On complete classes of valuated matroids*

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Abstract

We characterize a rich class of valuated matroids, called *R-minor valuated matroids* that includes the indicator functions of matroids, and is closed under operations such as taking minors, duality, and induction by network. We exhibit a family of valuated matroids that are not R-minor based on sparse paving matroids.

Valuated matroids are inherently related to gross substitute valuations in mathematical economics. By the same token we refute the Matroid Based Valuation Conjecture by Ostrovsky and Paes Leme (Theoretical Economics 2015) asserting that every gross substitute valuation arises from weighted matroid rank functions by repeated applications of merge and endowment operations. Our result also has implications in the context of Lorentzian polynomials: it reveals the limitations of known construction operations.

1 Introduction

Valuated (generalized) matroids capture a quantitative version of the exchange axiom(s) for matroids. They were first introduced by Dress and Wenzel [15], motivated by questions related to number theory and the greedy algorithm. Later, Murota [35] identified them as a fundamental building block for discrete convex analysis. They play important roles across different areas of mathematics and computer science, with particularly many applications in algorithmic game theory.

The study of valuated matroids and valuated generalized matroids is intimately connected and they can be defined in many different ways: in tropical geometry [18, Theorem 4.1.3], via the interplay of price and demands in economics [30], or with various exchange properties [38]. We follow [19, 39], and say that a function $f\colon 2^V \to \mathbb{R} \cup \{-\infty\}$ is a *valuated generalized matroid* if two properties hold:

$$\begin{aligned}&\forall X, Y \subseteq V \text{ with } |X| < |Y| : \\ &f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X + j) + f(Y - j)\}\end{aligned} \tag{1.1a}$$

$$\begin{aligned}&\forall X, Y \subseteq V \text{ with } |X| = |Y| \text{ and } \forall i \in X \setminus Y : \\ &f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X - i + j) + f(Y + i - j)\}.\end{aligned} \tag{1.1b}$$

For fixed $r \leq |V|$, those set functions $\binom{V}{r} \to \mathbb{R} \cup \{-\infty\}$ fulfilling (1.1b) are *valuated matroids*, the main objects of this work. This means that each layer of a valuated generalized matroid is a valuated matroid. Valuated matroids with codomain $\{0, -\infty\}$ coincide with usual matroids as the sets taking value 0 form the bases of a matroid; we call them *trivially valuated matroids*. In this context, (1.1b) corresponds to the strong basis exchange property.

R-minor valuated matroids We are interested in the following classes of valuated matroids arising from independent matchings in bipartite graphs. The name is inspired by the induction of matroids through bipartite graphs introduced by **R**ado [44].

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Definition 1.1. (R-minor, R-induced) *Let $G = (V \cup W, U; E)$ be a bipartite graph with edge weights $c \in \mathbb{R}^E$, and a matroid $\mathcal{M}$ on U of rank $d + |W|$. We define an* R-minor *valuated matroid $f\colon \binom{V}{d} \to \mathbb{R}$ for $X \in \binom{V}{d}$ as follows.*

The value $f(X)$ is the maximum weight of a matching in G whose endpoints in $V \cup W$ are $X \cup W$, and the endpoints in U form a basis in $\mathcal{M}$. For $W = \emptyset$, the function f is called an R-induced *valuated matroid.*

This concept naturally extends to valuated generalized matroids: the endpoints in U should not form a basis but a set in a *generalized matroid.*[1] In 2003, Frank [36, 37] asked if all valuated matroids arise as *R-induced* valuated matroids. The corresponding version of this question for valuated generalized matroids has been recently disproved by Garg et al. [20]. This is not surprising given that valuated (generalized) matroids are closed under contraction, whereas R-induced valuated (generalized) matroids are not.

Noting that R-minor valuated matroids are precisely the contractions of R-induced valuated matroids, this suggests a natural refinement of the original question:

Do all valuated matroids arise as R-minor valuated matroids?

The variant of this question for valuated generalized matroids was proposed in [20]. The main contributions of this paper are *(i)* showing that R-minor valuated matroids form a *complete class* of valuated matroids closed under several fundamental operations, yet *(ii)* not all valuated matroids arise in this form, answering the above question negatively. We then derive implications for gross substitute valuations and for Lorentzian polynomials.

Complete classes of matroids Let us consider R-induced and R-minor valuated matroids where $\mathcal{M}$ is the free matroid and $c \equiv 0$. Valuated matroids f arising in such forms are the $\{0, -\infty\}$ indicator functions of *transversal matroids* and *gammoids* respectively. In 1977, Ingleton [25] studied representations of transversal matroids and gammoids. He observed that gammoids arise via this simple construction yet form a rich class closed under several fundamental matroid operations. This motivated the definition of a *complete class* of matroids by requiring closure under the operations *restriction, dual, direct sum, principal extension*. Closure under principal extension combined with restriction implies closure under induction by bipartite graphs which encompasses many other natural matroid operations, including matroid union. Closure under this operation is what creates the rich structure of complete classes, even when one starts from very basic matroids; note that gammoids arise as the smallest complete class by taking the closure of the matroid on one element.

In 1971, Brualdi [12] showed that if a matroid is *strongly base-orderable*, then so is each matroid induced from it. Thus, gammoids are strongly base-orderable. A simple example of a matroid that is not strongly base-orderable is the graphical matroid of K_4, which is consequently not a gammoid.

The theory of complete classes was further developed in Bonin and Savitsky [9] who also collected the necessary properties to define a complete class. The quest for succinct representations of matroids is intimately connected to questions in parametrized complexity, see e.g. [28].

Gross Substitutes A somewhat surprising application of valuated generalized matroids arises in mathematical economics. Gross substitutability captures the following type of interaction between prices and demands for goods. At given prices, an agent would like to buy a certain amount of goods. If the price of a single good increases then we expect that demand for this good decreases. Consequently, more money can be spent on the goods with the unchanged price and thereby the demand for such goods should not decrease. This concept is crucial for equilibrium existence and computation [4, 5, 14], auction algorithms [21, 22, 29], and mechanism design [6, 24].

In the case of discrete (indivisible) goods, an agent determines her demand by maximizing a *valuation function*: a monotone set function taking value 0 on the empty set. Hence, gross substitutability is a property of a function. It turns out that the functions with the gross substitute property (GS functions) are exactly valuated generalized matroids [39].

For indivisible goods, the property was first formalized by Kelso and Crawford [26] to show that a natural auction-like price adjustment process converges to an equilibrium. We also point out that Gül and Stacchetti [22] showed that the so-called Walrasian equilibrium exists whenever agents' valuations satisfy the gross substitute property and that, in a sense, the converse also holds. For further results on gross substitutability, we refer to [40, Chapter 11] and a survey by Paes Leme [30].

[1] These are defined as the effective domain of a $\{0, -\infty\}$-valued valuated generalized matroid, see Section 4. The canonical examples are independent sets of matroids.

A classical example of GS functions (= valuated generalized matroids) are *assignment (OXS) valuations* introduced by Shapley [45]. For a graph $G = (V, U; E)$ with edge-weights $c \in \mathbb{R}^E_{\geq 0}$, the value $v(X)$ for $X \subseteq V$ is defined as the maximum weight matching with endpoints in X.[2]

Constructions of substitutes By the equivalence with valuated generalized matroids, functions with the gross substitute property can be described in many different ways. In fact, Balkanski and Paes Leme [7] mention eight characterizations of GS functions. Nevertheless, finding a constructive description of all GS functions/valuations remained elusive.

The first attempt to "construct" all GS valuations was by Hatfield and Milgrom [24]. After observing that most examples of GS valuations arising in applications are built from assignment valuations and the endowment operation, they asked if this is true for all GS valuations. Ostrovsky and Paes Leme [41] showed that this is not the case: some matroid rank functions cannot be constructed as endowed assignment valuations while all (weighted) matroid rank functions are GS valuations. Instead, Ostrovsky and Paes Leme proposed the *matroid based valuations (MBV)* conjecture. Matroid based valuations are those that arise from weighted matroid rank functions by repeatedly applying the operations of *merge* and *endowment.* Tran [47] showed that using only merge but no endowment operations does not suffice, but the conjecture remained open.

Originally, interest for such conjectures stemmed from auction design. They are an attempt at designing a language in which agents can represent their valuations in a compact and expressible way [30]. Moreover, valuations with a constructive description facilitate more algorithmic techniques, especially linear programming (see Section 3 and e.g., [20]). In this paper, we analyze and disprove the MBV conjecture through the lens of complete classes.

Sparse paving matroids A crucial tool for our counterexamples to the conjectures are valuated matroids arising from the well-known class of sparse paving matroids. A matroid of rank d is *paving* if all circuits are of size d or $d + 1$, and *sparse paving* if in addition the intersection of any two d-element circuits is of size at most $d - 2$. Knuth [27] gave an elegant construction of a doubly exponentially large family of sparse paving matroids; this is essentially the strongest lower bound on the number of matroids on n elements. In fact, it was conjectured in [33] that asymptotically almost all matroids are sparse paving; weaker versions were proved in [8] and [43]. Our main valuated matroid construction is based on sparse paving matroids that arise from Knuth's construction.

