# Improved hardness for $H$-colourings of $G$-colourable graphs* 

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We present new results on approximate colourings of graphs and, more generally, approximate $H$-colourings and promise constraint satisfaction problems.

First, we show NP-hardness of colouring $k$-colourable graphs with $\binom{k}{\lfloor k / 2\rfloor}-1$ colours for every $k \geq 4$. This improves the result of Bulín, Krokhin, and Opršal [STOC'19], who gave NP-hardness of colouring $k$-colourable graphs with $2 k-1$ colours for $k \geq 3$, and the result of Huang [APPROX-RANDOM'13], who gave NP-hardness of colouring $k$-colourable graphs with $2^{\Omega\left(k^{1 / 3}\right)}$ colours for sufficiently large $k$. Thus, for $k \geq 4$, we improve from known linear/subexponential gaps to exponential gaps.

Second, we show that the topology of the box complex of $H$ alone determines whether $H$-colouring of $G$-colourable graphs is NP-hard for all (non-bipartite, $H$-colourable) $G$. This formalises the topological intuition behind the result of Krokhin and Opršal [FOCS'19] that 3-colouring of $G$-colourable graphs is NP-hard for all (3-colourable, non-bipartite) $G$. We use this technique to establish NP-hardness of $H$-colouring of $G$-colourable graphs for $H$ that include but go beyond $K_{3}$, including square-free graphs and circular cliques (leaving $K_{4}$ and larger cliques open).

Underlying all of our proofs is a very general observation that adjoint functors give reductions between promise constraint satisfaction problems.

## 1. Introduction

Graph colouring is one of the most fundamental and studied problems in combinatorics and computer science. A graph $G$ is called $k$-colourable if there is an assignment of colours $\{1,2, \ldots, k\}$ to the vertices of $G$ so that any two adjacent vertices are assigned different colours. The chromatic number of $G$, denoted by $\chi(G)$, is the smallest integer $k$ for which $G$ is $k$-colourable. Deciding whether $\chi(G) \leq k$ appeared on Karp's original list of 21 NP-complete problems [Kar72], and is NP-hard for every $k \geq 3$. In particular,

[^0]it is NP-hard to decide whether $\chi(G) \leq 3$ or $\chi(G)>3$. Put differently (thanks to self-reducibility of graph colouring), it is NP-hard to find a 3-colouring of $G$ even if $G$ is promised to be 3-colourable.

In the approximate graph colouring problem, we are allowed to use more colours than needed. For instance, given a 3 -colourable graph $G$ on $n$ vertices, can we find a colouring of $G$ using significantly fewer than $n$ colours? On the positive side, the currently best polynomial-time algorithm of Kawarabayashi and Thorup [KT17] finds a colouring using $O\left(n^{0.19996}\right)$ colours. Their work continues a long line of research and is based on a semidefinite relaxation. On the negative side, it is believed that finding a $c$-colouring of a $k$-colourable graph is NP-hard for all constants $3 \leq k \leq c$. Already in this regime (let alone for non-constant $c$ ) our understanding remains rather limited, despite lots of work and the development of complex techniques, as we will survey in Section 1.1.

A natural and studied generalisation of graph colourings is that of graph homomorphisms and, more generally, constraint satisfaction problems [HN08].

Given two graphs $G$ and $H$, a map $h: V(G) \rightarrow V(H)$ is a homomorphism from $G$ to $H$ if $h$ preserves edges; that is, if $\{h(u), h(v)\} \in E(H)$ whenever $\{u, v\} \in E(G)$ [HN04]. A celebrated result of Hell and Nešetřil established a dichotomy for the homomorphism problem with a fixed target graph $H$, also known as the $H$-colouring problem: deciding whether an input graph $G$ has a homomorphism to $H$ is solvable in polynomial time if $H$ is bipartite or if $H$ has a loop; for all other $H$ this problem is NP-hard [HN90]. Note that the $H$-colouring problem for $H=K_{k}$, the complete graph on $k$ vertices, is precisely the graph colouring problem with $k$ colours.

The constraint satisfaction problem (CSP) is a generalisation of the graph homomorphism problem from graphs to arbitrary relational structures. One type of CSP that has attracted a lot of attention is the one with a fixed target structure, also known as the non-uniform CSP; see, e.g., the work of Jeavons, Cohen, and Gyssens [JCG97], Bulatov [Bul06; Bul11], and Barto and Kozik [BK14; BK16]. Following the above mentioned dichotomy of Hell and Nešetřil for the $H$-colouring [HN90] and a dichotomy result of Schaefer for Boolean CSPs [Sch78], Feder and Vardi famously conjectured a dichotomy for all non-uniform CSPs [FV98]. The Feder-Vardi conjecture was recently confirmed independently by Bulatov [Bul17] and Zhuk [Zhu17]. In fact, both proofs establish the so-called "algebraic dichotomy", conjectured by Bulatov, Jeavons, and Krokhin [BJK05], which delineates the tractability boundary in algebraic terms. A high-level idea of the tractability boundary is that of higher-order symmetries, called polymorphisms, which allow to combine several solutions to a CSP instance into a new solution. The lack of non-trivial ${ }^{1}$ polymorphisms guarantees NP-hardness, as shown already in [BJK05]. The work of Bulatov and Zhuk show that any non-trivial polymorphism guarantees tractability. We refer the reader to a recent accessible survey by Barto, Krokhin, and Willard on the algebraic approach to CSPs [BKW17].

Given two graphs $G$ an $H$ such that $G$ is $H$-colourable (i.e., there is a homomorphism from $G$ to $H$ ), the promise constraint satisfaction problem parametrised by $G$ and $H$, denoted by $\operatorname{PCSP}(G, H)$, is the following computational problem: given a $G$-colourable graph, find an $H$-colouring of this graph. ${ }^{2}$ More generally, $G$ and $H$ do not have to be

[^1]graphs but arbitrary relational structures. Note that if $G=H$ then we obtain the (search version of the) standard $H$-colouring and constraint satisfaction problem.

PCSPs have been studied as early as in the classic work of Garey and Johnson [GJ76] on approximate graph colouring but a systematic study originated in the paper of Austrin, Guruswami, and Håstad [AGH17], who studied a promise version of $(2 k+1)$-SAT, called $(2+\epsilon)$-SAT. In a series of papers [BG16; BG18; BG19], Brakensiek and Guruswami linked PCSPs to the universal-algebraic methods developed for the study of non-uniform CSPs [BKW17]. In particular, the notion of weak polymorphisms, identified in [AGH17], allowed for some ideas developed for CSPs to be be used in the context of PCSPs. The algebraic theory of PCSPs was then lifted to an abstract level by Bulín, Krokhin, and Opršal in [BKO19]. Consequently, this theory was used by Ficak, Kozik, Olšák, and Stankiewicz to obtain a dichotomy for symmetric Boolean PCSPs [Fic+19], thus improving on an earlier result from [BG18], which gave a dichotomy for symmetric Boolean PCSP with folding (negations allowed).

### 1.1. Prior and related work

While the NP-hardness of finding a 3 -colouring of a 3 -colourable graph was obtained by Karp [Kar72] in 1972, the NP-hardness of finding a 4 -colouring of a 3 -colourable graph was only proved in 2000 by Khanna, Linial, and Safra [KLS00] (see also the work of Guruswami and Khanna for a different proof [GK04]). This result implied NP-hardness of finding a ( $k+2\lfloor k / 3\rfloor-1$ )-colouring of a $k$-colourable graph for $k \geq 3$ [KLS00]. Early work of Garey and Johnson established NP-hardness of finding a $(2 k-5)$-colouring of a $k$-colourable graph for $k \geq 6$ [GJ76]. In 2016, Brakensiek and Guruswami proved NP-hardness of a $(2 k-2)$-colouring of a $k$-colourable graph for $k \geq 3$ [BG16]. Only very recently, Bulín, Krokhin, and Opršal showed that finding a 5 -colouring of a 3 -colourable graph, and more generally, finding a $(2 k-1)$-colouring of a $k$-colourable graph for any $k \geq 3$, is NP-hard [BKO19].
In 2001, Khot gave an asymptotic result - he showed that for sufficiently large $k$, finding a $k^{\frac{1}{25}(\log k)}$-colouring of a $k$-colourable graph is NP-hard [Kho01]. In 2013, Huang improved the gap. For sufficiently large $k$, he showed that finding a $2^{\Omega\left(k^{1 / 3}\right)}$-colouring of a $k$-colourable graph is NP-hard [Hua13].

The NP-hardness of colouring ( $k$-colourable graphs) with ( $2 k-1$ ) colours for $k \geq 3$ from [BKO19] and with $2^{\Omega\left(k^{1 / 3}\right)}$ colours for sufficiently large $k$ from [Hua13] constitute the currently strongest known NP-hardness results for approximate graph colouring.

Under stronger assumptions (Khot's 2-to-1 Conjecture [Kho02] for $k \geq 4$ and its nonstandard variant for $k=3$ ), Dinur, Mossel, and Regev showed that finding a $c$-colouring of a $k$-colourable graph is NP-hard for all constants $3 \leq k \leq c$ [DMR09] A variant of Khot's 2-to-1 Conjecture with imperfect completeness has recently been proved [Din+18; KMS18], which implies hardness for approximate colouring variants where most but not all of the graph is guaranteed to be $k$-colourable.

Hypergraphs colourings, a special case of PCSPs, is another line of work intensively studied. A $k$-colouring of a hypergraph is an assignment of colours $\{1,2, \ldots, k\}$ to its

[^2]vertices that leaves no hyperedge monochromatic. Dinur, Regev, and Smyth showed that for any constants $2 \leq k \leq c$, it is NP-hard to find a $c$-colouring of given 3-uniform $k$-colourable hypergraph [DRS05]. Other notions of colourings (such as different types of rainbow colourings) for hypergraphs were studied by Brakensiek and Guruswami [BG16; BG17], Guruswami and Lee [GL18], and Austrin, Bhangale, and Potukuchi [ABP20].

Some results are also known for colourings with a super-constant number of colours. For graphs, conditional hardness was obtained by Dinur and Shinkar [DS10]. For hypergraphs, NP-hardness results were obtained in recent work of Bhangale [Bha18] and Austrin, Bhangale, and Potukuchi [ABP19].

## 2. Results

For two graphs or digraphs $G, H$, we write $G \rightarrow H$ if there exists a homomorphism from $G$ to $H .{ }^{3}$ We are interested in the following computational problem.

Definition 2.1. Fix two graphs $G$ and $H$ with $G \rightarrow H$. The (decision variant of the) $\operatorname{PCSP}(G, H)$ is, given an input graph $I$, output YES if $I \rightarrow G$, and NO if $I \nrightarrow H$.

To state our results it will be convenient to use the following definition.
Definition 2.2. A graph $H$ is left-hard if for every non-bipartite graph $G$ with $G \rightarrow H$, $\operatorname{PCSP}(G, H)$ is NP-hard. A graph $G$ is right-hard if for every loop-less graph $H$ with $G \rightarrow H, \operatorname{PCSP}(G, H)$ is NP-hard.

If $G \rightarrow G^{\prime}$ and $H^{\prime} \rightarrow H$, then $\operatorname{PCSP}(G, H)$ trivially reduces to $\operatorname{PCSP}\left(G^{\prime}, H^{\prime}\right)$ (this is called homomorphic relaxation [BKO19]; intuitively, increasing the promise gap makes the problem easier). Therefore, if $H$ is a left-hard graph, then all graphs left of $H$ (that is, $H^{\prime}$ such that $\left.H^{\prime} \rightarrow H\right)$ are trivially left-hard. ${ }^{4}$ If $G$ is right-hard, then all graphs right of $G$ are right-hard.

For the same reason, since every non-bipartite graph admits a homomorphism from an odd cycle, to show that $H$ is left-hard it suffices to show that $\operatorname{PCSP}\left(C_{n}, H\right)$ is NP-hard for arbitrarily large odd $n$, where $C_{n}$ denotes the cycle on $n$ vertices. Dually, since every loop-less graph admits a homomorphism to a clique, to show that $G$ is right-hard it suffices to show that $\operatorname{PCSP}\left(G, K_{k}\right)$ is NP-hard for arbitrarily large $k$.

It is conjectured that all non-trivial PCSPs for (undirected) graphs are NP-hard, greatly extending Hell and Nešetřil's theorem:

Conjecture 2.3 (Brakensiek and Guruswami [BG18]). $\operatorname{PCSP}(G, H)$ is NP-hard for every non-bipartite loop-less $G, H$. Equivalently, every loop-less graph is left-hard. Equivalently, every non-bipartite graph is right-hard.

In addition to the results on classical colourings discussed above (the case where $G$ and $H$ are cliques), the following result was recently obtained in a novel application of topological ideas.

Theorem 2.4 (Krokhin and Opršal [KO19]). $K_{3}$ is left-hard.

[^3]
### 2.1. Improved hardness of classical colouring

In Section 3, we focus on right-hardness. We use a simple construction called the arc digraph or line digraph, which decreases the chromatic number of a graph in a controlled way. The construction allows to conclude the following, in a surprisingly simple way:

Proposition 2.5. There exists a right-hard graph if and only if $K_{4}$ is right-hard. ${ }^{5}$
More concretely, we show in particular that $\operatorname{PCSP}\left(K_{6}, K_{2^{k}}\right) \log$-space reduces to $\operatorname{PCSP}\left(K_{4}, K_{k}\right)$, for all $k \geq 4$. This contrasts with [Bar+19, Proposition 10.3], ${ }^{6}$ where it is shown to be impossible to obtain such a reduction with minion homomorphisms: an algebraic reduction, described briefly in Section 4.3, central to the framework of [BKO19; Bar +19 ] (in particular, there exists a $k$ such that $\operatorname{PCSP}\left(K_{4}, K_{k}\right)$ admits no minion homomorphism to any $\operatorname{PCSP}\left(K_{n^{\prime}}, K_{k^{\prime}}\right)$ for $\left.4<n^{\prime} \leq k^{\prime}\right)$.

Furthermore, we strengthen the best known asymptotic hardness: Huang [Hua13] showed that for all sufficiently large $n, \operatorname{PCSP}\left(K_{n}, K_{2^{n^{1 / 3}}}\right)$ is NP-hard. We improve this in two ways, using Huang's result as a black-box. First, we improve the asymptotics from sub-exponential $2^{n^{1 / 3}}$ to single-exponential $\binom{n}{\lfloor n / 2\rfloor} \sim \frac{2^{n}}{\sqrt{\pi n / 2}}$. Second, we show the claim holds for $n$ as low as 4 .

Theorem 2.6 (Main Result \#1). For all $n \geq 4, \operatorname{PCSP}\left(K_{n}, K_{\binom{n}{n / 2\rfloor}-1}\right)$ is NP-hard.
In comparison, the previous best result relevant for all integers $n$ was proved by Bulín, Krokhin, and Opršal [BKO19]: $\operatorname{PCSP}\left(K_{n}, K_{2 n-1}\right)$ is NP-hard for all $n \geq 3$. For $n=3$ we are unable to obtain any results; for $n=4$ the new bound $\binom{n}{\lfloor n / 2\rfloor}-1=5$ is worse than $2 n-1=7$, while for $n=5$ the two bounds coincide at 9 . However, already for $n=6$ we improve the bound from $2 n-1=11$ to $\binom{n}{\lfloor n / 2\rfloor}-1=19$.

### 2.2. Left-hardness and topology

In Section 4, we focus on left-hardness. The main idea behind Krokhin and Opršal's [KO19] proof that $K_{3}$ is left-hard is simple to state. To prove that $\operatorname{PCSP}\left(C_{n}, H\right)$ is NP-hard for all odd $n$, the algebraic framework of [BKO19] shows that it is sufficient to establish certain properties of polymorphisms: homomorphisms $f: C_{n}^{L} \rightarrow H$ for $L \in \mathbb{N}$ (where $G^{L}=$ $G \times \cdots \times G$ is the $L$-fold tensor product ${ }^{7}$ ). For large $n$ the graph $C_{n}^{L}$ looks like an $L$-torus: an $L$-fold product of circles, so the pertinent information about $f$ seems to be subsumed by its topological properties (such as winding numbers, when $H$ is a cycle). We refer to [KO19] for further details, but this general principle applies to any $H$ and in fact we prove (in Theorem 2.7 below) that whether $H$ is left-hard or not depends only on its topology.

