

Approximate Graph Colouring and Crystals*

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Abstract

We show that approximate graph colouring is not solved by any level of the affine integer programming (AIP) hierarchy. To establish the result, we translate the problem of exhibiting a graph fooling a level of the AIP hierarchy into the problem of constructing a highly symmetric *crystal* tensor. In order to prove the existence of crystals in arbitrary dimension, we provide a combinatorial characterisation for *realisable* systems of tensors; i.e., sets of low-dimensional tensors that can be realised as the projections of a single high-dimensional tensor.

1 Introduction

The *approximate graph colouring* problem (AGC) asks to find a d -colouring of a given c -colourable graph, where $3 \leq c \leq d$. There is a huge gap in our understanding of the computational complexity of this problem. For an n -vertex graph and $c = 3$, the currently best known polynomial-time algorithm finds a d -colouring of the graph with $d = O(n^{0.19996})$. It has been long conjectured [35] that the problem is NP-hard for any fixed constants $3 \leq c \leq d$ even in the decision variant: Given a graph, output YES if it is c -colourable and output NO if it is not d -colourable.

For $c = d$, the problem becomes the classic c -colouring problem, which appeared on Karp's original list of 21 NP-complete problems [41]. The case $c = 3$, $d = 4$ was only proved to be NP-hard in 2000 [42] (and a simpler proof was given in [37]); more generally, [42] showed hardness of the case $d = c + 2\lfloor c/3 \rfloor - 1$. This was improved to $d = 2c - 2$ in 2016 [12], and recently to $d = 2c - 1$ [6]. In particular, this last result implies hardness of the case $c = 3$, $d = 5$; the complexity of the case $c = 3$, $d = 6$ is still open. Building on [39, 43], NP-hardness was established for $d = \binom{c}{\lfloor c/2 \rfloor} - 1$ for $c \geq 4$ in [56]. NP-hardness of AGC was established for all constants $3 \leq c \leq d$ in [33] under a non-standard variant of the Unique Games Conjecture, in [38] under the d -to-1 conjecture [44] for any fixed d , and (an even stronger statement of

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distinguishing 3-colourability from not having an independent set of significant size) in [19] under the rich 2-to-1 conjecture [20].

AGC is an example of so called *promise constraint satisfaction problem* (PCSP). For a positive integer k , a k -uniform hypergraph \mathbf{H} consists of a set $\mathcal{V}(\mathbf{H})$ of elements called *vertices* and a set $\mathcal{E}(\mathbf{H}) \subseteq \mathcal{V}(\mathbf{H})^k$ of tuples of k vertices called *hyperedges*.¹ Given two k -uniform hypergraphs \mathbf{G} and \mathbf{H} , a map $f : \mathcal{V}(\mathbf{G}) \rightarrow \mathcal{V}(\mathbf{H})$ is a *homomorphism* from \mathbf{G} to \mathbf{H} if $f(\mathbf{g}) \in \mathcal{E}(\mathbf{H})$ for any $\mathbf{g} \in \mathcal{E}(\mathbf{G})$, where f is applied entrywise to the vertices in \mathbf{g} . We shall denote the existence of a homomorphism from \mathbf{G} to \mathbf{H} by the expression $\mathbf{G} \rightarrow \mathbf{H}$. The PCSP parameterised by two k -uniform hypergraphs \mathbf{H} and $\tilde{\mathbf{H}}$ such that $\mathbf{H} \rightarrow \tilde{\mathbf{H}}$, denoted by $\text{PCSP}(\mathbf{H}, \tilde{\mathbf{H}})$, is the following computational problem: Given a k -uniform hypergraph \mathbf{G} as input, answer YES if $\mathbf{G} \rightarrow \mathbf{H}$ and NO if $\mathbf{G} \not\rightarrow \tilde{\mathbf{H}}$. The requirement $\mathbf{H} \rightarrow \tilde{\mathbf{H}}$ implies that the two cases cannot happen simultaneously, as homomorphisms compose; the *promise* is that one of the two cases always happens.² A 2-uniform hypergraph is a *digraph*. Moreover, a p -colouring of a digraph \mathbf{G} is precisely a homomorphism from \mathbf{G} to the *clique* \mathbf{K}_p – i.e., the digraph on vertex set $\{1, \dots, p\}$ such that any pair of distinct vertices is a (directed) edge. Therefore, AGC is $\text{PCSP}(\mathbf{K}_c, \mathbf{K}_d)$.