Lorentzian polynomials Brändén and Huh [11] recently introduced *Lorentzian polynomials* generalizing stable polynomials in optimization theory and volume polynomials in algebraic geometry. They act as a bridge between discrete and continuous convexity. In particular, their domains form discrete convex sets, generalizing earlier work connecting matroids and polynomials, e.g., by Choe et al. [13]. Their connection to continuous convexity is via their equivalence to strongly log-concave polynomials discovered by Gurvits [23] and completely log-concave polynomials which were used by Anari et al. [1] in their breakthrough work for efficiently sampling bases of matroids. This connection has lead to numerous applications in combinatorial optimization and other areas [2, 3]. Furthermore, they are intimately connected to valuated matroids via tropical geometry: Brändén and Huh showed that the space of valuated matroids arises as the tropicalization of squarefree Lorentzian polynomials.

There is on-going research regarding the space of Lorentzian polynomials [10]. They are closed under many natural operations analogous to valuated matroids, therefore a natural question is whether one can construct the space of Lorentzian polynomials from certain "building block" functions closed under these operations. We use our techniques to deduce limitations on these constructions.

1.1 Our contributions

Complete classes of valuated matroids We introduce the notion of complete classes of valuated matroids. These are classes of valuated matroids closed under the valuated generalizations of the fundamental operations *restriction, dual, direct sum, principal extension.* The crucial ingredient going beyond the basic operations already introduced in [15] is (valuated) principal extension. This is a special case of *transformation by networks* [35, Theorem 9.27]. These operations appeared as 'linear maps' and 'linear extensions' in tropical geometry [18, 34]. Right from the definition, valuated gammoids are seen to form the smallest complete class of valuated matroids

[2]Shapley introduces the valuations as follows. Assume that V are workers and U is the set of jobs within a company. The edge set represents the possibilities (willingness) of assigning workers to jobs, and the weight c_{ij} is the value the company gets by assigning worker i to job j. Then the value of a subset $X \subseteq V$ of workers for the company is the maximum possible value the company gets by assigning workers X to jobs U.

(Theorem 2.3).

The study of complete classes gives rise to a common framework for R-minor valuated matroids and those arising from the MBV conjecture. We can also consider existing results from different fields in a unified manner: the proof of Ostrovsky and Paes Leme [41] that endowed assignment valuations do not exhaust all GS functions is based on a valuated analogue of strongly base orderable matroids. Also the work on *Stiefel tropical linear spaces* in tropical geometry [16, 17] can be considered as the study of representations in the complete class of valuated gammoids.

Complete class containing trivially valuated matroids After introducing complete classes, an immediate question arises: does the smallest complete class containing trivially valuated matroids cover all valuated matroids? Or in other words, does the smallest class containing all trivially valuated matroids and that is closed for deletion, contraction, duality, truncation, and principal extension exhaust all valuated matroids?

We show that the smallest class of valuated matroids containing all trivially valuated matroids and that is closed for the above operations is exactly the class of R-minor valuated matroids. Thus, the above question is equivalent to whether every valuated matroid is an R-minor valuated matroid. We can use an information-theoretic argument to show that not all valuated matroids are R-induced by constructing valuated matroids with many independent values (Proposition 2.1). However, such an arguments does not seem extendible to R-minor valuated matroids, as the size of the contracted set W may be arbitrarily large. Thus, the construction disproving the more general claim relies on a well-chosen family of valuated matroids.

Non-R-minor valuated matroids The most challenging part of our paper is proving that there are valuated matroids that are not R-minor valuated matroids. In particular, we show that none of the valuated matroids in the following family is R-minor.

Definition 1.2. *For $n \geq 2$, we define $\mathcal{F}_n$ as the following family of functions $\binom{[2n]}{4} \to \mathbb{R}$. Let $V = [2n]$, $P_i = \{2i-1, 2i\}$ for $i \in [n]$, and let*

$$\mathcal{H} = \{ P_i \cup P_j \mid ij \equiv 0 \mod 2\} \qquad (\mathcal{H}\text{-def})$$

i.e. we take pairs such that at least one of i, j is even. Let $X^* = P_1 \cup P_2 = \{1, 2, 3, 4\}$. A function $h : \binom{V}{4} \to \mathbb{R} \cup \{-\infty\}$ is in the family $\mathcal{F}_n$ if and only if the following hold:

- $h(X) = 0$ *if* $X \in \binom{V}{4} \setminus \mathcal{H}$,
- $h(X) < 0$ *if* $X \in \mathcal{H}$, *and*
- $h(X^*)$ *is the unique largest nonzero value of the function.*

Theorem 1.1. (Main) *If $n \geq 2$, then all functions in $\mathcal{F}_n$ are valuated matroids. If $n \geq 16$, then no function in $\mathcal{F}_n$ arises as an R-minor function.*

The functions in $\mathcal{F}_n$ are derived from sparse paving matroids; our construction was inspired by Knuth's [27] work. We note that if $\mathcal{B}$ is the family of bases of a sparse paving matroid of rank d, then any function $h : \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ with $h(X) = 0$ if $X \in \mathcal{B}$ and $h(X) < 0$ otherwise gives a valuated matroid. This implies in particular that all functions in $\mathcal{F}_n$ are valuated matroids.

As our family allows still for quite some flexibility and it is conjectured that almost all matroids are sparse paving [33], one could guess that even *almost all* valuated matroids might not be R-minor. But the development of the framework for making such a statement goes beyond the scope of this paper.

Refuting the Matroid Based Valuation Conjecture Building on Theorem 1.1, we also refute the MBV conjecture by Ostrovsky and Paes Leme [41]. This is done by considering valuated generalized matroids corresponding to R-minor valuated matroids and reduce to Theorem 1.1 by considering their layers.

First, we show that every function that can be obtained from weighted matroid rank functions by repeatedly applying merge and endowment is an R$^\natural$-minor valuated generalized matroid – the class of valuated generalized matroids arising by contraction and induction from generalized matroids. Garg et al. [20] proposed the conjecture that all valuated generalized matroids have an R$^\natural$-minor representation.

Then, we show that the function $h^\natural : 2^V \to \mathbb{R}_{\geq 0}$ defined as follows is a valuated generalized matroid but not R$^\natural$-minor. This disproves the conjecture in [20], as well as the MBV conjecture. For an arbitrary valuated matroid

$h \in \mathcal{F}_n$ taking values only in $(-1, 0]$ we define

$$h^{\natural}(X) := \begin{cases} |X| & \text{for } |X| \leq 3, \\ 4 + h(X) & \text{for } |X| = 4, \\ 4 & \text{for } |X| \geq 5. \end{cases}$$

We achieve this by focusing on the function restricted to all 4-subsets of V. This is an R-minor valuated matroid and therefore allows us to apply Theorem 1.1. Note that the function $h^{\natural}$ has the additional structure of being *monotone* and only taking *non-negative finite* values, as the MBV conjecture refers to valuations. Finally, we note that while matroid based valuations form a subset of $\mathrm{R}^{\natural}$-minor valuated generalized matroids f that are monotone and $f(\emptyset) = 0$, it is unclear whether the containment is strict or if these two classes coincide.

Lorentzian polynomials A fundamental operation which preserves Lorentzian polynomials is by the multiplicative action of non-negative matrices in the argument [11, Theorem 2.10]. This means that, given a Lorentzian polynomial p in n variables, a non-negative matrix $A \in \mathbb{R}^{n \times k}$ and a variable vector $(y_1, \ldots, y_k)$, the polynomial given by $p(A \cdot y) \in \mathbb{R}[y_1, \ldots, y_k]$ is also Lorentzian.

We demonstrate that several basic operations for Lorentzian polynomials translate to the basic operations considered for valuated matroids via *tropicalization*. Most notably, the action of the multiplicative semigroup of non-negative matrices on Lorentzian polynomials translates to induction via bipartite graphs for valuated matroids (Theorem 5.2).

Taking polynomials which correspond to our family of counterexamples given in Definition 1.2 via tropicalization, we can deduce limitations on constructions of Lorentzian polynomials. The proof is based on the relation between polynomials over real-closed fields via Tarski's principle. Explicitly, we show that not all Lorentzian polynomials can be realized by the action of non-negative matrices on *generating functions of matroids* (Theorem 5.3).

1.2 Preliminaries We briefly introduce the notation for the basic objects required. We denote a bipartite graph G by $G = (V, U; E)$, where V, U are the partitioned node sets and E the edge set. For $Y \subseteq U$ or $Y \subseteq V$, we denote the set of neighbours of Y by $\Gamma(Y) = \Gamma_G(Y)$. Given a subgraph μ of G, we let $\partial_V(\mu)$ and $\partial_U(\mu)$ denote the nodes incident to the subgraph in V and U respectively. If the bipartite graph is weighted, we denote the edges weights by $c \in \mathbb{R}^E$. We denote a network N by $N = (T, A)$ where T is the node set and A the arc set; if weighted then we again denote the weights by $c \in \mathbb{R}^A$. Given a set V, we denote its set of subsets of cardinality d by $\binom{V}{d}$.