The topology we associate with a graph is its box complex. See Appendix A for formal definitions and statements. Intuitively, the box complex $|\operatorname{Box}(H)|$ is a topological space built from $H$ by taking the tensor product $H \times K_{2}$ and then gluing faces to each four-cycle

[^4]and more generally, gluing higher-dimensional faces to complete bipartite subgraphs. The added faces ensure that the box complex of a product of graphs is the same as the product space of their box complexes: thanks to this, $\left|\operatorname{Box}\left(C_{n}^{L}\right)\right|$ is indeed equivalent to the $L$-torus. The product with $K_{2}$ equips the box complex with a symmetry that swaps the two sides of $H \times K_{2}$. This make the resulting space a $\mathbb{Z}_{2}$-space: a topological space together with a continuous involution from the space to itself, which we denote simply as - . $\mathbb{Z}_{2}$-map between two $\mathbb{Z}_{2}$-spaces is a continuous function which preserves this symmetry: $f(-x)=-f(x)$. This allows to concisely state that a given map is "non-trivial" (in contrast, there is always some continuous function from one space to another: just map everything to a single point). The main use of the box complex is then the statement that every graph homomorphism $G \rightarrow H$ induces a $\mathbb{Z}_{2}$-map from $|\operatorname{Box}(G)|$ to $|\operatorname{Box}(H)|$. Graph homomorphisms can thus be studied with tools from algebraic topology.

The classical example of this is an application of the Borsuk-Ulam theorem: there is no $\mathbb{Z}_{2}$-map from $\mathcal{S}^{n}$ to $\mathcal{S}^{m}$ for $n>m$, where $\mathcal{S}^{n}$ denotes the $n$-dimensional sphere with antipodal symmetry. Hence if $G$ and $H$ are graphs such that $|\operatorname{Box}(G)|$ and $|\operatorname{Box}(H)|$ are equivalent to $\mathcal{S}^{n}$ and $\mathcal{S}^{m}$, respectively, then there can be no graph homomorphism $G \rightarrow H$. See Figure 1.

This is essentially the idea in Lovász' proof [Lov78] of Kneser's conjecture that the chromatic number of Kneser graphs $K G(n, k)$ is $n-2 k+2$. In the language of box complexes, the proof amounts to showing that the box complex of a clique $K_{c}$ is equivalent to $\mathcal{S}^{c-2}$, while the box complex of a Kneser graph contains $\mathcal{S}^{n-2 k}$. We refer to [Mat08] for an in-depth, yet accessible reference.

We show that the left-hardness of a graph depends only on the topology of its box complex (in fact, it is only important what $\mathbb{Z}_{2}$-maps it admits, which is significantly coarser than $\mathbb{Z}_{2}$-homotopy equivalence):
Theorem 2.7 (Main Result \#2). If $H$ is left-hard and $H^{\prime}$ is a graph such that $\left|\operatorname{Box}\left(H^{\prime}\right)\right|$ admits a $\mathbb{Z}_{2}$-map to $|\operatorname{Box}(H)|$, then $H^{\prime}$ is left-hard.

Using Krokhin and Opršal's result that $K_{3}$ is left-hard (Theorem 2.4), since $\left|\operatorname{Box}\left(K_{3}\right)\right|$ is the circle $\mathcal{S}^{1}$ (up to $\mathbb{Z}_{2}$-homotopy equivalence), we immediately obtain the following:
Corollary 2.8. Every graph $H$ for which $|\operatorname{Box}(H)|$ admits a $\mathbb{Z}_{2}$-map to $\mathcal{S}^{1}$ is left-hard.
Two examples of such graphs (other than 3-colourable graphs) are loop-less square-free graphs and circular cliques $K_{p / q}$ with $2<\frac{p}{q}<4$ (see Lemma A. 1 for proofs), which we introduce next. Square-free graphs are graphs with no cycle of length exactly 4 . In particular, this includes all graphs of girth at least 5 and hence graphs of arbitrarily high chromatic number (but incomparable to $K_{4}$ and larger cliques, in terms of the homomorphism $\rightarrow$ relation). The circular clique $K_{p / q}$ (for $p, q \in \mathbb{N}, \frac{p}{q}>2$ ) is the graph with vertex set $\mathbb{Z}_{p}$ and an edge from $i$ to every integer at least $q$ apart: $i+q, i+q+$ $1, \ldots, i+p-q$. They generalise cliques $K_{n}=K_{n / 1}$ and odd cycles $C_{2 n+1} \simeq K_{(2 k+1) / k}$. Their basic property is that $K_{p / q} \rightarrow K_{p^{\prime} / q^{\prime}}$ if and only if $\frac{p}{q} \leq \frac{p^{\prime}}{q^{\prime}}$. Thus circular cliques refine the chain of cliques and odd cycles, corresponding to rational numbers between integers. For example:

$$
\cdots \rightarrow C_{7} \rightarrow C_{5} \rightarrow C_{3}=K_{3} \rightarrow K_{7 / 2} \rightarrow K_{4} \rightarrow K_{9 / 2} \rightarrow K_{5} \rightarrow \ldots
$$

The circular chromatic number $\chi_{c}(G)$ is the infimum over $\frac{p}{q}$ such that $G \rightarrow K_{p / q}$. Therefore:


Figure 1: The box complex of $K_{4}$ is the hollow cube (informally speaking; the drawing skips some irrelevant faces). It is equivalent ( $\mathbb{Z}_{2}$-homotopy equivalent) to the sphere. The box complex of the circular clique $K_{7 / 2}$ is equivalent to the circle. Thus there cannot be a homomorphism from $K_{4}$ to $K_{7 / 2}$ (of course in this case it is easier to show this directly).

Corollary 2.9. For every $2<r \leq r^{\prime}<4$, it is $N P$-hard to distinguish graphs $G$ with $\chi_{c}(G) \leq r$ from those with $\chi_{c}(G)>r^{\prime}$.

In this sense, we conclude that $K_{4-\varepsilon}$ is left-hard, thus extending the result for $K_{3}$. However, the closeness to $K_{4}$ is only deceptive and no conclusions on 4 -colourings follow. For $K_{4}$, since the box complex is equivalent to the standard 2-dimensional sphere, we can at least conclude that to prove left-hardness of $K_{4}$ it would be enough to prove left-hardness of any other graph with the same topology: these include all non-bipartite quadrangulations of the projective plane, in particular the Grötzsch graph, 4-chromatic generalised Mycielskians, and 4-chromatic Schrijver graphs [Mat08; BL03]. In this sense, the exact geometry of $K_{4}$ is irrelevant. However, the fact that it is a finite graph, with only finitely many possible maps from $C_{n}^{L}$ for any fixed $n, L$ should still be relevant, as it is for $K_{3}$. It is also quite probable that any proof for a "spherical" graph would apply just as well to $K_{4}$, where the proof could be just notationally much simpler.

Finally, in Appendix A we rephrase Krokhin and Opršal's [KO19] proof of Theorem 2.4 in terms of the box complex. In particular, left-hardness of $K_{3}$ follows from some general principles and the fact that $\left|\operatorname{Box}\left(K_{3}\right)\right|$ is a circle. The proof also extends to all graphs $H$ such that $|\operatorname{Box}(H)|$ admits a $\mathbb{Z}_{2}$-map to $\mathcal{S}^{1}$, giving an independent, self-contained proof of Corollary 2.8 (and Theorem 2.4 in particular).

The general principle is that a homomorphism $C_{n}^{L} \rightarrow H$ induces a $\mathbb{Z}_{2}$-map $\left(\mathcal{S}^{1}\right)^{L} \rightarrow$ $|\operatorname{Box}(H)|$, in a way that preserves minors (identifications within the $L$ variables) and automorphisms. (In the language of category theory, the box complex is a functor from the category of graphs to that of $\mathbb{Z}_{2}$-spaces, and the functor preserves products). In turn,
the $\mathbb{Z}_{2}$-map induces a group homomorphism between the fundamental group of $\left(\mathcal{S}^{1}\right)^{L}$, which is just $\mathbb{Z}^{L}$, and that of $|\operatorname{Box}(H)|$. This is essentially the map $\mathbb{Z}^{L} \rightarrow \mathbb{Z}$ obtained in [KO19]. While this rephrasing requires a bit more technical definitions, the main advantage is that it allows to replace a tedious combinatorial argument (about winding numbers preserving minors) with straightforward statements about preserving products.

### 2.3. Methodology - adjoint functors

While the proof of the first main result is given elementarily in Section 3, it fits together with the second main result in a much more general pattern. The underlying principle is that pairs of graph constructions satisfying a simple duality condition give reductions between PCSPs. To introduce them, let us consider a concrete example. For a graph $G$ and an odd integer $k, \Lambda_{k} G$ is the graph obtained by subdividing each edge into a path of $k$ edges; $\Gamma_{k} G$ is the graph obtained by taking the $k$-th power of the adjacency matrix (with zeroes on the diagonal); equivalently, the vertex set remains unchanged and two vertices are adjacent if and only if there is a walk of length exactly $k$ in $G$. (For example $\Gamma_{3} G$ has loops if $G$ has triangles).

We say a graph construction $\Lambda$ (a function from graphs to graphs) is a thin (graph) functor if $G \rightarrow H$ implies $\Lambda G \rightarrow \Lambda H$ (for all $G, H$ ). A pair of thin functors $(\Lambda, \Gamma)$ is a thin adjoint pair if

$$
\Lambda G \rightarrow H \text { if and only if } G \rightarrow \Gamma H
$$

We call $\Lambda$ the left adjoint of $\Gamma$ and $\Gamma$ the right adjoint of $\Lambda$.
For all odd $k,\left(\Lambda_{k}, \Gamma_{k}\right)$ are a thin adjoint pair. For example, since $\Gamma_{3} C_{5}=K_{5}$, we have $G \rightarrow K_{5}$ if and only if $\Lambda_{k} G \rightarrow C_{5}$. This is a basic reduction that shows the NP-hardness of $C_{5}$-colouring; in fact adjointness of various graph construction is the principal tool behind the original proof of Hell and Nešetriil's theorem (characterising the complexity of $H$-colouring) [HN90].

In category theory, there is a stronger and more technical notion of (non-thin) functors and adjoint pairs. A thin graph functor is in fact a functor in the thin category of graphs, that is, the category whose objects are graphs, and with at most one morphism from one graph to another, indicating whether a homomorphism exists or not. In other words, we are only interested in the existence of homomorphisms, and not in their identity and how they compose. Equivalently, we look only at the preorder of graphs by the $G \rightarrow H$ relation (we can also make this a poset by considering graphs up to homomorphic equivalence). In order-theoretic language, thin functors are just order-preserving maps, while thin adjoint functors are known as Galois connections. We prefer the categorical language as most of the constructions we consider are in fact functors (in the non-thin category of graphs), which is important for connections to the algebraic framework of [BKO19], as we discuss in Section 4.3. While unnecessary for our main results, we believe it may be important to understand these deeper connections to resolve the conjectures completely.

Thin adjoint functors give us a way to reduce one PCSP to another. We say that a graph functor $\Gamma$ is $\log$-space computable if, given a graph $G, \Gamma G$ can be computed in logarithmic space in the size of $G$.

Observation 2.10. Let $\Lambda, \Gamma$ be thin adjoint graph functors and let $\Lambda$ be log-space computable. Then $\operatorname{PCSP}(G, \Gamma H)$ reduces to $\operatorname{PCSP}(\Lambda G, H)$ in log-space, for all graphs $G, H$.

Proof. Let $F$ be an instance of $\operatorname{PCSP}(G, \Gamma H)$. Then $\Lambda F$ is an appropriate instance of $\operatorname{PCSP}(\Lambda G, H)$. Indeed, if $F \rightarrow G$, then $\Lambda F \rightarrow \Lambda G$ (because $\Lambda$ is a thin functor). If $\Lambda F \rightarrow H$, then $F \rightarrow \Gamma H$ by adjointness.

In some cases, a thin functor $\Gamma$ that is a thin right adjoint in a pair $(\Lambda, \Gamma)$ is also a thin left adjoint in a pair $(\Gamma, \Omega)$. This allows to get a reduction in the opposite direction:

Observation 2.11. Let $(\Lambda, \Gamma)$ and $(\Gamma, \Omega)$ be thin adjoint pairs of functors. Then $\operatorname{PCSP}(\Gamma G, H)$ and $\operatorname{PCSP}(G, \Omega H)$ are log-space equivalent (assuming $\Lambda$ and $\Gamma$ are logspace computable).

Proof. The previous observation gives a reduction from $\operatorname{PCSP}(G, \Omega H)$ to $\operatorname{PCSP}(\Gamma G, H)$. For the other direction, let $F$ be an instance of $\operatorname{PCSP}(\Gamma G, H)$. Then $\Lambda F$ is an appropriate instance of $\operatorname{PCSP}(G, \Omega H)$. Indeed, if $F \rightarrow \Gamma G$, then $\Lambda F \rightarrow G$. If $\Lambda F \rightarrow \Omega H$, then $F \rightarrow \Gamma \Omega H \rightarrow H$. The last arrow follows from the trivial $\Omega H \rightarrow \Omega H$.

The proofs of Observations 2.10 and 2.11 of course extend to digraphs and general relational structures. Note that the above proofs reduce decision problems; they work just as well for search problems: all the thin adjoint pairs $(\Lambda, \Gamma)$ we consider with $\Lambda \log$-space computable also have the property that a homomorphism $\Lambda F \rightarrow H$ can be computed from a homomorphism $F \rightarrow \Gamma H$ and vice versa, in space logarithmic in the size of $F$.

As we discuss in Section 4, all of our results follow from reductions that are either trivial (homomorphic relaxations) or instantiations of Observation 2.10. While for the first main result we prefer to first give a direct proof that avoids this formalism (in Section 3), it will be significantly more convenient for the second main result (in Section 4.1), where we use a certain right adjoint $\Omega_{k}$ to the $k$-th power $\Gamma_{k}$.

### 2.4. Hedetniemi's conjecture

Another leitmotif of this paper is the application of various tools developed in research around Hedetniemi's conjecture. A graph $K$ is multiplicative if $G \times H \rightarrow K$ implies $G \rightarrow K$ or $H \rightarrow K$. The conjecture states that all cliques $K=K_{n}$ are multiplicative. Equivalently, $\chi(G \times H)=\min (\chi(G), \chi(H))$; see [Zhu98; Sau01; Tar08] for surveys. In a very recent breakthrough, Shitov [Shi19] proved that the conjecture is false (for large $n$ ).

The arc digraph construction, which we will use in Section 3 to prove Theorem 2.6, was originally used by Poljak and Rödl [PR81] to show certain asymptotic bounds on chromatic numbers of products. The functors $\Lambda_{k}, \Gamma_{k}, \Omega_{k}$ were applied by Tardif [Tar05] to show that colourings to circular cliques $K_{p / q}\left(2<\frac{p}{q}<4\right)$ satisfy the conjecture. Matsushita [Mat19] used the box complex to show that Hedetniemi's conjecture would imply an analogous conjecture in topology. This was independently proved by the first author [Wro19] using $\Omega_{k}$ functors, while the box complex was used to show that squarefree graphs are multiplicative [Wro17]. See [FT18] for a survey on applications of adjoint functors to the conjecture.

The refutation of Hedetniemi's conjecture and the fact that methods for proving the multiplicativity of $K_{3}$ extend to $K_{4-\varepsilon}$ and square-free graphs, but fail to extend to $K_{4}$, might suggest that the Conjecture 2.3 is doomed to the same fate. However, it now seems clear that proving multiplicativity requires more than just topology [TW19]: known methods do not even extend to all graphs $H$ such that $|\operatorname{Box}(H)|$ is a circle. This contrasts with Theorem 2.7: topological tools work much more gracefully in the setting of PCSPs.

