}^{0}$.
- Finally, we use $D$ to obtain the contradiction and claim uniqueness of Nash equilibrium.

Per the outline, first we show $K_{\omega^{\prime}}=K_{\omega^{0}}$. Without loss of generality, we analyze identified lower boundaries, with identified upper boundaries by symmetry. The simple direction to prove is $K_{\omega^{0}} \subseteq K_{\omega^{\prime}}$. By contradiction, assume $\boldsymbol{\omega}_{i}^{0}$ is the lower bound of dimension $i$ with its lower boundary identified, but $\omega_{i}^{\prime}>\omega_{i}^{0}$. But then $\omega^{\prime}$ could not be an equilibrium point, because $\partial U / \partial \omega_{i}$ outputs 0 everywhere on the lower boundary in dimension $i$ (it is constant on the boundary and we know that it outputs 0 at point $\boldsymbol{\omega}^{0}$ by assumption), and $\partial U / \partial \omega_{i}$ is monotone decreasing in $\omega_{i}$.

We next show $K_{\omega^{\prime}} \subseteq K_{\omega^{0}}$, which uses a similar but more technical argument. Consider $\boldsymbol{\omega}^{\prime}$ to be in an identified lower boundary in dimension $i$, with general range $\left[a_{i}, b_{i}\right]$ for dimension $i$. The derivative $\partial U_{i} / \partial \omega_{i}$ is the element of the (output) psuedogradient function with index $i$. By definition of an identified boundary, the output of $\partial U_{i} / \partial \omega_{i}$ is constant for inputs ( $a_{i}, \boldsymbol{\omega}_{-i}$ ) for all $\boldsymbol{\omega}_{-i}$. At the lower boundary, it can not be that $\partial U_{i} / \partial \omega_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}^{\prime}\right)>0$ without contradicting $\boldsymbol{\omega}^{\prime}$ as an equilibrium, so it must be that $\partial U_{i} / \partial \omega_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}^{\prime}\right) \leq 0$.

However because $\partial U_{i} / \partial \omega_{i}\left(a_{i}, \cdot\right)$ is constant, then it must also be that $\partial U_{i} / \partial \omega_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}^{0}\right) \leq 0$, which implies that $\boldsymbol{\omega}^{0}$ must also have $\omega_{i}^{0}=a_{i}$ (and in fact by assumption $\partial U_{i} / \partial \omega_{i}\left(a_{i}=\omega_{i}^{0}, \boldsymbol{\omega}_{-i}^{0}\right)=$ 0 ). This follows because any other (larger) value of $\omega_{i}^{0}$ would contradict $\boldsymbol{\omega}^{0}$ as an equilibrium, from the non-positive derivative at $a_{i}$ and the strict concavity at all interior points from the pseudogradient being an interior $N$-matrix function.

So we have $\omega_{i}^{0}=\omega_{i}^{\prime}=a_{i}$ and $\partial U_{i} / \partial \omega_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}^{0}\right)=\partial U_{i} / \partial \omega_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}^{\prime}\right)=0$. The intermediate conclusion here is that dimension $i$ can not be used to maintain that $\boldsymbol{\omega}^{\prime}$ is distinct from $\boldsymbol{\omega}^{0}$. Further, the analysis so far has applied for general $i$. Therefore, it must be that for every dimension $i$ where $\boldsymbol{\omega}^{\prime}$ is in an identified boundary in dimension $i, \boldsymbol{\omega}^{0}$ must be in each of the same identified boundaries; i.e., it must be that $K_{\omega^{\prime}} \subseteq K_{\omega^{0}}$.

We continue to the second point of the outline. From this point on, we consider the dimensionallyreduced function $D$ applied to the pseudogradient function, $\boldsymbol{\omega}^{0}, \boldsymbol{\omega}^{\prime}$ and $K$ the (common) set of dimensions for which $\boldsymbol{\omega}^{\prime}$ and $\boldsymbol{\omega}^{0}$ each exist in identified boundaries. The reduction to $D$ in space $\Omega_{-K}$ is faithful for the following analysis because the coordinates of the pseudogradient fixed by $\mathbf{c}_{K}$ reflect both $\boldsymbol{\omega}^{\prime}$ and $\boldsymbol{\omega}^{0}$. Putting together the definitions of a dimensionally-reduced function (applied to $D$ ) and interior $N$-matrix function (applied to the pseudogradient), we have that $D$ is a strict $N$-matrix function, i.e., its Jacobian is an $N$-matrix everywhere on its (reduced) domain.

We now prove a contradiction. By the contrapositive of (the $N$-matrix version of) Theorem [11, for the pseudogradient function and our two distinct inputs, there must exist at least one dimension
$i$ such that

$$
\begin{aligned}
\left(\omega_{i}^{\prime}-\omega_{i}^{0}\right) & \cdot\left(\frac{\partial U_{i}}{\partial \omega_{i}}\left(\boldsymbol{\omega}^{\prime}\right)-\frac{\partial U_{i}}{\partial \omega_{i}}\left(\boldsymbol{\omega}^{0}\right)\right)<0 \\
& \Leftrightarrow \quad\left(\omega_{i}^{\prime}-\omega_{i}^{0}\right) \cdot \frac{\partial U_{i}}{\partial \omega_{i}}\left(\boldsymbol{\omega}^{\prime}\right)<0
\end{aligned}
$$

where the second line drops the derivative at $\boldsymbol{\omega}^{0}$ because it is known to be 0 .
If $\omega_{i}^{\prime}<\omega_{i}^{0}$, it must be that the pseudogradient at $\omega^{\prime}$ in dimension $i$ is greater than 0 ; alternatively if $\omega_{i}^{\prime}>\omega_{i}^{0}$, this pseudogradient element is less than 0 . But both cases contradict $\boldsymbol{\omega}^{\prime}$ as a Nash point because in either case, the gradient points back in the direction of $\boldsymbol{\omega}^{0}$, and the action space is convex which therefore guarantees that player $i$ has a better response than $\omega_{i}^{\prime}$ when others play $\boldsymbol{\omega}_{-i}^{\prime}$.

Corollary 2. A game with a compact and convex product action space and pseudogradient that is an $N$-matrix function has a unique Nash equilibrium, which is pure.

Proof. The description of the game here is sufficient to meet the conditions of Theorem 12 (below) from (Rosen, 1965), with existence of pure Nash gauranteed as a result. Intuitively, existence of pure Nash follows from the combination of continuity of the utility functions and resulting continuity of upper-countour sets, and fixed point theorems on compact and convex spaces.

The intuition for uniqueness is that it follows from Theorem 11, with structure and explanation mostly analogous to the proof of Theorem 4. In contrast to Theorem 4 however, because we have a strict $N$-matrix function as the pseudogradient, we do not need to make special arguments regarding identified boundaries.

Formally we argue uniqueness by contradiction. Assume there exist two distinct pure Nash equilibrium points. Theorem 2 says there exists a bijection between action space and the image of the pseudogradient function on utility (with the action space as domain). But Theorem 11 requires that there must exist a dimension in which one of the two supposed-distinct equilibrium points has a gradient pointing strictly in the direction of the other, a contradiction because the action space is convex so a preferred deviation much exist.

For completeness we give the theorem by Rosen referenced in Corollary 2,
Theorem 12 (Rosen, 1965). Consider a game with $n$ players and a compact and convex product action space $\Omega$. Assume the utility function $\boldsymbol{U}$ is continuous and for every player $i$ and vector of others actions' $\boldsymbol{\omega}_{-i}$, the function $U_{i}\left(\omega_{i}, \boldsymbol{\omega}_{-i}\right)$ is concave in $\omega_{i}$. There exists a pure Nash equilibrium.

## A. 2 Derivative Calculations; Proofs of Lemma 2, Lemma 10, Lemma 11

Allocation rule sub-calculations:

$$
\begin{aligned}
x_{i}(\boldsymbol{w}) & =\frac{w_{i}}{\sum_{k} w_{k}} \\
\frac{\partial x_{i}}{\partial w_{i}}(\boldsymbol{w}) & =\frac{\left(\sum_{k} w_{k}\right)-w_{i}}{\left(\sum_{k} w_{k}\right)^{2}} \\
\frac{\partial x_{i}}{\partial w_{j}}(\boldsymbol{w}) & =\frac{-w_{i}}{\left(\sum_{k} w_{k}\right)^{2}}=\frac{-x_{i}(\boldsymbol{w})}{\sum_{k} w_{k}}
\end{aligned}
$$

Re-stating the bid function:

$$
\pi_{i}(\boldsymbol{w})=v_{i}\left(w_{i}\right)-\frac{\int_{w_{i}(0)}^{w_{i}} x_{i}\left(z, \boldsymbol{w}_{-i}\right) v_{i}^{\prime}(z) d z}{x_{i}(\boldsymbol{w})}
$$

Self-partial:

$$
\begin{aligned}
\frac{\partial \pi_{i}}{\partial w_{i}}(\boldsymbol{w}) & =v_{i}^{\prime}\left(w_{i}\right)-\frac{x_{i}(\boldsymbol{w}) v_{i}^{\prime}\left(w_{i}\right)}{x_{i}(\boldsymbol{w})}+\frac{\int_{w_{i}(0)}^{w_{i}} x_{i}\left(z, \boldsymbol{w}_{-i}\right) v_{i}^{\prime}(z) d z \cdot \frac{\partial x_{i}}{\partial w_{i}}(\boldsymbol{w})}{x_{i}^{2}(\boldsymbol{w})} \\
& =\frac{\int_{w_{i}(0)}^{w_{i}} x_{i}\left(z, \boldsymbol{w}_{-i}\right) v_{i}^{\prime}(z) d z \cdot \frac{\partial x_{i}}{\partial w_{i}}(\boldsymbol{w})}{x_{i}^{2}(\boldsymbol{w})} \\
& =\frac{\int_{w_{i}(0)}^{w_{i}} x_{i}\left(z, \boldsymbol{w}_{-i}\right) v_{i}^{\prime}(z) d z \cdot \frac{\left(\sum_{k} w_{k}\right)-w_{i}}{\left(\sum_{k} w_{k}\right)^{2}}}{\left(\frac{w_{i}}{\sum_{k} w_{k}}\right)^{2}} \\
& =\int_{w_{i}(0)}^{w_{i}} x_{i}\left(z, \boldsymbol{w}_{-i}\right) v_{i}^{\prime}(z)\left[\frac{\left(\sum_{k} w_{k}\right)-w_{i}}{w_{i}^{2}}\right] d z \\
& =\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\left(\sum_{k} w_{k}\right)-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{w_{i}}-1\right] d z
\end{aligned}
$$

