Preprocessing Imprecise Points for the Pareto Front

Ivor van der Hoog

Department of Information and Computing Sciences, Utrecht University, the Netherlands i.d.vanderhoog@uu.nl

Irina Kostitsyna

Department of Mathematics and Computer Science, TU Eindhoven, the Netherlands i.kostitsyna@tue.nl

Maarten Löffler

Department of Information and Computing Sciences, Utrecht University, the Netherlands m.loffler@uu.nl

Department of Mathematics and Computer Science, TU Eindhoven, the Netherlands b.speckmann@tue.nl

Abstract

In the preprocessing model for uncertain data we are given a set of regions \mathcal{R} which model the uncertainty associated with an unknown set of points P. In this model there are two phases: a preprocessing phase, in which we have access only to \mathcal{R} , followed by a reconstruction phase, in which we have access to points in P at a certain retrieval cost C per point. We study the following algorithmic question: how fast can we construct the Pareto front of P in the preprocessing model?

We show that if \mathcal{R} is a set of pairwise-disjoint axis-aligned rectangles, then we can preprocess \mathcal{R} to reconstruct the Pareto front of P efficiently. To refine our algorithmic analysis, we introduce a new notion of algorithmic optimality which relates to the entropy of the uncertainty regions. Our proposed uncertainty-region optimality falls on the spectrum between worst-case optimality and instance optimality. We prove that instance optimality is unobtainable in the preprocessing model, whenever the classic algorithmic problem reduces to sorting. Our results are worst-case optimal in the preprocessing phase; in the reconstruction phase, our results are uncertainty-region optimal with respect to real RAM instructions, and instance optimal with respect to point retrievals.

2012 ACM Subject Classification Theory of computation \rightarrow Design and analysis of algorithms

Keywords and phrases preprocessing, imprecise points, geometric uncertainty, lower bounds, algorithmic optimality, Pareto front

Funding Ivor van der Hoog: Supported by the Dutch Research Council (NWO); 614.001.504. Maarten Löffler: Partially supported by the Dutch Research Council (NWO); 614.001.504. Bettina Speckmann: Partially supported by the Dutch Research Council (NWO); 639.023.208.

1 Introduction

In many applications of geometric algorithms to real-world problems the input is inherently imprecise. A classic example are GPS samples used in GIS applications, which have a significant error. Geometric imprecision can be caused by other factors as well. For example, if a measured object moves during measurement, it may have an error dependent on its speed [18]. Another example comes from I/O-sensitive computations: exact locations may be too costly to store in local memory [3]. Algorithms that can handle imprecise input well have received considerable attention in computational geometry. We continue this line of research by studying the efficient construction of the Pareto front of a collection of imprecise points.

Preprocessing model. Held and Mitchell [17] introduced the preprocessing model of uncertainty as a model to study the amount of geometric information contained in uncertain points. In this model, the input is a set of geometric (uncertainty) regions $\mathcal{R} = (R_1, R_2, \dots, R_n)$ with an associated "true" planar point set $P = (p_1, p_2, \dots, p_n)$. For any pair (\mathcal{R}, P) , we say that P respects \mathcal{R} if each p_i lies inside its associated region R_i ; we assume throughout the paper that P respects \mathcal{R} . The preprocessing model has two consecutive phases: a preprocessing phase where we have access only to the set of uncertainty regions \mathcal{R} and a reconstruction phase where we can for each $R_i \in \mathcal{R}$, request the true location p_i in (traditionally constant) C time. The value C can, for example, model the cost of disk retrievals for I/O-sensitive computations [3]. We typically want to preprocess \mathcal{R} in $O(n \log n)$ time to create some linear-size auxiliary datastructure Ξ . Afterwards, we want to reconstruct the desired output on P using Ξ faster than would be possible without preprocessing.

Löffler and Snoeyink [22] were the first to interpret \mathcal{R} as a collection of imprecise measurements of a true point set P. The size of Ξ and the running time of the reconstruction phase, together quantify the information about (the Delaunay triangulation of) P contained in \mathcal{R} . This interpretation was widely adopted within computational geometry and motivated many recent results for constructing Delaunay triangulations [4, 5, 11, 28], spanning trees [20, 30], convex hulls [15, 16, 23, 25] and other planar decompositions [21, 27] for imprecise points.

Output format. Classical work in the preprocessing model ultimately aims to preprocess the data in such a way that one can achieve a (near-)linear-time reconstruction phase. Indeed, if the final output structure has linear complexity and must explicitly contain the coordinates of each value in P, then returning the result takes $\Omega(nC)$ time. However, this point of view is limiting in two ways. First, certain geometric problems, such as the convex hull or the Pareto front, may have sub-linear output complexity. Second, even if the output has linear complexity, it may be possible to find its combinatorial structure without inspecting the true locations of all points. Consider the example in Figure 1: on the left, we do not need to retrieve any point; on the right, we do not need to retrieve p_3 after we retrieve p_4 . Van der Hoog et al. [27] propose an addition to the preprocessing model to enable a more fine-grained

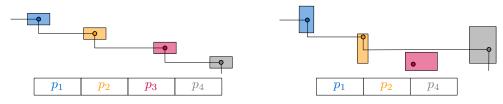


Figure 1 The Pareto front of P can be implied by the geometry of \mathcal{R} (left) or not (right).

analysis in these situations: instead of returning the desired structure on P explicitly, they instead return an *implicit representation* of the output. This implicit representation can take the form of a pointer structure which is guaranteed to be isomorphic to the desired output on P, but where each value is a pointer to either a certain (retrieved) point, or to an uncertain (unretrieved) point. In this paper, we study the efficient construction of the Pareto front of a set of imprecise points P, from pairwise-disjoint axis-aligned rectangles \mathcal{R} as uncertainty regions, in the preprocessing model with implicit representation.

Algorithmic efficiency. To assess the efficiency of any algorithm we generally want to compare its performance to a suitable lower bound. Two common types of lower bounds are worst-case and instance lower bounds. The classical worst-case lower bound takes the minimum over all algorithms A, of the maximal running time of A for any pair (\mathcal{R}, P) . The instance lower bound [1, 14] is the minimum over all A, for a fixed instance (\mathcal{R}, P) , of the running time of A on (\mathcal{R}, P) . For the Pareto front the worst-case lower bound is trivially $\Omega(nC)$; worst-case optimal performance (for us, in the reconstruction phase) is hence easily obtainable. Instance-optimality, on the other hand, is unobtainable in classical computational geometry [1]. Consider, for example, binary search for a value q amongst a set X of sorted numbers. For each instance (X,q), there exists a naive algorithm that guesses the correct answer in constant time. Thus the instance lower bound for binary search is constant, even though there is no algorithm that can perform binary search in constant time in a comparison-based RAM model [13]. Hence we introduce a new lower bound for the preprocessing model, whose granularity falls in between the instance and worst-case lower bound. Our uncertainty-region lower bound is the minimum over all algorithms A, for a fixed input \mathcal{R} , of the maximal running time of A on (\mathcal{R}, P) for any P that respects \mathcal{R} . A detailed discussion of algorithmic efficiency for the preprocessing model can be found in Section 2.

Related work. Bruce et al. [3] study the efficient construction of the Pareto front of two-dimensional pairwise disjoint axis-aligned uncertainty rectangles in what would later be the preprocessing model using implicit representation. As their paper is motivated by I/O-sensitive computation, they assume that the retrieval cost C dominates polynomial RAM running time and both their preprocessing and reconstruction phase use an unspecified polynomial number of RAM instructions. In the reconstruction phase they have a retrieval-strategy that iteratively selects a region R_i for which they retrieve p_i to construct Ξ^* (since Ξ^* is an implicit representation, they do not have to retrieve each $p_i \in P$). Their result is instance optimal under their assumption that C dominates the RAM running time of all parts of their algorithm. We study the same problem without their assumption on C.

Results and organization We discuss in Section 2 the three possible lower bounds for the preprocessing model: worst case, instance, and our new uncertainty-region lower bound. In Section 3 we present the necessary geometric preliminaries. Then, in Section 4, we prove an uncertainty-region lower bound on the time required for the reconstruction phase. In Section 5 we then show how to preprocess \mathcal{R} in $O(n \log n)$ time to create an auxiliary structure Ξ . We also explain how to reconstruct the Pareto front of P as an implicit representation Ξ^* from Ξ . Our results are worst-case optimal in the preprocessing phase; our reconstruction results are uncertainty-region optimal in the RAM instructions, instance optimal with respect to the retrieval cost C and an $O(\log n)$ factor removed from instance optimal with respect to both. This is the first two-dimensional result in the preprocessing model with better than worst-case optimal performance.

4 Preprocessing Imprecise Points for the Pareto Front

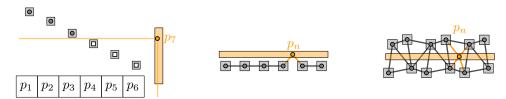


Figure 2 Thrice a collection of grey uncertainty regions where the Pareto front, EMST or Delaunay triangulation of the grey points is implied by the regions; plus an orange region R_n . Depending on the placement of p_n , it can neighbor any grey point in the final structure.

2 Algorithmic optimality

We briefly revisit the definitions of worst-case and instance lower bounds in the preprocessing model and then formally introduce our new uncertainty-region lower bound.

Worst-case lower bounds. The worst-case comparison-based lower bound of an algorithmic problem \mathcal{P} considers each algorithm¹ plus datastructure pair (A,Ξ) which solves \mathcal{P} in a competitive setting with respect to their maximal running time:

$$\text{Worst-case lower bound}(\mathcal{P}) := \min_{(A,\Xi)} \max_{(\mathcal{R},P)} \text{Runtime}(A,\Xi,\mathcal{R},P) \,.$$

The number L of distinct outcomes for all instances (\mathcal{R}, P) implies a lower bound on the maximal running time for any algorithm A: regardless of preprocessing, auxiliary datastructures and memory used, any comparison-based pointer machine algorithm A can be represented as a decision tree where at each algorithmic step, a binary decision is taken [2, 7, 13]. Since there are at least L different outcomes, there must exists a pair (\mathcal{R}, P) for which A takes $\log L$ steps before A terminates (this lower bound is often referred to as the information theoretic lower bound or sometimes the entropy of the problem [1, 6, 7]).

Instance lower bounds. A stronger lower bound, is an instance lower bound [14] (or *instance optimal in the random-order setting* in [1]). For an extensive overview of instance optimality we refer to Appendix A. For a given instance (\mathcal{R}, P) , its instance lower bound is:

Instance lower bound
$$(\mathcal{P}, \mathcal{R}, P) = \min_{(A,\Xi)} \text{Runtime}(A, \Xi, \mathcal{R}, P)$$
.

An algorithm A is instance optimal, if for every instance (\mathcal{R}, P) the runtime of A matches the instance lower bound. Löffler $et\ al.\ [21]$ define $proximity\ structures$ that include quadtrees, Delaunay triangulations, convex hulls, Pareto fronts and Euclidean minimum spanning trees. We prove the following:

▶ **Theorem 1.** Let the unspecified retrieval cost C not dominate $O(\log n)$ RAM instructions and R be any set of pairwise disjoint uncertainty rectangles. Then there exists no algorithm A in the preprocessing model with implicit representation that can construct a proximity data structure on the true points which is instance optimal.