## 3. The arc digraph construction

Let $D$ be a digraph. The arc digraph (or line digraph) of $D$, denoted $\delta D$, is the digraph whose vertices are arcs (directed edges) of $D$ and whose arcs are pairs of the form $((u, v),(v, w))$. We think of undirected graphs as symmetric relations: digraphs in which for every arc $(u, v)$ there is an $\operatorname{arc}(v, u)$. So for an undirected graph $G, \delta(G)$ has $2|E(G)|$ vertices and is a directed graph: the directions will not be important in this section, but will be in Section 4.2. The chromatic number of a digraph is the chromatic number of the underlying undirected graph (obtained by symmetrising each arc; so $\chi(D) \leq n$ if and only if $D \rightarrow K_{n}$ ).

The crucial property of the arc digraph construction is that it decreases the chromatic number in a controlled way (even though it is computable in log-space!). We include a short proof for completeness. We denote by $[n]$ the set $\{1,2, \ldots, n\}$.

Lemma 3.1 (Harner and Entringer [HE72]). For any graph G:

- if $\chi(\delta(G)) \leq n$, then $\chi(G) \leq 2^{n}$;
- if $\chi(G) \leq\binom{ n}{\lfloor n / 2\rfloor}$, then $\chi(\delta(G)) \leq n$.

Proof. Suppose $\delta G$ has an $n$-colouring. Recall that we think of $G$ as a digraph with two $\operatorname{arcs}(u, v)$ and $(v, u)$ for each edge $\{u, v\} \in E(G)$; thus $\delta G$ contains two vertices $(u, v)$ and ( $v, u$ ) , as well as (by definition of $\delta$ ) two arcs from one pair to the other. In particular, an $n$-colouring of $\delta G$ gives distinct colours to $(u, v)$ and $(v, u)$. Define a $2^{n}$-colouring $\phi$ of $G$ by assigning to each vertex $v$ the set $\phi(v)$ of colours of incoming arcs. For any edge $\{u, v\}$ of $G, \phi(v)$ contains the colour $c$ of the arc $(u, v)$. Since every arc incoming to $u$ gets a different colour from $(u, v)$, the set $\phi(u)$ does not contain $c$. Hence $\phi(u) \neq \phi(v)$, so $\phi$ is a proper colouring.

Suppose $G$ has a $\binom{n}{\lfloor n / 2\rfloor}$-colouring $\phi$. We interpret colours $\phi(v)$ as $\lfloor n / 2\rfloor$-element subsets of $[n]$. Define an $n$-colouring of $\delta G$ by assigning to each $\operatorname{arc}(u, v)$ an arbitrary colour in $\phi(u) \backslash \phi(v)$ (the minimum, say). Such a colour exists because $\phi(u) \neq \phi(v)$. For arcs $(u, v),(v, w)$ clearly $\phi(u) \backslash \phi(v)$ is disjoint from $\phi(v) \backslash \phi(w)$, so this is a proper colouring of $\delta(G)$.

The proofs in fact works for digraphs as well. For graphs, it is not much harder to show an exact correspondence (we note however that most conclusions only require the above approximate correspondence). Let us denote $b(n):=\binom{n}{\lfloor n / 2\rfloor}$.
Lemma 3.2 (Poljak and Rödl [PR81]). For a (symmetric) graph $G$,

$$
\chi(\delta(G))=\min \{n \mid \chi(G) \leq b(n)\} .
$$

In other words, $\delta G \rightarrow K_{n}$ if and only if $G \rightarrow K_{b(n)}$.
This immediately gives the following implication for approximate colouring:
Lemma 3.3. $\operatorname{PCSP}\left(K_{b(n)}, K_{b(k)}\right)$ log-space reduces to $\operatorname{PCSP}\left(K_{n}, K_{k}\right)$, for all $n, k \in \mathbb{N}$.
Proof. Let $G$ be an instance of the first problem. Then $\delta G$ is a suitable instance of $\operatorname{PCSP}\left(K_{n}, K_{k}\right)$ : if $G \rightarrow K_{b(n)}$, then $\delta G \rightarrow K_{n}$. If $\delta G \rightarrow K_{k}$, then $G \rightarrow K_{b(k)}$.

Remark 3.4. As a side note, adding a universal vertex gives the following obvious reduction: $\operatorname{PCSP}\left(K_{n}, K_{k}\right) \log$-space reduces to $\operatorname{PCSP}\left(K_{n+1}, K_{k+1}\right)$, for $n, k \in \mathbb{N}$.

Recall also that if $n \leq n^{\prime} \leq k^{\prime} \leq k$, then $\operatorname{PCSP}\left(K_{n}, K_{k}\right)$ trivially reduces to $\operatorname{PCSP}\left(K_{n^{\prime}}, K_{k^{\prime}}\right)$. One corollary of Lemma 3.3 is that if any clique of size at least 4 is right-hard, then all of them are:

Proposition 3.5. For all integers $n, n^{\prime} \geq 4, \operatorname{PCSP}\left(K_{n}, K_{k}\right)$ is NP-hard for all $k \geq n$ if and only if $\operatorname{PCSP}\left(K_{n^{\prime}}, K_{k^{\prime}}\right)$ is NP-hard for all $k^{\prime} \geq n^{\prime}$.

Proof. Let $n \leq n^{\prime}$. For one direction, right-hardness of $K_{n}$ trivially implies right-hardness of $K_{n^{\prime}}$.

On the other hand, we claim that if $K_{b(n)}$ is right-hard, then so is $K_{n}$. Indeed, suppose $\operatorname{PCSP}\left(K_{b(n)}, K_{k}\right)$ is hard for all $k \geq b(n)$. In particular it is hard for all $k$ of the form $k=b\left(k^{\prime}\right)$ for an integer $k^{\prime} \geq n$. Hence by Lemma 3.3, $\operatorname{PCSP}\left(K_{n}, K_{k^{\prime}}\right)$ is hard for all $k^{\prime} \geq n$.

Suppose $K_{n}$ is not right-hard. Then $K_{b(n)}$ is not right-hard, $K_{b(b(n))}$ is not right-hard and so on. Since starting with $n \geq 4$, the sequence $b(b(\ldots n \ldots))$ grows to infinity, we conclude that $K_{n^{\prime \prime}}$ is not right-hard for some $n^{\prime \prime} \geq n^{\prime}$. Therefore, trivially $K_{n^{\prime}}$ is not right-hard.

In other words if any loop-less graph $H$ is right-hard, then trivially some large enough clique $K_{\chi(H)}$ is right-hard; by the above, $K_{4}$ and all graphs right of it are right-hard. This proves Proposition 2.5. The proof fails to extend to $K_{3}$ because $b(3)=\binom{3}{\lfloor 3 / 2\rfloor}$ is not strictly greater than 3 .

The other consequence we derive from Lemma 3.3 is a strengthening of Huang's result:
Theorem 3.6 (Huang [Hua13]). For all sufficiently large $n, \operatorname{PCSP}\left(K_{n}, K_{2^{\Omega\left(n^{1 / 3}\right)}}\right)$ is NP-hard.

Theorem 2.6 (Main Result \#1). For all $n \geq 4, \operatorname{PCSP}\left(K_{n}, K_{\binom{n}{\lfloor n / 2\rfloor}-1}\right)$ is NP-hard.
We thus improve the asymptotics from sub-exponential $f(n):=2^{n^{1 / 3}}$ to singleexponential $b(n)=\binom{n}{\lfloor n / 2\rfloor} \sim \frac{2^{n}}{\sqrt{\pi n / 2}}$. The informal idea of the proof is that any $f(n)$ can be improved to $b^{-1}(f(b(n)))$. Since $b(n)$ is roughly exponential and $b^{-1}(n)$ is roughly logarithmic, starting from a function $f(n)$ of order $\exp ^{(i+1)}\left(\alpha \cdot \log ^{(i)}(n)\right)$ with $i$-fold compositions and a constant $\alpha>0$, such as $f(n)=2^{n^{1 / 3}}=2^{2^{\frac{1}{3} \log n}}$ from Huang's hardness, results in

$$
b^{-1}(f(b(n))) \approx \log \left(\exp ^{(i+1)}\left(\alpha \cdot \log ^{(i)}(\exp (n))\right)\right)=\exp ^{(i)}\left(\alpha \cdot \log ^{(i-1)}(n)\right),
$$

so a similar composition but with $i$ decreased. In a constant number of steps, this results in a single-exponential function. In fact using one more step, but without approximating the function $b(n)$, this results in exactly $b(n)-1$. We note it would not be sufficient to start from a quasi-polynomial $f(n)$, like $n^{\Theta(\log n)}$ in Khot's [Kho01] result.

Proof of Theorem 2.6. By Lemma 3.3:

$$
\operatorname{PCSP}\left(K_{b(n)}, K_{b(m)}\right) \log \text {-space reduces to } \operatorname{PCSP}\left(K_{n}, K_{m}\right) \text {, for all } n, m \in \mathbb{N} \text {. }
$$

For any $k \in \mathbb{N}$, let $m=\lfloor\log k\rfloor$ (all logarithms are base-2); then $b(m) \leq 2^{m} \leq k$, hence $\operatorname{PCSP}\left(K_{b(n)}, K_{k}\right)$ trivially reduces to $\operatorname{PCSP}\left(K_{b(n)}, K_{b(m)}\right)$.

Therefore, composing the two reductions:

$$
\operatorname{PCSP}\left(K_{b(n)}, K_{k}\right) \text { reduces to } \operatorname{PCSP}\left(K_{n}, K_{\lfloor\log k\rfloor}\right) \text {, for any } n, k \in \mathbb{N} \text {. }
$$

Starting from Theorem 3.6 we have a constant $C$ such that:

$$
\operatorname{PCSP}\left(K_{n}, K_{2\left\lfloor C \cdot n^{1 / 3}\right\rfloor}\right) \text { is NP-hard, for sufficiently large } n \text {. }
$$

Hence, substituting $n=b(k)$ :

$$
\operatorname{PCSP}\left(K_{b(k)}, K_{2\left\lfloor C \cdot b(k)^{1 / 3}\right\rfloor}\right) \text { is NP-hard, for sufficiently large } k \text {. }
$$

Applying the above reduction, since $\left\lfloor\log 2^{\left\lfloor C \cdot b(k)^{1 / 3}\right\rfloor}\right\rfloor=\left\lfloor C \cdot b(k)^{1 / 3}\right\rfloor \geq\left(\frac{2^{k}}{k}\right)^{1 / 3} \geq 2^{k / 4}$ for sufficiently large $k$, we conclude:

$$
\operatorname{PCSP}\left(K_{k}, K_{2^{k / 4}}\right) \text { is NP-hard, for sufficiently large } k .
$$

We repeat this process to bring the constant further "down". That is, we substitute $b(k)$ for $k$ and apply the above reduction again. Since $\left\lfloor\log 2^{b(k) / 4}\right\rfloor=\lfloor b(k) / 4\rfloor \geq 2^{k} / 4 k$ for sufficiently large $k$, we conclude:

$$
\operatorname{PCSP}\left(K_{k}, K_{2^{k} / 4 k}\right) \text { is NP-hard, for sufficiently large } k \text {. }
$$

To apply the reduction one more time, notice that for large $k, b(k) \geq \frac{3}{2} b(k-1)$ (because $b(2 k)=\binom{2 k}{k}=\binom{2 k-1}{k-1} \frac{2 k}{k}=2 \cdot b(2 k-1) \geq \frac{3}{2} b(2 k-1)$ and $b(2 k+1)=\binom{2 k+1}{k}=\binom{2 k}{k} \frac{2 k+1}{k+1}=$ $\left.b(2 k)\left(2-\frac{1}{k+1}\right) \geq \frac{3}{2} b(2 k)\right)$. Therefore $\left\lfloor\log \left(2^{b(k)} / 4 b(k)\right)\right\rfloor \geq b(k)-\log b(k) \geq \frac{2}{3} b(k) \geq b(k-1)$ for sufficiently large $k$, hence:

$$
\operatorname{PCSP}\left(K_{k}, K_{b(k-1)}\right) \text { is NP-hard, for sufficiently large } k \text {. }
$$

Substituting $b(k)$ for $k$ one last time:

$$
\operatorname{PCSP}\left(K_{b(k)}, K_{b(b(k)-1)}\right) \text { is NP-hard, for sufficiently large } k \text {. }
$$

Composing with Lemma 3.3 one last time:

$$
\operatorname{PCSP}\left(K_{k}, K_{b(k)-1}\right) \text { is NP-hard, for sufficiently large } k \text {. }
$$

This concludes the improvement in asymptotics. Moreover, one can notice that the requirements on "sufficiently large $k$ " gets relaxed whenever we substitute $b(k)$ for $k$. Formally, let $k$ be maximum such that $\operatorname{PCSP}\left(K_{k}, K_{b(k)-1}\right)$ is not NP-hard. Then because of Lemma 3.3, $\operatorname{PCSP}\left(K_{b(k)}, K_{b(b(k)-1)}\right)$ is not NP-hard, and because $b(b(k)-1) \leq b(b(k))-$ 1, trivially $\operatorname{PCSP}\left(K_{b(k)}, K_{b(b(k))-1}\right)$ is not NP-hard either. That is, $\operatorname{PCSP}\left(K_{n}, K_{b(n)-1}\right)$ is not NP-hard for $n=b(k)$. By maximality of $k, k \geq n$. But $k \geq b(k)$ is only possible when $k<4$. Hence hardness holds for all $k \geq 4$.

## 4. Adjoint functors and topology

### 4.1. Thin functors $\Lambda_{k}, \Gamma_{k}, \Omega_{k}$

Recall that $\Lambda_{k}$ denotes $k$-subdivision and $\Gamma_{k}$ denotes the $k$-th power of a graph. For all odd $k$, they are thin adjoint graph functors:

$$
\Lambda_{k} G \rightarrow H \text { if and only if } G \rightarrow \Gamma_{k} H
$$

More surprisingly, $\Gamma_{k}$ is itself the thin left adjoint of a certain thin functor $\Omega_{k}$ :

$$
\Gamma_{k} G \rightarrow H \text { if and only if } G \rightarrow \Omega_{k} H
$$

This characterizes $\Omega_{k} G$ up to homomorphic equivalence. The exact definition is irrelevant, but we state it for completeness: for $k=2 \ell+1$, the vertices of $\Omega_{k}$ are tuples $\left(A_{0}, \ldots, A_{\ell}\right)$ of vertex subsets $A_{i} \subseteq V(G)$ such that $A_{0}$ contains exactly one vertex. Two such tuples $\left(A_{0}, \ldots, A_{\ell}\right)$ and $\left(B_{0}, \ldots, B_{\ell}\right)$ are adjacent if $A_{i} \subseteq B_{i+1}, B_{i} \subseteq A_{i+1}$ (for $i=0 \ldots \ell-1$ ) and $A_{\ell}$ is fully adjacent to $B_{\ell}$ (meaning $a$ is adjacent to $b$ in $G$, for $a \in A_{k}, b \in B_{k}$ ). We note that $\Lambda_{k}$ and $\Gamma_{k}$ are log-space computable, for all odd $k$; however, $\Omega_{k}$ is not: $\Omega_{k} G$ is exponentially larger than $G$. See [Wro19] for more discussion about the thin functors $\Lambda_{k}, \Gamma_{k}, \Omega_{k}$ and their properties.

Observation 2.10 tells us that $\operatorname{PCSP}\left(G, \Omega_{k} H\right)$ log-space reduces to $\operatorname{PCSP}\left(\Gamma_{k} G, H\right)$ (in fact, by Observation 2.11, they are equivalent). To give conclusions on left-hardness, we will need to observe only two more facts about the functors $\Lambda_{k}, \Gamma_{k}, \Omega_{k}$. First, $\Omega_{k} G \rightarrow G$ for all $G$ (it suffices to map $\left(A_{0}, \ldots, A_{l-1}, A_{\ell}\right) \in V\left(\Omega_{2 \ell+1} G\right)$ to the unique vertex in $\left.A_{0}\right)$. Second, it is not hard to check that $\Gamma_{k} \Lambda_{k} G \rightarrow G$ and hence by adjointness $\Lambda_{k} G \rightarrow \Omega_{k} G$ for all $G$ and odd $k$ (see Lemma 2.3 in [Wro19]).