Cross-partials:

$$
\begin{aligned}
\frac{\partial \pi_{i}}{\partial w_{j}}(\boldsymbol{w}) & =-\frac{\int_{w_{i}(0)}^{w_{i}} \frac{\partial x_{i}}{\partial w_{j}}\left(z, \boldsymbol{w}_{-i}\right) v_{i}^{\prime}(z) d z}{x_{i}(\boldsymbol{w})}+\frac{\frac{\partial x_{i}}{\partial w_{j}}(\boldsymbol{w}) \int_{w_{i}(0)}^{w_{i}} x_{i}\left(z, \boldsymbol{w}_{-i}\right) v_{i}^{\prime}(z) d z}{x_{i}^{2}(\boldsymbol{w})} \\
& =\frac{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \cdot\left[x_{i}\left(z, \boldsymbol{w}_{-i}\right) \frac{\partial x_{i}}{\partial w_{j}}(\boldsymbol{w})-\frac{\partial x_{i}}{\partial w_{j}}\left(z, \boldsymbol{w}_{-i}\right) x_{i}(\boldsymbol{w})\right] d z}{x_{i}^{2}(\boldsymbol{w})} \\
& =\frac{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \cdot\left[x_{i}\left(z, \boldsymbol{w}_{-i}\right) \frac{-x_{i}(\boldsymbol{w})}{\sum_{k} w_{k}}-\frac{-x_{i}\left(z, \boldsymbol{w}_{-i}\right)}{\left(\sum_{k} w_{k}\right)-w_{i}+z} x_{i}(\boldsymbol{w})\right] d z}{x_{i}^{2}(\boldsymbol{w})} \\
& =\frac{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \cdot\left[x_{i}\left(z, \boldsymbol{w}_{-i}\right) \frac{-1}{\sum_{k} w_{k}}+\frac{x_{i}\left(z, \boldsymbol{w}_{-i}\right)}{\left(\sum_{k} w_{k}\right)-w_{i}+z}\right] d z}{\sum_{k}} \\
& =\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot\left[\frac{-z}{\left(\sum_{k} w_{k}\right)-w_{i}+z}+\frac{z}{\left(\left(\sum_{k} w_{k}\right)-w_{i}+z\right)^{2}} \cdot \sum_{k} w_{k}\right] d z \\
& =\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\left(\sum_{k} w_{k}\right)-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{\left(\sum_{k} w_{k}\right)-w_{i}+z}-1\right] d z
\end{aligned}
$$

Lemma 2. Given the price function $\boldsymbol{\pi}$ for proportional weights, for $j, k \neq i$, the cross derivatives are the same: $\frac{\partial \pi_{i}}{\partial w_{j}}=\frac{\partial \pi_{i}}{\partial w_{k}}$. Evaluating the Jacobian at $\boldsymbol{w}$, further, all elements of the Jacobian matrix $J_{\boldsymbol{\pi}}$ are positive, i.e., $\frac{\partial \pi_{i}}{\partial w_{i}}>0, \frac{\partial \pi_{i}}{\partial w_{j}}>0$, except at the $w_{i}(0)$ lower boundary in dimension $i$ where the elements of row $i$ are $\frac{\partial \pi_{i}}{\partial w_{i}}=\frac{\partial \pi_{i}}{\partial w_{j}}=0$. The lower boundaries are identified.

Proof. All cross-derivatives $\frac{\partial \pi_{i}}{\partial w_{j}}$ for fixed $i$ and $j \neq i$ are equal because a $d w_{j}$ increase in the weight of any other agent $j$ "looks the same" mathematically to the proportional weights allocation rule of agent $i$, which is $x_{i}\left(w_{i}\right)=\frac{w_{i}}{w_{i}+\sum_{j \neq i} w_{j}}$.

We continue by recalling our assumption that weights are strictly positive and strictly increasing in value. Then all terms in the derivative equations (6) and (7) within the integrals are non-negative everywhere by inspection. All denominator terms are strictly positive everywhere.

For any dimension $i$, consider $w_{i}>w_{i}(0)$. For integrand $z$ strictly interior to the endpoints in $\left(w_{i}(0), w_{i}\right)$, all terms in the derivative equations are strictly positive everywhere. With nonnegativity everywhere and positivity somewhere, all derivatives evaluate to be strictly positive.

Lemma 10. Each function $\Phi_{i}^{\rho}$ is strictly concave taking derivatives with respect to $i$, except at the lower end point of its domain where it is weakly concave.

Proof. We have $\Phi_{i}^{\boldsymbol{\rho}}\left(\tilde{w}_{i}, \tilde{\boldsymbol{w}}_{-i}\right)=\int_{w_{i}(0)}^{\tilde{w}_{i}} \phi_{i}^{\boldsymbol{\rho}}\left(z, \tilde{\boldsymbol{w}}_{-i}\right) \mathrm{d} z=\int_{w_{i}(0)}^{\tilde{w}_{i}} \rho_{i}-\pi_{i}\left(\tilde{w}_{i}, \tilde{\boldsymbol{w}}_{-i}\right) \mathrm{d} z$, i.e., the function $\Phi_{i}^{\boldsymbol{\rho}}$ is defined as the integral over the quantity which subtracts the price function $\pi_{i}$ from a constant price term $\rho_{i}$. In Lemma 2 (appearing immediately above), function $\pi_{i}$ is shown to be monotone strictly increasing on its domain except at the lower bound where its derivative is 0 . Such an integral is concave on its domain as stated.

Lemma 11. Given agents with (unknown) values $\boldsymbol{v} \in[0, h]^{n}$. Consider the price function $\boldsymbol{\pi}$ resulting from a strictly increasing, continuous and differentiable proportional weights social choice function $\boldsymbol{x}$, and dominant-strategy incentive-compatible mechanism implementing $\boldsymbol{x}$. The lower boundaries of weights space are identified boundaries (Definition 4).

Proof. By our assumptions in Section 2 for a proportional weights social choice function $\boldsymbol{x}$, its (parameter) weight functions are strictly positive, even for an agent with value 0 . Self-partials in equation (6) and cross-partials in equation (7) are well-defined. By Lemma 2, for each $i$ the cross-partials at the lower bound of weight space $w_{i}(0)$ are everywhere identically 0 , for all $i$, regardless of $\boldsymbol{w}_{-i}$, meeting the first requirement in the definition of an identified boundary. Again by Lemma 2, self-partials $\partial f_{i} / \partial \omega_{i}$ are strictly positive everywhere above the lower boundary ( $d_{i}>a_{i}$, c.f. proof of Lemma (8) : these are the diagonal element of the Jacobian at index $(i, i)$. Therefore $f_{i}\left(d_{i}, \boldsymbol{\omega}_{-i}\right)=f_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}\right)+\int_{a_{i}}^{d_{i}}\left[\partial f_{i} / \partial \omega_{i}\left(z, \boldsymbol{\omega}_{-i}\right)\right] d z>f_{i}\left(a_{i}, \boldsymbol{\omega}_{-i}\right)$.

## A. 3 Proof of Theorem 6 in Section 3.2

Theorem 6. Consider a $K \times K$ matrix $G$ with diagonal $g_{1}, g_{2}, \ldots, g_{K}$ and all other entries equal to 1 (and without loss of generality $g_{1} \leq g_{2} \leq \ldots \leq g_{K}$ ). The following is a complete characterization describing when $G$ is positive definite.

1. if $g_{1} \leq 0$, then the matrix $G$ is not positive definite;
2. if $g_{1} \geq 1$ and $g_{2}>1$, then $G$ is positive definite;
3. if $0<g_{1}, g_{2} \leq 1$, then $G$ is not positive definite;
4. if $0<g_{1}<1$ and $g_{2}>1$, then $G$ is positive definite if and only if $\sum_{k} \frac{1}{1-g_{k}}>1$.

Proof. To prove positive definiteness in cases (2) and (4), we will show that for any non-zero vector $\mathbf{z}$, it must be true that $\mathbf{z}^{\top} G \mathbf{z}>0$. For cases (1) and (3) we give counterexamples of $\mathbf{z}$ for which $\mathbf{z}^{\top} G \mathbf{z} \leq 0$. Given the structure of $G$ (as all ones except the diagonal), we have

$$
\begin{equation*}
\mathbf{z}^{\top} G \mathbf{z}=\left(\sum_{i} z_{i}\right)^{2}+\sum_{i}\left(g_{i}-1\right) z_{i}^{2} . \tag{12}
\end{equation*}
$$

We recall for use throughout this proof the assumption that, without loss of generality, the diagonal elements are such that $g_{1} \leq g_{2} \leq \ldots \leq g_{K}$. We prove each case of the characterization in turn.

Case (1) is correct by counter-example, setting $\mathbf{z}=(-1,0, \ldots, 0) .{ }^{12}$
Case (2) is correct by inspection of equation (12) in which all terms are non-negative. The vector $\mathbf{z}$ is non-zero, so either a $\left(g_{j}-1\right) z_{j}^{2}$ term for $j \neq 1$ in the second sum is strictly larger than 0 , or all such $z_{j}$ are 0 but then $z_{1} \neq 0$ and the first sum-squared is strictly larger than 0 .

Case (3) is correct by counter-example, setting $\mathbf{z}=(1,-1,0, \ldots, 0)$.
For case (4), we need to prove that when $0<g_{1}<1$ and $g_{2}>1$, then the matrix $G$ is positive definite if and only if $\sum_{k} \frac{1}{1-g_{k}}>1$.

For this last case, given the assumptions on the $g_{i}$ elements, only the $\left(g_{1}-1\right) z_{1}^{2}$ term from equation (12) is negative, all other terms are non-negative. Therefore, from this point on, we can ignore any sub-case where $z_{1}=0$, as some $\left(g_{j}-1\right) z_{j}^{2}$ term for $j \neq 1$ must be strictly positive.

Now consider fixing the value $z_{1}$ to any real number $\bar{z}_{1} \neq 0$. We will show that equation (12) is strictly positive for any $\mathbf{z}_{-1} \in \mathbb{R}^{n-1}$. Specifically, for any fixed $\bar{z}_{1} \neq 0$, equation (12) has a global minimum in variables $\mathbf{z}_{-1}$ that is strictly positive. This global minimum $\mathbf{z}_{-1}^{*}$ satisfies

$$
\begin{align*}
\mathbf{z}_{-1}^{*} & =\operatorname{argmin}_{\mathbf{z}_{-1}}\left(\bar{z}_{1}, \mathbf{z}_{-1}\right)^{\top} \cdot G \cdot\left(\bar{z}_{1}, \mathbf{z}_{-1}\right)  \tag{13}\\
& =\operatorname{argmin}_{\mathbf{z}_{-1}}\left(\bar{z}_{1}+\sum_{j \geq 2} z_{j}\right)^{2}+\sum_{j \geq 2}\left(g_{j}-1\right) z_{j}^{2} \tag{14}
\end{align*}
$$

where the second line substitutes equation (12) and drops the constant $\bar{z}_{1}$ term from the right hand sum. It will be convenient to denote the sum of the variables as $\mathcal{S}\left(\bar{z}_{1}\right)=\bar{z}_{1}+\sum_{j \geq 2} z_{j}^{*}$. After the brief argument that the minimizer $\mathbf{z}_{-1}^{*}$ exists and is characterized by its first-order conditions, we will use first-order conditions on $\mathbf{z}_{-1}^{*}$ to write all variables in terms of $\mathcal{S}\left(\bar{z}_{1}\right)$ which we substitute into (12) to analyze.

To show that $\mathbf{z}_{-1}^{*}$ exists and is characterized by its first-order conditions, observe that the polynomial $\left(\bar{z}_{1}, \mathbf{z}_{-1}\right)^{\top} G\left(\bar{z}_{1}, \mathbf{z}_{-1}\right)$ is a quadratic form with Hessian $2 \cdot G_{[2: K, 2: K]}$, i.e., twice the matrix $G$ without the first row and column:

$$
\operatorname{Hessian}\left(\left(\bar{z}_{1}, \mathbf{z}_{-1}\right)^{\top} G\left(\bar{z}_{1}, \mathbf{z}_{-1}\right)\right)=G_{[2: K, 2: K]}=\left[\begin{array}{cccc}
g_{2} & 1 & \ldots & 1 \\
1 & g_{3} & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & g_{K}
\end{array}\right]
$$

Matrix $G_{[2: K, 2: K]}$ is ones except by assumption we have $g_{j}>1$ for $j \geq 2$ in the diagonal; thus, by case (2) of the theorem, it is positive definite. A quadratic form with strictly positive definite Hessian has a unique local minimum which is characterized by its first-order conditions.