We denote a matroid $\mathcal{M}$ by $\mathcal{M} = (U, r)$ where U is the ground set of the matroid and $r = r_{\mathcal{M}}$ is the rank of the matroid. The notation of the major operations on matroids follows the notation of valuated matroids introduced in Section 2.1, as these are special cases of the valuated operations. For an introduction to matroids, we refer to Oxley's book [42].

2 Classes of valuated matroids

2.1 Operations on valuated matroids For a valuated matroid f, its *(effective) domain* $\mathrm{dom}(f)$ is formed by those sets X on which $f(X) > -\infty$. The exchange property implies that it forms the set of bases of a matroid. The *rank* $\mathrm{rk}(f)$ of a valuated matroid f is the rank of the underlying matroid $\mathrm{dom}(f)$.

Definition 2.1. *Let $f \colon \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ be a valuated matroid with $d = \mathrm{rk}(f)$, and $Y \subset V$ some subset of V.*

(i) If $V - Y$ has full rank in $\mathrm{dom}(f)$ then the deletion *of f by Y is the function $f \setminus Y \colon \binom{V-Y}{d} \to \mathbb{R} \cup \{-\infty\}$ defined as*

$$(f \setminus Y)(X) = f(X), \quad \forall X \in \binom{V-Y}{d}.$$

This is also called the restriction *to $V \setminus Y$ and denoted by $f|(V \setminus Y)$. If $V - Y$ does not have full rank in $\mathrm{dom}(f)$, the deletion is the function attaining only $-\infty$.*

(ii) If Y is independent in $\mathrm{dom}(f)$, then the contraction *of f by Y is the function $f/Y \colon \binom{V-Y}{d-|Y|} \to \mathbb{R} \cup \{-\infty\}$ defined as*

$$(f/Y)(X) = f(X \cup Y), \quad \forall X \in \binom{V-Y}{d-|Y|}.$$

If Y is not independent in $\operatorname{dom}(f)$*, the contraction is the function attaining only* $-\infty$.

(iii) *The* dual *of f is the function* $f^*\colon \binom{V}{|V|-d} \to \mathbb{R}\cup\{-\infty\}$ *defined as*

$$f^*(X) = f(V - X), \quad \forall X \in \binom{V}{|V|-d}.$$

(iv) *The* truncation *of f is the function* $f^{(1)}\colon \binom{V}{d-1} \to \mathbb{R}\cup\{-\infty\}$ *defined as*

$$f^{(1)}(X) = \max_{v \in V\setminus X} f(X \cup v), \quad \forall X \in \binom{V}{d-1}.$$

The iterated truncation for $1 \le r \le d-1$ *is given by* $f^{(r+1)} = (f^{(r)})^{(1)}$.

(v) *For* $w \in (\mathbb{R}\cup\{-\infty\})^V$*, the* principal extension f^w *of f with respect to w is the valuated matroid on $V \cup p$ of rank d, for an additional element p, with $f^w|V = f$ and*

$$f^w(X \cup p) = \max_{v \in V\setminus X} (f(X\cup v) + w_v) \quad \text{for all} \quad X \in \binom{V}{d-1}.$$

Remark 2.1. *Our definition of deletion and contraction differs from the usual definition, e.g. in [15], in that we impose these rank conditions. The usual definition of deletion (and dually contraction) for matroids could equally be formulated by first performing a truncation (to the rank of the remaining set) and then a deletion. While for unvaluated matroids this is the same, for valuated matroids the naive deletion, where the remaining set does not have full rank, would result in a function only taking $-\infty$ as value. Our reason to be more restrictive with deletion and contraction is that these definitions allow for simple 'layer-wise' extensions to valuated generalized matroids in Section 4 and they tie in more naturally with operations on polynomials as we demonstrate in Section 5.*

Example. The most basic examples of valuated matroids are those with *trivial valuation*, where only the values 0 and $-\infty$ are attained (following naming as in [16]). Such valuated matroids can be identified with the underlying matroid. Observe that the operations listed in Definition 2.1 agree with the usual matroid operations for trivially valuated matroids. □

Example. Valuated matroids corresponding to the layers of the assignment valuations are *transversally valuated matroids*. For a graph $G = (V, U; E)$ with edge weights $c \in \mathbb{R}^E$, we define transversally valuated matroid $f\colon \binom{V}{|U|} \to \mathbb{R}\cup\{-\infty\}$ for $X \in \binom{V}{d}$ as the maximum weight of a matching whose endpoints in V are exactly X; if no such matching exists then we set $f(X) = -\infty$.

Let $V = [4]$ and consider the valuated matroid $f\colon \binom{V}{2} \to \mathbb{R}\cup\{-\infty\}$ defined as

$$f(12) = -\infty\ ,\ f(13) = 0\ ,\ f(14) = 0\ ,\ f(23) = 1\ ,\ f(24) = 1\ ,\ f(34) = 1\,.$$

This valuated matroid is transversally valuated as it can be realized via the weighted bipartite graph shown in Figure 1. □

Example. One source of valuated matroids arises from matrices with polynomial entries. Let A be a matrix with d rows and columns labelled by V, whose entries are univariate polynomials over a field. For $J \subseteq V$, we denote by $A[J]$ the submatrix formed by the columns labelled by J. The valuated matroid induced by A is defined to be

$$f(J) = \deg\det A[J]\,,$$

where $f(J) = -\infty$ if $\det A[J] = 0$ or $A[J]$ is non-square, see [35, Section 2.4.2] for further details.

Recall the valuated matroid from the previous example. Observe that we can also represent this matrix via the polynomial matrix

$$A = \begin{bmatrix} 1 & t & t & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

e.g. $f(23) = \deg(t) = 1$. □

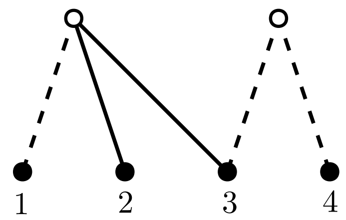

Figure 1: The bipartite graph realizing the transversally valuated matroid from Example 2.1. The dashed edges have weight zero and the solid edges have weight one.

Definition 2.2. (Direct sum, Valuated matroid union) *Let f_1 and f_2 be valuated matroids on ground sets V_1 and V_2 with ranks d_1 and d_2.*

- *For $V_1 \cap V_2 = \emptyset$, the* direct sum *of f_1 and f_2 is $(f_1 \oplus f_2)\colon \binom{V_1 \cup V_2}{d_1+d_2} \to \mathbb{R} \cup \{-\infty\}$, where*

$$(f_1 \oplus f_2)(X_1 \cup X_2) = f_1(X_1) + f_2(X_2) \text{ for all } X_1 \in \binom{V_1}{d_1}, X_2 \in \binom{V_2}{d_2}.$$

- *For $V := V_1 \cup V_2$, the* (valuated) matroid union *of f_1 and f_2 is $(f_1 \vee f_2)\colon \binom{V}{d_1+d_2} \to \mathbb{R} \cup \{-\infty\}$, where*

$$(f_1 \vee f_2)(X) = \max\left\{ f_1(Y) + f_2(X \setminus Y) \;\middle|\; Y \subseteq X\,,\, Y \in \binom{V_1}{d_1}\,,\, X \setminus Y \in \binom{V_2}{d_2} \right\}.$$

Undefined sets get the value $-\infty$.

Actually, the direct sum can be considered as valuated matroid union by embedding both ground sets in a common bigger ground set. We give both definitions for sake of explicitness.

2.2 Induction by networks The next operation is very powerful and can be seen as a vast generalization of Rado's theorem. Somewhat surprisingly, we show that it can be modelled by the basic operations defined so far.

Definition 2.3. *Let $N = (T, A)$ be a directed network with a weight function $c \in \mathbb{R}^A$. Let $V, U \subseteq T$ be two non-empty subsets of nodes of N. Let g be a valuated matroid on U of rank d. Then the* induction of g by N *is a function $\Phi(N, g, c)\colon \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$. For $X \in \binom{V}{d}$, one sets*

$$\Phi(N, g, c)(X) = \max\left\{ \sum_{a \in \mathcal{P}} c(a) + g(Y) \;\middle|\; \text{node-disjoint paths } \mathcal{P} \text{ in } N\colon \partial_V(\mathcal{P}) = X \wedge \partial_U(\mathcal{P}) = Y \right\}.$$

Note that the maximization can also result in $-\infty$ if there exists no node-disjoint paths from X to a set with finite value. It is even possible that $\operatorname{dom}(\Phi(N, g, c)) = \emptyset$.

