# Monotone edge flips to an orientation of maximum edge-connectivity à la Nash-Williams 

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#### Abstract

We initiate the study of $k$-edge-connected orientations of undirected graphs through edge flips for $k \geq 2$. We prove that in every orientation of an undirected $2 k$-edge-connected graph, there exists a sequence of edges such that flipping their directions one by one does not decrease the edge-connectivity, and the final orientation is $k$-edge-connected. This yields an "edge-flip based" new proof of Nash-Williams' theorem: an undirected graph $G$ has a $k$-edge-connected orientation if and only if $G$ is $2 k$-edge-connected. As another consequence of the theorem, we prove that the edge-flip graph of $k$-edge-connected orientations of an undirected graph $G$ is connected if $G$ is $(2 k+2)$-edge-connected. This has been known to be true only when $k=1$.


## 1 Introduction

An orientation of undirected graphs has been a subject of thorough studies over several decades. For an undirected graph $G=(V, E)$ with possible multiple edges, an orientation of $G$ is a directed graph $D=(V, A)$ obtained from $G$ by choosing a directed edge $(u, v) \in A$ or $(v, u) \in A$ for each undirected edge $\{u, v\} \in E$.

An old result by Robbins [27] states that an undirected graph $G$ has a strongly connected orientation if and only if $G$ is 2 -edge-connected. Robbins' theorem was extended by Nash-Williams 26 as an undirected graph $G$ has a $k$-edge-connected orientation if and only if $G$ is $2 k$-edge-connected.

This paper is concerned with reorientation. A basic question asks to find a smallest set $F$ of edges in an orientation of a 2-edge-connected graph such that flipping the directions of all edges in $F$ yields a strongly connected orientation. By a theorem of Lucchesi and Younger [24], the problem can be solved in polynomial time. A higher edge-connectedness version has also been studied,

[^0]

Figure 1: An example for Theorem 1. The underlying undirected graph is 4-edge-connected. The number under each orientation shows its edge-connectivity. Fat red edges depict flipped edges. The list of orientations show that a sequence of edge flips leads to an orientation of edge-connectivity two while all the intermediate orientations have edge-connectivity one.
which asks to find a smallest set $F$ of edges in an orientation of a $2 k$-edge-connected graph such that flipping the directions of all edges in $F$ yields a $k$-edge-connected orientation. By submodular flow, the problem can be solved in polynomial time [6]. For a faster algorithm, see Iwata and Kobayashi 20 .

We now want to investigate the situation where flips are performed one by one sequentially, while the results in the literature studied flipping the edges of a set at once. We also want each of the intermediate orientations in the process to maintain at least as high edge-connectivity as the previous orientations in the process. This has practical importance since simultaneous edge flips can be difficult to implement or control in some real-world situations such as traffic management [17], and the reduction of edge-connectivity in intermediate orientations may cause the loss of network quality.

To make the discussion more precise, we define an edge flip (or a flip for short) of a directed edge $(u, v)$ as an operation that replaces $(u, v)$ by $(v, u)$, i.e., reverses the direction of $(u, v)$. For directed graphs $D$ and $D^{\prime}$, we denote $D \rightarrow D^{\prime}$ if $D^{\prime}$ is obtained from $D$ by a single edge flip.

Our main theorem is the following. Remind that the edge-connectivity of a directed graph $D=(V, A)$ is the minimum integer $\lambda$ such that every non-empty subset $X \subsetneq V$ has at least $\lambda$ edges leaving $X$, and is denoted by $\lambda(D)$.

Theorem 1. Let $k$ be a non-negative integer. Let $G=(V, E)$ be an undirected $2 k$-edge-connected graph and $D=(V, A)$ be an orientation of $G$. Then, there exist orientations $D_{1}, D_{2}, \ldots, D_{\ell}$ of $G$ such that $\ell \leq k|V|^{3}, D \rightarrow D_{1} \rightarrow D_{2} \rightarrow \cdots \rightarrow D_{\ell}$, and $\lambda(D) \leq \lambda\left(D_{1}\right) \leq \lambda\left(D_{2}\right) \leq \cdots \leq \lambda\left(D_{\ell}\right)=k$. Furthermore, such orientations $D_{1}, \ldots, D_{\ell}$ can be found in polynomial time.

Theorem 1 states that for any orientation of a $2 k$-edge-connected undirected graph $G$, there exists a sequence of edge flips such that the orientations of $G$ obtained by the successive edge flips have non-decreasing edge-connectivity and the resulting orientation is $k$-edge-connected. Figure 1 shows an example.

Theorem 1 has several implications. First, it provides another (algorithmic) proof of NashWilliams' theorem [26] by edge flips. It should be emphasized that the edge-connectivity of the orientation does not decrease during the transformation in Theorem [1, while Nash-Williams' theorem itself does not provide any guarantee for the edge-connectivity of intermediate orientations.

The second implication is concerned with the connectedness of the edge-flip graph of $k$-edgeconnected orientations. For an undirected graph $G=(V, E)$, we define the edge-flip graph $\mathcal{G}_{k}(G)$ as the vertex set is all the $k$-edge-connected orientations of $G$, and two orientations are joined by
an edge in the edge-flip graph if and only if one is obtained from the other by a single edge flip. Figure 2 is an example of the edge-flip graph of the strongly connected orientations.


Figure 2: The edge-flip graph $\mathcal{G}_{1}\left(K_{4}\right)$ of the strongly connected (i.e., 1-edge-connected) orientations of a four-vertex complete graph $K_{4}$.

Then, we consider the following two questions.
Global Reachability: Given a connected undirected graph $G$, is the edge-flip graph $\mathcal{G}_{k}(G)$ connected?

Local Reachability: Given a connected undirected graph $G$ and two $k$-edge-connected orientations $D_{1}, D_{2}$ of $G$, is there a path connecting $D_{1}$ and $D_{2}$ in the edge-flip graph $\mathcal{G}_{k}(G)$ ?

When $k=1$, the Global Reachability question is completely answered. Greene and Zaslavsky [15] proved by hyperplane arrangements that the edge-flip graph $\mathcal{G}_{1}(G)$ is connected if and only if $G$ is 3 -edge-connected. Fukuda, Prodon, and Sakuma 12 gave a graph-theoretic proof for the same fact.

Our second result is a partial answer to the Global Reachability question for $k \geq 2$. This is a higher-edge-connectedness analogue of the result by Greene and Zaslavsky 15 and Fukuda, Prodon, and Sakuma [12].

Theorem 2. Let $k \geq 1$. If $G$ is $(2 k+2)$-edge-connected, then the edge-flip graph $\mathcal{G}_{k}(G)$ is connected.
Theorem 2 is obtained as a corollary of Theorem 1, combined with a result by Frank [7]. The proof implies that the diameter of $\mathcal{G}_{k}(G)$ is $O\left(k|V|^{3}+|E|^{2}\right)$ when $G$ is $(2 k+2)$-edge-connected. Note that $\mathcal{G}_{k}(G)$ has no vertex if the edge-connectivity of $G$ is less than $2 k$.

We do not know if the $(2 k+2)$-edge-connectedness can be replaced with the $(2 k+1)$-edgeconnectedness when $k \geq 2$. However, we know that we cannot replace it with the $2 k$-edgeconnectedness. Indeed, there exists a $2 k$-edge-connected graph $G$ such that $\mathcal{G}_{k}(G)$ is disconnected even when $k=1$ (e.g. consider the clockwise orientation and the counterclockwise orientation of a 3 -cycle).

For the Local Reachability question, we have the following characterization when $k=1$.


Figure 3: This example shows that an analogous statement to Theorem 3 does not hold for $k=2$.

Theorem 3. Let $G=(V, E)$ be a 2-edge-connected graph and $D_{1}, D_{2}$ be strongly connected orientations of $G$. Then, there exists a path connecting $D_{1}$ and $D_{2}$ in the edge-flip graph $\mathcal{G}_{1}(G)$ if and only if there exists no 2-edge-cut $\left\{\{u, v\},\left\{u^{\prime}, v^{\prime}\right\}\right\}$ such that $(u, v),\left(v^{\prime}, u^{\prime}\right)$ are edges of $D_{1}$ and $(v, u),\left(u^{\prime}, v^{\prime}\right)$ are edges of $D_{2}$. Furthermore, a shortest path between two strongly connected orientations can be found in polynomial time if one exists.

We note that an analogous statement for higher edge-connectedness does not hold. See the example in Figure 3. In this example, $D_{1}$ and $D_{2}$ are 2-edge-connected orientations of a 4-edgeconnected graph $G$. For any 4-edge-cut in $G$, their direction in $D_{1}$ is the same as in $D_{2}$. On the other hand, one can see that there is no edge flip of $D_{1}$ that maintains 2-edge-connectedness, which implies that $\mathcal{G}_{k}(G)$ contains no path connecting $D_{1}$ and $D_{2}$.

## Related Work

Orientations of graphs have been a subject of intensive studies in the literature of graph theory and combinatorial optimization.

Robbins 27] shows that an undirected graph $G$ has a strongly connected orientation if and only if $G$ is 2-edge-connected. Robbins' original proof is based on ear decompositions, which yields a linear-time algorithm [18]. The Global Reachability of the edge-flip graph of strongly connected orientations is investigated by Greene and Zaslavsky 15 and by Fukuda et al. [12: They proved that the edge-flip graph of the strongly connected orientations of an undirected graph $G$ is connected if and only if $G$ is 3 -edge-connected, and in this case, a shortest path between two strongly connected orientations can be found in polynomial time.

Nash-Williams [26] shows that an undirected graph $G$ has a $k$-edge-connected orientation if and only if $G$ is $2 k$-edge-connected, where $k \geq 1$ is an integer. Robbins' theorem [27] corresponds to the case where $k=1$. Nash-Williams' original proof is based on the so-called "odd-node pairing theorem" and Eulerian orientations. See also [22] for a simpler proof. Since an odd-node pairing with the desired property can be found in polynomial time [14, 25], this proof technique yields a polynomial-time algorithm to find a $k$-edge-connected orientation of a $2 k$-edge-connected graph. Other proofs are based on the "splitting-off theorem" by Lovász [23] and submodular flows [8]. Those two proofs also yield polynomial-time algorithms.

Nash-Williams' theorem can be generalized to the existence of an orientation satisfying a certain local connectivity constraint, which is called a well-balanced orientation [26]. A further extension is shown by Fukunaga [13]. Bernáth et al. [3] show some results on well-balanced orientations.

When $k \geq 2$, the edge-flip graph of $k$-edge-connected orientations has not been studied. Frank 7 proved that the path/cycle-flip graph of $k$-edge-connected orientations of an undirected graph $G$ is connected if and only if $G$ is $2 k$-edge-connected, where a path/cycle flip is an operation that flips all the edges of a directed path or a directed cycle simultaneously. Since this result will be used in our proof of Theorem 2, we highlight it in the following theorem, where we also include a bound for the length of a sequence that was implicit in his proof.
Theorem 4 (Frank $[7]$ ). Let $k \geq 1$ be an integer, $G=(V, E)$ be a $2 k$-edge-connected undirected graph, and $D_{1}, D_{2}$ be two $k$-edge-connected orientations of $G$. Then, $D_{1}$ and $D_{2}$ can be transformed with each other by a sequence of path/cycle flips in such a way that all the intermediate orientations are $k$-edge-connected. The length of such a sequence is bounded by $O(|E|)$ from above, and can be found in polynomial time.

Acyclic orientations are well-studied objects. An orientation is acyclic if it has no directed cycle. It is easy to see that every undirected graph has an acyclic orientation. The Global Reachability question is completely answered. Greene and Zaslavsky 15 gave a geometric proof that the edge-flip graph of acyclic orientations is connected. Fukuda, Prodon, and Sakuma 12 gave a graph-theoretic proof for the same fact. This trivially answers the Local Reachability question, too. Indeed, their proofs give a shortest path between two acyclic orientations in the edge-flip graph, which can be found in polynomial time.

