Hierarchies of Minion Tests for PCSPs through Tensors^{*}

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Abstract

We provide a unified framework to study hierarchies of relaxations for Constraint Satisfaction Problems and their Promise variant. The idea is to split the description of a hierarchy into an algebraic part, depending on a *minion* capturing the "base level", and a geometric part – which we call *tensorisation* – inspired by multilinear algebra. We exploit the geometry of the tensor spaces arising from our construction to prove general properties of hierarchies. We identify certain classes of minions, which we call *linear* and *conic*, whose corresponding hierarchies have particularly fine features. We establish that the (combinatorial) bounded width, Sherali-Adams LP, affine IP, Sum-of-Squares SDP, and combined "LP + affine IP" hierarchies are all captured by this framework. In particular, in order to analyse the Sum-of-Squares SDP hierarchy, we also characterise the solvability of the standard SDP relaxation through a new minion.

1 Introduction

What are the limits of efficient algorithms and where is the precise borderline of tractability? The constraint satisfaction problem (CSP) offers a general framework for studying such fundamental questions for a large class of computational problems [49, 50, 79] but yet for a class that is amenable to identifying the mathematical structure governing tractability. Canonical examples of CSPs are satisfiability or "not-all-equal" satisfiability of 3-CNF formulas (called 3-SAT and 3-NAE-SAT, respectively), linear equations, several variants of (hyper)graph colourings, and the graph clique problem. All CSPs can be seen as homomorphism problems between relational structures [61]: Given two relational structures **X** and **A**, is there a homomorphism from **X** to **A**? Intuitively, the structure **X** represents the variables of the CSP instance and their interactions, whereas the structure **A** represents the constraint language; i.e., the alphabet and the allowed constraint relations.

The most studied types of CSPs are so-called *non-uniform* CSPs [16, 61, 70, 76], in which the target structure **A** is fixed whereas the source structure **X** is given on input; this computational problem is denoted by CSP(**A**). From the examples above, 3-SAT, 3-NAE-SAT,

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(hyper)graph colourings with constantly many colours, linear equations of bounded width over finite fields, and linear equations of bounded width over the rationals are all examples of non-uniform CSPs, all on finite domains except the last one [18,22,23]. For instance, in the graph *c*-colouring problem the target structure **A** is a *c*-clique and the structure **X** is the input graph. The existence of a homomorphism from a graph to a *c*-clique is equivalent to the existence of a colouring of the graph with *c* colours. The graph clique problem is an example of a CSP with a fixed class of source structures [64,89] but an arbitrary target structure and, thus, it is not a non-uniform CSP.

We will be concerned with polynomial-time tractability of CSPs. Studied research directions include investigating questions such as: Is there a solution [37,97]? How many solutions are there, exactly [36, 48, 59] or approximately [38, 41]? What is the maximum number of simultaneously satisfied constraints, exactly [47, 69, 94] or approximately [7, 56, 91]? What is the minimum number of simultaneously unsatisfied constraints [54, 72]? Given an almost satisfiable instance, can one find a somewhat satisfying solution [14, 52, 53]? In this paper, we will focus on the following question:

Given a satisfiable instance, can one find a solution that is satisfying in a weaker sense [9, 12, 26]?

This was formalised as promise constraint satisfaction problems (PCSPs) by Austrin, Guruswami and Håstad [9] and Brakensiek and Guruswami [26]. Let **A** and **B** be two fixed relational structures¹ such that there is a homomorphism from **A** to **B**, indicated by $\mathbf{A} \to \mathbf{B}$. Intuitively, the structure **A** represents the "strict" constraints and the structure **B** represents the corresponding "weak" constraints. An instance of the PCSP over the template (**A**, **B**), denoted by PCSP(**A**, **B**), is a relational structure **X** such that there is a homomorphism from **X** to **A**. The task is to find a homomorphism from **X** to **B**, which exists by the composition of the two promised homomorphisms. What we described above is the *search* variant of the PCSP. In the *decision* variant, one is given a relational structure **X** and the task is to decide whether there is a homomorphism from **X** to **A** or whether there is not a homomorphism from **X** to **B**. Note that since homomorphisms compose, if $\mathbf{X} \to \mathbf{A}$ then also $\mathbf{X} \to \mathbf{B}$. Thus, the two cases cannot happen simultaneously. It is known that the decision variant of the PCSP reduces to the search variant [12], but it is not known whether there is a reduction in the other direction for all PCSPs. In this paper, we shall use the decision variant.

PCSPs are a vast generalisation of CSPs including problems that cannot be expressed as CSPs. The work of Barto, Bulín, Krokhin, and Opršal [12] lifted and greatly extended the algebraic framework developed for CSPs [17, 35, 70] to the realm of PCSPs. Subsequently, there has been a series of recent works on the computational complexity of PCSPs building on [12], including applicability of local consistency and convex relaxations [5,25,30,39,44] and complexity of fragments of PCSPs [2,11,15,27,31,66,80,90]. Strong results on PCSPs have also been established via other techniques than those in [12], mostly analytical methods, e.g., hardness of various (hyper)graph colourings [8,58,68,73] and other PCSPs [20,21,28,33].

An example of a PCSP, identified in [9], is (in the search variant) finding a satisfying assignment to a k-CNF formula given that a g-satisfying assignment exists; i.e., an assignment that satisfies at least g literals in each clause. Austrin et al. established that this problem is NP-hard if g/k < 1/2 and solvable via a constant level of the Sherali-Adams linear pro-

¹Unless otherwise stated, we shall use the word "structure" to mean finite-domain structures; if the domain is allowed to be infinite, we shall say it explicitly.

gramming relaxation otherwise [9]. This classification was later extended to problems over arbitrary finite domains by Brandts et al. [31].

A second example of a PCSP, identified in [26], is (in the search variant) finding a "notall-equal" assignment to a monotone 3-CNF formula given that a "1-in-3" assignment is promised to exist; i.e., given a 3-CNF formula with positive literals only and the promise that an assignment exists that satisfies exactly one literal in each clause, the task is to find an assignment that satisfies one or two literals in each clause. This problem is solvable in polynomial time via a constant level of the Sherali-Adams linear programming relaxation [26] but not via a reduction to finite-domain CSPs [12].

A third example of a PCSP is the well-known approximate graph colouring problem: Given a c-colourable graph, find a d-colouring of it, for constants c and d with $c \leq d$. This corresponds to PCSP($\mathbf{K}_c, \mathbf{K}_d$), where \mathbf{K}_p is the clique on p vertices. Despite a long history dating back to 1976 [62], the complexity of this problem is only understood under stronger assumptions [32, 57, 66] and for special cases [12, 24, 65, 68, 71, 73, 80]. It is believed that the problem is NP-hard already in the decision variant [62], i.e., deciding whether a graph is c-colourable or not even d-colourable, for each $3 \leq c \leq d$. By using the framework developed in the current work, non-solvability of approximate graph colouring through standard algorithmic techniques was established in the follow-up works [42, 43].

Like all decision problems, PCSPs can be solved by designing *tests*. If a test, applied to a given instance of the problem, is positive then the answer is YES; if it is negative then the answer is No. The challenge is then to find tests that are able to guarantee a low number – ideally, zero – of false positives and false negatives. Clearly, a test is itself a decision problem. However, its nature may be substantially different, and less complicated, than the nature of the original problem.

Given a PCSP template (\mathbf{A}, \mathbf{B}) , we may use any (potentially infinite) structure \mathbf{T} to make a test for $PCSP(\mathbf{A}, \mathbf{B})$: We simply let the outcome of the test on an instance structure \mathbf{X} be YES if $\mathbf{X} \to \mathbf{T}$, and No if $\mathbf{X} \not\to \mathbf{T}$. In other words, $CSP(\mathbf{T})$ is a test for $PCSP(\mathbf{A}, \mathbf{B})$. Let **X** be an instance of $PCSP(\mathbf{A}, \mathbf{B})$. If $\mathbf{X} \to \mathbf{T}$ whenever $\mathbf{X} \to \mathbf{A}$, the test is guaranteed not to generate false negatives, and we call it *complete*. Since homomorphisms compose, if $\mathbf{A} \to \mathbf{T}$ the test is automatically complete. If $\mathbf{X} \to \mathbf{B}$ whenever $\mathbf{X} \to \mathbf{T}$, the test is guaranteed not to generate false positives, and we call it *sound*. If both of these conditions hold, we say that the test $solves PCSP(\mathbf{A}, \mathbf{B})$. Notice that, in this case, one obtains a reduction from $PCSP(\mathbf{A}, \mathbf{B})$ to $CSP(\mathbf{T})$. These two types of computational problems – the original problem and the test – have different algebraic natures. The complexity of both CSPs and PCSPs was shown to be determined by higher-order symmetries of the solution sets of the problems, known as *polymorphisms*, denoted by $Pol(\mathbf{A})$ for $CSP(\mathbf{A})$ [35] and by $Pol(\mathbf{A}, \mathbf{B})$ for $PCSP(\mathbf{A}, \mathbf{B})$ [12]. For CSPs, polymorphisms form *clones*; in particular, they are closed under composition. This means that some symmetries may be obtainable through compositions of other symmetries, so that one can hope to capture properties of entire families of CSPs (e.g., bounded width, tractability, etc.) through the presence of a certain polymorphism and, more generally, to describe their complexity through universal-algebraic tools. A chief example of this approach is the positive resolution of the dichotomy conjecture for CSPs by Bulatov [37] and Zhuk [97], establishing that finite-domain non-uniform CSPs are either in P or are NP-complete. For PCSPs, however, polymorphisms are not closed under composition, and the algebraic structure they are endowed with - known as *minion* - is much less rich and, apparently, harder to understand through the lens of universal algebra.

To make a test **T** useful as a polynomial-time algorithm to solve a PCSP, one requires that $CSP(\mathbf{T})$ should be tractable. It was conjectured in [25] that every tractable (finite-domain) PCSP is solved by a tractable test. In other words, if the conjecture is true, *tests are the sole source of tractability for PCSPs*. For the conjecture to be true, one needs to admit tests on infinite domains: As shown in [12], the PCSP template (**1-in-3**, **NAE**) does not admit a finite-domain tractable test; i.e., there is no (finite) structure **T** such that **1-in-3** \rightarrow **T** \rightarrow **NAE** and $CSP(\mathbf{T})$ is tractable.

For a PCSP template (\mathbf{A}, \mathbf{B}) , one would ideally aim to build tests for PCSP (\mathbf{A}, \mathbf{B}) in a systematic way. One method to do so is by considering tests associated with minions and, in particular, their *free structures*. The free structure $\mathbb{F}_{\mathscr{M}}(\mathbf{A})$ of a minion \mathscr{M} generated by a structure \mathbf{A} [12] is a (potentially infinite) structure obtained, essentially, by simulating the relations in \mathbf{A} on a domain consisting of elements of \mathscr{M} . Then, we define $\text{Test}_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \text{YES}$ if $\mathbf{X} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A})$, and No otherwise. (Note that \mathbf{X} is the input to the problem; the minion \mathscr{M} and the relational structure \mathbf{A} , coming from a PCSP template, are (fixed) parameters of the test.)

For certain choices of \mathscr{M} , Test $_{\mathscr{M}}$ is a tractable test; i.e., $\operatorname{CSP}(\mathbb{F}_{\mathscr{M}}(\mathbf{A}))$ is tractable for any \mathbf{A} . This is the case for the minions $\mathscr{H} = \operatorname{Pol}(\operatorname{HORN-3-SAT})$ (whose elements are nonempty subsets of a given set), $\mathscr{Q}_{\operatorname{conv}}$ (whose elements are stochastic vectors), and $\mathscr{Z}_{\operatorname{aff}}$ (whose elements are affine integers vectors). As it was shown in [12], these three minions correspond to three well-studied algorithmic relaxations: $\operatorname{Test}_{\mathscr{H}}$ is Arc Consistency (AC) [87], $\operatorname{Test}_{\mathscr{Q}_{\operatorname{conv}}}$ is the Basic Linear Programming relaxation (BLP) [81], and $\operatorname{Test}_{\mathscr{Z}_{\operatorname{aff}}}$ is the Affine Integer Programming relaxation ($\mathscr{Z}_{\operatorname{aff}}$) [25]. In [30], the algorithm BLP + AIP (which we shall call BA in this work) corresponding to a combination of linear and integer programming was shown to be captured by a certain minion $\mathscr{M}_{\mathrm{BA}}$. In summary, several widely used algorithms for (P)CSPs are minion tests; in particular, Arc Consistency, which is the simplest example of consistency algorithms, and standard algorithms based on relaxations.

Convex relaxations have been instrumental in the understanding of the complexity of many variants of CSPs, including constant approximability of Min-CSPs [54,60] and Max-CSPs [74, 91], robust satisfiability of CSPs [14,81,98], and exact solvability of optimisation CSPs [77,95]. An important line of work focused on making convex relaxations stronger and stronger via the so-called "lift-and-project" method, which includes the Sherali-Adams LP hierarchy [93], the SDP hierarchy of Lovász and Schrijver [86], and the (stronger) SDP hierarchy of Lasserre [82], also known as the Sum-of-Squares hierarchy (see [83] for a comparison of these hierarchies). The study of the power of various convex hierarchies has led to several breakthroughs, e.g., [1, 40, 63, 78, 84, 96].

In the same spirit as lift-and-project hierarchies of convex relaxations, the (combinatorial) k-consistency algorithm (also known as the k-bounded width algorithm) has a central role in the study of tractability for constraint satisfaction problems [3, 61]. Here k is an integer bounding the number of variables considered in reasoning about partial solutions; the case k = 1 corresponds to Arc Consistency mentioned above. The notion of local consistency, in addition to being one of the key concepts in constraint satisfaction, has also emerged independently in finite model theory [75], graph theory [67], and proof complexity [4]. The power of local consistency for CSPs is now fully understood [10,13,34]. Recent work identified a necessary condition on local consistency to solve PCSPs [5].

Contributions The main contribution of this work is the introduction of a general framework for refining algorithmic relaxations of (P)CSPs. Given a minion \mathscr{M} , we present a technique to systematically turn Test \mathscr{M} into the corresponding hierarchy of minion tests: a sequence of increasingly tighter relaxations Test \mathscr{M} for $k \in \mathbb{N}$.

The technique we adopt to build hierarchies of minion tests is inspired by multilinear algebra. We describe a *tensorisation* construction that turns a given structure \mathbf{X} into a structure $\mathbf{X}^{(k)}$ on a different signature, where both the domain and the relations are multidimensional objects living in tensor spaces. Essentially, $\text{Test}_{\mathscr{M}}^{k}$ works by applying $\text{Test}_{\mathscr{M}}$ to *tensorised* versions of the structures \mathbf{X} and \mathbf{A} rather than to \mathbf{X} and \mathbf{A} themselves. This allows us to study the functioning of the algorithms in the hierarchy by describing the geometry of a space of tensors – which can be accomplished by using multilinear algebra. As far as we know, this approach has not appeared in the literature on Sherali-Adams, bounded width, Sum-of-Squares, hierarchies of integer programming, and related algorithmic techniques such as the high-dimensional Weisfeiler-Leman algorithm [6,39].²

One key feature of our framework is that it is *modular*, in that it allows splitting the description of a hierarchy of minion tests into an *algebraic* part, corresponding to the minion \mathcal{M} , and a *geometric* part, entirely dependent on the tensorisation construction and hence common to all hierarchies. By considering certain well-behaved families of minions, which we call *linear* and *conic*, we can then deduce general properties of the corresponding hierarchies by only focussing on the geometry of spaces of tensors.

Letting the minion \mathscr{M} be \mathscr{H} (resp., \mathscr{Q}_{conv} , \mathscr{Z}_{aff}), we shall retrieve in this way the bounded width hierarchy (resp., the Sherali-Adams LP hierarchy, the affine integer programming hierarchy). Additionally, we describe a new minion \mathscr{S} capturing the power of the basic semidefinite programming relaxation (SDP),³ and we show that $\operatorname{Test}^k_{\mathscr{S}}$ coincides with the Sum-of-Squares hierarchy. As a consequence, our framework is able to provide a unified description of all these four well-known hierarchies of algorithmic relaxations. In addition to casting *known* hierarchies of relaxations as hierarchies of minion tests, this approach can be used to design *new* hierarchies. In particular, we describe an operation that we call *semi-direct product* of minions, which consists in combining multiple minions to form a new minion associated with a stronger relaxation. In practice, this method can be used to design an algorithm that combines the features of different known algorithmic techniques. We show that the minion \mathscr{M}_{BA} associated with the BA relaxation from [30] is the semi-direct product of \mathscr{Q}_{conv} and \mathscr{Z}_{aff} , and we formally introduce the BA^k hierarchy as the hierarchy Test^k_{MBA}.

The scope of this framework is potentially not limited to constraint satisfaction: The multilinear pattern that we found at the core of different algorithmic hierarchies appears to be transversal to the constraint satisfaction setting and, instead, inherently connected to the algorithmic techniques themselves, which can be applied to classes of computational problems living beyond the realms of (P)CSPs.

Subsequent work The tensorisation methodology introduced in this paper has later been used by the authors in follow-up work on the applicability of relaxation hierarchies to specific

²Butti and Dalmau [39] recently characterised for CSPs when the k-th level of the Sherali-Adams LP programming hierarchy accepts in terms of a construction different from the one introduced in this work. Unlike the tensorisation, the construction considered in [39] yields a relational structure whose domain includes the set of constraints of the original structure.

³In the recent paper [29], Brakensiek, Guruswami, and Sandeep independently provided a characterisation for the power of SDP that is similar to the one we obtain in the current work.

problems. In particular, they have shown that the approximate graph colouring problem is not solved by the affine integer programming hierarchy [42] and not even by the (stronger) hierarchy for the combined basic linear programming and affine integer programming relaxation [43]. In the very recent paper [55], various concepts introduced in this work – in particular, the concept of conic minions – were shown to play an important role for the description of a certain type of reductions between (P)CSPs (cf. Remark 45).

2 Background

Notation We denote by \mathbb{N} the set of positive integers. For $k \in \mathbb{N}$, we denote by [k] the set $\{1, \ldots, k\}$. We indicate by \mathbf{e}_i the *i*-th standard unit vector of the appropriate size (which will be clear from the context); i.e., the *i*-th entry of \mathbf{e}_i is 1, and all other entries are 0. $\mathbf{0}_p$ and $\mathbf{1}_p$ denote the all-zero and all-one vector of size p, respectively, while I_p and $O_{p,q}$ denote the $p \times p$ identity matrix and the $p \times q$ all-zero matrix, respectively. Given a matrix M, we let tr(M) and csupp(M) be the trace and the set of indices of nonzero columns of M, respectively. The symbol \aleph_0 denotes the cardinality of \mathbb{N} .

Promise CSPs A signature σ is a finite set of relation symbols R, each with arity $\operatorname{ar}(R) \in \mathbb{N}$. A σ -structure \mathbf{A} consists of a domain (universe) A and, for each $R \in \sigma$, a relation $R^{\mathbf{A}} \subseteq A^{\operatorname{ar}(R)}$. A σ -structure \mathbf{A} is finite if the size |A| of its domain A is finite. In this case, we often assume that the domain of \mathbf{A} is A = [n].

Let **A** and **B** be σ -structures. A homomorphism from **A** to **B** is a map $h: A \to B$ such that, for each $R \in \sigma$ with $r = \operatorname{ar}(R)$ and for each $\mathbf{a} = (a_1, \ldots, a_r) \in A^r$, if $\mathbf{a} \in R^\mathbf{A}$ then $h(\mathbf{a}) = (h(a_1), \ldots, h(a_r)) \in R^\mathbf{B}$. We denote the existence of a homomorphism from **A** to **B** by $\mathbf{A} \to \mathbf{B}$. A pair of σ -structures (\mathbf{A}, \mathbf{B}) with $\mathbf{A} \to \mathbf{B}$ is called a *promise constraint* satisfaction problem (PCSP) template. The PCSP problem parameterised by the template (\mathbf{A}, \mathbf{B}) , denoted by PCSP (\mathbf{A}, \mathbf{B}) , is the following computational problem: The input is a σ -structure **X** and the goal is to answer YES if $\mathbf{X} \to \mathbf{A}$ and No if $\mathbf{X} \neq \mathbf{B}$. The promise is that it is not the case that $\mathbf{X} \neq \mathbf{A}$ and $\mathbf{X} \to \mathbf{B}$. We write $\operatorname{CSP}(\mathbf{A})$ for $\operatorname{PCSP}(\mathbf{A}, \mathbf{A})$, the classic (non-promise) constraint satisfaction problem.

Relaxations and hierarchies The following relaxations of (P)CSPs shall be mentioned in this paper: Arc Consistency (AC) is a propagation algorithm that checks for the existence of assignments satisfying the local constraints of the given (P)CSP instance [87]; the basic linear programming (BLP) relaxation looks for compatible probability distributions on assignments [81]; the affine integer programming (AIP) relaxation turns the constraints into linear equations, that can be solved over the integers using (a variant of) Gaussian elimination [25]; the basic semidefinite programming (SDP) relaxation is essentially a strengthening of BLP, where probabilities are replaced by vectors satisfying orthogonality requirements [91]; the combined basic linear programming and affine integer programming (BA) relaxation is a hybrid algorithm blending BLP and AIP [30].

In this work, we shall mainly focus on algorithmic *hierarchies*. The bounded width (BW^k) hierarchy (also known as local consistency checking algorithm) refines AC by propagating local solutions over bigger and bigger portions of the instance, while the Sherali-Adams LP (SA^k) , affine integer programming (AIP^k) , Sum-of-Squares (SoS^k) , and combined basic linear programming and affine integer programming (BA^k) hierarchies strengthen the BLP, AIP,

SDP, and BA relaxations, respectively, by looking for compatible distributions over bigger and bigger assignments.

The five hierarchies mentioned above, as well as the SDP relaxation, are described in Section 4.2 (see also the discussion in Appendix A). We refer to [12] for AC, BLP, and AIP, and to [30] for BA.

Algebraic approach to PCSPs The algebraic theory of PCSPs developed in [12] relies on the notions of polymorphism and minion. Let \mathbf{A} be a σ -structure. For $L \in \mathbb{N}$, the *L*-th power of \mathbf{A} is the σ -structure \mathbf{A}^L with domain A^L whose relations are defined as follows: Given $R \in \sigma$ and an $L \times \operatorname{ar}(R)$ matrix M such that all rows of M are tuples in $R^{\mathbf{A}}$, the columns of Mform a tuple in $R^{\mathbf{A}^L}$. An *L*-ary polymorphism of a PCSP template (\mathbf{A}, \mathbf{B}) is a homomorphism from \mathbf{A}^L to \mathbf{B} . Minions were defined in [12] as sets of functions with certain properties. We shall use here the abstract definition of minions, as first done in [30], cf. also [44]. A minion \mathscr{M} consists in the disjoint union of nonempty sets $\mathscr{M}^{(L)}$ for $L \in \mathbb{N}$ equipped with (so-called minor) operations $(\cdot)_{/\pi} : \mathscr{M}^{(L)} \to \mathscr{M}^{(L')}$ for all functions $\pi : [L] \to [L']$, which satisfy $M_{/\operatorname{id}} = M$ and, for $\pi : [L] \to [L']$ and $\tilde{\pi} : [L'] \to [L''], (M_{/\pi})_{/\tilde{\pi}} = M_{/\tilde{\pi}\circ\pi}$ for all $M \in \mathscr{M}^{(L)}$.

Example 1. The set $Pol(\mathbf{A}, \mathbf{B})$ of all polymorphisms of a PCSP template (\mathbf{A}, \mathbf{B}) is a minion with the minor operations defined by $f_{/\pi}(a_1, \ldots, a_{L'}) = f(a_{\pi(1)}, \ldots, a_{\pi(L)})$ for $f : \mathbf{A}^L \to \mathbf{B}$ and $\pi : [L] \to [L']$. In this minion, the minor operations correspond to identifying coordinates, permuting coordinates, and introducing dummy coordinates (of polymorphisms).

Example 2. Other examples of minions that shall appear frequently in this work are \mathcal{Q}_{conv} , \mathscr{Z}_{aff} , and \mathscr{H} , capturing the power of the algorithms BLP, AIP, and AC, respectively. The Lary elements of $\mathcal{Q}_{\text{conv}}$ are rational vectors of size L that are stochastic (i.e., whose entries are nonnegative and sum up to 1), with the minor operations defined as follows: For $\mathbf{q} \in \mathscr{Q}_{\text{conv}}^{(L)}$ and $\pi: [L] \to [L'], \mathbf{q}_{/\pi} = P\mathbf{q}$, where P is the $L' \times L$ matrix whose (i, j)-th entry is 1 if $\pi(j) = i$, and 0 otherwise. \mathscr{L}_{aff} is defined similarly to \mathscr{Q}_{conv} , the only difference being that its L-ary elements are affine integer vectors (i.e., their entries are integer – possibly negative – numbers and sum up to 1). \mathcal{H} is the minion of polymorphisms of the CSP template HORN-3-SAT, i.e., the Boolean structure whose four relations are " $x \wedge y \Rightarrow z$ ", " $x \wedge y \Rightarrow \neg z$ ", $\{0\}$, and $\{1\}$. Equivalently (cf. [12]), \mathscr{H} can be described as follows: For any $L \in \mathbb{N}$, the L-ary elements of \mathscr{H} are Boolean functions of the form $f_Z(x_1, \ldots, x_L) = \bigwedge_{z \in \mathbb{Z}} x_z$ for any $Z \subseteq [L], Z \neq \emptyset$; the minor operations are defined as in Example 1. We shall also mention the minion \mathcal{M}_{BA} capturing the algorithm BA. Its L-ary elements are $L \times 2$ matrices whose first column **u** belongs to $\mathscr{Q}_{\text{conv}}^{(L)}$ and whose second column **v** belongs to $\mathscr{Z}_{\text{aff}}^{(L)}$, and such that if the *i*-th entry of **u** is zero then the *i*-th entry of **v** is also zero, for each $i \in [L]$. The minor operation is defined on each column individually; i.e., $\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}_{/\pi} = \begin{bmatrix} \mathbf{u}_{/\pi} & \mathbf{v}_{/\pi} \end{bmatrix}$.

For two minions \mathscr{M} and \mathscr{N} , a minion homomorphism $\xi : \mathscr{M} \to \mathscr{N}$ is a map that preserves arities and minors: Given $M \in \mathscr{M}^{(L)}$ and $\pi : [L] \to [L'], \xi(M) \in \mathscr{N}^{(L)}$ and $\xi(M_{/\pi}) = \xi(M)_{/\pi}$. We denote the existence of a minion homomorphism from \mathscr{M} to \mathscr{N} by $\mathscr{M} \to \mathscr{N}$.

We will also need the concept of free structure from [12]. Let \mathscr{M} be a minion and let \mathbf{A} be a (finite) σ -structure. The *free structure* of \mathscr{M} generated by \mathbf{A} is a σ -structure $\mathbb{F}_{\mathscr{M}}(\mathbf{A})$ with domain $\mathscr{M}^{(|A|)}$ (potentially infinite). Given a relation symbol $R \in \sigma$ of arity r, a tuple (M_1, \ldots, M_r) of elements of $\mathscr{M}^{(|A|)}$ belongs to $R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A})}$ if and only if there is some $Q \in \mathscr{M}^{(|R^{\mathbf{A}}|)}$ such that $M_i = Q_{/\pi_i}$ for each $i \in [r]$, where $\pi_i : R^{\mathbf{A}} \to A$ maps $\mathbf{a} \in R^{\mathbf{A}}$ to its *i*-th coordinate a_i . The definition of free structure may at this point strike the reader as rather abstract. We

shall see that if we consider certain quite general classes of minions then this object unveils an interesting geometric description of linear and multilinear nature.