Proof. Let $\mathcal{R}' = (R_1, R_2, \dots R_{n-1})$ be a set of uncertainty regions for which the implicit data structure Ξ^* can be known in the preprocessing phase. Denote by R_n an uncertainty

We refer to comparison-based algorithms algorithms on an intuitive level: as RAM computations that do not make use of flooring. For a more formal definition we refer to any of [1, 2, 10, 13].

region for which p_n can neighbor any $p_i \in (p_1, \dots p_{n-1})$. See Figure 2 for an example of the Pareto front, the EMST and the Delaunay triangulation (with it, Voronoi diagrams) and Figure 3 for the convex hull. For the set of grey points $(p_1, \dots p_{n-1})$, their respective structure is known while the orange point p_n can neighbor any of the grey points. Via the information theoretic lower bound, there is no algorithm A that for every instance can decide the correct neighbor of p_n in O(C) time. Yet for every instance, there exists a naive algorithm that correctly guesses the constantly many neighbors of p_n and verifies this guess in O(C) time.

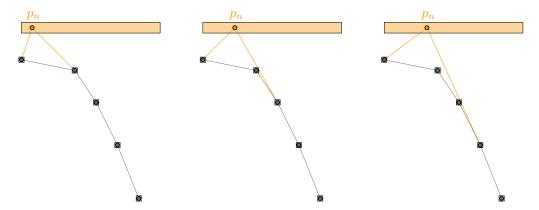


Figure 3 A collection of n-1 grey pairwise-disjoint uncertainty rectangles, for which the convex hull of their underlying points is implied by the convex hull of their bottom left vertices. The region R_n is shown in orange. Depending on the placement of p_n , it can neighbor any grey point in the convex hull of all the points.

Uncertainty-region lower bounds. Worst-case optimality is easily attainable by any algorithm and we proved that instance optimality is not attainable in the preprocessing model. Yet the examples in Figure 1 and 2 intuitively have a lower bound of $\Theta(1)$ and $\Theta(\log n + C)$, which is trivial to match via binary search. We capture this intuition for a fixed input \mathcal{R} :

$$\text{Uncertainty-region lower bound}(\mathcal{P},\mathcal{R}) := \min_{(A,\Xi)} \max_{(P \text{ respects } \mathcal{R})} \text{Runtime}(A,\Xi,\mathcal{R},P) \,,$$

and say an algorithm A is uncertainty-region optimal if for every \mathcal{R} , A has a running time that matches the uncertainty-region lower bound. Denote by $L(\mathcal{R})$ the number of distinct outcomes for all P that respect \mathcal{R} . Via the information theoretic lower bound we know:

$$\forall \mathcal{R}, \quad \log |L(\mathcal{R})| \leq \text{Uncertainty-region lower bound}(\mathcal{P}, \mathcal{R}).$$

For constructing proximity structures in the preprocessing model with implicit representations, the value of $\log L(\mathcal{R})$ can range from anywhere between 0 and $n \log n$. Consequently, an optimal algorithm cannot necessarily afford to explicitly retrieve the entire point set P.

3 Geometric preliminaries

Throughout the paper, we use the notation \mathcal{R}° , \mathcal{R}^{\times} for original and truncated regions respectively (which we define later). When the set is clear from context, we drop the superscript. Let $\mathcal{R} = (R_1, R_2, \dots, R_n)$ be a sequence of n pairwise disjoint closed axis-aligned

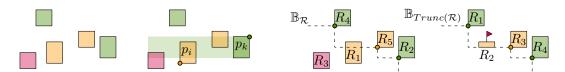


Figure 4 Left: a collection of uncertainty regions. Green is positive, red is negative and yellow is potential. The horizontal halfslab of a green region is shown. Right: A collection of uncertainty regions before and after truncation, note that we re-indexed the regions and flagged one.

uncertainty rectangles, with underlying point set P. For ease of exposition, we assume \mathcal{R} and P lie in general position (no points or region vertices share a coordinate). We denote by $[R_i, R_j] := (R_i, R_{i+1}, \ldots, R_j)$ a subsequence of j - i + 1 regions and similarly by $[p_i, p_j] = (p_i, p_{i+1}, \ldots, p_j)$ a subsequence of points. For brevity, with slight abuse of notation, we may refer to points as degenerate rectangles; hence any set \mathcal{R} may contain points. Whenever we place points on a vertex, we mean placing it arbitrarily close to said vertex. A region R_i precedes a region R_j if i < j. Conversely, R_j succeeds R_i .

For two points p and q, we say that p (Pareto) dominates q if both its x- and y-coordinates are greater than or equal to the respective coordinates of q. A point p (Pareto) dominates a rectangle R, if p dominates its top right vertex. We define the Pareto front of P as the boundary of the set of points that are dominated by a point in P. That is, the Pareto front is the set of points in P that are not dominated by any other point in P, connected by a rectilinear staircase. For any region or point R, we define its horizontal halfslab as the union of all horizontal halfslab symmetrically using downward vertical halfslaes. Given a set R without knowledge of P, we say a region $R_i \in R$ is (Figure 4, left):

- \blacksquare a negative region if for all choices of P, the point p_i is not part of the Pareto front of P;
- a positive region if for all choices of P, the point p_i is part of the Pareto front; or
- **a** potential region if it is neither positive nor negative.

▶ Lemma 2. A region $R_i \in \mathcal{R}$ is negative if and only if $\exists R_j \in \mathcal{R}$ such that the top right vertex of R_i is dominated by the bottom left vertex of R_j . A non-negative region R_i is positive if and only if $\not\exists R_k \in \mathcal{R}$ such that R_i intersects either halfslab of R_k .

Proof. Let R_i and R_j be two axis-aligned rectangular uncertainty regions where the top right vertex of R_i is dominated by the bottom left vertex of R_j . All choices of $p_i \in R_i$ are dominated by the top right vertex of R_i , similarly all choices of $p_j \in R_j$ dominate the bottom left vertex of R_j hence via transitivity p_j always dominates p_i which implies that R_i is a negative region. If there is no region whose bottom left vertex dominates the top right vertex of R_i , then p_i appears on the Pareto front of P if all regions have their point lie on the bottom left vertex and p_i lies on the top right vertex of R_i . Hence R_i is then not negative.

If R_i is non-negative, and there exists a region R_k that contains R_i in its horizontal or vertical halfslab then R_i cannot be positive since if p_k is placed on the top right vertex of R_k and p_i on the bottom left vertex, p_k must dominate p_i .

Suppose that R_i is not positive and not negative. Then per definition there exists a point placement of p_i , and another true point p_l , such that p_l dominates p_i . In this case, p_l also dominates the bottom left vertex of R_i , yet the uncertainty region R_l cannot be entirely contained in the quadrant that dominates the top right vertex of R_i , else R_i is negative. Hence R_l must have a halfslab that intersects R_i which proves the lemma.

Evans and Sember [15] and Nagai et al. [25] study convex hulls and Pareto fronts of imprecise points. They note that for a set of pairwise-disjoint convex regions \mathcal{R} , there is a connected area of negative points. They call this area the guaranteed dominated region. We refer to the boundary of the guaranteed dominated region as the guaranteed boundary $\mathbb{B}_{\mathcal{R}}$. We note that for Pareto fronts, the guaranteed boundary is the Pareto front of the bottom left vertices in \mathcal{R} . Intuitively, discovering the exact location of a point below $\mathbb{B}_{\mathcal{R}}$ does not provide additional useful information, only discovering that a point lies below $\mathbb{B}_{\mathcal{R}}$ does.

▶ **Lemma 3.** Let \mathcal{R} be a set of pairwise disjoint non-negative rectangles. The intersection of a region $R_i \in \mathcal{R}$ with $\mathbb{B}_{\mathcal{R}}$ is a staircase with no top right vertex.

Proof. Per definition, non-negative regions have a top right vertex that lies above $\mathbb{B}_{\mathcal{R}}$. Their bottom left vertex lies either on $\mathbb{B}_{\mathcal{R}}$, or below $\mathbb{B}_{\mathcal{R}}$ (since $\mathbb{B}_{\mathcal{R}}$ is the Pareto front of all bottom left vertices). Hence the closure of each uncertainty region intersects $\mathbb{B}_{\mathcal{R}}$. The intersection between a connected staircase and an axis-aligned rectangular region is always a connected staircase. Each top vertex of $\mathbb{B}_{\mathcal{R}}$ corresponds to a bottom left vertex of a region in \mathcal{R} . Each R_i cannot cannot contain such a top vertex since regions are pairwise disjoint.

We formalise the above intuition by defining a procedure Trunc. Given an original set \mathcal{R}° of n° pairwise disjoint axis-aligned rectangles, $Trunc(\mathcal{R}^{\circ})$ returns a truncated set \mathcal{R}^{\times} where some regions may be flagged (marked with a boolean). Refer to Figure 4. Specifically, each negative region in \mathcal{R}° gets removed, each potential region R_i , whose bottom left vertex is below $\mathbb{B}_{\mathcal{R}^{\circ}}$, gets flagged and replaced by the part of R_i above $\mathbb{B}_{\mathcal{R}^{\circ}}$. By Lemma 3 this results in a rectangular area. All remaining regions are rectangles which touch $\mathbb{B}_{\mathcal{R}^{\circ}}$. Since they are also disjoint, their intersections with $\mathbb{B}_{\mathcal{R}^{\circ}}$ induce a well-defined order, and Trunc re-indexes the remaining regions according to top left to bottom right ordering of their bottom left vertices. We obtain a set $\mathcal{R}^{\times} = (R_1, R_2, \dots R_{n^{\times}}) = Trunc(\mathcal{R}^{\circ})$ with $n^{\times} \leq n^{\circ}$. Observe that $\mathbb{B}_{\mathcal{R}^{\circ}} = \mathbb{B}_{\mathcal{R}^{\times}}$. We say \mathcal{R}^{\times} is a truncated set if it is the result of a truncation of some set \mathcal{R}° .

Dependency graphs. Given a truncated set $\mathcal{R} = \mathcal{R}^{\times}$, we define a *(directed) dependency graph* denoted by $G(\mathcal{R})$ as follows. The nodes of the graph correspond to the regions in \mathcal{R} . We have two types of directed edges which we refer to as horizontal and vertical arrows. A region R_i has a *vertical arrow* to R_j if R_j succeeds R_i and is vertically visible from R_i (that is, there exists a vertical segment connecting R_i and R_j that does not intersected any other region in \mathcal{R}). A region R_i has a horizontal arrow to R_j if R_j precedes R_i and is horizontally visible from R_i . Refer to Figure 5. Observe that, if \mathcal{R} is a truncated set, any point region $p \in \mathcal{R}$ has no outgoing arrows, since after truncation the halfslabs of p do not intersect the interior of any rectangle in \mathcal{R} . We note an important property of the dependency graph:

▶ Lemma 4. Let $R_i \in \mathcal{R}$ such that R_i is a source in $G(\mathcal{R})$. Then all $R_l \in \mathcal{R}$ with i < l cannot have an incoming dependency arrow from a region R_k with k < i and vice versa.

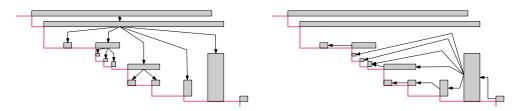


Figure 5 A truncated set and its horizontal and vertical arrows.

Proof. Consider such regions R_k , R_i and R_l . Per the ordering of \mathcal{R} , the bottom left vertex of R_k lies left and above the bottom left vertex of R_i . Per definition, R_k can only have a vertical arrow to R_l . The region R_k has a vertical arrow to R_l only if its bottom facet lies above R_l . However, then either its bottom facet intersects R_i (contradicting the assumption that the regions are pairwise disjoint) or it lies above R_i (contradicting the assumption that R_i is a source node in $G(\mathcal{R})$). The argument for arrows from R_l to R_k is symmetrical.