Degree-constrained orientations form another class of well-studied orientations. In this case, we are also given a non-negative number $m(v)$ for every vertex $v$ of an undirected graph $G=(V, E)$. Hakimi [16] proved that there exists an orientation of $G$ such that every vertex $v$ has the in-degree of $m(v)$ if and only if $|E|=\sum_{v \in V} m(v)$ and $|\{e \in E \mid e \subseteq X\}| \leq \sum_{v \in X} m(v)$ for all $X \subseteq V$ : such an orientation can be found in polynomial time if exists.

To define the flip graph of degree-constrained orientations, an edge flip is useless since a single edge flip does not keep the required degree property of the orientations. Instead, we consider a cycle flip that flips all the edges in a single directed cycle simultaneously. A cycle flip preserves the property that the in-degree of every vertex $v$ is $m(v)$. Therefore, the cycle-flip graph of degreeconstrained orientations has been studied in the literature.

The Global Reachability of the cycle-flip graph of degree-constrained orientations is known to hold as long as it is non-empty (see [10]). Thus, the Local Reachability question is again trivial. However, computing a shortest path in the cycle-flip graph of degree-constrained orientations is NP-hard [1,19].

Orientations with vertex-connectivity constraints are also studied in the literature. It is conjectured by Thomassen [28] that, for any positive integer $k$, there exists a smallest positive integer $f(k)$ such that every $f(k)$-connected graph has a $k$-connected orientation. Frank 9 proposed a stronger conjecture: for any positive integer $k$, a graph $G=(V, E)$ has a $k$-connected orientation if and only if $G-U$ is $2(k-|U|)$-edge-connected for any $U \subseteq V$ with $|U| \leq k$. Jordán [21] shows that $f(2) \leq 18$ based on a result by Berg and Jordán [2], and this upper bound is improved to 14 by Cheriyan, Durand de Gevigney, and Szigeti [4]. Thomassen [29] proves Frank's conjecture for $k=2$, that is, a graph $G=(V, E)$ admits a 2-connected orientation if and only if it is 4-edge-connected and $G-v$ is 2-edge-connected for every $v \in V$. This implies that $f(2)=4$. For general $k$, Frank's conjecture was disproved recently by Durand de Gevigney [5]. The existence of $f(k)$ is still open for $k \geq 3$.

Frank, Király, and Király [11] proved that many known graph orientation theorems can be extended to hypergraphs.

## Organization

The rest of this paper is organized as follows. In Section 2, we give preliminary definitions and useful facts. In Section 3, we prove Theorem 2 by using Theorem 1. A proof of Theorem 1, which is the main result in this paper, is given in Section 4. In Section 54, we give a proof of Theorem 3 . In Section 6, we conclude this paper with some remarks.

## 2 Preliminaries

### 2.1 Undirected Graphs

An undirected graph $G=(V, E)$ is a pair of its vertex set $V$ and its edge set $E$, where each edge $e \in E$ is specified by an unordered pair $\{u, v\}$ of vertices: in this case $u$ and $v$ are endpoints of the edge $e$. We allow multiple edges, and thus $E$ is considered a multiset.

A path in an undirected graph $G=(V, E)$ is a sequence $v_{0}, v_{1}, \ldots, v_{\ell}$ of distinct vertices such that $\left\{v_{i}, v_{i+1}\right\} \in E$ for every $i \in\{0,1, \ldots, \ell-1\}$ : in this case the path connects $v_{0}$ and $v_{\ell}$, and $\ell$ is the length of the path. A path that connects $v_{0}$ and $v_{\ell}$ is also called a $\left(v_{0}, v_{\ell}\right)$-path.

For an undirected graph $G=(V, E)$ and a vertex subset $S \subseteq V$, we denote by $E_{G}(S)$ the set of edges of $G$ that have one endpoint in $S$ and the other endpoint in $V-S$ :

$$
E_{G}(S):=\{e \in E| | e \cap S \mid=1\} .
$$

Define $\delta_{G}(S):=\left|E_{G}(S)\right|$. The set $E_{G}(S)$ is a $k$-edge-cut of $G$ if $\left|E_{G}(S)\right|=k$.
For $k \geq 1$, an undirected graph $G=(V, E)$ is $k$-edge-connected if $\delta_{G}(S) \geq k$ for every nonempty $S \subsetneq V$. By Menger's theorem, this is equivalent to the condition that there exists a set of $k$ edge-disjoint ( $s, t$ )-paths for every pair of distinct vertices $s, t \in V$. An undirected graph is connected if it is 1-edge-connected: an undirected graph is disconnected if it is not connected. It is easy to observe that $G$ is $k$-edge-connected if and only if there exists no $\ell$-edge-cut in $G$ for $\ell=0,1, \ldots, k-1$ except $E_{G}(\emptyset)$ and $E_{G}(V)$.

### 2.2 Directed Graphs

A directed graph $D=(V, A)$ is a pair of its vertex set $V$ and its edge set $A$, where each directed edge $e \in A$ is specified by an ordered pair $(u, v)$ of vertices: in this case $u$ is the tail of $e$ and $v$ is the head of $e$. We allow multiple edges, and thus $A$ is considered a multiset.

A path in a directed graph $D=(V, A)$ is a sequence $v_{0}, v_{1}, \ldots, v_{\ell}$ of distinct vertices such that $\left(v_{i}, v_{i+1}\right) \in A$ for every $i \in\{0,1, \ldots, \ell-1\}$ : in this case the path connects $v_{0}$ and $v_{\ell}$, and $\ell$ is the length of the path. A path that connects $v_{0}$ and $v_{\ell}$ is also called a $\left(v_{0}, v_{\ell}\right)$-path. For $i, j \in\{0,1, \ldots, \ell-1\}$ with $i \leq j$, let $P\left[v_{i}, v_{j}\right]$ denote the subpath of $P$ from $v_{i}$ to $v_{j}$, that is, $P\left[v_{i}, v_{j}\right]$ is the sequence $v_{i}, v_{i+1}, \ldots, v_{j}$. For a path $P$, the set of vertices (resp. edges) in $P$ is denoted by $V(P)$ (resp. $A(P)$ ).

Let $D=(V, A)$ be a directed graph and $S \subseteq V$ a vertex subset. The subgraph induced by $S$ is denoted by $D[S]$. An edge $e \in A$ leaves $S$ if the tail of $e$ belongs to $S$ but the head of $e$ does not belong to $S$. Similarly, an edge $e \in A$ enters $S$ if the head of $e$ belongs to $S$ but the tail of $e$ does not belong to $S$. We denote by $\Delta_{D}^{+}(S)$ the set of edges in $A$ that leave $S$, and similarly by $\Delta_{D}^{-}(S)$ the set of edges in $A$ that enter $S$ :

$$
\Delta_{D}^{+}(S):=\{(u, v) \in A \mid u \in S, v \notin S\}, \quad \Delta_{D}^{-}(S):=\{(u, v) \in A \mid u \notin S, v \in S\}
$$

Define $\delta_{D}^{+}(S):=\left|\Delta_{D}^{+}(S)\right|$ and $\delta_{D}^{-}(S):=\left|\Delta_{D}^{-}(S)\right|$.
For $k \geq 1$, a directed graph $D=(V, A)$ is $k$-edge-connected if $\delta_{D}^{+}(S) \geq k$ and $\delta_{D}^{-}(S) \geq k$ for every non-empty $S \subsetneq V$ By Menger's theorem, this is equivalent to the condition that there exists a set of $k$ edge-disjoint $(s, t)$-paths for every pair of distinct vertices $s, t \in V$. A directed graph is strongly connected if it is 1-edge-connected. The edge-connectivity of a directed graph $D$ is the maximum integer $k$ such that $D$ is $k$-edge-connected, and is denoted by $\lambda(D)$.

Simple counting shows that the functions $\delta_{D}^{+}, \delta_{D}^{-}$satisfy the following inequalities: for all $X, Y \subseteq$ V,

$$
\delta_{D}^{+}(X)+\delta_{D}^{+}(Y) \geq \delta_{D}^{+}(X \cap Y)+\delta_{D}^{+}(X \cup Y), \quad \delta_{D}^{-}(X)+\delta_{D}^{-}(Y) \geq \delta_{D}^{-}(X \cap Y)+\delta_{D}^{-}(X \cup Y)
$$

The two inequalities are referred to as submodularity.

## 3 Proof of Theorem 2

Let $k \geq 1$ and $G$ be an undirected $(2 k+2)$-edge-connected graph. We will show that the edge-flip graph $\mathcal{G}_{k}(G)$ is connected by using Theorem 1 .

Let $D_{1}, D_{2}$ be $k$-edge-connected orientations of $G$. We want to find a sequence of edge flips that transforms $D_{1}$ to $D_{2}$ in such a way that all the intermediate orientations are $k$-edge-connected.

Below is our strategy.

1. We apply Theorem 1 to transform $D_{1}$ to a $(k+1)$-edge-connected orientation $D_{1}^{\prime}$ by edge flips so that all the intermediate orientations are $k$-edge-connected. This can be done by the assumption that $G$ is $(2 k+2)$-edge-connected. We apply the same procedure to $D_{2}$ to obtain a $(k+1)$-edge-connected orientation $D_{2}^{\prime}$.
2. Then, we apply Theorem 4 due to Frank $[7]$ to transform $D_{1}^{\prime}$ to $D_{2}^{\prime}$. Since operations in Theorem 4 are path/cycle flips, we need to turn them into sequences of edge flips. We emphasize that all the intermediate orientations will be $k$-edge-connected, but not necessarily $(k+1)$-edge-connected.
3. Finally, we consider the reverse sequence of edge flips that transformed $D_{2}$ to $D_{2}^{\prime}$ from the first step. Combining them, we obtain a sequence of edge flips that transforms $D_{1}$ to $D_{2}$ such that all the intermediate orientations are $k$-edge-connected.

In the strategy above, the first and the third steps are clear, and the numbers of necessary steps are $O\left(k|V|^{3}\right)$ by Theorem 1 . We concentrate on the second step. Let $D_{1}^{\prime}, D_{2}^{\prime}$ be ( $k+1$ )-edgeconnected orientations of $G$. Then, by Theorem 4, there exists a sequence of path/cycle flips that transforms $D_{1}^{\prime}$ to $D_{2}^{\prime}$ in such a way that all the intermediate orientations are ( $k+1$ )-edge-connected. Let $D_{1}^{\prime}=\hat{D}_{0}, \hat{D}_{1}, \ldots, \hat{D}_{\ell}=D_{2}^{\prime}$ be a sequence of orientations of $G$ such that $\hat{D}_{i}$ is obtained from $\hat{D}_{i-1}$ by a path/cycle flip and $\hat{D}_{i}$ is $(k+1)$-edge-connected for $i \in\{1, \ldots, \ell\}$.

We now fix $i \in\{1, \ldots, \ell\}$, and we will construct a sequence of orientations from $\hat{D}_{i-1}$ to $\hat{D}_{i}$ by edge flips. Let $F$ be a directed path/cycle in $\hat{D}_{i-1}$ such that flipping the edges in $F$ yields $\hat{D}_{i}$. Suppose that $F$ traverses arcs $e_{1}, e_{2}, \ldots, e_{m}$ in this order, where $m$ is the number of edges in $F$. Then, we flip $e_{1}, e_{2}, \ldots, e_{m}$ one by one in this order. The obtained sequence of orientations is

[^1]denoted by $\hat{D}_{i-1}=\tilde{D}_{0}, \tilde{D}_{1}, \ldots, \tilde{D}_{m}=\hat{D}_{i}$. Note that $\tilde{D}_{j}$ is obtained from $\tilde{D}_{0}$ by flipping the edges of the path $\left\{e_{1}, \ldots, e_{j}\right\}$.

We prove $\tilde{D}_{j}$ is $k$-edge-connected for any $j \in\{1, \ldots, m-1\}$. Let $X$ be a non-empty proper subset of $V$. We already know $\tilde{D}_{0}$ is $(k+1)$-edge-connected, that is, $\delta_{\tilde{D}_{0}}^{+}(X) \geq k+1$ and $\delta_{\tilde{D}_{0}}^{\tilde{D}_{0}}(X) \geq k+1$. Furthermore, since $\left\{e_{1}, e_{2}, \ldots, e_{j}\right\}$ forms a path in $\tilde{D}_{0}$,

$$
\left|\left|\Delta_{\tilde{D}_{0}}^{+}(X) \cap\left\{e_{1}, e_{2}, \ldots, e_{j}\right\}\right|-\left|\Delta_{\tilde{D}_{0}}^{-}(X) \cap\left\{e_{1}, e_{2}, \ldots, e_{j}\right\}\right|\right| \leq 1 .
$$

This implies that $\delta_{\tilde{D}_{j}}^{+}(X) \geq k$ and $\delta_{\tilde{D}_{j}}^{-}(X) \geq k$. Hence, $\tilde{D}_{j}$ is $k$-edge-connected for any $j \in$ $\{0, \ldots, m\}$.