Lemma 4.1. For every odd $k, \Omega_{k} H$ is left-hard if and only if $H$ is left-hard.
Proof. If $H$ is left-hard, then trivially so is $\Omega_{k} H$ because $\Omega_{k} H \rightarrow H$. For the other implication, suppose $\Omega_{k} H$ is left-hard, that is, $\operatorname{PCSP}\left(G, \Omega_{k} H\right)$ is hard for every nonbipartite $G$ such that $G \rightarrow \Omega_{k} H$. By Observation 2.10 , this implies $\operatorname{PCSP}\left(\Gamma_{k} G, H\right)$ is hard. Let $G^{\prime}$ be any non-bipartite graph such that $G^{\prime} \rightarrow H$. We want to show that $\operatorname{PCSP}\left(G^{\prime}, H\right)$ is hard. Observe that $\Omega_{k} G^{\prime}$ is non-bipartite, because $\Lambda_{k} G^{\prime} \rightarrow \Omega_{k} G^{\prime}$ and $\Lambda_{k}$ subdivides each edge of $G^{\prime}$ an odd number of times. Since $\Omega_{k} G^{\prime} \rightarrow \Omega_{k} H$, using $G:=\Omega_{k} G^{\prime}$ we conclude that $\operatorname{PCSP}\left(\Gamma_{k} \Omega_{k} G^{\prime}, H\right)$ is hard. Since $\Gamma_{k} \Omega_{k} G^{\prime} \rightarrow G^{\prime}$, this implies $\operatorname{PCSP}\left(G^{\prime}, H\right)$ is hard.

As an example, consider the circular clique $K_{7 / 2}$ (we have $K_{3} \rightarrow K_{7 / 2} \rightarrow K_{4}$ ). Knowing that $K_{3}$ is left-hard, one could check that $\Omega_{3}\left(K_{7 / 2}\right)$ is 3 -colorable and hence left-hard as well; the above lemma then allows to conclude that $K_{7 / 2}$ is left-hard.

What other graphs could one use in place of $K_{7 / 2}$ ? The answer turns out to be topological. Intuitively, while the operation $\Gamma_{k}$ gives a "thicker" graph, the operation $\Omega_{k}$ gives a "thinner" one. In fact, $\Omega_{k}$ behaves like barycentric subdivision in topology: it preserves the topology of a graph (formally: its box complex is $\mathbb{Z}_{2}$-homotopy equivalent to the original graph's box complex) but refines its geometry. With increasing $k$, this eventually allows to model any continuous map with a graph homomorphism; in particular:

Theorem 4.2 ([Wro19]). There exists a $\mathbb{Z}_{2}$-map $|\operatorname{Box}(G)| \rightarrow_{\mathbb{Z}_{2}}|\operatorname{Box}(H)|$ if and only if for some odd $k, \Omega_{k} G \rightarrow H$.

This concludes our second main result:
Proof of Theorem 2.7. Let $H$ be left-hard and let $H^{\prime}$ be a graph such that $\left|\operatorname{Box}\left(H^{\prime}\right)\right|$ admits a $\mathbb{Z}_{2}$-map to $|\operatorname{Box}(H)|$. By Theorem $4.2, \Omega_{k} H^{\prime} \rightarrow H$ for some odd $k$. Trivially then, $\Omega_{k} H^{\prime}$ is left-hard. By Lemma 4.1, $H^{\prime}$ is left-hard.

### 4.2. Other examples of adjoint functors

The arc construction $\delta$ is also an example of a digraph functor which admits both a thin left adjoint $\delta_{L}$ and a thin right adjoint $\delta_{R} ;^{8}$ this adjointness essentially gives a proof of Lemma 3.1, see [FT18, Proposition 3.3]. In fact, Lemma 3.3, and hence all results of Section 3, can be deduced as instantiations of Observation 2.10 and homomorphic relaxations as follows. Let $\operatorname{sym}(D)$ be the symmetric closure of a digraph $D$ and let $\operatorname{sub}(D)$ be the maximal symmetric subgraph of $D$; note $\operatorname{sub}(D) \rightarrow D \rightarrow \operatorname{sym}(D)$. Observe that they are thin adjoint functors: $\operatorname{sym}(D) \rightarrow D^{\prime}$ if and only if $D \rightarrow \operatorname{sub}\left(D^{\prime}\right)$, for all digraphs $D, D^{\prime} .{ }^{9}$ Poljak and Rödl [PR81] showed that $\operatorname{sub}\left(\delta_{R}\left(K_{k}\right)\right) \rightarrow K_{b(k)}$ (the sub is essential here); recall also that $\delta\left(\operatorname{sym}\left(K_{b(n)}\right)\right) \rightarrow K_{n}$. Therefore, $\operatorname{PCSP}\left(K_{b(n)}, K_{b(k)}\right)$ trivially reduces to $\operatorname{PCSP}\left(K_{b(n)}, \operatorname{sub}\left(\delta_{R}\left(K_{k}\right)\right)\right)$, which by Observation $2.10 \log$-space reduces to $\operatorname{PCSP}\left(\delta\left(\operatorname{sym}\left(K_{b(n)}\right)\right), K_{k}\right)$, which trivially reduces to $\operatorname{PCSP}\left(K_{n}, K_{k}\right)$, proving Lemma 3.3. From Observation 2.11 we also have:

Corollary 4.3. $\operatorname{PCSP}(\delta(G), H)$ is log-space equivalent to $\operatorname{PCSP}\left(G, \delta_{R}(H)\right)$, for all digraphs $G, H$.

Another example of a thin adjoint pair (but not triple) of functors is given by products and exponential graphs (see e.g. [FT13] for definitions): for any graphs $F, G, H$, we have $F \times G \rightarrow H$ if and only if $G \rightarrow H^{F}$. That is, for any graph $F$, the operations $G \mapsto F \times G$ and $H \mapsto H^{F}$ are left and right adjoints, respectively. By Observation 2.10:

Corollary 4.4. $\operatorname{PCSP}\left(G, H^{F}\right)$ reduces to $\operatorname{PCSP}(F \times G, H)$ in log-space.
Here $\times$ is the tensor (or categorical) product, in particular $G \rightarrow H_{1} \times H_{2}$ if and only if $G \rightarrow H_{1}$ and $G \rightarrow H_{2}$. Nevertheless, a few other products have an associated exponentiation as well. These and other examples fall into a pattern known as Pultr functors - see [FT13] for an extended discussion (we note here that central Pultr functors, like $\Gamma_{k}$ or $\delta$, are a kind of pp-interpretation). Foniok and Tardif [FT15] studied which digraph functors admit both thin left and right adjoints.

The box complex also admits a left adjoint, though they involve two categories. More precisely, the functor $G \mapsto \operatorname{Hom}\left(K_{2}, G\right)$ (see definitions in Appendix A) gives a $\mathbb{Z}_{2^{-}}$ simplicial complex that is $\mathbb{Z}_{2}$-homotopy equivalent to the box complex. As proved by Matsushita [Mat19], it admits a left adjoint $A$ from the category of $\mathbb{Z}_{2}$-simplicial complexes (with $\mathbb{Z}_{2}$-simplicial maps as morphisms) to the category of graphs.

[^5]
### 4.3. Relation to the algebraic framework

We will need basic concepts from the algebraic approach to (P)CSPs, such as polymorphisms [AGH17; BG18], minions, and minion homomorphisms [BKO19]. We shall define them only for graphs as we do not need them for relational structures. We refer the reader to [BKW17; BKO19] for more details, examples, and general definitions.

An $n$-ary polymorphism of two graphs $G$ and $H$ is a homomorphism from $G^{n}$ to $H$; that is, a map $f: V(G)^{n} \rightarrow V(H)$ such that, for all edges $\left(u_{1}, v_{1}\right), \ldots,\left(u_{n}, v_{n}\right)$ in $G$, $\left(f\left(u_{1}, \ldots, u_{n}\right), f\left(v_{1}, \ldots, v_{n}\right)\right)$ is an edge in $H$. We denote by $\operatorname{Pol}(G, H)$ the set of all polymorphisms of $G$ and $H$.

Given an $n$-ary function $f: A^{n} \rightarrow B$, the, say, first coordinate is called essential if there exist $a, a^{\prime} \in A$ and $\vec{a} \in A^{n-1}$ such that $f(a, \vec{a}) \neq f\left(a^{\prime}, \vec{a}\right)$; otherwise, the first coordinate is called inessential or dummy. Analogously, one defines the $i$-th coordinate to be (in)essential. The essential arity of $f$ is the number of essential coordinates.

Let $f: A^{n} \rightarrow B$ and $g: A^{m} \rightarrow B$ be $n$-ary and $m$-ary functions, respectively. We call $f$ a minor of $g$ if $f$ can be obtained from $g$ by identifying variables, permuting variables, and introducing inessential variables. More formally, $f$ is a minor of $g$ given by a map $\pi:[m] \rightarrow[n]$ if $f\left(x_{1}, \ldots, x_{n}\right)=g\left(x_{\pi(1)}, \ldots, x_{\pi(m)}\right)$.

A minion on a pair of sets $(A, B)$ is a non-empty set of functions (of possibly different arities) from $A$ to $B$ that is closed under taking minors. A minion is said to have bounded essential arity if there is some $k$ such that every function from the minion has essential arity at most $k$.

Let $\mathscr{M}$ and $\mathscr{N}$ be two minions, not necessarily on the same pairs of sets. A map $\xi: \mathscr{M} \rightarrow \mathscr{N}$ is called a minion homomorphism if (1) it preserves arities; i.e., maps $n$-ary functions to $n$-ary functions, for all $n$; and (2) it preserves taking minors; i.e., for each $\pi$ : $[m] \rightarrow[n]$ and each $m$-ary $g \in \mathscr{M}$, we have $\xi(g)\left(x_{\pi(1)}, \ldots, x_{\pi(m)}\right)=\xi\left(g\left(x_{\pi(1)}, \ldots, x_{\pi(m)}\right)\right)$. Minion homomorphisms provide an algebraic way to give reductions between PCSPs.

Theorem 4.5 ([BKO19]). If there is a minion homomorphism $\xi: \operatorname{Pol}\left(G_{1}, H_{1}\right) \rightarrow$ $\operatorname{Pol}\left(G_{2}, H_{2}\right)$, then $\operatorname{PCSP}\left(G_{2}, H_{2}\right)$ is log-space reducible to $\operatorname{PCSP}\left(G_{1}, H_{1}\right)$.

The following hardness result is a special case of a result obtained in [BKO19] via a reduction from Gap Label Cover. It gives an algebraic tool to prove hardness for PCSPs.

Theorem 4.6 ([BKO19]). Let $G$ and $H$ be two graphs with $G \rightarrow H$. Assume that there exists a minion homomorphism $\xi: \operatorname{Pol}(G, H) \rightarrow \mathscr{M}$ for some minion $\mathscr{M}$ on a pair of (possibly infinite) sets such that $\mathscr{M}$ has bounded essential arity and does not contain a constant function (i.e., a function without essential variables). Then $\operatorname{PCSP}(G, H)$ is NP-hard.

Our methods do not give minion homomorphisms in general: while Observation 2.10 gives a reduction from $\operatorname{PCSP}(G, \Gamma H)$ to $\operatorname{PCSP}(\Lambda G, H)$, it does not give a minion homomorphism from which the reduction would follow $($ from $\operatorname{Pol}(\Lambda G, H)$ to $\operatorname{Pol}(G, \Gamma H))$. Indeed it cannot, as discussed below Proposition 2.5. However, adjoint functors in the (non-thin) category of graphs do imply such a minion homomorphism.

In the remainder of this section, we assume knowledge of basic definitions in category theory. One can define minions in any Cartesian category $\mathcal{C}$ (i.e. a category with all finite products), using morphisms of $\mathcal{C}$ in place of functions. For objects $G, H \in \mathcal{C}, \operatorname{Pol}_{\mathcal{C}}(G, H)$ is the minion of morphisms from $G^{L}$ (the $L$-fold categorical product of $G$ ) to $H$. A function
$\pi:[L] \rightarrow\left[L^{\prime}\right]$ induces a morphism $\pi_{G}: G^{L^{\prime}} \rightarrow G^{L}$. For a graph $G$, it maps $\left(v_{1}, \ldots, v_{L^{\prime}}\right)$ to $\left(v_{\pi(1)}, \ldots, v_{\pi(L)}\right)$. In general, it can be defined as the product morphism $\left\langle p_{\pi(1)}, \ldots, p_{\pi(L)}\right\rangle$ of appropriate projections $p_{i}: G^{L} \rightarrow G$. For a polymorphism $f: G^{L} \rightarrow H$, the minor of $f$ by $\pi$ is then simply $f \circ \pi_{G}: G^{L^{\prime}} \rightarrow H$.

For objects $G$ and $H$ of a category, we denote by $\operatorname{hom}(G, H)$ the set of morphisms from $G$ to $H$.

Lemma 4.7. Let $\Gamma: \mathcal{C} \rightarrow \mathcal{D}$ and $\Omega: \mathcal{D} \rightarrow \mathcal{C}$ be adjoint functors between Cartesian categories $\mathcal{C}, \mathcal{D}$. Then for all objects $G$ in $\mathcal{C}$ and $H$ in $\mathcal{D}$, there is a minion homomorphism from $\operatorname{Pol}_{\mathcal{D}}(\Gamma G, H)$ to $\operatorname{Pol}_{\mathcal{C}}(G, \Omega H)$. If, moreover, $\Gamma$ preserves products then this is a minion isomorphism.

Proof. This essentially amounts to checking definitions. We have a natural morphism $\psi_{L}: \Gamma\left(G^{L}\right) \rightarrow(\Gamma G)^{L}$ defined as the product morphism $\left\langle\Gamma p_{1}, \ldots, \Gamma p_{L}\right\rangle$ for projections $p_{i}: G^{L} \rightarrow G$. It is natural in the following sense: for every function $\pi:[L] \rightarrow\left[L^{\prime}\right]$, the following diagram commutes:

$$
\begin{array}{crr}
\Gamma\left(G^{L^{\prime}}\right) \xrightarrow{\psi_{L^{\prime}}} & (\Gamma G)^{L^{\prime}} \\
\stackrel{\downarrow \pi_{G}}{ } & \downarrow \pi_{\Gamma G} \\
\Gamma\left(G^{L}\right) \xrightarrow{\psi_{L}} & (\Gamma G)^{L}
\end{array}
$$

Indeed, $\psi_{L} \circ \Gamma \pi_{G}=\pi_{\Gamma G} \circ \psi_{L^{\prime}}$, because it is the unique morphism whose composition with $p_{i}^{\prime}:(\Gamma G)^{L} \rightarrow \Gamma G$ is $\Gamma p_{\pi(i)}$ (in other words, it is the product morphism $\left.\left\langle\Gamma p_{\pi(1)}, \ldots, \Gamma p_{\pi(L)}\right\rangle\right)$.

Let $\Omega$ be a right adjoint of $\Gamma$. Let $\Phi_{G^{L}, H}: \operatorname{hom}\left(\Gamma\left(G^{L}\right), H\right) \rightarrow \operatorname{hom}\left(G^{L}, \Omega H\right)$ be the natural isomorphism given by definition of adjunction. Naturality here means that in particular the right square in the following diagram commutes:


That is, for $f: \Gamma\left(G^{L}\right) \rightarrow \Omega H$, we have $\Phi_{G^{L}, H}(f) \circ \pi_{G}=\Phi_{G^{L^{\prime}, H}}\left(f \circ \Gamma \pi_{G}\right)$. The left square also commutes because of the previously discussed commutation. Therefore, we can define a minion homomorphism $\xi: \operatorname{hom}\left((\Gamma G)^{L}, H\right) \rightarrow \operatorname{hom}\left(G^{L}, \Omega H\right)$ as $\xi(f):=\Phi_{G^{L}, H}\left(f \circ \psi_{L}\right)$. Indeed, $\xi$ preserves minors, because $\xi\left(f \circ \pi_{\Gamma G}\right)=\xi(f) \circ \pi_{G}$ as seen on the perimeter of the above diagram.