In the special case that the directed network is bipartite with the edges directed from V to U, we can also consider this as an undirected weighted bipartite graph and call the corresponding operation induction by bipartite graphs.

Theorem 2.1. (Special case of [35, Theorem 9.27]) *Let N, g and c as in Definition 2.3. Then if $\Phi(N, g, c) \not\equiv -\infty$ the induced function is a valuated matroid.*

While it is a special case of induction by networks, induction by bipartite graphs is an extremely powerful operation. Many of the operations introduced so far can be modelled using induction by bipartite graphs, which is a key observation in the proof of Theorem 2.2. For example, principal extension is equivalent to induction by the bipartite graph displayed in Figure 2. Furthermore, the following lemma shows we can realize induction by a network as induction by a bipartite graph followed by a contraction. Given the power of this operation, it shall be a key construction throughout.

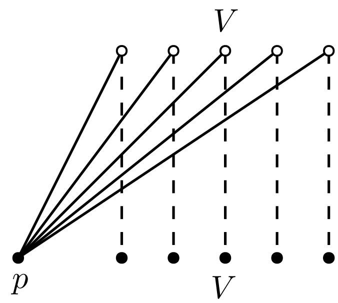

Figure 2: Given a valuated matroid f on V and $(w \in \mathbb{R} \cup \{-\infty\})^V$, the principal extension f^w is realized as the induction of f by the above bipartite graph. The dashed edges are weighted zero, while the solid edges (p, v) are weighted w_v.

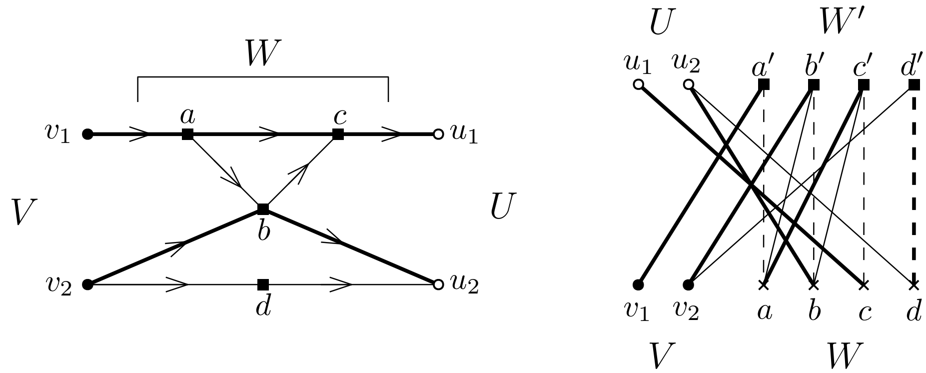

Figure 3: An example of the construction from Lemma 2.1: a network N and the corresponding bipartite graph G. A set of node-disjoint paths in N correspond to a matching in G, both displayed in bold.

Lemma 2.1. *Let N be a directed network with weight function d and g a valuated matroid such that $f = \Phi(N, g, d)$ is again a valuated matroid. Then there is a bipartite graph G with weight function c, a valuated matroid h and a subset of the nodes of G such that $f = (\Phi(G, h, c))/W$.*

The proof of this lemma is constructive. Let $N = (T, A)$ be the weighted directed network such that the valuated matroid f on the subset V of T is the induction of the valuated matroid g on the subset U of T through N. Let $W = T \setminus (V \cup U)$ and W' a disjoint copy of W. We define the bipartite graph $G = (V \cup W, U \cup W'; E)$ with weight function $c \in \mathbb{R}^E$ where for each arc $(a, b) \in A$, we add the edge (a, b) if $b \in U$ or (a, b') if $b \in W$ to E with weight $d(a, b)$. Furthermore, we add the zero weight edges (w, w') for all $w \in W$ with copy w'. An example of this construction is displayed in Figure 3.

We end this section by stating that valuated matroids are closed under all the operations introduced so far.

Theorem 2.2. *The class of valuated matroids is closed under the operations deletion, contraction, dualization, truncation, principal extension, direct sum, matroid union.*

2.3 Classes of valuated matroids In the following, we consider certain classes of valuated matroids that arise naturally in combinatorial optimization.

(i) The class of *transversally valuated matroids* are those valuated matroids arising from trivially valuated free matroids by induction by bipartite graphs.

(ii) The class of *valuated gammoids* are those valuated matroids arising as contractions of transversally valuated matroids.

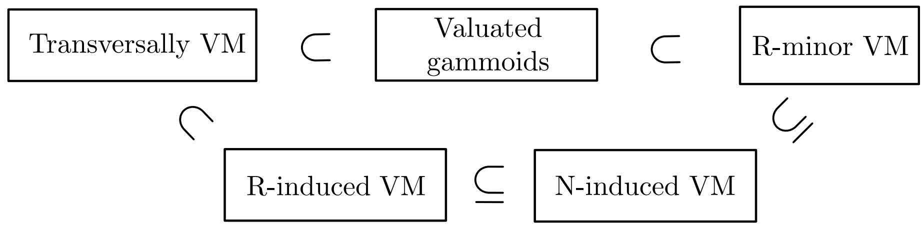

Figure 4: The inclusion relationship between classes of valuated matroids.

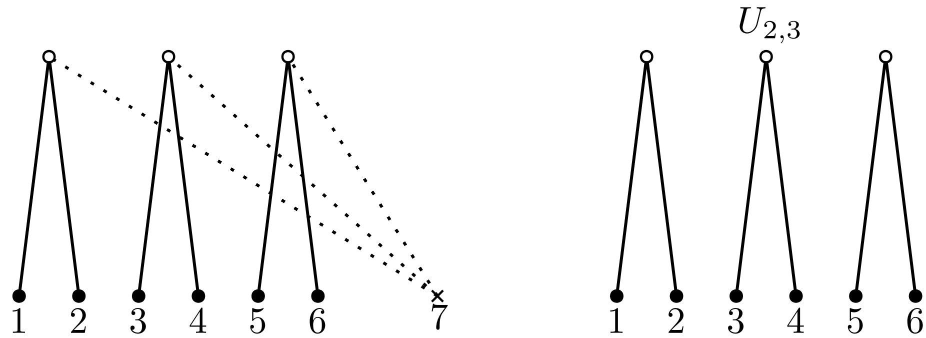

Figure 5: Two representations of the Snowflake. The left is a valuated gammoid representation, where the element 7 is contracted. The right is an R-induced representation with induced matroid $U_{2,3}$. All edges are weighted zero.

(iii) The class of *R-induced valuated matroids* are those valuated matroids arising from trivially valuated matroids by induction by bipartite graphs.

(iv) The class of *N-induced valuated matroids* are those valuated matroids arising from trivially valuated matroids by induction by networks.

(v) The class of *R-minor valuated matroids* are those valuated matroids arising as contractions of R-induced valuated matroids.

Transversally valuated matroids are essentially the layers of assignment valuations. They were extensively studied in [16], which also considered the class of valuated strict gammoids, a subclass of valuated gammoids, from the perspective of tropical geometry.

The inclusion relationship between these classes is shown in Figure 4, and can be deduced as follows. The inclusion of R-induced within N-induced and R-minor are immediate from definition. Furthermore, Lemma 2.1 shows how to represent an N-induced valuated matroid as an R-minor valuated matroid. The inclusion of valuated gammoids within R-minor is also immediate from definition, as valuated gammoids are a special case of the contractions of R-induced. Strict containment can be obtained by considering *strictly base-orderable* matroids: valuated gammoids are strictly base-orderable by [41, Lemma 1], while any trivially valuated matroid that is not strictly base-orderable is an R-induced valuated matroid. Finally, strict containment of transversally valuated matroids within valuated gammoids and R-induced is given by the "Snowflake". This is the valuated matroid on six elements of rank two that takes the value $-\infty$ on $\{12, 34, 56\}$, and 0 on all other pairs of elements. It was shown in [16, Example 3.10] to not be a transversally valuated matroid, however it is both a valuated gammoid and an R-induced valuated matroid, as given by the representations in Figure 5.

As we show in the following lemma, R-induced valuated matroids have a polynomial size representation.

Lemma 2.2. *Let $f : \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ be an R-induced valuated matroid with representation $G = (V, U; E)$, $\mathcal{M} = (U, r)$ and $c \in \mathbb{R}^E$. Then, there is an R-induced representation of f with $G' = (V, U'; E')$, $\mathcal{M}' = (U', r')$ and $c' \in \mathbb{R}^{E'}$ such that $|\Gamma_{G'}(v)| \leq d$ for all $v \in V$. In particular, $|E'| + |U'| + |V| \in O(|V| \cdot d)$.*

This lemma gives a short, information theoretic proof that not all valuated matroids are R-induced. We can consider a valuated matroid $h \in \mathcal{F}_n$ where $A = \{h(X) : X \in \mathcal{H}\}$ are linearly independent over $\mathbb{Q}$: in particular,

$|A|$ grows quadratically with n while the set of edge weights in an R-induced representation grows linearly. For sufficiently large n, there are more elements of A than the edge weights can realise, and so h is not R-induced.