Therefore, we can construct a sequence of $k$-edge-connected orientations from $D_{1}^{\prime}$ to $D_{2}^{\prime}$ by edge flips. The length of the sequence for the second step is $O\left(|E|^{2}\right)$ by Theorem 4 . This completes the proof.

## 4 Proof of the Main Theorem (Theorem 1)

Remind that for two orientations $D$ and $D^{\prime}$ of $G$, we denote $D \rightarrow D^{\prime}$ if $D^{\prime}$ is obtained from $D$ by a single edge flip. In this section, we show that, given a $k$-edge-connected orientation $D$ of $G$, we can increase the edge-connectivity of $D$ via a sequence of edge flips without losing the $k$-edge-connectedness.

Theorem 5. Let $k$ be a non-negative integer. Let $G=(V, E)$ be an undirected $(2 k+2)$-edgeconnected graph and $D=(V, A)$ be a $k$-edge-connected orientation of $G$. Then, there exist orientations $D_{1}, D_{2}, \ldots, D_{\ell}$ of $G$ such that $\ell \leq|V|^{3}, D \rightarrow D_{1} \rightarrow D_{2} \rightarrow \cdots \rightarrow D_{\ell}, \lambda\left(D_{i}\right) \geq k$ for $i \in\{1, \ldots, \ell-1\}$, and $\lambda\left(D_{\ell}\right) \geq k+1$. Furthermore, such $D_{1}, \ldots, D_{\ell}$ can be found in polynomial time.

Note that the $(2 k+2)$-edge-connectedness is necessary for an undirected graph $G$ to have a ( $k+1$ )-edge-connected orientation.

Theorem 1 is then a simple corollary of Theorem 5 as exhibited in the proof below.
Proof of Theorem 1. If $\lambda(D)=k$, then the claim is obvious. Otherwise, let $p:=\lambda(D)<k$ and $D^{p}:=D$. Since $G$ is $(2 p+2)$-edge-connected, by applying Theorem 5 with $D^{p}$, we obtain orientations $D_{1}^{p}, D_{2}^{p}, \ldots, D_{\ell_{p}}^{p}$ of $G$ such that $\ell_{p} \leq|V|^{3}, D^{p} \rightarrow D_{1}^{p} \rightarrow D_{2}^{p} \rightarrow \cdots \rightarrow D_{\ell_{p}}^{p}, \lambda\left(D_{i}^{p}\right) \geq p$ for $i \in\left\{1, \ldots, \ell_{p}-1\right\}$, and $\lambda\left(D_{\ell_{p}}^{p}\right) \geq p+1$. By taking a subsequence if necessary, we may assume that $\lambda\left(D_{i}^{p}\right)=p$ for $i \in\left\{1, \ldots, \ell_{p}-1\right\}$. Note that, since $D_{\ell_{p}}^{p}$ is obtained from $D_{\ell_{p}-1}^{p}$ by flipping exactly one edge, we obtain $\lambda\left(D_{\ell_{p}}^{p}\right)-\lambda\left(D_{\ell_{p}-1}^{p}\right) \leq 1$, and hence $\lambda\left(D_{\ell_{p}}^{p}\right)=p+1$. Then, set $D^{p+1}:=D_{\ell_{p}}^{p}$ and apply Theorem 5 again with $D^{p+1}$ to obtain a sequence $D_{1}^{p+1}, D_{2}^{p+1}, \ldots, D_{\ell_{p+1}}^{p+1}=: D^{p+2}$. We repeat this procedure until the edge-connectivity becomes $k$. Then,

$$
D_{1}^{p}, D_{2}^{p}, \ldots, D_{\ell_{p}}^{p}, D_{1}^{p+1}, D_{2}^{p+1}, \ldots, D_{\ell_{p+1}}^{p+1}, \ldots, D_{1}^{k-1}, D_{2}^{k-1}, \ldots, D_{\ell_{k-1}}^{k-1}
$$

is a desired sequence, because $\lambda\left(D_{1}^{i}\right)=\cdots=\lambda\left(D_{\ell_{i}-1}^{i}\right)=i$ and $\lambda\left(D_{\ell_{i}}^{i}\right)=i+1$ for $i \in\{p, p+$ $1, \ldots, k-1\}$, and the length of the sequence is $\sum_{i=p}^{k-1} \ell_{i} \leq k|V|^{3}$. Such a sequence can be found in polynomial time by Theorem 5 .

In the rest of this section, we give a proof of Theorem 5 .

### 4.1 Proof Outline

We first consider the case of $k=0$. Since $G$ is 2-edge-connected, it has a strongly connected orientation $D^{\prime}=\left(V, A^{\prime}\right)$ (see $[27)$. Let $r \in V$ be an arbitrary vertex. Then, by considering the union of an in-tree an out-tree both rooted at $r$, one can see that there exists a subgraph $D^{\prime \prime}=\left(V, A^{\prime \prime}\right)$ of $D^{\prime}$ such that $D^{\prime \prime}$ is strongly connected and $\left|A^{\prime \prime}\right| \leq 2|V|-2$. Let $F \subseteq A^{\prime \prime}$ be the set of arcs whose directions are different in $D^{\prime \prime}$ and $D$. Then, by flipping edges in $F$ one by one in an arbitrary order, we obtain a sequence of orientations satisfying the conditions in Theorem 5 because we have no constraint on the intermediate orientations when $k=0$, and the length of this sequence is at most $2|V|-2$.

In what follows in this section, we suppose that $k$ is a positive integer. Let $G=(V, E)$ be an undirected $(2 k+2)$-edge-connected graph and $D=(V, A)$ be a $k$-edge-connected orientation of $G$. Throughout this section, we fix a vertex $r \in V$ arbitrarily. To simplify the notation, for $v \in V,\{v\}$ is sometimes denoted by $v$ if no confusion may arise. Define $\mathcal{F}_{\text {out }}(D)$ and $\mathcal{F}_{\text {in }}(D)$ as

$$
\begin{aligned}
\mathcal{F}_{\text {out }}(D) & :=\left\{X \subseteq V-r \mid \delta_{D}^{+}(X)=k\right\} \cup\{V\}, \\
\mathcal{F}_{\text {in }}(D) & :=\left\{X \subseteq V-r \mid \delta_{D}^{-}(X)=k\right\} \cup\{V\} .
\end{aligned}
$$

Throughout this paper, a set in $\mathcal{F}_{\text {out }}(D)$ (resp. $\mathcal{F}_{\text {in }}(D)$ ) is shown by a blue (resp. red) oval in the figures. Note that, for a vertex set $X \subseteq V$ with $r \in X, \delta_{D}^{+}(X)=k$ if and only if $V-X \in \mathcal{F}_{\text {in }}(D)$. With this observation, we see that $D$ is $(k+1)$-edge-connected if and only if $\mathcal{F}_{\text {out }}(D)=\mathcal{F}_{\text {in }}(D)=$ $\{V\}$. We also note that $\mathcal{F}_{\text {out }}(D) \cap \mathcal{F}_{\text {in }}(D)=\{V\}$, because $\delta_{D}^{+}(X)+\delta_{D}^{-}(X)=\delta_{G}(X) \geq 2 k+2$ for any non-empty subset $X \subseteq V-r$. Define $\mathcal{F}_{\min }(D)$ as the set of all inclusionwise minimal sets in $\mathcal{F}_{\text {out }}(D) \cup \mathcal{F}_{\text {in }}(D)$. As we will see in Corollary 10, $\mathcal{F}_{\min }(D)$ consists of disjoint sets. If $D$ is clear from the context, $\mathcal{F}_{\text {out }}(D), \mathcal{F}_{\text {in }}(D)$, and $\mathcal{F}_{\text {min }}(D)$ are simply denoted by $\mathcal{F}_{\text {out }}, \mathcal{F}_{\text {in }}$, and $\mathcal{F}_{\text {min }}$, respectively.

In our proof of Theorem 5. by flipping some edges in $D$, we decrease the value of

$$
\operatorname{val}(D):=\sum_{X \in \mathcal{F}_{\min }(D)}(|V|-|X|) .
$$

We repeat this procedure as long as $\operatorname{val}(D)$ is positive. If this value becomes 0 , then $\mathcal{F}_{\min }=\{V\}$. This means that $\mathcal{F}_{\text {out }}=\mathcal{F}_{\text {in }}=\{V\}$, and hence $D$ is $(k+1)$-edge-connected. Note that we decrease the value of $\operatorname{val}(D)$ at most $|V|^{2}$ times, because $\operatorname{val}(D)$ is integral and $\operatorname{val}(D) \leq|V|^{2}$. Therefore, to prove Theorem 5, it suffices to show the following proposition.

Proposition 6. Suppose that $\mathcal{F}_{\min } \neq\{V\}$. Then, there exist orientations $D_{1}, D_{2}, \ldots, D_{\ell}$ of $G$ such that $\ell \leq|V|, D \rightarrow D_{1} \rightarrow D_{2} \rightarrow \cdots \rightarrow D_{\ell}, \lambda\left(D_{i}\right) \geq k$ for $i \in\{1, \ldots, \ell\}$, and $\operatorname{val}\left(D_{\ell}\right)<\operatorname{val}(D)$. Furthermore, such $D_{1}, \ldots, D_{\ell}$ can be found in polynomial time.

In what follows in this section, we assume that $\mathcal{F}_{\text {min }} \neq\{V\}$ and give an algorithm for finding such orientations as in Proposition 6. In our algorithm, we find an inclusionwise minimal set $S$ in $\mathcal{F}_{\text {in }}$, an inclusionwise minimal set $T$ in $\mathcal{F}_{\text {out }}$, and a path $P$ from $S$ to $T$. Then, we flip the edges in $P$ one by one from one end to the other. In order to obtain orientations with the conditions in Proposition 6, we have to choose $P$ carefully. First, it is necessary that $P$ does not enter a set in $\mathcal{F}_{\text {in }}$ (or does not leave a set in $\mathcal{F}_{\text {out }}$ ), as otherwise flipping edges violates $k$-edge-connectivity. Moreover, to decrease $\operatorname{val}(D)$, we choose a path $P$ so that it is from a safe source in $S$ to a safe sink in $T$ (see Section 4.3 for definitions).

We note that $\operatorname{val}(D)$, safe sources, and safe sinks are first introduced in this paper, and they are key ingredients in our arguments.

After describing basic properties of $\mathcal{F}_{\text {out }}, \mathcal{F}_{\text {in }}$, and $\mathcal{F}_{\text {min }}$ in Section 4.2, we introduce safe sources and safe sinks in Section 4.3. Then, we describe our algorithm in Section 4.4, and prove its validity in Section 4.5. A proof of a key lemma in our algorithm is shown in Section 4.6.

### 4.2 Basic Properties

In this subsection, we show some basic properties of $\mathcal{F}_{\text {out }}, \mathcal{F}_{\text {in }}$, and $\mathcal{F}_{\text {min }}$.
Lemma 7. For $X, Y \subseteq V$ with $X \cap Y \neq \emptyset$, we have the following.

1. If $X, Y \in \mathcal{F}_{\text {out }}$, then $X \cap Y, X \cup Y \in \mathcal{F}_{\text {out }}$.
2. If $X, Y \in \mathcal{F}_{\text {in }}$, then $X \cap Y, X \cup Y \in \mathcal{F}_{\text {in }}$.