If $\Gamma$ preserves products, then $\psi_{L}$ is an isomorphism. Since $\Phi_{G^{L}, H}$ is a bijection, this means $\xi$ is a minion isomorphism.

A basic lemma in category theory says that if a functor $\Gamma$ admits a left adjoint, then it preserves products (indeed, all limits). So a pair of adjoint pairs $(\Lambda, \Gamma),(\Gamma, \Omega)$ implies a minion isomorphism. Hence the first part of Lemma 4.7 is analogous to Observation 2.10, while the second part is analogous to Observation 2.11. We can also derive the second direction as a corollary to the following lemma.

Lemma 4.8. Let $\Gamma: \mathcal{C} \rightarrow \mathcal{D}$ be a functor which preserves products. Then there is a minion homomorphism $\operatorname{Pol}_{\mathcal{C}}(G, H) \rightarrow \operatorname{Pol}_{\mathcal{D}}(\Gamma G, \Gamma H)$, for all $G, H \in \mathcal{C}$.

Proof. Recall from the proof of Lemma 4.7 the following diagram, for $G \in \mathcal{C}, L, L^{\prime} \in \mathbb{N}$, and $\pi:[L] \rightarrow\left[L^{\prime}\right]:$


Since $\Gamma$ preserves products, $\psi_{L}$ is an isomorphism, so we can define a minion homomor$\operatorname{phism} \xi: \operatorname{Pol}_{\mathcal{C}}(G, H) \rightarrow \operatorname{Pol}_{\mathcal{D}}(\Gamma G, \Gamma H)$ as follows: $\xi(f):=\Gamma(f) \circ \psi_{L}^{-1}$, for $f: G^{L} \rightarrow H$. This preserves minors, because from the diagram's commutation we have:

$$
\xi\left(f \circ \pi_{G}\right)=\Gamma\left(f \circ \pi_{G}\right) \circ \psi_{L^{\prime}}^{-1}=\Gamma(f) \circ \Gamma\left(\pi_{G}\right) \circ \psi_{L^{\prime}}^{-1}=\Gamma(f) \circ \psi_{L}^{-1} \circ \pi_{\Gamma G}=\xi(f) \circ \pi_{\Gamma G}
$$

Corollary 4.9. Let $\Gamma: \mathcal{C} \rightarrow \mathcal{D}$ be a functor which preserves products. Let $\Omega$ be a thin right adjoint to $\Gamma$. Then there is a minion homomorphism $\operatorname{Pol}_{\mathcal{C}}(G, \Omega H) \rightarrow \operatorname{Pol}_{\mathcal{D}}(\Gamma G, H)$ for all $G \in \mathcal{C}, H \in \mathcal{D}$.

Proof. Since $\Gamma$ has a thin right adjoint $\Omega$, there exists a morphism $\varepsilon_{H}: \Gamma \Omega H \rightarrow H$ for all $H$ (we don't need it to be natural in any way). Hence we can compose the minion homomorphism $\operatorname{Pol}_{\mathcal{C}}(G, \Omega H) \rightarrow \operatorname{Pol}_{\mathcal{D}}(\Gamma G, \Gamma \Omega H)$ from Lemma 4.8 with the trivial minion homomorphism $\operatorname{Pol}_{\mathcal{D}}(\Gamma G, \Gamma \Omega H) \rightarrow \operatorname{Pol}_{\mathcal{D}}(\Gamma G, H)$ obtained by composing with $\varepsilon_{H}$.

If we have adjoint functors in the (non-thin) category of graphs (or multigraphs), then Lemma 4.7 implies a minion homomorphism between the standard polymorphism minions (because a morphism is associated with a function between vertex sets). One could also apply Lemma 4.7 to the thin category of graphs, but the conclusion is then about minions of polymorphisms in that thin category, which is useless, since it does not distinguish between different projections $G^{L} \rightarrow G$.

All the thin functors we have considered are in fact functors in the category of graphs or digraphs: in particular $\Lambda_{k}, \Gamma_{k}, \Omega_{k}, \delta_{L}, \delta, \delta_{R}$. The definitions can also be extended to give functors in the category of multi(di)graphs. The pairs $\left(\Lambda_{k}, \Gamma_{k}\right)$ and $\left(\delta_{L}, \delta\right)$ are adjoint pairs in the categories of multi(di)graphs (this fails in the category of (di)graphs; e.g. the number of homomorphisms $\Lambda_{3} G \rightarrow H$ is not always equal to the number of homomorphisms $\left.G \rightarrow \Gamma_{3} H\right)$. This implies minion homomorphisms $\operatorname{Pol}\left(\Lambda_{k} G, H\right) \rightarrow \operatorname{Pol}\left(G, \Gamma_{k} H\right)$ and $\operatorname{Pol}\left(\delta_{L} G, H\right) \rightarrow \operatorname{Pol}(G, \delta H)$.

In contrast, the pairs ( $\Gamma_{k}, \Omega_{k}$ ) and ( $\delta, \delta_{R}$ ) are not adjoint pairs; they are only thin adjoints. Since $\Gamma_{k}$ and $\delta$ are right adjoints (of $\Lambda_{k}$ and $\delta_{L}$ ), they preserve products. Applying Corollary 4.9 hence at least gives minion homomorphisms $\operatorname{Pol}\left(G, \Omega_{k} H\right) \rightarrow \operatorname{Pol}\left(\Gamma_{k} G, H\right)$ and $\operatorname{Pol}\left(G, \delta_{R} H\right) \rightarrow \operatorname{Pol}(\delta G, H)$. However, our results would only follow from the opposite direction. This is impossible to obtain in general: a minion homomorphism $\operatorname{Pol}(\delta G, H) \xrightarrow{?}$ $\operatorname{Pol}\left(G, \delta_{R} H\right)$ would imply the following minion homomorphism

$$
\operatorname{Pol}\left(K_{4}, K_{k}\right) \rightarrow \operatorname{Pol}\left(\delta K_{6}, K_{k}\right) \xrightarrow{?} \operatorname{Pol}\left(K_{6}, \delta_{R} K_{k}\right) \rightarrow \operatorname{Pol}\left(K_{6}, K_{2^{k}}\right)
$$

(trivially from $\delta K_{6} \rightarrow K_{4}$ and $\delta_{R} K_{k} \rightarrow K_{2^{k}}$ ), which is impossible by [Bar+19, Proposition 10.3]. Thus the seemingly technical difference between adjoints and thin adjoints turns out to be crucial.

As proved by Matsushita [Mat19], the hom complex $\operatorname{Hom}\left(K_{2},-\right)$ has a left adjoint from the category of $\mathbb{Z}_{2}$-simplicial complexes with $\mathbb{Z}_{2}$-simplicial maps to the category of graphs; the left adjoint preserves products.

## 5. Conclusions

The reduction in Lemma 3.3, on which our first main result relies, does not have a corresponding minion homomorphism. Given the simplicity of the reduction itself, this contrasts with the success of minion homomorphism in explaining other reductions between promise constraint satisfaction problems. It is to been seen whether this notion can be extended to a more general relation between polymorphism sets in a way that would imply Lemma 3.3.

The question of whether $K_{4}$ is left-hard stands open. In principle, it may be possible to extend the proof in Appendix A using more tools from algebraic topology to analyse $\mathbb{Z}_{2}$-maps $\left(\mathcal{S}^{1}\right)^{L} \rightarrow \mathcal{S}^{2}$ and deduce an appropriate minion homomorphism. It could also be interesting to consider how $\delta$ or $\delta_{R}$ affect the topology of a graph, cliques in particular.

Another direction could be to look at Huang's Theorem 3.6 not as a black-box: could constructions like $\delta$ be useful to say something directly about PCPs?

## A. Left-hardness using the box complex

## Basic definitions in topology

For topological spaces $X, Y$, we call a continuous function $f: X \rightarrow Y$ a map, for short. Two maps $f, g: X \rightarrow Y$ are homotopic if they can be continuously transformed into one another; formally: there is a family of maps $\phi_{t}: X \rightarrow Y$ for $t \in[0,1]$ (called a homotopy) such that $\phi_{0}=f, \phi_{1}=g$ and such that the function $(t, x) \mapsto \phi_{t}(x)$ from $[0,1] \times X$ to $Y$ is continuous. Two spaces $X, Y$ are homotopy equivalent if there are maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f$ and $f \circ g$ are homotopic to identity maps on $X$ and on $Y$.

We shall only consider topological spaces described in the following simple combinatorial way. A (simplicial) complex $K$ is a family of non-empty finite sets that is downward closed, in the sense that $\emptyset \neq \sigma^{\prime} \subseteq \sigma \in K$ implies $\sigma^{\prime} \in K$. The sets in $K$ are called faces (or simplices) of the complex, while their elements $V(K):=\bigcup_{\sigma \in K} \sigma$ are the vertices of the complex. The geometric realisation $|\sigma|$ of a face $\sigma \in K$ is the subset of $\mathbb{R}^{V(K)}$ defined as the convex hull of $\left\{e_{v} \mid v \in \sigma\right\}$, where $e_{v}$ is the standard basis vector corresponding to the $v$ coordinate in $\mathbb{R}^{V(K)}$. The geometric realisation $|K|$ of $K$ is the topological space obtained as the subspace $\bigcup_{\sigma \in K}|\sigma| \subseteq \mathbb{R}^{V(K)}$. We represent the points of $|K|$ as linear combinations of vertices $\lambda_{1} v_{1}+\ldots \lambda_{n} v_{n}$ such that $\left\{v_{1}, \ldots, v_{n}\right\} \in K$ and $\lambda_{i}$ are non-negative reals summing to 1 . We often refer to $K$ itself as a topological space, meaning $|K|$. A simplicial map $K \rightarrow K^{\prime}$ is a function $f: V(K) \rightarrow V\left(K^{\prime}\right)$ such that $f(\sigma):=\{f(v) \mid v \in \sigma\}$ is a face of $K^{\prime}$ whenever $\sigma$ is a face of $K$. It induces a map $|f|:|K| \rightarrow\left|K^{\prime}\right|$ by extending it linearly from vertices on each face: $|f|\left(\sum_{i} \lambda_{i} v_{i}\right):=\sum \lambda_{i} f\left(v_{i}\right)$.

For example, the circle may be represented as the triangle $K=\{\{1\},\{2\},\{3\}$, $\{1,2\},\{2,3\},\{3,1\}\}$, meaning that $|K|$, which is the sum of three intervals in $\mathbb{R}^{3}$, is homotopy equivalent to the unit circle $\mathcal{S}^{1}$ in $\mathbb{R}^{2}$. Adding the face $\{1,2,3\}$ to $K$ would make $|K|$ contractible, that is, homotopy equivalent to the one-point space.

## Equivariant topology - topology with symmetries

Rather than asking about "non-trivial maps" (maps not homotopic to a constant map) it is easier to work with equivariant topology, that is, considering topological spaces together with their symmetries and symmetry-preserving maps. A $\mathbb{Z}_{2}$-space is a topological space $X$ equipped with a map $-: X \rightarrow X$, called a $\mathbb{Z}_{2}$-action on $X$, satisfying $-(-x)=x$ (for all $x \in X$ ). We will call $-x$ the antipode of $x$. The main example is the $n$-dimensional sphere: the $\mathbb{Z}_{2}$-space defined as the unit sphere in $\mathbb{R}^{n+1}$ with $\mathbb{Z}_{2}$-action $x \mapsto-x$ as vectors. A $\mathbb{Z}_{2}$-map from $\left(X,-_{X}\right)$ to $\left(Y,-_{Y}\right)$ is a map $f: X \rightarrow Y$ that preserves the symmetry: $f\left(-_{X} x\right)={ }_{Y} f(x)$ (this is also called an equivariant map). We write $X \rightarrow_{\mathbb{Z}_{2}} Y$ if such a map exists (the $\mathbb{Z}_{2}$-actions being clear from context).

Standard notions extend in a fairly straightforward way to equivariant notions. A $\mathbb{Z}_{2}$ complex is a simplicial complex $K$ together with a function $-: V(K) \rightarrow V(K)$ such that $-(-v)=v$ and $-\sigma:=\{-v \mid v \in \sigma\} \in K$ for $\sigma \in K$; this induces a $\mathbb{Z}_{2}$-action on $|K|$. The product of two $\mathbb{Z}_{2}$-spaces $X, Y$ is $X \times Y$ with "simultaneous" $\mathbb{Z}_{2}$-action $(x, y) \mapsto(-x,-y)$. A homotopy $\phi_{t}$ between $\mathbb{Z}_{2}$-maps $f, g: X \rightarrow Y$ is called a $\mathbb{Z}_{2}$-homotopy if $\phi_{t}$ is a $\mathbb{Z}_{2}$-map for all $t \in[0,1]$. We say that two $\mathbb{Z}_{2}$-spaces $X, Y$ are $\mathbb{Z}_{2}$-homotopy equivalent, denoted $X \simeq_{\mathbb{Z}_{2}} Y$, if there are $\mathbb{Z}_{2}$-maps $f: X \rightarrow_{\mathbb{Z}_{2}} Y$ and $g: Y \rightarrow_{\mathbb{Z}_{2}} X$ such that $g \circ f$ and $f \circ g$ are $\mathbb{Z}_{2}$-homotopic to the identity. Note this is stronger than just requiring $X \rightarrow_{\mathbb{Z}_{2}} Y$ and $Y \rightarrow_{\mathbb{Z}_{2}} X$; homotopy equivalence is more similar to graph isomorphism than to homomorphic equivalence of graphs.

## The box complex - the topology of a graph

The box complex $\operatorname{Box}(G)$ of a graph $G$ is a $\mathbb{Z}_{2}$-complex defined as the family of vertex sets of complete bipartite subgraphs of $G \times K_{2}$ (with both sides non-empty) and their subsets. In particular it contains all edges of $G \times K_{2}$ and every $K_{2,2}=C_{4}$ subgraph. The topology of box complexes of the following graphs is folklore.

Lemma A.1. The following spaces are $\mathbb{Z}_{2}$-homotopy equivalent:
(i)
$\left|\operatorname{Box}\left(K_{n}\right)\right| \simeq_{\mathbb{Z}_{2}} \mathcal{S}^{n-2}$ for $n \geq 2$,
(ii) $\left|\operatorname{Box}\left(C_{n}\right)\right| \simeq_{\mathbb{Z}_{2}} \mathcal{S}^{1}$ for odd $n \geq 3$,
(iii) $\left|\operatorname{Box}\left(K_{p / q}\right)\right| \simeq_{\mathbb{Z}_{2}} \mathcal{S}^{1}$ for $2<\frac{p}{q}<4$,
(iv) for every loop-less square-free graph $K,|\operatorname{Box}(K)|$ is $\mathbb{Z}_{2}$-homotopy-equivalent to a 1-dimensional complex (a complex in which every face has at most 2 vertices).

Proof. For (i), see Proposition 19.8 in [Koz08], Proposition 4.3 in [BK06], or Lemma 5.9.2 in [Mat08]. Informally, the vertices of $\operatorname{Box}\left(K_{n}\right)$ can be mapped bijectively to points in $\mathbb{R}^{n}$ of the form $\pm e_{i}:=(0, \ldots, 0, \pm 1,0, \ldots, 0)$. These are vertices of the cross-polytope in $\mathbb{R}^{n}$ (the $n$-dimensional counterpart of the octahedron). Faces of $\operatorname{Box}\left(K_{n}\right)$ are exactly those subsets of $\left\{ \pm e_{1}, \ldots, \pm e_{n}\right\}$ that do not contain repeated indices ( $+e_{i}$ and $-e_{i}$ for any $i$ ), except for the two sets $\left\{+e_{1}, \ldots,+e_{n}\right\}$ and $\left\{-e_{1}, \ldots,-e_{n}\right\}$ (since a bipartite complete graph containing all $n$ vertices on one side cannot contain any vertex on the other side). The complex is thus isomorphic to the cross-polytope (the $n$-dimensional counterpart to the octahedron) in $\mathbb{R}^{n}$, but with the interior and two opposite facets removed. The cross-polytope after removing the interior is $\mathbb{Z}_{2}$-homotopy equivalent to $\mathcal{S}^{n-1}$ and after removing two opposite facets it is $\mathbb{Z}_{2}$-homotopy equivalent to $\mathcal{S}^{n-2}$.