▶ Corollary 5. Let \mathcal{R} be a truncated set and let R_i and R_j be source nodes in $G(\mathcal{R})$. There is no region in $\mathcal{R}\setminus[R_i,R_j]$ that has a directed path in $G(\mathcal{R})$ to any region in $[R_i,R_j]$.

The Pareto cost function. We show that for any set \mathcal{R}° , we can construct the Pareto front of the underlying point set using only $\mathcal{R}^{\bowtie} = Trunc(\mathcal{R}^{\circ})$. To show that we can use \mathcal{R}^{\bowtie} to construct Ξ^{*} in uncertainty-region optimal time, we define the *Pareto cost function* denoted by $CP(\mathcal{R}^{\bowtie}, P)$. In Section 4 we show that $CP(\mathcal{R}^{\bowtie}, P)$ is the uncertainty-region lower bound for constructing Ξ^{*} and in Section 5 we show that this lower bound is tight.

Before we can define the Pareto cost function, we define additional concepts (Figure 6). By C we denote the unspecified cost for a retrieval. Whenever we write log we refer to the logarithm base 2. Let $\mathcal{R} = \mathcal{R}^*$ be a truncated set. For all regions $R_i \in \mathcal{R}$, we denote by V_i the subset of $[R_i, R_n]$ that is vertically visible from R_i (including R_i itself) and by H_i the subset of $[R_1, R_i]$ that is horizontally visible from R_i (including R_i itself). Given P, we denote by $V_i(P) \subseteq V_i$: the union of $\{R_i\}$ with the subset of V_i of regions that are dominated by a point p_j with $j \leq i$. The set $H_i(P)$ is defined symmetrically taking points p_j with $i \leq j$.

Intuitively, the truncation operator represents the foresight about the Pareto front of P. Now, given a truncated set \mathcal{R} and P we construct a set $\tilde{\mathcal{R}}(P) \subset \mathcal{R}$ that intuitively represents which regions of \mathcal{R} were geometrically interesting in hindsight. Consider for a given P, all regions that are intersected by the Pareto front of P. Let R_j be such a region, then given the Pareto front of $P\setminus\{p_j\}$, R_j covers some area above this Pareto front. Hence, the point p_j could be part of the Pareto front of P if it lies in this area. Intuitively, all regions intersected by the Pareto front of P are hereby suitable for further inspection; however, if the regions are positive regions this further inspection might not be required to construct Ξ^* . Similarly, if the region R_j lies above the Pareto front of the points $P\setminus\{p_j\}$, the point p_j cannot be dominated by a point in $P\setminus\{p_j\}$ and hence we can conclude it lies on the Pareto front of P without further inspection. This is why we define $\tilde{\mathcal{R}}(P)$ as the subset of \mathcal{R} where each region $R_i \in \tilde{\mathcal{R}}(P)$ is intersected by the Pareto front of P and one of three conditions holds:

- 1. R_i is flagged;
- **2.** R_i intersects and edge e with endpoint $p_j \in P$ and $i \neq j$; and/or
- **3.** R_i is not a sink in $G(\mathcal{R})$.

We define the Pareto cost function as: $CP(\mathcal{R}, P) = \sum_{R_i \in \tilde{\mathcal{R}}(P)} C + \log |V_i(P)| + \log |H_i(P)|$.

4 Lower bounds

One is free to compute any auxiliary Ξ in the preprocessing phase, in order to reconstruct a structure Ξ^* , isomorphic to the Pareto front, as efficiently as possible. There exists a choice of input \mathcal{R}° where all regions are positive: namely whenever $\mathcal{R}^{\circ} = \mathcal{R}^{\times} = Trunc(\mathcal{R}^{\circ})$ and $G(\mathcal{R}^{\circ})$ is a graph with no edges. In this case, for every choice of P that respects \mathcal{R}° , the Pareto front of P is isomorphic to $\mathbb{B}_{\mathcal{R}^{\circ}}$ hence it is possible to construct Ξ^* in the preprocessing phase. If \mathcal{R}° has m elements, constructing $\mathbb{B}_{\mathcal{R}^{\circ}}$ has a well-known $O(m \log m)$ worst case lower bound.

In the reconstruction phase an algorithm can use any auxiliary structure Ξ to aid its computation. In the remainder of this section we consider any truncated set $\mathcal{R} = \mathcal{R}^* = Trunc(\mathcal{R}^{\circ})$ of n elements, together with any auxiliary datastructure. We provide an information-theoretical lower bound, which depends on \mathcal{R} and P, for both the number of RAM instructions and disk retrievals required to construct Ξ^* regardless of Ξ .

4.1 A lower bound for disk retrievals

Bruce $et\ al.$ study in their paper the reconstruction of the Pareto front of P in a variant of (what would later be) the preprocessing model with implicit representation. Bruce $et\ al.$ present an iterative retrieval strategy that is instance optimal. Their strategy performs at most three times more retrievals than any algorithm must use to discover the Pareto front of P and they prove that this factor-3 redundancy is the best anyone can do. Their strategy describes the regions that must be considered in a geometric sense, not an algorithmic sense. That is, at each iteration they can identify a triplet of regions to query. But they have no algorithmic procedure to identify these three regions as such, nor a way to beforehand specify which regions should be considered. In their model this is justifiable as they assume that the retrieval cost C vastly dominates any RAM instructions and hence identifying the triple each iteration is trivial. In this paper, we drop the assumption that C is enormous and are interested in a retrieval strategy which not only minimizes the number of retrievals, but which can also elect which points to retrieve efficiently.

We note that the query strategy of Bruce *et al.* produces a result of the same quality as the lemma below and naturally, our proofs share some elements which we fully wish to attribute to the work of [3]. The novelty in our result is that for each pair (\mathcal{R}, P) we are able to characterize the regions which require a disk retrieval using $\tilde{\mathcal{R}}(P)$. Which will help us in the reconstruction phase, when we want to identify these regions efficiently.

▶ **Lemma 6.** Let \mathcal{R} be a truncated set and let P be any point set that respects \mathcal{R} . Any algorithm that constructs Ξ^* of P must perform at least $\frac{1}{3}|\tilde{\mathcal{R}}(P)|$ retrievals.

Proof. Let $R_i \in \tilde{\mathcal{R}}(P)$. Per definition, R_i is not dominated by a point in P. Hence given $P \setminus p_i$, there exists a choice of p_i such that p_i appears on the Pareto front of P. Any algorithm A must spend a disk retrieval on p_i , if there also exists a choice of p_i such that it does not appear on the Pareto front, given $P \setminus p_i$. We consider the three cases for when $R_i \in \tilde{\mathcal{R}}(P)$:

Let R_i be flagged. Then there exists a choice of p_i such that p_i lies below $\mathbb{B}_{\mathcal{R}}$ and hence does not appear on the Pareto front of P. Else let R_i be intersected by an edge that has as an endpoint a point p_j with $j \neq i$. Then e is either a vertical edge whose top vertex is p_j or a horizontal edge whose right vertex is p_j . In both cases, there exists a choice of p_i for which it does not appear on the Pareto front of P since it would be dominated by p_j (this

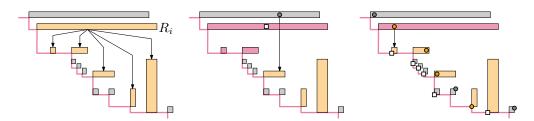


Figure 6 A region R_i and the set V_i in orange. Middle: for a given set of points, the set $V_i(P)$ is shown in red. Right: the set $V_i(P)$ changes for different P, but always includes R_i .

is achieved by placing p_i left of the vertical edge, or below the horizontal edge). Lastly let neither first two cases apply and R_i have at least one outgoing edge in $G(\mathcal{R})$. Then there is at least one region $R' \in H_i \cup V_i$, the argument for this case is illustrated by Figure 7. Denote by R' a region in H_i (the case for V_i is symmetrical). Moreover, let R' be the region in H_i with the highest index. We 'charge' the region R' one disk retrieval. First we show that each region in \mathcal{R} gets charged at most twice, then we show this charge is justified.

Suppose that R' gets charged by two regions R_i , R_j with $R' \in H_i$ and $R' \in H_j$ (the argument for when R' lies in two vertical halfslabs is symmetrical) and let i < j. If R'lies in H_i and H_j , then R_i must lie in the horizontal halfslab of R_j , which contradicts the assumption that R' was the region in H_j with the highest index (see Figure 7, middle).

Second we show that this charge is justified. Consider R' and the two regions R_i and R_l (l < i) that charge R' and all points in $P \setminus \{p', p_i, p_l\}$. Since case (2) does not apply to R_i and R_l , there is no point $p \in P \setminus \{p_i, p_l\}$ whose horizontal or vertical halfslab intersects R_i or R_l , thus no point in $P \setminus \{p_i, p_l\}$ can dominate R', R_i or R_l . This implies that regardless of all other points, there a choice for p_i, p_l, p' where all three points appear on the Pareto front of P (the point placement where p_i and p_l appear on the bottom left vertex of their respective regions and R' appears on the top right vertex). However, there also exists a choice where p'is dominated by p_l or p_i . Any algorithm must therefore consider at least p', p_i or p_l in order to find out and this is why the charge is justified.

4.2 A lower bound on RAM instructions

In Section 2 we defined the uncertainty-region lower bound. By an information-theoretical lower bound (algebraic decision tree or entropy [1, 7]), we have, for any \mathcal{R} , that the Uncertainty-region lower bound is at least $\log L(\mathcal{R})$, where $L(\mathcal{R})$ is the number of combinatorially different Pareto fronts of point sets that respect \mathcal{R} . We prove the following:

Lemma 7. Let \mathcal{R} be a truncated set and P be any point set that respects \mathcal{R} . Then

$$\sum_{R_i \in \tilde{\mathcal{R}}(P)} \log |V_i(P)| + \log |H_i(P)| \le 2 \cdot \log L(\mathcal{R}).$$

Proof. We show that $\sum_{R_i \in \tilde{\mathcal{R}}(P)} \log |V_i(P)| \le \log L(\mathcal{R})$. By a symmetric argument we have $\sum_{R_i \in \tilde{\mathcal{R}}(P)} \log |H_i(P)| \le \log L(\mathcal{R})$ and the lemma follows. Consider for a fixed set P all regions $R_i \in \tilde{\mathcal{R}}(P)$ for which $|V_i(P)| \geq 2$ (recall that $R_i \in V_i(P)$) and sort them from lowest index to highest. For ease of exposition we denote these regions as (R^1, R^2, \dots, R^m) . We create m different, pairwise disjoint vertical slabs as follows: the first slab is bound by the left facets of R^1 and R^2 , the second by facets of R^2 and R^3 and the m'th slab is a halfplane (Figure 8). In the degenerate case that a slab has width 0 (this can occur, when after truncation regions can have left vertices that share a coordinate) we give it width ε .

Let $R_i = R^1$ and $R_j = R^2$. For all regions $R_k \in V_i(P)$, per definition $i \leq k < j$. Each of these truncated regions has thus a bottom left endpoint that lies left of the bottom left

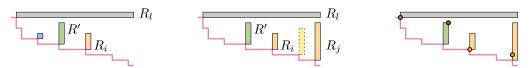


Figure 7 Left: The region R_l charges the blue region and R_i the green. Middle: for R_i , either $R_i \in H_j$ or there is another region (yellow) with higher index in H_j .