3 Overview of results and techniques

Let (\mathbf{A}, \mathbf{B}) be a PCSP template. As discussed in Section 1, any (potentially infinite) structure \mathbf{T} on the same signature as \mathbf{A} and \mathbf{B} can be viewed as a test for the computational problem PCSP (\mathbf{A}, \mathbf{B}) : Given an instance \mathbf{X} , the test returns YES if $\mathbf{X} \to \mathbf{T}$, and No otherwise. As the next definition illustrates, minions provide a systematic method to build tests for PCSPs.

Definition 3. Let \mathscr{M} be a minion. The minion test $\operatorname{Test}_{\mathscr{M}}$ is the computational problem defined as follows: Given two σ -structures **X** and **A**, return YES if $\mathbf{X} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A})$, and No otherwise.

If **X** is an instance of PCSP(**A**, **B**) for some template (**A**, **B**), we write $\text{Test}_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \text{YES}$ if $\text{Test}_{\mathscr{M}}$ applied to **X** and **A** returns YES (i.e., if $\mathbf{X} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A})$), and we write $\text{Test}_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) =$ No otherwise. Note that, in the expression "Test_ $\mathscr{M}(\mathbf{X}, \mathbf{A})$ ", **X** is the input structure of the PCSP, while **A** is the fixed structure from the PCSP template.

Leaving SDP aside for the moment, it turns out that the algebraic structure lying at the core of all relaxations mentioned in Section 2, of seemingly different nature, is the same, as all of them are minion tests for specific minions.

Theorem 4 ([12,30]). AC = Test_{\mathscr{H}}, BLP = Test_{\mathscr{Q}_{conv}}, AIP = Test_{\mathscr{Q}_{aff}}, BA = Test_{\mathscr{M}_{BA}}.

One reason why minion tests are an interesting type of tests is that they are always complete.

Proposition 5. Test_{*M*} *is complete for any minion* \mathcal{M} *; i.e., for any* \mathbf{X} *and* \mathbf{A} *with* $\mathbf{X} \to \mathbf{A}$ *, we have* $\mathbf{X} \to \mathbb{F}_{\mathcal{M}}(\mathbf{A})$ *.*

A second feature of minion tests is that their soundness can be characterised algebraically, as stated in the next proposition and shown easily using a compactness argument from [88], cf. [12].

Proposition 6. Let \mathscr{M} be a minion and let (\mathbf{A}, \mathbf{B}) be a PCSP template. Then, Test_ \mathscr{M} solves PCSP (\mathbf{A}, \mathbf{B}) if and only if $\mathscr{M} \to \operatorname{Pol}(\mathbf{A}, \mathbf{B})$.

3.1 A minion for SDP

The first contribution of this work is to design a minion \mathscr{S} capturing the power of SDP, thus showing that, similarly to AC, BLP, AIP, and BLP + AIP, also SDP is a minion test.

Definition 7. For $L \in \mathbb{N}$, let $\mathscr{S}^{(L)}$ be the set of real $L \times \aleph_0$ matrices M such that

(C1) $\operatorname{csupp}(M)$ is finite (C2) MM^T is a diagonal matrix (C3) $\operatorname{tr}(MM^T) = 1$. (1)

Given a function $\pi : [L] \to [L']$ and a matrix $M \in \mathscr{S}^{(L)}$, we let $M_{/\pi} = PM$, where P is the $L' \times L$ matrix whose (i, j)-th entry is 1 if $\pi(j) = i$, and 0 otherwise. We set $\mathscr{S} = \bigsqcup_{L \in \mathbb{N}} \mathscr{S}^{(L)}$.

In Section 6, we shall prove that the object defined above is indeed a minion and that it captures the power of the SDP relaxation, as stated below.

Proposition 8. SDP = Test \mathscr{S} . In other words, given two σ -structures \mathbf{X} and \mathbf{A} , SDP(\mathbf{X}, \mathbf{A}) = YES if and only if $\mathbf{X} \to \mathbb{F}_{\mathscr{S}}(\mathbf{A})$.

Using Proposition 6, we obtain a characterisation of the power of the SDP relaxation.

Theorem 9. Let (\mathbf{A}, \mathbf{B}) be a PCSP template. Then, SDP solves PCSP (\mathbf{A}, \mathbf{B}) if and only if $\mathscr{S} \to \operatorname{Pol}(\mathbf{A}, \mathbf{B})$.

3.2 Tensorisation

As discussed earlier, minions give a systematic method for designing tests for (P)CSPs. We now describe a construction, which we call *tensorisation*, that provides a technique to systematically refine minion tests, thus creating hierarchies of progressively stronger algorithms.

Let S be a set and let $k \in \mathbb{N}$. For $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, $\mathcal{T}^{\mathbf{n}}(S)$ denotes the set of all functions from $[n_1] \times \cdots \times [n_k]$ to S, which we visualise as hypermatrices or tensors. Many of the tensors appearing in this paper are *cubical*, which means that $\mathbf{n} = n \cdot \mathbf{1}_k = (n, \ldots, n)$ is a constant tuple.⁴

For $k \in \mathbb{N}$ and a signature σ , $\sigma^{(k)}$ is the signature consisting of the same symbols as σ such that each symbol R of arity r in σ has arity r^k in $\sigma^{(k)}$.

Definition 10. The *k*-th tensor power of a σ -structure **A** is the $\sigma^{(k)}$ -structure $\mathbf{A}^{(k)}$ having domain A^k and relations defined as follows: For each symbol $R \in \sigma$ of arity r in σ , we set $R^{\mathbf{A}^{(k)}} = \{\mathbf{a}^{(k)} : \mathbf{a} \in R^{\mathbf{A}}\}$, where, for $\mathbf{a} \in R^{\mathbf{A}}$, $\mathbf{a}^{(k)}$ is the tensor in $\mathcal{T}^{r \cdot \mathbf{1}_k}(A^k)$ defined as follows: For any $(i_1, i_2, \ldots, i_k) \in [r]^k$, the (i_1, i_2, \ldots, i_k) -th element of $\mathbf{a}^{(k)}$ is $(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$.^{5,6}

Notice that $\mathbf{A}^{(1)} = \mathbf{A}$. Also, the function $R^{\mathbf{A}} \to R^{\mathbf{A}^{(k)}}$ given by $\mathbf{a} \mapsto \mathbf{a}^{(k)}$ is a bijection, so the cardinality of $R^{\mathbf{A}^{(k)}}$ equals the cardinality of $R^{\mathbf{A}}$.

Example 11. Let us describe the third tensor power of the 3-clique – i.e., the structure $\mathbf{K}_{3}^{(3)}$. The domain of $\mathbf{K}_{3}^{(3)}$ is [3]³, i.e., the set of tuples of elements in [3] having length 3. Let R be the symbol corresponding to the binary edge relation in \mathbf{K}_{3} , so that $R^{\mathbf{K}_{3}} = \{(1,2), (2,1), (2,3), (3,2), (3,1), (1,3)\}$. Then, $R^{\mathbf{K}_{3}^{(3)}}$ has arity $2^{3} = 8$ and it is a subset of $\mathcal{T}^{2\cdot\mathbf{1}_{3}}([3]^{3})$. Specifically, $R^{\mathbf{K}_{3}^{(3)}} = \{(1,2)^{(3)}, (2,1)^{(3)}, (2,3)^{(3)}, (3,2)^{(3)}, (3,1)^{(3)}, (1,3)^{(3)}\}$ where, e.g., $(2,3)^{(3)} = \begin{bmatrix} (2,2,2) & (2,2,3) \\ (2,3,2) & (2,3,3) \end{bmatrix} \begin{bmatrix} (3,2,2) & (3,2,3) \\ (3,3,2) & (3,3,3) \end{bmatrix}$.

We say that a σ -structure **A** is *k*-enhanced if σ contains a *k*-ary symbol R_k such that $R_k^{\mathbf{A}} = A^k$. Observe that any two σ -structures **A** and **B** are homomorphic if and only if the structures $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ obtained by adding R_k to their signatures are homomorphic. Hence,

 $^{^{4}}$ See Section 4.1 for further details on the terminology for tensors.

⁵Using the terminology for tensors that we shall introduce in Section 4.1, $\mathbf{a}^{(k)}$ can be more compactly defined as follows: $E_{\mathbf{i}} * \mathbf{a}^{(k)} = \mathbf{a}_{\mathbf{i}}$ for any $\mathbf{i} \in [r]^{k}$.

⁶We can visualise $\mathbf{a}^{(k)}$ as the formal Segre outer product of k copies of \mathbf{a} (cf. [85]).

⁷The vertical line separates the two 2×2 layers of the $2 \times 2 \times 2$ tensor.

 $PCSP(\mathbf{A}, \mathbf{B})$ is equivalent to $PCSP(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$, and considering k-enhanced structures results in no loss of generality. We now give the main definition of this work.

Definition 12. For a minion \mathscr{M} and an integer $k \in \mathbb{N}$, the k-th level of the minion test Test_ \mathscr{M} , denoted by Test_ \mathscr{M} , is the computational problem defined as follows: Given two k-enhanced σ -structures **X** and **A**, return YES if $\mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$, and No otherwise.

Comparing Definition 12 with Definition 3, we see that $\text{Test}_{\mathscr{M}}^{k}(\mathbf{X}, \mathbf{A}) = \text{Test}_{\mathscr{M}}(\mathbf{X}^{(\Bbbk)}, \mathbf{A}^{(\Bbbk)})$. In other words, the *k*-th level of a minion test is just the minion test applied to the tensor power of the structures. We have seen (cf. Proposition 5) that a minion test is always complete. It turns out that this property keeps holding for any level of a minion test.

Proposition 13. Test^k_{\mathcal{M}} is complete for any minion \mathcal{M} and any integer $k \in \mathbb{N}$.

The proof of Proposition 13 relies on the fact that homomorphisms between structures are in some sense invariant under the tensorisation construction, as formally stated in Proposition 28.

It is well known that each of the hierarchies of relaxations mentioned in Section 2 has the property that higher levels are at least as powerful as lower levels. As the next result shows, this is in fact a property of all hierarchies of minion tests.

Proposition 14. Let \mathscr{M} be a minion, let $k, p \in \mathbb{N}$ be such that k > p, and let \mathbf{X}, \mathbf{A} be two k- and p-enhanced σ -structures. If $\operatorname{Test}^k_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$ then $\operatorname{Test}^p_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$.

It follows from Proposition 14 that, if some level of a minion test is sound for a template (\mathbf{A}, \mathbf{B}) (equivalently, if it solves $PCSP(\mathbf{A}, \mathbf{B})$), then any higher level is sound for (\mathbf{A}, \mathbf{B}) (equivalently, it solves $PCSP(\mathbf{A}, \mathbf{B})$).

The next theorem is the second main result of this paper. It shows that the framework defined above is general enough to capture each of the five hierarchies for (P)CSPs mentioned in Section 2.

Theorem 15 (Informal). If $k \in \mathbb{N}$ is at least the maximum arity of the template,

3.3 Linear minions

Certain features of the hierarchies of minion tests from Definition 12 – in particular, the fact that they are complete (Proposition 13) and progressively stronger (Proposition 14) – hold true for any minion, as they only depend on basic properties of the tensorisation construction. In order to prove Theorem 15, however, it is necessary to dig deeper by investigating how the tensorisation construction interacts with the free structure. In other words, we need

to understand the object $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$. To that end, we isolate a property shared by all minions mentioned in this work: Their objects can be interpreted as matrices, and their minor operations can be expressed as matrix multiplications. We call such minions *linear*.

Definition 16. A minion \mathscr{M} is *linear* if there exists a semiring \mathcal{S} with additive identity $0_{\mathcal{S}}$ and multiplicative identity $1_{\mathcal{S}}$ and a number $d \in \mathbb{N} \cup \{\aleph_0\}$ (called *depth*) such that

- 1. the elements of $\mathcal{M}^{(L)}$ are $L \times d$ matrices whose entries belong to \mathcal{S} , for each $L \in \mathbb{N}$;
- 2. given $L, L' \in \mathbb{N}, \pi : [L] \to [L']$, and $M \in \mathscr{M}^{(L)}, M_{/\pi} = PM$, where P is the $L' \times L$ matrix such that, for $i \in [L']$ and $j \in [L]$, the (i, j)-th entry of P is $1_{\mathcal{S}}$ if $\pi(j) = i$, and $0_{\mathcal{S}}$ otherwise.

Observe that pre-multiplying a matrix M by P amounts to performing a combination of the following three elementary operations to the rows of M: swapping two rows, replacing two rows with their sum, and inserting a zero row. Hence, we may equivalently define a linear minion as a collection of matrices over S that is closed under such elementary operations.

As illustrated in the next proposition, the family of linear minions is rich enough to include the minions associated with all minion tests studied in the literature on PCSPs, including SDP.

Proposition 17. The following minions are linear:⁸

- \mathscr{H} , with $\mathcal{S} = (\{0,1\}, \lor, \land)$ and d = 1 $\mathscr{Q}_{\text{conv}}$, with $\mathcal{S} = \mathbb{Q}$ and d = 1
- \mathscr{Z}_{aff} , with $\mathcal{S} = \mathbb{Z}$ and d = 1
- \mathcal{M}_{BA} , with $\mathcal{S} = \mathbb{Q}$ and d = 2.
- \mathscr{S} , with $\mathcal{S} = \mathbb{R}$ and $d = \aleph_0$

Recall that, as per Definition 3, the minion test associated with a minion \mathscr{M} works by checking whether a given instance is homomorphic to the free structure of \mathscr{M} ; in other words, Test \mathscr{M} for a template (\mathbf{A}, \mathbf{B}) is essentially $\operatorname{CSP}(\mathbb{F}_{\mathscr{M}}(\mathbf{A}))$. It is then worth checking what the latter object looks like in the case that \mathscr{M} is linear. The next remark shows that, in this case, $\mathbb{F}_{\mathscr{M}}(\mathbf{A})$ has a simple matrix-theoretic description.

Remark 18. Given a linear minion \mathcal{M} with semiring \mathcal{S} and depth d, and a σ -structure \mathbf{A} , the free structure $\mathbb{F}_{\mathcal{M}}(\mathbf{A})$ of \mathcal{M} generated by \mathbf{A} has the following description:

- The elements of its domain $\mathscr{M}^{(|A|)}$ are $|A| \times d$ matrices having entries in \mathcal{S} .
- For $R \in \sigma$ of arity r, the elements of $R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A})}$ are tuples of the form (P_1Q, \ldots, P_rQ) , where $Q \in \mathscr{M}^{(|R^{\mathbf{A}}|)}$ is a $|R^{\mathbf{A}}| \times d$ matrix having entries in \mathcal{S} and, for $i \in [r]$, P_i is the $|A| \times |R^{\mathbf{A}}|$ matrix whose (a, \mathbf{a}) -th entry is $1_{\mathcal{S}}$ if $a_i = a$, and $0_{\mathcal{S}}$ otherwise.⁹

⁸It is not hard to verify that also the minion \mathscr{C} capturing the power of the CLAP algorithm from [44] is linear, with $\mathcal{S} = \mathbb{Q}$ and $d = \aleph_0$.

⁹We shall often write 0 and 1 for 0_S and 1_S to avoid cumbersome notation. The relevant semiring S will always be clear from the context.



Figure 1: A tensor $M \in \mathbb{R}^{\mathbf{F}}$ from Example 19, corresponding to the uniform distribution on the set of edges of \mathbf{K}_3 . The opacity of a cell is proportional to the value of the corresponding entry: $\mathbf{T} = \frac{1}{3}$, $\mathbf{T} = \frac{1}{6}$, $\mathbf{T} = 0$.

3.4 Multilinear tests

We say that a test is *multilinear* if it can be expressed as $\text{Test}_{\mathscr{M}}^k$ for some linear minion \mathscr{M} and some integer k. In the same way as, for a template (\mathbf{A}, \mathbf{B}) , $\text{Test}_{\mathscr{M}}$ is essentially $\text{CSP}(\mathbb{F}_{\mathscr{M}}(\mathbf{A}))$, it follows from Definition 12 that $\text{Test}_{\mathscr{M}}^k$ is essentially $\text{CSP}(\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)}))$, as it checks for the existence of a homomorphism between the tensor power of the instance and the free structure of \mathscr{M} generated by the tensor power of \mathbf{A} .

In Section 8, we show that, if \mathscr{M} is linear, $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ is a space of tensors endowed with relations that can be described through a tensor operation called *contraction*. Hence, the matrix-theoretic description in Remark 18 is naturally extended to a tensor-theoretic description. To give a first glance of this object, we illustrate below the structure of $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ in the case that $\mathscr{M} = \mathscr{Q}_{\text{conv}}, k = 3$, and $\mathbf{A} = \mathbf{K}_3$.

Example 19. Let us denote $\mathbb{F}_{\mathscr{Q}_{\text{conv}}}(\mathbf{K}_3^{(3)})$ by **F**. The domain of **F** is the set of nonnegative tensors in $\mathcal{T}^{3\cdot\mathbf{1}_3}(\mathbb{Q})$ whose entries sum up to 1. The relation $R^{\mathbf{F}}$ is the set of those tensors $M \in \mathcal{T}^{2\cdot\mathbf{1}_3}(\mathcal{T}^{3\cdot\mathbf{1}_3}(\mathbb{Q})) = \mathcal{T}^{6\cdot\mathbf{1}_3}(\mathbb{Q})$ such that there exists a stochastic vector $\mathbf{q} = (q_1, \ldots, q_6) \in \mathscr{Q}_{\text{conv}}^{(6)}$ (which should be interpreted as a probability distribution over the elements of $R^{\mathbf{K}_3}$, i.e., over the directed edges in \mathbf{K}_3) for which the **i**-th block $M_{\mathbf{i}}$ of M satisfies $M_{\mathbf{i}} = \mathbf{q}_{/\pi_{\mathbf{i}}}$ for each $\mathbf{i} \in [2]^3$. It will follow from the results in Section 8 that, for example,

Figure 1 illustrates the tensor $M \in R^{\mathbf{F}}$ corresponding to the uniform distribution $\mathbf{q} = \frac{1}{6} \cdot \mathbf{1}_{6}$.

In Section 8, we investigate the geometry of $\mathbb{F}_{\mathscr{M}}(\mathbf{A})$ for a linear minion \mathscr{M} . As hinted by Example 19, we shall see that this object is a space of sparse tensors, whose nonzero entries form regular patterns (cf. Lemma 31). This feature becomes more evident for higher values of the level k. In turn, the geometry of $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ is reflected in the properties of the homomorphisms ξ from $\mathbf{X}^{(k)}$ to it – which, by virtue of Definition 12, are precisely the solutions sought by $\text{Test}_{\mathscr{M}}^k$. For instance, Proposition 36 distils the requirements of the BW^k, SA^k , AIP^k , SoS^k , and BA^k hierarchies enforcing compatibility between partial assignments¹⁰ from \mathbf{X} to \mathbf{A} into the single tensor equation

$$\xi(\mathbf{x}_{\mathbf{i}}) = \prod_{\mathbf{i}} \overset{k}{*} \xi(\mathbf{x})$$

For k = 1, the equation is vacuous, since in this case $\Pi_{\mathbf{i}}$ is the identity matrix and $\mathbf{x}_{\mathbf{i}} = \mathbf{x}$ (cf. Section 4.1 for the notation). As k increases, it produces a progressively richer system of symmetries that must be satisfied by ξ , which corresponds to a progressively stronger relaxation. Concretely, we shall use all results obtained in Section 8 on the geometry of $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ to prove Theorem 15.

3.5 Conic minions

A primary message of this work is that the tensorisation construction establishes a correspondence between the algebraic properties of a minion and the algorithmic properties of the hierarchy of tests built on the minion. For example, we have seen that if the minion is linear some general properties of the solutions of the hierarchy can be deduced by studying the geometry of $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$. Now, the bounded width hierarchy has the property that it only seeks assignments that are partial homomorphisms; similarly, the Sherali-Adams, Sum-of-Squares, and BA^k hierarchies only assign a positive weight to solutions satisfying local constraints. The next definition identifies the minion property guaranteeing this algorithmic feature.

Definition 20. A linear minion \mathscr{M} of depth d is *conic* if, for any $L \in \mathbb{N}$ and for any $M \in \mathscr{M}^{(L)}$, (i) $M \neq O_{L,d}$, and (ii) for any $V \subseteq [L]$, the following implication is true:¹¹

$$\sum_{i \in V} M^T \mathbf{e}_i = \mathbf{0}_d \quad \Rightarrow \quad M^T \mathbf{e}_i = \mathbf{0}_d \quad \forall i \in V.$$

Paraphrasing Definition 20, a linear minion \mathscr{M} is conic if any matrix in \mathscr{M} is nonzero and has the property that, whenever some of its rows sum up to the zero vector, each of those rows is the zero vector. All minions appearing in Proposition 17 are conic, with the exception of \mathscr{Z}_{aff} .

Proposition 21. \mathscr{H} , \mathscr{Q}_{conv} , \mathscr{S} , and \mathscr{M}_{BA} are conic minions,¹² while \mathscr{Z}_{aff} is not.

It turns out that this simple property guarantees that the hierarchies of tests built on conic minions only look at assignments yielding partial homomorphisms (cf. Proposition 38). It also follows that conic hierarchies are not fooled by small instances: Proposition 39 establishes that the k-th level of such hierarchies is able to correctly classify instances on k (or fewer) elements¹³ – as it is well known for the bounded width, Sherali-Adams, and Sum-of-Squares hierarchies. Moreover, we shall see that any linear minion can be transformed into a conic minion – whose hierarchy enjoys the features mentioned above – via the *semi-direct product* construction (cf. Proposition 40).

 $^{^{10}}$ Cf. the "closure under restriction" property of BW^k and the requirements a^2 and a^3 in Section 4.2.

¹¹As usual, the sum, product, 0, and 1 operations appearing in this definition are to be meant in the semiring S associated with the linear minion \mathcal{M} .

 $^{^{12}}$ The minion \mathscr{C} associated with the CLAP algorithm from [44] can be easily shown to be conic as well. 13 We may informally express this fact by saying that conic hierarchies are "sound in the limit".

Organisation The rest of the paper contains full details and proofs of the statements presented in this Overview. Section 4 includes relevant terminology for tensors as well as a description of the relaxations and hierarchies used throughout the paper (see also Appendix A). Section 5 gives some basic results on minion tests. Section 6 shows that the power of SDP is captured by the minion \mathscr{S} . Sections 7, 8, and 9 are the technical core of the paper; they provide a description of hierarchies of tests built on arbitrary minions, linear minions, and conic minions, respectively. Section 10 describes the semi-direct product of minions, needed to capture the BA^k hierarchy. The machinery assembled in the previous sections is finally used in Section 11 to prove Theorem 15.

4 Preliminaries

4.1 Terminology for tensors

Tuples Given a set S, two integers $k, \ell \in \mathbb{N}$, a tuple $\mathbf{s} = (s_1, \ldots, s_k) \in S^k$, and a tuple $\mathbf{i} = (i_1, \ldots, i_\ell) \in [k]^\ell$, \mathbf{s}_i shall denote the *projection* of \mathbf{s} onto \mathbf{i} , i.e., the tuple in S^ℓ defined by $\mathbf{s}_i = (s_{i_1}, \ldots, s_{i_\ell})$. Given two tuples $\mathbf{s} = (s_1, \ldots, s_k) \in S^k$ and $\tilde{\mathbf{s}} = (\tilde{s}_1, \ldots, \tilde{s}_\ell) \in S^\ell$, their *concatenation* is the tuple $(\mathbf{s}, \tilde{\mathbf{s}}) = (s_1, \ldots, s_k, \tilde{s}_1, \ldots, \tilde{s}_\ell) \in S^{k+\ell}$. We also define $\{\mathbf{s}\} = \{s_1, \ldots, s_k\}$. Given two sets S, \tilde{S} and two tuples $\mathbf{s} = (s_1, \ldots, s_k) \in S^k$, $\tilde{\mathbf{s}} = (\tilde{s}_1, \ldots, \tilde{s}_k) \in \tilde{S}^k$, we write $\mathbf{s} \prec \tilde{\mathbf{s}}$ if, for any $\alpha, \beta \in [k], s_\alpha = s_\beta$ implies $\tilde{s}_\alpha = \tilde{s}_\beta$. The expression $\mathbf{s} \not\prec \tilde{\mathbf{s}}$ shall mean the negation of $\mathbf{s} \prec \tilde{\mathbf{s}}$. Notice that the relation " \prec " is preserved under projections: If $\mathbf{s} \prec \tilde{\mathbf{s}}$ and $\mathbf{i} \in [k]^\ell$, then $\mathbf{s}_i \prec \tilde{\mathbf{s}}_i$.

Semirings A semiring S consists of a set S equipped with two binary operations "+" and "." such that

- (S, +) is a commutative monoid with an identity element " 0_S " (i.e., (r+s)+t = r+(s+t), $0_S + r = r + 0_S = r$, and r + s = s + r);
- (S, \cdot) is a monoid with an identity element " $1_{\mathcal{S}}$ " (i.e., $(r \cdot s) \cdot t = r \cdot (s \cdot t)$ and $1_{\mathcal{S}} \cdot r = r \cdot 1_{\mathcal{S}} = r$);
- "." distributes over "+" (i.e., $r \cdot (s+t) = (r \cdot s) + (r \cdot t)$ and $(r+s) \cdot t = (r \cdot t) + (s \cdot t)$);
- " $0_{\mathcal{S}}$ " is a multiplicative absorbing element (i.e., $0_{\mathcal{S}} \cdot r = r \cdot 0_{\mathcal{S}} = 0_{\mathcal{S}}$).

Examples of semirings are \mathbb{Z} , \mathbb{Q} , and \mathbb{R} with the usual addition and multiplication operations, or the Boolean semiring $(\{0,1\}, \lor, \land)$.

Let V be a finite set, and choose an element $s_v \in S$ for each $v \in V$. We let the formal expression $\sum_{v \in V} s_v$ equal 0_S if $V = \emptyset$. To increase the readability, we shall usually write 0 and 1 for 0_S and 1_S ; the relevant semiring S will always be clear from the context.

Tensors As anticipated in Section 3, given a set S, an integer $k \in \mathbb{N}$, and a tuple $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, by $\mathcal{T}^{\mathbf{n}}(S)$ we denote the set of functions from $[n_1] \times \cdots \times [n_k]$ to S, which we visualise as hypermatrices or tensors having k modes, where the *i*-th mode has size n_i for $i \in [k]$. If $\mathbf{n} = n \cdot \mathbf{1}_k = (n, \ldots, n)$ is a constant tuple, $\mathcal{T}^{\mathbf{n}}(S)$ is a set of *cubical* tensors, each of whose modes has the same length n. For example, if $\mathbf{n} = n \cdot \mathbf{1}_2 = (n, n)$, $\mathcal{T}^{\mathbf{n}}(S)$ is the set of $n \times n$ matrices having entries in S. We sometimes denote an element of $\mathcal{T}^{\mathbf{n}}(S)$ by $T = (t_i)$,

where $\mathbf{i} \in [n_1] \times \cdots \times [n_k]$ and $t_{\mathbf{i}}$ is the image of \mathbf{i} under T. Moreover, given two tuples $\mathbf{n} \in \mathbb{N}^k$ and $\tilde{\mathbf{n}} \in \mathbb{N}^\ell$, we sometimes write $\mathcal{T}^{\mathbf{n},\tilde{\mathbf{n}}}(S)$ for $\mathcal{T}^{(\mathbf{n},\tilde{\mathbf{n}})}(S)$, where $(\mathbf{n},\tilde{\mathbf{n}})$ is the concatenation of \mathbf{n} and $\tilde{\mathbf{n}}$. Whenever $k \geq 2$ and $n_i = 1$ for some $i \in [k]$, we can (and will) identify $\mathcal{T}^{\mathbf{n}}(S)$ with $\mathcal{T}^{\hat{\mathbf{n}}}(S)$, where $\hat{\mathbf{n}} \in \mathbb{N}^{k-1}$ is obtained from \mathbf{n} by deleting the *i*-th entry.