Proposition 2.1. *There exist valuated matroids that are not R-induced.*

We note that the information-theoretic argument given does not extend to N-induced and R-minor valuated matroids as it cannot control the size of the contracted set. However, we conjecture that N-induced valuated matroids should also have a polynomial size representation. This suggests that several of the inclusions in Figure 4 should indeed be strict.

2.4 Complete Classes

Definition 2.4. (Complete class) *Let $\mathcal{V}$ be a subset of the set of valuated matroids. We call $\mathcal{V}$ a* complete class *if it is closed under taking restriction, duals, direct sum and principal extension.*

The following is an extension of results in [9] from unvaluated to valuated matroids. The closedness under the listed operations can be shown by explicit constructions.

Theorem 2.3. *A complete class of valuated matroids is closed under taking contraction, truncation, induction by bipartite graphs, induction by directed graph and valuated union.*

Furthermore, valuated gammoids forms the smallest complete class. Hence, they are contained in all complete classes.

To obtain the latter statement, observe that a non-empty complete class must contain the free matroid on one element. By taking iterated direct sum, this yields all free matroids. Then closure under induction by bipartite graphs and minors yields valuated gammoids.

The classes of valuated matroids discussed in the beginning of this section arising from induction through a network may only be induced by trivially valuated matroids. As discussed in Example 2.1, a trivially valuated matroid g can be identified with its underlying matroid $\mathcal{M}$, where $g(X)$ takes the value zero on bases of $\mathcal{M}$ and $-\infty$ otherwise. Working with this underlying matroid shall be more convenient much of the time, therefore we extend the notation of Definition 2.3 to define $\Phi(N, \mathcal{M}, c) := \Phi(N, g, c)$.

Let f be an R-minor valuated matroid on V. By definition, there exists an R-induced valuated matroid $\tilde{f}$ on $V \cup W$ such that $f = \tilde{f}/W$. By definition, there exists some bipartite graph $G = (V \cup W, U; E)$ with edge weights $c \in \mathbb{R}^E$ and matroid $\mathcal{M} = (U, r)$ such that $\tilde{f} = \Phi(G, \mathcal{M}, c)$; we say $\tilde{f}$ has an *R-induced representation* $(G, \mathcal{M}, c)$. As $f = \Phi(G, \mathcal{M}, c)/W$, we extend this notation to say that f has an *R-minor representation* $(G, \mathcal{M}, c, W)$, where W is the set to be contracted.

The following theorem shows that R-minor valuated matroids are closed under deletion, principal extension, duality and direct sum, making them a complete class. Furthermore, they are the smallest complete class to contain all trivially valuated matroids.

Theorem 2.4. *The set of R-minor valuated matroids forms a complete class of valuated matroids.*

The proof shows that R-minor valuated matroids are closed under deletion, direct sum, duals and principal extension by manipulation of R-minor representations. For an R-minor matroid f with representation $(G, \mathcal{M}, c, W)$ and a subset $X \subseteq V$, the deletion $f \setminus X$ is represented by $(G \setminus X, \mathcal{M}, c, W)$ where $G \setminus X$ is the graph obtained from G by deleting the nodes X and all edges adjacent. For two R-minor valuated matroids f_1 and f_2 represented by $(G_1, \mathcal{M}_1, c_1, W_1)$ and $(G_2, \mathcal{M}_2, c_2, W_2)$, the direct sum $f_1 \oplus f_2$ is represented by $(G', \mathcal{M}_1 \oplus \mathcal{M}_2, c', W_1 \cup W_2)$, where G' and its weight function c' arises by taking the union of the weighted graphs G_1 and G_2. The proof for duals and principal extension are slightly more subtle, but resemble the usual techniques for (fundamental) transversal matroids.

3 R-minor functions do not cover valuated matroids

We now give an overview of the proof of Theorem 1.1 showing that functions in $\mathcal{F}_n$ (Definition 1.2) are not R-minor. Recall that the domain of each of these functions contains $\mathcal{B}_0 := \binom{V}{4} \setminus \mathcal{H}$. We reduce the study of the family to the combinatorial and matroid structure of $\mathcal{B}_0$ and the domain of the function. To achieve this, we use a canonical linear programming formulation and the submodularity of the rank of the neighbourhood function arising in the bipartite graph of an R-minor representation. Finally, we impose several extremality assumptions on a potential representation which we exploit by applying local modifications. Now, we elaborate on these steps.

3.1 Rado representations of matroids The key combinatorial tool is to specialize R-induced and R-minor representations to matroids without valuation.

Definition 3.1. (Rado-minor representation) *Let $G = (V \cup W, U; E)$ be a bipartite graph and $\mathcal{M} = (U, r_{\mathcal{M}})$ be a matroid. We define a matroid $\mathcal{N}$ on V as follows. A set $X \subseteq V$ is independent in $\mathcal{N}$ if there exists $S \subseteq U$ such that there is a perfect matching in the subgraph induced by $(X \cup W, S)$ and S is independent in $\mathcal{M}$. We say that $(G, \mathcal{M}, W)$ is* Rado-minor representation *of $\mathcal{N}$. If $W = \emptyset$, we say that $(G, \mathcal{M})$ is a* Rado representation *of $\mathcal{N}$.*

The following generalization of Rado's Theorem [44, 42] verifies that this construction indeed defines a matroid, and characterizes its rank function.

Proposition 3.1. *Let $\mathcal{N}$ be as in Definition 3.1. Then $\mathcal{N}$ is a matroid. Moreover, $X \subset V$ is independent in $\mathcal{N}$ if and only if for all $Z \subseteq X \cup W$ it holds $r_{\mathcal{M}}(\Gamma(Z)) \geq |Z|$.*

Working with Rado representations allows us to forget edge weights and work solely with the underlying matroids. One structural property we will be particularly interested in is when these matroids are *fully reducible*.

Definition 3.2. (Matroid union, Fully reducible) *For $i \in [k]$, let $\mathcal{B}_i$ be the bases of matroid $\mathcal{M}_i$ on U. We define the* matroid union *$\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$ as a matroid $\mathcal{M}$ on U with bases $\mathcal{B} = \{\cup_{i=1}^k B_i : B_i \in \mathcal{B}_k\}$.*

We say that a matroid $\mathcal{M}$ is reducible *if and only if it is a* matroid union *of at least two non-empty matroids. Further, $\mathcal{M}$ is* fully reducible*, if $\mathcal{M} = \mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$ for non-empty matroids $\mathcal{M}_1, \ldots, \mathcal{M}_k$, $k \geq 2$, and $r(\mathcal{M}) = \sum_{i=1}^k r(\mathcal{M}_i)$. Given the latter condition, this is a* full-rank matroid union *[13].*

3.2 Linear Programming representation of R-minor function The first main ingredient is the linear programming dual of the R-minor representation. Let h be a function of rank d on $V = [2n]$ represented by a bipartite graph $G = (V \cup W, U; E)$, matroid $\mathcal{M} = (U, r)$, and weights $c \in \mathbb{R}^E$, such that $r(\mathcal{M}) = |W| + d$. (In our construction, $d = 4$.) The maximum weight independent matching problem of size $|W| + d$ can be formulated as a linear program.

$$
\begin{aligned}
\max \quad & \textstyle\sum_{(i,j)\in E} c_{ij} x_{ij} & \\
\text{s.t.:} \quad & \textstyle\sum_{j \in U} x_{ij} \leq 1 & \forall i \in V \\
& \textstyle\sum_{j \in U} x_{ij} = 1 & \forall i \in W \\
& \textstyle\sum_{i \in V \cup W, j \in S} x_{ij} \leq r(S) & \forall S \subset U \\
& \textstyle\sum_{i \in V \cup W, j \in U} x_{ij} = r(U) & \\
& x_{ij} \geq 0 & \forall i \in V \cup W, \forall j \in U\,.
\end{aligned}
\tag{3.2}
$$

The dual program can be equivalently written in the following convex form, using the Lovász-extension $\hat{r} : \mathbb{R}^U \to \mathbb{R}$, i.e., $\hat{r}(\tau)$ is the maximum τ-weight of any basis. The variables of the program are $\pi \in \mathbb{R}^{V \cup W}$ and $\tau \in \mathbb{R}^U$.

$$
\begin{aligned}
\min \quad & \pi(V \cup W) + \hat{r}(\tau) & \\
\text{s.t.:} \quad & \pi_i + \tau_j \geq c_{ij} & \forall (i,j) \in E \\
& \pi_i \geq 0 & \forall i \in V \\
& \pi_i - \text{free} & \forall i \in W \\
& \tau - \text{free}. &
\end{aligned}
\tag{3.3}
$$

Note for $h \in \mathcal{F}_n$, the maximum is 0 and the set of maximizers equals $\mathcal{B}_0$. For an optimal dual solution (π, τ) let $E_0 \subseteq E$ denote the tight edges ($\pi_i + \tau_j = c_{ij}$), $G_0 = (V \cup W, U; E_0)$ the tight subgraph, and $\mathcal{M}_\tau$ the matroid formed by the maximum τ-weight bases. We also let $E^* \subseteq E$ denote the union of all maximum weight independent matchings. By complementary slackness, $E^* \subseteq E_0$ for any dual optimal solution.