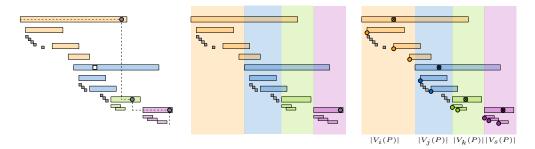


Figure 8 Left: A pair (\mathcal{R}, P) such that the grey points form the Pareto front. Given the Pareto front, we can extract $V_i(P)$ for each i. Middle: based on the sets $V_i(P)$, we create vertical slabs irrespective of the original points P. Right: In each vertical slab, we can create $V_i(P)$ combinatorially distinct (partial) Pareto fronts using only points in the vertical slab.

vertex of R_j and right of the bottom vertex of R_i which implies that their bottom left vertex lies in the first vertical slab. The result of this observation is, that given \mathcal{R} , there are at least $|V_i(P)|$ combinatorially different Pareto fronts contained within the first vertical slab. These Pareto fronts are obtained by placing the points of the regions in $V_i(P)\backslash R_i$ on their respective bottom left endpoints, and by letting p_i dominate any prefix of these points.

Let $R_j = R^2$ and $R_k = R^3$. Via the same argument each region in $V_j(P)$ has its bottom endpoint in the second vertical halfslab. Hence with the same argument as above, there are at least $|V_j(P)|$ combinatorially different Pareto fronts contained within the second halfslab. Moreover, we created $|V_i(P)|$ different combinatorial outcomes by placing only points in the first vertical halfslab, using only points preceding p_j . This means that these combinations can be generated, whilst no point preceding p_j dominates any point following p_j . This implies that the total number of combinatorially different Pareto fronts contained in both the first and second halfslab is $|V_i(P)| \cdot |V_j(P)|$. By applying this argument recursively it follows that: $\prod_{R_i \in \tilde{\mathcal{R}}(P)} |V_i(P)| \le L(\mathcal{R})$, which concludes the proof.

Given Lemma 6 and Lemma 7 we can immediately conclude the following:

▶ **Theorem 8.** Let \mathcal{R} be a truncated set and P be any set that respects \mathcal{R} . Then $CP(\mathcal{R}, P)$ is fewer than three times the uncertainty-region lower bound of \mathcal{R} .

We wish to briefly note that for each i, $V_i(P)$ and $H_i(P)$ have at most n elements and thus by Lemma 6, $CP(\mathcal{R}, P)$ is a factor $\log n$ removed from the instance lower bound.

5 Reconstructing a Pareto front

Theorem 8 gives an uncertainty-region lower bound for any truncated set \mathcal{R} . In this section, we show that this lower bound is tight. To that end, we first define additional geometric concepts. First, we introduce the notion of *canonical* rectangles. Then we define the notion of *subproblems*. Finally, we show how to use the subproblems of a canonical set to quickly select only regions which lie in $\tilde{\mathcal{R}}(P)$. We wish to emphasise that in the reconstruction phase we have *implicit* access to the point set P, meaning that for each region R_i , we can request p_i in O(C) time. Thus reading all points in P takes $\Omega(nC)$ time, which we aim to avoid.

5.1 Geometric preliminaries for reconstruction

Let \mathcal{R} be a truncated set of n regions and let P respect \mathcal{R} . Denote by V_i^{next} the region strictly right of the vertical slab of R_i with the lowest index; H_i^{prev} is defined symmetrically

Figure 9 Left: \mathcal{R}° with $\mathbb{B}_{\mathcal{R}^{\odot}}$ in red. Middle: the set of regions after truncation. The yellow region is a source and a sink, it splits the problem into two. Right: The canonical set.

using the highest index (refer to Figure 11). For each i, let p_i^{xMax} (respectively p_i^{yMax}) be the point in P with maximal x-coordinate (y-coordinate) among points p_k with $k \leq i$ (with $k \geq i$). Throughout this section, we denote by $f_i(P)$ the region succeeding R_i with the lowest index that is not dominated by a point p_k with $k \leq i$. The region $g_i(P)$ is the region preceding R_i with highest index not dominated by a point p_k with $k \geq i$.

Let $R_i \in \mathcal{R}$ be both a source and sink in $G(\mathcal{R})$. By Lemma 4, p_i appears on the Pareto front and connects the Pareto front of $[p_1, p_{i-1}]$ and $[p_{i+1}, p_n]$. Thus, we can split the problem of computing the Pareto front of P into two, and solve each half independently. We say that a truncated set \mathcal{R} is culled if $G(\mathcal{R})$ contains no region that is both a source and a sink. Let $[R_i, R_j]$ be a sequence of sinks in $G(\mathcal{R})$, and R^* be the smallest rectangle that contains R_i and R_j . Note that R^* is disjoint from regions in $\mathcal{R} \setminus [R_i, R_j]$ and contains all $[R_i, R_j]$. We can use R^* to capture a "streak" of points which do, or do not, appear on the Pareto front:

▶ Lemma 9. Let $[R_i, R_j]$ be a sequence of sinks in $G(\mathcal{R})$. If there is no $p_k \in P$ preceding p_i that dominates p_i then there is no point preceding p_i that dominates any point in $[p_i, p_j]$. If some p_k preceding p_i dominates p_j , then p_k dominates all points in $[p_i, p_j]$. Similar statements hold for p_k succeeding p_j .

Proof. Any p_k that dominates any point p_s with $s \in \langle i, j \rangle$, but not p_i or p_j itself must lie in the interior of R^* , but R^* contains only points whose regions are sinks in $G(\mathcal{R})$. This contradiction implies all claims of the lemma.

This lemma implies that if both p_i and p_j are not dominated by other points in P then all the points in $[p_i, p_j]$ appear on the Pareto front of P as a contiguous subsequence, and all regions $R_k \in [R_i, R_j]$ are not part of $\tilde{\mathcal{R}}(P)$. Theorem 8 states we cannot "afford" to spend any disk retrievals on $(p_i, p_{i+1}, \ldots, p_j)$. Instead, we should add a pre-stored chain referencing $[p_i, p_j]$ to Ξ^* in constant time. This is why for any maximal sequence of sinks $[R_i, R_j]$ in a truncated and culled set \mathcal{R} , we define their compound region R^* and we replace $[R_i, R_j]$ in \mathcal{R} with R^* (refer to Figure 9 (right)). Let \mathcal{R}^* be the resulting set of regions. The region R^* is a sink in $G(\mathcal{R}^*)$ and a region R has an outgoing arrow to R^* in $G(\mathcal{R}^*)$ if and only if it had an outgoing arrow in $G(\mathcal{R}^*)$ to at least one region in $[R_i, R_j]$. Since R^* is just another rectangle disjoint from all other rectangles in \mathcal{R}^{comp} , the definition of truncated and culled still applies to \mathcal{R}^{comp} . We say a set \mathcal{R}^* is a canonical set if it is truncated, culled, and if there are no two consecutive regions that are sinks in $G(\mathcal{R}^*)$. In the remainder, we assume \mathcal{R} is a truncated set and $R^0 = \mathcal{R}^*$ is its respective canonical set as the reconstruction input.

Subproblems. Let \mathcal{R} be a truncated set. We say two indices i < j form a subproblem with respect to a dependency graph $G(\mathcal{R})$ if R_i and R_j are sources in $G(\mathcal{R})$ and if there does

not exist a region R_k with i < k < j that is also a source. With slight abuse of notation, we say that $[R_i, R_j]$ is a subproblem of $G(\mathcal{R})$. At later stages we will consider some altered dependency graph $G(\mathcal{R}')$ and will refer to subproblems $[R_l, R_m]$ of $G(\mathcal{R}')$.

The algorithm sketch. The core of our algorithm is rather straightforward: it is an iterative strategy, where at each iteration t we have an (implicitly truncated) set \mathcal{R}^t and a queue of subproblems of $G(\mathcal{R}^t)$. Each iteration, we dequeue a subproblem $[R_i, R_j]$ of $G(\mathcal{R}^t)$, retrieve p_i, p_j to replace R_i and R_j and (implicitly) re-truncate. We maintain the following invariant:

▶ Invariant 1. For each iteration, when we consider a subproblem $[R_i, R_j]$ we have a pointer to the region R which stores p_{i-1}^{xMax} and the region R' which stores p_{i+1}^{yMax} .

Observe that for all subproblems $[R_i, R_j]$ of $G(\mathcal{R}^0 = \mathcal{R}^*)$, the point $p_{i-1}^{xMax} = p_{i-1}$ and $p_{j+1}^{yMax} = p_{j+1}$. We sketch Algorithm 1. We want to prove that its runtime matches the value $CP(\mathcal{R}, P)$ of Theorem 8. This would trivially be true, if for each subproblem $[R_i, R_j]$ of $G(\mathcal{R}^t)$, $R_i, R_j \in \tilde{\mathcal{R}}(P)$. Unfortunately that is not always the case, and thus we resort to a more involved argument to prove the following theorem. In the remainder of this section, we show that the algorithm's running time is $O(A(\mathcal{R}, \mathcal{R}^*, P))$.

▶ Theorem 10. Let \mathcal{R} be a truncated set and let \mathcal{R}^* be its respective canonical set, Ξ be built on \mathcal{R}^* and Algorithm 1 run on \mathcal{R}^* as input. Let Algorithm 1 consider for each iteration t, a subproblem $[R_{i(t)}, R_{j(t)}]$ with i(t) < j(t) - 1. Let $\mathcal{R}^{A1}(\mathcal{R}^*, P) = \bigcup_t \{R_{i(t)}, R_{j(t)}\}$. Let $V_i(P)$ and $H_i(P)$ refer to subsets of \mathcal{R} , not \mathcal{R}^* . Then:

$$A(\mathcal{R}, \mathcal{R}^{\star}, P) = \sum_{R_i \in \mathcal{R}^{A1}(\mathcal{R}^{\star}, P)} \left(\frac{1}{2} C + \log |V_i(P)| + \log |H_i(P)| \right) \le \mathrm{CP}(\mathcal{R}, P).$$

```
Algorithm 1: Algorithm sketch, assuming \mathbb{R}^0 is canonical.
  Result: The pointer structure \Xi^*.
                                                                                                   (Runtime)
  Q \leftarrow \text{subproblems} (G(\mathcal{R}^0))
   (Preprocessing)
  while Q \neq \emptyset do
       [R_i, R_j] \leftarrow \text{Q.DeQueue}()
                                                                                                    (O(1))
      p_i, p_j \leftarrow \text{Retrieve}(R_i, R_j)

p_i^{xMax}, p_j^{yMax} \leftarrow \text{Compare}((p_i, p_{i-1}^{xMax}), (p_j, p_{j+1}^{yMax}))
                                                                                                (2C + O(1))
                                                                                                (2C + O(1))
      if p_i not dominated by i^{xMax}, p_j^{yMax} then
          \Xi^*. Append (p_i \text{ after } p_{i-1}^{xMax})
                                                                                                   (O(1))
      if p_j not dominated by p_i^{xMax}, p_i^{yMax} then
      (O(1))
       (O(\log |V_i(P)|))
      g_j(P) \leftarrow \text{gallopingSearch}(p_i^{yMax}, H_j)
       (O(\log |H_j(P)|))
       R^{t+1} \leftarrow \text{ImplicitTruncate}(R^t - R_i - R_j + p_i + p_j)
                                                                                                    (O(1))
       DetermineSubproblems(\mathbb{R}^{t+1}, f_i(P), g_j(P))
                                                                                                     (O(1))
       for each subproblem [R_c, R_d] of G(\mathcal{R}^{t+1} \cap [R_i = p_i, R_j = p_j]) do
                                                                               (O(1), charged to
           Q.Queue([R_c, R_d])
             [R_c, R_d]
```

Proving Theorem 10. This theorem describes an intuitive "runtime allowance" that Algorithm 1 has. We first prove 3 Lemmas about subproblems encountered by Algoritm 1.

▶ Lemma 11. Let \mathcal{R} be a canonical set and $R_i \in \mathcal{R}$. Algorithm 1 encounters a subproblem $[R_i, \cdot]$ or $[\cdot, R_i]$ if and only if R_i is intersected by the Pareto front of P.