Contraction Take a semiring \mathcal{S} . For $k, \ell, m \in \mathbb{N}$, take $\mathbf{n} \in \mathbb{N}^k$, $\mathbf{p} \in \mathbb{N}^\ell$, and $\mathbf{q} \in \mathbb{N}^m$. The contraction of two tensors $T = (t_i) \in \mathcal{T}^{\mathbf{n},\mathbf{p}}(\mathcal{S})$ and $\tilde{T} = (\tilde{t}_i) \in \mathcal{T}^{\mathbf{p},\mathbf{q}}(\mathcal{S})$, denoted by $T \stackrel{\ell}{*} \tilde{T}$, is the tensor in $\mathcal{T}^{\mathbf{n},\mathbf{q}}(\mathcal{S})$ such that, for $\mathbf{i} \in [n_1] \times \cdots \times [n_k]$ and $\mathbf{j} \in [q_1] \times \cdots \times [q_m]$, the (\mathbf{i},\mathbf{j}) -th entry of $T \stackrel{\ell}{*} \tilde{T}$ is given by

$$\sum_{\mathbf{z} \in [p_1] \times \dots \times [p_\ell]} t_{(\mathbf{i}, \mathbf{z})} \tilde{t}_{(\mathbf{z}, \mathbf{j})}$$

(where the addition and multiplication are meant in the semiring S). This notation straightforwardly extends to the cases when k = 0 or m = 0, i.e., when we are contracting over all modes of T or \tilde{T} . In such cases, we write $T * \tilde{T}$ for $T \stackrel{\ell}{*} \tilde{T}$. We shall use the convention of always contracting from the left to the right, so that the expression $T_1 * T_2 * T_3$ is a synonym for the expression $(T_1 * T_2) * T_3$. The reason for this is that the contraction operation is not associative in general. For instance, for T and \tilde{T} as above and $\hat{T} \in \mathcal{T}^{\mathbf{n},\mathbf{q}}(S)$, the expression $(T \stackrel{\ell}{*} \tilde{T}) \stackrel{k+m}{*} \hat{T} = (T \stackrel{\ell}{*} \tilde{T}) * \hat{T}$ is well defined, while changing the order of the contractions results in an expression that is not well defined in general. On the other hand, it is not hard to check that the order in which the contractions are performed is irrelevant if the contractions are taken over disjoint sets of modes.

Example 22. Given two vectors $\mathbf{u}, \mathbf{v} \in \mathcal{T}^p(\mathbb{R})$ and two matrices $M \in \mathcal{T}^{n,p}(\mathbb{R}), N \in \mathcal{T}^{p,q}(\mathbb{R})$, we have that $\mathbf{u} \stackrel{1}{*} \mathbf{v} = \mathbf{u} * \mathbf{v} = \mathbf{u}^T \mathbf{v}$ (the dot product of \mathbf{u} and \mathbf{v}), $M \stackrel{1}{*} \mathbf{u} = M * \mathbf{u} = M \mathbf{u}$ (the matrix-vector product of M and \mathbf{u}), and $M \stackrel{1}{*} N = MN$ (the matrix product of M and N).

For $\mathbf{i} \in [n_1] \times \cdots \times [n_k]$, we denote by $E_{\mathbf{i}}$ the \mathbf{i} -th standard unit tensor; i.e., the tensor in $\mathcal{T}^{\mathbf{n}}(\mathcal{S})$ all of whose entries are $0_{\mathcal{S}}$, except the \mathbf{i} -th entry that is $1_{\mathcal{S}}$. Given $T \in \mathcal{T}^{\mathbf{n}}(\mathcal{S})$, notice that $E_{\mathbf{i}} * T = E_{\mathbf{i}} * T$ is the \mathbf{i} -th entry of T. The support of T is the set of indices of all nonzero entries of T; i.e., the set $\operatorname{supp}(T) = {\mathbf{i} \in [n_1] \times \cdots \times [n_k] : E_{\mathbf{i}} * T \neq 0_{\mathcal{S}}}.$

Remark 23. The definition of tensors straightforwardly extends to the case that $n_i = \aleph_0$ for some $i \in [k]$.

4.2 Relaxations and hierarchies

In this section, we define the standard semidefinite programming relaxation as well as the bounded width, Sherali-Adams LP, affine integer programming, Sum-of-Squares, and combined basic linear programming and affine integer programming hierarchies. We refer to Appendix A for a more detailed discussion, as well as for a comparison with different formulations of these algorithms appearing in the literature on (P)CSPs.

Given two σ -structures **A** and **B** and a subset $S \subseteq A$, a partial homomorphism from **A** to **B** with domain S is a homomorphism from $\mathbf{A}[S]$ to **B**, where $\mathbf{A}[S]$ is the substructure of **A** induced by S – i.e., it is the σ -structure whose domain is S and, for any $R \in \sigma$, $R^{\mathbf{A}[S]} = R^{\mathbf{A}} \cap S^{\operatorname{ar}(R)}$. Recall that, for $k \in \mathbb{N}$, we say that a σ -structure **A** is k-enhanced if the signature σ contains a k-ary symbol R_k such that $R_k^{\mathbf{A}} = A^k$. **BW^k** Given two σ -structures **X**, **A**, we say that the k-th level of the bounded width algorithm accepts when applied to **X** and **A**, and we write BW^k(**X**, **A**) = YES, if there exists a nonempty collection \mathcal{F} of partial homomorphisms from **X** to **A** with at most k-element domains such that (i) \mathcal{F} is closed under restrictions, i.e., for every $f \in \mathcal{F}$ and every $V \subseteq \text{dom}(f), f|_V \in \mathcal{F}$, and (ii) \mathcal{F} has the extension property up to k, i.e., for every $f \in \mathcal{F}$ and every $V \subseteq X$ with $|V| \leq k$ and $\text{dom}(f) \subseteq V$, there exists $g \in \mathcal{F}$ such that g extends f and dom(g) = V.

We say that BW^k solves a PCSP template (\mathbf{A}, \mathbf{B}) if $\mathbf{X} \to \mathbf{B}$ whenever $BW^k(\mathbf{X}, \mathbf{A}) = YES$. (Note that the algorithm is always complete: If $\mathbf{X} \to \mathbf{A}$ then $BW^k(\mathbf{X}, \mathbf{A}) = YES$.)

 $\mathbf{SA}^{\mathbf{k}}$, $\mathbf{AIP}^{\mathbf{k}}$, and $\mathbf{BA}^{\mathbf{k}}$ Given two k-enhanced σ -structures \mathbf{X} , \mathbf{A} , we introduce a variable $\lambda_{R,\mathbf{x},\mathbf{a}}$ for every $R \in \sigma$, $\mathbf{x} \in R^{\mathbf{X}}$, $\mathbf{a} \in R^{\mathbf{A}}$. Consider the following system of equations:

$$(\clubsuit1) \sum_{\mathbf{a}\in R^{\mathbf{A}}} \lambda_{R,\mathbf{x},\mathbf{a}} = 1 \qquad R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}$$
$$(\clubsuit2) \sum_{\mathbf{a}\in R^{\mathbf{A}},\mathbf{a}_{\mathbf{i}}=\mathbf{b}} \lambda_{R,\mathbf{x},\mathbf{a}} = \lambda_{R_{k},\mathbf{x}_{\mathbf{i}},\mathbf{b}} \quad R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, \mathbf{i} \in [\mathrm{ar}(R)]^{k}, \mathbf{b} \in A^{k}$$
(2)
$$(\clubsuit3) \quad \lambda_{R,\mathbf{x},\mathbf{a}} = 0 \qquad R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, \mathbf{a} \in R^{\mathbf{A}}, \mathbf{x} \not\prec \mathbf{a}.^{14}$$

We say that the k-th level of the Sherali-Adams linear programming hierarchy accepts when applied to **X** and **A**, and we write $SA^k(\mathbf{X}, \mathbf{A}) = YES$, if the system (2) admits a solution such that all variables take rational nonnegative values. Similarly, we say that the k-th level of the affine integer programming hierarchy accepts when applied to **X** and **A**, and we write $AIP^k(\mathbf{X}, \mathbf{A}) = YES$, if the system above admits a solution such that all variables take integer values. Moreover, we say that the k-th level of the combined basic linear programming and affine integer programming hierarchy accepts when applied to **X** and **A**, and we write $BA^k(\mathbf{X}, \mathbf{A}) = YES$, if the system above admits both a solution such that all variables take rational nonnegative values and a solution such that all variables take integer values, and the following refinement condition holds: Denoting the rational nonnegative and the integer solutions by the superscripts (B) and (A), respectively, we require that $\lambda_{R,\mathbf{x},\mathbf{a}}^{(A)} = 0$ whenever $\lambda_{R,\mathbf{x},\mathbf{a}}^{(B)} = 0$, for each $R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, \mathbf{a} \in R^{\mathbf{A}}$.

We say that SA^k solves a PCSP template (\mathbf{A}, \mathbf{B}) if $\mathbf{X} \to \mathbf{B}$ whenever $SA^k(\mathbf{X}, \mathbf{A}) = YES$. The definition for AIP^k and BA^k is analogous. (Note that the three algorithms are always complete: If $\mathbf{X} \to \mathbf{A}$ then $SA^k(\mathbf{X}, \mathbf{A}) = AIP^k(\mathbf{X}, \mathbf{A}) = BA^k(\mathbf{X}, \mathbf{A}) = YES$.)

SDP Given two σ -structures \mathbf{X}, \mathbf{A} , let $\omega = |X| \cdot |A| + \sum_{R \in \sigma} |R^{\mathbf{X}}| \cdot |R^{\mathbf{A}}|$. We introduce a variable $\lambda_{x,a}$ taking values in \mathbb{R}^{ω} for every $x \in X$, $a \in A$, and a variable $\lambda_{R,\mathbf{x},\mathbf{a}}$ taking values

¹⁴The condition $\mathbf{x} \not\prec \mathbf{a}$ formalises the requirement that the same variables (elements of \mathbf{x}) should not be assigned different values (elements of \mathbf{a}). Some papers avoid this requirement by imposing that (P)CSP instances should have no repetition of variables in constraints scopes; i.e., elements of \mathbf{x} are all distinct.

in \mathbb{R}^{ω} for every $R \in \sigma, \mathbf{x} \in \mathbb{R}^{\mathbf{X}}, \mathbf{a} \in \mathbb{R}^{\mathbf{A}}$. Consider the following system of equations:

$$(\bigstar 1) \sum_{a \in A} \|\lambda_{x,a}\|^2 = 1 \qquad x \in X$$

$$(\bigstar 2) \quad \lambda_{x,a} \cdot \lambda_{x,a'} = 0 \qquad x \in X, a \neq a' \in A$$

$$(\bigstar 3) \quad \lambda_{R,\mathbf{x},\mathbf{a}} \cdot \lambda_{R,\mathbf{x},\mathbf{a}'} = 0 \qquad R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, \mathbf{a} \neq \mathbf{a}' \in R^{\mathbf{A}}$$

$$(\bigstar 4) \quad \sum_{\substack{\mathbf{a} \in R^{\mathbf{A}} \\ a_i = a}} \lambda_{R,\mathbf{x},\mathbf{a}} = \lambda_{x_i,a} \qquad R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, a \in A, i \in [\operatorname{ar}(R)].$$

$$(\bigstar 4) \quad \sum_{\substack{\mathbf{a} \in R^{\mathbf{A}} \\ a_i = a}} \lambda_{R,\mathbf{x},\mathbf{a}} = \lambda_{x_i,a} \qquad R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, a \in A, i \in [\operatorname{ar}(R)].$$

We say that the standard semidefinite programming relaxation accepts when applied to \mathbf{X} and \mathbf{A} , and we write $\text{SDP}(\mathbf{X}, \mathbf{A}) = \text{YES}$, if the system (3) admits a solution. We say that SDP solves a PCSP template (\mathbf{A}, \mathbf{B}) if $\mathbf{X} \to \mathbf{B}$ whenever $\text{SDP}(\mathbf{X}, \mathbf{A}) = \text{YES}$. (Note that the algorithm is always complete: If $\mathbf{X} \to \mathbf{A}$ then $\text{SDP}(\mathbf{X}, \mathbf{A}) = \text{YES}$.)

SoS^k Given two k-enhanced σ -structures **X**, **A**, let $\omega = \sum_{R \in \sigma} |R^{\mathbf{X}}| \cdot |R^{\mathbf{A}}|$. We introduce a variable $\lambda_{R,\mathbf{x},\mathbf{a}}$ taking values in \mathbb{R}^{ω} for every $R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, \mathbf{a} \in R^{\mathbf{A}}$. Consider the following system of equations:

We say that the k-th level of the Sum-of-Squares semidefinite programming hierarchy accepts when applied to \mathbf{X} and \mathbf{A} , and we write $\mathrm{SoS}^k(\mathbf{X}, \mathbf{A}) = \mathrm{YES}$, if the system (4) admits a solution. We say that SoS^k solves a PCSP template (\mathbf{A}, \mathbf{B}) if $\mathbf{X} \to \mathbf{B}$ whenever $\mathrm{SoS}^k(\mathbf{X}, \mathbf{A}) =$ YES. (Note that the algorithm is always complete: If $\mathbf{X} \to \mathbf{A}$ then $\mathrm{SoS}^k(\mathbf{X}, \mathbf{A}) = \mathrm{YES}$.)

5 Minion tests

In this section, we present some basic results on minion tests (introduced in Definition 3). In particular, we show that a minion test $\text{Test}_{\mathscr{M}}$ is always complete, and its power can be characterised algebraically through the existence of a homomorphism from \mathscr{M} to the polymorphism minion of the template.

We start off with a simple lemma, implicitly proved in [12] for the case of minions of operations.

Lemma 24. Let \mathscr{M} be a minion and let \mathbf{A} be a σ -structure. Then, $\mathbf{A} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A})$.

Proof. Take a unary element $M \in \mathcal{M}^{(1)}$, and consider the map

$$f: A \to \mathscr{M}^{(n)}$$
$$a \mapsto M_{/\rho_a}$$

where $\rho_a : [1] \to [n] = A$ is defined by $\rho_a(1) = a$. Take $R \in \sigma$ of arity r, and consider a tuple $\mathbf{a} = (a_1, \ldots, a_r) \in R^{\mathbf{A}}$. Let $m = |R^{\mathbf{A}}|$, and consider the function $\pi : [1] \to [m]$ defined

by $\pi(1) = \mathbf{a}$. Let $Q = M_{/\pi} \in \mathscr{M}^{(m)}$. For each $i \in [r]$, consider the function $\pi_i : [m] \to [n]$ defined by $\pi_i(\mathbf{b}) = b_i$, where $\mathbf{b} = (b_1, \ldots, b_r) \in \mathbb{R}^{\mathbf{A}}$. Observe that $\rho_{a_i} = \pi_i \circ \pi$ for each $i \in [r]$. We obtain

$$f(\mathbf{a}) = (f(a_1), \dots, f(a_r)) = (M_{/\rho_{a_1}}, \dots, M_{/\rho_{a_r}}) = (M_{/\pi_1 \circ \pi}, \dots, M_{/\pi_r \circ \pi})$$
$$= ((M_{/\pi})_{/\pi_1}, \dots, (M_{/\pi})_{/\pi_r}) = (Q_{/\pi_1}, \dots, Q_{/\pi_r}) \in R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A})},$$

thus showing that f is a homomorphism from A to $\mathbb{F}_{\mathscr{M}}(\mathbf{A})$.

Proposition (Proposition 5 restated). Test_{*M*} is complete for any minion *M*; i.e., for any **X** and **A** with $\mathbf{X} \to \mathbf{A}$, we have $\mathbf{X} \to \mathbb{F}_{\mathcal{M}}(\mathbf{A})$.

Proof. It immediately follows from Lemma 24 that, if **X** and **A** are two σ -structures such that $\mathbf{X} \to \mathbf{A}$, then $\mathbf{X} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A})$, thus witnessing that $\operatorname{Test}_{\mathscr{M}}(\mathbf{X}, \mathbf{A})$ accepts. \Box

We now prove Proposition 6, which establishes that the power of minion tests can be characterised algebraically. Our proof follows the lines of [12, Remark 7.13], where the same result is derived from Kőnig's Lemma for *locally countable minions* – i.e., minions \mathscr{M} having the property that $\mathscr{M}^{(L)}$ is countable for any L. Since the minion \mathscr{S} described in Definition 7 is not locally countable, we shall need this stronger version of the result when proving that \mathscr{S} provides an algebraic characterisation of the power of SDP (cf. Theorem 9).

First, we need the following result on compact structures, which is a consequence of the (uncountable version of the) compactness theorem of logic (cf. [88]). We say that a potentially infinite σ -structure **B** is compact if, for any potentially infinite σ -structure **A**, $\mathbf{A} \to \mathbf{B}$ if and only if $\mathbf{A}' \to \mathbf{B}$ for every finite substructure \mathbf{A}' of \mathbf{A} .

Theorem 25 ([92]). Every (finite) σ -structure is compact.

Proposition (Proposition 6 restated). Let \mathscr{M} be a minima and let (\mathbf{A}, \mathbf{B}) be a PCSP template. Then, Test \mathscr{M} solves PCSP (\mathbf{A}, \mathbf{B}) if and only if $\mathscr{M} \to \text{Pol}(\mathbf{A}, \mathbf{B})$.

Proof. We first observe that the condition $\mathcal{M} \to \text{Pol}(\mathbf{A}, \mathbf{B})$ is equivalent to the condition $\mathbb{F}_{\mathcal{M}}(\mathbf{A}) \to \mathbf{B}$ by [12, Lemma 4.4] (see also [44] for the proof for abstract minions).

Suppose that $\mathbb{F}_{\mathscr{M}}(\mathbf{A}) \to \mathbf{B}$. Given an instance \mathbf{X} , if $\operatorname{Test}_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$ then $\mathbf{X} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A})$, and composing the two homomorphisms yields $\mathbf{X} \to \mathbf{B}$. Hence, $\operatorname{Test}_{\mathscr{M}}$ is sound on the template (\mathbf{A}, \mathbf{B}) . Since, as noted above, $\operatorname{Test}_{\mathscr{M}}$ is always complete, we deduce that $\operatorname{Test}_{\mathscr{M}}$ solves $\operatorname{PCSP}(\mathbf{A}, \mathbf{B})$.

Conversely, suppose that $\text{Test}_{\mathscr{M}}$ solves $\text{PCSP}(\mathbf{A}, \mathbf{B})$. Let \mathbf{F} be a finite substructure of $\mathbb{F}_{\mathscr{M}}(\mathbf{A})$, and notice that the inclusion map yields a homomorphism from \mathbf{F} to $\mathbb{F}_{\mathscr{M}}(\mathbf{A})$. Hence, $\text{Test}_{\mathscr{M}}(\mathbf{F}, \mathbf{A}) = \text{YES}$, so $\mathbf{F} \to \mathbf{B}$. Since \mathbf{B} is compact by Theorem 25, we deduce that $\mathbb{F}_{\mathscr{M}}(\mathbf{A}) \to \mathbf{B}$, as required.

6 A minion for SDP

The goal of this section is to show that the relaxation SDP is a minion test and that its power can be captured algebraically.¹⁴ To this end, we start by showing that the object introduced in Definition 7 is indeed a minion. Then, it will easily follow from its construction that it is in fact a linear minion.

¹⁴We remark that a characterisation for the power of SDP, similar to the one we describe in this section, was also obtained independently by Brakensiek, Guruswami, and Sandeep in [29].

Proposition 26. \mathscr{S} is a linear minion.

Proof. We first show that \mathscr{S} is a minion. Observe that, for $\pi : [L] \to [L']$ and $M \in \mathscr{S}^{(L)}$, $M_{/\pi} = PM \in \mathcal{T}^{L',\aleph_0}(\mathbb{R})$ and $\operatorname{csupp}(PM)$ is finite. One easily checks that (i) $P^T \mathbf{1}_{L'} = \mathbf{1}_L$, and (ii) PP^T is a diagonal matrix. Using that both MM^T and PP^T are diagonal, we find that $M_{/\pi}(M_{/\pi})^T = PMM^TP^T$ is diagonal, too. Moreover, since the trace of a diagonal matrix equals the sum of its entries, we obtain

$$\operatorname{tr}(M_{/\pi}(M_{/\pi})^{T}) = \mathbf{1}_{L'}^{T} M_{/\pi}(M_{/\pi})^{T} \mathbf{1}_{L'} = \mathbf{1}_{L'}^{T} P M M^{T} P^{T} \mathbf{1}_{L'} = \mathbf{1}_{L}^{T} M M^{T} \mathbf{1}_{L} = \operatorname{tr}(M M^{T}) = 1.$$

It follows that $M_{/\pi} \in \mathscr{S}^{(L')}$. Furthermore, one easily checks that $M_{/id} = M$ and, given $\tilde{\pi} : [L'] \to [L''], (M_{/\pi})_{/\tilde{\pi}} = M_{/\tilde{\pi}\circ\pi}$, which concludes the proof that \mathscr{S} is a minion. The fact that \mathscr{S} is linear directly follows from Definition 16.

Recall from Remark 18 that, for a σ -structure **A**, a symbol $R \in \sigma$ of arity r, and a number $i \in [r]$, P_i is the $|A| \times |R^{\mathbf{A}}|$ matrix whose (a, \mathbf{a}) -th entry is 1 if $a_i = a$, and 0 otherwise. In order to show that SDP is a minion test corresponding to the minion \mathscr{S} , we shall use the following simple description of the entries of P_i .

Lemma 27 ([44]). Let A be a σ -structure, let $R \in \sigma$ of arity r, and let $i \in [r], a \in A$. Then,

$$\mathbf{e}_a^T P_i = \sum_{\substack{\mathbf{a} \in R^{\mathbf{A}} \\ a_i = a}} \mathbf{e}_a^T.$$

In order to prove that SDP = Test \mathscr{S} , we essentially need to encode the vectors λ witnessing an SDP solution as rows of matrices belonging to \mathscr{S} . To this end, we need to solve the following "size problem": The vectors λ live in a vector space having a fixed, finite dimension – namely, the number $\omega = |X| \cdot |A| + \sum_{R \in \sigma} |R^{\mathbf{X}}| \cdot |R^{\mathbf{A}}|$. On the other hand, the matrices in \mathscr{S} have rows of infinite size, living in \mathbb{R}^{\aleph_0} . This issue is easily solved by considering an ω dimensional subspace of \mathbb{R}^{\aleph_0} and working in an orthonormal basis of such subspace, through a standard orthonormalisation argument.

Proposition (Proposition 8 restated). SDP = Test \mathscr{S} . In other words, given two σ -structures **X** and **A**, SDP(**X**, **A**) = YES if and only if **X** $\rightarrow \mathbb{F}_{\mathscr{S}}(\mathbf{A})$.

Proof. Suppose that SDP(**X**, **A**) = YES, and let the family of vectors $\lambda_{x,a}, \lambda_{R,\mathbf{x},\mathbf{a}} \in \mathbb{R}^{\omega}$ witness it, for $x \in X, a \in A, R \in \sigma, \mathbf{x} \in \mathbb{R}^{\mathbf{X}}, \mathbf{a} \in \mathbb{R}^{\mathbf{A}}$, where $\omega = |X| \cdot |A| + \sum_{R \in \sigma} |\mathbb{R}^{\mathbf{X}}| \cdot |\mathbb{R}^{\mathbf{A}}|$. Consider the map $\xi : X \to \mathscr{S}^{(n)}$ defined by

$$\mathbf{e}_{a}^{T}\xi(x)\mathbf{e}_{j} = \begin{cases} \mathbf{e}_{j}^{T}\boldsymbol{\lambda}_{x,a} & \text{if } j \leq \omega \\ 0 & \text{otherwise} \end{cases} \qquad x \in X, a \in A, j \in \mathbb{N}.$$

We claim that ξ is well defined. For each $x \in X$, we have $\xi(x) \in \mathcal{T}^{n,\aleph_0}(\mathbb{R})$. Moreover, $\xi(x)\mathbf{e}_j = \mathbf{0}$ for each $j > \omega$, so C1 is satisfied. Given $a, a' \in A$,

$$\mathbf{e}_{a}^{T}\xi(x)\xi(x)^{T}\mathbf{e}_{a'} = \sum_{j\in\mathbb{N}}\mathbf{e}_{a}^{T}\xi(x)\mathbf{e}_{j}\mathbf{e}_{j}^{T}\xi(x)^{T}\mathbf{e}_{a'} = \sum_{j\leq\omega}\mathbf{e}_{j}^{T}\boldsymbol{\lambda}_{x,a}\boldsymbol{\lambda}_{x,a'}^{T}\mathbf{e}_{j} = \boldsymbol{\lambda}_{x,a}\cdot\boldsymbol{\lambda}_{x,a'}.$$

If $a \neq a'$, this quantity is zero by $\blacklozenge 2$, so C2 is satisfied. Finally,

$$\operatorname{tr}(\xi(x)\xi(x)^T) = \sum_{a \in A} \mathbf{e}_a^T \xi(x)\xi(x)^T \mathbf{e}_a = \sum_{a \in A} \|\boldsymbol{\lambda}_{x,a}\|^2 = 1$$

by $\blacklozenge 1$, so C3 is also satisfied and the claim is true. We now show that ξ yields a homomorphism from **X** to $\mathbb{F}_{\mathscr{S}}(\mathbf{A})$. Take $R \in \sigma$ of arity r and $\mathbf{x} = (x_1, \ldots, x_r) \in R^{\mathbf{X}}$. Consider the matrix $Q \in \mathcal{T}^{|R^{\mathbf{A}}|,\aleph_0}(\mathbb{R})$ defined by

$$\mathbf{e}_{\mathbf{a}}^{T} Q \mathbf{e}_{j} = \begin{cases} \mathbf{e}_{j}^{T} \boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}} & \text{if } j \leq \omega \\ 0 & \text{otherwise} \end{cases} \qquad \mathbf{a} \in R^{\mathbf{A}}, j \in \mathbb{N}.$$

Using the same arguments as above, we check that Q satisfies C1 and that $\mathbf{e}_{\mathbf{a}}^{T}QQ^{T}\mathbf{e}_{\mathbf{a}'} = \lambda_{R,\mathbf{x},\mathbf{a}} \cdot \lambda_{R,\mathbf{x},\mathbf{a}'}$, so C2 follows from $\blacklozenge 3$ and C3 from point (*ii*) of Proposition 55. Therefore, $Q \in \mathscr{S}^{(|R^{\mathbf{A}}|)}$. We now claim that $\xi(x_{i}) = Q_{/\pi_{i}}$ for each $i \in [r]$. Indeed, for $a \in A$ and $j \in [\omega]$, we have

$$\mathbf{e}_{a}^{T}\xi(x_{i})\mathbf{e}_{j} = \mathbf{e}_{j}^{T}\boldsymbol{\lambda}_{x_{i},a} = \sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\a_{i}=a}}\mathbf{e}_{j}^{T}\boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}} = \sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\a_{i}=a}}\mathbf{e}_{a}^{T}Q\mathbf{e}_{j} = \mathbf{e}_{a}^{T}P_{i}Q\mathbf{e}_{j} = \mathbf{e}_{a}^{T}Q_{/\pi_{i}}\mathbf{e}_{j},$$

where the second and fourth equalities follow from $\blacklozenge 4$ and Lemma 27, respectively. Also, clearly,

$$\mathbf{e}_a^T \xi(x_i) \mathbf{e}_j = \mathbf{e}_a^T Q_{/\pi_i} \mathbf{e}_j = 0$$

if $j \in \mathbb{N} \setminus [\omega]$. As a consequence, the claim holds. It follows that $\xi(\mathbf{x}) \in R^{\mathbb{F}_{\mathscr{S}}(\mathbf{A})}$, so that ξ is a homomorphism.