For (iv), let use denote the two vertices of $\operatorname{Box}(K)$ corresponding to $v \in V(K)$ as $v^{\circ}$ and $v^{\bullet}$. Observe that $\operatorname{Box}(K)$ would be isomorphic to $K \times K_{2}$ (meaning the 1-dimensional simplicial complex with $V\left(K \times K_{2}\right)$ as vertices and with $E\left(K \times K_{2}\right)$ and their subsets as faces), except that it also contains $N\left[v^{\circ}\right]:=\left\{v^{\circ}\right\} \cup\left\{w^{\bullet}: w \in N(v)\right\}$ and $N\left[v^{\bullet}\right]$ for each $v \in V(K)$ (except those with empty neighbourhood). However, these additional faces can be collapsed. Formally, every face not in $E\left(K \times K_{2}\right)$ is either of the form $\left\{v^{\circ}, w_{1}^{\bullet}, \ldots, w_{n}^{\bullet}\right\}$ or $\left\{w_{1}^{\bullet}, \ldots, w_{n}^{\bullet}\right\}$ for some $w_{i} \in N(v)$ and $n \geq 2$, or the same with $\circ$ and • swapped. Since $K$ is square-free, even in the second case $v$ is uniquely determined by the $w_{i}$. Hence we can match these faces in pairs. This matching is easily checked to satisfy the definitions of a so-called acyclic $\mathbb{Z}_{2}$-matching in Discrete Morse Theory, which allows to show that removing these faces gives a $\mathbb{Z}_{2}$-homotopy equivalent complex: see Section 3 in [Wro19] for definitions and details.

For (ii), observe that by the above, $\operatorname{Box}\left(C_{n}\right)$ is $\mathbb{Z}_{2}$-homotopy equivalent to $C_{n} \times K_{2}=$ $C_{2 n}$ as a simplicial complex (for odd $n$ ). It is straightforward to give a $\mathbb{Z}_{2}$-homotopy equivalence (in fact a homeomorphism) to $\mathcal{S}^{1}$.

For (iii), we first consider the case when $p$ is odd. Then, $K_{p / q} \times K_{2}$ is isomorphic to the Caley graph $K^{\prime}$ of $\mathbb{Z}_{2 p}$ with generators $\{ \pm 1, \pm 3, \ldots, \pm p-2 q\}$ (the isomorphism maps $(i, 0)$ to $2 i$ and $(i, 1)$ to $2 i+p)$. In particular, $K^{\prime}$ includes a cycle $C_{2 p}$ on $0,1, \ldots, 2 p-1$ and the $\mathbb{Z}_{2}$-action on $K_{p / q} \times K_{2}$ correspond to point reflection on $C_{2 p}$. We thus have an inclusion map $\iota:\left|C_{2 p}\right| \rightarrow\left|K^{\prime}\right|$ (where $\left|K^{\prime}\right|$ is is shorthand for $\left|\operatorname{Box}\left(K_{p / q} \times K_{2}\right)\right|$ and $C_{2 p}$ is meant as a subcomplex). Note that $\frac{p}{q}<4$ is equivalent to $p-2 q<\frac{p}{2}$, so two adjacent vertices of $K^{\prime}$ are at distance at $<\frac{p}{2}$ in $C_{2 p}$. Therefore, every face of the box complex (a complete bipartite subgraph of $K^{\prime}$ ) is contained in an interval of length $<p$ in $\mathbb{Z}_{2 p}$. Every point in the geometric realization of such a face can be unambiguously mapped by linear extension in the interval to a point in the geometric realization of $C_{2 p}$, giving a $\mathbb{Z}_{2}$-map $f:\left|K^{\prime}\right| \rightarrow\left|C_{2 p}\right|$. The maps $\iota, f$ give a $\mathbb{Z}_{2}$-homotopy equivalence $\left(f \circ \iota:\left|C_{2 p}\right| \rightarrow\left|C_{2 p}\right|\right.$ is equal to the identity, while $\iota \circ f$ is $\mathbb{Z}_{2}$-homotopic to the identity, since one can also linearly extrapolate between the definition of $f$ and the identity map). The proof for even $p$ is similar, the main difference being that $K^{\prime}$ should be the graph on $\mathbb{Z}_{p} \times\{0,1\}$ with $(i, a)$ adjacent to $(j, b)$ if $a \neq b$ and $i, j$ are at distance $\leq \frac{p-2 q}{2}$.

Note that for a loop-less graph $K, \operatorname{Box}(K)$ is a free $\mathbb{Z}_{2}$-complex, which means every face $\sigma$ is disjoint from $-\sigma$. This in turn implies that $|\operatorname{Box}(K)|$ is a free $\mathbb{Z}_{2}$-space, which means that a point is never its own antipode. Proposition 5.3.2.(v) in [Mat08] shows that a free $\mathbb{Z}_{2}$-complex of dimension $n$ admits a $\mathbb{Z}_{2}$-map to $\mathcal{S}^{n}$. Hence for loop-less, square-free graphs $K$, we have $|\operatorname{Box}(K)| \rightarrow_{\mathbb{Z}_{2}} \mathcal{S}^{1}$.

## The hom complex - preserving products

Instead, we will use the Hom complex $\operatorname{Hom}\left(K_{2}, G\right)$, which is $\mathbb{Z}_{2}$-homotopy equivalent to $\operatorname{Box}(G)$, as proved by Csorba [Cso08]. Its vertices are homomorphisms $K_{2} \rightarrow G$, that is, oriented edges $(u, v)$ of $G$. For every $U, V \subseteq V(G)$ such that $U \times V \subseteq E(G), U \times V$ and its subsets are faces of $\operatorname{Hom}\left(K_{2}, G\right)$. In other words, a set $\sigma$ of oriented edges is a face if for every two $(u, v),\left(u^{\prime}, v^{\prime}\right) \in \sigma,\left(u, v^{\prime}\right)$ is an oriented edge of $G$. The $\mathbb{Z}_{2}$-action swaps $(u, v)$ to $(v, u)$.

This definition has the advantage that it respects products trivially (and exactly, not just up to homotopy equivalence): $\operatorname{Hom}\left(K_{2}, G \times H\right)$ is isomorphic to $\operatorname{Hom}\left(K_{2}, G\right) \times$
$\operatorname{Hom}\left(K_{2}, H\right)$ (as $\mathbb{Z}_{2}$-simplicial complexes). The isomorphism simply maps the oriented edge between pairs $\left(g_{1}, h_{1}\right)$ and $\left(g_{2}, h_{2}\right) \in V(G) \times V(H)$ to the pair of oriented edges $\left(\left(g_{1}, h_{1}\right),\left(g_{2}, h_{2}\right)\right)$. In the same way, $\operatorname{Hom}\left(K_{2}, G^{L}\right)$ is isomorphic $\operatorname{Hom}\left(K_{2}, G\right)^{L}$, mapping pairs of $L$-tuples to $L$-tuples of pairs.
Lemma A.2. Let $f: G^{L} \rightarrow H$ be a graph homomorphism. Let $f^{\prime}: \operatorname{Hom}\left(K_{2}, G\right)^{L} \rightarrow$ $\operatorname{Hom}\left(K_{2}, H\right)$ be the induced simplical $\mathbb{Z}_{2}$-map, defined as:

$$
f^{\prime}\left(\left(u_{1}, v_{1}\right), \ldots,\left(u_{L}, v_{L}\right)\right):=\left(f\left(u_{1}, \ldots, u_{L}\right), f\left(v_{1}, \ldots, v_{L}\right)\right) .
$$

Then the transformation $f \mapsto f^{\prime}$ preserves minors and composition.
This is straightforward from the definitions. Here by compositions we mean functions of the form $h\left(f\left(g_{1}\left(x_{1}\right), \ldots, g_{L}\left(x_{L}\right)\right)\right)$ for $g_{i}: G^{\prime} \rightarrow G$ and $h: H \rightarrow H^{\prime}$; the graph homomorphisms $g_{i}$ and $h$ induce simplicial maps just as above for $L=1$. Preserving compositions means in particular that if $\mu$ is an automorphism of $G$ and $\mu^{\prime}$ is the automorphism of $\operatorname{Hom}\left(K_{2}, G\right)$ it induces, then $f\left(x_{1}, \ldots, \mu\left(x_{i}\right), \ldots, x_{L}\right)$ induces $\left.f^{\prime}\left(x_{1}, \ldots, \mu^{\prime}\left(x_{i}\right), \ldots, x_{L}\right)\right)$.

In the geometric realisation, the above-mentioned isomorphism induces (by linear extension) an isomorphism from $\left|\operatorname{Hom}\left(K_{2}, G \times H\right)\right|$ to $\left|\operatorname{Hom}\left(K_{2}, G\right) \times \operatorname{Hom}\left(K_{2}, H\right)\right|$. The latter has a natural $\mathbb{Z}_{2}$-homotopy equivalence to $\left|\operatorname{Hom}\left(K_{2}, G\right)\right| \times\left|\operatorname{Hom}\left(K_{2}, H\right)\right|$, implicit in the following claim:
Lemma A.3. Let $f: X^{L} \rightarrow Y$ be a $\mathbb{Z}_{2}$-simplicial map and let $x_{0} \in V(X)$. Let $|f|:|X|^{L} \rightarrow$ $|Y|$ be the induced $\mathbb{Z}_{2}$-map, defined as:

$$
|f|\left(\sum_{i} \lambda_{i}^{(1)} v_{i}^{(1)}, \ldots, \sum_{i} \lambda_{i}^{(L)} v_{i}^{(L)}\right):=\sum_{i_{1}, \ldots, i_{L}} \lambda_{i_{1}} \cdots \lambda_{i_{L}} f\left(v_{i_{1}}^{(1)}, \ldots, v_{i_{L}}^{(L)}\right)
$$

for faces $\left\{v_{i}^{(1)} \mid i\right\}, \ldots,\left\{v_{i}^{(L)} \mid i\right\} \in X$. Then the transformation $f \mapsto|f|$ preserves minors up to $\mathbb{Z}_{2}$-homotopy rel $x_{0}$ and preserves composition exactly.

Proof. Preservation of composition is again straightforward.
To see that the transformation preserves minors, consider for example the contraction (identification) of two coordinates. The general case is entirely analogous. Let $f: X^{2} \rightarrow Y$ and let $f_{/ 2}: X \rightarrow Y$ be the minor obtained by contracting the two coordinates. Then

$$
\left|f_{/ 2}\right|\left(\sum_{i} \lambda_{i} v_{i}\right)=\sum_{i} \lambda_{i} f_{/ 2}\left(v_{i}\right)=\sum_{i} \lambda_{i} f\left(v_{i}, v_{i}\right) .
$$

On the other hand, if we take the induced map first and only then contract, we obtain:

$$
|f|_{/ 2}\left(\sum_{i} \lambda_{i} v_{i}\right)=|f|\left(\sum_{i} \lambda_{i} v_{i}, \sum_{i} \lambda_{i} v_{i}\right)=\sum_{i, j} \lambda_{i} \lambda_{j} f\left(v_{i}, v_{j}\right) .
$$

The first point is in the face $\left\{f\left(v_{i}, v_{i}\right) \mid i\right\}$ of $Y$, the second is in the face $\left\{f\left(v_{i}, v_{j}\right) \mid i, j\right\}$ of $Y$ which contains the former. We can thus continuously move from one to the other. Formally, let $\mu_{i, j}:=\lambda_{i}$ if $i=j$ and 0 otherwise. Then the functions (for $t \in[0,1]$ )

$$
f_{t}\left(\sum_{i} \lambda_{i} v_{i}\right):=\left(t \cdot \mu_{i, j}+(1-t) \cdot \lambda_{i} \lambda_{j}\right) f\left(v_{i}, v_{j}\right)
$$

are always well-defined and give a $\mathbb{Z}_{2}$-homotopy between $\left|f_{/ 2}\right|$ and $|f|_{/ 2}$. For any vertex $x_{0}$ (i.e. $\lambda_{1}=1$ ) $f_{t}\left(x_{0}\right)$ is constantly equal to $f\left(x_{0}\right)$.

We thus have a minion homomorphism from $\operatorname{Pol}(G, H)$ to the minion of maps-up-tohomotopy $\left|\operatorname{Hom}\left(K_{2}, G\right)\right|^{L} \rightarrow\left|\operatorname{Hom}\left(K_{2}, H\right)\right|$, which preserves automorphisms of $G$. This, as well as the minion homomorphism in the following subsection, can be interpreted as an instance of Lemma 4.8.

## The fundamental group

For a topological space $|X|$ and a point $x_{0} \in|X|$, two maps from $|X|$ to some topological space are homotopic rel $x_{0}$ if there are homotopies that do not move the image of $x_{0}$. In the fundamental group $\pi_{1}\left(|X|, x_{0}\right)$, the elements are equivalence classes of loops at $x_{0}$ (maps $[0,1] \rightarrow|X|$ mapping 0 and 1 to $x_{0}$ ) under homotopy rel $x_{0}$, the group operation is concatenation. We skip $x_{0}$ when it is not important, since $\pi_{1}\left(|X|, x_{0}\right)$ is always isomorphic to $\pi_{1}\left(|X|, x_{0}^{\prime}\right)$ if $|X|$ is path-connected ${ }^{10}$ which we implicitly assume throughout.

Including information about the $\mathbb{Z}_{2}$-symmetry in the fundamental group is a bit less obvious. For a $\mathbb{Z}_{2}$-space $|X|$ we can look at the fundamental group of $|X|$ but also the fundamental group of the quotient $|X|_{\mathbb{Z}_{2}}$ (where every point is identified with its antipode; a.k.a. the orbit space or base space; we denote the equivalence class of $x$ by $\pm x$ ). One way to think of elements of $\pi_{1}\left(|X|_{/ \mathbb{Z}_{2}}, \pm x_{0}\right)$ is as paths from $x_{0}$ to either $x_{0}$ or $-x_{0}$, with concatenation defined using the $\mathbb{Z}_{2}$-action if necessary. Observe that $\pi_{1}\left(|X|_{\mathbb{Z}_{2}}\right)$ contains $\pi_{1}(|X|)$ as a subgroup, consisting of paths from $x_{0}$ to $x_{0}$.

Another way to describe the subgroup is by a group homomorphism to $\nu_{X}: \pi_{1}\left(|X|_{\mathbb{Z}_{2}}\right) \rightarrow$ $\mathbb{Z}_{2}$ mapping the subgroup (paths $x_{0}$ to $x_{0}$ ) to 0 and everything else (paths $x_{0}$ to $-x_{0}$ ) to 1 . Thus $\pi_{1}(|X|)$ is the subgroup given by the kernel of $\nu_{X} .{ }^{11}$

For example, consider $\mathcal{S}^{1}$. The quotient $\mathcal{S}_{/ \mathbb{Z}_{2}}^{1}$ is again a circle, so $\pi_{1}\left(\mathcal{S}_{/ \mathbb{Z}_{2}}^{1}\right)$ is isomorphic to $\mathbb{Z}$ (a loop in the quotient is represented by its winding number); $\nu$ is the remainder $\bmod 2$ (loops with odd winding number in the quotient correspond to paths from a point to its antipode in $\left.\mathcal{S}^{1}\right)$ and $\pi_{1}\left(\mathcal{S}^{1}\right)$ is the subgroup $2 \mathbb{Z}$ of even integers. In contrast, the quotient $\mathcal{S}_{/ \mathbb{Z}_{2}}^{2}$ is the projective plane, so $\pi_{1}\left(\mathcal{S}_{/ \mathbb{Z}_{2}}^{2}\right)$ is isomorphic to $\mathbb{Z}_{2} ; \nu$ is the identity and the subgroup $\pi_{1}\left(\mathcal{S}^{2}\right)$ is the trivial group.