Proof. The region R_i is not intersected by the Pareto front of P if and only if R_i is dominated by a point $p_i \in P$. Let p_i appear on the Pareto front of P (via transitivity of domination, we can always obtain such a p_j). The iterative procedure must consider p_j before p_i since R_i prevents R_i from being a source in the dependency graph. But when R_i is considered, R_i is truncated. The graph must always have at least one source. Thus, since R_i will never be removed after truncation, it must eventually become a source.

▶ **Lemma 12.** Let \mathcal{R}^0 be a canonical set. Algorithm 1 encounters only subproblems $[R_i, R_i]$ where either: j = i + 1 or $R_i \in \tilde{\mathcal{R}}(P)$ or $R_j \in \tilde{\mathcal{R}}(P)$, and $R_i \notin \tilde{\mathcal{R}}(P)$ if and only if $|V_i(P)| = |H_i(P)| = 1$ (the same holds for R_j).

Proof. If \mathbb{R}^0 is a canonical set, then there cannot by any subproblem $[R_i, R_j]$ of $G(\mathbb{R}^0)$ where R_i and R_j are both sinks in $G(\mathcal{R}^0)$. As a consequence, for each $[R_i, R_j]$ either $R_i \in \tilde{\mathcal{R}}(P)$ or $R_i \in RP$ and $R_i \notin \tilde{\mathcal{R}}(P)$ implies $V_i(P) = H_i(P) = \{R_i\}$.

In later iterations, we cannot immediately guarantee that \mathcal{R}^t is canonical, and the allowance for spending computation time is hence lost. Via Lemma 11 we know that R_i and R_i are both intersected by the Pareto front of P. Thus, the regions $R_i, R_i \notin \tilde{\mathcal{R}}(P)$ implies that R_i and R_j are both sinks in the original graph $G(\mathcal{R})$ (as $\tilde{\mathcal{R}}(P)$ is defined on the original truncated set). Thus $R_i \notin \mathcal{R}(P)$ implies $V_i(P) = H_i(P) = \{R_i\}$.

What remains is to show that for each subproblem either R_i or R_j does lie in $\mathcal{R}(P)$. Let i < j-1. Then if R_i and R_j are both sinks, then by Lemma 4 the region R_{i+1} or R_{j-1} must also be a source which contradicts the assumption that $[R_i, R_j]$ is a subproblem.

Lemma 13. Let \mathcal{R} be a canonical set. Algorithm 1 encounters only subproblems $[R_i, R_j]$ followed by $[R_i = \{p_i\}, R_k]$ if $R_k \in \tilde{\mathcal{R}}(P)$.

Proof. By the argument of Lemma 12, R_k is intersected by the Pareto front of P. Moreover after the iteration t where the algorithm considers $[R_i, R_j]$, the region R_i has no outgoing edges in each iteration t' with t < t'. Hence if $[R_i = \{p_k\}, R_k]$ is a subproblem, the region R_k has at least one outgoing arrow and thus $R_k \in \mathcal{R}(P)$.

These three Lemmas imply the following theorem that we later use for a charging scheme: when we relate algorithm runtime to $CP(\mathcal{R}, P)$:

Proof of Theorem 10. Recall that $CP(\mathcal{R}, P) = \sum_{R_k \in \tilde{\mathcal{R}}(P)} C + \log |V_k(P)| + \log |H_k(P)|$. Let $[R_i, R_j]$ be the first subproblem considered that has R_i as its left boundary. By Lemma 12, at least R_i or R_j is in $\tilde{\mathcal{R}}(P)$ hence we charge $\frac{1}{2}C$ time to either the term $(C + \log |V_i(P)| + \log |H_i(P)|)$ or $(C + \log |V_i(P)| + \log |H_i(P)|)$ in the sum of $CP(\mathcal{R}, P)$. Moreover, $R_i \notin \tilde{\mathcal{R}}(P)$ implies $\log |V_i(P)| = \log |V_i(P)| = 0$ hence including these two terms, does not increase the sum's value. For subsequent subproblems $[R_i, R_k]$, Lemma 13 guarantees that $R_k \in \tilde{\mathcal{R}}(P)$. Hence the term: $(\frac{1}{2}C + \log |V_k(P)| + \log |H_k(P)|)$ in the sum of $A(\mathcal{R}, \mathcal{R}^*, P)$ can be charged to the term $(C + \log |V_k(P)| + \log |H_k(P)|)$ in the sum of $CP(\mathcal{R}, P)$.

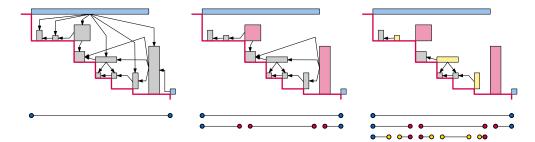


Figure 10 The construction of the subproblem tree. Left we see a subproblem of a canonical set with the vertical arrows drawn. In the middle we see the children of this subproblem with the horizontal arrows drawn. On the right we continued the recursion one additional step.

The subproblem tree. Theorem 10 shows that if we are able to execute our described algorithm in the specified running time, then we prove that $\operatorname{CP}(\mathcal{R},P)$ is tight and we have obtained an uncertainty-region optimal algorithm. However, in order to achieve this running time, in each iteration we must determine the new subproblems efficiently. This is why we define a subproblem tree on the original dependency graph $G(\mathcal{R})$. The subproblem tree, denoted by $T_{\mathcal{R}}$, is a range tree on the interval $[1,n] \subset \mathbb{Z}$ (Figure 10). The root node of the subproblem tree stores the interval [1,n]. If \mathcal{R} is a canonical set, the subproblems of \mathcal{R} partition \mathcal{R} , and the root node has a child for each subproblem $[R_i, R_j]$ where the child stores the interval [i,j] and a pointer to R_i and R_j . We construct the subsequent children as follows: for each node [i,j], we remove all outgoing arrows from R_i and R_j and we create a child node for each subproblem of $G([R_i, R_j])$ without these arrows. Note that each node has at least two children: as removing the outgoing arrows from R_i and R_j creates at least one additional source R_k with $k \in \langle i,j \rangle$ and R_i and R_j remain sources in $G([R_i, R_j])$.

5.2 Preprocessing phase

Here, we elaborate on the preprocessing procedure. First, we transform a set \mathcal{R}° of m axis-aligned pairwise disjoint rectangles into a truncated set \mathcal{R}^{\times} with n elements in $O(m \log m)$ total time. Next, we construct a canonical set \mathcal{R}^{\star} and the auxiliary datastructure Ξ (which consists of the subproblem tree $T_{\mathcal{R}^{\times}}$ and some additional pointers) in $O(n \log n)$ time. Specifically, we define Ξ as follows:

Defining Ξ . Given a canonical set \mathcal{R}^* , let Ξ consist of $G(\mathcal{R}^*)$ and the tree $T_{\mathcal{R}^*}$ augmented with the following *attributes* stored for every region $R_i \in \mathcal{R}$ (Figure 11):

- 1. A binary search tree on V_i and H_i from $G(\mathcal{R}^*)$.
- **2.** A pointer to V_i^{next} and H_i^{prev} in \mathcal{R}^* .
- **3.** A pointer to the region R_j with highest j, such that $R_i \in V_j$ (the back pointer) and a pointer to the region R_j with lowest index j, such that $R_i \in H_j$ (the forward pointer).
- **4.** A pointer to the highest node in $T_{\mathcal{R}^*}$ that stores an interval $[\cdot, i]$, and a pointer to the highest node in $T_{\mathcal{R}^*}$ that stores an interval $[i, \cdot]$.
- **5.** If R_i is a compound region, an array of all the regions compound in R_i .

Creating a truncated set. We consider the bottom left vertices of all regions in \mathcal{R}° , construct $\mathbb{B}_{\mathcal{R}^{\circ}}$, together with a range tree on the horizontal edges of $\mathbb{B}_{\mathcal{R}^{\circ}}$ [9] in $O(m \log m)$ time. For each region $R \in \mathcal{R}^{\circ}$ we detect whether R is negative by performing a point location with its top right vertex on the interior of $\mathbb{B}_{\mathcal{R}^{\circ}}$; if it is negative then it is discarded. If a region

 $R \in \mathcal{R}^{\circ}$ is not negative then by Lemma 3 we know that $R \cap \mathbb{B}_{\mathcal{R}^{\circ}}$ is a staircase of constant complexity which we compute in logarithmic time using binary search on $\mathbb{B}_{\mathcal{R}^{\circ}}$. We flag each non-negative $R \in \mathcal{R}^{\circ}$ whose interior intersects $\mathbb{B}_{\mathcal{R}}$, and store its region after truncation. This results in a set \mathcal{R}^{\times} of n pairwise disjoint axis-aligned rectangles, which we sort and re-index based on their intersection with $\mathbb{B}_{\mathcal{R}^{\times}}$ in $O(m \log m)$ time and conclude:

▶ **Lemma 14.** For any set \mathcal{R}^{\otimes} of m axis-aligned, pairwise disjoint axis-aligned rectangles we can construct its truncated set \mathcal{R}^{\times} of n rectangles in $O(m \log m)$ time.

Recall that for any truncated set \mathcal{R}^* we denote by H_i the set of regions R_j in \mathcal{R} with j < i which are horizontally visible from R_i and by V_i the set of regions R_j with j > i which are vertically visible from R_i . In the remainder of the preprocessing phase, we spend $O(n \log n)$ time to transform \mathcal{R}^* into a canonical set \mathcal{R}^* , construct $G(\mathcal{R}^*)$ and $G(\mathcal{R}^*)$ and construct the datastructure Ξ .

▶ **Observation 1.** For any truncated set \mathcal{R}^* , a region $R_j \in \mathcal{R}^*$ is vertically visible from a region $R_i \in \mathcal{R}^*$ if and only if there exists a face or edge in the vertical decomposition of \mathcal{R} which is vertically adjacent to both R_i and R_j .

Using Observation 1 we obtain the following through standard Computational Geometry:

▶ **Lemma 15.** For any truncated set \mathcal{R}^* of n axis-aligned, pairwise disjoint rectangles we can construct its canonical set \mathcal{R}^* and Ξ in $O(n \log n)$ time.

Proof. A vertical or horizontal decomposition has a number of faces and edges which is linear in the number of input vertices and can be constructed in $O(n \log n)$ time [9]. Given the vertical decomposition of \mathcal{R}^* , we can traverse it in linear time to store for each region R_i the set V_i . Similarly we can identify and store H_i for each R_i , and in $O(n \log n)$ total time we construct a binary search tree on each set H_i and V_i to obtain Attribute 1. For each set V_i , we identify V_i^{next} in logarithmic time by searching by searching for the left-most bottom-left endpoint right of the vertical slab through R_i to obtain Attribute 2.

Through this procedure, we construct the dependency graph $G(\mathcal{R}^{\times})$ in $O(n \log n)$ time by iterating over all nodes in this graph. In linear time, we can identify the connected components of $G(\mathcal{R}^{\times})$ and the regions which are both a source and sink in $G(\mathcal{R}^{\times})$. From Lemma 4 we know that we can solve each connected component of $G(\mathcal{R}^{\times})$ independently and that the solutions must be concatenated through the regions that are both a source and sink. We store the connected components of $G(\mathcal{R}^{\times})$ as a doubly linked list and remove all sources and sinks from \mathcal{R}^{\times} to create a *culled* set.