Conversely, let $\xi : \mathbf{X} \to \mathbb{F}_{\mathscr{S}}(\mathbf{A})$ be a homomorphism. For $R \in \sigma$ of arity r and $\mathbf{x} = (x_1, \ldots, x_r) \in R^{\mathbf{X}}$, we can fix a matrix $Q_{R,\mathbf{x}} \in \mathscr{S}^{(|R^{\mathbf{A}}|)}$ satisfying $\xi(x_i) = Q_{R,\mathbf{x}/\pi_i}$ for each $i \in [r]$. Consider the sets $S_1 = \{\xi(x)^T \mathbf{e}_a : x \in X, a \in A\}$ and $S_2 = \{Q_{R,\mathbf{x}}^T \mathbf{e}_a : R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, \mathbf{a} \in R^{\mathbf{A}}\}$, and the vector space $\mathcal{U} = \operatorname{span}(S_1 \cup S_2) \subseteq \mathbb{R}^{\aleph_0}$. Observe that dim $\mathcal{U} \leq |S_1 \cup S_2| \leq |S_1| + |S_2| \leq |X| \cdot |A| + \sum_{R \in \sigma} |R^{\mathbf{X}}| \cdot |R^{\mathbf{A}}| = \omega$. Consider a vector space \mathcal{V} of dimension ω such that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathbb{R}^{\aleph_0}$. Using the Gram-Schmidt process,¹⁵ we find a projection matrix $Z \in \mathcal{T}^{\aleph_0,\omega}(\mathbb{R})$ such that $Z^T Z = I_\omega$ and $ZZ^T \mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in \mathcal{V}$. Consider the family of vectors

$$\lambda_{x,a} = Z^T \xi(x)^T \mathbf{e}_a \qquad x \in X, a \in A, \lambda_{R,\mathbf{x},\mathbf{a}} = Z^T Q_{R,\mathbf{x}}^T \mathbf{e}_\mathbf{a} \qquad R \in \sigma, \mathbf{x} \in R^\mathbf{X}, \mathbf{a} \in R^\mathbf{A}.$$
(5)

We claim that (5) witnesses that $SDP(\mathbf{X}, \mathbf{A}) = YES$. To check $\blacklozenge 1$, observe that

$$\sum_{a\in A} \|\boldsymbol{\lambda}_{x,a}\|^2 = \sum_{a\in A} \mathbf{e}_a^T \xi(x) Z Z^T \xi(x)^T \mathbf{e}_a = \sum_{a\in A} \mathbf{e}_a^T \xi(x) \xi(x)^T \mathbf{e}_a = \operatorname{tr}(\xi(x)\xi(x)^T) = 1,$$

where the second equality follows from the fact that $\xi(x)^T \mathbf{e}_a \in S_1 \subseteq \mathcal{U} \subseteq \mathcal{V}$ and the fourth from C3. In a similar way, using C2, we obtain

$$\boldsymbol{\lambda}_{x,a} \cdot \boldsymbol{\lambda}_{x,a'} = \mathbf{e}_a^T \xi(x) Z Z^T \xi(x)^T \mathbf{e}_{a'} = \mathbf{e}_a^T \xi(x) \xi(x)^T \mathbf{e}_{a'} = 0,$$

$$\boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}} \cdot \boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}'} = \mathbf{e}_a^T Q_{R,\mathbf{x}} Z Z^T Q_{R,\mathbf{x}}^T \mathbf{e}_{\mathbf{a}'} = \mathbf{e}_a^T Q_{R,\mathbf{x}} Q_{R,\mathbf{x}}^T \mathbf{e}_{\mathbf{a}'} = 0$$

¹⁵We note that the Gram-Schmidt process also applies to vector spaces of countably infinite dimension.

if $a \neq a' \in A$ and $\mathbf{a} \neq \mathbf{a}' \in R^{\mathbf{A}}$. This shows that $\blacklozenge 2$ and $\blacklozenge 3$ hold. Finally, to prove $\blacklozenge 4$, we observe that

$$\sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\a_{i}=a}} \lambda_{R,\mathbf{x},\mathbf{a}} = \sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\a_{i}=a}} Z^{T}Q_{R,\mathbf{x}}^{T}\mathbf{e}_{\mathbf{a}} = \left(\sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\a_{i}=a}} \mathbf{e}_{\mathbf{a}}^{T}Q_{R,\mathbf{x}}Z\right)^{T} = \left(\mathbf{e}_{a}^{T}P_{i}Q_{R,\mathbf{x}}Z\right)^{T} = \left(\mathbf{e}_{a}^{T}\xi(x_{i})Z\right)^{T}$$
$$= Z^{T}\xi(x_{i})^{T}\mathbf{e}_{a} = \boldsymbol{\lambda}_{x_{i},a},$$

where the third equality follows from Lemma 27. Therefore, the claim is true and the proof is complete. $\hfill \Box$

Using Propositions 8 and 6, we immediately have the following algebraic characterisation of the power of SDP.

Theorem (Theorem 9 restated). Let (\mathbf{A}, \mathbf{B}) be a PCSP template. Then, SDP solves PCSP (\mathbf{A}, \mathbf{B}) if and only if $\mathscr{S} \to \text{Pol}(\mathbf{A}, \mathbf{B})$.

7 Hierarchies of minion tests

In this section, we start studying the *hierarchies of minion tests* introduced in Definition 12, by describing two properties that are common to all such hierarchies: They are complete, and they become tighter as the level increases. A finer description will be available in the next sections, where we will focus on hierarchies built on minions having a *linear* or *conic* form.

The next proposition shows that the tensorisation construction described in Definition 10 does not alter whether two structures are homomorphic or not. We let $\text{Hom}(\mathbf{A}, \mathbf{B})$ denote the set of homomorphisms from \mathbf{A} to \mathbf{B} .

Proposition 28. Let $k \in \mathbb{N}$ and let \mathbf{A}, \mathbf{B} be two σ -structures. Then

- (i) $\mathbf{A} \to \mathbf{B}$ if and only if $\mathbf{A}^{(k)} \to \mathbf{B}^{(k)}$;
- (*ii*) if **A** is k-enhanced, there is a bijection $\rho : \operatorname{Hom}(\mathbf{A}, \mathbf{B}) \to \operatorname{Hom}(\mathbf{A}^{(\Bbbk)}, \mathbf{B}^{(\Bbbk)})$.

Proof. Let $f : \mathbf{A} \to \mathbf{B}$ be a homomorphism, and consider the function $f^* : A^k \to B^k$ defined by $f^*((a_1, \ldots, a_k)) = (f(a_1), \ldots, f(a_k))$. Take $R \in \sigma$ of arity r, and consider $\mathbf{a}^{(k)} \in R^{\mathbf{A}^{(k)}}$, where $\mathbf{a} \in R^{\mathbf{A}}$. Let $\mathbf{b} = f(\mathbf{a})$. Since f is a homomorphism, $\mathbf{b} \in R^{\mathbf{B}}$, so $\mathbf{b}^{(k)} \in R^{\mathbf{B}^{(k)}}$. For any $\mathbf{i} \in [r]^k$, we have

$$E_{\mathbf{i}} * f^*\left(\mathbf{a}^{(\mathbf{k})}\right) = f^*\left(E_{\mathbf{i}} * \mathbf{a}^{(\mathbf{k})}\right) = f^*(\mathbf{a}_{\mathbf{i}}) = f(\mathbf{a}_{\mathbf{i}}) = (f(\mathbf{a}))_{\mathbf{i}} = \mathbf{b}_{\mathbf{i}} = E_{\mathbf{i}} * \mathbf{b}^{(\mathbf{k})},$$

which yields $f^*(\mathbf{a}^{(k)}) = \mathbf{b}^{(k)} \in R^{\mathbf{B}^{(k)}}$. Hence, $f^* : \mathbf{A}^{(k)} \to \mathbf{B}^{(k)}$ is a homomorphism. Conversely, let $g : \mathbf{A}^{(k)} \to \mathbf{B}^{(k)}$ be a homomorphism. We define the function $g_* : A \to B$

Conversely, let $g: \mathbf{A}^{\textcircled{B}} \to \mathbf{B}^{\textcircled{B}}$ be a homomorphism. We define the function $g_*: A \to B$ by setting $g_*(a) = \mathbf{e}_1^T g((a, \ldots, a))$ for each $a \in A$. Take $R \in \sigma$ of arity r, and consider a tuple $\mathbf{a} = (a_1, \ldots, a_r) \in R^{\mathbf{A}}$. Since $\mathbf{a}^{\textcircled{k}} \in R^{\mathbf{A}^{\textcircled{k}}}$ and g is a homomorphism, we have that $g(\mathbf{a}^{\textcircled{k}}) \in R^{\mathbf{B}^{\textcircled{k}}}$. Therefore, $g(\mathbf{a}^{\textcircled{k}}) = \mathbf{b}^{\textcircled{k}}$ for some $\mathbf{b} = (b_1, \ldots, b_r) \in R^{\mathbf{B}}$. For each $j \in [r]$, consider the tuple $\mathbf{i} = (j, \ldots, j) \in [r]^k$ and observe that

$$g((a_j,\ldots,a_j)) = g(\mathbf{a}_i) = g\left(E_i * \mathbf{a}^{(k)}\right) = E_i * g(\mathbf{a}^{(k)}) = E_i * \mathbf{b}^{(k)} = \mathbf{b}_i = (b_j,\ldots,b_j).$$

Hence, we find

$$g_*(\mathbf{a}) = \left(\mathbf{e}_1^T g((a_1, \dots, a_1)), \dots, \mathbf{e}_1^T g((a_r, \dots, a_r))\right) = (b_1, \dots, b_r) = \mathbf{b} \in R^{\mathbf{B}}.$$

Therefore, $g_* : \mathbf{A} \to \mathbf{B}$ is a homomorphism. This concludes the proof of (i).

To prove (*ii*), observe first that, if $\mathbf{A} \not\rightarrow \mathbf{B}$, then $\operatorname{Hom}(\mathbf{A}, \mathbf{B}) = \operatorname{Hom}(\mathbf{A}^{(k)}, \mathbf{B}^{(k)}) = \emptyset$, so there is a trivial bijection in this case. If $\mathbf{A} \rightarrow \mathbf{B}$, consider the map $\rho : \operatorname{Hom}(\mathbf{A}, \mathbf{B}) \rightarrow$ $\operatorname{Hom}(\mathbf{A}^{(k)}, \mathbf{B}^{(k)})$ defined by $f \mapsto f^*$ and the map $\rho' : \operatorname{Hom}(\mathbf{A}^{(k)}, \mathbf{B}^{(k)}) \rightarrow$ $\operatorname{Hom}(\mathbf{A}, \mathbf{B})$ defined by $g \mapsto g_*$. For $f : \mathbf{A} \rightarrow \mathbf{B}$ and $a \in A$, we have

$$(f^*)_*(a) = \mathbf{e}_1^T f^*((a, \dots, a)) = \mathbf{e}_1^T (f(a), \dots, f(a)) = f(a)$$

so that $\rho' \circ \rho = \operatorname{id}_{\operatorname{Hom}(\mathbf{A},\mathbf{B})}$. Consider now $g : \mathbf{A}^{(k)} \to \mathbf{B}^{(k)}$, and take $\mathbf{a} = (a_1, \ldots, a_k) \in A^k$. Using the assumption that \mathbf{A} is k-enhanced, we have $\mathbf{a} \in R_k^{\mathbf{A}}$, which implies $\mathbf{a}^{(k)} \in R_k^{\mathbf{A}^{(k)}}$. Hence, $g(\mathbf{a}^{(k)}) \in R_k^{\mathbf{B}^{(k)}}$, so $g(\mathbf{a}^{(k)}) = \mathbf{b}^{(k)}$ for some $\mathbf{b} = (b_1, \ldots, b_k) \in R_k^{\mathbf{B}} \subseteq B^k$. For $j \in [k]$ and $\mathbf{i} = (j, \ldots, j) \in [k]^k$, we have

$$g((a_j,\ldots,a_j)) = g(\mathbf{a}_i) = g\left(E_i * \mathbf{a}^{(k)}\right) = E_i * g\left(\mathbf{a}^{(k)}\right) = E_i * \mathbf{b}^{(k)} = \mathbf{b}_i = (b_j,\ldots,b_j).$$

Letting $\mathbf{i}' = (1, \dots, k) \in [k]^k$, we obtain

$$(g_*)^*(\mathbf{a}) = (g_*(a_1), \dots, g_*(a_k)) = \left(\mathbf{e}_1^T g((a_1, \dots, a_1)), \dots, \mathbf{e}_1^T g((a_k, \dots, a_k))\right) = (b_1, \dots, b_k)$$

= $\mathbf{b} = \mathbf{b}_{\mathbf{i}'} = E_{\mathbf{i}'} * \mathbf{b}^{(\mathbf{k})} = E_{\mathbf{i}'} * g\left(\mathbf{a}^{(\mathbf{k})}\right) = g\left(E_{\mathbf{i}'} * \mathbf{a}^{(\mathbf{k})}\right) = g(\mathbf{a}_{\mathbf{i}'}) = g(\mathbf{a}),$

so that $\rho \circ \rho' = \operatorname{id}_{\operatorname{Hom}(\mathbf{A}^{(k)}, \mathbf{B}^{(k)})}$, which concludes the proof of (ii).

Remark 29. Part (*ii*) of Proposition 28 does not hold in general if we relax the requirement that **A** be *k*-enhanced. More precisely, in this case, the function ρ : Hom(\mathbf{A}, \mathbf{B}) \rightarrow Hom($\mathbf{A}^{(k)}, \mathbf{B}^{(k)}$) defined in the proof of Proposition 28 still needs to be injective, but may not be surjective. Therefore, we have $|\text{Hom}(\mathbf{A}, \mathbf{B})| \leq |\text{Hom}(\mathbf{A}^{(k)}, \mathbf{B}^{(k)})|$, and the inequality may be strict.

For example, consider the Boolean structure **A** having a unique unary relation $R_1^{\mathbf{A}} = A = \{0, 1\}$. So, **A** is 1-enhanced but not 2-enhanced. Observe that $|\operatorname{Hom}(\mathbf{A}, \mathbf{A})| = 4$. The tensorised structure $\mathbf{A}^{(2)}$ has domain $\{0, 1\}^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, and its (unary) relation is $R_1^{\mathbf{A}^{(2)}} = \{0^{(2)}, 1^{(2)}\} = \{(0, 0), (1, 1)\}$. Therefore, each map $f : \{0, 1\}^2 \to \{0, 1\}^2$ such that $f((0, 0)) \in \{(0, 0), (1, 1)\}$ and $f((1, 1)) \in \{(0, 0), (1, 1)\}$ yields a proper homomorphism $\mathbf{A}^{(2)} \to \mathbf{A}^{(2)}$. It follows that $|\operatorname{Hom}(\mathbf{A}^{(2)}, \mathbf{A}^{(2)})| = 64$, so $\operatorname{Hom}(\mathbf{A}, \mathbf{A})$ and $\operatorname{Hom}(\mathbf{A}^{(2)}, \mathbf{A}^{(2)})$ are not in bijection.

It readily follows from Proposition 28 that hierarchies of minion tests are always complete.

Proposition (Proposition 13 restated). Test^k_{\mathcal{M}} is complete for any minion \mathcal{M} and any integer $k \in \mathbb{N}$.

Proof. Let **X** and **A** be two k-enhanced σ -structures and suppose that $\mathbf{X} \to \mathbf{A}$. Proposition 28 yields $\mathbf{X}^{(k)} \to \mathbf{A}^{(k)}$, while Lemma 24 yields $\mathbf{A}^{(k)} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$. The composition of the two homomorphisms witnesses that $\operatorname{Test}^{k}_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$, as required.

We conclude the section by showing that hierarchies of minion tests become tighter as the level increases.

Proposition (Proposition 14 restated). Let \mathcal{M} be a minion, let $k, p \in \mathbb{N}$ be such that k > p, and let \mathbf{X}, \mathbf{A} be two k- and p-enhanced σ -structures. If $\operatorname{Test}^{k}_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$ then $\operatorname{Test}^{p}_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}.$

Proof. Let $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ be a homomorphism witnessing that $\operatorname{Test}_{\mathscr{M}}^{k}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$. Since k > p, we can choose two tuples $\mathbf{v} \in [k]^p$ and $\mathbf{w} \in [p]^k$ such that $\mathbf{w}_{\mathbf{v}} = (1, \ldots, p)$. (For instance, we may take $\mathbf{v} = (1, \dots, p)$ and $\mathbf{w} = (1, \dots, p, \dots, p)$.) Consider the map $\tau : A^k \to \mathbf{v}$ A^p defined by $\mathbf{a} \mapsto \mathbf{a}_{\mathbf{v}}$. We claim that the map $\vartheta : X^p \to \mathscr{M}^{(n^p)}$ defined by $\mathbf{x} \mapsto \xi(\mathbf{x}_{\mathbf{w}})_{/\tau}$ yields a homomorphism from $\mathbf{X}^{\textcircled{p}}$ to $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{\textcircled{p}})$, thus witnessing that $\operatorname{Test}^p_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$. To that end, for $R \in \sigma$, take $\mathbf{x} \in R^{\mathbf{X}}$ and observe that, since ξ is a homomorphism and $\mathbf{x}^{(k)} \in R^{\mathbf{X}^{(k)}}$, $\xi(\mathbf{x}^{(k)}) \in R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})}$. Therefore, there exists $Q \in \mathscr{M}^{(|R^{\mathbf{A}}|)}$ satisfying $\xi(\mathbf{x}_{\mathbf{i}}) = Q_{/\pi_{\mathbf{i}}}$ for each $\mathbf{i} \in [r]^k$. If we manage to show that $\vartheta(\mathbf{x_j}) = Q_{/\pi_i}$ for each $\mathbf{j} \in [r]^p$, we would deduce that $\vartheta(\mathbf{x}^{(p)}) \in R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(p)})}$, thus proving the claim. Observe that

$$\vartheta(\mathbf{x}_{\mathbf{j}}) = \xi(\mathbf{x}_{\mathbf{j}_{\mathbf{w}}})_{/\tau} = (Q_{/\pi_{\mathbf{j}_{\mathbf{w}}}})_{/\tau} = Q_{/\tau \circ \pi_{\mathbf{j}_{\mathbf{w}}}},$$

so we are left to show that $\tau \circ \pi_{\mathbf{j}_{\mathbf{w}}} = \pi_{\mathbf{j}}$. Indeed, given any $\mathbf{a} \in \mathbb{R}^{\mathbf{A}}$,

$$(\tau \circ \pi_{\mathbf{j}_{\mathbf{w}}})(\mathbf{a}) = \tau(\pi_{\mathbf{j}_{\mathbf{w}}}(\mathbf{a})) = \tau(\mathbf{a}_{\mathbf{j}_{\mathbf{w}}}) = \mathbf{a}_{\mathbf{j}_{\mathbf{w}_{\mathbf{v}}}} = \mathbf{a}_{\mathbf{j}} = \pi_{\mathbf{j}}(\mathbf{a}),$$

as required.

As noted in Section 3, it follows from Proposition 14 that if a PCSP template is solved by some level of a minion test, then it is also solved by any higher level.

8 Hierarchies of linear minion tests

By the Definition 12 of a hierarchy of minion tests, $\text{Test}_{\mathscr{M}}^k$ applied to an instance **X** of $PCSP(\mathbf{A}, \mathbf{B})$ checks for the existence of a homomorphism from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$. Therefore, to describe the hierarchy and get knowledge on its functioning it is necessary to study the structure $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$. This is done by investigating how the tensorisation construction interacts with the free structure of the minion \mathcal{M} on which the hierarchy is built. In this section, we show that, if we consider minions having a "matrix form" – namely, the *linear minions* of Definition $16 - \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ is a space of highly structured tensors, which we can describe using the tools of multilinear algebra. Restricting the focus on this class of minions may strike as artificial. The next result argues that the choice is in fact quite natural, as all the minions that have been hitherto utilised in the literature on (P)CSPs to capture the power of algorithms (including the minion \mathscr{S} introduced in this work) are linear. In particular, the machinery we build in this section (and in Section 9, where we consider an even more specialised minion class) shall be crucial to show that the framework of hierarchies of minion tests captures various well-known hierarchies of relaxations (cf. Theorem 15).

Proposition (Proposition 17 restated). The following minimum are linear:

- \mathscr{H} , with $\mathcal{S} = (\{0,1\}, \lor, \land)$ and d = 1
- \mathscr{Z}_{aff} , with $\mathcal{S} = \mathbb{Z}$ and d = 1
- $\mathscr{Q}_{\text{conv}}$, with $\mathcal{S} = \mathbb{Q}$ and d = 1
- \mathcal{M}_{BA} , with $\mathcal{S} = \mathbb{Q}$ and d = 2.

Proof. The result for $\mathscr{Q}_{\text{conv}}$, \mathscr{Z}_{aff} , and \mathscr{M}_{BA} directly follows from their definitions in Example 2, while the result for \mathscr{S} is clear from Proposition 26.

As for \mathscr{H} , recall its description given in Example 2. Given $L \in \mathbb{N}$ and $\emptyset \neq Z \subseteq [L]$, we identify the Boolean function $f_Z = \bigwedge_{z \in Z} x_z \in \mathscr{H}^{(L)}$ with the indicator vector $\mathbf{v}_Z \in \{0, 1\}^L$ whose *i*-th entry, for $i \in [L]$, is 1 if $i \in Z$, and 0 otherwise. To conclude, we need to show that, under this identification, the minor operations of \mathscr{H} correspond to the minor operations given in part 3 of Definition 16. In other words, we claim that the function $f_{Z/\pi}$ corresponds to the vector $P\mathbf{v}_Z$ for any $L' \in \mathbb{N}$ and any $\pi : [L] \to [L']$, where P is the $L' \times L$ matrix given in Definition 16. First, observe that

$$f_{Z/\pi}(x_1,\ldots,x_{L'}) = f_Z(x_{\pi(1)},\ldots,x_{\pi(L)}) = \bigwedge_{z \in Z} x_{\pi(z)} = \bigwedge_{t \in \pi(Z)} x_t = f_{\pi(Z)}(x_1,\ldots,x_{L'}),$$

so $f_{Z/\pi} = f_{\pi(Z)}$. To conclude, we need to show that $P\mathbf{v}_Z = \mathbf{v}_{\pi(Z)}$, where $\mathbf{v}_{\pi(Z)}$ is the indicator vector of the nonempty set $\pi(Z) \subseteq [L']$. Notice that the matrix multiplication is performed in the semiring $\mathcal{S} = (\{0, 1\}, \vee, \wedge)$. For any $i \in [L']$, we have

$$\mathbf{e}_i^T P \mathbf{v}_Z = \bigvee_{j \in [L]} \left((\mathbf{e}_i^T P \mathbf{e}_j) \land (\mathbf{e}_j^T \mathbf{v}_Z) \right) = \bigvee_{\substack{j \in Z \\ \pi(j) = i}} 1 = \bigvee_{i \in \pi(Z)} 1 = \mathbf{e}_i^T \mathbf{v}_{\pi(Z)},$$

as required.

Let \mathscr{M} be a linear minion. We have seen in Section 3 (cf. Remark 18) that in this case the structure lying at the core of $\operatorname{Test}_{\mathscr{M}}$ – i.e., the free structure $\mathbb{F}_{\mathscr{M}}(\mathbf{A})$ generated by a σ -structure \mathbf{A} – consists in a space of matrices with relations defined through specific matrix products. We now describe the geometry of the structure lying at the core of the *multilinear* test $\operatorname{Test}_{\mathscr{M}}^k$ – which, by virtue of Definition 12, is $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$. Given a semiring \mathcal{S} , a symbol $R \in \sigma$ of arity r, and a tuple $\mathbf{i} \in [r]^k$, consider the tensor $P_{\mathbf{i}} \in \mathcal{T}^{n \cdot \mathbf{1}_k, |R^{\mathbf{A}}|}(\mathcal{S})$ defined by

$$E_{\mathbf{a}} * P_{\mathbf{i}} * E_{\mathbf{a}'} = \begin{cases} 1 & \text{if } \mathbf{a}'_{\mathbf{i}} = \mathbf{a} \\ 0 & \text{otherwise} \end{cases} \quad \forall \mathbf{a} \in A^k, \mathbf{a}' \in R^{\mathbf{A}}.$$
(6)

Observe that the tensor P_i is the multilinear equivalent of the matrix P_i from Remark 18.

Let \mathscr{M} be a linear minion with semiring \mathscr{S} and depth d. The domain of $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(\Bbbk)})$ is $\mathscr{M}^{(n^k)}$, which we visualise as a subset of $\mathcal{T}^{n\cdot\mathbf{1}_k,d}(\mathscr{S})$. Given a symbol $R \in \sigma$ of arity r, consider a block tensor $M = (M_{\mathbf{i}})_{\mathbf{i}\in[r]^k} \in \mathcal{T}^{r\cdot\mathbf{1}_k}(\mathcal{T}^{n\cdot\mathbf{1}_k,d}(\mathscr{S})) = \mathcal{T}^{rn\cdot\mathbf{1}_k,d}(\mathscr{S})$. From the definition of free structure, we have that $M \in R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(\Bbbk)})}$ if and only if there exists $Q \in \mathscr{M}^{(|R^{\mathbf{A}}|)}$ such that $M_{\mathbf{i}} = Q_{/\pi_{\mathbf{i}}} = P_{\mathbf{i}}^{-1} Q$ for each $\mathbf{i} \in [r]^k$.

We now show that the entries of $P_{\mathbf{i}}$ satisfy the following simple equality, analogous to Lemma 27.

Lemma 30. Let $k \in \mathbb{N}$, let \mathbf{A} be a σ -structure, let $R \in \sigma$ of arity r, and consider the tuples $\mathbf{a} \in A^k$ and $\mathbf{i} \in [r]^k$. Then

$$E_{\mathbf{a}} * P_{\mathbf{i}} = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a}}} E_{\mathbf{b}}.$$

Proof. For any $\mathbf{a}' \in R^{\mathbf{A}}$, we have

$$\left(\sum_{\substack{\mathbf{b}\in R^{\mathbf{A}}\\\mathbf{b}_{\mathbf{i}}=\mathbf{a}}} E_{\mathbf{b}}\right) * E_{\mathbf{a}'} = \sum_{\substack{\mathbf{b}\in R^{\mathbf{A}}\\\mathbf{b}_{\mathbf{i}}=\mathbf{a}}} (E_{\mathbf{b}} * E_{\mathbf{a}'}) = \sum_{\substack{\mathbf{b}\in R^{\mathbf{A}}\\\mathbf{b}_{\mathbf{i}}=\mathbf{a}}} 1 = \begin{cases} 1 & \text{if } \mathbf{a}'_{\mathbf{i}} = \mathbf{a} \\ 0 & \text{otherwise} \end{cases} = (E_{\mathbf{a}} * P_{\mathbf{i}}) * E_{\mathbf{a}'},$$

from which the result follows.

The next lemma shows that certain entries of a tensor in the relation $R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})}$ (i.e., the interpretation of R in the free structure of \mathscr{M} generated by $\mathbf{A}^{(k)}$) need to be zero.

Lemma 31. Let \mathscr{M} be a linear minion, let $k \in \mathbb{N}$, let \mathbf{A} be a σ -structure, let $R \in \sigma$ of arity r, and suppose $M = (M_{\mathbf{i}})_{\mathbf{i} \in [r]^k} \in R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(\mathbf{k})})}$. Then $E_{\mathbf{a}} * M_{\mathbf{i}} = \mathbf{0}_d$ for any $\mathbf{i} \in [r]^k$, $\mathbf{a} \in A^k$ such that $\mathbf{i} \not\prec \mathbf{a}$.