[^8]We now use the first-order conditions to write optimizer $\mathbf{z}_{-1}^{*}$ of equation (14) in terms of $\mathcal{S}\left(\bar{z}_{1}\right) .{ }^{13}$

$$
\begin{equation*}
0=2\left(\bar{z}_{1}+\left(\sum_{k \geq 2, k \neq j} z_{k}^{*}\right)+\left(g_{j}-1\right) \cdot z_{j}^{*}\right) \quad \text { for each } j \geq 2 \tag{15}
\end{equation*}
$$

and re-arranging:

$$
\begin{equation*}
z_{j}^{*}=\frac{1}{1-g_{j}} \mathcal{S}\left(\bar{z}_{1}\right) \quad \text { for each } j \geq 2 \tag{16}
\end{equation*}
$$

We now similarly identify a substitution of $\bar{z}_{1}$ in terms of $\mathcal{S}\left(\bar{z}_{1}\right)$. Starting from equation (16), sum the $z_{j}^{*}$ first-order condition equalities over all $j \geq 2$ :

$$
\begin{equation*}
\sum_{j \geq 2} z_{j}^{*}=\sum_{j \geq 2}\left(\frac{1}{1-g_{j}} \mathcal{S}\left(\bar{z}_{1}\right)\right) \tag{17}
\end{equation*}
$$

Add $\bar{z}_{1}$ to both sides of the equation:

$$
\begin{equation*}
1 \cdot\left(\bar{z}_{1}+\sum_{j \geq 2} z_{j}^{*}\right)=\bar{z}_{1}+\left(\sum_{j \geq 2} \frac{1}{1-g_{j}}\right) \mathcal{S}\left(\bar{z}_{1}\right) \tag{18}
\end{equation*}
$$

Substitute $\mathcal{S}\left(\bar{z}_{1}\right)$ on the left and solve for the right-hand side $\bar{z}_{1}$ term:

$$
\begin{equation*}
\bar{z}_{1}=\left(1-\sum_{j \geq 2} \frac{1}{1-g_{j}}\right) \cdot \mathcal{S}\left(\bar{z}_{1}\right) \tag{19}
\end{equation*}
$$

Notice that equation (19) and the definition of $\bar{z}_{1} \neq 0$ excludes the possibility that $\mathcal{S}\left(\bar{z}_{1}\right)=0$.
In the analysis below, the first line re-writes the objective function in (121). The second line substitutes equations (16) and (19). Subsequent lines are elementary manipulations.

$$
\begin{aligned}
& \left(\bar{z}_{1}, \mathbf{z}_{-1}^{*}\right)^{\top} \cdot G \cdot\left(\bar{z}_{1}, \mathbf{z}_{-1}^{*}\right) \\
& \quad=\left(g_{1}-1\right) \bar{z}_{1}+\mathcal{S}\left(\bar{z}_{1}\right)^{2}+\sum_{j \geq 2}\left(g_{j}-1\right) z_{j}^{*} \\
& \quad=\left(g_{1}-1\right)\left(1-\sum_{j \geq 2} \frac{1}{1-g_{j}}\right)^{2} \mathcal{S}\left(\bar{z}_{1}\right)^{2}+\left(1-\sum_{j \geq 2} \frac{1}{1-g_{j}}\right) \mathcal{S}\left(\bar{z}_{1}\right)^{2} \\
& \quad=\mathcal{S}\left(\bar{z}_{1}\right)^{2}\left[\left(g_{1}-1\right)\left(1-\sum_{j \geq 2} \frac{1}{1-g_{j}}\right)^{2}+1-\sum_{j \geq 2} \frac{1}{1-g_{j}}\right] \\
& \quad=\mathcal{S}\left(\bar{z}_{1}\right)^{2}\left(1-\sum_{j \geq 2} \frac{1}{1-g_{j}}\right)\left[\left(1-g_{1}\right)\left(\sum_{j \geq 2} \frac{1}{1-g_{j}}-1\right)+\frac{1-g_{1}}{1-g_{1}}\right] \\
& \quad=\mathcal{S}\left(\bar{z}_{1}\right)^{2}\left(1-\sum_{j \geq 2} \frac{1}{1-g_{j}}\right)\left(1-g_{1}\right)\left[\sum_{k} \frac{1}{1-g_{k}}-1\right] .
\end{aligned}
$$

Given the assumptions on the $g_{i}$ for current case (4), the first three terms of this product are strictly positive (recalling $0<g_{1}<1$ and $g_{j}>1$ for $j>1$, and $\bar{z}_{1} \neq 0$ and $\mathcal{S}\left(\bar{z}_{1}\right) \neq 0$, so $\left(\mathcal{S}\left(\bar{z}_{1}\right)\right)^{2}>0$ ).

[^9]To finish, we observe that the exact dependence of positive definiteness of the matrix $G$ is on the bracketed fourth term (where the first term $k=1$ of the sum is positive and all of the other terms are negative):

$$
\text { For } 0<g_{1}<1 \text { and } g_{j}>1 \forall j \geq 2, G \text { is positive definite iff }\left[\sum_{k} \frac{1}{1-g_{k}}-1\right]>0 \text {. }
$$

## A. 4 Lemmas Supporting Theorem 7 in Section 3.2

Lemma 3. If $h_{i} \leq 1$, then $w_{i}>0.5 \sum_{k} w_{k}$, and all other weights must have $w_{j}<0.5 \sum_{k} w_{k}$, and all other $h_{j}>1$.

Proof. Writing out $h_{i}$ from its definition as the ratio of partial derivatives,

$$
h_{i}=\frac{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\sum_{k} w_{k}-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{w_{i}}-1\right] d z}{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\sum_{k} w_{k}-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{\sum_{k} w_{k}-w_{i}+z}-1\right] d z}
$$

If $h_{i} \leq 1$, by implication it is well-defined so the denominator can not disappear and $w_{i}>w_{i}(0)$. There must exist $z \in\left(0, w_{i}\right]$, such that

$$
\begin{equation*}
\frac{\sum_{k} w_{k}}{\sum_{k} w_{k}-w_{i}+z} \geq \frac{\sum_{k} w_{k}}{w_{i}} \tag{20}
\end{equation*}
$$

which implies $w_{i}>0.5 \sum_{k} w_{k}$ by noting equal numerators and comparison of denominators. The rest of the claim follows as $w_{i}$ is obviously the only weight more than half the total, and claiming $h_{j}>1$ for other $j$ is simply an explicit statement of the contrapositive.

Lemma 4 proves the necessary and sufficient lower bound to show that $\boldsymbol{\pi}$ meets the conditions of Theorem 7 Case (4). Technical Lemma 12 below it is used by Lemma 4 ,
Lemma 4. When $h_{1}<1$ and $h_{j}>1 \forall j \neq 1$, we have $\sum_{k} \frac{1}{1-h_{k}}>1$.
Proof. With $h_{1}<1$ by assumption, then $w_{1}>0.5 \sum_{k} w_{k}$ by Lemma 3, and $x_{1}>0.5$. Thus $x_{j}<0.5$ for $j \neq 1$ and we can apply Lemma 12 (below), to get the first inequality in the following analysis:

$$
\begin{aligned}
\sum_{k} \frac{1}{1-h_{k}} & >\frac{x_{1}^{2}}{2 x_{1}-1}+\sum_{k>1} \frac{x_{k}^{2}}{2 x_{k}-1} \\
& \geq \frac{x_{1}^{2}}{2 x_{1}-1}+\frac{\left(1-x_{1}\right)^{2}}{2\left(1-x_{1}\right)-1} \\
& =1
\end{aligned}
$$

and with the second step following because $\left.\frac{x_{k}^{2}}{2 x_{k}-1}\right|_{0}=0$ and is a concave function when $0<x_{k}<0.5$ and $\sum_{k>1} x_{k}=\left(1-x_{1}\right)$ (its second derivative is $\frac{2}{\left(2 x_{k}-1\right)^{3}}$ and it acts submodular).

Lemma 12. When $h_{1}<1$ and $h_{j}>1 \forall j \neq 1$, then $\forall i \in\{1, \ldots, n\}$, we have $\frac{1}{1-h_{i}}>\frac{x_{i}^{2}}{2 x_{i}-1}$.

Proof. By subtracting 1 from both sides, it is equivalent to prove the inequality on the right:

$$
\frac{1}{1-h_{i}}>\frac{x_{i}^{2}}{2 x_{i}-1} \quad \Longleftrightarrow \quad \frac{h_{i}}{1-h_{i}}>\frac{x_{i}^{2}-2 x_{i}+1}{2 x_{i}-1}=\frac{\left(1-x_{i}\right)^{2}}{2 x_{i}-1}
$$

Working from the definition of $h_{i}$ :

$$
\frac{h_{i}}{1-h_{i}}=\frac{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\sum_{k} w_{k}-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{w_{i}}-1\right] d z}{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\sum_{k} w_{k}-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{\sum_{k} w_{k}-w_{i}+z}-\frac{\sum_{k} w_{k}}{w_{i}}\right] d z}
$$

The numerator is always positive.
For the denominator, we would like to get a less complex upper bound on it by dropping the $z$ term within the brackets. Generally we can do this but we have to be careful that the overall sign of the denominator does not change.

For $i \neq 1$ and $h_{i}>1$, then the denominator is negative by simple inspection of the left hand side. For $i=1, h_{1}<1$, then the denominator is positive. We relax the denominator and increase it, arguing after the calculations that doing this does not change the sign of the expression.

$$
\begin{aligned}
\int_{w_{i}(0)}^{w_{i}} & v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\sum_{k} w_{k}-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{\sum_{k} w_{k}-w_{i}+z}-\frac{\sum_{k} w_{k}}{w_{i}}\right] d z \\
& <\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\sum_{k} w_{k}-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{\sum_{k} w_{k}-w_{i}}-\frac{\sum_{k} w_{k}}{w_{i}}\right] d z \\
& =\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\sum_{k} w_{k}-w_{i}+z} \cdot\left[\frac{\left(\sum_{k} w_{k}\right)\left(2 w_{i}-\sum_{k} w_{k}\right)}{w_{i}\left(\sum_{k} w_{k}-w_{i}\right)}\right] d z
\end{aligned}
$$

The important term is $\left(2 w_{i}-\sum_{k} w_{k}\right)$. For $i=1, w_{1}>0.5 \sum_{k} w_{k}$ by Lemma 3, and also for $j \neq 1$, $w_{j}<0.5 \sum_{k} w_{k}$ by Lemma 3. Then clearly the denominator is still positive for $i=1$; and still negative for agents $i \neq 1$. So we give a lower bound on the fraction using the proved upper bound on the denominator.

$$
\begin{aligned}
\frac{h_{i}}{1-h_{i}} & >\frac{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\sum_{k} w_{k}-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{w_{i}}-1\right] d z}{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\sum_{k} w_{k}-w_{i}+z} \cdot\left[\frac{\sum_{k} w_{k}}{\sum_{k} w_{k}-w_{i}}-\frac{\sum_{k} w_{k}}{w_{i}}\right] d z} \\
& =\frac{\frac{\sum_{k} w_{k}}{w_{i}}-1}{\frac{\sum_{k} w_{k}}{\sum_{k} w_{k}-w_{i}}-\frac{\sum_{k} w_{k}}{w_{i}}} \\
& =\frac{\left(1-x_{i}\right)^{2}}{2 x_{i}-1}
\end{aligned}
$$

## B Supporting Material for Section 4

The goal of this section is to show in detail how to reduce the price inversion question to binary search. We do this by showing that the analysis is largely many-to-one separable: we can make meaningful observations about each agent individually, in particular by treating the (initially unknown) sum total of all weights $s=\sum_{k} w_{k}$ as an independent variable used as input to the analysis of each agent.