Proof. If $X=V$ or $Y=V$, then the claim is obvious. Otherwise, $X, Y \subseteq V-r$. Since $D$ is $k$-edge-connected, if $X, Y \in \mathcal{F}_{\text {out }}$, then

$$
2 k=\delta_{D}^{+}(X)+\delta_{D}^{+}(Y) \geq \delta_{D}^{+}(X \cap Y)+\delta_{D}^{+}(X \cup Y) \geq 2 k
$$

by the submodularity of $\delta_{D}^{+}$. Here, we note that $X \cup Y \neq V$ since $r \notin X \cup Y$. Therefore, $\delta_{D}^{+}(X \cap Y)=\delta_{D}^{+}(X \cup Y)=k$, which means that $X \cap Y, X \cup Y \in \mathcal{F}_{\text {out }}$. The same argument can be applied to $\mathcal{F}_{\text {in }}$.

Lemma 8. For $X, Y \subseteq V$, it holds that $\delta_{D}^{+}(X)+\delta_{D}^{-}(Y) \geq \delta_{D}^{+}(X-Y)+\delta_{D}^{-}(Y-X)$.
Proof. Since $\delta_{D}^{-}(S)=\delta_{D}^{+}(V-S)$ for any $S \subseteq V$, we obtain

$$
\begin{aligned}
\delta_{D}^{+}(X)+\delta_{D}^{-}(Y) & =\delta_{D}^{+}(X)+\delta_{D}^{+}(V-Y) \\
& \geq \delta_{D}^{+}(X \cap(V-Y))+\delta_{D}^{+}(X \cup(V-Y)) \\
& =\delta_{D}^{+}(X-Y)+\delta_{D}^{-}(Y-X)
\end{aligned}
$$

by the submodularity of $\delta_{D}^{+}$.
Lemma 9. Suppose that $X \in \mathcal{F}_{\text {out }}, Y \in \mathcal{F}_{\text {in }}, X-Y \neq \emptyset$, and $Y-X \neq \emptyset$. Then, $X-Y \in \mathcal{F}_{\text {out }}$ and $Y-X \in \mathcal{F}_{\text {in }}$.

Proof. Since $X-Y \neq \emptyset$ and $Y-X \neq \emptyset$, we have $X \neq V$ and $Y \neq V$, and hence $X, Y \subseteq V-r$. Since $D$ is $k$-edge-connected, we obtain

$$
2 k=\delta_{D}^{+}(X)+\delta_{D}^{-}(Y) \geq \delta_{D}^{+}(X-Y)+\delta_{D}^{-}(Y-X) \geq 2 k
$$

by Lemma 8. Therefore, $\delta_{D}^{+}(X-Y)=\delta_{D}^{-}(Y-X)=k$, which means that $X-Y \in \mathcal{F}_{\text {out }}$ and $Y-X \in \mathcal{F}_{\text {in }}$.

By these lemmas, we obtain the following corollary. Recall that $\mathcal{F}_{\text {min }}$ is the family of all inclusionwise minimal sets in $\mathcal{F}_{\text {in }} \cup \mathcal{F}_{\text {out }}$.

Corollary 10. $\mathcal{F}_{\min }$ consists of disjoint sets.


Figure 4: The second condition.

Proof. Assume to the contrary that $\mathcal{F}_{\text {min }}$ contains two distinct sets $X$ and $Y$ with $X \cap Y \neq \emptyset$. Then, Lemmas 7 and 9 show that $X \cap Y$ or $X-Y$ is in $\mathcal{F}_{\text {in }} \cup \mathcal{F}_{\text {out }}$, which contradicts the minimality of $X$.

We note that $\mathcal{F}_{\text {min }}$ can be computed in polynomial time, because each inclusionwise minimal element of $\mathcal{F}_{\text {out }}$ and $\mathcal{F}_{\text {in }}$ can be computed by using a minimum cut algorithm.

### 4.3 Safe Source and Safe Sink

As described in Section 4.1, we choose a path $P$ from a safe source to a safe sink in our algorithm. In this subsection, we introduce safe sources and safe sinks.

Let $S$ be an inclusionwise minimal vertex set in $\mathcal{F}_{\text {in }}$ (or $\mathcal{F}_{\text {out }}$, respectively). A vertex $s \in S$ is called a safe source in $S$ (resp. a safe sink in $S$ ) if, for any $X \subseteq V-r$ with $s \in X$ and $S-X \neq \emptyset$,

1. $\delta_{D}^{+}(X) \geq k+1$ (resp. $\delta_{D}^{-}(X) \geq k+1$ ) holds, and
2. if $\delta_{D}^{+}(X)=k+1$ (resp. $\delta_{D}^{-}(X)=k+1$ ), then there exists a vertex set $X^{\prime} \subseteq X-s$ with $X^{\prime} \in \mathcal{F}_{\text {out }}$ (resp. $X^{\prime} \in \mathcal{F}_{\text {in }}$ ); see Figure 4 .

Here is an intuition of the definition. In our algorithm (see Section 4.4), we will find a path from a safe source $s$ in $S$ to a safe $\operatorname{sink} t$ in $T$, and flip the edges of the path one by one from $t$ to $s$. By the edge flips, $S$ and $T$ are removed from $\mathcal{F}_{\text {min }}$, but new sets may be added to $\mathcal{F}_{\text {min }}$. The definition of safety guarantees that a set $X$ newly becomes a member of $\mathcal{F}_{\min }$ only if $X \supseteq S$ or $X \supseteq T$. For example, if a set $X$ with $s \in X$ and $t \notin X$ satisfies $\delta_{D}^{+}(X)=k+1$, then $X$ may newly become a member of $\mathcal{F}_{\text {min }}$. However, the definition of a safe sink guarantees that $X$ has a proper subset $X^{\prime} \in \mathcal{F}_{\text {out }}$ contained in $X-s$, which implies that $X$ cannot become inclusionwise minimal after edge flips. A similar argument holds for a safe sink. Therefore, a set $X$ newly becomes a member of $\mathcal{F}_{\text {min }}$ only if $X \in \mathcal{F}_{\text {out }} \cup \mathcal{F}_{\text {in }}$. This shows that $\operatorname{val}(D)$ decreases by at least one. See the proof of Lemma 13 for the details.

As we will see in Lemma 14, a safe source (resp. a safe sink) always exists in any inclusionwise minimal vertex set in $\mathcal{F}_{\text {in }}\left(\right.$ resp. $\left.\mathcal{F}_{\text {out }}\right)$.

### 4.4 Our Algorithm

In this subsection, we describe our algorithm for finding orientations $D_{1}, \ldots, D_{\ell}$ with the conditions in Proposition 6.

Let $R \subseteq V$ be an inclusionwise minimal vertex set satisfying either
(a) $R \in \mathcal{F}_{\text {in }}$ and there exists a vertex set $X \subsetneq R$ with $X \in \mathcal{F}_{\text {out }}$, or


Figure 5: Conditions in Lemma 11.
(b) $R \in \mathcal{F}_{\text {out }}$ and there exists a vertex set $X \subsetneq R$ with $X \in \mathcal{F}_{\text {in }}$.

Note that such a vertex set $R$ always exists, since $\mathcal{F}_{\text {min }} \neq\{V\}$ implies that $R=V$ satisfies (a) or (b). Furthermore, for each inclusionwise minimal set $X$ in $\mathcal{F}_{\text {out }}$ (resp. $\mathcal{F}_{\text {in }}$ ), we can compute the unique minimal set $R^{\prime}$ satisfying $R^{\prime} \supsetneq X$ and $R^{\prime} \in \mathcal{F}_{\text {in }}$ (resp. $R^{\prime} \in \mathcal{F}_{\text {out }}$ ) by a minimum cut algorithm, which shows that $R$ can be found in polynomial time. We also note that $X \subsetneq R$ in the conditions can be replaced with $X \subseteq R$ unless $R=V$, because $\mathcal{F}_{\text {in }} \cap \mathcal{F}_{\text {out }}=\{V\}$. By symmetry, we may assume that $R$ satisfies (a), i.e., $R \in \mathcal{F}_{\text {in }}$ and there exists a vertex set $X \subsetneq R$ with $X \in \mathcal{F}_{\text {out }}$. Define $\mathcal{F}_{\text {out }}^{R}$ as

$$
\mathcal{F}_{\text {out }}^{R}:=\left\{X \in \mathcal{F}_{\text {out }} \mid X \subsetneq R\right\},
$$

which is nonempty by the choice of $R$.
We use the following key lemma, whose proof is given in Section 4.6.
Lemma 11. Let $R \subseteq V$ be an inclusionwise minimal vertex set satisfying either (a) or (b). If $R$ satisfies (a), then we can find in polynomial time an ( $s, t$ )-path $P$ in $D[R]$ that consists of an $\left(s, t^{\prime}\right)$-path $Q_{1}$ and a $\left(t^{\prime}, t\right)$-path $Q_{2}$ for some $t^{\prime} \in R$ satisfying the following (Figure 5).

1. The vertex $s$ is a safe source in some inclusionwise minimal set $S \in \mathcal{F}_{\text {in }}$ with $S \subseteq R$, and $t$ is a safe sink in some inclusionwise minimal set $T \in \mathcal{F}_{\text {out }}$ with $T \subsetneq R$.
2. $\left(V\left(Q_{1}\right)-t^{\prime}\right) \cap X=\emptyset$ for every $X \in \mathcal{F}_{\text {out }}^{R}$. That is, the subpath $Q_{1}$ is disjoint from any set in $\mathcal{F}_{\text {out }}^{R}$, except for the end vertex $t^{\prime}$.
3. $A\left(Q_{2}\right) \cap \Delta_{D}^{+}(X)=\emptyset$ for every $X \in \mathcal{F}_{\text {out }}$. That is, the subpath $Q_{2}$ does not intersect with $\Delta_{D}^{+}(X)$ for any $X \in \mathcal{F}_{\text {out }}$.

Let $P$ be a path satisfying the conditions in Lemma 11. Suppose that $P$ traverses arcs $e_{\ell}, e_{\ell-1}, \ldots, e_{2}$, and $e_{1}$ in this order from $s$ to $t$. Then, we flip $e_{1}, e_{2}, \ldots, e_{\ell-1}$, and $e_{\ell}$ in this order. For $i=1,2, \ldots, \ell$, let $D_{i}$ be the directed graph obtained from $D$ by flipping $e_{1}, e_{2}, \ldots, e_{i-1}$, and $e_{i}$. Then, our algorithm returns $D_{1}, D_{2}, \ldots, D_{\ell}$.

We now show that $D_{1}, D_{2}, \ldots, D_{\ell-1}$, and $D_{\ell}$ satisfy the conditions in Proposition 6. By Lemma 11, $D_{1}, D_{2}, \ldots, D_{\ell-1}$, and $D_{\ell}$ can be computed in polynomial time. Furthermore, $\ell \leq|V|$ and $D \rightarrow D_{1} \rightarrow D_{2} \rightarrow \cdots \rightarrow D_{\ell}$ are obvious by definition. In the next subsection, we will prove $\lambda\left(D_{i}\right) \geq k$ for each $i \in\{1,2, \ldots, \ell\}$ and $\operatorname{val}\left(D_{\ell}\right)<\operatorname{val}(D)$.

### 4.5 Validity of the Algorithm

Suppose that the algorithm described in the previous subsection finds a path $P$ satisfying the conditions in Lemma 11, and returns $D_{1}, D_{2}, \ldots, D_{\ell-1}$, and $D_{\ell}$. The following two lemmas show that they satisfy the conditions in Proposition 6 .


Figure 6: Case 1 of Lemma 12.