Proof. Observe that there exists $Q \in \mathscr{M}^{(|R^{\mathbf{A}}|)}$ such that $M_{\mathbf{i}} = Q_{/\pi_{\mathbf{i}}}$ for each $\mathbf{i} \in [r]^{k}$. Using Lemma 30, we obtain

$$E_{\mathbf{a}} * M_{\mathbf{i}} = E_{\mathbf{a}} * Q_{/\pi_{\mathbf{i}}} = E_{\mathbf{a}} * (P_{\mathbf{i}} \stackrel{1}{*} Q) = E_{\mathbf{a}} * P_{\mathbf{i}} * Q = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a}}} E_{\mathbf{b}} * Q = \mathbf{0}_{d},$$

where the last equality follows from the fact that $\mathbf{b_i} = \mathbf{a}$ implies $\mathbf{i} \prec \mathbf{a}$; indeed, in that case, $i_{\alpha} = i_{\beta}$ implies $a_{\alpha} = b_{i_{\alpha}} = b_{i_{\beta}} = a_{\beta}$.

It follows from Lemma 31 and the previous discussion that, if \mathscr{M} is linear, $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ can be visualised as a space of sparse and highly symmetric tensors (see also Example 19 and Figure 1). The geometry of this space is reflected in the properties of the homomorphisms from the k-th tensor power of an instance \mathbf{X} to $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ – which correspond to solutions of Test^k_{\mathscr{M}}, cf. Definition 12. Lemma 32, Lemma 33, and Proposition 36 highlight certain features of such homomorphisms that will be used to prove Theorem 15 in Section 11.

Lemma 32. Let \mathscr{M} be a linear minion, let $k \in \mathbb{N}$, let \mathbf{X}, \mathbf{A} be two k-enhanced σ -structures, and let $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ be a homomorphism. Then $E_{\mathbf{a}} * \xi(\mathbf{x}) = \mathbf{0}_d$ for any $\mathbf{x} \in X^k$, $\mathbf{a} \in A^k$ such that $\mathbf{x} \not\prec \mathbf{a}$.

Proof. From $\mathbf{x} \in X^k = R_k^{\mathbf{X}}$, we derive $\mathbf{x}^{(k)} \in R_k^{\mathbf{X}^{(k)}}$; since ξ is a homomorphism, this yields $\xi(\mathbf{x}^{(k)}) \in R_k^{\mathbb{F}_{\mathcal{M}}(\mathbf{A}^{(k)})}$. Writing $\xi(\mathbf{x}^{(k)})$ in block form as $\xi(\mathbf{x}^{(k)}) = (\xi(\mathbf{x}_i))_{i \in [k]^k}$ and applying Lemma 31, we obtain $E_{\mathbf{a}} * \xi(\mathbf{x}_i) = \mathbf{0}_d$ for any $\mathbf{i} \in [k]^k$ such that $\mathbf{i} \not\prec \mathbf{a}$. Write $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{a} = (a_1, \ldots, a_k)$. Since $\mathbf{x} \not\prec \mathbf{a}$, there exist $\alpha, \beta \in [k]$ such that $x_\alpha = x_\beta$ and $a_\alpha \neq a_\beta$. Let $\mathbf{i}' \in [k]^k$ be the tuple obtained from $(1, \ldots, k)$ by replacing the β -th entry with α . Observe that $\mathbf{x}_{\mathbf{i}'} = \mathbf{x}$ and $\mathbf{i'} \not\prec \mathbf{a}$. Hence,

$$\mathbf{0}_d = E_{\mathbf{a}} * \xi(\mathbf{x}_{\mathbf{i}'}) = E_{\mathbf{a}} * \xi(\mathbf{x}),$$

as required.

Using Lemma 32, we obtain some more information on the image of a homomorphism from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$. Given a signature σ , we let $\operatorname{armax}(\sigma)$ denote the maximum arity of a relation symbol in σ .

Lemma 33. Let \mathscr{M} be a linear minion, let $k \in \mathbb{N}$, let \mathbf{X}, \mathbf{A} be two k-enhanced σ -structures such that $k \geq \operatorname{armax}(\sigma)$, and let $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ be a homomorphism. For $R \in \sigma$ of arity r, let $\mathbf{x} \in R^{\mathbf{X}}$ and $\mathbf{a} \in R^{\mathbf{A}}$ be such that $\mathbf{x} \not\prec \mathbf{a}$. Let $Q \in \mathscr{M}^{(|R^{\mathbf{A}}|)}$ be such that $Q_{/\pi_{\mathbf{i}}} = \xi(\mathbf{x}_{\mathbf{i}})$ for each $\mathbf{i} \in [r]^{k}$. Then $E_{\mathbf{a}} * Q = \mathbf{0}_{d}$.

Proof. Write $\mathbf{x} = (x_1, \ldots, x_r)$ and $\mathbf{a} = (a_1, \ldots, a_r)$. Since $\mathbf{x} \not\prec \mathbf{a}$, there exist $\alpha, \beta \in [r]$ such that $x_{\alpha} = x_{\beta}$ and $a_{\alpha} \neq a_{\beta}$. Using that $k \geq r$, we can take the tuple $\mathbf{i} = (1, 2, \ldots, r, r, r, \ldots, r) \in [r]^k$. Consider $\mathbf{x}_{\mathbf{i}} \in X^k$, $\mathbf{a}_{\mathbf{i}} \in A^k$. Notice that $x_{i_{\alpha}} = x_{\alpha} = x_{\beta} = x_{i_{\beta}}$ and $a_{i_{\alpha}} = a_{\alpha} \neq a_{\beta} = a_{i_{\beta}}$, so $\mathbf{x}_{\mathbf{i}} \not\prec \mathbf{a}_{\mathbf{i}}$. Applying Lemma 32, we find

$$\mathbf{0}_d = E_{\mathbf{a}_i} * \xi(\mathbf{x}_i) = E_{\mathbf{a}_i} * Q_{/\pi_i} = E_{\mathbf{a}_i} * P_i * Q.$$

Using Lemma 30, we conclude that

$$\mathbf{0}_d = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_i = \mathbf{a}_i}} E_{\mathbf{b}} * Q = E_{\mathbf{a}} * Q,$$

where the last equality follows from the fact that $\mathbf{b_i} = \mathbf{a_i}$ if and only if $\mathbf{b} = \mathbf{a}$.

If **X** and **A** are k-enhanced σ -structures, any homomorphism ξ from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ must satisfy certain symmetries that ultimately depend on the fact that ξ preserves R_k . As shown below in Proposition 36, these symmetries can be concisely expressed through a tensor equation. Given a tuple $\mathbf{i} \in [k]^k$, we let $\Pi_{\mathbf{i}} \in \mathcal{T}^{n \cdot \mathbf{1}_{2k}}(\mathcal{S})$ be the tensor defined by

$$E_{\mathbf{a}} * \Pi_{\mathbf{i}} * E_{\mathbf{a}'} = \begin{cases} 1 & \text{if } \mathbf{a}'_{\mathbf{i}} = \mathbf{a} \\ 0 & \text{otherwise} \end{cases} \quad \forall \mathbf{a}, \mathbf{a}' \in A^k.$$
(7)

The tensor defined above satisfies the following simple identity, which should be compared to the one in Lemma 30 concerning the tensor P_i .

Lemma 34. For any $\mathbf{a} \in A^k$ and $\mathbf{i} \in [k]^k$,

$$E_{\mathbf{a}} * \Pi_{\mathbf{i}} = \sum_{\substack{\mathbf{b} \in A^k \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a}}} E_{\mathbf{b}}$$

Proof. For any $\mathbf{a}' \in A^k$, we have

$$\left(\sum_{\substack{\mathbf{b}\in A^k\\\mathbf{b}_{\mathbf{i}}=\mathbf{a}}} E_{\mathbf{b}}\right) * E_{\mathbf{a}'} = \sum_{\substack{\mathbf{b}\in A^k\\\mathbf{b}_{\mathbf{i}}=\mathbf{a}}} (E_{\mathbf{b}} * E_{\mathbf{a}'}) = \sum_{\substack{\mathbf{b}\in A^k\\\mathbf{b}_{\mathbf{i}}=\mathbf{a}\\\mathbf{b}=\mathbf{a}'}} 1 = \begin{cases} 1 & \text{if } \mathbf{a}'_{\mathbf{i}} = \mathbf{a} \\ 0 & \text{otherwise} \end{cases} = (E_{\mathbf{a}} * \Pi_{\mathbf{i}}) * E_{\mathbf{a}'},$$

from which the result follows.

Remark 35. It is clear from the expressions (6) and (7) that, for any $\mathbf{i} \in [k]^k$, $\Pi_{\mathbf{i}}$ coincides with the tensor $P_{\mathbf{i}}$ associated with the relation symbol R_k .

Proposition 36. Let \mathscr{M} be a linear minion, let $k \in \mathbb{N}$, let \mathbf{X}, \mathbf{A} be two k-enhanced σ structures, and let $\xi: X^k \to \mathscr{M}^{(n^k)}$ be a map. Then, ξ preserves R_k if and only if

$$\xi(\mathbf{x}_{\mathbf{i}}) = \Pi_{\mathbf{i}} * \xi(\mathbf{x}) \qquad \qquad for \ any \ \mathbf{x} \in X^k, \mathbf{i} \in [k]^k.$$
(8)

Proof. Suppose that ξ preserves R_k , and take $\mathbf{x} \in X^k = R_k^{\mathbf{X}}$. It follows that $\mathbf{x}^{(k)} \in R_k^{\mathbf{X}^{(k)}}$, so $\xi(\mathbf{x}^{(k)}) \in R_k^{\mathbb{F}_{\mathcal{M}}(\mathbf{A}^{(k)})}$. This means that there exists $Q \in \mathcal{M}^{(|R_k^{\mathbf{A}}|)} = \mathcal{M}^{(n^k)}$ such that $\xi(\mathbf{x_i}) = Q_{/\pi_i} = \prod_i {\stackrel{k}{*}} Q$ for each $\mathbf{i} \in [k]^k$ (where we have used Remark 35). Consider now the tuple $\mathbf{j} = (1, \dots, k) \in [k]^k$, and observe that $\mathbf{x}_{\mathbf{j}} = \mathbf{x}$. Noticing that the contraction by $\Pi_{\mathbf{i}}$ acts as the identity, we conclude that

$$\xi(\mathbf{x}) = \xi(\mathbf{x}_{\mathbf{j}}) = \Pi_{\mathbf{j}} \overset{k}{*} Q = Q,$$

which concludes the proof of (8).

Conversely, suppose (8) holds and take $\mathbf{x} \in R_k^{\mathbf{X}} = X^k$. We need to show that $\xi(\mathbf{x}^{(k)}) \in R_k^{\mathbb{F}_{\mathcal{M}}(\mathbf{A}^{(k)})}$. Take $Q = \xi(\mathbf{x}) \in \mathcal{M}^{(n^k)} = \mathcal{M}^{(|R_k^{\mathbf{A}}|)}$. Using again Remark 35, we observe that, for any $\mathbf{i} \in [k]^k$ any $\mathbf{i} \in [k]^k$,

$$\xi(\mathbf{x}_{\mathbf{i}}) = \Pi_{\mathbf{i}} \stackrel{\kappa}{*} \xi(\mathbf{x}) = \xi(\mathbf{x})_{/\pi_{\mathbf{i}}} = Q_{/\pi_{\mathbf{i}}}$$

whence the result follows.

9 Hierarchies of conic minion tests

The machinery developed in Section 8 applies to hierarchies of tests built on any linear minion. It turns out that certain finer features of hierarchies of minion tests can be deduced if we assume that the minion is *conic*; i.e., that, in addition to being linear, it enjoys a sort of nonnegativity requirement described in Definition 20. As we establish next, all minions considered in this work are conic, with the notable exception of \mathscr{Z}_{aff} .

Proposition (Proposition 21 restated). \mathcal{H} , \mathcal{Q}_{conv} , \mathcal{S} , and \mathcal{M}_{BA} are conic minions, while $\mathscr{Z}_{\mathrm{aff}}$ is not.

Proof. The fact that \mathscr{Q}_{conv} is conic trivially follows by noting that its elements are nonnegative vectors whose entries sum up to 1. Similarly, using the description of \mathcal{H} as a linear minion on the semiring $(\{0,1\}, \lor, \land)$ (cf. the proof of Proposition 17), the fact that \mathscr{H} is conic follows from the fact that $\bigvee_{i \in V} x_i = 0$ means that $x_i = 0$ for each $i \in V$. (Observe also that the vectors in \mathscr{H} are nonzero, as they are the indicator vectors of nonempty sets.) To show that \mathscr{S} is conic, take $L \in \mathbb{N}$ and $M \in \mathscr{S}^{(L)}$, and notice first that $M \neq O_{L,\aleph_0}$ by C3 in Definition 7. Take now $V \subseteq [L]$. If $\sum_{i \in V} M^T \mathbf{e}_i = \mathbf{0}_{\aleph_0}$, using that MM^T is a diagonal matrix by C2, we find

$$0 = (\sum_{i \in V} M^T \mathbf{e}_i)^T (\sum_{j \in V} M^T \mathbf{e}_j) = \sum_{i,j \in V} \mathbf{e}_i^T M M^T \mathbf{e}_j = \sum_{i \in V} \mathbf{e}_i^T M M^T \mathbf{e}_i = \sum_{i \in V} \|M^T \mathbf{e}_i\|^2,$$

which means that $M^T \mathbf{e}_i = \mathbf{0}_{\aleph_0}$ for any $i \in V$, as required. As for \mathscr{M}_{BA} , we shall see in Section 10 (cf. Example 43) that this minion can be obtained as the semi-direct product of

 $\mathscr{Q}_{\text{conv}}$ and \mathscr{Z}_{aff} . Then, the fact that \mathscr{M}_{BA} is conic is a direct consequence of the fact that semi-direct products of minions are always conic (cf. Proposition 40). Finally, the element $(1, -1, 1) \in \mathscr{Z}_{\text{aff}}$ witnesses that \mathscr{Z}_{aff} is not conic.

It turns out that Lemma 33 can be slightly strengthened if we are dealing with conic minions, in that the level k for which it holds can be decreased down to 2. As a consequence, the algebraic description of the Sherali-Adams and Sum-of-Squares hierarchies in terms of the tensorisation construction can be extended to lower levels, cf. Remark 51. As in Section 8, the letter d shall denote the depth of the relevant minion in all results of this section. (Recall that a conic minion is linear, by definition).

Lemma 37. Let \mathscr{M} be a conic minion, let $2 \leq k \in \mathbb{N}$, let \mathbf{X}, \mathbf{A} be two k-enhanced σ structures, and let $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ be a homomorphism. For $R \in \sigma$ of arity r, let $\mathbf{x} \in R^{\mathbf{X}}$ and $\mathbf{a} \in R^{\mathbf{A}}$ be such that $\mathbf{x} \not\prec \mathbf{a}$. Let $Q \in \mathscr{M}^{(|R^{\mathbf{A}}|)}$ be such that $Q_{/\pi_{\mathbf{i}}} = \xi(\mathbf{x}_{\mathbf{i}})$ for each $\mathbf{i} \in [r]^{k}$. Then $E_{\mathbf{a}} * Q = \mathbf{0}_{d}$.

Proof. Take $\alpha, \beta \in [r]$ such that $x_{\alpha} = x_{\beta}$ and $a_{\alpha} \neq a_{\beta}$, and consider the tuple $\mathbf{j} = (\alpha, \ldots, \alpha, \beta) \in [r]^k$. Using that $k \geq 2$, we have $\mathbf{x}_{\mathbf{j}} \not\prec \mathbf{a}_{\mathbf{j}}$, since $x_{j_{k-1}} = x_{\alpha} = x_{\beta} = x_{j_k}$ and $a_{j_{k-1}} = a_{\alpha} \neq a_{\beta} = a_{j_k}$. From Lemma 32 and Lemma 30 we obtain

$$\mathbf{0}_d = E_{\mathbf{a}_j} * \xi(\mathbf{x}_j) = E_{\mathbf{a}_j} * Q_{/\pi_j} = E_{\mathbf{a}_j} * P_j * Q = \sum_{\substack{\mathbf{b} \in R^\mathbf{A} \\ \mathbf{b}_i = \mathbf{a}_i}} E_{\mathbf{b}} * Q.$$

Using that \mathscr{M} is a conic minion, we deduce that $E_{\mathbf{b}} * Q = \mathbf{0}_d$ for any $\mathbf{b} \in R^{\mathbf{A}}$ such that $\mathbf{b}_{\mathbf{j}} = \mathbf{a}_{\mathbf{j}}$. In particular, $E_{\mathbf{a}} * Q = \mathbf{0}_d$, as required.

The following result shows that hierarchies of tests built on conic minions only give a nonzero weight to those assignments that yield partial homomorphisms.

Proposition 38. Let \mathscr{M} be a conic minion, let $k \in \mathbb{N}$, let \mathbf{X}, \mathbf{A} be two k-enhanced σ structures such that $k \geq \min(2, \operatorname{armax}(\sigma))$, and let $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ be a homomorphism. Let $R \in \sigma$ have arity r, and take $\mathbf{x} \in X^k$, $\mathbf{a} \in A^k$, and $\mathbf{i} \in [k]^r$. If $\mathbf{x}_{\mathbf{i}} \in R^{\mathbf{X}}$ and $\mathbf{a}_{\mathbf{i}} \notin R^{\mathbf{A}}$, then $E_{\mathbf{a}} * \xi(\mathbf{x}) = \mathbf{0}_d$.

Proof. From $\mathbf{x_i} \in R^{\mathbf{X}}$ we have $\mathbf{x_i^{(k)}} \in R^{\mathbf{X}^{(k)}}$ and, thus, $\xi(\mathbf{x_i^{(k)}}) \in R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})}$. It follows that there exists $Q \in \mathscr{M}^{(|R^{\mathbf{A}}|)}$ such that $\xi(\mathbf{x_{i_j}}) = Q_{/\pi_j}$ for each $\mathbf{j} \in [r]^k$. Proposition 36 then yields

$$\Pi_{\mathbf{i}_{\mathbf{j}}}^{k} * \xi(\mathbf{x}) = \xi(\mathbf{x}_{\mathbf{i}_{\mathbf{j}}}) = Q_{/\pi_{\mathbf{j}}} = P_{\mathbf{j}}^{-1} * Q.$$
(9)

Consider, for each $\alpha \in [k]$, the set $S_{\alpha} = \{\beta \in [r] : i_{\beta} = \alpha\}$, and fix an element $\hat{\beta} \in [r]$. The tuple $\mathbf{j} \in [r]^k$ defined by setting $j_{\alpha} = \min S_{\alpha}$ if $S_{\alpha} \neq \emptyset$, $j_{\alpha} = \hat{\beta}$ otherwise satisfies $\mathbf{i}_{\mathbf{j}_i} = \mathbf{i}$. Indeed, for any $\beta \in [r]$, we have $S_{i_{\beta}} \neq \emptyset$ since $\beta \in S_{i_{\beta}}$, so $j_{i_{\beta}} = \min S_{i_{\beta}} \in S_{i_{\beta}}$, which means that $i_{j_{i_{\beta}}} = i_{\beta}$, as required. We obtain

$$E_{\mathbf{a}_{\mathbf{i}_{\mathbf{j}}}} * P_{\mathbf{j}} * Q = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{j}} = \mathbf{a}_{\mathbf{i}_{\mathbf{j}}}}} E_{\mathbf{b}} * Q = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{x}_{\mathbf{i}} \prec \mathbf{b} \\ \mathbf{b}_{\mathbf{j}} = \mathbf{a}_{\mathbf{i}_{\mathbf{j}}}}} E_{\mathbf{b}} * Q,$$
(10)

where the first equality comes from Lemma 30 and the second from Lemma 33 or Lemma 37 (depending on whether $k \ge \operatorname{armax}(\sigma)$ or $k \ge 2$). We claim that the sum on the right-hand side of (10) equals $\mathbf{0}_d$. Indeed, let $\mathbf{b} \in A^r$ satisfy $\mathbf{x}_i \prec \mathbf{b}$ and $\mathbf{b}_j = \mathbf{a}_{\mathbf{i}_j}$. Since $\mathbf{i}_{\mathbf{j}_i} = \mathbf{i}$, for any $\alpha \in [r]$ we have $x_{i_\alpha} = x_{i_{j_{i_\alpha}}}$ and, hence, $b_\alpha = b_{j_{i_\alpha}}$. It follows that $\mathbf{b} = \mathbf{b}_{\mathbf{j}_i} = \mathbf{a}_{\mathbf{i}_{\mathbf{j}_i}} = \mathbf{a}_i \notin R^\mathbf{A}$, which proves the claim. Combining this with (9), (10), and Lemma 34, we find

$$\mathbf{0}_{d} = E_{\mathbf{a}_{\mathbf{i}_{\mathbf{j}}}} * P_{\mathbf{j}} * Q = E_{\mathbf{a}_{\mathbf{i}_{\mathbf{j}}}} * (P_{\mathbf{j}}^{-1} * Q) = E_{\mathbf{a}_{\mathbf{i}_{\mathbf{j}}}} * (\Pi_{\mathbf{i}_{\mathbf{j}}}^{-k} * \xi(\mathbf{x})) = E_{\mathbf{a}_{\mathbf{i}_{\mathbf{j}}}} * \Pi_{\mathbf{i}_{\mathbf{j}}} * \xi(\mathbf{x})$$
$$= \sum_{\substack{\mathbf{b} \in A^{k} \\ \mathbf{b}_{\mathbf{i}_{\mathbf{j}}} = \mathbf{a}_{\mathbf{i}_{\mathbf{j}}}} E_{\mathbf{b}} * \xi(\mathbf{x})$$

so, in particular, $E_{\mathbf{a}} * \xi(\mathbf{x}) = \mathbf{0}_d$ since \mathscr{M} is a conic minion.

The next result shows that hierarchies of tests built on conic minions are "sound in the limit", in the sense that they correctly classify all instances whose domain size is less than or equal to the hierarchy level.

Proposition 39. Let \mathscr{M} be a conic minion, let $2 \leq k \in \mathbb{N}$, let \mathbf{X}, \mathbf{A} be two k-enhanced σ -structures such that $|X| \leq k$, and suppose that $\operatorname{Test}^k_{\mathscr{M}}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$. Then $\mathbf{X} \to \mathbf{A}$.

Proof. Let $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ be a homomorphism witnessing that $\operatorname{Test}_{\mathscr{M}}^{k}(\mathbf{X}, \mathbf{A}) = \operatorname{Yes}$, and assume without loss of generality that $X = [\ell]$ with $\ell \in [k]$. Take the tuple $\mathbf{v} = (1, \ldots, \ell, \ell, \ldots, \ell) \in [\ell]^{k}$, and notice that $\xi(\mathbf{v}) \neq O_{n^{k}, d}$ since \mathscr{M} is a conic minion. Therefore, there exists some $\mathbf{a} \in A^{k}$ such that $E_{\mathbf{a}} * \xi(\mathbf{v}) \neq \mathbf{0}_{d}$. Consider the function $f : X \to A$ defined by $x \to a_{x}$ for each $x \in X$. We claim that f yields a homomorphism from \mathbf{X} to \mathbf{A} .

Let $R \in \sigma$ be a relation symbol of arity r, take a tuple $\mathbf{x} \in R^{\mathbf{X}}$, and suppose, for the sake of contradiction, that $f(\mathbf{x}) \notin R^{\mathbf{A}}$. Notice that $\mathbf{x} \in [\ell]^r \subseteq [k]^r$ and that $f(\mathbf{x}) = \mathbf{a}_{\mathbf{x}}$. Since $\mathbf{x} \in R^{\mathbf{X}}$, we have $\mathbf{x}^{(k)} \in R^{\mathbf{X}^{(k)}}$. Being ξ a homomorphism, this implies that $\xi(\mathbf{x}^{(k)}) \in R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})}$. Therefore, $\exists Q \in \mathscr{M}^{(|R^{\mathbf{A}}|)}$ such that $\xi(\mathbf{x}_{\mathbf{i}}) = Q_{/\pi_{\mathbf{i}}}$ for each $\mathbf{i} \in [r]^k$. Observe that

$$E_{\mathbf{a}_{\mathbf{x}_{\mathbf{i}}}} * \xi(\mathbf{x}_{\mathbf{i}}) = E_{\mathbf{a}_{\mathbf{x}_{\mathbf{i}}}} * Q_{/\pi_{\mathbf{i}}} = E_{\mathbf{a}_{\mathbf{x}_{\mathbf{i}}}} * (P_{\mathbf{i}} * Q) = E_{\mathbf{a}_{\mathbf{x}_{\mathbf{i}}}} * P_{\mathbf{i}} * Q = \sum_{\substack{\mathbf{c} \in R^{\mathbf{A}} \\ \mathbf{c}_{\mathbf{i}} = \mathbf{a}_{\mathbf{x}_{\mathbf{i}}}}} E_{\mathbf{c}} * Q = \sum_{\substack{\mathbf{c} \in R^{\mathbf{A}} \\ \mathbf{c}_{\mathbf{i}} = \mathbf{a}_{\mathbf{x}_{\mathbf{i}}}}} E_{\mathbf{c}} * Q = \sum_{\substack{\mathbf{c} \in R^{\mathbf{A}} \\ \mathbf{c}_{\mathbf{i}} = \mathbf{a}_{\mathbf{x}_{\mathbf{i}}}}} E_{\mathbf{c}} * Q,$$

$$(11)$$

where the fourth equality comes from Lemma 30 and the fifth from Lemma 37. On the other hand, noting that $\mathbf{v_x} = \mathbf{x}$, Proposition 36 and Lemma 34 yield

$$E_{\mathbf{a}_{\mathbf{x}_{i}}} * \xi(\mathbf{x}_{i}) = E_{\mathbf{a}_{\mathbf{x}_{i}}} * \xi(\mathbf{v}_{\mathbf{x}_{i}}) = E_{\mathbf{a}_{\mathbf{x}_{i}}} * (\Pi_{\mathbf{x}_{i}} * \xi(\mathbf{v}))$$
$$= E_{\mathbf{a}_{\mathbf{x}_{i}}} * \Pi_{\mathbf{x}_{i}} * \xi(\mathbf{v}) = \sum_{\substack{\mathbf{b} \in A^{k} \\ \mathbf{b}_{\mathbf{x}_{i}} = \mathbf{a}_{\mathbf{x}_{i}}}} E_{\mathbf{b}} * \xi(\mathbf{v}).$$
(12)

If $E_{\mathbf{a}_{\mathbf{x}_{\mathbf{i}}}} * \xi(\mathbf{x}_{\mathbf{i}}) = \mathbf{0}_d$, using that \mathscr{M} is a conic minion, we would deduce from (12) that $E_{\mathbf{b}} * \xi(\mathbf{v}) = \mathbf{0}_d$ for any $\mathbf{b} \in A^k$ such that $\mathbf{b}_{\mathbf{x}_{\mathbf{i}}} = \mathbf{a}_{\mathbf{x}_{\mathbf{i}}}$; in particular, $E_{\mathbf{a}} * \xi(\mathbf{v}) = \mathbf{0}_d$, a contradiction. Therefore, $E_{\mathbf{a}_{\mathbf{x}_{\mathbf{i}}}} * \xi(\mathbf{x}_{\mathbf{i}}) \neq \mathbf{0}_d$. Then, it follows from (11) that, for each $\mathbf{i} \in [r]^k$, there exists some $\mathbf{c} = (c_1, \ldots, c_r) \in R^{\mathbf{A}}$ such that $\mathbf{c}_{\mathbf{i}} = \mathbf{a}_{\mathbf{x}_{\mathbf{i}}}$ and $\mathbf{x} \prec \mathbf{c}$. Choose \mathbf{i} so that

 $\{\mathbf{x}\} = \{\mathbf{x}_{\mathbf{i}}\}$. Since $\mathbf{c} \in R^{\mathbf{A}}$ and $\mathbf{a}_{\mathbf{x}} \notin R^{\mathbf{A}}$, we have $\mathbf{c} \neq \mathbf{a}_{\mathbf{x}}$; so, $c_p \neq a_{x_p}$ for some $p \in [r]$. From $x_p \in \{\mathbf{x}\} = \{\mathbf{x}_{\mathbf{i}}\}$, we obtain $x_p = x_{i_t}$ for some $t \in [k]$. Since $\mathbf{x} \prec \mathbf{c}$, this yields $c_p = c_{i_t}$. Therefore, $c_{i_t} = c_p \neq a_{x_p} = a_{x_{i_t}}$, so $\mathbf{c}_{\mathbf{i}} \neq \mathbf{a}_{\mathbf{x}_{\mathbf{i}}}$, a contradiction. We conclude that $f(\mathbf{x}) \in R^{\mathbf{A}}$, so f yields a homomorphism from \mathbf{X} to \mathbf{A} , as claimed.