Before getting to the key results, we use a more measured pace than is possible in the main body of the paper to give some preliminary analysis of the problem regarding price functions and structure of search spaces, in particular for "small" agents with weight at most half the total. We do this in Appendix B. 1 and then the rest of this section is laid out as follows: Appendix B. 2 gives both intuition and the fully detailed version of the algorithm; Appendix B.3 gives the proofs of the critical lemmas and Theorem 8 from Section 4; and finally technical Appendix B. 4 is used to support Appendix B. 3 and to describe within the algorithm how we set up "oracle checks" to find the correct sub-space of weight space to search for a solution, and the endpoints of binary search.

## B. 1 First Computations and Analysis of the Search Space

This section exhibits the fundamentals of a reduced, separated, one-agent analysis of the price inversion question. Note the following explicit conversion of the function $\pi_{i}(\cdot)$ to accept sum $s=\sum_{k} w_{k}$ as an input variable in place of $\boldsymbol{w}_{-i}$. We recall equation (3):

$$
\pi_{i}(\boldsymbol{w})=v_{i}\left(w_{i}\right)-\frac{\int_{w_{i}(0)}^{w_{i}} x_{i}\left(z, \boldsymbol{w}_{-i}\right) v_{i}^{\prime}(z) d z}{x_{i}(\boldsymbol{w})}
$$

where we also recall $v_{i}(\cdot)$ is overloaded to be the function that maps from buyer $i$ 's weight back to buyer $i$ 's value (well-defined by the assumption that $w_{i}(\cdot)$ is strictly increasing). Re-arranging we have:

$$
\begin{equation*}
\bar{\pi}_{i}\left(s, w_{i}\right)=v_{i}\left(w_{i}\right)-\frac{s}{w_{i}} \int_{w_{i}(0)}^{w_{i}} \frac{z}{s-w_{i}+z} v_{i}^{\prime}(z) d z \tag{21}
\end{equation*}
$$

The form of equation (21) illustrates the critical relationships between $\bar{\pi}_{i}, s$, and $w_{i}$. Our high level goal will be to understand the behavior of the function $\bar{\pi}_{i}$ in the space ranging over feasible $s$ and $w_{i}$, starting with the technical computations of the partials on $\bar{\pi}_{i}$. Recall from Lemma 2 that functions $\pi_{i}$ have the same cross-partials with respect to $w_{j} \forall j \neq i$. This property extends to $\bar{\pi}_{i}$ :

$$
\begin{aligned}
& \bar{\pi}_{i}\left(s+d w_{i}, w_{i}+d w_{i}\right)=\pi_{i}\left(w_{i}+d w_{i}, \boldsymbol{w}_{-i}\right) \\
& \quad \Rightarrow \quad \frac{\partial \bar{\pi}_{i}}{\partial s} d w_{i}+\frac{\partial \bar{\pi}_{i}}{\partial w_{i}} d w_{i}=\frac{\partial \pi_{i}}{\partial w_{i}} d w_{i} \\
& \bar{\pi}_{i}\left(s, w_{i}+d w_{i}\right)=\pi_{i}\left(w_{i}+d w_{i}, w_{j}-d w_{i}, \boldsymbol{w}_{-i, j}\right) \\
& \quad \Rightarrow \quad \frac{\partial \bar{\pi}_{i}}{\partial w_{i}} d w_{i}=\frac{\partial \pi_{i}}{\partial w_{i}} d w_{i}-\frac{\partial \pi_{i}}{\partial w_{j}} d w_{i}
\end{aligned}
$$

Combining the above equations together, and any $j \neq i$ we get

$$
\begin{align*}
\frac{\partial \bar{\pi}_{i}}{\partial w_{i}} & =\frac{\partial \pi_{i}}{\partial w_{i}}-\frac{\partial \pi_{i}}{\partial w_{j}}  \tag{22}\\
\frac{\partial \bar{\pi}_{i}}{\partial s} & =\frac{\partial \pi_{i}}{\partial w_{j}} \tag{23}
\end{align*}
$$

We give the intuition for these calculations. If $w_{i}$ increases unilaterally without a change in $s$, then it must be that some other $w_{j}$ decreases by an equal amount. If we increase $s$ without an observed change in $w_{i}$, then it must be some other $w_{j}$ that increased ${ }^{14}$ The result is the symbolic

[^10]

Figure 3: The price level set curve $\mathcal{Q}_{i}^{\boldsymbol{\rho}}=\left\{\left(s, w_{i}\right): \bar{\pi}_{i}\left(s, w_{i}\right)=\rho_{i}\right\}$ (thick, gray, dashed), is decreasing below the $w_{i}=s / 2$ line (Lemma (6) where it is defined by its subset $\mathcal{P}_{i}^{\rho}$ (thin, black, solid). It is bounded above by the $w_{i}=s$ line (trivially as $s$ sums over all weights) and the $w_{i}=w_{i}(h)$ line (the maximum weight in the support of the values), and below by the $w_{i}=w_{i}(0)$ line which we have assumed to be strictly positive. $r_{i}^{\rho}$ is the minimum weight-sum consistent with observed price $\rho_{i}$ and weights $w_{i} \leq s / 2$. This is an exact replica of Figure 圆, copied here for convenience.
identities as given in equations (22) and (23) above. We will evaluate them in more detail in Lemma 13 below.

We formally identify three objects of interest (initially discussed in Section 4, see Figure (3). These quantities are defined for each agent $i$, weight function $w_{i}$, and the observed price $\rho_{i}$ of this agent. Importantly, though the notation includes the whole profile of observed prices $\boldsymbol{\rho}$, these objects only depend on its $i$ th coordinate $\rho_{i}$.

- First, the price level set $\mathcal{Q}_{i}^{\rho}$ is defined as $\left\{\left(s, w_{i}\right) \mid \bar{\pi}_{i}\left(s, w_{i}\right)=\rho_{i}\right\}$, i.e., these are the $\rho_{i}$ level-sets of $\bar{\pi}_{i}\left(s, w_{i}\right)$. The pertinent subset of $\mathcal{Q}_{i}^{\boldsymbol{\rho}}$ is $\mathcal{P}_{i}^{\boldsymbol{\rho}}=\left\{\left(s, w_{i}\right) \mid \bar{\pi}_{i}\left(s, w_{i}\right)=\rho_{i}\right.$ and $\left.w_{i} \leq s / 2\right\} \subseteq \mathcal{Q}_{i}^{\boldsymbol{\rho}}$, i.e., the subset which restricts the set $\mathcal{Q}_{i}^{\rho}$ to the region where $w_{i}$ is at most half the total weight $s{ }^{15}$ These sets are illustrated respectively by the dashed and solid lines in Figure 3.
- Second, the elements of the price level-set $\mathcal{P}_{i}^{\boldsymbol{\rho}}$ each have unique $s$ coordinate (see Lemma (13). It will be convenient to describe it as a function mapping sum $s$ to weight $w_{i}$ of agent $i$, parameterized by the price $\rho_{i}$. Denote this function $w_{i}^{\rho}(s)$. This function is illustrated in Figure 3 where below the dotted line $w_{i}=s / 2$, the curve is a function in $s$. Qualitatively, it is monotone decreasing and not necessarily convex.
- Third, $\mathcal{P}_{i}^{\rho}$ is non-empty and possesses a smallest total weights coordinate $s$ which we define as $r_{i}^{\boldsymbol{\rho}}=\min \left\{s:\left(s, w_{i}\right) \in \mathcal{P}_{i}^{\boldsymbol{\rho}}\right\}$. In the example of Figure 3, $r_{i}^{\boldsymbol{\rho}}$ is the $s$-coordinate of the point where the level-set $\mathcal{P}_{i}^{\boldsymbol{\rho}}$ intersects the $w_{i}=s / 2$ line. In the case that the entire set $\mathcal{Q}_{i}^{\boldsymbol{\rho}}$ is below the $w_{i}=s / 2$ line, $\mathcal{P}_{i}^{\boldsymbol{\rho}}=\mathcal{Q}_{i}^{\boldsymbol{\rho}}$ and $r_{i}^{\boldsymbol{\rho}}$ is the sum $s$ that uniquely satisfies $\bar{\pi}_{i}\left(s, w_{i}(h)\right)=\rho_{i}{ }^{16}$

[^11]Continuing, consider price level set $\mathcal{Q}_{i}^{\rho}$. We note again that $\bar{\pi}_{i}(\cdot)$ can be used to map a domain of $\left(s, w_{i}\right)$ to price level sets (as depicted in Figure (3). In this context we return to analyzing the partial derivatives of $\bar{\pi}_{i}(\cdot)$, formally with Lemma 13 (immediately to follow). Intuitively, the statement of Lemma 13 claims the following, with relation to Figure 3:

- Part 1 of Lemma 13: below the $w_{i}=s / 2$ line, starting at any point $\left(\hat{s}, \hat{w}_{i}\right)$, we strictly "move up" fixed-price level sets as we move up to ( $\hat{s}, \hat{w}_{i}+\delta$ ), or to the right to $\left(\hat{s}+\delta, \hat{w}_{i}\right)$.
- Part 2 of Lemma [13) below the $w_{i}=s / 2$ line, price level sets are necessarily decreasing curves; further they are defined for arbitrarily large $s$, which reflects the many-to-one nature of this analysis: other than the summary statistic $s$, nothing specific is known about the other agents, for example we do not need to know the number of other agents or their weights functions or bounds on their weights.
- Additionally, above the $w_{i}=s / 2$ line, we "move up" fixed-price level sets with an increase in $s$ but not necessarily with an increase in $w_{i}$.

For use in Lemma 13 and the rest of this Appendix B we overload the notation $h_{i}$ as defined in equation (8) to be a function of $w_{i}$ and $s$ rather than $\boldsymbol{w}$, with the obvious substitution in its definition to replace $\sum_{k} w_{k}$ with $s$.

Lemma 13. Assume $w_{i} \leq s / 2$ and fix the price of agent $i$ to be $\rho_{i}>0$. Let $\mathcal{Q}_{i}^{\boldsymbol{\rho}}, \mathcal{P}_{i}^{\boldsymbol{\rho}}, w_{i}^{\boldsymbol{\rho}}(s)$ and $r_{i}^{\rho}$ be defined as above, and $h_{i}=\frac{\partial \pi_{i}}{\partial w_{i}} / \frac{\partial \pi_{i}}{\partial w_{j}}$ extended from equation (8). Then restricting analysis to the cone described by $w_{i} \leq s / 2$ and non-negative weight $w_{i}$ :

1. $\bar{\pi}_{i}\left(s, w_{i}\right)$ is a continuous and strictly increasing function in both variables $s$ and $w_{i}$, with specifically $\frac{\partial \bar{\pi}_{i}\left(s, w_{i}\right)}{\partial s}=\frac{\partial \pi_{i}}{\partial w_{j}}$ and $\frac{\partial \bar{\pi}_{i}\left(s, w_{i}\right)}{\partial w_{i}}=\frac{\partial \pi_{i}}{\partial w_{j}} \cdot\left(h_{i}-1\right)$;
2. $w_{i}^{\rho}(s)$ is a well-defined and strictly decreasing function on $s \in\left[r_{i}^{\rho}, \infty\right)$ with $\frac{d w_{i}^{\rho}(s)}{d s}=\frac{1}{1-h_{i}}$; in particular the function is well-defined for arbitrarily large $s$ independent of the number of other agents or their weight functions;
3. $w_{i}^{\rho}(s)$ can be computed to arbitrary precision using binary search.