A map $f:|X| \rightarrow|Y|$ induces a group homomorphism $f_{*}: \pi_{1}\left(|X|_{/ \mathbb{Z}_{2}}\right) \rightarrow \pi_{1}\left(|Y|_{/ \mathbb{Z}_{2}}\right)$, simply by composing a loop with $f$. This homomorphism preserves the subgroup: $\nu_{Y}\left(f_{*}(x)\right)=\nu_{X}(x)$. Equivalently, $f_{*}^{-1}\left(\pi_{1}(|Y|)\right)=\pi_{1}(|X|)$.

The fundamental group of a product $\pi_{1}\left(|X| \times|Y|,\left(x_{0}, y_{0}\right)\right)$ is isomorphic to the direct product of fundamental groups $\pi_{1}\left(|X|, x_{0}\right) \times \pi_{1}\left(|Y|, y_{0}\right)$. The isomorphism just maps a loop $\mathcal{S}^{1} \rightarrow|X| \times|Y|$ to the pair of loops obtained by composing with projections; the inverse maps a pair of loops $p: \mathcal{S}^{1} \rightarrow|X|$ and $q: \mathcal{S}^{1} \rightarrow|Y|$ to the "simultaneous" loop $(p, q): t \mapsto(p(t), q(t))$.

However, $\pi_{1}\left((|X| \times|Y|)_{\mathbb{Z}_{2}}\right)$ is not isomorphic to $\pi_{1}\left(|X|_{\mathbb{Z}_{2}}\right) \times \pi_{1}\left(|Y|_{/ \mathbb{Z}_{2}}\right)$, but to the subgroup of it given by elements $(x, y)$ such that $\nu_{X}(x)=\nu_{Y}(y)$. Indeed, it contains paths from $\left(x_{0}, y_{0}\right)$ to either $\left(x_{0}, y_{0}\right)$ or ( $-x_{0},-y_{0}$ ) but not to $\left(x_{0},-y_{0}\right)$.

In other words, to a $\mathbb{Z}_{2}$-space $|X|$ we assign a group $\pi_{1}\left(|X|_{\mathbb{Z}_{2}}\right)$ together with a group homomorphism $\nu_{X}$ to $\mathbb{Z}_{2}$. Consider the category whose objects are such pairs $(G, \nu)$ (a group with a homomorphism to $\mathbb{Z}_{2}$ ), while morphisms $\left(G, \nu_{G}\right) \rightarrow\left(H, \nu_{H}\right)$ are group homomorphisms $G \rightarrow H$ preserving $\nu$. The categorical product of ( $G, \nu_{G}$ ) and $\left(H, \nu_{H}\right)$ is $\left\{(g, h) \in G \times H: \nu_{G}(g)=\nu_{H}(h)\right\}$ with coordinate-wise multiplication and the homomorphism to $\mathbb{Z}_{2}$ defined in an obvious way $\left(\nu(g, h):=\nu_{G}(g)=\nu_{H}(h)\right)$.

[^6]Let us denote this product as $\star$ for clarity ${ }^{12}$ and the $L$-fold product of $\left(G, \nu_{G}\right)$ as $G^{\star L}$. Then a $\mathbb{Z}_{2}$-map $f:|X|^{L} \rightarrow|Y|$ mapping a point $x_{0}$ to $y_{0}$ induces a morphism $f_{*}: \pi_{1}\left(|X|_{\mathbb{Z}_{2}}, \pm x_{0}\right)^{\star L} \rightarrow \pi_{1}\left(|Y|_{/_{2}}, \pm y_{0}\right)$ (a group homomorphism preserving $\nu$ ):

$$
f_{*}\left(\left[p_{1}\right], \ldots,\left[p_{L}\right]\right):=\left[t \mapsto f\left(p_{1}(t), \ldots, p_{L}(t)\right)\right]
$$

(where $[p]$ denotes the equivalence class of a loop $\mathcal{S}^{1} \rightarrow|X|_{/ \mathbb{Z}_{2}}$ under homotopy rel $\pm x_{0}$ ).
The following is straightforward to check from the definition of $f_{*}$ :
Lemma A.4. Let $f:|X|^{L} \rightarrow|Y|$ be $\mathbb{Z}_{2}$-map, let $x_{0} \in|X|$ be an arbitrary point and let $f\left(x_{0}\right)=y_{0}$. Let $f_{*}: \pi_{1}\left(|X|_{/ \mathbb{Z}_{2}}{ }^{\star L} \rightarrow \pi_{1}\left(|Y|_{/ \mathbb{Z}_{2}}\right)\right.$ be the induced group homomorphism. Then the transformation $f \mapsto f_{*}$ preserves minors and preserves automorphisms of $|X|$ that fix $x_{0} .{ }^{13}$

## Wrapping it up

Let us denote $\pi_{1}\left(\left|\operatorname{Hom}\left(K_{2}, G\right)\right|\right)$ and $\pi_{1}\left(\left|\operatorname{Hom}\left(K_{2}, G\right)\right| / \mathbb{Z}_{2}\right)$ as respectively $\pi_{1}(G)$ and $\pi_{1}\left(G_{/ \mathbb{Z}_{2}}\right)$, for short.

Consider a graph homomorphism $f: C_{n}^{L} \rightarrow H$ ( $n$ odd). We have $\left|\operatorname{Hom}\left(K_{2}, C_{n}\right)\right| \simeq_{\mathbb{Z}_{2}} \mathcal{S}^{1}$ and hence $\pi_{1}\left(C_{n / \mathbb{Z}_{2}}\right)$ is $\mathbb{Z}$ with a group homomorphism $\nu_{C_{n}}: i \mapsto(i \bmod 2)$. In particular $\mathbb{Z}^{\star L}$ is the subgroup of $\mathbb{Z}^{L}$ given by $L$-tuples in which the integers are all even or all odd and $\pi_{1}\left(C_{n}\right)$ is the subgroup $2 \mathbb{Z}$ of even integers in $\mathbb{Z}$. For an arbitrarily fixed edge $e_{0}$ of $C_{n}$, the automorphism $\mu_{C_{n}}$ that mirrors the graph and fixes $e_{0}$ induces the automorphism of $\mathbb{Z}$ which maps $i$ to $-i$.

Therefore, composing the transformations from Lemmas A.2, A.3, and A.4, we obtain a group homomorphism $f_{*}: \mathbb{Z}^{\star L} \rightarrow \pi_{1}\left(H_{/ \mathbb{Z}_{2}}\right)$ which preserves the homomorphism to $\mathbb{Z}_{2}$ and the mirror automorphism on each coordinate.

Suppose that $\left|\operatorname{Hom}\left(K_{2}, H\right)\right| \simeq_{\mathbb{Z}_{2}} \mathcal{S}_{1}$, so again $\pi_{1}\left(H_{/ \mathbb{Z}_{2}}\right)=\mathbb{Z}$ with the same homomorphism to $\mathbb{Z}_{2}(i \bmod 2)$ and the same mirror automorphism $(-i)$. Since $f_{*}$ preserves the homomorphism to $\mathbb{Z}_{2}, d:=f_{*}(1,1, \ldots, 1) \in \mathbb{Z}$ is an odd number, which means $f_{*}(2,2, \ldots, 2)=2 d$ is non-zero. This is why we needed the $\mathbb{Z}_{2}$-action: to conclude that $f_{*}$ is non-trivial. We can now focus on what $f_{*}$ does on the subgroup of even integers.

Let $a_{\ell}:=f_{*}(0, \ldots, 0,2,0, \ldots, 0) \in \mathbb{Z}$ with a 2 in the $\ell$-th coordinate. Then $f_{*}$ on even numbers is completely determined by these elements: $f_{*}\left(2 i_{1}, \ldots, 2 i_{L}\right)=a_{1} \cdot i_{1}+$ $\cdots+a_{L} \cdot i_{L}$ (because it is a group homomorphism). By the above, $\sum_{\ell=1}^{L} a_{\ell}$ is nonzero. Since $f \mapsto f_{*}$ preserves minors, we know that the minor $i \mapsto f_{*}(i, i, \ldots, i)$ is a group homomorphism induced by some graph homomorphism $C_{n} \rightarrow H$ (namely by the corresponding minor $v \mapsto f(v, \ldots, v)$ ), hence the integer $f_{*}(2,2, \ldots, 2)$ belongs to a set of at most $|H|^{n}$ possibilities. The same holds for compositions with mirror symmetries: the group homomorphism $i \mapsto f_{*}(i, \ldots,-i, \ldots, i)$ with a minus on any subset of coordinates is induced by the graph homomorphism $C_{n} \rightarrow H$ defined as $f\left(v, \ldots, \mu_{C_{n}}(v), \ldots, v\right)$ with $\mu_{C_{n}}$ on the same set of coordinates. Hence for $i_{1}, \ldots, i_{L} \in\{+1,-1\}$, the values $f_{*}\left(2 i_{1}, 2 i_{2}, \ldots, 2 i_{L}\right)=a_{1} \cdot i_{1}+\cdots+a_{L} \cdot i_{L}$ belong to a set of at most $|H|^{n}$ possibilities. This implies less than $|H|^{n}$ of the integers $a_{\ell}$ are non-zero. Indeed, if there are $L^{\prime}$ coordinates

[^7]$\ell$ for which $a_{\ell}$ is non-zero, then one can set the corresponding $i_{\ell}$ to make $a_{\ell} \cdot i_{\ell}$ positive, and then swap $i_{\ell}$ one-by-one in any order, resulting in a strictly decreasing sequence of values $a_{1} \cdot i_{1}+\cdots a_{L} \cdot i_{L}$, hence in $L^{\prime}+1$ distinct values. Hence $L^{\prime}+1 \leq|H|^{n}$.

Therefore, the group homomorphism $\left(2 i_{1}, \ldots, 2 i_{L}\right) \mapsto f_{*}\left(2 i_{1}, \ldots, 2 i_{L}\right):(2 \mathbb{Z})^{L} \rightarrow(2 \mathbb{Z})$ has bounded (but non-zero) essential arity. Note that this is exactly the homomorphism $\left.f_{*}\right|_{\pi_{1}\left(C_{n}\right)^{L}}$, from the subgroup $\pi_{1}\left(C_{n}\right)^{L}$ to the subgroup $\pi_{1}(H)$. Therefore, the transformation $\left.f \mapsto f_{*}\right|_{\pi_{1}\left(C_{n}\right)^{L}}$ is a minion homomorphism from $\operatorname{Pol}\left(C_{n}, H\right)$ to a minion of functions of bounded essential arity.

The same argument would work if instead of $\left|\operatorname{Hom}\left(K_{2}, H\right)\right| \simeq_{\mathbb{Z}_{2}} \mathcal{S}_{1}$ we only assumed we had a $\mathbb{Z}_{2}$-map $g:\left|\operatorname{Hom}\left(K_{2}, H\right)\right| \rightarrow \mathcal{S}_{1}$, since it would induce a group homomorphism $g_{*}: \pi_{1}\left(H_{/ \mathbb{Z}_{2}}\right) \rightarrow \mathbb{Z}$ which preserves the homomorphism to $\mathbb{Z}_{2}$, in a way that preserves mirror automorphisms of $\mathcal{S}^{1}$; it then suffices to compose $g_{*}$ with $f_{*}$ and continue as above.

This concludes the proof of the following:
Theorem A.5. Let $H$ be a graph such that $\left|\operatorname{Hom}\left(K_{2}, H\right)\right| \rightarrow_{\mathbb{Z}_{2}} \mathcal{S}_{1}$. Then for all odd $n$, $\operatorname{Pol}\left(C_{n}, H\right)$ admits a minion homomorphism to a minion of bounded essential arity with no constant functions.

By Theorem 4.6, this concludes the direct proof that $\operatorname{PCSP}\left(C_{n}, H\right)$ is NP-hard for all odd $n$ :

Corollary A.6. Let $H$ be a graph such that $\left|\operatorname{Hom}\left(K_{2}, H\right)\right| \rightarrow_{\mathbb{Z}_{2}} \mathcal{S}_{1}$. Then $H$ is left-hard.
Since $\left|\operatorname{Hom}\left(K_{2}, H\right)\right|$ is $\mathbb{Z}_{2}$-homotopy equivalent to $|\operatorname{Box}(H)|$ (hence they admit the same $\mathbb{Z}_{2}$-maps), this is exactly equivalent to Corollary 2.8; in particular it gives a proof of Theorem 2.4.

## Further remarks

In the case of $\mathcal{S}^{1}$, the fact that a $\mathbb{Z}_{2}$-map $g:\left|\operatorname{Hom}\left(K_{2}, H\right)\right| \rightarrow \mathcal{S}_{1}$ induces a group homomorphism $g_{*}: \pi_{1}\left(H_{/ \mathbb{Z}_{2}}\right) \rightarrow \mathbb{Z}$ which preserves the homomorphism to $\mathbb{Z}_{2}$ is in fact an exact characterisation. That is, as stated by Matsushita [Mat19], standard covering space theory yields the following:

Lemma A.7. A connected $\mathbb{Z}_{2}$-space $|X|$ admits a $\mathbb{Z}_{2}$-map to $\mathcal{S}^{1}$ if and only if there exists a group homomorphism $f: \pi_{1}\left(|X|_{\mathbb{Z}_{2}}\right) \rightarrow \mathbb{Z}$ which preserves the action (that is, $\left.f^{-1}(2 \mathbb{Z})=\pi_{1}(X)\right)$.

In the above proof, one could go directly from graphs to fundamental groups, avoiding simplicial complexes and topological spaces (though they remain the simplest way to prove that these fundamental groups preserve products). A direct definition of the fundamental group of the quotient space $|\operatorname{Box}(H)|_{\mathbb{Z}_{2}}$ is as follows. We consider closed walks (cycles that are allowed to self-intersect) from an arbitrary fixed vertex $v_{0} \in V(H)$. Two such walks are consider equivalent if one can be obtained from the other by adding/removing backtracks (a pair of consecutive edges going back and forth on the same edge of $H$ ) and 4 -cycles (subwalks around a cycle of length 4). The elements of the group are equivalence classes of walks, with concatenation as multiplication. The resulting group is isomorphic to $\pi_{1}\left(H_{/ K_{2}}\right)$ (this combinatorial definition is known as the edge-path group; see [Mat17] or Section 3.6 and 3.7 in [Spa66]). Considering walks in $H \times K_{2}$ instead would yield a group isomorphic to $\pi_{1}(H)$.

For example, for odd cycles and more generally circular cliques $<4$ the group is just $\mathbb{Z}$ (Lemma 4.1 in [Wro17] has a direct but technical proof), for square-free graphs the group is a free (non-Abelian) group. For $K_{4}$, the resulting group is just $\mathbb{Z}_{2}$ (all walks of the same parity are equivalent), which corresponds to the fact that $\left|\operatorname{Box}\left(K_{4}\right)\right|$ is the 2-sphere and $\left|\operatorname{Box}\left(K_{4}\right)\right|_{\mathbb{Z}_{2}}$ is the projective plane.

Unfortunately, this makes the fundamental group useless for the question of whether $K_{4}$ is left-hard. Indeed, there is only one possible induced group homomorphism $f_{*}: \mathbb{Z}^{\star L} \rightarrow$ $\pi_{1}\left(K_{4 / \mathbb{Z}_{2}}\right)=\pi_{1}\left(\mathbb{R} P_{2}\right)=\mathbb{Z}_{2}$ : it maps $L$-tuples of even integers to 0 and $L$-tuples of odd integers to 1 (because it has to preserve the homomorphism to $\mathbb{Z}_{2}$, which is the identity). Whether other tools of algebraic topology can be useful remains to be seen.