10 The semi-direct product of minions

Is it possible for multiple linear minions to "join forces", to obtain a new linear minion corresponding to a stronger relaxation? The natural way to do so is to take as the elements of the new linear minion block matrices, whose blocks are the elements of the original minions. Let \mathscr{M} and \mathscr{N} be two linear minions that we wish to "merge". A zero row in a matrix of \mathscr{M} corresponds to zero weight assigned to the variable associated with the row by the relaxation given by \mathscr{M} . Ideally, we would like to preserve this information when we run the relaxation given by the second minion \mathscr{N} . In other words, we require that zero rows in \mathscr{M} should be associated with zero rows in \mathscr{N} . For this to make sense (i.e., for the resulting object to be a linear minion), we need to assume that \mathscr{M} is conic. Under this assumption, it turns out that the new linear minion is conic, too. Therefore, this construction yields a method to transform a linear minion into a conic one, by taking its product with a fixed conic minion (for instance, $\mathscr{Q}_{\rm conv}$). Equivalently, the semi-direct product provides a way to turn a hierarchy of linear tests into a more powerful hierarchy of conic tests – which enjoys the appealing properties described in Section 9.

Proposition 40. Let \mathscr{M} be a conic minion with semiring \mathscr{S} and depth d, let \mathscr{N} be a linear minion with semiring \mathscr{S} and depth d', and consider, for each $L \in \mathbb{N}$, the set $(\mathscr{M} \ltimes \mathscr{N})^{(L)} = \{[\mathcal{M} \ \mathcal{N}] : \mathcal{M} \in \mathscr{M}^{(L)}, \mathcal{N} \in \mathscr{N}^{(L)}, \text{ and } \mathcal{N}^T \mathbf{e}_i = \mathbf{0}_{d'} \text{ for any } i \in [L] \text{ such that } \mathcal{M}^T \mathbf{e}_i = \mathbf{0}_d\}.$ Then $\mathscr{M} \ltimes \mathscr{N} = \bigcup_{L \in \mathbb{N}} (\mathscr{M} \ltimes \mathscr{N})^{(L)}$ is a conic minion with semiring \mathscr{S} and depth d + d'.

Definition 41. Let \mathscr{M} and \mathscr{N} be a conic minion and a linear minion, respectively, over the same semiring. The *semi-direct product* of \mathscr{M} and \mathscr{N} is the conic minion $\mathscr{M} \ltimes \mathscr{N}$ described in Proposition 40.

Remark 42. If the minions \mathscr{M} and \mathscr{N} in Definition 41 have different semirings \mathcal{S} and \mathcal{S}' , we cannot in general use the definition to build their semi-direct product. However, it is immediate to check that a linear minion over \mathcal{S} is also a linear minion over any semiring of which \mathcal{S} is a sub-semiring. Hence, if \mathcal{S} is a sub-semiring of \mathcal{S}' (or vice-versa), $\mathscr{M} \ltimes \mathscr{N}$ is well defined (see Example 43 below). In general, however, we might not be able to find a common semiring of which \mathcal{S} and \mathcal{S}' are both sub-semirings. In particular, it is not true in general that the direct sum of semirings admits homomorphic injections from the components, see [51]. To be able to define $\mathscr{M} \ltimes \mathscr{N}$ also in this case, we would need to redefine linear minions by allowing each of the d columns of the matrices in a linear minion of depth d to contain entries from a possibly different semiring. In this way, the requirement in Definition 41 can be circumvented – which, for example, makes it possible to define the minion $\mathscr{H} \ltimes \mathscr{L}_{aff}$ (see Remark 52).

Example 43. Since \mathbb{Z} is a sub-semiring of \mathbb{Q} , we can view $\mathscr{Q}_{\text{conv}}$ and \mathscr{Z}_{aff} as a conic minion and a linear minion over the same semiring \mathbb{Q} , respectively. It is easy to check that their semi-direct product $\mathscr{Q}_{\text{conv}} \ltimes \mathscr{Z}_{\text{aff}}$ is precisely the minion \mathscr{M}_{BA} from [30].

Proof of Proposition 40. We start by showing that $\mathscr{M} \ltimes \mathscr{N}$ is a linear minion of depth d+d'. Notice that each set $(\mathscr{M} \ltimes \mathscr{N})^{(L)}$ consists of $L \times (d+d')$ matrices having entries in \mathcal{S} . Given $\pi : [L] \to [L']$ and $[\mathcal{M} \ N] \in (\mathscr{M} \ltimes \mathscr{N})^{(L)}$, we claim that $P[\mathcal{M} \ N] = [\mathcal{PM} \ \mathcal{PN}]$ belongs to $(\mathscr{M} \ltimes \mathscr{N})^{(L')}$, where P is the $L' \times L$ matrix corresponding to π as per Definition 16. First, since \mathscr{M} and \mathscr{N} are both linear minions, we have that $PM = M_{/\pi} \in \mathscr{M}^{(L')}$ and $PN = N_{/\pi} \in \mathscr{N}^{(L')}$. Let $j \in [L']$ be such that $(PM)^T \mathbf{e}_j = \mathbf{0}_d$. We find

$$\mathbf{0}_d = M^T P^T \mathbf{e}_j = \sum_{i \in \pi^{-1}(j)} M^T \mathbf{e}_i$$

Using that \mathscr{M} is conic, we obtain $M^T \mathbf{e}_i = \mathbf{0}_d$ for each $i \in \pi^{-1}(j)$. By the definition of $(\mathscr{M} \ltimes \mathscr{N})^{(L)}$, this means that $N^T \mathbf{e}_i = \mathbf{0}_{d'}$ for each $i \in \pi^{-1}(j)$. Therefore,

$$\mathbf{0}_{d'} = \sum_{i \in \pi^{-1}(j)} N^T \mathbf{e}_i = N^T P^T \mathbf{e}_j = (PN)^T \mathbf{e}_j,$$

which proves the claim. It follows that $\mathcal{M} \ltimes \mathcal{N}$ is indeed a linear minion.

To show that $\mathcal{M} \ltimes \mathcal{N}$ is conic, we first note that no element of $\mathcal{M} \ltimes \mathcal{N}$ is the all-zero matrix, since \mathcal{M} is conic. If $\sum_{i \in V} [M \ N]^T \mathbf{e}_i = \mathbf{0}_{d+d'}$ for some $L \in \mathbb{N}$, $[M \ N] \in (\mathcal{M} \ltimes \mathcal{N})^{(L)}$, and $V \subseteq [L]$, we find in particular that $\sum_{i \in V} \mathcal{M}^T \mathbf{e}_i = \mathbf{0}_d$, which implies that $\mathcal{M}^T \mathbf{e}_i = \mathbf{0}_d$ for each $i \in V$ by the fact that \mathcal{M} is conic. By the definition of $(\mathcal{M} \ltimes \mathcal{N})^{(L)}$, this yields $\mathcal{N}^T \mathbf{e}_i = \mathbf{0}_{d'}$ for each $i \in V$, so

$$\begin{bmatrix} M & N \end{bmatrix}^T \mathbf{e}_i = \begin{bmatrix} M^T \mathbf{e}_i \\ N^T \mathbf{e}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0}_d \\ \mathbf{0}_{d'} \end{bmatrix} = \mathbf{0}_{d+d'}$$

for each $i \in V$, as required.

The next result – crucial for the characterisation of the BA^k hierarchy in Theorem 15, cf. the proof of Proposition 50 – shows that homomorphisms corresponding to the semi-direct product of two minions factor into homomorphisms corresponding to the components.

Proposition 44. Let \mathscr{M} be a conic minion with semiring \mathscr{S} and depth d, let \mathscr{N} be a linear minion with semiring \mathscr{S} and depth d', let $k \in \mathbb{N}$, and let \mathbf{X}, \mathbf{A} be k-enhanced σ -structures such that $k \geq \operatorname{armax}(\sigma)$. Then there exists a homomorphism $\vartheta : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M} \ltimes \mathscr{N}}(\mathbf{A}^{(k)})$ if and only if there exist homomorphisms $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ and $\zeta : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{N}}(\mathbf{A}^{(k)})$ such that, for any $\mathbf{x} \in X^k$ and $\mathbf{a} \in A^k$, $E_{\mathbf{a}} * \xi(\mathbf{x}) = \mathbf{0}_d$ implies $E_{\mathbf{a}} * \zeta(\mathbf{x}) = \mathbf{0}_{d'}$.

Proof. To prove the "if" part, take two homomorphisms ξ and ζ as in the statement of the proposition, and consider the map

$$\vartheta: X^k \to (\mathscr{M} \ltimes \mathscr{N})^{(n^k)}$$
$$\mathbf{x} \mapsto [\xi(\mathbf{x}) \quad \zeta(\mathbf{x})].$$

Observe that ϑ is well defined since we are assuming that $E_{\mathbf{a}} * \xi(\mathbf{x}) = \mathbf{0}_d$ implies $E_{\mathbf{a}} * \zeta(\mathbf{x}) = \mathbf{0}_{d'}$ for any $\mathbf{x} \in X^k$ and $\mathbf{a} \in A^k$. We claim that ϑ yields a homomorphism from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathcal{M} \ltimes \mathcal{N}}(\mathbf{A}^{(k)})$. To this end, take $R \in \sigma$ of arity r and $\mathbf{x} \in R^{\mathbf{X}}$, so $\mathbf{x}^{(k)} \in R^{\mathbf{X}^{(k)}}$. We need to show that $\vartheta(\mathbf{x}^{(k)}) \in R^{\mathbb{F}_{\mathcal{M} \ltimes \mathcal{N}}(\mathbf{A}^{(k)})}$; equivalently, we need to find some $W \in (\mathcal{M} \ltimes \mathcal{N})^{(m)}$

such that $\vartheta(\mathbf{x_i}) = W_{/\pi_i}$ for each $\mathbf{i} \in [r]^k$, where $m = |R^{\mathbf{A}}|$. Using that ξ is a homomorphism, we have that $\xi(\mathbf{x}^{(k)}) \in R^{\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})}$, so there exists $Q \in \mathscr{M}^{(m)}$ for which $\xi(\mathbf{x}_{\mathbf{i}}) = Q_{/\pi_{\mathbf{i}}}$ for each $\mathbf{i} \in [r]^k$. Similarly, using that ζ is a homomorphism, we can find $Z \in \mathcal{N}^{(m)}$ for which $\zeta(\mathbf{x_i}) = Z_{/\pi_i}$ for each $\mathbf{i} \in [r]^k$. We now show that $\begin{bmatrix} Q & Z \end{bmatrix} \in (\mathcal{M} \ltimes \mathcal{N})^{(m)}$. To this end, take $\mathbf{a} \in R^{\mathbf{A}}$ such that $E_{\mathbf{a}} * Q = \mathbf{0}_d$; we need to prove that $E_{\mathbf{a}} * Z = \mathbf{0}_{d'}$. Using the assumption that $k \geq r$, let us pick the tuple $\mathbf{j} = (1, 2, \dots, r, 1, 1, \dots, 1) \in [r]^k$. Notice that this choice guarantees that $\{\mathbf{b} \in R^{\mathbf{A}} : \mathbf{b}_{\mathbf{j}} = \mathbf{a}_{\mathbf{j}}\} = \{\mathbf{a}\}$. Hence,

$$E_{\mathbf{a}_{\mathbf{j}}} * \xi(\mathbf{x}_{\mathbf{j}}) = E_{\mathbf{a}_{\mathbf{j}}} * Q_{/\pi_{\mathbf{j}}} = E_{\mathbf{a}_{\mathbf{j}}} * P_{\mathbf{j}} * Q = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{j}} = \mathbf{a}_{\mathbf{j}}}} E_{\mathbf{b}} * Q = E_{\mathbf{a}} * Q,$$

where the third equality follows from Lemma 30. Similarly, $E_{\mathbf{a}_{j}} * \zeta(\mathbf{x}_{j}) = E_{\mathbf{a}} * Z$. Then, from our assumption $E_{\mathbf{a}} * Q = \mathbf{0}_{d}$ it follows that $E_{\mathbf{a}_{j}} * \xi(\mathbf{x}_{j}) = \mathbf{0}_{d}$. Using the hypothesis of the proposition, we deduce that $E_{\mathbf{a}_{j}} * \zeta(\mathbf{x}_{j}) = \mathbf{0}_{d'}$, and we thus conclude that $E_{\mathbf{a}} * Z = \mathbf{0}_{d'}$, as wanted. Call $W = \begin{bmatrix} Q & Z \end{bmatrix}$. For each $\mathbf{i} \in [r]^k$, we find

$$\vartheta(\mathbf{x_i}) = \begin{bmatrix} \xi(\mathbf{x_i}) & \zeta(\mathbf{x_i}) \end{bmatrix} = \begin{bmatrix} Q_{/\pi_i} & Z_{/\pi_i} \end{bmatrix} = \begin{bmatrix} P_i \stackrel{1}{*}Q & P_i \stackrel{1}{*}Z \end{bmatrix} = P_i \stackrel{1}{*} \begin{bmatrix} Q & Z \end{bmatrix} = W_{/\pi_i},$$

as required. This concludes the proof that ϑ is a homomorphism. Conversely, let $\vartheta : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M} \ltimes \mathscr{N}}(\mathbf{A}^{(k)})$ be a homomorphism. For each $\mathbf{x} \in X^k$, write $\vartheta(\mathbf{x}) \in (\mathscr{M} \ltimes \mathscr{N})^{(n^k)}$ as $\vartheta(\mathbf{x}) = [M_{(\mathbf{x})} \ N_{(\mathbf{x})}]$, where $M_{(\mathbf{x})} \in \mathscr{M}^{(n^k)}$ and $N_{(\mathbf{x})} \in \mathscr{N}^{(n^k)}$. Using the same argument as in the previous part of the proof, we check that the assignment $\mathbf{x} \mapsto M_{(\mathbf{x})}$ (resp. $\mathbf{x} \mapsto N_{(\mathbf{x})}$) yields a homomorphism from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathscr{M}}(\mathbf{A}^{(k)})$ (resp. to $\mathbb{F}_{\mathscr{N}}(\mathbf{A}^{(\Bbbk)})$, and that the implication $E_{\mathbf{a}} * \xi(\mathbf{x}) = \mathbf{0}_d \implies E_{\mathbf{a}} * \zeta(\mathbf{x}) = \mathbf{0}_{d'}$ is met for each $\mathbf{x} \in X^k$ and $\mathbf{a} \in A^k$.

Remark 45. In recent work [55], Dalmau and Opršal used the notion of conic minions in the context of reductions between PCSPs. For a minion \mathcal{M} , they construct a new minion $\omega(\mathcal{M})$ that they use to characterise the applicability of arc-consistency reductions. If \mathcal{M} is linear, $\omega(\mathcal{M})$ coincides with the semi-direct product between \mathcal{H} and \mathcal{M} (cf. Remark 42). It is not hard to show that a linear minion \mathscr{M} satisfies $\mathscr{M} \to \mathscr{H} \ltimes \mathscr{M}$ if and only if it is homomorphically equivalent to a conic minion. Using this fact, it can be shown¹⁶ that the k-consistency reductions from [55] and the k-th level of a hierarchy of minion tests as defined in this paper are equivalent for conic minions.

A proof of Theorem 15 11

In this section, we prove Theorem 15 using the machinery developed in Sections 7, 8, 9, and 10.

Theorem (Theorem 15 restated – Informal). If $k \in \mathbb{N}$ is at least the maximum arity of the

¹⁶Personal communication with Jakub Opršal.

template,

• $BW^k = Test^k_{\mathscr{H}}$ • $SA^k = Test^k_{\mathscr{D}_{conv}}$ • $AIP^k = Test^k_{\mathscr{D}_{aff}}$ • $SoS^k = Test^k_{\mathscr{H}_{PA}}$ • $BA^k = Test^k_{\mathscr{M}_{PA}}$

The five parts of the theorem will be formally stated and proved separately, in Propositions 46, 47, 48, 49, and 50. The statements in Propositions 47 and 49, concerning SA^k and SoS^k , are actually slightly stronger than Theorem 15, as they do not require that the level k of the hierarchy be at least the maximum arity of the template (cf. Remark 51).

Proposition 46. Let $k \in \mathbb{N}$ and let \mathbf{X} , \mathbf{A} be k-enhanced σ -structures such that $k \geq \operatorname{armax}(\sigma)$. Then $\operatorname{BW}^k(\mathbf{X}, \mathbf{A}) = \operatorname{Test}^k_{\mathscr{H}}(\mathbf{X}, \mathbf{A})$.

Proof. Given two sets S, T, an integer $p \in \mathbb{N}$, and two tuples $\mathbf{s} = (s_1, \ldots, s_p) \in S^p, \mathbf{t} = (t_1, \ldots, t_p) \in T^p$ such that $\mathbf{s} \prec \mathbf{t}$, we shall consider the function $f_{\mathbf{s}, \mathbf{t}} : {\mathbf{s}} \to T$ defined by $f_{\mathbf{s}, \mathbf{t}}(s_\alpha) = t_\alpha$ for each $\alpha \in [p]$. Also, we denote by $\epsilon : \emptyset \to \mathbf{A}$ the empty mapping.

Suppose BW^k(**X**, **A**) = YES and let \mathcal{F} be a nonempty collection of partial homomorphisms from **X** to **A** witnessing it. Recall from Section 4.1 that the space of tensors $\mathcal{T}^{n\cdot\mathbf{1}_k,\mathbf{1}}(\{0,1\})$ can be identified with $\mathcal{T}^{n\cdot\mathbf{1}_k}(\{0,1\})$. Define the map $\xi: X^k \to \mathcal{T}^{n\cdot\mathbf{1}_k}(\{0,1\})$ by setting, for $\mathbf{x} \in X^k$ and $\mathbf{a} \in A^k$, $E_{\mathbf{a}} * \xi(\mathbf{x}) = 1$ if $\mathbf{x} \prec \mathbf{a}$ and $f_{\mathbf{x},\mathbf{a}} \in \mathcal{F}$, $E_{\mathbf{a}} * \xi(\mathbf{x}) = 0$ otherwise. We claim that ξ yields a homomorphism from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathscr{H}}(\mathbf{A}^{(k)})$. Take $R \in \sigma$ of arity r and $\mathbf{y} \in R^{\mathbf{X}}$, so $\mathbf{y}^{(k)} \in R^{\mathbf{X}^{(k)}}$. We need to show that $\xi(\mathbf{y}^{(k)}) \in R^{\mathbb{F}_{\mathscr{H}}(\mathbf{A}^{(k)})}$. Since $k \ge r$, we can write $\mathbf{y} = \mathbf{x}_i$ for some $\mathbf{x} \in X^k$, $\mathbf{i} \in [k]^r$. Given $\mathbf{a} \in R^{\mathbf{A}}$, consider the set $B_{\mathbf{a}} = \{\mathbf{b} \in A^k : \mathbf{b}_i = \mathbf{a} \text{ and } \mathbf{x} \prec \mathbf{b}\}$. We define a vector $\mathbf{q} \in \mathcal{T}^{|R^{\mathbf{A}|}(\{0,1\})$ by letting, for each $\mathbf{a} \in R^{\mathbf{A}}$, the **a**-th entry of \mathbf{q} be 1 if $f_{\mathbf{x},\mathbf{b} \in \mathcal{F}$ for some $\mathbf{b} \in B_{\mathbf{a}}$, 0 otherwise. We now show that $\mathbf{q} \in \mathscr{H}^{(|R^{\mathbf{A}|)}|$; i.e., that \mathbf{q} is not identically zero. Observe first that, since \mathcal{F} is nonempty and closed under restrictions, it contains the empty mapping ϵ . Applying the extension property to ϵ , we find that there exists some $f \in \mathcal{F}$ whose domain is $\{\mathbf{x}\}$ – that is, there exists some $\mathbf{c} \in A^k$ such that $\mathbf{x} \prec \mathbf{c}$ and $f_{\mathbf{x},\mathbf{c}} = f \in \mathcal{F}$. Notice that $\mathbf{y} \in R^{\mathbf{X}} \cap \{\mathbf{x}\}^r = R^{\mathbf{X}[\{\mathbf{x}\}]}$ (where we recall that $\mathbf{X}[\{\mathbf{x}\}]$ is the substructure of \mathbf{X} induced by $\{\mathbf{x}\}$). Using that $f_{\mathbf{x},\mathbf{c}}$ is a partial homomorphism, we obtain $\mathbf{c}_i = f_{\mathbf{x},\mathbf{c}}(\mathbf{x}_i) = f_{\mathbf{x},\mathbf{c}}(\mathbf{y}) \in R^{\mathbf{A}}$. We then conclude that $\mathbf{e}_{\mathbf{c}_i} * \mathbf{q} = 1$, so $\mathbf{q} \in \mathscr{H}^{(|R^{\mathbf{A}|)}$, as required. If we manage to show that $\mathbf{q}_{/\pi_\ell} = \xi(\mathbf{y}_\ell)$ for any $\ell \in [r]^k$, we can conclude that $\xi(\mathbf{y}^{(k)}) \in R^{\mathbb{F}_{\mathscr{H}}(\mathbf{A}^{(k)})}$, thus proving the claim. Recall from Proposition

$$E_{\mathbf{a}} * \mathbf{q}_{/\pi_{\ell}} = E_{\mathbf{a}} * P_{\ell} * \mathbf{q} = \sum_{\substack{\mathbf{c} \in R^{\mathbf{A}} \\ \mathbf{c}_{\ell} = \mathbf{a}}} E_{\mathbf{c}} * \mathbf{q} = \bigvee_{\substack{\mathbf{c} \in R^{\mathbf{A}} \\ \mathbf{c}_{\ell} = \mathbf{a}}} E_{\mathbf{c}} * \mathbf{q}.$$
 (13)

It follows that the expression in (13) equals 1 if

$$\exists \mathbf{c} \in R^{\mathbf{A}} \text{ s.t. } \mathbf{c}_{\ell} = \mathbf{a} \text{ and } f_{\mathbf{x},\mathbf{b}} \in \mathcal{F} \text{ for some } \mathbf{b} \in B_{\mathbf{c}}, \qquad (\star)$$

0 otherwise. On the other hand, $E_{\mathbf{a}} * \xi(\mathbf{y}_{\ell})$ equals 1 if

$$\mathbf{y}_{\ell} \prec \mathbf{a} \text{ and } f_{\mathbf{y}_{\ell}, \mathbf{a}} \in \mathcal{F},$$
 (•)

0 otherwise. We now show that the conditions (*) and (•) are equivalent, which concludes the proof of the claim. Suppose that (*) holds. Since $\mathbf{b} \in B_{\mathbf{c}}$, we have $\mathbf{x} \prec \mathbf{b}$, which yields $\mathbf{x}_{i_{\ell}} \prec \mathbf{b}_{i_{\ell}}$ as " \prec " is preserved under projections. Using the restriction property applied to $f_{\mathbf{x},\mathbf{b}}$, we find that $f_{\mathbf{x}_{i_{\ell}},\mathbf{b}_{i_{\ell}}} \in \mathcal{F}$. Then, (•) follows by observing that $\mathbf{x}_{i_{\ell}} = \mathbf{y}_{\ell}$ and, since $\mathbf{b} \in B_{\mathbf{c}}$ and $\mathbf{c}_{\ell} = \mathbf{a}$, $\mathbf{b}_{i_{\ell}} = \mathbf{a}$. Suppose now that (•) holds. Using the extension property applied to $f_{\mathbf{y}_{\ell},\mathbf{a}} = f_{\mathbf{x}_{i_{\ell}},\mathbf{a}}$ we find that $f_{\mathbf{x},\mathbf{b}} \in \mathcal{F}$ for some $\mathbf{b} \in A^k$ such that $\mathbf{x} \prec \mathbf{b}$ and $\mathbf{b}_{i_{\ell}} = \mathbf{a}$. Since $f_{\mathbf{x},\mathbf{b}}$ is a partial homomorphism from \mathbf{X} to \mathbf{A} , $R^{\mathbf{A}} \ni f_{\mathbf{x},\mathbf{b}}(\mathbf{y}) = f_{\mathbf{x},\mathbf{b}}(\mathbf{x}_{\mathbf{i}}) = \mathbf{b}_{\mathbf{i}}$. Calling $\mathbf{c} = \mathbf{b}_{\mathbf{i}}$, we obtain (*).