Further, (1) partially extends such that $\bar{\pi}_{i}\left(s, w_{i}\right)$ is increasing in $s$ with $\frac{\partial \bar{\pi}_{i}\left(s, w_{i}\right)}{\partial s}=\frac{\partial \pi_{i}}{\partial w_{j}}$ holding everywhere, (so including above the line $w_{i}=s / 2$ ).

Proof. For (1), as in Section 3.2, we set $h_{i}=\frac{\partial \pi_{i}}{\partial w_{i}} / \frac{\partial \pi_{i}}{\partial w_{j}}$, i.e., the diagonal entry in the Jacobian matrix after the normalization (divide each row by its common cross-partial term), and with the substitution $s=\sum_{k} w_{k}$.

From equation (22), $\frac{\partial \bar{\pi}_{i}}{\partial w_{i}}=\frac{\partial \pi_{i}}{\partial w_{i}}-\frac{\partial \pi_{i}}{\partial w_{j}}=\frac{\partial \pi_{i}}{\partial w_{j}} \cdot\left(h_{i}-1\right)$. By Lemma 3, $h_{i}$ is larger than 1 when $w_{i} \leq s / 2$. By Lemma 2, $\frac{\partial \pi_{i}}{\partial w_{j}}>0$. Hence $\frac{\partial \pi_{i}}{\partial w_{j}} \cdot\left(h_{i}-1\right)>0$ when $w_{i} \leq s / 2$. The $\frac{\partial \bar{\pi}_{i}}{\partial s}$ direction follows directly from equation (23) with Lemma 2 applying to $\frac{\partial \pi_{i}}{\partial w_{j}}$. This argument is also sufficient to prove the last claim of the lemma statement extending (1).

For (2), we first observe that the function $w_{i}^{\rho}(s)$ is well-defined (on an appropriate domain) because $w_{i}^{\rho}(s)$ uses fixed $\rho_{i}$, otherwise it would contradict the monotonicity properties proved in (1) which requires we "move up" price level sets whenever we unilaterally increase $w_{i}$. Therefore
we can take the derivative with respect to $s$. We get $\frac{d w_{i}^{\rho}(s)}{d s}$ is negative for $w_{i} \leq s / 2$ by the following calculation (from first-order conditions as we move along the fixed curve resulting from $\bar{\pi}_{i}(\cdot)$ having constant output $\rho_{i}$ ):

$$
\begin{aligned}
0= & \frac{\partial \bar{\pi}_{i}}{\partial w_{i}^{\rho}(s)} d w_{i}^{\rho}(s)+\frac{\partial \bar{\pi}_{i}}{\partial s} d s \\
& \Rightarrow \quad \frac{d w_{i}^{\rho}(s)}{d s}=\frac{-\frac{\partial \bar{\pi}_{i}}{\partial s}}{\frac{\partial \bar{\pi}_{i}}{\partial w_{i}^{\rho}(s)}}=-\frac{\frac{\partial \pi_{i}}{\partial w_{j}}}{\frac{\partial \pi_{i}}{\partial w_{i}}-\frac{\partial \pi_{i}}{\partial w_{j}}}=\frac{1}{1-h_{i}}<0
\end{aligned}
$$

The last inequality uses Lemma 3 from which $w_{i} \leq s / 2$ implies $h_{i}>1$. We next prove for (2) that $w_{i}^{\rho}(\cdot)$ and its domain are well-defined.

Technical Lemma 16 (deferred to Appendix (B.4) will show that $\mathcal{P}_{i}^{\rho}$ is non-empty. Consider starting at any of its elements. We can theoretically use its continuous derivative to "trace out" the curve of the function $w_{i}^{\rho}(s)$. As $s$ increases from the starting point, we note that positive prices can never be consistent with non-positive weights, such that the continuous and negative derivative implies that the function converges to some positive infimum as $s \rightarrow \infty$. As $s$ decreases, the function increases until either we reach a maximum feasible point with $\left(s, w_{i}^{\boldsymbol{\rho}}(s)=w_{i}(h)\right)$ from the maximum value type $h$, or otherwise the input-output pair $\left(s, w_{i}^{\rho}(s)\right)$ intersects the line $w_{i}=s / 2$, and minimum total weight $r_{i}^{\rho}$ is realized at the point of intersection.

This shows that "reals at least $r_{i}^{\boldsymbol{\rho}}$ " is a valid domain for $w_{i}^{\boldsymbol{\rho}}(s)$, and this completes the first statement in (2). The second statement of (2) follows because the construction of the set $\mathcal{Q}_{i}^{\rho}$ is independent of other agents: for any realization of the set of other agents, their effect is summarized with the variable $s$.

For (3), we note that the output of function $w_{i}^{\rho}(s)$ has constant lower-bound $w_{i}(0)$ and is upper-bounded by $s / 2$, so we can indeed run binary search.

Within Lemma 13 we explicitly note the significance of the $h_{i}$ terms in derivative calculations. As the last part of the statement shows, these derivative calculations also hold for the space $w_{i}>s / 2$ (with a carefully extended interpretation of the $w_{i}^{\rho}$ function to be sure to apply the mapping from $s$ at the correct $w_{i}$ ); but we do not get the contrapositive of Lemma 3 in this region to guarantee the sign of $\left(1-h_{i}\right)$, and so we do not get the monotonicity property of (2) everywhere.

Further, recall the statement of Lemma 4 (originally given on page (9):
Lemma 4. When $h_{1}<1$ and $h_{j}>1 \forall j \neq 1$, we have $\sum_{k} \frac{1}{1-h_{k}}>1$.
As economic intuition for this result, we now see that the terms in the sum are exactly the derivatives $\frac{\partial \bar{\pi}_{i}\left(s, w_{i}\right)}{\partial s}$, i.e., derivatives of the respective agents' $\rho_{i}$ level set curves. We will see the importance of Lemma 4 below as the key final step in the proof of Lemma 7 .

## B. 2 The Full Algorithm

Because the observed profile of prices $\boldsymbol{\rho}$ is invertible to a unique profile of weights (from Section 3.2), the quantity $s=\sum_{k} w_{k}$ is uniquely determined by observed prices. The intuitive description of the algorithmic strategy to compute the inversion from prices to weights is as follows.

Motivated by Appendix B.1, we intend to split the search space for the unique $s$. Clearly at most one of the agents can have strictly more than half the weight. We cover the entirety of weights
space by considering $n$ subspaces, representing the $n$ possibilities that any one agent $i^{*}$ is allowed but not required to have strictly more than half the weight. (The region where all agents have at most half the weight is covered by all subspaces, without introducing a conflict.) Explicitly, define

$$
\begin{equation*}
\text { Space-i }=\left\{\boldsymbol{w} \mid w_{i} \text { unrestricted } \wedge w_{j} \leq \sum_{k} w_{k} / 2, \forall j \neq i\right\} \quad \text { for } i \in\{1, \ldots, n\} \tag{24}
\end{equation*}
$$

Recall the definition of $\pi_{i^{*}}^{\rho}$ from line (10):

$$
\pi_{i^{*}}^{\rho}(s):=\pi_{i^{*}}\left(\max \left\{s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s), w_{i^{*}}(0)\right\}, \boldsymbol{w}_{-i^{*}}^{\rho}(s)\right)
$$

A specific monotonicity property within each Space-i (see Lemma 7 and its proof) will allow the algorithm to use a natural binary search for the solution. Considering such a search in each of $n$ spaces will deterministically find $\tilde{s}$ to yield a vector of weights $\tilde{\boldsymbol{w}}$ as $\left(\left(\tilde{s}-\sum_{i \neq i^{*}} \bar{w}_{i}(\tilde{s})\right), \boldsymbol{w}_{-i}^{\rho}(\tilde{s})\right)$, which are arbitrarily close to the true $s^{*}$ and true $\boldsymbol{w}^{*}$ (i.e., the $\boldsymbol{w}^{*}$ which maps to $\boldsymbol{\rho}$ under $\boldsymbol{\pi}$ ).

The goal of the algorithm is to find the agent $i^{*}$ and unique $s$ such that $\pi_{i^{*}}^{\rho}(s)$ outputs $\rho_{i^{*}}$, the true payment. I.e., we search for the equality of $\pi_{i^{*}}^{\rho}(s)=\rho_{i^{*}}$.

We now give the full version of the algorithm. Beyond the outline in the main body of the paper, the most significant new technical piece in the expanded description is the use of $r_{j}^{\rho}$ variables $(j \neq i)$ to lower bound the search for $s$ in any given candidate Space-i. The $r_{j}^{\rho}$ variables were described as the third item of interest in Appendix B.1. They are used in the expanded descriptions of new pre-process step 0 , and steps $1(\mathrm{a})(\mathrm{b})(\mathrm{c})$. We also newly use $s(h)=\sum_{k} w_{k}(h)$ to denote the maximum sum of weights possible.

The full algorithm (with intuitive remarks):
0. Pre-process: For each $i$, compute $r_{i}^{\rho} \sqrt{17}$
(a) (general case: $\mathcal{P}_{i}^{\boldsymbol{\rho}} \neq \mathcal{Q}_{i}^{\boldsymbol{\rho}}$ ) if $\bar{\pi}_{i}\left(2 w_{i}(h), w_{i}(h)\right) \geq \rho_{i}$, run binary search "diagonally" on the line segment of $w_{i}=s / 2$ between $(0,0)$ and $\left(2 w_{i}(h), w_{i}(h)\right)$ to find an element of $\mathcal{Q}_{i}^{\boldsymbol{\rho}}$ and use its $s$ coordinate as $r_{i}^{\rho}$ (which we can do because $\bar{\pi}_{i}(\cdot)$ is strictly increasing on this domain);
(b) (edge case: $\mathcal{P}_{i}^{\rho}=\mathcal{Q}_{i}^{\rho}$ ) otherwise, fix $w_{i}$ coordinate to its maximum $w_{i}(h)$ and run binary search "horizontally" to find $\hat{s} \in\left[2 w_{i}(h), s(h)\right]$ representing $\left(\hat{s}, w_{i}(h)\right) \in \mathcal{Q}_{i}^{\boldsymbol{\rho}}$ (which we can do because $\bar{\pi}_{i}(\cdot)$ is strictly increasing in $s$ for constant $\left.w_{i}\right)$; set minimum total weight $r_{i}^{\rho}=\hat{s}$.