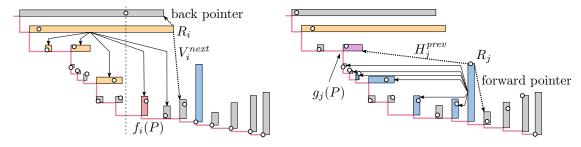


Figure 11 Two choices of P for the same set \mathcal{R} . The sets $R_i(P)$ and $R_j(P)$ are shown in orange and blue respectively. Left: we show V_i^{next} and the backward pointer and $f_i(P)$. Right: we show H_j^{prev} and the forward pointer and $g_j(P)$.

To transform a culled set into a canonical set, we identify all sinks in the graph in linear time (by checking if $|V_i| = |H_i| = 1$) and we iterate over all regions in order of their index. Neighboring sinks get recursively grouped into a compound region and this procedure creates a canonical set in linear time. For each region compounding k regions, we construct Attribute 5 in O(k) time. After having compound all regions, we do a linear-time scan to re-index all the (compound) regions so that all indices are consecutive and we obtain a canonical set \mathcal{R}^* . During this linear time scan, we identify for each R_i the region of its back pointer and forward pointer (Attribute 3) in logarithmic time, through searching through the vertical and horizontal decomposition. Moreover, whenever we compound a set $[R_i, R_{i+k}]$ into a region R, we make sure to remove $[R_i, R_{i+k}]$ from $G(\mathcal{R}^*)$ and replace it with R (where all arrows pointing to a region in $[R_i, R_{i+k}]$ now point to R). In this way, we simultaneously create $G(\mathcal{R}^*)$.

Lastly, we want to obtain from a canonical set \mathcal{R}^* its subproblem tree $T_{\mathcal{R}^*}$ in O(n) time using prior constructed $G(\mathcal{R}^*)$. This can be done as follows: first we identify the subproblems of $G(\mathcal{R}^*)$ in linear time. Then for each subproblem $[R_i, R_j]$ of $G(\mathcal{R})$ we (temporarily) remove all outgoing arrows from R_i and R_j from the graph and for each node that has an arrow from R_i or R_j we check if it becomes a source node in constant time. This gives us the child nodes of the node that stores [i,j] in the $T_{\mathcal{R}^*}$. During this process, we store for each region R_i a pointer to the largest interval $[i,\cdot]$ in the $T_{\mathcal{R}^*}$ (which must always exist) in constant additional time per region (Attribute 4). Applying this procedure recursively takes time linear in the number of edges in $G(\mathcal{R})$, which itself is linear in the number of cells of the vertical and horizontal decomposition of \mathcal{R}^* , which concludes the lemma.

Lemma 14 and 15 and the observation that $n \leq m$ immediately imply Theorem 16.

▶ **Theorem 16.** For any set \mathcal{R}° of m axis-aligned, pairwise disjoint axis-aligned rectangles we can construct its trucated set \mathcal{R} and its canonical set \mathcal{R}^{\star} and Ξ in $O(m \log m)$ time.

5.3 Reconstruction phase

We want to run Algorithm 1 whilst maintaining Invariant 1, in $O(A(\mathcal{R}, \mathcal{R}^*, P))$ time (Theorem 10). First, we argue that the reporting (appending) step of the algorithm is correct:

▶ Lemma 17. For any iteration t, for any subproblem $[R_i, R_j]$ of $G(\mathcal{R}^t)$, the point p_i appears on the Pareto front of P if and only if p_i is not dominated by p_i^{xMax} or p_j^{yMax} .

Proof. Let p_i be not dominated by p_i^{xMax} and p_j^{yMax} , but dominated by some point p_k . Then k < i or k > j, because R_i and R_j are both sources in $G(\mathcal{R}^t)$. If k < i then the x-coordinate of p_k is greater than of p_i , and thus $p_i^{xMax} \neq p_i$. Then, the point p_i^{xMax} has greater x-coordinate than p_i , it lies in some region $R' \neq R_i$, and since R' precedes R_i and contains p_i^{xMax} , its bottom facet must lie above the top facet of R_i . Thus p_i^{xMax} dominates p_i which is a contradiction. If j < k then $p_j^{yMax} \neq p_j$ and the symmetrical argument applies. \blacksquare

The previous lemma implies that if Invariant 1 is maintained, we can iteratively identify points that appear on the Pareto front. Lemma 4 guarantees that for each iteration t, for each subproblem $[R_i, R_j]$, the Pareto front of $\{p_i^{xMax}\} \cup [p_i, p_j] \cup \{p_j^{yMax}\}$ is a connected subchain of the Pareto front of P. Hence we can safely append p_i after p_i^{xMax} . What remains to show is that we can maintain Invariant 1 and identify the subproblems of \mathcal{R}^t efficiently.

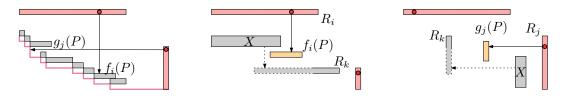


Figure 12 An illustration of the argument of Lemma 20. If R_k loses the incoming arrow from X, there must be a directed path from $f_i(P)$ or $g_i(P)$ to R_k , or either $R_k = f_i(P)$, $R_k = g_j(P)$.

Identifying subproblems. Consider an iteration t in which we handle subproblem $[R_i, R_j]$, and let $[R_k, R_l]$ be any subproblem of $G(\mathcal{R}^{t+1})$ that is not already a subproblem of $G(\mathcal{R}^t)$. It must be that $i \leq k \leq l \leq j$ (Lemma 4). We need to quickly identify these new subproblems.

▶ **Lemma 18.** For any truncated set \mathcal{R}^t , for any subproblem $[R_i, R_j]$ of $G(\mathcal{R}^t)$, either $f_i(P) \in V_i$ or $f_i(P) = V_i^{next}$.

Proof. Any region in $[R_i, R_j]$ that is dominated by a point preceding p_i is dominated by p_i^{xMax} . The point p_{i-1}^{xMax} cannot dominate R_i , as else R_i would have been removed during a truncation. Hence, $f_i(P)$ is V_i^{next} or a region preceding it. Suppose for the sake of contradiction that $f_i(P)$ is a region preceding V_i^{next} and not in V_i . Consider any vertical ray from a point in R_i , right of p_i^{xMax} that intersects $f_i(P)$ (such a ray must always exist, since $f_i(P)$ precedes V_i^{next} and is not dominated by p_i^{xMax}). Since $f_i(P) \notin V_i$, this ray must also intersect a region $R' \in V_i$ (else this ray would be a line of sight to $f_i(P)$, which would imply $f_i(P) \in V_i$). However, then R' must precede $f_i(P)$ which contradicts the assumption that $f_i(P)$ was the lowest-indexed region succeeding R_i , not dominated by p_i^{xMax} .

▶ Corollary 19. Let \mathcal{R}^t be a truncated set, $[R_i, R_j]$ be a subproblem. Given Invariant 1 and Ξ , we can identify $f_i(P)$ in $O(\log |V_i(P)|)$ time using the folklore galloping search.

Proof. The datastructure Ξ stores for R_i the set V_i as a balanced binary search tree (Attribute 1). The set $V_i(P)$ is a prefix of V_i which ends at $f_i(P) \in V_i$ (or, in the case that $V_i(P) = V_i$, $f_i(P) = V_i^{next}$). Thus, given Invariant 1, we can use p_i^{xMax} to identify $V_i(P)$ in $O(\log |V_i(P)|)$ time by using the folklore galloping (exponential) search by Bentley and Chi-Chih Yao. If $V_i(P) = V_i$, we refer to V_i^{next} which is stored in Ξ (Attribute 2).

Next, we prove a lemma that helps us to identify the subproblems of $G(\mathbb{R}^{t+1})$:

▶ Lemma 20. Let $[R_i, R_j]$ be a subproblem of $G(\mathcal{R}^t)$ and denote by v the lowest node in $T_{\mathcal{R}}$ such that the interval [i, j] is stored in v. For any descendent [a, b] of v, there is no region $R' \in [R_a, R_b]$ that is a source node in $G(\mathcal{R}^{t+1})$ other than possibly $R_a, R_b, f_i(P)$ or $g_i(P)$.

Proof. If $f_i(P)$ equals or succeeds $g_j(P)$ then per definition of $f_i(P)$ and $g_j(P)$ all regions in (R_i, R_j) apart from $f_i(P) = g_j(P)$ are dominated and therefore removed after truncation of \mathcal{R}^{t+1} . Hence, they cannot be sources in $G(\mathcal{R}^{t+1})$ (Figure 12). Let [a, b] be a descendent of v, R_k be a region with $k \in \langle a, b \rangle$ succeeding $f_i(P)$ and preceding $g_j(P)$. Per construction of $T_{\mathcal{R}}$ each such R_k has at least one incoming arrow from a region $X \in [R_a, R_b]$. The region R_k can only become a source in $G(\mathcal{R}^{t+1})$ if either p_i or p_j dominates X (else, X was dominated by p_i^{xMax} or p_j^{yMax} before iteration t and does not exist in $G(\mathcal{R}^t)$).

We consider the case where p_i dominates X (Figure 12). If p_i dominates X, then X lies strictly left of the vertical line through p_i , and R_k intersects the vertical halfslab of X.

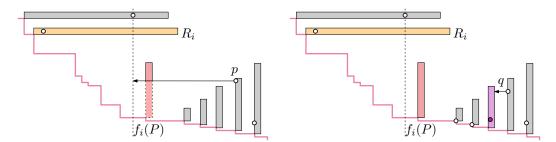


Figure 13 Left: the first case of the proof of Lemma 21, where p must dominate the remaining regions with an arrow to $f_i(P)$. Right: the second case, where either q sees $f_i(P)$, dominates $f_i(P)$ or the purple region keeps its horizontal arrow to $f_i(P)$.

Similarly if $f_i(P) \neq R_k$ then R_k must lie at least partly right of the vertical line through p_i and below the bottom facet of $f_i(P)$. This means that if R_k lies in the vertical halfslab of X then it must also lie in the vertical halfslab of $f_i(P)$. The region $f_i(P)$ is therefore a node in $G(\mathbb{R}^t)$ with a directed path to R_k , so R_k is not a source node in $G(\mathbb{R}^{t+1})$.

Algorithm 1 runtime. We further specify the iterative procedure of our algorithm. Our algorithm maintains a queue of subproblems. In iteration t, we dequeue a subproblem $[R_i, R_j]$ of $G(\mathcal{R}^t)$ and we denote by v the lowest node in $T_{\mathcal{R}}$ such that the interval [i,j] is stored in v. We can obtain v in constant time via Attribute 4. By Lemma 4, processing $[R_i, R_j]$ does not affect other subproblems which are in the queue before we process $[R_i, R_j]$. If the algorithm has not yet retrieved p_i nor p_{i-1}^{xMax} , it retrieves both points using Invariant 1 in 2C time and computes p_i^{xMax} in constant time. Similarly we compute p_j^{yMax} in with at most 2C additional time. By Lemma 17, we check in O(1) time if p_i and p_j appear on the Pareto front, and if so we add them as the respective successor of p_{i-1}^{xMax} or predecessor p_{j+1}^{yMax} . If we have just retrieved p_i , we use galloping search to identify $f_i(P)$ in $O(\log |V_i(P)|)$ time (Corollary 19), we set the back pointer (Attribute 3) to null and (for later use) we store a reference in R_i to $f_i(P)$. If we did not retrieve p_i this iteration, we retrieved it in a prior iteration and we use the pre-stored result $f_i(P)$ in O(1) time. We do the same for $g_j(P)$ in $O(C + \log |H_j(P)|)$ time. We briefly remark the following claim.