Conversely, suppose that $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{H}}(\mathbf{A}^{(k)})$ is a homomorphism witnessing that $\operatorname{Test}_{\mathscr{H}}^{k}(\mathbf{X}, \mathbf{A})$ accepts, and consider the collection $\mathcal{F} = \{f_{\mathbf{x},\mathbf{a}} : \mathbf{x} \in X^{k}, \mathbf{a} \in \operatorname{supp}(\xi(\mathbf{x}))\} \cup \{\epsilon\}$. Notice that \mathcal{F} is well defined by virtue of Lemma 32, as $\mathbf{a} \in \operatorname{supp}(\xi(\mathbf{x}))$ implies that $\mathbf{x} \prec \mathbf{a}$, and it is nonempty. We claim that any function $f_{\mathbf{x},\mathbf{a}} \in \mathcal{F}$ is a partial homomorphism from \mathbf{X} to \mathbf{A} . (Notice that ϵ is trivially a partial homomorphism.) Indeed, given $R \in \sigma$ of arity r and $\mathbf{y} \in R^{\mathbf{X}[\{\mathbf{x}\}]} = R^{\mathbf{X}} \cap \{\mathbf{x}\}^{r}$, we can write $\mathbf{y} = \mathbf{x}_{\mathbf{i}}$ for some $\mathbf{i} \in [k]^{r}$. Then, $f_{\mathbf{x},\mathbf{a}}(\mathbf{y}) = f_{\mathbf{x},\mathbf{a}}(\mathbf{x}_{\mathbf{i}}) = \mathbf{a}_{\mathbf{i}} \in R^{\mathbf{A}}$, where we have used Proposition 38 (which applies to \mathscr{H} since, by Proposition 21, \mathscr{H} is a conic minion). To show that \mathcal{F} is closed under restrictions, take $f \in \mathcal{F}$ and $V \subseteq \operatorname{dom}(f)$; we need to show that $f|_{V} \in \mathcal{F}$. The cases $f = \epsilon$ or $V = \emptyset$ are trivial, so we can assume $f = f_{\mathbf{x},\mathbf{a}}$ (which means that $\operatorname{dom}(f) = \{\mathbf{x}\}$) and write $V = \{\mathbf{x}_{\ell}\}$ for some $\ell \in [k]^{k}$. Observe that $f_{\mathbf{x},\mathbf{a}|_{V}} = f_{\mathbf{x}_{\ell},\mathbf{a}_{\ell}}$. We claim that $E_{\mathbf{a}_{\ell}} * \xi(\mathbf{x}_{\ell}) = 1$. Otherwise, using Proposition 36 and Lemma 34, we would have

$$0 = E_{\mathbf{a}_{\ell}} * \xi(\mathbf{x}_{\ell}) = E_{\mathbf{a}_{\ell}} * (\Pi_{\ell} * \xi(\mathbf{x})) = E_{\mathbf{a}_{\ell}} * \Pi_{\ell} * \xi(\mathbf{x}) = \bigvee_{\substack{\mathbf{b} \in A^{k} \\ \mathbf{b}_{\ell} = \mathbf{a}_{\ell}}} E_{\mathbf{b}} * \xi(\mathbf{x}).$$

This would imply that $E_{\mathbf{b}} * \xi(\mathbf{x}) = 0$ whenever $\mathbf{b} \in A^k$ is such that $\mathbf{b}_{\ell} = \mathbf{a}_{\ell}$; in particular, $E_{\mathbf{a}} * \xi(\mathbf{x}) = 0$, a contradiction. So $E_{\mathbf{a}_{\ell}} * \xi(\mathbf{x}_{\ell}) = 1$, as claimed, and it follows that $f_{\mathbf{x}_{\ell},\mathbf{a}_{\ell}} \in \mathcal{F}$. We now claim that \mathcal{F} has the extension property up to k. Take $f \in \mathcal{F}$ and $V \subseteq X$ such that $|V| \leq k$ and dom $(f) \subseteq V$; we need to show that there exists $g \in \mathcal{F}$ such that g extends f and dom(g) = V. If $f = \epsilon$ and $V = \emptyset$, the claim is trivial; if $f = \epsilon$ and $V \neq \emptyset$, we can write $V = \{\mathbf{x}\}$ for some $\mathbf{x} \in X^k$, and the claim follows by noticing that, by the definition of \mathscr{H} , $\mathrm{supp}(\xi(\mathbf{x})) \neq \emptyset$. Therefore, we can assume that $f \neq \epsilon$, so $f = f_{\mathbf{x},\mathbf{a}}$ for some $\mathbf{x} \in X^k$, $\mathbf{a} \in \mathrm{supp}(\xi(\mathbf{x}))$. Since $V \neq \emptyset$, we can write $V = \{\mathbf{y}\}$ for some $\mathbf{y} \in X^k$. Then, dom $(f) \subseteq V$ becomes $\{\mathbf{x}\} \subseteq \{\mathbf{y}\}$, so $\mathbf{x} = \mathbf{y}_{\ell}$ for some $\ell \in [k]^k$. Using Proposition 36 and Lemma 34, we find

$$1 = E_{\mathbf{a}} * \xi(\mathbf{x}) = E_{\mathbf{a}} * \xi(\mathbf{y}_{\ell}) = E_{\mathbf{a}} * \Pi_{\ell} * \xi(\mathbf{y}) = \bigvee_{\substack{\mathbf{b} \in A^k \\ \mathbf{b}_{\ell} = \mathbf{a}}} E_{\mathbf{b}} * \xi(\mathbf{y}),$$

which implies that $E_{\mathbf{b}} * \xi(\mathbf{y}) = 1$ for some $\mathbf{b} \in A^k$ such that $\mathbf{b}_{\ell} = \mathbf{a}$. It follows that $f_{\mathbf{y},\mathbf{b}} \in \mathcal{F}$. Notice that $\operatorname{dom}(f_{\mathbf{y},\mathbf{b}}) = \{\mathbf{y}\} = V$, and $f_{\mathbf{y},\mathbf{b}}|_{\{\mathbf{x}\}} = f_{\mathbf{x},\mathbf{a}}$, so the claim is true. Hence, \mathcal{F} witnesses that $\operatorname{BW}^k(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$.

Proposition 47. Let $k \in \mathbb{N}$ and let \mathbf{X} , \mathbf{A} be k-enhanced σ -structures such that $k \geq \min(2, \operatorname{armax}(\sigma))$. Then $\operatorname{SA}^{k}(\mathbf{X}, \mathbf{A}) = \operatorname{Test}_{\mathscr{Q}_{\operatorname{conv}}}^{k}(\mathbf{X}, \mathbf{A})$.

Proof. Suppose $\mathrm{SA}^k(\mathbf{X}, \mathbf{A}) = \mathrm{YES}$ and let the rational numbers $\lambda_{R,\mathbf{x},\mathbf{a}}$ witness it, for $R \in \sigma$, $\mathbf{x} \in R^{\mathbf{X}}$, and $\mathbf{a} \in R^{\mathbf{A}}$. Consider the map $\xi : X^k \to \mathcal{T}^{n \cdot \mathbf{1}_k}(\mathbb{Q})$ defined by $E_{\mathbf{a}} * \xi(\mathbf{x}) = \lambda_{R_k,\mathbf{x},\mathbf{a}}$

for any $\mathbf{x} \in X^k$, $\mathbf{a} \in A^k$. We claim that ξ yields a homomorphism from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathscr{Q}_{\text{conv}}}(\mathbf{A}^{(k)})$. Notice first that, for any $\mathbf{x} \in X^k$, $\xi(\mathbf{x})$ is an entrywise nonnegative tensor in the space $\mathcal{T}^{n \cdot \mathbf{1}_k}(\mathbb{Q})$ (which can be identified with $\mathcal{T}^{n \cdot \mathbf{1}_k, 1}(\mathbb{Q})$). Moreover, using \clubsuit 1, we find

$$\sum_{\mathbf{a}\in A^k} E_{\mathbf{a}} * \xi(\mathbf{x}) = \sum_{\mathbf{a}\in R_k^{\mathbf{A}}} \lambda_{R_k,\mathbf{x},\mathbf{a}} = 1.$$

It follows that $\xi(\mathbf{x}) \in \mathscr{Q}_{\text{conv}}^{(n^k)}$. We now prove that ξ yields a homomorphism from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathscr{Q}_{\text{conv}}}(\mathbf{A}^{(k)})$. Take a symbol $R \in \sigma$ of arity r and a tuple $\mathbf{x} \in R^{\mathbf{X}}$, so that $\mathbf{x}^{(k)} \in R^{\mathbf{X}^{(k)}}$. We need to show that $\xi(\mathbf{x}^{(k)}) \in R^{\mathbb{F}_{\mathscr{Q}_{\text{conv}}}(\mathbf{A}^{(k)})}$. Equivalently, we seek some vector $\mathbf{q} \in \mathscr{Q}_{\text{conv}}^{(|R^{\mathbf{A}}|)}$ such that $\xi(\mathbf{x}_{\mathbf{i}}) = \mathbf{q}_{/\pi_{\mathbf{i}}}$ for any $\mathbf{i} \in [r]^{k}$. Consider the vector $\mathbf{q} \in \mathcal{T}^{|R^{\mathbf{A}}|}(\mathbb{Q})$ defined by $E_{\mathbf{a}} * \mathbf{q} = \lambda_{R,\mathbf{x},\mathbf{a}}$ for any $\mathbf{a} \in R^{\mathbf{A}}$. Similarly as before, $\clubsuit 1$ implies that $\mathbf{q} \in \mathscr{Q}_{\text{conv}}^{(|R^{\mathbf{A}}|)}$. For $\mathbf{i} \in [r]^{k}$ and $\mathbf{a} \in A^{k}$, we have

$$E_{\mathbf{a}} * \xi(\mathbf{x}_{\mathbf{i}}) = \lambda_{R_k, \mathbf{x}_{\mathbf{i}}, \mathbf{a}} = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a}}} \lambda_{R, \mathbf{x}, \mathbf{b}} = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a}}} E_{\mathbf{b}} * \mathbf{q} = E_{\mathbf{a}} * P_{\mathbf{i}} * \mathbf{q} = E_{\mathbf{a}} * \mathbf{q}_{/\pi_{\mathbf{i}}},$$

where the second equality is $\clubsuit 2$ and the fourth follows from Lemma 30. We deduce that ξ is a homomorphism, as claimed, which means that $\text{Test}_{\mathscr{Q}_{\text{conv}}}^k(\mathbf{X}, \mathbf{A}) = \text{YES}$.

Conversely, suppose that ξ is a homomorphism from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathcal{Q}_{conv}}(\mathbf{A}^{(k)})$ witnessing that $\operatorname{Test}_{\mathcal{Q}_{conv}}^{k}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$. We associate with any pair (R, \mathbf{x}) such that $R \in \sigma$ and $\mathbf{x} \in R^{\mathbf{X}}$ a vector $\mathbf{q}_{R,\mathbf{x}} \in \mathcal{Q}_{conv}^{(|R^{\mathbf{A}}|)}$ defined as follows. Using that $\mathbf{x}^{(k)} \in R^{\mathbf{X}^{(k)}}$ and ξ is a homomorphism, we deduce that $\xi(\mathbf{x}^{(k)}) \in R^{\mathbb{F}_{\mathcal{Q}_{conv}}(\mathbf{A}^{(k)})}$ – i.e., there exists $\mathbf{q} \in \mathcal{Q}_{conv}^{(|R^{\mathbf{A}}|)}$ such that $\xi(\mathbf{x}_{\mathbf{i}}) = \mathbf{q}_{/\pi_{\mathbf{i}}} = P_{\mathbf{i}} * \mathbf{q}$ for each $\mathbf{i} \in [r]^{k}$, where r is the arity of R. We set $\mathbf{q}_{R,\mathbf{x}} = \mathbf{q}$. We now build a solution to $\operatorname{SA}^{k}(\mathbf{X}, \mathbf{A})$ as follows: For any $R \in \sigma$, $\mathbf{x} \in R^{\mathbf{X}}$, and $\mathbf{a} \in R^{\mathbf{A}}$, we set $\lambda_{R,\mathbf{x},\mathbf{a}} = E_{\mathbf{a}} * \mathbf{q}_{R,\mathbf{x}}$. Notice that each $\lambda_{R,\mathbf{x},\mathbf{a}}$ is a rational number in the interval [0, 1]. Moreover, for $R \in \sigma$ and $\mathbf{x} \in R^{\mathbf{X}}$, we have

$$\sum_{\mathbf{a}\in R^{\mathbf{A}}}\lambda_{R,\mathbf{x},\mathbf{a}} = \sum_{\mathbf{a}\in R^{\mathbf{A}}}E_{\mathbf{a}}*\mathbf{q}_{R,\mathbf{x}} = 1$$

since $\mathbf{q}_{R,\mathbf{x}} \in \mathscr{Q}_{\text{conv}}$, thus yielding $\clubsuit 1$. Observe that, for any $\mathbf{y} \in X^k$, we have $\mathbf{q}_{R_k,\mathbf{y}} = \xi(\mathbf{y})$. Indeed, letting $\mathbf{j} = (1, \ldots, k) \in [k]^k$, we have

$$\mathbf{q}_{R_k,\mathbf{y}} = \Pi_{\mathbf{j}} * \mathbf{q}_{R_k,\mathbf{y}} = P_{\mathbf{j}} * \mathbf{q}_{R_k,\mathbf{y}} = \mathbf{q}_{R_k,\mathbf{y}/\pi_{\mathbf{j}}} = \xi(\mathbf{y}_{\mathbf{j}}) = \xi(\mathbf{y}), \tag{14}$$

where the first equality follows from the fact that the contraction by $\Pi_{\mathbf{j}}$ acts as the identity (cf. the proof of Proposition 36) and the second from Remark 35. Then, $\clubsuit 2$ follows by noticing that, for $\mathbf{i} \in [r]^k$ and $\mathbf{b} \in A^k$,

$$\sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\\mathbf{a}_{\mathbf{i}}=\mathbf{b}}} \lambda_{R,\mathbf{x},\mathbf{a}} = \sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\\mathbf{a}_{\mathbf{i}}=\mathbf{b}}} E_{\mathbf{a}} * \mathbf{q}_{R,\mathbf{x}} = E_{\mathbf{b}} * P_{\mathbf{i}} * \mathbf{q}_{R,\mathbf{x}} = E_{\mathbf{b}} * \xi(\mathbf{x}_{\mathbf{i}}) = E_{\mathbf{b}} * \mathbf{q}_{R_{k},\mathbf{x}_{\mathbf{i}}} = \lambda_{R_{k},\mathbf{x}_{\mathbf{i}},\mathbf{b}},$$

where the second and fourth equalities follow from Lemma 30 and (14), respectively. Recall from Proposition 21 that $\mathscr{Q}_{\text{conv}}$ is a conic minion. Using either Lemma 33 or Lemma 37 (depending on whether $k \ge \operatorname{armax}(\sigma)$ or $k \ge 2$), if $\mathbf{a} \in \mathbb{R}^{\mathbf{A}}$ is such that $\mathbf{x} \not\prec \mathbf{a}$, we obtain

$$\lambda_{R,\mathbf{x},\mathbf{a}} = E_{\mathbf{a}} * \mathbf{q}_{R,\mathbf{x}} = 0,$$

thus showing that \$3 is satisfied, too. It follows that $SA^k(\mathbf{X}, \mathbf{A}) = YES$, as required. \Box

Proposition 48. Let $k \in \mathbb{N}$ and let \mathbf{X}, \mathbf{A} be k-enhanced σ -structures such that $k \geq \operatorname{armax}(\sigma)$. Then $\operatorname{AIP}^k(\mathbf{X}, \mathbf{A}) = \operatorname{Test}^k_{\mathscr{Z}_{\operatorname{off}}}(\mathbf{X}, \mathbf{A})$.

Proof. The proof is analogous to that of Proposition 47, the only difference being that Lemma 37 cannot be applied in this case since \mathscr{Z}_{aff} is not a conic minion (cf. Proposition 21). As a consequence, unlike in Proposition 47, we need to assume that $k \geq \operatorname{armax}(\sigma)$.

The proof of Theorem 15 for SoS^k , given in Proposition 49 below, follows the same scheme as that of Proposition 47. There is, however, one additional complication due to the fact that the objects in the minion \mathscr{S} are matrices having infinitely many columns and finite, but arbitrarily large, csupp. We deal with this technical issue through the orthonormalisation argument we already used for the proof of Proposition 8: We find an orthonormal basis for the finitely generated vector space defined as the sum of the row-spaces of the matrices in \mathscr{S} appearing as images of a given homomorphism.

Proposition 49. Let $k \in \mathbb{N}$ and let \mathbf{X} , \mathbf{A} be k-enhanced σ -structures such that $k \ge \min(2, \operatorname{armax}(\sigma))$. Then $\operatorname{SoS}^{k}(\mathbf{X}, \mathbf{A}) = \operatorname{Test}^{k}_{\mathscr{S}}(\mathbf{X}, \mathbf{A})$.

Proof. Suppose that $\operatorname{SoS}^{k}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$ and let the family of vectors $\lambda_{R, \mathbf{x}, \mathbf{a}} \in \mathbb{R}^{\omega}$ witness it, where $\omega = \sum_{R \in \sigma} |R^{\mathbf{X}}| \cdot |R^{\mathbf{A}}|$. Consider the map $\xi : X^{k} \to \mathcal{T}^{n \cdot \mathbf{1}_{k}, \aleph_{0}}(\mathbb{R})$ defined by

$$E_{\mathbf{a}} * \xi(\mathbf{x}) = \begin{bmatrix} \boldsymbol{\lambda}_{R_k, \mathbf{x}, \mathbf{a}} \\ \mathbf{0}_{\aleph_0} \end{bmatrix} \qquad \mathbf{x} \in X^k, \mathbf{a} \in A^k.$$

We claim that ξ yields a homomorphism from $\mathbf{X}^{(k)}$ to $\mathbb{F}_{\mathscr{S}}(\mathbf{A}^{(k)})$. First of all, we need to show that $\xi(\mathbf{x}) \in \mathscr{S}^{(n^k)}$ for each $\mathbf{x} \in X^k$. The requirement C1 is trivially satisfied since, by construction, the *j*-th entry of $E_{\mathbf{a}} * \xi(\mathbf{x})$ is zero whenever $j > \omega$. Given $\mathbf{a}, \mathbf{a}' \in A^k$,

$$(E_{\mathbf{a}} * \xi(\mathbf{x}))^T (E_{\mathbf{a}'} * \xi(\mathbf{x})) = \boldsymbol{\lambda}_{R_k, \mathbf{x}, \mathbf{a}} \cdot \boldsymbol{\lambda}_{R_k, \mathbf{x}, \mathbf{a}'}.$$

If $\mathbf{a} \neq \mathbf{a}'$, this quantity is zero by \mathbf{a}^2 , so C2 is satisfied. Finally,

$$\sum_{\mathbf{a}\in A^k} (E_{\mathbf{a}} * \xi(\mathbf{x}))^T (E_{\mathbf{a}} * \xi(\mathbf{x})) = \sum_{\mathbf{a}\in A^k} \|\boldsymbol{\lambda}_{R_k,\mathbf{x},\mathbf{a}}\|^2 = 1$$

by $\blacklozenge 1$, so C3 is also satisfied. Therefore, $\xi(\mathbf{x}) \in \mathscr{S}^{(n^k)}$. To show that ξ is in fact a homomorphism, take $R \in \sigma$ of arity r and $\mathbf{x} \in R^{\mathbf{X}}$, so $\mathbf{x}^{(k)} \in R^{\mathbf{X}^{(k)}}$. We need to show that $\xi(\mathbf{x}^{(k)}) \in R^{\mathbb{F}_{\mathscr{S}}(\mathbf{A}^{(k)})}$. Consider the matrix $Q \in \mathcal{T}^{|R^{\mathbf{A}}|,\aleph_0}(\mathbb{R})$ defined by

$$Q^T \mathbf{e_a} = \begin{bmatrix} \boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}} \\ \mathbf{0}_{\aleph_0} \end{bmatrix} \qquad \mathbf{a} \in R^{\mathbf{A}}.$$

Using the same arguments as above, we check that Q satisfies C1 and that $\mathbf{e}_{\mathbf{a}}^{T}QQ^{T}\mathbf{e}_{\mathbf{a}'} = \lambda_{R,\mathbf{x}\mathbf{a}} \cdot \lambda_{R,\mathbf{x},\mathbf{a}'}$, so C2 follows from $\blacklozenge 2$ and C3 from $\blacklozenge 1$. Therefore, $Q \in \mathscr{S}^{(|R^{\mathbf{A}}|)}$. We now claim that $\xi(\mathbf{x}_{\mathbf{i}}) = Q_{/\pi_{\mathbf{i}}}$ for each $\mathbf{i} \in [r]^{k}$. Indeed, observe that, for each $\mathbf{a} \in A^{k}$,

$$\begin{split} E_{\mathbf{a}} * Q_{/\pi_{\mathbf{i}}} &= E_{\mathbf{a}} * P_{\mathbf{i}} * Q = \left(\sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a}}} E_{\mathbf{b}}\right) * Q = Q^{T} \left(\sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a}}} \mathbf{e}_{\mathbf{b}}\right) = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a}}} Q^{T} \mathbf{e}_{\mathbf{b}} \\ &= \begin{bmatrix} \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a}}} \lambda_{R, \mathbf{x}, \mathbf{b}} \\ \mathbf{b}_{\mathbf{i}} = \mathbf{a} \\ \mathbf{0}_{\aleph_{0}} \end{bmatrix} = \begin{bmatrix} \lambda_{R_{k}, \mathbf{x}_{\mathbf{i}}, \mathbf{a}} \\ \mathbf{0}_{\aleph_{0}} \end{bmatrix} = E_{\mathbf{a}} * \xi(\mathbf{x}_{\mathbf{i}}), \end{split}$$

where the second and sixth equalities are obtained using Lemma 30 and \bigstar 3, respectively. It follows that the claim is true, so $\xi(\mathbf{x}^{(k)}) \in R^{\mathbb{F}_{\mathscr{S}}(\mathbf{A}^{(k)})}$, which concludes the proof that ξ is a homomorphism and that $\operatorname{Test}_{\mathscr{S}}^{k}(\mathbf{X}, \mathbf{A}) = \operatorname{YES}$. Conversely, let $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{S}}(\mathbf{A}^{(k)})$ be a homomorphism witnessing that $\operatorname{Test}_{\mathscr{S}}^{k}(\mathbf{X}, \mathbf{A}) =$ YES. Take $R \in \sigma$ of arity r and $\mathbf{x} \in R^{\mathbf{X}}$. We have that $\mathbf{x}^{(k)} \in R^{\mathbf{X}^{(k)}}$, so $\xi(\mathbf{x}^{(k)}) \in R^{\mathbb{F}_{\mathscr{S}}(\mathbf{A}^{(k)})}$

Conversely, let $\xi : \mathbf{X}^{(\Bbbk)} \to \mathbb{F}_{\mathscr{S}}(\mathbf{A}^{(\Bbbk)})$ be a homomorphism witnessing that $\operatorname{Test}_{\mathscr{S}}^{k}(\mathbf{X}, \mathbf{A}) =$ YES. Take $R \in \sigma$ of arity r and $\mathbf{x} \in R^{\mathbf{X}}$. We have that $\mathbf{x}^{(\Bbbk)} \in R^{\mathbf{X}^{(\Bbbk)}}$, so $\xi(\mathbf{x}^{(\Bbbk)}) \in R^{\mathbb{F}_{\mathscr{S}}(\mathbf{A}^{(\Bbbk)})}$ since ξ is a homomorphism. As a consequence, we can fix a matrix $Q_{R,\mathbf{x}} \in \mathscr{S}^{(|R^{\mathbf{A}}|)}$ satisfying $\xi(\mathbf{x}_{\mathbf{i}}) = Q_{R,\mathbf{x}/\pi_{\mathbf{i}}}$ for each $\mathbf{i} \in [r]^{k}$. Consider the set $S = \{Q_{R,\mathbf{x}}^{T}\mathbf{e}_{\mathbf{a}} : R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, \mathbf{a} \in R^{\mathbf{A}}\}$ and the vector space $\mathcal{U} = \operatorname{span}(S) \subseteq \mathbb{R}^{\aleph_{0}}$, and observe that $\dim(\mathcal{U}) \leq |S| \leq \sum_{R \in \sigma} |R^{\mathbf{X}}| \cdot |R^{\mathbf{A}}| = \omega$. Choose a vector space \mathcal{V} of dimension ω such that $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathbb{R}^{\aleph_{0}}$. Using the Gram–Schmidt process, we find a projection matrix $Z \in \mathbb{R}^{\aleph_{0},\omega}$ such that $Z^{T}Z = I_{\omega}$ and $ZZ^{T}\mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in \mathcal{V}$. Consider the family of vectors

$$\boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}} = Z^T Q_{R,\mathbf{x}}^T \mathbf{e}_{\mathbf{a}} \in \mathbb{R}^{\omega} \qquad R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, \mathbf{a} \in R^{\mathbf{A}}.$$
 (15)

We claim that (15) witnesses that $SoS^k(\mathbf{X}, \mathbf{A}) = YES$. Take $R \in \sigma$ of arity r and $\mathbf{x} \in R^{\mathbf{X}}$. Recall from Proposition 21 that \mathscr{S} is a conic minion. Using either Lemma 33 or Lemma 37 (depending on whether $k \ge \operatorname{armax}(\sigma)$ or $k \ge 2$), given $\mathbf{a} \in R^{\mathbf{A}}$ such that $\mathbf{x} \not\prec \mathbf{a}$, we find

$$\boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}} = Z^T Q_{R,\mathbf{x}}^T \mathbf{e}_{\mathbf{a}} = Z^T \mathbf{0}_{\aleph_0} = \mathbf{0}_{\omega},$$

so 4 holds. 1 follows from

$$\sum_{\mathbf{a}\in R^{\mathbf{A}}} \|\boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}}\|^2 = \sum_{\mathbf{a}\in R^{\mathbf{A}}} \mathbf{e}_{\mathbf{a}}^T Q_{R,\mathbf{x}} Z Z^T Q_{R,\mathbf{x}}^T \mathbf{e}_{\mathbf{a}} = \sum_{\mathbf{a}\in R^{\mathbf{A}}} \mathbf{e}_{\mathbf{a}}^T Q_{R,\mathbf{x}} Q_{R,\mathbf{x}}^T \mathbf{e}_{\mathbf{a}} = \operatorname{tr}(Q_{R,\mathbf{x}} Q_{R,\mathbf{x}}^T) = 1,$$

where the second equality is true since $Q_{R,\mathbf{x}}^T \mathbf{e}_{\mathbf{a}} \in S \subseteq \mathcal{U} \subseteq \mathcal{V}$ and the fourth follows from C3. Similarly, using C2, we find that, if $\mathbf{a} \neq \mathbf{a}' \in R^{\mathbf{A}}$,

$$\boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}} \cdot \boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}'} = \mathbf{e}_{\mathbf{a}}^T Q_{R,\mathbf{x}} Z Z^T Q_{R,\mathbf{x}}^T \mathbf{e}_{\mathbf{a}'} = \mathbf{e}_{\mathbf{a}}^T Q_{R,\mathbf{x}} Q_{R,\mathbf{x}}^T \mathbf{e}_{\mathbf{a}'} = 0$$

so $\bigstar 2$ holds. We now show that $Q_{R_k,\mathbf{y}} = \xi(\mathbf{y})$ for any $\mathbf{y} \in X^k$. Indeed, using the same argument as in (14), letting $\mathbf{j} = (1, \ldots, k) \in [k]^k$, we have

$$Q_{R_k,\mathbf{y}} = \Pi_{\mathbf{j}} \stackrel{k}{*} Q_{R_k,\mathbf{y}} = Q_{R_k,\mathbf{y}/\pi_{\mathbf{j}}} = \xi(\mathbf{y}_{\mathbf{j}}) = \xi(\mathbf{y}).$$
(16)

Given $\mathbf{i} \in [r]^k$ and $\mathbf{b} \in A^k = R_k^{\mathbf{A}}$, we have

$$\sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\\mathbf{a}_{i}=\mathbf{b}}} \lambda_{R,\mathbf{x},\mathbf{a}} = \sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\\mathbf{a}_{i}=\mathbf{b}}} Z^{T} Q_{R,\mathbf{x}}^{T} \mathbf{e}_{\mathbf{a}} = \sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\\mathbf{a}_{i}=\mathbf{b}}} E_{\mathbf{a}} * Q_{R,\mathbf{x}} * Z = E_{\mathbf{b}} * P_{\mathbf{i}} * Q_{R,\mathbf{x}} * Z = E_{\mathbf{b}} * Q_{R,\mathbf{x}_{/\pi_{\mathbf{i}}}} * Z$$
$$= E_{\mathbf{b}} * \xi(\mathbf{x}_{\mathbf{i}}) * Z = E_{\mathbf{b}} * Q_{R_{k},\mathbf{x}_{\mathbf{i}}} * Z = Z^{T} Q_{R_{k},\mathbf{x}_{\mathbf{i}}}^{T} \mathbf{e}_{\mathbf{b}} = \lambda_{R_{k},\mathbf{x}_{\mathbf{i}},\mathbf{b}},$$

where the third and sixth equalities follow from Lemma 30 and (16), respectively. This shows that $\bigstar 3$ holds, too, so that (15) yields a solution for $SoS^k(\mathbf{X}, \mathbf{A})$, as claimed.