1. find an agent $i^{*}$ and search a range $\left[s_{L}, s_{H}\right]$ over possible $s$ by iteratively running the following for each fixed assignment of agent $i \in\{1, \ldots, n\}$ :
(a) temporarily set $i^{*}=i$;
(b) determine the range $\left[s_{L}, s_{H}\right]$ on which $\pi_{i^{*}}^{\boldsymbol{\alpha}}$ is well-defined and searching is appropriate:

- identify a candidate lower bound $s_{L}=\max _{j \neq i^{*}} r_{j}^{\rho}$ (because any smaller $s \in\left[0, s_{L}\right.$ ) is outside the domain of $w_{j}^{\rho}$, for some $j$ );

[^12]- run a "validation check" on the lower bound, specifically, exit this iteration of the for-loop if we do not observe:

$$
\pi_{i^{*}}^{\rho}\left(s_{L}\right)=\bar{\pi}_{i^{*}}\left(s_{L}, \max \left\{s_{L}-\sum_{k \neq i^{*}} w_{k}^{\rho}\left(s_{L}\right), w_{i^{*}}(0)\right\}\right) \leq \rho_{i^{*}}
$$

(because recall the goal of the algorithm, to search for equality of $\pi_{i^{*}}^{\rho}(s)=\rho_{i^{*}}$; but by Lemma 13, $\pi_{i^{*}}^{\rho}$ is increasing, then if the inequality here does not hold at the lower bound, the left hand side is already too big and will never decrease);

- identify a candidate upper bound $s_{H}$ using binary search to find $s_{H}$ as the largest total weight consistent with the maximum weight of agent $i^{*}$, i.e, such that $s_{H}$ $\sum_{k \neq i^{*}} w_{k}^{\rho}\left(s_{H}\right)=w_{i^{*}}(h):$
- search for $s_{H} \in\left[s_{L}, s(h)\right]\left(s>s_{H}\right.$ will "guess" impossible weights $w_{i^{*}}>w_{i^{*}}(h)$ as input to $\bar{\pi}_{i^{*}}$, because $w_{i^{*}}$ gets the balance of $s$ after subtracting the decreasing functions in $\sum_{k \neq i^{*}} w_{k}(s)$, see Lemma 17);
- run a "validation check" on the upper bound, specifically, exit this iteration of the for-loop immediately after either of the following fail (in order):

$$
\begin{aligned}
w_{i^{*}}(0) & \leq s_{H}-\sum_{k \neq i^{*}} \pi_{k}^{\rho}\left(s_{H}\right) \\
\rho_{i^{*}} & \leq \pi_{i^{*}}^{\rho}\left(s_{H}\right)=\bar{\pi}_{i^{*}}\left(s_{H}, s_{H}-\sum_{k \neq i^{*}} w_{k}^{\rho}\left(s_{H}\right)\right)
\end{aligned}
$$

(with the first line checking the rationality of the interim guess of weight $w_{i^{*}}$ and the second applying reasoning symmetric to the justification of the oracle on the lower bound);
(c) permanently fix $i^{*}=i, s_{L}, s_{H}$ and break the for-loop (if this step is reached, then the range $\left[s_{L}, s_{H}\right]$ over $s$ is non-empty and in fact it definitively contains a solution by passing the checks at both end points, which is why they are "validation" checks);
2. use the monotonicity of $\pi_{i^{*}}^{\boldsymbol{\rho}}$ to binary search on $s$ for the true $s^{*}$, converging $\pi_{i^{*}}^{\boldsymbol{\rho}}(s)$ to $\rho_{i^{*}}$;
3. when the binary search has been run to satisfactory precision and reached a final estimate $\tilde{s}$, output weights $\tilde{\boldsymbol{w}}=\left(\tilde{s}-\sum_{i \neq i^{*}} w_{i}^{\rho}(\tilde{s}), \boldsymbol{w}_{-i^{*}}^{\boldsymbol{\rho}}(\tilde{s})\right)$ which invert to values $\tilde{\boldsymbol{v}}$ via respective $v_{i}(\cdot)$ functions.

## B. 3 Proofs of Lemma 6, Lemma 7, and Theorem 8 (Algorithm Correctness)

We now prove the key lemmas claimed in the main body of the paper. The purpose of Lemma 6 is to show that if we fix the "large weight candidate agent" $i^{*}$ putting us in Space-i*, then all other agents have weights that are a precise, monotonically decreasing function of $s$. Critically, recall that we can set $i^{*}$ to be any agent, it is not restricted to be the agent (if any) who actually has more than half the weight (according to the true weights of any specific problem instance).

Lemma 6. The price level set $\mathcal{Q}_{i}^{\rho}$ is a curve; further, restricting $\mathcal{Q}_{i}^{\rho}$ to the region $w_{i} \leq s / 2$, the resulting subset $\mathcal{P}_{i}^{\boldsymbol{\rho}}$ can be written as $\left\{\left(s, w_{i}^{\boldsymbol{\rho}}(s)\right): s \in\left[r_{i}^{\boldsymbol{\rho}}, \infty\right)\right\}$ for a real-valued decreasing function $w_{i}^{\rho}$ mapping sum $s$ to a weight $w_{i}$ that is parameterized by the observed price $\rho_{i}$.

Proof. This lemma follows as a special case of Lemma 13 ,

The purpose of Lemma 7 is to prove that function $\pi_{i^{*}}^{\rho}$ is monotone increasing in $s$ within Space-i*; setting up our ability to identify end points $s_{L}$ and $s_{H}$ where we run oracle checks to identify if a solution exists between them, i.e., setting up our ability to run binary search for the unique solution $s^{*}$ in a correct space.

Lemma 7. For any agent $i^{*}$ and $s \in\left[\max _{j \neq i^{*}} r_{j}^{\rho}, \infty\right)$, function $\pi_{i^{*}}^{\rho}$ is weakly increasing; specifically, $\pi_{i^{*}}^{\rho}$ is constant when $s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s) \leq w_{i^{*}}(0)$ and strictly increasing otherwise.

Proof. The quantity $s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s)$ is monotone increasing in $s$ as every term in the negated sum is decreasing in $s$ (Lemma 13). Therefore there are two cases: the "guess" of weight $w_{i^{*}}^{g}:=$ $\max \left\{s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s), w_{i^{*}}(0)\right\}$ lies in one of two ranges that are delineated by the threshold where the increasing quantity $s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s)$ crosses the constant $w_{i^{*}}(0)$.

For small weight sums $s$ (below the threshold), the guess $w_{i^{*}}$ evaluates to $w_{i^{*}}(0)$. In this region we have

$$
\pi_{i^{*}}^{\rho}(s)=\pi_{i^{*}}\left(w_{i^{*}}(0), \boldsymbol{w}_{-i^{*}}^{\rho}(s)\right)=0
$$

because an agent with minimum weight (from value 0 ) uniquely inverts back to value of 0 ; and an agent with value 0 pays 0 , from the definition of $\pi_{i^{*}}$, see equation (2).

The remainder of this proof is devoted to showing the second case, that the function $\pi_{i^{*}}^{\rho}(s)$ is strictly increasing for large weight sums $s$ where the guess $w_{i^{*}}^{g}$ for the weight of $i^{*}$ evaluates to $s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s)$. For the following, we use the result of Lemma 3 and the definition of $h_{i}$ in equation (8). Note that when we are in Space-i ${ }^{*}$, we have $h_{k}>1$ for $k \neq i^{*}$.

$$
\begin{aligned}
\frac{d \pi_{i^{*}}^{\rho}(s)}{d s} & =\frac{d \pi_{i^{*}}\left(\left(s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s)\right), \boldsymbol{w}_{-i^{*}}^{\rho}(s)\right)}{d s} \\
& =\frac{\partial \pi_{i^{*}}}{\partial w_{i^{*}}}\left(1-\sum_{i \neq i^{*}} \frac{d w_{i}^{\rho}(s)}{d s}\right)+\frac{\partial \pi_{i^{*}}}{\partial w_{j \neq i^{*}}} \sum_{i \neq i^{*}} \frac{d w_{i}^{\rho}(s)}{d s} \\
& =\frac{\partial \pi_{i^{*}}}{\partial w_{i^{*}}}\left(1-\sum_{i \neq i^{*}} \frac{1}{1-h_{i}}+\frac{1}{h_{i^{*}}} \sum_{i \neq i^{*}} \frac{1}{1-h_{i}}\right) \\
& =\frac{\partial \pi_{i^{*}}}{\partial w_{i^{*}}}\left[1+\left(\frac{1}{h_{i^{*}}}-1\right) \sum_{i \neq i^{*}} \frac{1}{1-h_{i}}\right]
\end{aligned}
$$

In the second line here, the notation $\frac{\partial \pi_{i^{*}}}{\partial w_{j \neq i^{*}}}$ recalls that all cross-partials are the same for other agents $j$; moving from the second line to the third line, we replaced $\frac{d w_{i}^{\rho}(s)}{d s}=1 /\left(1-h_{i}\right)$ from Part 2 of Lemma 13, which also guarantees that each of these terms is strictly negative. When $h_{i^{*}} \geq 1$, the total bracketed term is positive, and $\frac{d \pi_{i^{*}}\left(\left(s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s)\right), \boldsymbol{w}_{-i^{*}}^{\rho}(s)\right)}{d s}>0$.

Alternatively to make an argument when $h_{i^{*}}<1$, we further rearrange the algebra of the
partial. Continuing from the last line:

$$
\begin{aligned}
\frac{d \pi_{i^{*}}^{\rho}(s)}{d s} & =\frac{\partial \pi_{i^{*}}}{\partial w_{i^{*}}}\left[1+\left(\frac{1}{h_{i^{*}}}-1\right) \sum_{i \neq i^{*}} \frac{1}{1-h_{i}}\right] \\
& =\frac{\partial \pi_{i^{*}}}{\partial w_{i^{*}}}\left[\frac{\frac{1}{h_{i^{*}}}-1}{\frac{1}{h_{i^{*}}}-1}+\left(\frac{1}{h_{i^{*}}}-1\right) \sum_{i \neq i^{*}} \frac{1}{1-h_{i}}\right] \\
& =\frac{\partial \pi_{i^{*}}}{\partial w_{i^{*}}}\left[\left(\frac{1}{h_{i^{*}}}-1\right)\left(\left(\sum_{i \neq i^{*}} \frac{1}{1-h_{i}}\right)+\frac{h_{i^{*}}}{1-h_{i^{*}}}+\frac{1}{1-h_{i^{*}}}-\frac{1}{1-h_{i^{*}}}\right)\right] \\
& =\frac{\partial \pi_{i^{*}}}{\partial w_{i^{*}}}\left[\left(\frac{1}{h_{i^{*}}}-1\right)\left(\left(\sum_{k} \frac{1}{1-h_{k}}\right)+\frac{h_{i^{*}}}{1-h_{i^{*}}}-\frac{1}{1-h_{i^{*}}}\right)\right] \\
& =\frac{\partial \pi_{i^{*}}}{\partial w_{i^{*}}}\left(\frac{1}{h_{i^{*}}}-1\right)\left(\sum_{k} \frac{1}{1-h_{k}}-1\right)
\end{aligned}
$$

When $h_{i^{*}}<1$, this quantity is again necessarily positive (with the last term positive by Lemma(4) 18 So again $\frac{d \pi_{i^{*}}\left(\left(s-\sum_{i \neq i^{*}} \bar{w}_{i}(s)\right), \boldsymbol{w}_{-i^{*}}^{\boldsymbol{\rho}}(s)\right)}{d s}>0$. We conclude that $\pi_{i^{*}}^{\boldsymbol{\rho}}(s):=\pi_{i^{*}}\left(\left(s-\sum_{i \neq i^{*}} w_{i}^{\boldsymbol{\rho}}(s)\right), \boldsymbol{w}_{-i^{*}}^{\rho}(s)\right)$ is strictly increasing in $s$ for this second case, i.e., the region of large $s$ for which

$$
w_{i^{*}}^{g}:=\max \left\{s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s), w_{i^{*}}(0)\right\}=s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s)
$$

Finally we argue the correctness of the algorithm. However, correctness of the technical computations in pre-processing step 0 will be delayed to Appendix B. 4 .

Theorem 8, Given weights $\boldsymbol{w}$ and payments $\boldsymbol{\rho}=\boldsymbol{\pi}(\boldsymbol{w})$ according to a proportional weights social choice function, the algorithm identifies weights $\tilde{\boldsymbol{w}}$ to within $\epsilon$ of the true weights $\boldsymbol{w}$ in time polynomial in the number of agents $n$, the logarithm of the ratio of high to low weights $\max _{i} \ln \left(w_{i}(h) / w_{i}(0)\right)$, and the logarithm of the desired precision $\ln 1 / \epsilon$.