By letting $\mathbf{H} = \tilde{\mathbf{H}}$ in the definition of a PCSP, one obtains the standard (non-promise) *constraint satisfaction problem* (CSP). PCSPs were introduced in [4, 14] as a general framework for studying approximability of perfectly satisfiable CSPs and have emerged as a new direction in constraint satisfaction that requires different techniques than CSPs. Recent works on PCSPs include those using analytical methods [10, 11, 15, 20] and those building on algebraic methods [2, 5, 7, 13, 16, 17, 25, 28, 38, 50, 56] developed in [6]. However, most basic questions are still left open, including applicability of different types of algorithms. Remarkably, most algorithmic techniques in constraint satisfaction can be broadly classified into two distinct classes: Algorithms enforcing some type of local consistency, and algorithms related to (generalisations of) linear equations.

The first class of algorithms is based on the following idea: Rather than directly checking for the existence of a global map between \mathbf{G} and \mathbf{H} satisfying constraints (i.e., a homomorphism), which may not be doable in polynomial time, one tries to draw an *atlas* of *charts* covering each region of the instance \mathbf{G} . The charts are partial homomorphisms, i.e., homomorphisms from a substructure of \mathbf{G} to \mathbf{H} ; the atlas must have the property that the maps are consistent, i.e., whenever two regions overlap, there exist charts of the regions that agree on the intersection. The *bounded width* (or *local-consistency checking*) algorithm outputs YES if and only if such an atlas exists – which can be checked in polynomial time provided that the size of each chart is bounded [34]. More powerful versions of this technique require that the charts of each region should be sampled according to some probability distribution. In this case, the consistency requirement of the atlas is stronger, as it asks that, whenever two regions overlap, the probability distribution over the charts of the intersection should be exactly the *marginal* of the distributions over the charts of the two regions. Concretely, checking for the existence of such a “random atlas” amounts to solving a linear program, and results in the so-called Sherali-Adams LP hierarchy [53], which is provably more powerful than bounded width [3]. Treating probabilities as vectors satisfying certain orthogonality requirements gives an even stronger algorithm based on semidefinite programming, known

¹Unless otherwise stated, all hypergraphs appearing in this paper are finite, meaning that their vertex set is finite.

²It is customary to study PCSPs on more general objects known as *relational structures*, which consist of a collection of relations of arbitrary arities on a vertex set.

as the sum-of-squares or Lasserre SDP hierarchy [48, 51, 54]. In general, the existence of a (random) atlas is not sufficient to deduce that a *planisphere* (i.e., a global map satisfying all constraints) exists. In fact, if $P \neq NP$, we do not expect polynomial-time algorithms to solve NP-hard problems. Thus, a well-established line of work has sought to prove lower bounds on the efficacy of these consistency algorithms; see [1, 18, 27, 36, 47] for lower bounds on LPs arising from lift-and-project hierarchies such as that of Sherali-Adams, and [26, 49, 55] for lower bounds on SDPs.

Any PCSP can be formulated as a system of linear equations over $\{0, 1\}$. The second class of algorithms essentially consists in solving the equations using (some variant or a generalisation of) Gaussian elimination. This requires relaxing the problem by admitting a larger range for the variables in the equations (as, in general, the system cannot be efficiently solved over $\{0, 1\}$). In particular, it is possible to solve the system in polynomial time over \mathbb{Z} ([40], cf. also [13]) – which results in the *affine integer programming* (AIP) relaxation, also known as *linear Diophantine equations*,³ that we consider in this work. The “base level” of this algorithm was studied in [13, 16] in the context of PCSPs, and its power was characterised algebraically in [6]. This algorithmic technique is substantially different from the first class of algorithms: Instead of looking for an atlas of charts faithfully describing regions of the world – i.e., a system of *local assignments satisfying the constraints* – the algorithms of the second class aim to draw a possibly imprecise planisphere – i.e., a *global assignment satisfying a relaxed version of the constraints*. In the context of CSPs, the elusive interaction between consistency-checking and methods based on (generalisations of) Gaussian-elimination was the major obstacle to the proof of the Feder-Vardi dichotomy conjecture [34], that was finally settled in [24] and, independently, in [58].