Lemma 12. For each $i \in\{1,2, \ldots, \ell\}, D_{i}$ is $k$-edge-connected.
Proof. Assume to the contrary that $D_{i}$ is not $k$-edge-connected for some $i \in\{1,2, \ldots, \ell\}$, that is, $\delta_{D_{i}}^{+}(X)<k$ for some $X \subseteq V-r$ or $\delta_{D_{i}}^{-}(X)<k$ for some $X \subseteq V-r$. Let $p \in V$ be the tail of $e_{i}$. Then, since $D_{i}$ is obtained from $D$ by reversing the direction of the subpath of $P$ from $p$ to $t$, we obtain

$$
\begin{align*}
& \delta_{D_{i}}^{+}(X)= \begin{cases}\delta_{D}^{+}(X)-1 & \text { if } t \notin X \text { and } p \in X, \\
\delta_{D}^{+}(X)+1 & \text { if } t \in X \text { and } p \notin X, \\
\delta_{D}^{+}(X) & \text { otherwise },\end{cases}  \tag{1}\\
& \delta_{D_{i}}^{-}(X)= \begin{cases}\delta_{D}^{-}(X)-1 & \text { if } t \in X \text { and } p \notin X, \\
\delta_{D}^{-}(X)+1 & \text { if } t \notin X \text { and } p \in X, \\
\delta_{D}^{-}(X) & \text { otherwise }\end{cases} \tag{2}
\end{align*}
$$

for any $X \subseteq V$. Since $D$ is $k$-edge-connected and $D_{i}$ is not $k$-edge-connected, there exists a vertex set $X^{*} \subseteq V-r$ such that either

- $\delta_{D}^{+}\left(X^{*}\right)=k$ (equivalently, $X^{*} \in \mathcal{F}_{\text {out }}$ ), $t \notin X^{*}$, and $p \in X^{*}$, or
- $\delta_{D}^{-}\left(X^{*}\right)=k$ (equivalently, $X^{*} \in \mathcal{F}_{\text {in }}$ ), $t \in X^{*}$, and $p \notin X^{*}$.

We treat the two cases separately. Recall that $t \in T$, where $T$ is a minimal vertex set in $\mathcal{F}_{\text {out }}$.
Case 1: $X^{*} \in \mathcal{F}_{\text {out }}, t \notin X^{*}$, and $p \in X^{*}$ (see Figure 6).
Recall that $P$ is the concatenation of $Q_{1}$ and $Q_{2}$ as in Lemma 11. We can derive a contradiction if $p \in V\left(Q_{2}\right)$ or $X^{*} \cap T \neq \emptyset$ as follows.

- If $p \in V\left(Q_{2}\right)$, then $Q_{2}$ contains the $(p, t)$-path and hence it contains an edge in $\Delta_{D}^{+}\left(X^{*}\right)$, which contradicts that $A\left(Q_{2}\right) \cap \Delta_{D}^{+}(X)=\emptyset$ for any $X \in \mathcal{F}_{\text {out }}$.
- If $X^{*} \cap T \neq \emptyset$, then $X^{*} \cap T \in \mathcal{F}_{\text {out }}$ by Lemma 7. Since $X^{*} \cap T \subseteq T-t \subsetneq T$, this contradicts that $T$ is a minimal vertex set in $\mathcal{F}_{\text {out }}$.

Therefore, $p \in V(P)-V\left(Q_{2}\right)=V\left(Q_{1}\right)-t^{\prime}$ and $X^{*} \cap T=\emptyset$ hold. By the second condition of Lemma 11, we have $X^{*} \notin \mathcal{F}_{\text {out }}^{R}$. This together with $X^{*} \in \mathcal{F}_{\text {out }}$ shows that $X^{*}-R \neq \emptyset$. We also see that $T \subseteq R-X^{*}$ holds by $X^{*} \cap T=\emptyset$, in particular, $R-X^{*} \neq \emptyset$ holds. Then, by applying Lemma 9 to $R$ and $X^{*}$, we obtain $R-X^{*} \in \mathcal{F}_{\text {in }}$. Since $T \subseteq R-X^{*}$ and $T \in \mathcal{F}_{\text {out }}$, this shows that $R-X^{*}$ satisfies the condition (a). This contradicts the minimality of $R$, since $R-X^{*} \subseteq R-p \subsetneq R$.


Figure 7: Case 2 (i).


Figure 8: Case 2 (ii).


Figure 9: Case 2 (iii).

Case 2: $X^{*} \in \mathcal{F}_{\text {in }}, t \in X^{*}$, and $p \notin X^{*}$.
In this case, we derive a contradiction as follows.
(i) If $X^{*} \subseteq T$, then $T$ satisfies the condition (b), which contradicts the minimality of $R$ (Figure 7 ).
(ii) If $T \subseteq X^{*}$, then $R \cap X^{*} \in \mathcal{F}_{\text {in }}$ by Lemma 7 . This together with $T \subseteq R \cap X^{*}$ shows that $R \cap X^{*}$ satisfies the condition (a). Since $R \cap X^{*} \subseteq R-p \subsetneq R$, this contradicts the minimality of $R$ (Figure 8).
(iii) If $T-X^{*} \neq \emptyset$ and $X^{*}-T \neq \emptyset$, then $T-X^{*} \in \mathcal{F}_{\text {out }}$ by Lemma 9 . Since $T-X^{*} \subseteq T-t \subsetneq T$, this contradicts that $T$ is a minimal vertex set in $\mathcal{F}_{\text {out }}$ (Figure 9 ).

By Cases 1 and $2, D_{i}$ is $k$-edge-connected for each $i \in\{1,2, \ldots, \ell\}$.
We next show that $\operatorname{val}(D)$ is decreased by the procedure, where we recall that $\operatorname{val}(D):=$ $\sum_{X \in \mathcal{F}_{\text {min }}(D)}(|V|-|X|)$.
Lemma 13. $\operatorname{val}\left(D_{\ell}\right)<\operatorname{val}(D)$.
Proof. To simplify the notation, we denote $\mathcal{F}_{\text {out }}:=\mathcal{F}_{\text {out }}(D), \mathcal{F}_{\text {in }}:=\mathcal{F}_{\text {in }}(D), \mathcal{F}_{\text {min }}:=\mathcal{F}_{\text {min }}(D)$, $\mathcal{F}_{\text {out }}^{\prime}:=\mathcal{F}_{\text {out }}\left(D_{\ell}\right), \mathcal{F}_{\text {in }}^{\prime}:=\mathcal{F}_{\text {in }}\left(D_{\ell}\right)$, and $\mathcal{F}_{\text {min }}^{\prime}:=\mathcal{F}_{\text {min }}\left(D_{\ell}\right)$. Recall that $D_{\ell}$ is obtained from $D$ by reversing the direction of an $(s, t)$-path. In the same way as (1) and (2), we see that

$$
\begin{align*}
& \delta_{D_{\ell}}^{+}(X)= \begin{cases}\delta_{D}^{+}(X)-1 & \text { if } t \notin X \text { and } s \in X, \\
\delta_{D}^{+}(X)+1 & \text { if } t \in X \text { and } s \notin X, \\
\delta_{D}^{+}(X) & \text { otherwise },\end{cases}  \tag{3}\\
& \delta_{D_{\ell}}^{-}(X)= \begin{cases}\delta_{D}^{-}(X)-1 & \text { if } t \in X \text { and } s \notin X, \\
\delta_{D}^{-}(X)+1 & \text { if } t \notin X \text { and } s \in X, \\
\delta_{D}^{-}(X) & \text { otherwise }\end{cases} \tag{4}
\end{align*}
$$

for any $X \subseteq V$. This shows that, to investigate the gap between $\mathcal{F}_{\text {min }}$ and $\mathcal{F}_{\text {min }}^{\prime}$, it suffices to focus on sets containing $s$ or $t$. We treat the following two cases separately.

Case 1: $S=R$.
Recall that $t \in T$ and $T$ is an inclusionwise minimal vertex set in $\mathcal{F}_{\text {out }}$. Since $S-T \neq \emptyset$, $\delta_{D}^{+}(T)=k$, and $s$ is a safe source in $S$, we obtain $s \notin T$. The minimality of $R$ implies that $T$ does not contain a set in $\mathcal{F}_{\text {in }}$, and hence $T \in \mathcal{F}_{\text {min }}$. Since $\delta_{D_{\ell}}^{+}(T)=\delta_{D}^{+}(T)+1=k+1$ and $\delta_{D_{\ell}}^{-}(T) \geq(2 k+2)-\delta_{D_{\ell}}^{+}(T)=k+1$, it holds that $T \in \mathcal{F}_{\text {min }}-\left(\mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}\right)$.

The following claim asserts that only the set $T$ is removed from $\mathcal{F}_{\min }$ and, if some set $X$ is newly added to $\mathcal{F}_{\min }^{\prime}$, then $X \supsetneq T$ holds.


Figure 10: Case (ii) of Claim 1.


Figure 11: Case (iii) of Claim 1.

Claim 1. If $S=R$, then it holds that $\mathcal{F}_{\text {min }}^{\prime}=\mathcal{F}_{\text {min }}-\{T\}$ or $\mathcal{F}_{\text {min }}^{\prime}=\left(\mathcal{F}_{\min }-\{T\}\right) \cup\{X\}$ for some $X \supsetneq T$.

Proof of Claim 1. We first show that $\mathcal{F}_{\text {min }}-\left(\mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}\right)=\{T\}$. Assume to the contrary that there exists a set $X \in \mathcal{F}_{\text {min }}-\left(\mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}\right)$ with $X \neq T$. Since $X \in \mathcal{F}_{\text {min }}-\left(\mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}\right) \subseteq$ $\left(\mathcal{F}_{\text {out }} \cup \mathcal{F}_{\text {in }}\right)-\left(\mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}\right)$, by (3) and $(4)$, it holds that $|X \cap\{s, t\}|=1$. This shows that $X \in \mathcal{F}_{\text {in }}$, $t \notin X$, and $s \in X$, because $T$ is the unique element in $\mathcal{F}_{\min }$ containing $t$. Since $s \in X \cap S$, $S \in \mathcal{F}_{\text {in }}$, and $X \in \mathcal{F}_{\text {in }} \cap \mathcal{F}_{\text {min }}$, we obtain $X \subseteq S$, and hence $X \subseteq S-t \subsetneq S$. This contradicts the fact that $S$ is an inclusionwise minimal vertex set in $\mathcal{F}_{\text {in }}$ with $s \in S$. Therefore, we obtain $\mathcal{F}_{\text {min }}-\left(\mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}\right)=\{T\}$.

We now claim that $X \supsetneq T$ holds for any $X \in \mathcal{F}_{\text {min }}^{\prime}-\mathcal{F}_{\text {min }}$. Let $X$ be a set in $\mathcal{F}_{\min }^{\prime}-\mathcal{F}_{\text {min }}$. Since $X \neq T$ is obvious, it suffices to show $X \supseteq T$. Since $X \in \mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\mathrm{in}}^{\prime}$, by (3) and (4), we have one of the following: (i) $X \in \mathcal{F}_{\text {out }} \cup \mathcal{F}_{\text {in }}$, (ii) $s \notin X, t \in X$, and $\delta_{D}^{-}(X)=k+1$, or (iii) $s \in X$, $t \notin X$, and $\delta_{D}^{+}(X)=k+1$. Then, for each case, $X \supseteq T$ holds if such a set $X$ exists as follows.
(i) If $X \in \mathcal{F}_{\text {out }} \cup \mathcal{F}_{\text {in }}$, then $X \notin \mathcal{F}_{\text {min }}$ implies that there exists a set $Y \subsetneq X$ with $Y \in \mathcal{F}_{\text {min }}$. Since $X \in \mathcal{F}_{\min }^{\prime}$, it holds that $Y \in \mathcal{F}_{\min }-\left(\mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}\right)=\{T\}$. Therefore, $Y$ must be equal to $T$, and hence $X \supsetneq T$.
(ii) Suppose that $s \notin X, t \in X$, and $\delta_{D}^{-}(X)=k+1$ (see Figure 10). Assume to the contrary that $X \supseteq T$ does not hold, i.e., $T-X \neq \emptyset$. Since $t$ is a safe sink in $T$, there exists a vertex set $X^{\prime} \subseteq X-t$ with $X^{\prime} \in \mathcal{F}_{\text {in }}$ by the definition of a safe sink. This shows that $X^{\prime} \subsetneq X$ and $X^{\prime} \in \mathcal{F}_{\text {in }}^{\prime}$ as $s, t \notin X^{\prime}$, which contradicts $X \in \mathcal{F}_{\text {min }}^{\prime}$. Therefore, $X \supseteq T$ holds.
(iii) Suppose that $s \in X, t \notin X$, and $\delta_{D}^{+}(X)=k+1$ (see Figure 11). Then, $R-X \neq \emptyset$, because it contains $t$. Since $s$ is a safe source in $R$, there exists a vertex set $X^{\prime} \subseteq X-s$ with $X^{\prime} \in \mathcal{F}_{\text {out }}$ by the definition of a safe source. This shows that $X^{\prime} \subsetneq X$ and $X^{\prime} \in \mathcal{F}_{\text {out }}^{\prime}$ as $s, t \notin X^{\prime}$, which contradicts $X \in \mathcal{F}_{\text {min }}^{\prime}$. Therefore, such $X$ does not exist.
By the above argument, $X \supsetneq T$ holds for any $X \in \mathcal{F}_{\min }^{\prime}-\mathcal{F}_{\text {min }}$. Since there exists at most one set in $\mathcal{F}_{\text {min }}^{\prime}$ containing $T$, we obtain $\mathcal{F}_{\text {min }}^{\prime}=\mathcal{F}_{\text {min }}-\{T\}$ or $\mathcal{F}_{\text {min }}^{\prime}=\left(\mathcal{F}_{\text {min }}-\{T\}\right) \cup\{X\}$ for some $X \supsetneq T$. Thus, Claim 1 holds.