A key proof strategy is to work with the purely combinatorial structure of Rado-minor representations of two matroids: the one with bases $\mathcal{B}_0$ and the larger one with bases $\mathcal{B}_1$, where $\mathcal{B}_1 := \mathrm{dom}(h)$ is the *effective domain*, i.e., where $h(X) > -\infty$. The following lemma allows us to obtain Rado-minor representations for them from the R-minor representation and its LP.

Lemma 3.1. *$\mathcal{B}_0$ and $\mathcal{B}_1$ are matroids with Rado-minor representations $(G_0, \mathcal{M}_\tau, W)$ and $(G, \mathcal{M}, W)$ respectively. Furthermore, if $\mathcal{M} \neq \mathcal{M}_\tau$ then $\mathcal{B}_0$ is fully reducible.*

3.3 A guide through the core proof We fix $n \geq 16$, and prove by contradiction that no function in $h \in \mathcal{F}_n$ can be represented. We carefully select a counterexample that satisfies certain minimality criteria. Most importantly, we require that *(a)* $\mathcal{B}_1$ is minimal; subject to this, that *(b)* the contracted set $|W|$ is minimal, and finally, that *(c)* $|E \setminus E^*|$ is minimal. From these, we can easily deduce that one of two main cases holds:

(CI) There exist a dual optimal solution (π, τ) such that $E = E_0 \cup \{(i', j')\}$ for an edge (i', j'), $\mathcal{M}_\tau = \mathcal{M}$, and $\mathcal{B}_1 = \mathcal{B}_0 \cup \{X^*\}$.

(CII) $E = E^*$ and $\mathcal{M}_\tau \neq \mathcal{M}$ for any dual optimal solution (π, τ).

Thus, in case (CI), all bases in $\mathcal{M}$ have the same τ-weight, and there is a single non-tight edge. Further, $h(X^*)$ is the only finite value outside $\mathcal{B}_0$. In contrast, in case (CII), all edges are tight, but we need to work with two different matroids on U.

We now explain the proof for the base case $W = \emptyset$, i.e., that h is not R-induced. We show that case (CI) must apply. Otherwise, $\mathcal{M}_\tau \neq \mathcal{M}$ and so $\mathcal{B}_0$ is fully reducible by Lemma 3.1. By exploiting the combinatorics of the pairs P_i, we show that this is not the case for $\mathcal{B}_0$.

To complete the proof of the base case $W = \emptyset$, we note that the set $X^* = P_1 \cup P_2$ does not have an independent matching in E_0 but has one in $E_1 = E_0 \cup \{(i', j')\}$. Hence, (i', j') is incident to X^*; say, $i' \in P_1$. With an uncrossing argument using the submodularity of the rank of the neighbourhood function, we show that (i', j') should create an independent matching also for another set $X = P_1 \cup P_k \notin \mathcal{B}_0$. Since $\mathcal{M} = \mathcal{M}_\tau$ and this is the single non-tight edge, it follows that $0 > h(Z) \geq h(X^*)$, a contradiction that $h(X^*)$ is the unique largest negative function value.

To prove the general case where $W \neq \emptyset$, we first analyze Rado representations of *robust* matroids: a common generalization of $\mathcal{B}_0$ and $\mathcal{B}_0 \cup \{X^*\}$, sparse paving matroids with elements arranged in pairs P_i. It turns out that the structure of the pairs P_i forces itself on the full representation; in particular, for each pair P_i there exists a unique largest 'extension set' $Z_i \subseteq V \cup W$ such that $Z_i \cap V = P_i$, and these are tight with respect to Rado's condition. Moreover, the Z_i's are pairwise disjoint, and encode all relevant information of the robust matroid, $\mathcal{B}_0$ or $\mathcal{B}_1$. The structural analysis is based on careful uncrossing arguments of the rank of the neighbourhood function in the Rado representation.

Finally, we apply this structure to first show that $\mathcal{B}_1 = \mathcal{B}_0 \cup \{X^*\}$ is as small as it can be. We also show that the sets Z_i^0 and Z_i^1, obtained for each pair P_i from the robust matroid analysis for $\mathcal{B}_0$ and $\mathcal{B}_1$, are closely related: $Z_i^0 = Z_i^1 \cup Q_0$ for a certain set Q_0. Both cases (CI) and (CII) can be derived by exposing the discrepancy between two near-identical representations of two near-identical (yet different) matroids.

4 Valuated generalized matroids

In this section, we build on Theorem 1.1 to refute the matroid based valuation conjecture. To do this, we extend the class of R-minor valuated matroids to R$^\natural$-minor valuated generalized matroids, and show this contains matroid based valuations as a subclass. Furthermore, we extend our main counterexample to a valuated generalized matroid that is not R$^\natural$-minor and therefore not a matroid based valuation, refuting the MBV conjecture.

Recall from (1.1a) and (1.1b) the properties of valuated generalized matroids. An important class are the trivially valuated generalized matroids, those taking only values 0 and $-\infty$. This includes the characteristic functions of the family of independent sets of a matroid. Indeed, if $g(\emptyset) > -\infty$ for a valuated generalized matroid, then $\mathrm{dom}(g)$ is the family of independent sets of a matroid [39, Corollary 1.4].

For a valuated generalized matroid g on V and $k \leq |V|$, let $\ell^k(g)$ denote the restriction of g to $\binom{V}{k}$; recall that this is a valuated matroid as it satisfies (1.1b). Recall the constructions defined on valuated matroids in Definition 2.1. It turns out that all these operations extend essentially layer-wise to valuated generalized matroids.

Definition 4.1. *Let $N = (T, A)$ be a directed network with a weight function $c \in \mathbb{R}^A$. Let $V, U \subseteq T$ be two non-empty subsets of nodes of N. Let g be a valuated generalized matroid on U. Then the* induction of g by N *is the function $\Phi(N, g, c) \colon 2^V \to \mathbb{R} \cup \{-\infty\}$ such that*

$$\ell^k(\Phi(N, g, c)) = \Phi(N, \ell^k(g), c),$$

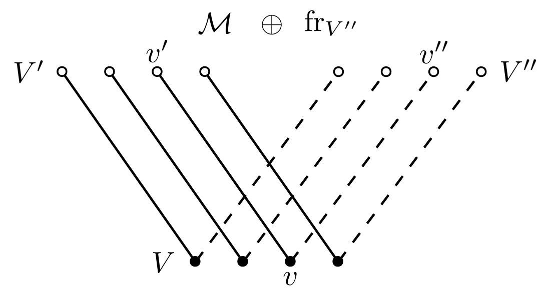

Figure 6: The graph $G = (V, V' \cup V''; E)$ realising the weighted matroid rank function from Example 4. Edges of weight w_v are solid while edges of weight zero are dashed.

where $\Phi(N, g, c)(\emptyset) = g(\emptyset)$.

In the special case that the directed network is bipartite with the edges directed from V to U, we can also consider this as an undirected weighted bipartite graph and call the corresponding operation induction by bipartite graphs.

Analogous to Theorem 2.1 this is just a special case of transformation by networks.

Theorem 4.1. (Special case of [35, Theorem 9.27]) *Let N, g, c as in Definition 4.1. Then if $\Phi(N, g, c) \not\equiv -\infty$ the induced function is a valuated generalized matroid.*

As with induction of valuated matroids, we are interested in the induction of trivially valuated generalized matroids. A trivially valuated generalized matroid g can be identified with its underlying domain $\mathcal{I}$, where $g(I) = 0$ if $I \in \mathcal{I}$ and $-\infty$ otherwise. As stated previously, if $\emptyset \in \mathcal{I}$ then $\mathcal{I}$ forms the set of independent sets of a matroid; however this does not have to be the case, $\mathcal{I}$ only has to satisfy the independent set exchange axiom (the unvaluated equivalent of (1.1a)). We call such an $\mathcal{I}$ a *generalized matroid*. As working with $\mathcal{I}$ directly will be convenient in some situations, we extend the notation of Definition 4.1 to define $\Phi(N, \mathcal{I}, c) := \Phi(N, g, c)$.

Example. Let $\mathcal{I}$ be the independent sets of a matroid $\mathcal{M}$ on ground set V. A *weighted rank function* $r^w : 2^V \to \mathbb{R}_{\geq 0}$ with weight $w \in \mathbb{R}^n_{\geq 0}$ is

$$r^w(X) = \max\left\{ \sum_{i \in I} w_i \;\middle|\; I \subseteq X \,,\, I \in \mathcal{I} \right\} \;.$$

Note that if w is the vector of all ones, then r^w is precisely the rank function of $\mathcal{M}$.