If, as conjectured, AGC is an NP-hard problem and $P \neq NP$, neither of the two algorithmic techniques should be able to solve it. In a striking sequence of works [31, 32, 45, 46], the 2-to-2 conjecture of Khot [44] (with imperfect completeness) was resolved. As detailed in [46], this implies (together with [38]) that polynomially many levels of the sum-of-squares hierarchy do not solve AGC, which implies the same result for polynomially many levels of the weaker Sherali-Adams and bounded width algorithms. Recent work [3] established that even sublinear levels of bounded width do not solve AGC.

Contributions In this paper, we focus on the second class of algorithms and show that no level of the affine integer programming hierarchy solves AGC. Recently, [29] described a linear-algebraic characterisation of the algorithm in terms of a geometric construction called *tensorisation*. Using this characterisation as a black box, we translate the problem of finding an instance of AGC fooling the algorithm into the problem of finding a tensor with many symmetries, which we call a *crystal*. Our main technical contribution is the construction of crystals. More precisely, we prove the following result: Given a collection of low-dimensional tensors (“pictures”) satisfying a compatibility requirement, it is possible to build a high-dimensional tensor such that by projecting it onto low-dimensional hyperplanes one recovers the pictures. Variants of this problem have appeared in the literature in combinatorial matrix theory. In particular, the problem of constructing a matrix (i.e., a 2-dimensional tensor) having prescribed row-sum and column-sum vectors (i.e., 1-dimensional projections) has been studied for different classes of matrices, such as nonnegative integer matrices [23], $(0, 1)$

³A hierarchy based on AIP (with additional local-consistency conditions) was considered in [9], where a lower bound on its power was shown for the graph isomorphism problem.

matrices [30, 52], alternating-sign matrices [57], and sign-restricted matrices [22], see also the survey [8]. For example, the Gale-Ryser theorem [52] provides a characterisation, based on the notion of majorisation, of the pairs of vectors \mathbf{r}, \mathbf{c} for which there exists a $(0, 1)$ matrix whose row-sum and column-sum vectors are \mathbf{r} and \mathbf{c} , respectively. In a similar fashion, we not only show that a tensor having prescribed low-dimensional projections exists, but we also prove that a natural necessary combinatorial condition is in fact also sufficient for a system of low-dimensional “picture” tensors in order to be the set of projections of a high-dimensional tensor.

We point out that our proof is constructive, as it allows to explicitly find a tensor with the desired characteristics. As far as we know, the problem of reconstructing a tensor from low-dimensional projections has hitherto only been studied for matrices (but cf. [21], where a related problem is investigated in three dimensions in the restricted setting of alternating-sign 3-dimensional tensors). However, in order to rule out affine integer programming as an algorithm to solve AGC for all numbers of colours, we need to build crystals of arbitrarily high dimension and hence approach the reconstruction problem for arbitrarily high-dimensional tensors. In addition to its direct application to the non-solvability of AGC, we believe that our result might be of independent interest to the linear algebra and tensor theory communities.

2 Overview

Let $k \geq 2$ be an integer. Given a set V , we define $\binom{V}{\leq k} = \{S \subseteq V : 1 \leq |S| \leq k\}$. Let \mathbf{G} and \mathbf{H} be two digraphs. We introduce a variable $\lambda_S(f)$ for every $S \in \binom{\mathcal{V}(\mathbf{G})}{\leq k}$ and every function $f : S \rightarrow \mathcal{V}(\mathbf{H})$, and a variable $\lambda_{\mathbf{g}}(f)$ for every $\mathbf{g} = (g_1, g_2) \in \mathcal{E}(\mathbf{G})$ and every $f : \{g_1, g_2\} \rightarrow \mathcal{V}(\mathbf{H})$. The k -th level of the AIP hierarchy is given by the following constraints:

$$\begin{aligned}
(\text{AIP1}) \quad & \sum_{f:S \rightarrow \mathcal{V}(\mathbf{H})} \lambda_S(f) = 1 & S \in \binom{\mathcal{V}(\mathbf{G})}{\leq k} \\
(\text{AIP2}) \quad & \lambda_R(f) = \sum_{\tilde{f}:S \rightarrow \mathcal{V}(\mathbf{H}), \tilde{f}|_R=f} \lambda_S(\tilde{f}) & \emptyset \neq R \subseteq S \in \binom{\mathcal{V}(\mathbf{G})}{\leq k}, f : R \rightarrow \mathcal{V}(\mathbf{H}) \\
(\text{AIP3}) \quad & \lambda_R(f) = \sum_{\tilde{f}:\{g_1, g_2\} \rightarrow \mathcal{V}(\mathbf{H}), \tilde{f}|_R=f} \lambda_{\mathbf{g}}(\tilde{f}) & \mathbf{g} \in \mathcal{E}(\mathbf{G}), \emptyset \neq R \subseteq \{g_1, g_2\}, f : R \rightarrow \mathcal{V}(\mathbf{H}) \\
(\text{AIP4}) \quad & \lambda_{\mathbf{g}}(f) = 0 & \mathbf{g} \in \mathcal{E}(\mathbf{G}), f : \{g_1, g_2\} \rightarrow \mathcal{V}(\mathbf{H}) \text{ with } f(\mathbf{g}) \notin \mathcal{E}(\mathbf{H}).
\end{aligned}$$

We say that $\text{AIP}^k(\mathbf{G}, \mathbf{H}) = \text{YES}$ if the system above admits a solution such that all variables take integer values. For a fixed k , this can be checked in polynomial time in the number of vertices of the input digraph \mathbf{G} by solving a polynomial-sized system of linear equations over the integers [40]. (For the “base level” of the hierarchy $k = 1$, cf. Appendix A.)

Let $\tilde{\mathbf{H}}$ be a digraph such that $\mathbf{H} \rightarrow \tilde{\mathbf{H}}$. One easily checks that $\text{AIP}^k(\mathbf{G}, \mathbf{H}) = \text{YES}$ if $\mathbf{G} \rightarrow \mathbf{H}$; we say that the k -th level of AIP *solves* $\text{PCSP}(\mathbf{H}, \tilde{\mathbf{H}})$ if $\mathbf{G} \rightarrow \tilde{\mathbf{H}}$ whenever $\text{AIP}^k(\mathbf{G}, \mathbf{H}) = \text{YES}$. Clearly, if $\text{AIP}^k(\mathbf{G}, \mathbf{H}) = \text{YES}$ then $\text{AIP}^{k'}(\mathbf{G}, \mathbf{H}) = \text{YES}$ for any level k' lower than k . It follows that if some level of the hierarchy solves $\text{PCSP}(\mathbf{H}, \tilde{\mathbf{H}})$ then any *higher* level of the hierarchy also solves it. It is worth noticing that the AIP hierarchy does not enforce consistency, in the sense that it is possible that a partial assignment is given nonzero weight without being a partial homomorphism. This is in sharp contrast to the “consistency-enforcing” algorithms mentioned in the Introduction, such as the bounded-width, Sherali-Adams LP, and Lasserre SDP hierarchies. We now state the first main result of this work.

Theorem 1. *No level of the AIP hierarchy solves approximate graph colouring; i.e., for any fixed $3 \leq c \leq d$, there is no k such that the k -th level of AIP solves $\text{PCSP}(\mathbf{K}_c, \mathbf{K}_d)$.*

2.1 Affine integer programming and tensors

In order to prove Theorem 1, we need to find instances of AGC that fool the AIP hierarchy. Rather than working with the hierarchy itself, we shall lift the analysis to a tensor-theoretic framework. Next, we define some terminology on tensors that will be used throughout the paper.