By Claim 1, if $\mathcal{F}_{\text {min }}^{\prime}=\mathcal{F}_{\text {min }}-\{T\}$, then $\operatorname{val}(D)-\operatorname{val}\left(D_{\ell}\right)=|V|-|T|>0$, and, if $\mathcal{F}_{\text {min }}^{\prime}=$ $\left(\mathcal{F}_{\text {min }}-\{T\}\right) \cup\{X\}$ for some $X \supsetneq T$, then $\operatorname{val}(D)-\operatorname{val}\left(D_{\ell}\right)=|X|-|T|>0$ holds. This completes the proof for the case of $S=R$.

Case 2: $S \subsetneq R$.


Figure 12: Case (iii) of Claim 2.

Recall that $S$ is a minimal vertex set in $\mathcal{F}_{\text {in }}$ and $s$ is a safe source in $S$. The minimality of $R$ implies that $S$ (resp. $T$ ) does not contain a set in $\mathcal{F}_{\text {out }}$ (resp. $\mathcal{F}_{\text {in }}$ ), and hence $S, T \in \mathcal{F}_{\min }$. This implies that $S$ and $T$ are disjoint. Since $\delta_{D_{\ell}}^{-}(S)=\delta_{D}^{-}(S)+1=k+1, \delta_{D_{\ell}}^{+}(S) \geq(2 k+2)-\delta_{D}^{-}(S)=$ $k+1, \delta_{D_{\ell}}^{+}(T)=\delta_{D}^{+}(T)+1=k+1$, and $\delta_{D_{\ell}}^{-}(T) \geq(2 k+2)-\delta_{D_{\ell}}^{+}(T)=k+1$, it holds that $S, T \in \mathcal{F}_{\min }-\left(\mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}\right)$. Therefore, by (3) and (4), we obtain $\mathcal{F}_{\min }-\left(\mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}\right)=\{S, T\}$. Moreover, we have the following claim.
Claim 2. If $S \subsetneq R$, then one of the following cases holds:

- $\mathcal{F}_{\text {min }}^{\prime}=\mathcal{F}_{\text {min }}-\{S, T\}$,
- $\mathcal{F}_{\text {min }}^{\prime}=\left(\mathcal{F}_{\text {min }}-\{S, T\}\right) \cup\{X\}$ for some set $X$ with $X \supsetneq S$ or $X \supsetneq T$,
- $\mathcal{F}_{\text {min }}^{\prime}=\left(\mathcal{F}_{\min }-\{S, T\}\right) \cup\left\{X_{s}, X_{t}\right\}$ for some sets $X_{s} \supsetneq S$ and $X_{t} \supsetneq T$.

Proof of Claim 2. We claim that $X \supsetneq S$ or $X \supsetneq T$ holds for any $X \in \mathcal{F}_{\min }^{\prime}-\mathcal{F}_{\min }$. Let $X$ be a set in $\mathcal{F}_{\min }^{\prime}-\mathcal{F}_{\min }$. Since $X \neq S, T$ is obvious, it suffices to show $X \supseteq S$ or $X \supseteq T$. Since $X \in \mathcal{F}_{\text {out }}^{\prime} \cup \mathcal{F}_{\text {in }}^{\prime}$, by (3) and (4), we have one of the following: (i) $X \in \mathcal{F}_{\text {out }} \cup \mathcal{F}_{\text {in }}$, (ii) $s \notin X, t \in X$, and $\delta_{D}^{-}(X)=k+1$, or (iii) $s \in X, t \notin X$, and $\delta_{D}^{+}(X)=k+1$. For the cases (i) and (ii), we see that $X \supseteq S$ or $X \supseteq T$ holds in the same way as Case 1. For the case (iii), we can show that $X \supseteq S$ holds in the same way as (ii) as follows.

Suppose that $s \in X, t \notin X$, and $\delta_{D}^{+}(X)=k+1$ (Figure 12). Assume to the contrary that $X \supseteq S$ does not hold, i.e., $S-X \neq \emptyset$. Since $s$ is a safe source in $S$, there exists a vertex set $X^{\prime} \subseteq X-s$ with $X^{\prime} \in \mathcal{F}_{\text {out }}$ by the definition of a safe source. This shows that $X^{\prime} \subsetneq X$ and $X^{\prime} \in \mathcal{F}_{\text {out }}^{\prime}$ as $s, t \notin X^{\prime}$, which contradicts $X \in \mathcal{F}_{\min }^{\prime}$. Therefore, $X \supseteq S$ holds.

By the above argument, $X \supsetneq S$ or $X \supsetneq T$ holds for any $X \in \mathcal{F}_{\min }^{\prime}-\mathcal{F}_{\text {min }}$. Note that there exists at most one set $X_{s}\left(\right.$ resp. $\left.X_{t}\right)$ in $\mathcal{F}_{\min }^{\prime}$ containing $S$ (resp. $T$ ). Thus the claim holds.

It follows from Claim 2 that $\operatorname{val}(D)-\operatorname{val}\left(D_{\ell}\right)>0$. Indeed, for the first case, $\operatorname{val}(D)-\operatorname{val}\left(D_{\ell}\right)=$ $2|V|-|S|-|T|>0$; for the second case, $\operatorname{val}(D)-\operatorname{val}\left(D_{\ell}\right)=|V|+|X|-|S|-|T|>0$; for the last case, $\operatorname{val}(D)-\operatorname{val}\left(D_{\ell}\right)=\left|X_{s}\right|+\left|X_{t}\right|-|S|-|T|>0$. This completes the proof for the case of $S \subsetneq R$, and closes the whole proof of Lemma 13 .

Lemmas 12 and 13 show that the output of our algorithm in Section 4.4 satisfies the conditions in Proposition 6. By repeatedly applying Proposition 6 at most $|V|^{2}$ times, we obtain Theorem 5 .


Figure 13: $Y_{i}, Z_{j}$, and $s$.

### 4.6 Construction of a Path (Proof of Lemma 11)

In this section, we first show some useful lemmas in Sections 4.6.1 and 4.6.2, and then give a proof of Lemma 11 in Section 4.6.3.

### 4.6.1 Existence of a Safe Source and a Safe Sink

Lemma 14. For any inclusionwise minimal vertex $\operatorname{set} S$ in $\mathcal{F}_{\text {in }}$ (or $\mathcal{F}_{\text {out }}$, respectively), there exists a safe source (resp. a safe sink) s in $S$. Furthermore, such a vertex s can be found in polynomial time.

Proof. Let $S$ be an inclusionwise minimal vertex set in $\mathcal{F}_{\text {in }}$. If $S=V$, then $s=r$ satisfies the conditions. Hence, it suffices to consider the case of $S \subseteq V-r$. In this case, we obtain $\delta_{D}^{+}(S)=\delta_{G}(S)-\delta_{D}^{-}(S) \geq(2 k+2)-k=k+2$. Let

$$
\mathcal{F}_{\text {out }}^{S}:=\left\{X \in \mathcal{F}_{\text {out }} \mid X \subsetneq S\right\}=\left\{X \subseteq S \mid \delta_{D}^{+}(X)=k\right\}
$$

and let $Y_{1}, Y_{2}, \ldots, Y_{\alpha-1}$, and $Y_{\alpha}$ be the inclusionwise maximal vertex sets in $\mathcal{F}_{\text {out }}^{S}$. Note that these sets are mutually disjoint, because $Y_{i} \cap Y_{j} \neq \emptyset$ implies $Y_{i} \cup Y_{j} \in \mathcal{F}_{\text {out }}^{S}$ by Lemma 7. Let

$$
\mathcal{G}:=\left\{Z \subseteq S-\bigcup_{i=1}^{\alpha} Y_{i} \mid \delta_{D}^{+}(Z)=k+1\right\}
$$

and let $Z_{1}, Z_{2}, \ldots, Z_{\beta-1}$, and $Z_{\beta}$ be the inclusionwise maximal vertex sets in $\mathcal{G}$ (see Figure 13). We show the following two claims.
Claim 3. $Z_{1}, Z_{2}, \ldots, Z_{\beta-1}$, and $Z_{\beta}$ are mutually disjoint.
Proof of Claim 3. Assume to the contrary that $Z_{i} \cap Z_{j} \neq \emptyset$ for some distinct $i, j \in\{1, \ldots, \beta\}$. Since $Z_{i} \cap Z_{j} \notin \mathcal{F}_{\text {out }}^{S}$ and $Z_{i} \cup Z_{j} \notin \mathcal{F}_{\text {out }}^{S}$, we obtain $\delta_{D}^{+}\left(Z_{i} \cap Z_{j}\right) \geq k+1$ and $\delta_{D}^{+}\left(Z_{i} \cup Z_{j}\right) \geq k+1$. Then, it holds that

$$
2(k+1)=\delta_{D}^{+}\left(Z_{i}\right)+\delta_{D}^{+}\left(Z_{j}\right) \geq \delta_{D}^{+}\left(Z_{i} \cap Z_{j}\right)+\delta_{D}^{+}\left(Z_{i} \cup Z_{j}\right) \geq 2(k+1)
$$

Therefore, $\delta_{D}^{+}\left(Z_{i} \cup Z_{j}\right)=k+1$, and hence $Z_{i} \cup Z_{j} \in \mathcal{G}$. This contradicts the maximality of $Z_{i}$ and $Z_{j}$.

Claim 4. $S-\bigcup_{i=1}^{\alpha} Y_{i}-\bigcup_{j=1}^{\beta} Z_{j} \neq \emptyset$.

Proof of Claim \& Assume to the contrary that $S-\bigcup_{i=1}^{\alpha} Y_{i}-\bigcup_{j=1}^{\beta} Z_{j}=\emptyset$. Then,

$$
\begin{aligned}
k \alpha+(k+1) \beta-(k+2) & \geq \sum_{i=1}^{\alpha} \delta_{D}^{+}\left(Y_{i}\right)+\sum_{j=1}^{\beta} \delta_{D}^{+}\left(Z_{j}\right)-\delta_{D}^{+}(S) \\
& =\sum_{i=1}^{\alpha} \delta_{D}^{-}\left(Y_{i}\right)+\sum_{j=1}^{\beta} \delta_{D}^{-}\left(Z_{j}\right)-\delta_{D}^{-}(S) \\
& \geq \sum_{i=1}^{\alpha}\left(2 k+2-\delta_{D}^{+}\left(Y_{i}\right)\right)+\sum_{j=1}^{\beta}\left(2 k+2-\delta_{D}^{+}\left(Z_{j}\right)\right)-k \\
& =(k+2) \alpha+(k+1) \beta-k \\
& >k \alpha+(k+1) \beta-(k+2)
\end{aligned}
$$

which is a contradiction.
By Claim 4, we can choose a vertex $s \in S-\bigcup_{i=1}^{\alpha} Y_{i}-\bigcup_{j=1}^{\beta} Z_{j}$ (see Figure 13). We now show that $s$ is a safe source in $S$. Recall that a vertex $s \in S$ is called a safe source in $S$ if, for any $X \subseteq V-r$ with $s \in X$ and $S-X \neq \emptyset$,

1. $\delta_{D}^{+}(X) \geq k+1$ holds, and
2. if $\delta_{D}^{+}(X)=k+1$, then there exists a vertex set $X^{\prime} \subseteq X-s$ with $X^{\prime} \in \mathcal{F}_{\text {out }}$.