Proposition 50. Let $k \in \mathbb{N}$ and let \mathbf{X}, \mathbf{A} be k-enhanced σ -structures such that $k \geq \operatorname{armax}(\sigma)$. Then $\operatorname{BA}^k(\mathbf{X}, \mathbf{A}) = \operatorname{Test}^k_{\mathscr{M}_{BA}}(\mathbf{X}, \mathbf{A})$. *Proof.* Recall from Section 4.2 that $BA^k(\mathbf{X}, \mathbf{A}) = YES$ is equivalent to the existence of a rational nonnegative solution (denoted by the superscript (B)) and an integer solution (denoted by the superscript (A)) to the system (2), such that

$$\lambda_{R,\mathbf{x},\mathbf{a}}^{(\mathrm{B})} = 0 \quad \Rightarrow \quad \lambda_{R,\mathbf{x},\mathbf{a}}^{(\mathrm{A})} = 0 \tag{17}$$

for each $R \in \sigma$, $\mathbf{x} \in R^{\mathbf{X}}$, and $\mathbf{a} \in R^{\mathbf{A}}$. Note that requiring (17) for each $R \in \sigma$ is equivalent to only requiring it for $R = R_k$. Indeed, take some $R \in \sigma$ of arity r, $\mathbf{x} \in R^{\mathbf{X}}$, and $\mathbf{a} \in R^{\mathbf{A}}$, and consider the tuple $\mathbf{i} = (1, 2, \ldots, r, 1, 1, \ldots, 1) \in [r]^k$, which is well defined as $k \geq r$. Noting that $\{\mathbf{b} \in R^{\mathbf{A}} : \mathbf{b_i} = \mathbf{a_i}\} = \{\mathbf{a}\}$, we find from \mathbf{a} 2 that

$$\lambda_{R,\mathbf{x},\mathbf{a}}^{(\mathrm{B})} = \sum_{\substack{\mathbf{b} \in R^{\mathbf{A}} \\ \mathbf{b}_{i} = \mathbf{a}_{i}}} \lambda_{R,\mathbf{x},\mathbf{b}}^{(\mathrm{B})} = \lambda_{R_{k},\mathbf{x}_{i},\mathbf{a}_{i}}^{(\mathrm{B})}$$

and, similarly, $\lambda_{R,\mathbf{x},\mathbf{a}}^{(A)} = \lambda_{R_k,\mathbf{x}_i,\mathbf{a}_i}^{(A)}$. As a consequence, if (17) holds for R_k , it also holds for R. Therefore, it follows from Propositions 47 and 48 that $BA^k(\mathbf{X}, \mathbf{A}) = YES$ is equivalent to the existence of homomorphisms $\xi : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{Q}_{conv}}(\mathbf{A}^{(k)})$ and $\zeta : \mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{Z}_{aff}}(\mathbf{A}^{(k)})$ such that $\operatorname{supp}(\zeta(\mathbf{x})) \subseteq \operatorname{supp}(\xi(\mathbf{x}))$ for each $\mathbf{x} \in X^k$. By virtue of Proposition 44, this happens precisely when $\mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{Q}_{conv} \ltimes \mathscr{Z}_{aff}}(\mathbf{A}^{(k)})$. Since $\mathscr{M}_{BA} = \mathscr{Q}_{conv} \ltimes \mathscr{Z}_{aff}$ (cf. Example 43), this is equivalent to $\mathbf{X}^{(k)} \to \mathbb{F}_{\mathscr{M}_{BA}}(\mathbf{A}^{(k)})$; i.e., to $\operatorname{Test}_{\mathscr{M}_{BA}}^k(\mathbf{X}, \mathbf{A}) = YES$.

Remark 51. The characterisations of SA^k and SoS^k in Propositions 47 and 49 hold for any higher level than the first, unlike the characterisation of AIP^k in Proposition 48. This is due to the fact that \mathscr{Q}_{conv} and \mathscr{S} are conic minions, so Lemma 37 applies, while \mathscr{Z}_{aff} is not. As for BW^k , Proposition 46 requires $k \ge \operatorname{armax}(\sigma)$ even if \mathscr{H} is a conic minion. The reason for this lies in the definition of the bounded width hierarchy. Essentially, any constraint whose scope has more than k distinct variables does not appear among the constraints of the partial homomorphisms witnessing acceptance of BW^k , while it does appear in the requirements of SA^k and SoS^k . Finally, assuming $k \ge \operatorname{armax}(\sigma)$ is also required in the characterisation of BA^k in Proposition 50, in order to make use of Proposition 44 and of the characterisation of AIP^k .

Remark 52. As it was shown in Section 9, hierarchies of relaxations built on conic minions (such as BW^k, SA^k, SoS^k, and BA^k) are "sound in the limit", in that their k-th level correctly classifies instances **X** with $|X| \leq k$ (cf. Proposition 39). This is not the case for the nonconic hierarchy AIP^k, as it was established in the follow-up work [42]. In [19], a stronger affine hierarchy was proposed, which – contrary to AIP^k – requires that the variables in the relaxation should be partial homomorphisms and is thus sound in the limit. By virtue of Proposition 38, this requirement can be captured by taking the semi-direct product of any conic minion and \mathscr{Z}_{aff} . In particular, it follows that the hierarchy in [19] is not stronger than the hierarchy built on the minion $\mathscr{H} \ltimes \mathscr{Z}_{aff}$ (cf. Remark 42). In recent work [46], a different algorithm for (P)CSPs has been proposed. The relationship of [46] with our work is an interesting direction for future research.

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A Notes on relaxations and hierarchies

In this appendix, we discuss some basic properties of the relaxations, and hierarchies thereof, presented in Section 4.2.

A.1 SA^k

The hierarchy defining SA^k given by the system (2) slightly differs from the one described in [39]. For completeness, we report below the hierarchy in [39] and we show that it is equivalent to the one adopted in this work.

Given two σ -structures \mathbf{X}, \mathbf{A} , introduce a variable $\mu_V(f)$ for every subset $V \subseteq X$ with $1 \leq |V| \leq k$ and every function $f: V \to A$, and a variable $\mu_{R,\mathbf{x}}(f)$ for every $R \in \sigma$, every $\mathbf{x} \in R^{\mathbf{X}}$, and every $f: {\mathbf{x}} \to A$. The k-th level of the hierarchy defined in [39] is given by the following constraints:

$$\begin{aligned} (\bigstar 1) \quad & \sum_{f:V \to A} \mu_V(f) = 1 & V \subseteq X \text{ such that } 1 \leq |V| \leq k \\ (\bigstar 2) \quad & \mu_U(f) = \sum_{g:V \to A, |g|_U = f} \mu_V(g) & U \subseteq V \subseteq X \text{ such that } 1 \leq |V| \leq k, U \neq \emptyset, f:U \to A \\ (\bigstar 3) \quad & \mu_U(f) = \sum_{g:\{\mathbf{x}\} \to A, |g|_U = f} \mu_{R,\mathbf{x}}(g) & R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, U \subseteq \{\mathbf{x}\} \text{ such that } 1 \leq |U| \leq k, f:U \to A \\ (\bigstar 4) \quad & \mu_{R,\mathbf{x}}(f) = 0 & R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, f:\{\mathbf{x}\} \to A \text{ such that } f(\mathbf{x}) \notin R^{\mathbf{A}}. \end{aligned}$$

$$(18)$$

Lemma 53. Let $k \in \mathbb{N}$, let \mathbf{X} , \mathbf{A} be two σ -structures, and let $\tilde{\mathbf{X}}$ (resp. $\tilde{\mathbf{A}}$) be the structure obtained from \mathbf{X} (resp. \mathbf{A}) by adding the relation $R_k^{\tilde{\mathbf{X}}} = X^k$ (resp. $R_k^{\tilde{\mathbf{A}}} = A^k$). Then the system (18) applied to \mathbf{X} and \mathbf{A} is equivalent to the system (2) applied to $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{A}}$.

Proof. Let λ be a solution to (2) applied to $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{A}}$. Given $V \subseteq X$ with $1 \leq |V| \leq k$ and $f: V \to A$, let $\mathbf{x} \in X^k$ be such that $V = \{\mathbf{x}\}$ and set $\mu_V(f) = \lambda_{R_k, \mathbf{x}, f(\mathbf{x})}$. We claim that this assignment does not depend on the choice of \mathbf{x} ; i.e., we claim that $\lambda_{R_k, \mathbf{x}, f(\mathbf{x})} = \lambda_{R_k, \mathbf{y}, f(\mathbf{y})}$ whenever $\mathbf{x}, \mathbf{y} \in X^k$ are such that $\{\mathbf{x}\} = \{\mathbf{y}\}$. The latter condition implies that $\mathbf{x} = \mathbf{y}_i$ and $\mathbf{y} = \mathbf{x}_j$ for some $\mathbf{i}, \mathbf{j} \in [k]^k$. Using $\clubsuit 2$ and $\clubsuit 3$, we find

$$\lambda_{R_k,\mathbf{y},f(\mathbf{y})} = \lambda_{R_k,\mathbf{x_j},f(\mathbf{x_j})} = \sum_{\substack{\mathbf{a} \in A^k \\ \mathbf{a_j} = f(\mathbf{x_j})}} \lambda_{R_k,\mathbf{x},\mathbf{a}} = \sum_{\substack{\mathbf{a} \in A^k \\ \mathbf{a_j} = f(\mathbf{x_j}) \\ \mathbf{x} \prec \mathbf{a}}} \lambda_{R_k,\mathbf{x},\mathbf{a}} = \lambda_{R_k,\mathbf{x},f(\mathbf{x})} + \sum_{\substack{\mathbf{a} \in A^k \\ \mathbf{a_j} = f(\mathbf{x_j}) \\ \mathbf{x} \prec \mathbf{a}}} \lambda_{R_k,\mathbf{x},\mathbf{a}}.$$

The claim then follows if we show that there is no $\mathbf{a} \in A^k$ such that $\mathbf{a}_j = f(\mathbf{x}_j)$, $\mathbf{x} \prec \mathbf{a}$, and $\mathbf{a} \neq f(\mathbf{x})$. If such \mathbf{a} exists, using that $\mathbf{x} = \mathbf{x}_{\mathbf{j}_i}$, we find that for some $p \in [k]$

$$a_p \neq f(x_p) = f(x_{j_{i_p}}) = a_{j_{i_p}}.$$

Since $\mathbf{x} \prec \mathbf{a}$, this implies that $x_p \neq x_{j_{i_p}}$, a contradiction. Therefore, the claim is true. Additionally, given $R \in \sigma$, $\mathbf{x} \in R^{\mathbf{X}}$, and $f : \{\mathbf{x}\} \to A$, we set $\mu_{R,\mathbf{x}}(f) = \lambda_{R,\mathbf{x},f(\mathbf{x})}$ if $f(\mathbf{x}) \in R^{\mathbf{A}}$, $\mu_{R,\mathbf{x}}(f) = 0$ otherwise. It is straightforward to check that μ satisfies all constraints in the system (18) applied to \mathbf{X} and \mathbf{A} . Conversely, let μ be a solution to (18) applied to \mathbf{X} and \mathbf{A} . As in the proof of Proposition 46, given two sets S, T, an integer $p \in \mathbb{N}$, and two tuples $\mathbf{s} \in S^p, \mathbf{t} \in T^p$ such that $\mathbf{s} \prec \mathbf{t}$, we define the map $f_{\mathbf{s},\mathbf{t}} : \{\mathbf{s}\} \to T$ by $f_{\mathbf{s},\mathbf{t}}(s_\alpha) = t_\alpha$ for each $\alpha \in [p]$. For every $R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}$, and $\mathbf{a} \in R^{\mathbf{A}}$, we set $\lambda_{R,\mathbf{x},\mathbf{a}} = \mu_{R,\mathbf{x}}(f_{\mathbf{x},\mathbf{a}})$ if $\mathbf{x} \prec \mathbf{a}, \lambda_{R,\mathbf{x},\mathbf{a}} = 0$ otherwise. Additionally, for every $\mathbf{x} \in X^k = R_k^{\mathbf{X}}$ and $\mathbf{a} \in A^k = R_k^{\mathbf{A}}$, we set $\lambda_{R_k,\mathbf{x},\mathbf{a}} = \mu_{\{\mathbf{x}\}}(f_{\mathbf{x},\mathbf{a}})$ if $\mathbf{x} \prec \mathbf{a}, \lambda_{R_k,\mathbf{x},\mathbf{a}} = 0$ otherwise. It is easily verified that λ yields a solution to (2) applied to \mathbf{X} and \mathbf{A} .

We also note that [5] has yet another definition of the Sherali-Adams hierarchy. However, it was shown in [39, Appendix A] that the hierarchy given in [5] interleaves with the one in [39] and, by virtue of Lemma 53, with the hierarchy used in this work. In particular, the class of PCSPs solved by constant levels of the hierarchy is the same for all definitions.

A.2 SDP

Remark 54. The relaxation defined by (3) is not in semidefinite programming form, because of the constraint $\blacklozenge 4$. However, it can be easily translated into a semidefinite program by introducing ω additional variables $\mu_1, \ldots, \mu_{\omega}$ taking values in \mathbb{R}^{ω} , and requiring that the following constraints are met:

where $\delta_{p,q}$ is the Kronecker delta. One easily checks that the requirements $\mathbf{4}'$ and $\mathbf{4}''$ are together equivalent to the requirement $\mathbf{4}$, and they are expressed in semidefinite programming form.

Proposition 55. Let \mathbf{X}, \mathbf{A} be two σ -structures. The system (3) implies the following facts:

(i)
$$\|\sum_{a \in A} \lambda_{x,a}\|^2 = 1$$
 $x \in X;$
(ii) $\sum_{a \in A} \|\lambda_{R,\mathbf{x},\mathbf{a}}\|^2 = \|\sum_{a \in A} \lambda_{R,\mathbf{x},\mathbf{a}}\|^2 = 1$ $R \in \sigma, \mathbf{x} \in R^{\mathbf{X}};$

(*iii*)
$$\sum_{\substack{\mathbf{a}\in R^{\mathbf{A}}\\a_{i}=a,\ a_{j}=a'}}^{\mathbf{a}\in R^{\mathbf{A}}} \|\boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}}\|^{2} = \boldsymbol{\lambda}_{x_{i},a} \cdot \boldsymbol{\lambda}_{x_{j},a'} \qquad R \in \sigma, \mathbf{x} \in R^{\mathbf{X}}, a, a' \in A, i, j \in [\operatorname{ar}(R)].$$

If, in addition, \mathbf{X} and \mathbf{A} are 2-enhanced,¹⁷

(*iv*)
$$\sum_{a \in A} \lambda_{x,a} = \sum_{a \in A} \lambda_{x',a}$$
 $x, x' \in X.$

Proof. (i) We have

$$\|\sum_{a\in A}\boldsymbol{\lambda}_{x,a}\|^2 = \left(\sum_{a\in A}\boldsymbol{\lambda}_{x,a}\right) \cdot \left(\sum_{a'\in A}\boldsymbol{\lambda}_{x,a'}\right) = \sum_{a,a'\in A}\boldsymbol{\lambda}_{x,a} \cdot \boldsymbol{\lambda}_{x,a'} = \sum_{a\in A}\|\boldsymbol{\lambda}_{x,a}\|^2 = 1,$$

where the third equality comes from $\blacklozenge 2$ and the fourth from $\blacklozenge 1$.

¹⁷Assuming that **X** and **A** are 2-enhanced does not result in a loss of generality. Indeed, $\mathbf{X} \to \mathbf{A}$ if and only if $\tilde{\mathbf{X}} \to \tilde{\mathbf{A}}$, where $\tilde{\mathbf{X}}$ (resp. $\tilde{\mathbf{A}}$) is obtained from **X** (resp. \mathbf{A}) by adding the relation $R_2^{\tilde{\mathbf{X}}} = X^2$ (resp. $R_2^{\tilde{\mathbf{A}}} = A^2$).

(ii) We have

$$\sum_{\mathbf{a}\in R^{\mathbf{A}}} \|\lambda_{R,\mathbf{x},\mathbf{a}}\|^{2} = \sum_{\mathbf{a},\mathbf{a}'\in R^{\mathbf{A}}} \lambda_{R,\mathbf{x},\mathbf{a}} \cdot \lambda_{R,\mathbf{x},\mathbf{a}'} = \left(\sum_{\mathbf{a}\in R^{\mathbf{A}}} \lambda_{R,\mathbf{x},\mathbf{a}}\right) \cdot \left(\sum_{\mathbf{a}'\in R^{\mathbf{A}}} \lambda_{R,\mathbf{x},\mathbf{a}'}\right) = \|\sum_{\mathbf{a}\in R^{\mathbf{A}}} \lambda_{R,\mathbf{x},\mathbf{a}}\|^{2}$$
$$= \|\sum_{a\in A} \sum_{\mathbf{a}\in R^{\mathbf{A}}} \lambda_{R,\mathbf{x},\mathbf{a}}\|^{2} = \|\sum_{a\in A} \lambda_{x_{1},a}\|^{2} = 1,$$

where the first equality comes from $\blacklozenge 3$, the fifth from $\blacklozenge 4$, and the sixth from part (i) of this proposition.

(*iii*) We have

$$\boldsymbol{\lambda}_{x_{i},a} \cdot \boldsymbol{\lambda}_{x_{j},a'} = \left(\sum_{\substack{\mathbf{a} \in R^{\mathbf{A}} \\ a_{i}=a}} \boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}}\right) \cdot \left(\sum_{\substack{\mathbf{a}' \in R^{\mathbf{A}} \\ a'_{j}=a'}} \boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}'}\right) = \sum_{\substack{\mathbf{a},\mathbf{a}' \in R^{\mathbf{A}} \\ a_{i}=a, a'_{j}=a'}} \boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}} \cdot \boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}'}$$
$$= \sum_{\substack{\mathbf{a} \in R^{\mathbf{A}} \\ a_{i}=a, a_{j}=a'}} \|\boldsymbol{\lambda}_{R,\mathbf{x},\mathbf{a}}\|^{2},$$

where the first equality comes from $\blacklozenge 4$ and the third from $\blacklozenge 3$.

(iv) If **X** and **A** are 2-enhanced, we have

$$\sum_{a \in A} \boldsymbol{\lambda}_{x,a} = \sum_{a \in A} \sum_{\substack{\mathbf{a} \in R_{\mathbf{a}}^{\mathbf{a}} \\ a_1 = a}} \boldsymbol{\lambda}_{R_2,(x,x'),\mathbf{a}} = \sum_{\mathbf{a} \in R_{\mathbf{a}}^{\mathbf{a}}} \boldsymbol{\lambda}_{R_2,(x,x'),\mathbf{a}} = \sum_{a \in A} \sum_{\substack{\mathbf{a} \in R_{\mathbf{a}}^{\mathbf{a}} \\ a_2 = a}} \boldsymbol{\lambda}_{R_2,(x,x'),\mathbf{a}} = \sum_{a \in A} \boldsymbol{\lambda}_{x',a},$$

where the first and fourth equalities come from $\blacklozenge 4$.

We point out that slightly different versions of the "standard SDP relaxation" appeared in the literature on CSPs, some of which use parts (i) through (iii) of Proposition 55 as constraints defining the relaxation. In particular, certain versions require that the scalar products $\lambda_{x,a} \cdot \lambda_{y,b}$ should be nonnegative for all choices of $x, y \in X$ and $a, b \in A$. For example, this is the case of the SDP relaxation used in [14]. It follows from Proposition 57, proved in Appendix A.3, that one can enforce nonnegativity of the scalar products by taking the second level of the SoS hierarchy of the SDP relaxation as defined in this work.

A.3 SoS^k

Remark 56. The relaxation defined by (4) can be easily translated into a semidefinite program through the procedure described in Remark 54.

Proposition 57. Let $k \in \mathbb{N}$, let \mathbf{X} , \mathbf{A} be 2k-enhanced σ -structures, suppose that $\mathrm{SoS}^{2k}(\mathbf{X}, \mathbf{A}) = \mathrm{YES}$, and let λ denote a solution. Then λ satisfies the following additional constraints:

- (i) $\lambda_{R_{2k},(\mathbf{x},\mathbf{x}),(\mathbf{a},\mathbf{a})} \cdot \lambda_{R_{2k},(\mathbf{y},\mathbf{y}),(\mathbf{b},\mathbf{b})} \ge 0$ $\mathbf{x}, \mathbf{y} \in X^k, \mathbf{a}, \mathbf{b} \in A^k$
- $\begin{array}{ll} (ii) \quad \boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{x}),(\mathbf{a},\mathbf{a})} \cdot \boldsymbol{\lambda}_{R_{2k},(\mathbf{y},\mathbf{y}),(\mathbf{b},\mathbf{b})} = 0 \\ \qquad \qquad \mathbf{x},\mathbf{y} \in X^k, \mathbf{a}, \mathbf{b} \in A^k, \\ \mathbf{a}_{\mathbf{i}} \neq \mathbf{b}_{\mathbf{j}} \ for \ some \ \mathbf{i}, \mathbf{j} \in [k]^k \ such \ that \ \mathbf{x}_{\mathbf{i}} = \mathbf{y}_{\mathbf{j}} \end{array}$

(*iii*)
$$\begin{aligned} \lambda_{R_{2k},(\mathbf{x},\mathbf{x}),(\mathbf{a},\mathbf{a})} \cdot \lambda_{R_{2k},(\mathbf{y},\mathbf{y}),(\mathbf{b},\mathbf{b})} &= \\ \lambda_{R_{2k},(\hat{\mathbf{x}},\hat{\mathbf{x}}),(\hat{\mathbf{a}},\hat{\mathbf{a}})} \cdot \lambda_{R_{2k},(\hat{\mathbf{y}},\hat{\mathbf{y}}),(\hat{\mathbf{b}},\hat{\mathbf{b}})} \end{aligned} \qquad \begin{aligned} \mathbf{x}, \hat{\mathbf{x}}, \mathbf{y}, \hat{\mathbf{y}} \in X^k, \mathbf{a}, \hat{\mathbf{a}}, \mathbf{b}, \hat{\mathbf{b}} \in A^k, \\ (\hat{\mathbf{x}}, \hat{\mathbf{y}})_{\ell} &= (\mathbf{x}, \mathbf{y}), (\hat{\mathbf{a}}, \hat{\mathbf{b}})_{\ell} = (\mathbf{a}, \mathbf{b}) \\ for \ some \ \ell \in [2k]^{2k} \ such \ that \ |\{\ell\}| = 2k. \end{aligned}$$

Proof. Observe that, for $\mathbf{x}, \mathbf{y} \in X^k$ and $\mathbf{a}, \mathbf{b} \in A^k$,

$$\begin{aligned} \boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{x}),(\mathbf{a},\mathbf{a})} \cdot \boldsymbol{\lambda}_{R_{2k},(\mathbf{y},\mathbf{y}),(\mathbf{b},\mathbf{b})} &= \left(\sum_{\mathbf{c}\in A^k} \boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{y}),(\mathbf{a},\mathbf{c})} \right) \cdot \left(\sum_{\mathbf{c}'\in A^k} \boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{y}),(\mathbf{c}',\mathbf{b})} \right) \\ &= \sum_{\mathbf{c},\mathbf{c}'\in A^k} \boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{y}),(\mathbf{a},\mathbf{c})} \cdot \boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{y}),(\mathbf{c}',\mathbf{b})} \\ &= \|\boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{y}),(\mathbf{a},\mathbf{b})}\|^2, \end{aligned}$$
(19)

where the first and third equalities come from $\diamondsuit 3$ and $\bigstar 2$, respectively. Hence, (*i*) holds. If, in addition, $\mathbf{a}_{\mathbf{i}} \neq \mathbf{b}_{\mathbf{j}}$ for some $\mathbf{i}, \mathbf{j} \in [k]^k$ such that $\mathbf{x}_{\mathbf{i}} = \mathbf{y}_{\mathbf{j}}$, we deduce that $(\mathbf{x}_{\mathbf{i}}, \mathbf{y}_{\mathbf{j}}) \not\prec (\mathbf{a}_{\mathbf{i}}, \mathbf{b}_{\mathbf{j}})$ and, therefore,

$$0 = \|\boldsymbol{\lambda}_{R_{2k},(\mathbf{x}_{i},\mathbf{y}_{j}),(\mathbf{a}_{i},\mathbf{b}_{j})}\|^{2} = \|\sum_{\substack{(\mathbf{c},\mathbf{d})\in A^{2k}\\\mathbf{c}_{i}=\mathbf{a}_{i},\mathbf{d}_{j}=\mathbf{b}_{j}}} \boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{y}),(\mathbf{c},\mathbf{d})}\|^{2}$$
$$= \sum_{\substack{(\mathbf{c},\mathbf{d})\in A^{2k}\\\mathbf{c}_{i}=\mathbf{a}_{i},\mathbf{d}_{j}=\mathbf{b}_{j}}} \|\boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{y}),(\mathbf{c},\mathbf{d})}\|^{2} \ge \|\boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{y}),(\mathbf{a},\mathbf{b})}\|^{2},$$
(20)

where the first, second, and third equalities come from $\mathbf{A}4$, $\mathbf{A}3$, and $\mathbf{A}2$, respectively. Combining (19) and (20), we obtain (*ii*). Suppose now that $\mathbf{x}, \hat{\mathbf{x}}, \mathbf{y}, \hat{\mathbf{y}} \in X^k$ and $\mathbf{a}, \hat{\mathbf{a}}, \mathbf{b}, \hat{\mathbf{b}} \in A^k$ are such that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})_{\ell} = (\mathbf{x}, \mathbf{y})$ and $(\hat{\mathbf{a}}, \hat{\mathbf{b}})_{\ell} = (\mathbf{a}, \mathbf{b})$ for some $\ell \in [2k]^{2k}$ such that $|\{\ell\}| = 2k$. Using $\mathbf{A}3$, we find

$$\boldsymbol{\lambda}_{R_{2k},(\mathbf{x},\mathbf{y}),(\mathbf{a},\mathbf{b})} = \boldsymbol{\lambda}_{R_{2k},(\hat{\mathbf{x}},\hat{\mathbf{y}})_{\boldsymbol{\ell}},(\hat{\mathbf{a}},\hat{\mathbf{b}})_{\boldsymbol{\ell}}} = \sum_{\substack{(\mathbf{c},\mathbf{d})\in A^{2k}\\ (\mathbf{c},\mathbf{d})_{\boldsymbol{\ell}} = (\hat{\mathbf{a}},\hat{\mathbf{b}})_{\boldsymbol{\ell}}}} \boldsymbol{\lambda}_{R_{2k},(\hat{\mathbf{x}},\hat{\mathbf{y}}),(\mathbf{c},\mathbf{d})} = \boldsymbol{\lambda}_{R_{2k},(\hat{\mathbf{x}},\hat{\mathbf{y}}),(\hat{\mathbf{a}},\hat{\mathbf{b}})},$$

where the last equality is due to the fact that $|\{\ell\}| = 2k$. Hence, (*iii*) follows from (19).

We observe that the relaxation in (4) is formally different from the one described in [96]. However, it can be shown that the 2k-th level of the hierarchy as defined here is at least as tight as the k-th level of the hierarchy as defined in [96]. Let us denote the two relaxations by SoS and SoS', respectively. First of all, each variable in SoS' corresponds to a subset S of \mathbf{X} and an assignment $f : S \to A$, while in SoS the variables correspond to pairs of tuples $\mathbf{x} \in R^{\mathbf{X}}, \mathbf{a} \in R^{\mathbf{A}}$. This is an inessential difference, as one can check through the same argument used to prove Lemma 53 – in particular, $\bigstar 4$ ensures that the only variables having nonzero weight are those corresponding to well-defined assignments (cf. Footnote 14). The *k*-th level of SoS' contains constraints that, in our language, are expressed as $\bigstar 4$, $\bigstar 1$, and parts (*i*), (*ii*), (*iii*) of Proposition 57. By virtue of Proposition 57, therefore, any solution λ to the 2*k*-th level of SoS yields a solution to the *k*-th level of SoS' – which means that the 2*k*-th level of SoS is at least as tight as the *k*-th level of SoS'.

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