▶ Lemma 21. Let $[R_i, R_j]$ be a subproblem of $G(\mathcal{R}^t)$ and $f_i(P)$ precede $g_j(P)$. Then the region $f_i(P)$ is a source in $G(\mathcal{R}^{t+1})$ if and only if: (1) the forward pointer of $f_i(P)$ is null or (2) the region resulting from the forward pointer has been retrieved in an iteration t' < t.

Proof. Suppose that the pointer is null and suppose that there is no region R_k for which $f_i(P) \in H_k(P)$. Then $f_i(P)$ has no incoming horizontal arrows. If there is a region R_k for which $f_i(P) \in H_k(P)$ then there is a point p retrieved in an iteration earlier such that p is horizontally visible from $f_i(P)$ that set the pointer to null Figure 13, Left. The point p dominates all remaining regions with a horizontal arrow to $f_i(P)$. If the region resulting from the forward pointer has been retrieved in an iteration t' < t, all regions with a horizontal pointer to $f_i(P)$ must have been considered by the algorithm, so $f_i(P)$ is not dominated. By definition, all regions preceding $f_i(P)$ in $[R_i, R_j]$ are dominated by p_i^{xMax} , thus, if $f_i(P)$ has no incoming horizontal arrows it must be a source in $G(\mathcal{R}^{t+1})$.

If the pointer is not null and the region resulting from the forward pointer has not yet been retrieved in an earlier iteration then $f_i(P)$ must have at least one incoming horizontal arrow. Indeed, suppose that all regions with a horizontal pointer to $f_i(P)$ that are not yet

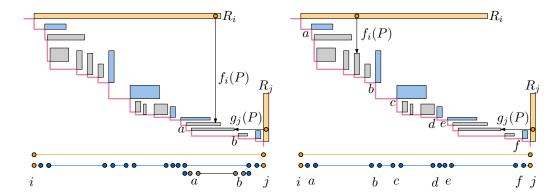


Figure 14 Left: Case 1 where $f_i(P), g_i(P)$ lie in the same grandchild [a, b]. Right they don't.

retrieved are dominated by a point q retrieved prior to the current iteration. Then either q dominates $f_i(P)$, contradicting the assumption that $f_i(P)$ precedes $g_j(P)$, or the retrieval of q would have set the forward pointer of $f_i(P)$ to null.

For ease of exposition, we assume $f_i(P)$ and $g_j(P)$ are not compound regions. For compound regions, we refer to Appendix B. We distinguish between two cases based on which children of v contain $f_i(P)$ and $g_j(P)$ (Figure 14). Note that we never add a subproblem $[R_a, R_b]$ if a = b + 1 (as such a subproblem does not satisfy the premise of Theorem 10). Instead, we charge retrieving and comparing p_a and p_b immediately with at most 4C overhead.

Case 1: $f_i(P)$ and $g_j(P)$ are contained in the same grandchild [a,b] of v. We check in constant time whether $f_i(P)$ and $g_j(P)$ are sources in $G(\mathcal{R}^t)$ (by Lemma 21). Note that either $f_i(P)$ or $g_j(P)$ must be a source. Let $R_k = f_i(P)$ and $R_l = g_j(P)$.

- If both $f_i(P)$ and $g_j(P)$ are sources, then by Lemma 20 the only three subproblems in $G(\mathcal{R}^{t+1})$ and $[R_i, R_j]$ are: $[R_i = p_i, R_k]$, $[R_k, R_l]$ and $[R_l, R_j = p_j]$. In this case $p_{k-1}^{xMax} = p_i^{xMax}$ and $p_{l+1}^{yMax} = p_j^{yMax}$. If k = l 1, we immediately retrieve p_k and p_l in 2C time as the aforementioned overhead. Else we add to $[R_k, R_l]$ a reference to p_{k-1}^{xMax} and p_{l+1}^{yMax} to maintain Invariant 1 and add the subproblem $[R_k, R_l]$ to the queue.
- If $f_i(P)$ is a source and $g_j(P)$ is not, by the same reasoning the only subproblems are $[R_i, R_k]$ and $[R_k, R_j]$. We check if k = j 1 as before. If not, we maintain Invariant 1 in constant time just as above by adding $[R_k, R_k]$ to the queue with a reference to p_i^{xMax} .
- This case is symmetric to the previous, as $f_i(P)$ is not a source and $g_j(P)$ is.

Case 2: $f_i(P) \in [R_a, R_b]$ and $g_j(P) \in [R_e, R_f]$ for distinct children [a, b] and [e, f] of v. In this case, per construction of $T_{\mathcal{R}^*}$, each child [c, d] of v with $b \leq c < d \leq e$ is a subproblem of $G(\mathcal{R}^{t+1})$. We wish to briefly note, that either c < d-1, or [c, d] neighbors a child of v for which this is true (else, regions could have been compounded). Hence by Theorem 10 if c = d-1 we charge 2C time to the neighbor to immediately retrieve p_c and p_d and possibly add them to Ξ^* (again as the aforementioned overhead). If c < d-1, then per construction of $T_{\mathcal{R}^*}$, the point p_c appears on the Pareto front of P. Note that since [c, d] is a child of v, p_{c-1}^{xMax} can only be p_{c-1} or p_i^{xMax} . We charge O(1) time to the future processing of $[R_c, R_d]$ to provide four pointers to $[R_c, R_d]$ (to maintain Invariant 1) and add $[R_c, R_d]$ to the queue.

What remains is to handle [a, b] and [e, f] and we describe the procedure for [a, b]. We check in constant time if $f_i(P)$ is a source using Lemma 21. If it is, then by Lemma 20 the only subproblems of $G(\mathcal{R}^{t+1})$ contained in $[R_i, R_b]$ are $[R_i, f_i(P)]$ and $[f_i(P), R_b]$. We briefly

check if $[f_i(P), R_b]$ is a subproblem of length 2. If so we retrieve the corresponding points to see if they appear on the Pareto front. Else we add $[f_i(P), R_b]$ to the queue in constant time via the same procedure as Case 1. If $f_i(P)$ is not a source, then $[R_i, R_b]$ is the only subproblem of $G(\mathcal{R}^{t+1})$ in $[R_i, R_b]$ and we handle it similarly. We conclude:

▶ Theorem 22. Algorithm 1 constructs Ξ^* in $O(A(\mathcal{R}, \mathcal{R}^*, P)) = \Theta(CP(\mathcal{R}, P))$ time.

- References -

- 1 Peyman Afshani, Jérémy Barbay, and Timothy M Chan. Instance-optimal geometric algorithms. Journal of the ACM (JACM), 64(1):1–38, 2017.
- 2 Michael Ben-Or. Lower bounds for algebraic computation trees. In *Proc. 15th annual ACM Symposium on Theory of Computing*, pages 80–86, 1983.
- 3 Richard Bruce, Michael Hoffmann, Danny Krizanc, and Rajeev Raman. Efficient update strategies for geometric computing with uncertainty. *Theory of Computing Systems*, 38(4):411–423, 2005.
- 4 Kevin Buchin, Maarten Löffler, Pat Morin, and Wolfgang Mulzer. Delaunay triangulation of imprecise points simplified and extended. *Algorithmica*, 61:674–693, 2011. doi:http://dx.doi.org/10.1007/s00453-010-9430-0.
- 5 Kevin Buchin and Wolfgang Mulzer. Delaunay triangulations in O(sort(n)) time and more. Journal of the ACM (JACM), 58(2):6, 2011.
- 6 Jean Cardinal, Samuel Fiorini, and Gwenaël Joret. Minimum entropy coloring. In Proc. 16th International Symposium on Algorithms and Computation (ISAAC), pages 819–828. Springer, 2005.
- 7 Jean Cardinal, Gwenaël Joret, and Jérémie Roland. Information-theoretic lower bounds for quantum sorting. arXiv preprint:1902.06473, 2019.
- 8 Timothy M Chan. Comparison-based time-space lower bounds for selection. ACM Transactions on Algorithms (TALG), 6(2):1–16, 2010.
- **9** Mark de Berg, Otfried Cheong, Marc Van Kreveld, and Mark Overmars. *Computational Geometry: Introduction*. Springer, 2008.
- 10 Erik D Demaine, Adam C Hesterberg, and Jason S Ku. Finding closed quasigeodesics on convex polyhedra. In Proc. 36th International Symposium on Computational Geometry (SoCG). Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.
- Olivier Devillers. Delaunay triangulation of imprecise points, preprocess and actually get a fast query time. *Journal of Computational Geometry*, 2(1):30–45, 2011.
- 12 Jeff Erickson et al. Lower bounds for linear satisfiability problems. In SODA, pages 388–395, 1995.
- 13 Jeff Erickson, Ivor van der Hoog, and Tillmann Miltzow. Smoothing the gap between np and er. In *Proc. IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE, 2020.
- William Evans, David Kirkpatrick, Maarten Löffler, and Frank Staals. Competitive query strategies for minimising the ply of the potential locations of moving points. In Proc. 29th Annual Symposium on Computational Geometry, pages 155–164. ACM, 2013.
- William Evans and Jeff Sember. The possible hull of imprecise points. In *Proc. 23rd Canadian Conference on Computational Geometry*, 2011.
- Esther Ezra and Wolfgang Mulzer. Convex hull of points lying on lines in $o(n \log n)$ time after preprocessing. Computational Geometry, 46(4):417-434, 2013.
- 17 Martin Held and Joseph SB Mitchell. Triangulating input-constrained planar point sets. *Information Processing Letters*, 109(1):54–56, 2008.
- 18 Simon H Kahan. Real-time processing of moving data. 1992.
- David G Kirkpatrick and Raimund Seidel. Output-size sensitive algorithms for finding maximal vectors. In Proceedings of the first annual symposium on Computational geometry, pages 89–96, 1985.

22 Preprocessing Imprecise Points for the Pareto Front

- 20 Chih-Hung Liu and Sandro Montanari. Minimizing the diameter of a spanning tree for imprecise points. Algorithmica, 80(2):801–826, 2018.
- 21 Maarten Löffler and Wolfgang Mulzer. Unions of onions: Preprocessing imprecise points for fast onion decomposition. *Journal of Computational Geometry*, 5:1–13, 2014.
- 22 Maarten Löffler and Jack Snoeyink. Delaunay triangulation of imprecise points in linear time after preprocessing. *Computational Geometry*, 43(3):234–242, 2010.
- Maarten Löffler and Marc van Kreveld. Largest and smallest convex hulls for imprecise points. Algorithmica, 56(2):235, 2010.
- 24 Shlomo Moran, Marc Snir, and Udi Manber. Applications of ramsey's theorem to decision tree complexity. *Journal of the ACM (JACM)*, 32(4):938–949, 1985.
- Takayuki Nagai, Seigo Yasutome, and Nobuki Tokura. Convex hull problem with imprecise input and its solution. Systems and Computers in Japan, 30(3):31–42, 1999.
- Arnold Schönhage. On the power of random access machines. In *International Colloquium on Automata, Languages, and Programming*, pages 520–529. Springer, 1979.
- 27 Ivor van der Hoog, Irina Kostitsyna, Maarten Löffler, and Bettina Speckmann. Preprocessing ambiguous imprecise points. In 35th International Symposium on Computational Geometry (SoCG). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2019.
- Marc van Kreveld, Maarten Löffler, and Joseph SB Mitchell. Preprocessing imprecise points and splitting triangulations. SIAM Journal on Computing, 39(7):2990–3000, 2010.
- Andrew Chi-Chih Yao. A lower bound to finding convex hulls. *Journal of the ACM (JACM)*, 28(4):780–787, 1981.
- 30 Jiemin Zeng. Integrating Mobile Agents and Distributed Sensors in Wireless Sensor Networks. PhD thesis, The Graduate School, Stony Brook University: Stony Brook, NY., 2016.