## References

[ABP19] P. Austrin, A. Bhangale, and A. Potukuchi. Simplified inpproximability of hypergraph coloring via $t$-agreeing families. Tech. rep. 2019. arXiv: 1904.01163.
[ABP20] P. Austrin, A. Bhangale, and A. Potukuchi. "Improved Inapproximability of Rainbow Coloring". Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'20). (to appear). 2020. arXiv: 1810.02784.
[AGH17] P. Austrin, V. Guruswami, and J. Håstad. "(2+e)-SAT Is NP-hard". SIAM J. Comput. 46.5 (2017), pp. 1554-1573.
[Bar+19] L. Barto, J. Bulín, A. Krokhin, and J. Opršal. Algebraic approach to promise constraint satisfaction. Tech. rep. 2019. arXiv: 1811.00970v3.
[BG16] J. Brakensiek and V. Guruswami. "New Hardness Results for Graph and Hypergraph Colorings". Proceedings of the 31st Conference on Computational Complexity (CCC'16). Vol. 50. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016, 14:1-14:27.
[BG17] J. Brakensiek and V. Guruswami. "The Quest for Strong Inapproximability Results with Perfect Completeness". Proceedings of the 21st International Workshop on Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques and the 20th International Workshop on Randomization and Computation (APPROX-RANDOM'17). Vol. 81. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017, 4:1-4:20.
[BG18] J. Brakensiek and V. Guruswami. "Promise Constraint Satisfaction: Structure Theory and a Symmetric Boolean Dichotomy". Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'18). SIAM, 2018, pp. 1782-1801. arXiv: 1704.01937.
[BG19] J. Brakensiek and V. Guruswami. "An Algorithmic Blend of LPs and Ring Equations for Promise CSPs". Proceedings of the 30th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'19). SIAM, 2019, pp. 436-455. arXiv: 1807.05194.
[Bha18] A. Bhangale. "NP-Hardness of Coloring 2-Colorable Hypergraph with Poly-Logarithmically Many Colors". Proceedings of the 45th International Colloquium on Automata, Languages, and Programming (ICALP'18). Vol. 107. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018, 15:1-15:11.
[BJK05] A. Bulatov, P. Jeavons, and A. Krokhin. "Classifying the Complexity of Constraints using Finite Algebras". SIAM Journal on Computing 34.3 (2005), pp. 720-742.
[BK06] E. Babson and D. N. Kozlov. "Complexes of graph homomorphisms". Israel Journal of Mathematics 152.1 (2006), pp. 285-312. arXiv: math/0310056.
[BK14] L. Barto and M. Kozik. "Constraint Satisfaction Problems Solvable by Local Consistency Methods". Journal of the ACM 61.1 (2014).
[BK16] L. Barto and M. Kozik. "Robustly Solvable Constraint Satisfaction Problems". SIAM Journal on Computing 45.4 (2016), pp. 1646-1669. arXiv: 1512.01157.
[BKO19] J. Bulín, A. Krokhin, and J. Opršal. "Algebraic approach to promise constraint satisfaction". Proceedings of the 51st Annual ACM Symposium on Theory of Computing (STOC'19). ACM, 2019, pp. 602-613. arXiv: 1811.00970.
[BKW17] L. Barto, A. Krokhin, and R. Willard. "Polymorphisms, and how to use them". Complexity and approximability of Constraint Satisfaction Problems. Vol. 7. Dagstuhl Follow-Ups. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017, pp. 1-44.
[BL03] A. Björner and M. de Longueville. "Neighborhood Complexes of Stable Kneser Graphs". Combinatorica 23.1 (2003), pp. 23-34.
[Bul06] A. Bulatov. "A dichotomy theorem for constraint satisfaction problems on a 3-element set". Journal of the ACM 53.1 (2006), pp. 66-120.
[Bul11] A. A. Bulatov. "Complexity of conservative constraint satisfaction problems". ACM Transactions on Computational Logic 12.4 (2011). Article 24.
[Bul17] A. A. Bulatov. "A Dichotomy Theorem for Nonuniform CSPs". Proceedings of the 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS'17). IEEE, 2017, pp. 319-330. arXiv: 1703.03021.
[Cso08] P. Csorba. "On the Simple $\mathbb{Z}_{2}$-homotopy Types of Graph Complexes and Their Simple $\mathbb{Z}_{2}$-universality". Canad. Math. Bull. 51.4 (2008), pp. 535-544.
[Din+18] I. Dinur, S. Khot, G. Kindler, D. Minzer, and M. Safra. "Towards a proof of the 2-to-1 games conjecture?" Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing (STOC'18). 2018, pp. 376-389.
[DMR09] I. Dinur, E. Mossel, and O. Regev. "Conditional Hardness for Approximate Coloring". SIAM J. Comput. 39.3 (2009), pp. 843-873. arXiv: cs/0504062.
[DRS05] I. Dinur, O. Regev, and C. D. Smyth. "The Hardness of 3-Uniform Hypergraph Coloring". Combinatorica 25.5 (2005), pp. 519-535.
[DS10] I. Dinur and I. Shinkar. "On the Conditional Hardness of Coloring a 4-Colorable Graph with Super-Constant Number of Colors". Proceedings of the 13th International Workshop on Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques and the 14 th International Workshop on Randomization and Computation (APPROXRANDOM'10). Vol. 6302. Lecture Notes in Computer Science. Springer, 2010, pp. 138151.
[Fic+19] M. Ficak, M. Kozik, M. Olšák, and S. Stankiewicz. "Dichotomy for symmetric Boolean PCSPs". Proceedings of the 46 th International Colloquium on Automata, Languages, and Programming (ICALP'19). LIPIcs. (to appear). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019. arXiv: 1904.12424.
[FT13] J. Foniok and C. Tardif. "Adjoint functors in graph theory" (2013). arXiv: 1304.2215.
[FT15] J. Foniok and C. Tardif. "Digraph functors which admit both left and right adjoints". Discrete Math. 338.4 (2015), pp. 527-535. arXiv: 1304.2204.
[FT18] J. Foniok and C. Tardif. "Hedetniemi's Conjecture and Adjoint Functors in Thin Categories". Applied Categorical Structures 26 (1 2018), pp. 113-128. arXiv: 1608.02918.
[FV98] T. Feder and M. Y. Vardi. "The Computational Structure of Monotone Monadic SNP and Constraint Satisfaction: A Study through Datalog and Group Theory". SIAM Journal on Computing 28.1 (1998), pp. 57-104.
[GJ76] M. R. Garey and D. S. Johnson. "The Complexity of Near-Optimal Graph Coloring". J. ACM 23.1 (1976), pp. 43-49.
[GK04] V. Guruswami and S. Khanna. "On the Hardness of 4-Coloring a 3-Colorable Graph". SIAM J. Discrete Math. 18.1 (2004), pp. 30-40.
[GL18] V. Guruswami and E. Lee. "Strong Inapproximability Results on Balanced Rainbow-Colorable Hypergraphs". Combinatorica 38.3 (2018), pp. 547-599.
[HE72] C. Harner and R. Entringer. "Arc Colorings of Digraphs". J. Comb. Theory, Ser. B 13.3 (1972), pp. 219-225.
[HN04] P. Hell and J. Nešetřil. Graphs and Homomorphisms. Vol. 28. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, 2004.
[HN08] P. Hell and J. Nešetřil. "Colouring, constraint satisfaction, and complexity". Computer Science Review 2.3 (2008), pp. 143-163.
[HN90] P. Hell and J. Nešetřil. "On the complexity of H-coloring". Journal of Combinatorial Theory, Series B 48.1 (1990), pp. 92-110.
[Hua13] S. Huang. "Improved Hardness of Approximating Chromatic Number". Proceedings of the 16th International Workshop on Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques and the 17th International Workshop on Randomization and Computation (APPROX-RANDOM'13). Springer, 2013, pp. 233-243. arXiv: 1301.5216.
[JCG97] P. Jeavons, D. A. Cohen, and M. Gyssens. "Closure properties of constraints". Journal of the ACM 44.4 (1997), pp. 527-548.
[Kar72] R. M. Karp. "Reducibility Among Combinatorial Problems". Proceedings of a Symposium on the Complexity of Computer Computations. 1972, pp. 85-103.
[Kho01] S. Khot. "Improved Inaproximability Results for MaxClique, Chromatic Number and Approximate Graph Coloring". Proceedings of the 42nd Annual Symposium on Foundations of Computer Science (FOCS'01). IEEE Computer Society, 2001, pp. 600-609.
[Kho02] S. Khot. "On the power of unique 2-prover 1-round games". Proceedings of the 34th Annual ACM Symposium on Theory of Computing (STOC'02). ACM, 2002, pp. 767-775.
[KLS00] S. Khanna, N. Linial, and S. Safra. "On the Hardness of Approximating the Chromatic Number". Combinatorica 20.3 (2000), pp. 393-415.
[KMS18] S. Khot, D. Minzer, and M. Safra. "Pseudorandom Sets in Grassmann Graph Have NearPerfect Expansion". Proceedings of the 59th IEEE Annual Symposium on Foundations of Computer Science (FOCS'18). IEEE Computer Society, 2018, pp. 592-601.
[KO19] A. Krokhin and J. Opršal. "The complexity of 3-colouring $H$-colourable graphs". Proceedings of the 60th Annual IEEE Symposium on Foundations of Computer Science (FOCS'19). (to appear). 2019. arXiv: 1904.03214.
[Koz08] D. Kozlov. Combinatorial algebraic topology. Vol. 21. Algorithms and Computation in Mathematics. Springer, 2008, pp. XX, 390.
[KT17] K. Kawarabayashi and M. Thorup. "Coloring 3-Colorable Graphs with Less than $n^{1 / 5}$ Colors". J. ACM 64.1 (2017), 4:1-4:23.
[Lov78] L. Lovász. "Kneser's Conjecture, Chromatic Number, and Homotopy". J. Comb. Theory, Ser. A 25.3 (1978), pp. 319-324.
[Mat08] J. Matoušek. Using the Borsuk-Ulam theorem: lectures on topological methods in combinatorics and geometry. Universitext. Springer, 2008.
[Mat17] T. Matsushita. "Fundamental groups of neighborhood complexes". J. Math. Sci. Univ. Tokyo 24 (3 2017), pp. 321-353. arXiv: 1210. 2803.
[Mat19] T. Matsushita. " $\mathbb{Z}_{2}$-indices and Hedetniemi's conjecture". Discrete \& Computational Geometry (2019). arXiv: 1710.05290.
[PR81] S. Poljak and V. Rödl. "On the arc-chromatic number of a digraph". J. Comb. Theory, Ser. B 31.2 (1981), pp. 190-198.
[Ror +16 ] D. Rorabaugh, C. Tardif, D. Wehlau, and I. Zaguia. Iterated Arc Graphs. Tech. rep. 2016. arXiv: 1610.01259.
[Sau01] N. Sauer. "Hedetniemi's conjecture - a survey". Discrete Math. 229.1-3 (2001), pp. 261-292.
[Sch78] T. J. Schaefer. "The Complexity of Satisfiability Problems". Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC'78). ACM, 1978, pp. 216-226.
[Shi19] Y. Shitov. "Counterexamples to Hedetniemi's conjecture". Annals of Mathematics 190.2 (2019), pp. 663-667. arXiv: 1905.02167.
[Spa66] E. H. Spanier. Algebraic Topology. McGraw-Hill, 1966.
[Tar05] C. Tardif. "Multiplicative graphs and semi-lattice endomorphisms in the category of graphs". J. Comb. Theory, Ser. B 95.2 (2005), pp. 338-345.
[Tar08] C. Tardif. "Hedetniemi's conjecture, 40 years later". Graph Theory Notes NY 54.46-57 (2008), p. 2.
[TW19] C. Tardif and M. Wrochna. "Hedetniemi's conjecture and strongly multiplicative graphs". SIAM J. Discrete Math. (2019). (to appear). arXiv: 1808.04778.
[Wro17] M. Wrochna. "Square-free graphs are multiplicative". J. Comb. Theory, Ser. B 122 (2017), pp. 479-507. arXiv: 1601.04551.
[Wro19] M. Wrochna. "On inverse powers of graphs and topological implications of Hedetniemi's conjecture". J. Comb. Theory, Ser. B (2019). arXiv: 1712.03196.
[Zhu17] D. Zhuk. "A Proof of CSP Dichotomy Conjecture". Proceedings of the 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS'17). IEEE, 2017, pp. 331-342. arXiv: 1704.01914.
[Zhu98] X. Zhu. "A survey on Hedetniemi's conjecture". Taiwanese J. Math. 2.1 (1998), pp. 1-24.


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[^1]:    ${ }^{1}$ We note that projections/dictators are not the only trivial polymorphims, cf. [BKW17, Example 41].
    ${ }^{2}$ What we described is the "search version" of PCSPs. In the "decision version", the goal is to say YES if the input graph is $G$-colourable and NO if the input graph is not $H$-colourable. The decision PCSP

[^2]:    reduces to the search PCSP but they are not known to be equivalent in general. However, as far as we know, all known positive results are for the search version, while all known negative results, including the new results from this paper, are for the decision version.

[^3]:    ${ }^{3}$ In this paper, we allow graphs to have loops: the existence of homomorphisms for such graphs is trivial, but this allows us to make statements about graph constructions that will work without exceptions.
    ${ }^{4}$ Note that by our definition, bipartite graphs are vacuously left-hard.

[^4]:    ${ }^{5}$ Jakub Opršal and Andrei Krokhin realised that in this Proposition, 4 can be improved to 3 by using the fact that $\delta\left(\delta\left(K_{4}\right)\right)$ is 3-colourable, as proved by Rorabaugh, Tardif, Wehlau, and Zaguia [Ror+16]. Details will appear in a future journal version.
    ${ }^{6}[\mathrm{Bar}+19]$ is a full version of [BKO19]. Proposition 10.3 in [Bar +19$]$ is Proposition 5.31 in the previous two versions of [Bar+19].
    ${ }^{7}$ The tensor (or categorical) product $G \times H$ of graphs $G, H$ has pairs $(g, h) \in V(G) \times V(H)$ as vertices and $(g, h)$ is adjacent to $\left(g^{\prime}, h^{\prime}\right)$ whenever $g$ is adjacent to $g^{\prime}($ in $G)$ and $h$ is adjacent to $h^{\prime}$ (in $H$ ).

[^5]:    ${ }^{8}$ For the interested reader: $\delta_{L} D$ is obtained by making a new arc $\left(s_{v}, t_{v}\right)$ for each vertex of $D$ and then for each arc $(u, v)$ of $D$, gluing $t_{u}$ with $s_{v}$ (which results in many transitive gluings); $\delta_{R} D$ has a vertex for each pair $S, T \subseteq V(D)$ such that $S \times T \subseteq E(D)$, and an arc from $(S, T)$ to $\left(S^{\prime}, T^{\prime}\right)$ iff $T \cap S^{\prime} \neq \emptyset$.
    ${ }^{9}$ As Jakub Opršal observed, this is in fact the composition of two adjoint pairs: taking sym and sub as functors from digraphs to graphs and the inclusion functor $\iota$ from graphs to digraphs, we have $\operatorname{sym}(D) \rightarrow G$ iff $D \rightarrow \iota(G)$ and $\iota(G) \rightarrow D$ iff $G \rightarrow \operatorname{sub}(D)$.

[^6]:    ${ }^{10}$ All the spaces we consider come from finite simplicial complexes, so connectivity in the topological sense is equivalent to path-connectivity (every two points being connected by a path) and to connectivity of the complex (as in a graph).
    ${ }^{11}$ In group theory, one would say $\pi_{1}(|X|)$ is a normal subgroup of index 2 , or that $\pi_{1}(|X|) \rightarrow \pi_{1}\left(|X|_{\left|\left|\mathbb{Z}_{2}\right|\right.}\right) \rightarrow$ $\mathbb{Z}_{2}$ is a short exact sequence. In topology, one would say that $|X|$ is a degree- 2 covering, or double cover, of $|X|_{\mathbb{Z}_{2}}$; the group homomorphism $\nu_{X}$ is the monodromy action, acting on the set $\left\{x_{0},-x_{0}\right\}$.

[^7]:    ${ }^{12}$ In category theory, $\left(G, \nu_{G}\right) \star\left(H, \nu_{H}\right)$ is called the pullback of $\nu_{G}$ and $\nu_{H}$, and may be denoted $G \times_{\mathbb{Z}_{2}} H$.
    ${ }^{13}$ More generally, one could consider pointed spaces (pairs $\left.\left(|X|, x_{0}\right)\right)$ and pointed $\mathbb{Z}_{2}$-maps $\left(|X|, x_{0}\right) \rightarrow$ $\left(|Y|, y_{0}\right)$ (maps that map $x_{0}$ to $y_{0}$ ). Then $f \mapsto f_{*}$ preserves composition with pointed maps; automorphisms fixing $x_{0}$ are a special case.