Let V' and V'' be copies of V and let $\overline{\mathcal{I}}$ be the independent sets of the matroid $\overline{\mathcal{M}} = \mathcal{M} \oplus \mathrm{fr}_{V''}$ on $V' \cup V''$, where $\mathrm{fr}_{V''}$ is the free matroid on V''. Furthermore, we define the bipartite graph $G = (V, V' \cup V''; E)$ where E consists of the edges (v, v') and (v, v'') connected each node in V its copies in V' and V''. We attach weights $c \in \mathbb{R}^E$ where the edge (v, v') gets the weight w_v and the edge (v, v'') gets the weight 0.

Let $I \subseteq X$ be the max weight independent set contained in X. The value of $\Phi(G, \overline{\mathcal{I}}, c)(X)$ is obtained by connecting elements of I to $I' \subseteq V'$ via edges of weight w_i, and then connecting elements of $X \setminus I$ to their copy in V'' by edges of weight zero. In this way $r^w = \Phi(G, \overline{\mathcal{I}}, c)$ arises from a trivially valuated generalized matroid by induction by bipartite graphs. □

It was shown in [7] that valuated generalized matroids are not covered by the cone of matroid rank functions; note that not even all non-negative combinations of matroid rank functions are valuated generalized matroids. In particular, not every valuated generalized matroid can be represented as a weighted matroid rank function [46, Theorem 4]. However, it was conjectured that allowing two operations, merge and endowment, would suffice to construct all.

Definition 4.2. *Let $f, g \colon 2^V \to \mathbb{R} \cup \{-\infty\}$. The* merge *of f and g is the function $f * g \colon 2^V \to \mathbb{R} \cup \{-\infty\}$ defined as*

$$(f * g)(X) = \max\{\, f(Y) + g(X \setminus Y) \mid Y \subseteq X \}, \quad \forall X \subseteq V.$$

The endowment of f by $T \subseteq V$ is the function $\Delta_T(f)\colon 2^{V\setminus T} \to \mathbb{R} \cup \{-\infty\}$ defined as

$$\Delta_T(f)(X) = f(X \cup T) - f(T), \quad \forall X \subseteq V \setminus T.$$

With this, the class of *matroid based valuations* are those functions arising from the class of weighted matroid rank functions by arbitrary application of merge and endowment.

Conjecture 4.1. (MBV conjecture [41]) *The class of matroid based valuations is equal to the class of monotone valuated generalized matroids taking value zero on the empty set and not attaining the value $-\infty$.*

We study a subclass of valuated generalized matroids which is an extension of the class of R-minor valuated matroids. This allows us to use similar results to those in the proof of Theorem 1.1.

Definition 4.3. *The class of* $\mathrm{R}^\natural$-induced functions *are valuated generalized matroids arising from trivially valuated generalized matroids via induction by bipartite graphs.*

The class of $\mathrm{R}^\natural$-minor functions *are valuated generalized matroids arising from contractions of $R^\natural$-induced functions.*

$\mathrm{R}^\natural$-minor functions are closed under merge and endowment. Combining this with Example 4, we see that matroid based valuations are a subclass of $\mathrm{R}^\natural$-minor functions.

Corollary 4.1. *Matroid based valuations form a subclass of $R^\natural$-minor functions with the properties that they are monotone, real-valued and have value* 0 *on the empty set.*

In the following, we construct a real-valued, monotone valuated generalized matroid that has no $\mathrm{R}^\natural$-minor representation. Let h be an arbitrary function in the class $\mathcal{F}_n$ which takes only values in $(-1, 0]$. We define a function $h^\natural\colon 2^V \to \mathbb{R}$ by

$$h^\natural(X) = \begin{cases} |X| & \text{for } |X| \le 3 \\ 4 + h(X) & \text{for } |X| = 4 \\ 4 & \text{for } |X| \ge 5 \end{cases}$$

Note that $h^\natural$ is a perturbed rank function of the uniform matroid on V of rank 4.

The way $h^\natural$ is constructed in such a way that if it has an $\mathrm{R}^\natural$-minor representation, its restriction to cardinality four subsets gives rise to an R-minor representation of h. For sufficiently large n, Theorem 1.1 states that this is not possible. It is monotone and real-valued however, and therefore a counterexample to the MBV conjecture.

Theorem 4.2. *For $n \ge 2$, the function $h^\natural$ is a valuated generalized matroid. For $n \ge 16$, the function $h^\natural$ is not an $R^\natural$-minor function. In particular, Conjecture 4.1 is false.*

5 Lorentzian polynomials

In this section, we recall basic concepts of Lorentzian polynomials and their connection to valuated matroids, and more generally M-concave functions, via tropicalization. We strengthen this connection by reframing operations on Lorentzian polynomials as natural operations on valuated matroids. In particular, we show the action of the semigroup of non-negative matrices on Lorentzian polynomials is equivalent to induction by networks for valuated matroids. As an application of our main counterexample we demonstrate the limitation of this operation, showing it does not generate the space of Lorentzian polynomials from generating functions of matroids.

5.1 Background We recall the basic properties of M-concave functions; see [35] for further details. A function $f\colon \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is *M-concave* if and only if

$$\begin{aligned} &\forall x, y \in \mathbb{Z}^n \text{ and all } i \in \operatorname{supp}^+(x-y) : \\ &f(x) + f(y) \le \max_{j \in \operatorname{supp}^-(x-y)} \{f(x - e_i + e_j) + f(y + e_i - e_j)\}, \end{aligned} \tag{5.4}$$

where $\operatorname{supp}^+(z) = \{i \in [n]\colon z_i > 0\}$ and $\operatorname{supp}^-(z) = \{i \in V\colon z_i < 0\}$ for $z \in \mathbb{Z}^n$, and e_ℓ is the ℓ-th unit vector. This extends (1.1b) from points in $\{0,1\}^n$ to $\mathbb{Z}^n$. An M-concave function has $\sum_{i=1}^n z_i = d$ for some fixed $d \in \mathbb{Z}$

for all $z \in \mathrm{dom}(f)$; we call d the *rank* of f. Observe that an M-concave function with $\mathrm{dom}(f) \subseteq \{0,1\}^n$ is a valuated matroid.

A set $B \subset \mathbb{Z}^n$ is *M-convex* if its characteristic function, taking value 0 on elements of B and $-\infty$ otherwise, is an M-concave function.

Let $\mathbb{K}$ be an arbitrary ordered field. Furthermore, let Δ_n^d be the set of lattice points $\{x \in \mathbb{Z}_{\geq 0}^n \colon \sum_{i=1}^n x_i = d\}$. Given a multivariate polynomial $p(w) = \sum_{\alpha \in \Delta_n^d} c_\alpha w^\alpha \in \mathbb{K}[w_1, \ldots, w_n]$, its *support* $\mathrm{supp}(p)$ is the set $\{\alpha \in \Delta_n^d \colon c_\alpha \neq 0\}$.

Several characterizations of Lorentzian polynomials were given in [11]; we follow their exposition. Let $\mathrm{M}_n^d(\mathbb{K})$ denote the homogeneous polynomials over $\mathbb{K}$ of degree d on n variables with non-negative coefficients whose support is an M-convex set. The set of Lorentzian polynomials over $\mathbb{K}$ of degree d on n variables is denoted by $\mathrm{L}_n^d(\mathbb{K})$ and is defined recursively.

Definition 5.1. ([11, Definition 3.18]) $\mathrm{L}_n^0(\mathbb{K}) = M_n^0(\mathbb{K})$ *and* $\mathrm{L}_n^1(\mathbb{K}) = M_n^1(\mathbb{K})$,

$$\mathrm{L}_n^2(\mathbb{K}) = \{p \in M_n^2(\mathbb{K}) \colon \textit{Hessian of } p \textit{ has at most one eigenvalue in } \mathbb{K}_{>0}\}.$$

For $d \geq 3$

$$\mathrm{L}_n^d(\mathbb{K}) = \{p \in M_n^d(\mathbb{K}) \colon \partial^\alpha p \in \mathrm{L}_n^2(\mathbb{K}) \textit{ for all } \alpha \in \Delta_n^{d-2}\}\,,$$

where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ *denotes the composition of* α_i*-th partial derivative with respect to* w_i*.*

Brändén and Huh give several other characterizations of Lorentzian polynomials when $\mathbb{K} = \mathbb{R}$, see [11, Definitions 2.1 & 2.6]. These definitions require taking limits, while the Hessian of a polynomial can be defined independently of a limit process, hence we only need to require $\mathbb{K}$ to be ordered.

We also note that while this definition holds for arbitrary ordered fields, many key results concerning Lorentzian polynomials were only proved over the real numbers. We can extend these results to the larger class of *real closed fields* via Tarski's principle; this states that first-order sentences of ordered fields hold over a real closed field $\mathbb{K}$ if and only if they hold over $\mathbb{R}$. We therefore will restrict to working with real closed fields from now, and construct explicit fields from Section 5.2 onwards. For further model theoretic details, see [31, Section 3.3].