To show the first condition, assume to the contrary that $\delta_{D}^{+}(X)=k$ holds for some $X \subseteq V-r$ with $s \in X$ and $S-X \neq \emptyset$. Since $s \in X$ implies that $X \notin \mathcal{F}_{\text {out }}^{S}, X$ is not a subset of $S$, i.e., $X-S \neq \emptyset$. Then, $S-X \in \mathcal{F}_{\text {in }}$ by Lemma 9, which contradicts the minimality of $S$ as $S-X \subseteq S-s \subsetneq S$. Therefore, the first condition is satisfied.

To show the second condition, suppose that $\delta_{D}^{+}(X)=k+1$ holds for some $X \subseteq V-r$ with $s \in X$ and $S-X \neq \emptyset$. We treat the case of $X \subseteq S$ and that of $X-S \neq \emptyset$, separately.

- Suppose that $X \subseteq S$. Since $X$ and its supersets are not in $\mathcal{G}$ by the choice of $s, X$ is not contained in $S-\bigcup_{i=1}^{\alpha} Y_{i}$, that is, $X \cap Y_{i} \neq \emptyset$ for some $i \in\{1, \ldots, \alpha\}$. By the $k$-edgeconnectedness of $D$, it holds that $\delta_{D}^{+}\left(X \cap Y_{i}\right) \geq k$. Since $X \cup Y_{i} \supseteq Y_{i}+s$, we obtain $X \cup Y_{i} \notin \mathcal{F}_{\text {out }}^{S}$ by the maximality of $Y_{i}$, which implies that $\delta_{D}^{+}\left(X \cup Y_{i}\right) \geq k+1$. Then, we obtain

$$
2 k+1=\delta_{D}^{+}(X)+\delta_{D}^{+}\left(Y_{i}\right) \geq \delta_{D}^{+}\left(X \cap Y_{i}\right)+\delta_{D}^{+}\left(X \cup Y_{i}\right) \geq 2 k+1,
$$

and hence $\delta_{D}^{+}\left(X \cap Y_{i}\right)=k$ and $\delta_{D}^{+}\left(X \cup Y_{i}\right)=k+1$. Therefore, $X^{\prime}:=X \cap Y_{i}$ satisfies that $X^{\prime} \subseteq X-s$ and $X^{\prime} \in \mathcal{F}_{\text {out }}$ (see Figure 14).

- Suppose that $X-S \neq \emptyset$. By the $k$-edge-connectedness of $D$, it holds that $\delta_{D}^{+}(X-S) \geq k$. Since $S-X \subseteq S-s \subsetneq S$, we obtain $S-X \notin \mathcal{F}_{\text {in }}$ by the minimality of $S$, which implies that $\delta_{D}^{-}(S-X) \geq k+1$. Then, by Lemma 8 , we obtain

$$
2 k+1=\delta_{D}^{+}(X)+\delta_{D}^{-}(S) \geq \delta_{D}^{+}(X-S)+\delta_{D}^{-}(S-X) \geq 2 k+1,
$$

and hence $\delta_{D}^{+}(X-S)=k$ and $\delta_{D}^{-}(S-X)=k+1$. Therefore, $X^{\prime}:=X-S$ satisfies that $X^{\prime} \subseteq X-s$ and $X^{\prime} \in \mathcal{F}_{\text {out }}$ (see Figure 15).


Figure 14: Case of $X \subseteq S$.


Figure 15: Case of $X-S \neq \emptyset$.

By this argument, $s$ is a safe source in $S$. Furthermore, since $Y_{1}, Y_{2}, \ldots, Y_{\alpha}, Z_{1}, Z_{2}, \ldots, Z_{\beta-1}$, and $Z_{\beta}$ can be computed by using a minimum cut algorithm, a vertex $s \in S-\bigcup_{i=1}^{\alpha} Y_{i}-\bigcup_{j=1}^{\beta} Z_{j}$ can be found in polynomial time.

By the same argument, if $S$ is an inclusionwise minimal vertex set in $\mathcal{F}_{\text {out }}$, then a safe sink $s$ in $S$ can be found in polynomial time.

### 4.6.2 Path to a Minimal Vertex Set

The goal of this sub-subsection is to show Lemma 15 below, saying that, for any vertex $s \in V$, $D$ has a path $P$ from $s$ to some inclusionwise minimal set $T$ in $\mathcal{F}_{\text {out }}$ such that $P$ leaves no set in $\mathcal{F}_{\text {out }}$. Analogously to Lemma 15, we can obtain a path to any vertex $t \in V$ from some inclusionwise minimal set $S$ in $\mathcal{F}_{\text {in }}$ (Lemma 17). These paths will be used in our proof of Lemma 11 .

Lemma 15. For any vertex $s \in V$, there exists a vertex set $T \in \mathcal{F}_{\text {out }}$ satisfying the following conditions:

- $T$ is inclusionwise minimal in $\mathcal{F}_{\text {out }}$, and
- for any vertex $t \in T, D$ contains an $(s, t)$-path $P_{t}$ such that $A\left(P_{t}\right) \cap \Delta_{D}^{+}(X)=\emptyset$ for any $X \in \mathcal{F}_{\text {out }}$.

Furthermore, such $T$ and $P_{t}$ can be found in polynomial time.
To prove the lemma, we need more definitions. For a vertex $s \in V$, let $X_{\text {out }}(s)$ denote the inclusionwise minimal vertex set subject to $s \in X_{\text {out }}(s) \in \mathcal{F}_{\text {out }}$. Note that such a vertex set always exists as $s \in V \in \mathcal{F}_{\text {out }}$. Note also that the minimal one is uniquely determined, because if $s \in X \in \mathcal{F}_{\text {out }}$ and $s \in Y \in \mathcal{F}_{\text {out }}$, then $s \in X \cap Y \in \mathcal{F}_{\text {out }}$ by Lemma 7. For each $s \in V$, we can easily compute $X_{\text {out }}(s)$ in polynomial time by using a minimum cut algorithm.

Lemma 16. Let $s \in V$. For any vertex $t \in X_{\text {out }}(s), D\left[X_{\text {out }}(s)\right]$ contains a path from $s$ to $t$.
Proof. If $X_{\text {out }}(s)=V$, then the lemma holds since $D$ is strongly connected, where we note that $k \geq 1$. Thus, we consider the case when $X_{\text {out }}(s) \neq V$. Assume to the contrary that $D\left[X_{\text {out }}(s)\right]$ does not contain a path from $s$ to $t$. Then, there exists a vertex set $S \subseteq X_{\text {out }}(s)$ such that $s \in S$, $t \in X_{\text {out }}(s)-S$, and $D$ has no edge from $S$ to $X_{\text {out }}(s)-S$ (Figure 16). Since $X_{\text {out }}(s) \in \mathcal{F}_{\text {out }}$ and $D$ is $k$-edge-connected, we obtain

$$
k=\delta_{D}^{+}\left(X_{\mathrm{out}}(s)\right) \geq \delta_{D}^{+}(S) \geq k
$$

and hence $S \in \mathcal{F}_{\text {out }}$. Since $s \in S \subsetneq X_{\text {out }}(s)$, this contradicts the minimality of $X_{\text {out }}(s)$.


Figure 16: Proof of Lemma 16.


Figure 17: Proof of Lemma 15.

We are ready to prove Lemma 15 .
Proof of Lemma 15. We prove the lemma by induction on $\left|X_{\text {out }}(s)\right|$.
If $X_{\text {out }}(s)$ is an inclusionwise minimal vertex set in $\mathcal{F}_{\text {out }}$, then $T:=X_{\text {out }}(s)$ satisfies the condition. This is because the existence of $P_{t}$ is guaranteed by Lemma 16 and $A\left(P_{t}\right) \cap \Delta_{D}^{+}(X)=\emptyset$ for any $X \in \mathcal{F}_{\text {out }}$ follows from the minimality of $T=X_{\text {out }}(s)$ and Lemma 7. This is the base case of the induction.

Suppose that $X_{\text {out }}(s)$ is not an inclusionwise minimal vertex set in $\mathcal{F}_{\text {out }}$, that is, there exists a vertex set $Y$ in $\mathcal{F}_{\text {out }}$ that is strictly contained in $X_{\text {out }}(s)$. We can take a vertex $u \in X_{\text {out }}(s)$ such that $X_{\text {out }}(u)=Y$. By Lemma 16, $D\left[X_{\text {out }}(s)\right]$ contains a path $Q$ from $s$ to $u$. Traverse along $Q$ from $s$ to $u$ and let $s^{\prime}$ be the first vertex on $Q$ such that $X_{\text {out }}\left(s^{\prime}\right) \subsetneq X_{\text {out }}(s)$ (see Figure 17). Note that such $s^{\prime}$ always exists as $u$ satisfies the condition. Recall that $Q\left[s, s^{\prime}\right]$ denotes the subpath of $Q$ between $s$ and $s^{\prime}$. We show the following claim.
Claim 5. $A\left(Q\left[s, s^{\prime}\right]\right) \cap \Delta_{D}^{+}(X)=\emptyset$ for any $X \in \mathcal{F}_{\text {out }}$.
Proof of Claim 5. Assume to the contrary that there exists an edge $(x, y) \in A\left(Q\left[s, s^{\prime}\right]\right) \cap \Delta_{D}^{+}\left(X^{*}\right)$ for some $X^{*} \in \mathcal{F}_{\text {out }}$. Then $X^{*} \cap X_{\text {out }}(s) \in \mathcal{F}_{\text {out }}$ by Lemma 7. Since $x \in X^{*} \cap X_{\text {out }}(s)$, we see that $X_{\text {out }}(x) \subseteq X^{*} \cap X_{\text {out }}(s)$. Therefore, since $X^{*} \cap X_{\text {out }}(s) \subseteq X_{\text {out }}(s)-y \subsetneq X_{\text {out }}(s)$, we obtain $X_{\text {out }}(x) \subsetneq X_{\text {out }}(s)$, which contradicts the choice of $s^{\prime}$. Thus, Claim 5 follows.

Since $\left|X_{\text {out }}\left(s^{\prime}\right)\right|<\left|X_{\text {out }}(s)\right|$, by the induction hypothesis, there exists an inclusionwise minimal vertex set $T$ of $\mathcal{F}_{\text {out }}$ satisfying the following condition: for any vertex $t \in T, D$ contains an $\left(s^{\prime}, t\right)$ path $P_{t}^{\prime}$ such that $A\left(P_{t}^{\prime}\right) \cap \Delta_{D}^{+}(X)=\emptyset$ for any $X \in \mathcal{F}_{\text {out }}$. We remark that $P_{t}^{\prime}$ is contained in $X_{\text {out }}\left(s^{\prime}\right)$ as it contains no edge in $\Delta_{D}^{+}\left(X_{\text {out }}\left(s^{\prime}\right)\right)$, noting that $X_{\text {out }}\left(s^{\prime}\right) \in \mathcal{F}_{\text {out }}$.

We now show that $T$ is a desired set also for $s$. For each $t \in T$, let $P_{t}$ be the $(s, t)$-path obtained by concatenating $Q\left[s, s^{\prime}\right]$ and $P_{t}^{\prime}$. Note that $P_{t}$ is indeed a path (i.e., it goes through each vertex at most once), since $V\left(Q\left[s, s^{\prime}\right]\right) \cap X_{\text {out }}\left(s^{\prime}\right)=\left\{s^{\prime}\right\}$ and $V\left(P_{t}^{\prime}\right) \subseteq X_{\text {out }}\left(s^{\prime}\right)$. Then, Claim 5 shows that $A\left(P_{t}\right) \cap \Delta_{D}^{+}(X)=\left(A\left(Q\left[s, s^{\prime}\right]\right) \cap \Delta_{D}^{+}(X)\right) \cup\left(A\left(P_{t}^{\prime}\right) \cap \Delta_{D}^{+}(X)\right)=\emptyset$ for any $X \in \mathcal{F}_{\text {out }}$. Hence $T$ is a desired set for $s$.

Since the above inductive proof can be converted to a recursive algorithm, $T$ and $P_{t}$ satisfying the conditions can be computed in polynomial time.