A Reviewing lower bounds

The folklore worst-case lower bound definition of an algorithmic problem \mathcal{P} with input X is:

 $\text{Worst-case lower bound}(\mathcal{P}) := \min_{A} \max_{X} \text{Runtime}(A, X) \,,$

where each A is an algorithm that solves \mathcal{P} for some definition of solving. Afshani, Barbay and Chan [1] observe that there are three common techniques to prove lower bounds within computational geometry:

- direct arguments based on counting, or information theory;
- topological arguments, as used by e.g. Yao [29] or Ben-Or [2] (sometimes referred to as algebraic decision tree arguments); or
- arguments based on Ramsey theory, as used by e.g. Moran, Snir and Manber [24].

The latter two techniques decompose algorithms into decision trees and reason about their depth. In traditional computation models decisions are binary; therefore, without additional information about the decision tree structure of the specific problem \mathcal{P} , the best possible lower bound on its tree depth is $\Omega(\log(\#\text{leaves}))$, which is equivalent to the information-theoretic bound. We mention that an additional technique for obtaining lower bounds is an adversarial argument as by Erickson [12] or the more recent Chan [8]. Here, we restrict our attention to information-theoretic arguments.

Models of computation. Applying these techniques to bound the running time of the algorithms A, requires a precise definition of the model of computation used for the algorithmic analysis. The classical argument by Ben-Or [2] assumes that the computation can be modeled by an algebraic decision tree, where in each node a binary decision is taken at which the algorithm branches based on an algebraic test.

Afshani, Barbay and Chan investigate a stronger definition for an algorithmic lower bound. They reason that the computational power that comes from the abstract algebraic decision tree model, where algebraic test functions are only bounded in the number of arguments and not their degree, is too large for a more fine-grained analysis of algorithmic running time. They restrict the class of algorithms that they consider for their competitive analysis to algebraic decision trees where each test is a multilinear function (a function that is linear, separate in each of its variables) with a constant number of variables. We share the sentiment that a computational model that allows arbitrary algebraic computations in constant time is unrealistically powerful, but note that the alternative model is perhaps too restrictive, as it becomes difficult, if not impossible, to express computations such as higher-dimensional range searching using only multilinear functions.

Recently, Erickson, van der Hoog and Miltzow [13] note that computations that involve data structures do not only need to make decisions, but also need to be able to access memory. Memory is inherently discrete: a model that supports only real-valued algebraic decisions can either not access memory, or has the ability to access discrete values with real-valued computations which would imply that P = PSPACE [26]. Fueled by the desire to analyse algorithms within computational geometry, they (re)define the real RAM. We use their definition of RAM to be able to define lower bounds for the preprocessing model (as the preprocessing model inherently can access memory as it needs to be able to use an auxiliary data structure Ξ). For completeness, we summarize their definition and how it enables an information theoretic lower bound, even when dealing with a pre-stored structure Ξ at the end of this section.

Better than worst-case optimality. A natural more refined lower bound than the worst-case lower bound is the instance lower bound. Given an algorithmic problem \mathcal{P} with input X, the *instance lower bound* is defined as:

```
Instance lower bound(\mathcal{P}, X) := \min_{A} \text{Runtime}(A, X).
```

We recall the example in the introduction where we perform a binary search to see whether a value q is contained in a sorted sequence of numbers X. For each instance (X,q), there exists a "lucky" algorithm that guesses the location of q in X in constant time. Thus, the instance lower bound for binary search is constant, even though there is no algorithm that can perform binary search in constant time in a comparison-based RAM model. Fine-grained algorithmic analysis is desirable, yet instance optimality is unobtainable. It is therefore unsurprising that there is a rich tradition of finding algorithmic analyses that capture an algorithmic performance that is better than worst-case optimality. Many attempts parametrize the algorithmic problem, to better enable its analysis. For example, there is output-sensitive analysis as used by Kirkpatrick and Seidel [19] where the algorithm runtime depends on the size k of the output. Other parameters can include geometric restrictions such as fatness, the spread of the input, or the number of reflex vertices in a (simple) polygon. Such parameters are hard to apply in the preprocessing model with implicit representation, as the auxiliary structure Ξ allows one to bypass the natural lower bound that these parameters bring. For example: an output-sensitive lower bound is not applicable, as output of any size can be computed in the preprocessing phase to be referred to in the reconstruction phase in O(1)time.

Better than worst-case optimality without additional parameters. Afshani, Barbay and Chan propose an alternative definition of instance optimality which is not inherently unobtainable. They restrict the algorithms A that solve \mathcal{P} and consider the input I together with a permutation σ . They analyse the running time of A, conditioned on that it receives input X in the order given by σ . They then compare algorithmic running time based on the worst choice of σ :

```
Instance lower bound in the \mathit{order\ oblivious}\ \mathsf{setting}(\mathcal{P},X) := \min_{A} \max_{\sigma} \mathsf{Runtime}(A,X,\sigma)\,.
```

Intuitively, a permutation σ can force the algorithm to make poor decisions by placing the input in a bad order and they assume that an algorithm receives "the worst order of processing the input" to avoid the unreasonable computational power that a guessing algorithm has. The instance lower bound in the order oblivious setting for our binary search example would be $\Omega(n)$, as there exists a σ for which X is not a sorted set. Given q and (X,σ) , any algorithm then has to spend linear time to check if q is in X.

This definition of lower bound would strictly speaking be applicable to the preprocessing model: given P and a permutation σ an algorithm can then only retrieve points in the order σ . However, we would argue that this lower bound is not very compatible with the spirit of the model. Per definition, one is free to preprocess \mathcal{R} , Therefore, during preprocessing it would not be unreasonable for an algorithm to decide on a favourable order to retrieve the points in P. This is why, amongst many alternative stricter-than-worst-case lower bound definitions, we propose another, specifically for the preprocessing model.

```
\text{Uncertainty-region lower bound}(\mathcal{P},\mathcal{R}) := \min_{(A,\Xi)} \max_{(P \text{ respects } \mathcal{R})} \text{Runtime}(A,\Xi,\mathcal{R},P) \,,
```

Denote for any fixed algorithmic problem \mathcal{P} , by $L(\mathcal{R})$ the number of combinatorially distinct outcomes of P given \mathcal{R} . In the remainder of this section we recall the RAM definition of [13]

to show that regardless of (A, Ξ) , $\Omega(\log L(\mathcal{R}))$ is an uncertainty region lower bound for the time required by A to solve \mathcal{P} .

Recalling the real RAM definition. If the reader is confident in the ability of the RAM model to support such a lower bound, we advise the reader skips ahead. Erickson, van der Hoog and Miltzow define the real RAM in two steps. First, they define computations based on the (discrete) word RAM, so that discrete memory can be accessed without unreasonable computational power. Then, they augment the word RAM with separate real-valued computations that only work on values stored within the discrete memory cells. Their operations include memory manipulation, real arithmetic and comparisons (which verifies if the real value stored in a memory cell is greater than 0). For an extensive overview of the computations that they allow, we refer to Table 1 in [13]. They say a program on the real RAM consists of a fixed, finite indexed sequence of read-only instructions. The machine maintains an integer program counter, which is initially equal to 1. At each time step, the machine executes the instruction indicated by the program counter. Every real RAM operation increases the program counter by one, apart from a comparison operation which ends in a goto statement that can set the program counter to any discrete value. This model thereby immediately allows the classical information theoretic lower bound argument, even if there is some pre-stored data Ξ within memory. Indeed, let \mathcal{P} be an algorithmic problem such that there are L distinct outcomes and fix a program (algorithm) that reports the correct outcome. Each outcome may be described by the sequence of instructions that lead to it, together with a halt instruction that tells the program to stop and output the result. Hence, the program only terminates on the correct outcome, if it arrived there via a goto statement from a comparison instruction (all other instructions only increase the program counter by 1, hence without comparisons the algorithm terminates at the first outcome in the sequence). It follows, that any sequence of instructions can be converted into a binary tree where each node is a comparison instruction and where the leaves of the tree are lines in the sequence that store an outcome with a halt instruction. Hence regardless of Ξ , there is an outcome stored as a leaf in the tree where the program that requires $\Omega(\log L)$ comparison instructions until it arrives at that leaf.

B Handling compound regions

We describe the algorithmic procedure for when Algorithm 1 encounters a subproblem $[R_i, R_j]$ where $f_i(P)$ or $g_i(P)$ is a compound region. Let $f_i(P)$ be a compound region. Then per definition $f_i(P)$ is a sink in the original graph: $G(\mathcal{R}^0)$. Consequently, the region R' in the canonical set \mathcal{R}^0 that succeeds R must have no more remaining incoming vertical arrows (as else, R would not have been visible from the just processed R_i). The region R' itself cannot be a compound region, since else R and R' could have been compounded together. We set $f_i(P)$ to be R' instead, and continue as normal.

We set the compound region R aside, with a reference to p_i^{xMax} and add it to a separate queue that we handle at the algorithm's termination in O(1) time. We charge this O(1) time to this iteration t where we added it to the special queue. Per definition, for each region R_i , there is a unique $f_i(P)$, so R_i gets charged at most once in this manner. It is possible that in a later iteration t', when a subproblem $[R_{i'}, R_{j'}]$ is considered by Algorithm 1, the region R is $g_{j'}(P)$. In this case, we do not add R to the queue again but we do store a reference to $p_{i'}^{yMax}$ and we charge $[R_{i'}, R_{j'}]$, O(1) time for storing this reference.

For any compound region R, that is not dominated by a point in P, there must be an

iteration t where a subproblem is considered such that $f_i(P) = R$ or $g_i(P) = R$ and thus it must be in the special queue. When we process the special queue, we do the following: we use p_i^{xMax} to identify the prefix of the original regions stored in R that are dominated by points preceding R in $O(\log |V_i(P)|)$ time using galloping search (we charge the prior $f_i(P)$, and just as above a region can only get charged once).

At this point, we wish to briefly remark upon any possible ambiguity regarding the runtime $O(\log |V_i(P)|)$. In the premise of Theorem 10 we defined the sets $V_i(P)$ as subsets of the truncated set \mathcal{R} , not the canonical set $\mathcal{R}^0 = \mathcal{R}^*$ that serves as the input of the algorithm. Note that $O(\log |V_i(P)|)$ is smaller than $O(\log |V_i^*(P)|)$ where $V_i^*(P)$ is a subset of \mathcal{R}^* since \mathcal{R}^* can compound regions in $V_i(P)$ together. Throughout Section 5.3, we performed a galloping search over the outgoing edges in the graph $G(\mathcal{R}^*)$, hence we spent $O(\log |V_i^*(P)|) \le O(\log |V_i(P)|)$ time per search. Here, we perform a galloping search over regions in V_i that are compounded (not in \mathcal{R}^*), and this is the first point where we use the larger $O(\log |V_i(P)|)$ runtime. We wish to emphasise that the runtime of Section 4.1 is hereby correct: as $O(\log |V_i(P)|)$ is an over-estimation of the actual time spent on the galloping search. We continue the argument:

Whenever $g_j(P) = R$, we similarly use p_j^{yMax} to identify the suffix of the original regions stored in R that are dominated by points in P succeeding R. For the at most 2 regions that are intersected by the vertical line through p_i^{xMax} and the horizontal line through p_i^{yMax} respectively, we explicitly retrieve their points in order to determine whether they are dominated or not. We charge this 2C retrieval time to R_i and R_j . By Lemma 9, the remaining sequence of original regions (if any) must appear on the Pareto front, and we do not need to retrieve their points. When the algorithm terminates, we append the non-dominated interval in constant time by providing the pointers in the array of Attribute 5, and we charge this constant time to the aforementioned iteration.