Proof. Fix observed prices $\boldsymbol{\rho}$ that correspond to true weights $\boldsymbol{w}$ with sum $s=\sum_{i} w_{i}$. Fix an agent $i^{*}$ with $w_{i^{*}}>s / 2$ if one exists or $i^{*}=1$ if none exists. Set $s_{L}=\max _{i \neq i^{*}} r_{i}^{\rho}$, and $s_{H}$ as calculated in the algorithm for Space-i*. It must be that $\pi_{i^{*}}^{\boldsymbol{\rho}}\left(s_{L}\right) \leq \rho_{i^{*}} \leq \pi_{i^{*}}^{\boldsymbol{\rho}}\left(s_{H}\right)$. The bounds follow by $w_{i} \leq s / 2$ for all $i \neq i^{*}$, and Lemma 16 and Lemma 17 (stated and proved in the next section). Monotonicity of $\pi_{i^{*}}^{\rho}(\cdot)$ then implies binary search will identify a sum $\tilde{s}$ arbitrarily close to satisfying $\pi_{i^{*}}^{\rho}(\tilde{s})=\rho_{i^{*}}$.

By the definition of $\pi_{i}^{\boldsymbol{\rho}}(\cdot)$ and the convergence $\tilde{s} \rightarrow s$, the weights $\tilde{\boldsymbol{w}}=\boldsymbol{w}^{\boldsymbol{\rho}}(\tilde{s})$ satisfy $\boldsymbol{\pi}(\tilde{\boldsymbol{w}}) \approx \boldsymbol{\rho}$. We discuss rates of convergence below, but this follows because $w_{i \neq i^{*}}^{\rho}$ functions are decreasing in input $\tilde{s}$, so as the range of possible total weight decreases, their range of output also decreases (while still containing the solution). The range of the guess $w_{i^{*}}^{\boldsymbol{\rho}}$ for the weight of $i^{*}$ is upper-bounded by a simple additive function of the ranges of possible $s$ and $\boldsymbol{w}_{-i}$ (see Lemma 15), so it is also decreasing with each binary search iteration. By uniqueness of the inverse $\boldsymbol{\pi}^{-1}$, these weights are converging to the original weights, i.e., $\tilde{\boldsymbol{w}} \approx \boldsymbol{w}$.

[^13]In the case where $w_{i^{*}}>s / 2$, the iterative searches of Space-i for $i \neq i^{*}$ will fail as these searches only consider points ( $s, w_{i^{*}}$ ) where $w_{i^{*}}<s / 2$, but the weights $\boldsymbol{w}$ that corresponds to $\boldsymbol{\rho}$ are unique (by Theorem 5) and do not satisfy $w_{i^{*}}<s / 2$. When $w_{i^{*}} \leq s$ then all searches, in particular $i^{*}=1$, will converge to the same result of $\boldsymbol{w}$.

Lastly, we show that binary search over $s$-coordinates within Space- $i^{*}$ is sufficient to converge the algorithm's approximate $\tilde{\boldsymbol{w}}$ to $\boldsymbol{w}$ (measured by $\mathcal{L}_{1}$-norm distance) at the same assymptotic rate of the binary search on $s$, a rate which has only polynomial dependence on $n, \max _{i} \ln \left(w_{i}(h) / w_{i}(0)\right)$, and $\ln 1 / \epsilon$.

By Lemma 14 below, for each agent $k \neq i^{*}$ there is a bound $B_{k}$ on the magnitude of the slope of $\frac{\partial \lambda_{i}^{P}}{\partial s}$ as a function of the value space and weight functions inputs to the problem. $B_{k}$ depends on the factor $w_{i} / w_{i}(0) \leq w_{i}(h) / w_{i}(0)$ leading to the running time dependence.

Given a binary-search-step range on $s$ with size $S$, for every agent $k \neq i^{*}$, the size of the range containing $w_{k}$ can not be larger than $s \cdot B_{k}$. Every time the range of $s$ gets cut in half, this upper bound on the range of $w_{k}$ also gets cut in half. The convergence of $\tilde{w}_{i^{*}}$ to $w_{i^{*}}$ follows from the convergence in coordinates $s, \boldsymbol{w}_{-i^{*}}$ and Lemma 15 ,

We conclude this section with the lemmas supporting the convergence rate claims of Theorem 8 , Within the statement of Lemma 14 recall that the definition of the derivative was proved by Lemma 13 ,
Lemma 14. Given agent $i$ with $w_{i} \leq s / 2$ and function $\pi_{i}^{\rho}$, the slope $\frac{\partial \pi_{i}^{\rho}}{\partial s}=\frac{1}{1-h_{i}}<0$ has magnitude bounded by $\frac{w_{i}}{2 w_{i}(0)} \leq \frac{w_{i}(h)}{2 w_{i}(0)}$.
Proof. We will show $\left|\frac{1}{1-h_{i}}\right| \leq \frac{w_{i}}{2 w_{i}(0)}$. To upper bound $\left|\frac{1}{1-h_{i}}\right|$, we lower bound $h_{i}>1$. Note that a lower bound on $h_{i}$ will only be useful for us if it strictly separates $h_{i}$ above 1 . Substitute $s=\sum_{k} w_{k}$ into the definition of $h_{i}$ in equation (8) and bound, with justification to follow, as:

$$
\begin{aligned}
h_{i} & =\frac{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{s-w_{i}+z} \cdot\left[\frac{s}{w_{i}}-1\right] d z}{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{s-w_{i}+z} \cdot\left[\frac{s}{s-w_{i}+z}-1\right] d z} \\
& \geq \frac{\left[\frac{s}{w_{i}}-1\right]}{\left[\frac{s}{s-w_{i}+w_{i}(0)}-1\right]} \cdot \frac{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}} \cdot \frac{z}{\int_{w_{i}(0)}^{w_{i}} v_{i}^{\prime}(z) \frac{1}{w_{i}+z} \cdot \frac{z}{s-w_{i}+z} d z}}{} \\
& \geq \frac{s-w_{i}+w_{i}(0)}{w_{i}-w_{i}(0)}>1+\frac{2 w_{i}(0)}{w_{i}}>1
\end{aligned}
$$

The first inequality replaces the integrand $z$ in the bracketted term in the denominator with its constant lower bound $w_{i}(0)$ (which only decreases a denominator, in the denominator); thereafter both bracketted terms can be brought outside of their respective integrals. The second inequality replaces the numerator with 1 because $w_{i} \leq s / 2$ by statement assumption. The third (strict) inequality both replaces $s$ with $2 w_{i}$ by the same reason, and adds $w_{i}(0)$ to both numerator and denominator, which makes the fraction smaller because it was originally larger than 1 .

Using this bound we get:

$$
\left|\frac{1}{1-h_{i}}\right| \leq\left|\frac{1}{1-\left(1+\frac{2 w_{i}(0)}{w_{i}}\right)}\right|=\frac{w_{i}}{2 w_{i}(0)} \leq \frac{w_{i}(h)}{2 w_{i}(0)}
$$

Lemma 15. Given agent $i^{*}$, true $s^{*} \in\left[s^{-}, s^{+}\right]$, and true weights $w_{k} \in\left[w_{k}^{-}, w_{k}^{+}\right]$for agents $k \neq i^{*}$, which induce the range for $i^{*}$, weight of $w_{i^{*}} \in\left[s^{-}-\sum_{k \neq i^{*}} w_{k}^{+}, s^{+}-\sum_{k \neq i^{*}} w_{k}^{-}\right]$. If the sizes of the ranges $\left[s^{-}, s^{+}\right]$and $\left[w_{k}^{-}, w_{k}^{+}\right]$are each individually reduced by (at least) a constant factor $\alpha$, then the size of the range of $w_{i^{*}}$ is also reduced by (at least) $\alpha$.

Proof. The statement follows immediately from the induced range of $w_{i^{*}}$. Its size is exactly equal to the sum of the $n$ other ranges, i.e.,

$$
\left|\left[s^{-}-\sum_{k \neq i^{*}} w_{k}^{+}, s^{+}-\sum_{k \neq i^{*}} w_{k}^{-}\right]\right|=\left|\left[s^{-}, s^{+}\right]\right|+\sum_{k \neq i^{*}}\left|\left[w_{k}^{-}, w_{k}^{+}\right]\right|
$$

## B. 4 Correctness of Algorithm Search End Points as Oracle Checks

This section has four purposes:

- analyze the structure of $r_{i}^{\rho}$ corresponding to level set $\mathcal{Q}_{i}^{\rho}$;
- prove the correctness and run-time of the pre-process step 0 of the algorithm, which precomputes $r_{i}^{\rho}$ for all $i$;
- conclude that the lower bounds $s_{L}$ of search in any given Space-i, determined within each iteration of step 1 of the algorithm, are the correct lower bounds of feasibility;
- conclude that the upper bounds $s_{H}$ calculated within each iteration of step 1 are the correct upper bounds of feasibility.

For strictly positive observed payment $\rho_{i}>0$, the level set $\mathcal{Q}_{i}^{\rho}$ takes on the full range of weights $w_{i} \in\left(w_{i}\left(\rho_{i}\right), w_{i}(h)\right]$ (the lower bound of $w_{i}\left(\rho_{i}\right)$ will not play an important role, our algorithms will use the less restrictive bound of $w_{i}(0)$ instead). Our search for the minimum $s$-coordinate of $\mathcal{P}_{i}^{\boldsymbol{\rho}}$, i.e., $r_{i}^{\boldsymbol{\rho}}$, which is the intersection of $\mathcal{Q}_{i}^{\boldsymbol{\rho}}$ with the points below the $w_{i}=s / 2$ line is either on the $w_{i}=s / 2$ boundary or on the $w_{i}=w_{i}(h)$ boundary. This follows because constrained to $w_{i} \leq s / 2$ the level set is given by a decreasing function (Lemma 6) and all level sets extend to $s=\infty$ (this second fact is true, but will not need to be explicitly proven). The two cases are depicted in Figure 4 . For convenience, we restate the preprocessing step of the algorithm:
0. Pre-process: For each $i$, compute $r_{i}^{\rho}$ :
(a) (general case: $\mathcal{P}_{i}^{\boldsymbol{\rho}} \neq \mathcal{Q}_{i}^{\boldsymbol{\rho}}$ ) if $\bar{\pi}_{i}\left(2 w_{i}(h), w_{i}(h)\right) \geq \rho_{i}$, run binary search "diagonally" on the line segment of $w_{i}=s / 2$ between $(0,0)$ and $\left(2 w_{i}(h), w_{i}(h)\right)$ to find an element of $\mathcal{Q}_{i}^{\rho}$ and use its $s$ coordinate as $r_{i}^{\rho}$ (which we can do because $\bar{\pi}_{i}(\cdot)$ is strictly increasing on this domain);
(b) (edge case: $\mathcal{P}_{i}^{\boldsymbol{\rho}}=\mathcal{Q}_{i}^{\boldsymbol{\rho}}$ ) otherwise, fix $w_{i}$ coordinate to its maximum $w_{i}(h)$ and run binary search "horizontally" to find $\hat{s} \in\left[2 w_{i}(h), s(h)\right]$ representing $\left(\hat{s}, w_{i}(h)\right) \in \mathcal{Q}_{i}^{\rho}$ (which we can do because $\bar{\pi}_{i}(\cdot)$ is strictly increasing in $s$ for constant $\left.w_{i}\right)$; set minimum total weight $r_{i}^{\rho}=\hat{s}$.