We note that the vertex set $T$ in Lemma 15 is not necessarily an element of $\mathcal{F}_{\text {min }}$, because there might exist a vertex set $U \subsetneq T$ with $U \in \mathcal{F}_{\text {in }}$. By changing the roles of $\mathcal{F}_{\text {out }}$ and $\mathcal{F}_{\text {in }}$, we obtain the following lemma in the same way as Lemma 15 .

Lemma 17. For any vertex $t \in V$, there exists a vertex set $S \in \mathcal{F}_{\text {in }}$ satisfying the following conditions:

- $S$ is inclusionwise minimal in $\mathcal{F}_{\text {in }}$, and
- for any vertex $s \in S, D$ contains an $(s, t)$-path $P_{s}$ such that $A\left(P_{s}\right) \cap \Delta_{D}^{-}(X)=\emptyset$ for any $X \in \mathcal{F}_{\text {in }}$.
Furthermore, such $S$ and $P_{s}$ can be found in polynomial time.


### 4.6.3 Proof of Lemma 11

We are now ready to prove Lemma 11 .
Let $T^{*} \in \mathcal{F}_{\text {out }}^{R}$ and $t^{*} \in T^{*}$. By applying Lemma 17 to $t^{*}$, we obtain an inclusionwise minimal vertex set $S$ in $\mathcal{F}_{\text {in }}$. By Lemma 14, $S$ has a safe source $s$. Lemma 17 guarantees that there exists an $\left(s, t^{*}\right)$-path $P_{s}$ such that $A\left(P_{s}\right) \cap \Delta_{D}^{-}(X)=\emptyset$ for any $X \in \mathcal{F}_{\text {in }}$. This implies that $P_{s}$ is in $D[R]$, since $A\left(P_{s}\right) \cap \Delta_{D}^{-}(R)=\emptyset$ as $R \in \mathcal{F}_{\text {in }}$. In particular, $s \in R$. By the minimality of $S$, this shows that $S \subseteq R$ (possibly, $S=R$ ).

Traverse along $P_{s}$ from $s$ to $t^{*}$ and let $t^{\prime}$ be the first vertex on $P_{s}$ such that there exists a vertex set $T^{\prime} \in \mathcal{F}_{\text {out }}^{R}$ with $t^{\prime} \in T^{\prime}$. Note that such $t^{\prime}$ always exists, because $t^{\prime}=t^{*}$ satisfies the condition. We also note that, for each $x \in V$, we can check the existence of a vertex set $X \in \mathcal{F}_{\text {out }}^{R}$ with $x \in X$ by a minimum cut algorithm. Let $Q_{1}$ denote the subpath of $P_{s}$ between $s$ and $t^{\prime}$, i.e., $Q_{1}:=P_{s}\left[s, t^{\prime}\right]$. Then we see by the choice of $t^{\prime}$ that $V\left(Q_{1}\right)-t^{\prime}$ is disjoint from $X$ for every $X \in \mathcal{F}_{\text {out }}^{R}$.

By applying Lemma 15 to $t^{\prime}$, we obtain an inclusionwise minimal vertex set $T$ in $\mathcal{F}_{\text {out }}$. Let $t$ be a safe sink in $T$ as in Lemma 14, and let $Q_{2}$ be a $\left(t^{\prime}, t\right)$-path such that $A\left(Q_{2}\right) \cap \Delta_{D}^{+}(X)=\emptyset$ for any $X \in \mathcal{F}_{\text {out }}$, whose existence is guaranteed by Lemma 15. We note that $V\left(Q_{2}\right) \subseteq T^{\prime} \subsetneq R$, since $A\left(Q_{2}\right) \cap \Delta_{D}^{+}\left(T^{\prime}\right)=\emptyset$ as $T^{\prime} \in \mathcal{F}_{\text {out }}$. In particular, $t \in T^{\prime} \subsetneq R$. By the minimality of $T$, this shows that $T \subseteq T^{\prime} \subsetneq R$ (possibly, $T=T^{\prime}$ ).

Let $P$ be the ( $s, t$ )-path obtained by concatenating $Q_{1}$ and $Q_{2}$ (see Figure 18). Note that $P$ goes through each vertex at most once, since $V\left(Q_{1}\right) \cap T^{\prime}=\left\{t^{\prime}\right\}$ and $V\left(Q_{2}\right) \subseteq T^{\prime}$. Then, $P$ satisfies the conditions in the lemma. By Lemmas 14, 15, and 17 , the above procedure can be executed in polynomial time, which completes the proof.

## 5 Proof of Theorem 3

In this section, we prove Theorem 3; for two strongly connected orientations $D_{1}, D_{2}$, there exists a path connecting $D_{1}$ and $D_{2}$ in the edge-flip graph $\mathcal{G}_{1}(G)$ if and only if there exists no 2-edge-cut $\left\{\{u, v\},\left\{u^{\prime}, v^{\prime}\right\}\right\}$ such that $(u, v),\left(v^{\prime}, u^{\prime}\right)$ are edges of $D_{1}$ and $(v, u),\left(u^{\prime}, v^{\prime}\right)$ are edges of $D_{2}$.

If there exists a 2-edge-cut $\left\{\{u, v\},\left\{u^{\prime}, v^{\prime}\right\}\right\}$ such that $(u, v),\left(v^{\prime}, u^{\prime}\right)$ are edges of $D_{1}$ and $(v, u),\left(u^{\prime}, v^{\prime}\right)$ are edges of $D_{2}$, then we cannot flip $(u, v)$ or $\left(v^{\prime}, u^{\prime}\right)$ in $D_{1}$ one-by-one. Hence $D_{1}$ and $D_{2}$ are not connected in the edge-flip graph $\mathcal{G}_{1}(G)$.


Figure 18: Proof of Lemma 11

Suppose that no such 2-edge-cut exists. Let $D_{1}=\left(V, A_{1}\right)$ and $D_{2}=\left(V, A_{2}\right)$. We will show that there exists an edge $e$ in $A_{1}-A_{2}$ such that we can flip $e$ in $D_{1}$ keeping strong edge-connectedness.

Let $e=(u, v)$ be an arbitrary edge in $D_{1}$ but not in $D_{2}$ (i.e., $(v, u)$ is in $D_{2}$ ). We may assume that we cannot flip $e$. Then there exists a vertex subset $X$ such that $\Delta_{D_{1}}^{+}(X)=\{e\}$. Since $D_{1}$ is strongly connected, $\delta_{D_{1}}^{-}(X) \geq 1$. We take such a vertex subset $X$ so that $\delta_{D_{1}}^{-}(X)$ is maximized.

Since $D_{2}$ has an edge $(v, u)$ and $\delta_{D_{2}}^{+}(X) \geq 1$, there exists an edge $f$ in $\Delta_{D_{1}}^{-}(X)$ such that the reverse of $f$ is in $A_{2}$. We will show that the edge $f$ can be flipped keeping strong connectivity.

Assume to the contrary that $f$ cannot be flipped. Then there exists a vertex subset $Y$ such that $\Delta_{D_{1}}^{+}(Y)=\{f\}$ and $\delta_{D_{1}}^{-}(Y) \geq 1$. Moreover, we see that $\delta_{D_{1}}^{-}(Y) \geq 2$. In fact, if $\delta_{D_{1}}^{-}(Y)=1$, then $E_{G}(Y)$ is a 2-edge cut. Since $D_{2}$ is strongly connected, the (unique) edge in $\Delta_{D_{1}}^{-}(Y)$ must be flipped in $D_{2}$. This, however, contradicts the assumption of the theorem.

First consider the case when $X \cap Y=\emptyset$. If $X \cup Y=V$, then $\Delta_{D_{1}}^{+}(X)=\{e\}$ and $\Delta_{D_{1}}^{-}(X)=$ $\Delta_{D_{1}}^{+}(Y)=\{f\}$, which contradicts the assumption of the theorem. Thus we have $X \cup Y \subsetneq V$. Define $X^{\prime}=X \cup Y$. Then, since the edge $f$ only enters from $Y$ to $X$, it follows that $\delta_{D_{1}}^{-}\left(X^{\prime}\right)=$ $\delta_{D_{1}}^{-}(X)-1+\delta_{D_{1}}^{-}(Y)$. Since $\delta_{D_{1}}^{-}(Y) \geq 2$, we have $\delta_{D_{1}}^{-}\left(X^{\prime}\right)>\delta_{D_{1}}^{-}(X)$, which contradicts the maximality of $\delta_{D_{1}}^{-}(X)$.

Thus we may suppose that $X \cap Y \neq \emptyset$. We first claim that $X \cup Y=V$. Indeed, if $X \cup Y \subsetneq V$, then we have

$$
2=\delta_{D_{1}}^{+}(X)+\delta_{D_{1}}^{+}(Y) \geq \delta_{D_{1}}^{+}(X \cap Y)+\delta_{D_{1}}^{+}(X \cup Y) \geq 2
$$

and hence $\delta_{D_{1}}^{+}(X \cap Y)=\delta_{D_{1}}^{+}(X \cup Y)=1$. However, we see that $\Delta_{D_{1}}^{+}(X \cap Y) \cup \Delta_{D_{1}}^{+}(X \cup Y)=\{e\}$ and $e \notin \Delta_{D_{1}}^{+}(X \cap Y) \cap \Delta_{D_{1}}^{+}(X \cup Y)$, which is a contradiction.

Define $X^{\prime}=X \cap Y$. Since the edge $f$ only enters from $Y-X$ to $X-Y$, it follows that $\delta_{D_{1}}^{-}\left(X^{\prime}\right)=\delta_{D_{1}}^{-}(X)-1+\delta_{D_{1}}^{-}(Y)$. Since $\delta_{D_{1}}^{-}(Y) \geq 2$, we have $\delta_{D_{1}}^{-}\left(X^{\prime}\right)>\delta_{D_{1}}^{-}(X)$, which contradicts the maximality of $\delta_{D_{1}}^{-}(X)$. Therefore, the edge $f$ can be flipped.

From the above discussion, we can find in polynomial time an edge $e$ in $A_{1}-A_{2}$ such that flipping the edge $e$ in $D_{1}$ does not violate strong edge-connectedness. By repeatedly finding such edges, we obtain a sequence of orientations from $D_{1}$ to $D_{2}$ by edge flips. In each edge flip, $\left|A_{1}-A_{2}\right|$ decreases by one. Since the length of a sequence is at least $\left|A_{1}-A_{2}\right|$, the obtained sequence turns out to be the shortest. This completes the proof of Theorem 3.

## 6 Concluding Remarks

This paper initiates the study of $k$-edge-connected orientations through edge flips for $k \geq 2$. As a showcase, we give a new edge-flip-based proof (Theorem 1) of Nash-Williams' theorem [26]: an
undirected graph $G$ has a $k$-edge-connected orientation if and only if $G$ is $2 k$-edge-connected. Our new proof has another useful property that all the intermediate orientations have non-decreasing edge-connectivity in the process. Using Theorem 11, we prove that the edge-flip graph of $k$-edgeconnected orientations of an undirected graph $G$ is connected if $G$ is $(2 k+2)$-edge-connected (Theorem 2).

Several questions remain open. In Theorem 1, we showed that the length of an edge-flip sequence is bounded by $k|V|^{3}$. However, we do not know this bound is tight. It is not clear how to find such a shortest sequence in polynomial time. We do not know the tightness of Theorem 2, either: we do not know whether the edge-flip graph of $k$-edge-connected orientations is connected when the underlying undirected graph is $(2 k+1)$-edge-connected. We do not know the $k$-edge-connectedness counterpart of Theorem 3 when $k \geq 2$.

It is not clear how to find a shortest path in the edge-flip graph of $k$-edge-connected orientations in polynomial time when $k \geq 2$. When $k=1$, we can find a shortest path by looking at the "symmetric difference" of two given strongly connected orientations [12, 15]. However, when $k=2$, there exists an example for which the symmetric difference does not determine a shortest path.

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[^1]:    ${ }^{1}$ One of the conditions is actually redundant since $\delta_{D}^{-}(S) \geq k$ implies $\delta_{D}^{+}(V-S) \geq k$.