Given n in the set \mathbb{N} of positive integer numbers, we let $[n] = \{1, \dots, n\}$. We also let $[0] = \emptyset$. Given a tuple $\mathbf{n} = (n_1, \dots, n_q) \in \mathbb{N}^q$ for some $q \in \mathbb{N}$, we let $[\mathbf{n}] = [n_1] \times \dots \times [n_q]$. Given a tuple $\mathbf{b} = (b_1, \dots, b_q) \in [\mathbf{n}]$ and a tuple $\mathbf{i} = (i_1, \dots, i_p) \in [q]^p$ for $p, q \in \mathbb{N}$, the *projection* of \mathbf{b} onto \mathbf{i} is the tuple $\mathbf{b}_{\mathbf{i}} = (b_{i_1}, \dots, b_{i_p})$. Notice that $\mathbf{b}_{\mathbf{i}} \in [\mathbf{n}_{\mathbf{i}}]$. For $\tilde{\mathbf{n}} \in \mathbb{N}^p$, the *concatenation* of two tuples $\mathbf{b} = (b_1, \dots, b_q) \in [\mathbf{n}]$ and $\mathbf{c} = (c_1, \dots, c_p) \in [\tilde{\mathbf{n}}]$ is the tuple $(\mathbf{b}, \mathbf{c}) = (b_1, \dots, b_q, c_1, \dots, c_p)$. Notice that $(\mathbf{b}, \mathbf{c}) \in [(\mathbf{n}, \tilde{\mathbf{n}})]$. It will be handy to extend the notation above to include tuples of length zero. For any set S , we define $S^0 = \{\epsilon\}$, where ϵ denotes the empty tuple. For any tuple \mathbf{x} , we let $\mathbf{x}_{\epsilon} = \epsilon$ and $(\mathbf{x}, \epsilon) = (\epsilon, \mathbf{x}) = \mathbf{x}$. We also define $[\epsilon] = \{\epsilon\}$. For $n \in \mathbb{N}$, define the tuple $\langle n \rangle = (1, \dots, n)$. Also, let $\langle 0 \rangle = \epsilon$. Given a tuple \mathbf{x} , $\#(\mathbf{x})$ is the cardinality of the set of elements appearing in \mathbf{x} .

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Take a set S , an integer $q \in \mathbb{N}_0$, and a tuple $\mathbf{n} \in \mathbb{N}^q$. We denote by $\mathcal{T}^{\mathbf{n}}(S)$ the set of *tensors* on q *modes* of sizes n_1, \dots, n_q whose entries are in S ; formally, $\mathcal{T}^{\mathbf{n}}(S)$ is the set of functions from $[\mathbf{n}]$ to S . We sometimes denote a tensor in $\mathcal{T}^{\mathbf{n}}(S)$ by $T = (t_{\mathbf{i}})_{\mathbf{i} \in [\mathbf{n}]}$, where $t_{\mathbf{i}} \in S$ is the image of \mathbf{i} under T . For example, $\mathcal{T}^{\mathbf{n}}(S)$ and $\mathcal{T}^{(m,n)}(S)$ are the sets of n -vectors and $m \times n$ matrices, respectively, having entries in S . Notice that $\mathcal{T}^{\epsilon}(S)$ is the set of functions from $[\epsilon] = \{\epsilon\}$ to S , which we identify with S . We will often consider *cubical* tensors, all of whose modes have equal size; i.e., tensors in the set $\mathcal{T}^{\mathbf{n} \cdot \mathbf{1}_q}$ for some $n \in \mathbb{N}$, where $\mathbf{1}_q$ is the all-one tuple of length q .