There is an intuitive explanation to the order of operations in the pre-processing step 0 . First we check if we are in the general case. We can do this because price level-sets are strictly increasing


Figure 4: The cases for the initialization of lower bound $r_{i}^{\rho}$ are depicted. When $\rho_{i}=0$ both of the corresponding price level sets $\mathcal{P}_{i}$ and $\mathcal{Q}_{i}$ are on the line $w_{i}=w_{i}(0)$ (depicted, but not labeled). For observed price $\rho_{i} \leq \bar{\pi}_{i}\left(2 w_{i}(h), w_{i}(h)\right)$ the intermediate level sets look like the depicted $\mathcal{P}_{i}^{\rho} \neq \mathcal{Q}_{i}^{\rho}$, and $r_{i}^{\boldsymbol{\rho}}$ corresponds to the $s$-coordinate at the intersection with the $w_{i}=s / 2$ line. For observed price $\rho_{i} \geq \bar{\pi}_{i}\left(2 w_{i}(h), w_{i}(h)\right)$ the high level sets look like the depicted $\mathcal{P}_{i}^{\rho^{\prime}}=\mathcal{Q}_{i}^{\rho^{\prime}}$, and $r_{i}^{\rho^{\prime}}$ corresponds to the $s$-coordinate at the intersection with the $w_{i}=w_{i}(h)$ line.
on the line $w_{i}=s / 2$ (see Lemma 16 below, extending Lemma 13). So we can check the largest the price at the largest possible point as $\bar{\pi}_{i}\left(2 w_{i}(h), w_{i}(h)\right)$; if it is too big, we can run binary search down to $\bar{\pi}_{i}\left(2 w_{i}(0), w_{i}(0)\right)=0$; otherwise we are in the edge case where $r_{i}^{\boldsymbol{\rho}}$ corresponds to $w_{i}(h)$. In this case, we can binary search the line $w_{i}=w_{i}(h)$ for the point with payment $\rho_{i}$ as, again, price level-sets are strictly increasing (Lemma 13). The formal proof is given as Lemma 16 ,

Lemma 16. For any realizable payment $\rho_{i}$, price level set $\mathcal{P}_{i}^{\boldsymbol{\rho}}$ is non-empty and its $s$-coordinates are lower bounded by $r_{i}^{\rho}$ which can be computed to arbitrary precision by a binary search.

Proof. As mentioned previously, denote the maximum sum of weights possible by $s(h)=\sum_{i} w_{i}(h)$. To find $r_{i}^{\boldsymbol{\rho}}$, we first focus attention on the horizontal line with constant weight $w_{i}(h)$.

A point $\left(\hat{s}, w_{i}(h)\right)$ on price level set $\mathcal{Q}_{i}^{\rho}$, i.e., with $\bar{\pi}_{i}\left(\hat{s}, w_{i}(h)\right)=\rho_{i}$, can be found to arbitrary precision with binary search over $s \in\left(w_{i}(h), s(h)\right]$. Correctness of this binary search follows because a realizable payment $\rho_{i}$ must satisfy $0=\bar{\pi}_{i}\left(w_{i}(h), w_{i}(h)\right) \leq \rho_{i} \leq \bar{\pi}_{i}\left(s(h), w_{i}(h)\right)$ and because increasing $s$-coordinate corresponds to increasing price-level set on any line with fixed weight $w_{i}$ by Lemma 13. For the lower bound on the range, an agent wins with certainty and makes no payment when the sum of the other agent weights is zero; the upper bound is from the natural upper bound $s \leq s(h)$.

There are now two cases depending on whether this point $\left(\hat{s}, w_{i}(h)\right)$ is above or below the $w_{i}=s / 2$ line 19 If below, then $r_{i}^{\rho}=\hat{s}$ because this point is tight to the maximum weight $w_{i}(h)$ (see Figure (4), and (again by Lemma 13) the slope of curve $\mathcal{P}_{i}^{\boldsymbol{\rho}}$ is strictly negative and all smaller $s$ are infeasible.

Alternatively suppose $\left(w_{i}(h), \hat{s}\right)$ is above the $w_{i}=s / 2$ line, then $r_{i}^{\boldsymbol{\rho}}$ can be found by searching the $w_{i}=s / 2$ line. Part (1) of Lemma 13 guarantees that points on this line are consistent with unique and increasing observed prices (partials of the price function are strictly positive in both dimensions

[^14]$w_{i}$ and $s$, we can first move right $d s$, and then move up $d w_{i}$, with the price function strictly increasing as a result of both "moves"). On this line we have $0=\bar{\pi}_{i}\left(2 w_{i}(0), w_{i}(0)\right) \leq \rho_{i} \leq \bar{\pi}_{i}\left(2 w_{i}(h), w_{i}(h)\right)$ where the lower bound observes an agent with value 0 to always pay 0 , and the upper bound follows from the supposition $w_{i}(h) \geq \hat{s} / 2$ of this case. Thus, a binary search of the $w_{i}=s / 2$ line with $w_{i} \in\left[w_{i}(0), w_{i}(h)\right]$ is guaranteed to find a point with price arbitrarily close to $\rho_{i}$. Since $\mathcal{P}_{i}^{\boldsymbol{\rho}}$ as a curve is decreasing in $s$, the identified point, which is in $\mathcal{P}_{i}^{\rho}$, has the minimum $s$-coordinate.

The two cases are exhaustive and so $r_{i}^{\boldsymbol{\rho}}$ is identified and $\mathcal{P}_{i}^{\boldsymbol{\rho}}$ is non-empty.
We finish the section with the lemma showing the correctness of the search range of sum $s$ within $\left[s_{L}, s_{H}\right]$.

Lemma 17. For true weights $\boldsymbol{w}$, true weight $\operatorname{sum} s=\sum w_{i}$, and $i^{*}$ with $w_{i} \leq s / 2$ for $i \neq i^{*}$, sum $s$ is contained in interval $\left[s_{L}, s_{H}\right]$ (defined in step 1 of the algorithm for $i^{*}$ ).

Proof. First for the lower bound $s_{L}$, the assumption of the lemma requires $i \neq i^{*}$ satisfy $w_{i} \leq s / 2$. Therefore the true pair $\left(s, w_{i}\right)$ must be a point in $\mathcal{P}_{i}^{\boldsymbol{\rho}}$, and the true sum $s$ must be at least the lower bound $r_{i}^{\boldsymbol{\rho}}$ for each $i \neq i^{*}$.

Second for the upper bound $s_{H}$, recall the definition

$$
\pi_{i^{*}}^{\boldsymbol{\rho}}(s):=\pi_{i^{*}}\left(\max \left\{s-\sum_{i \neq i^{*}} w_{i}^{\boldsymbol{\rho}}(s), w_{i^{*}}(0)\right\}, \boldsymbol{w}_{-i^{*}}^{\boldsymbol{\rho}}(s)\right)
$$

which uses a guess at the total weights $\tilde{s}$ to guess the corresponding the weight of agent $i^{*}$ as $\left(\tilde{s}-\sum_{i \neq i^{*}} w_{i}^{\rho}(\tilde{s})\right)$. In fact, this guessed weight is strictly increasing in $\tilde{s}$ as each term in the negated sum is strictly decreasing (Lemma [13). Our choice of $s_{H}$ equates this guessed weight with its highest possible value $w_{i^{*}}(h)$. By monotonicity of the guessed weight the true $s$ must be at most $s_{H}$.


[^0]:    *Arxiv: https://arxiv.org/abs/1902.06908
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[^1]:    ${ }^{1}$ Our methods are written assuming that per-unit prices are observed rather than expected payments. These prices are more natural for mechanisms usually considered in algorithmic mechanism design as they arise in mechanisms where losers pay nothing, i.e., ex post individually rational mechanism. If instead the realized expected payments and realized allocation probabilities are observed, then these per-unit prices can be easily calculated and our methods applied to the result.

[^2]:    ${ }^{2}$ For simplicity, we assume that all weights functions are everywhere strictly positive for all agents, even at $v_{i}=0$.
    ${ }^{3}$ The full importance of the proxy game construction is realized when "erroneous" prices are used as inputs, as the proxy game is still defined with action space corresponding to weights space, is still continuous, and will still have a unique pure Nash equilibrium which can be output.

[^3]:    ${ }^{4}$ Further supporting results given in Appendix A. 1 are also from Gale and Nikaido (1965).
    ${ }^{5}$ We make frequent use of input space $\Omega$ in this paper as a compact and convex product space. Unless noted specifically otherwise, we let the dimension-wise ranges be $\Omega=\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right] \times \cdots \times\left[a_{n}, b_{n}\right]$ as in this definition.

[^4]:    ${ }^{6}$ Technically the $h_{i}$ terms are functions，each of input $\boldsymbol{w}$ ，but we suppress this in the notation．

[^5]:    ${ }^{7}$ See equation (21) in Appendix B for its formal definition.
    ${ }^{8}$ Regarding functions $w_{i}^{\rho}$ and $\pi_{i^{*}}^{\rho}$. We write them both parameterized by vector $\rho$ to demark them with a simple notation, in a common way because their usage is always related. Note however, $\boldsymbol{\rho}$ implies an over-dependence on parameters. $w_{i}^{\boldsymbol{\rho}}$ only uses $\rho_{i}$, and $\pi_{i^{*}}^{\rho}$ uses all of $\boldsymbol{\rho}_{-i^{*}}$ but not $\rho_{i^{*}}$.
    ${ }^{9}$ When the guess $w_{i^{*}}=s-\sum_{i \neq i^{*}} w_{i}^{\rho}(s)$ is irrationally small or even negative, the structure of the problem allows us to round it up to constant $w_{i^{*}}(0)$, sufficiently preserving monotonicity. See the proof of Lemma 7 in Appendix B.3,

[^6]:    ${ }^{10}$ In the proper algorithm and proof, we will give better bounds on the range of the search; for now, as a simple indication that bounds exist, note that there exists a solution for some appropriate $i^{*}$ within the general bounds on $s$ as $\sum_{k} w_{k}\left(\rho_{k}\right) \leq \sum_{k} w_{k}\left(v_{k}\right)=s \leq \sum_{k} w_{k}(h)$ for known $\rho_{i}$ and max value $h$, because $\rho_{i} \leq v_{i} \leq h$.

[^7]:    ${ }^{11}$ An organizational note on numbering of theorems: our Theorem 2 is given as Theorem 4 in the Gale and Nikaido paper; our Theorem 10 is their Theorem 3. Our Theorem 11 is their result but is not an explicit statement.

[^8]:    ${ }^{12}$ Of course, it is a well-known property of positive definite matrices $G$ that all diagonal elements must be strictly positive, otherwise they have $\mathbf{z}^{\top} G \mathbf{z} \leq 0$ with a simple counter-example $\mathbf{z}$ described by all zeroes except -1 in the index of the matrix's non-positive diagonal element.

[^9]:    ${ }^{13}$ Note that line (16) is not a definition for $z_{j}^{*}$, which appears on both sides of the equation. The goal is substitution of $z_{j}^{*}$ from necessary first-order conditions, not to define it.

[^10]:    ${ }^{14}$ Note that because all the cross-derivatives are the same, it is without loss of generality that we assume that changes $\partial s$ are entirely attributable to one other particular agent $j \neq i$ as $\partial w_{j}$.

[^11]:    ${ }^{15}$ We can not assume that the set $\mathcal{P}_{i}^{\rho}$ is non-empty without proof. We prove that it is non-empty in Lemma 16.
    ${ }^{16}$ Further discussion will be given in Appendix B. 4 where we show that $r_{i}^{\rho}$ can be computed via a binary search, between starting lower and upper bounds which are easy to find.

[^12]:    ${ }^{17}$ See Appendix B. 4 for further explanation.

[^13]:    ${ }^{18}$ The significance of Lemma 4 here was discussed after the original statement of Lemma 7 at the end of Section 4.1 , and after Lemma 13 at the end of Appendix B. 1 .

[^14]:    ${ }^{19}$ If on the line, the cases are equal and either suffices.