Lorentzian polynomials are closed under several basic operations, see [13, 11], among which we emphasize one here.

Proposition 5.1. *Let* $\mathbb{K}$ *be a real closed field, and let* $p \in \mathrm{L}_n^d(\mathbb{K})$ *and* $A \in \mathbb{K}_{\geq 0}^{n \times k}$*. Then the polynomial* $(A \curvearrowright p)(w) := p(Aw) \in \mathrm{L}_k^d(\mathbb{K})$ *where* $w = (w_1, \ldots, w_k)$ *arising from the* matrix semi-group action *is also Lorentzian.*

5.2 Tropicalization In this section, we focus on Lorentzian polynomials over $\mathbb{K} = \mathbb{R}\{\{t\}\}$, the field of *(generalized) Puiseux series*, see [32] for further details. The field $\mathbb{R}\{\{t\}\}$ consists of formal series of the form

$$c(t) = \sum_{k \in A} a_k t^k \ , \ a_k \in \mathbb{R}$$

where $A \subset \mathbb{R}$ has no accumulation point and a well defined maximal element. The leading term of a Puiseux series is the term with largest exponent. We say a Puiseux series is *positive* if its leading term has a positive coefficient, and denote the semiring of *non-negative* Puiseux series (with zero) by $\mathbb{R}\{\{t\}\}_{\geq 0}$. We can extend this to make $\mathbb{R}\{\{t\}\}$ an ordered field by defining $c > d$ if and only if $c - d$ is a positive Puiseux series. Crucially, $\mathbb{R}\{\{t\}\}$ is also real closed and therefore we can invoke Tarski's principle.

This ordered field is equipped with a non-archimedean valuation deg (an extension of the degree map) which maps all non-zero elements to their leading exponent and zero to $-\infty$. The valuation deg extends entry-wise to vectors and matrices. It is enough to think of Puiseux series as polynomials in t with arbitrary exponents and coefficients in $\mathbb{R}$.

Observation 1. *For* $x, y \in \mathbb{R}\{\{t\}\}_{\geq 0}$ *the map* deg *is a semiring homomorphism, this means* $\deg(x + y) = \max(\deg(x), \deg(y))$ *and* $\deg(x \cdot y) = \deg(x) + \deg(y)$*. Note that this does not hold for general Puiseux series, as the sum of a positive and negative series may cause the leading terms to cancel.*

Recall that by definition, Lorentzian polynomials have non-negative coefficients. As deg is a semiring homomorphism on these coefficients, this motivates the study of Lorentzian polynomials under the degree map.

Definition 5.2. *For a polynomial* $p(w) = \sum_{\alpha \in \Delta_n^d} c_\alpha(t) w^\alpha \in \mathbb{R}\{\{t\}\}[w]$*, its* tropicalization *is the function* $\operatorname{trop}(p) \colon \Delta_n^d \to \mathbb{R} \cup \{-\infty\}$ *with* $\operatorname{trop}(p)(\alpha) = \deg(c_\alpha(t))$.

Theorem 5.1. ([11, Theorem 3.20]) *For* $f \colon \Delta_n^d \to \mathbb{R} \cup \{-\infty\}$*, the following are equivalent:*

(i) the function f *is M-concave,*

(ii) there is a Lorentzian polynomial $p \in \mathbb{R}\{\{t\}\}[w_1, \ldots, w_n]$ *with* $\operatorname{trop}(p) = f$.

Remark 5.1. *Lorentzian polynomials are usually associated with M-convex functions, which are the negatives of M-concave functions. However, this is merely a matter of how we choose the tropicalization as highest or lowest term, or actually its negative. It translates to the choice of convention between* max *and* min *and one can easily switch between them via the relation* $\max(x, y) = \min(-x, -y)$.

We give a more general version of induction by bipartite graph than introduced in Definition 2.3, allowing for M-concave functions and more general subgraphs. Note this is still a special case of transformation by networks derived from [35, Theorem 9.27].

Proposition 5.2. *Let* $G = (V, U; E)$ *be a bipartite graph with weight function* $c \in \mathbb{R}^E$*. Let* g *be an M-concave function on* $\mathbb{Z}_{\geq 0}^U$ *of rank* d*. Then the* transformation of g by G *is the function* $\Psi(G, g, c) \colon \mathbb{Z}_{\geq 0}^V \to \mathbb{R} \cup \{-\infty\}$ *with*

$$\Psi(G, g, c)(x) = \max\left\{ \sum_{e \in \mu} c(e) + g(y) \;\middle|\; \mu \text{ subgraph of } G \text{ with } \delta_V(\mu) = x \text{ and } \delta_U(\mu) = y \right\},$$

where $\delta_V(\mu)$ *and* $\delta_U(\mu)$ *are the degree vectors of* μ *on* V *and* U*. Furthermore,* $\Psi(G, g, c)(x)$ *is an M-concave function.*

For consistency of notation with Lorentzian polynomials, we will use the node sets $V = [n]$ and $U = [k]$. Note that the multiplication of the argument vectors leads to products of linear forms in the polynomial. By expanding these products and using the translation from sums of products to the maximum of sums via tropicalization one gets the following.

Theorem 5.2. *Let* $q \in \mathrm{L}_n^d(\mathbb{R}\{\{t\}\})$ *and let* $A \in \mathbb{R}\{\{t\}\}_{\geq 0}^{n \times k}$*. Let* $G = ([n], [k]; E)$ *be the bipartite graph with weight function* $\deg(A) \in \mathbb{R}^E$ *that weights* (i, j) *by* $\deg(a_{ij})$*. Then* $\operatorname{trop}(A \curvearrowright q)$ *is the M-concave function* $\Psi(G, \operatorname{trop}(q), \deg(A))$ *arising from* $\operatorname{trop}(q)$ *by transformation via* G.

5.3 Limitations of basic constructions In this section, we will allow $\mathbb{K}$ to be both $\mathbb{R}$ and $\mathbb{R}\{\{t\}\}$ unless explicitly stated.

For an M-convex set $\mathcal{B} \subset \mathbb{Z}_{\geq 0}^n$, its *generating function* is the Lorentzian polynomial

$$q_{\mathcal{B}} = \sum_{\alpha \in \mathcal{B}} \frac{1}{\alpha!} w^\alpha \ , \ \alpha! = \prod_{i=1}^n \alpha_i! \, .$$

Of particular interest for us is when $\mathcal{B} \subseteq \{0, 1\}^n$ i.e., $\mathcal{B}$ forms the set of bases of a matroid. Let $\mathcal{G}_n^d \subset \mathrm{L}_n^d(\mathbb{K})$ be the set of all generating functions corresponding to rank d matroids on n elements. For each $k \in \mathbb{Z}_{\geq 0}$, the multiplicative semigroup $\mathbb{K}_{\geq 0}^{n \times k}$ acts on $\mathcal{G}_n^d$ by $A \curvearrowright q(w) = q(Av) \in \mathrm{L}_k^d(\mathbb{K})$ where $A \in \mathbb{R}_{\geq 0}^{n \times k}, q \in \mathcal{G}_n^d$. We denote the orbit of this action by $\mathbb{K}_{\geq 0}^{n \times k} \curvearrowright \mathcal{G}_n^d \subseteq \mathrm{L}_k^d(\mathbb{K})$.

Definition 5.3. *We say a Lorentzian polynomial is* matroid induced *if it is contained in the orbit* $\mathbb{K}_{\geq 0}^{n \times k} \curvearrowright \mathcal{G}_n^d$ *for some* $n \geq d$.

Our main theorem of this section is that the class of matroid induced Lorentzian polynomials is a strict subclass of Lorentzian polynomials, over both the reals and Puiseux series.

We achieve this in a few steps by combining the results obtained so far. From Theorem 5.2, we know that the semigroup action of multiplication with non-negative matrices translates to induction by networks via tropicalization. We construct a Lorentzian polynomial over $\mathbb{R}\{\{t\}\}$ from the valuated matroid on the ground set $[2k]$ defined in Definition 1.2 by taking some pre-image under the degree map. Now, Theorem 1.1 shows that this Lorentzian polynomial cannot arise from the semigroup action of multiplication with non-negative matrices.

Finally, we use the correspondence between first order sentences over the reals and over Puiseux series to tranfer our result to general real Lorentzian polynomials.

THEOREM 5.3. *For $k \geq 10$ and arbitrary $N \in \mathbb{N}$, we have*

$$\bigcup_{n \geq d}^{N} (\mathbb{K}_{\geq 0}^{n \times 2k} \curvearrowright \mathcal{G}_n^d) \subsetneq \mathrm{L}_{2k}^d(\mathbb{K}),$$

for both $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{R}\{\{t\}\}$.

In particular, not all Lorentzian polynomial are matroid induced.

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