We shall usually consider tensors having entries in the ring of integers \mathbb{Z} . For $k, \ell, m \in \mathbb{N}_0$, take $\mathbf{n} \in \mathbb{N}^k$, $\mathbf{p} \in \mathbb{N}^{\ell}$, and $\mathbf{q} \in \mathbb{N}^m$. The *contraction* of two tensors $T = (t_{\mathbf{i}})_{\mathbf{i} \in [(\mathbf{n}, \mathbf{p})]} \in \mathcal{T}^{(\mathbf{n}, \mathbf{p})}(\mathbb{Z})$ and $\tilde{T} = (\tilde{t}_{\mathbf{j}})_{\mathbf{j} \in [(\mathbf{p}, \mathbf{q})]} \in \mathcal{T}^{(\mathbf{p}, \mathbf{q})}(\mathbb{Z})$, denoted by $T \overset{\ell}{*} \tilde{T}$, is the tensor in $\mathcal{T}^{(\mathbf{n}, \mathbf{q})}(\mathbb{Z})$ such that, for $\mathbf{i} \in [\mathbf{n}]$ and $\mathbf{j} \in [\mathbf{q}]$, the (\mathbf{i}, \mathbf{j}) -th entry of $T \overset{\ell}{*} \tilde{T}$ is given by $\sum_{\mathbf{z} \in [\mathbf{p}]} t_{(\mathbf{i}, \mathbf{z})} \tilde{t}_{(\mathbf{z}, \mathbf{j})}$. If at least one of k and m equals zero – i.e., if we are contracting over all modes of T or \tilde{T} , we write $T \overset{\ell}{*} \tilde{T}$ for $T \overset{\ell}{*} \tilde{T}$, to increase readability. It is not hard to see that tensor contraction is associative, in the sense that $(T \overset{\ell}{*} \tilde{T}) \overset{m}{*} \hat{T} = T \overset{\ell}{*} (\tilde{T} \overset{m}{*} \hat{T})$ for any $\hat{T} \in \mathcal{T}^{(\mathbf{q}, \mathbf{r})}(\mathbb{Z})$, where $\mathbf{r} \in \mathbb{N}^n$ for some $n \in \mathbb{N}_0$. On the other hand, the order of operations matters for the “ $*$ ” operator. For example, if $\bar{T} \in \mathcal{T}^{\mathbf{n}}(\mathbb{Z})$, the expression $(\bar{T} \overset{\ell}{*} T) \overset{m}{*} \tilde{T}$ is well defined but the expression $\bar{T} \overset{\ell}{*} (T \overset{m}{*} \tilde{T})$ is not, in general. For this reason, we define “ $*$ ” to be left-associative; i.e., $T_1 \overset{\ell}{*} T_2 \overset{m}{*} T_3$ means $(T_1 \overset{\ell}{*} T_2) \overset{m}{*} T_3$. The next example shows that contraction generalises various linear-algebraic products.

Example 2. For $m, n, p \in \mathbb{N}$, consider the tensors $c \in \mathcal{T}^{\epsilon}(\mathbb{Z}) = \mathbb{Z}$, $\mathbf{u}, \mathbf{v} \in \mathcal{T}^m(\mathbb{Z})$, $\mathbf{w} \in \mathcal{T}^n(\mathbb{Z})$, $M, N \in \mathcal{T}^{(m,n)}(\mathbb{Z})$, and $Q \in \mathcal{T}^{(n,p)}(\mathbb{Z})$. The following products can be seen as examples of contraction: $c \overset{0}{*} \mathbf{u} = c * \mathbf{u} = \mathbf{c}\mathbf{u}$, $c \overset{0}{*} M = c * M = cM$ (multiplication times scalar); $\mathbf{u} \overset{1}{*} \mathbf{v} = \mathbf{u} * \mathbf{v} = \mathbf{u}^T \mathbf{v}$ (inner product); $\mathbf{u} \overset{0}{*} \mathbf{w} = \mathbf{u}\mathbf{w}^T$ (outer product); $M \overset{1}{*} Q = MQ$ (standard matrix product); $M \overset{2}{*} N = \text{tr}(M^T N)$ (Frobenius inner product).

Let $q \in \mathbb{N}_0$ and $\mathbf{n} \in \mathbb{N}^q$. Given $\mathbf{i} \in [\mathbf{n}]$, we denote by $E_{\mathbf{i}}$ the \mathbf{i} -th *standard unit tensor*; i.e.,

the tensor in $\mathcal{T}^{\mathbf{n}}(\mathbb{Z})$ all of whose entries are 0, except the \mathbf{i} -th entry that is 1. Observe that, for any $T \in \mathcal{T}^{\mathbf{n}}(\mathbb{Z})$, we may express the \mathbf{i} -th entry of T as $E_{\mathbf{i}} * T$. If $q = 1$, $n \in \mathbb{N}$, and $i \in [n]$, notice that E_i is the i -th standard unit vector of length n . Let $\mathbf{i} \in [q]^p$ for some $p \in \mathbb{N}_0$. We associate with \mathbf{n} and \mathbf{i} the tensor $\Pi_{\mathbf{i}}^{\mathbf{n}} \in \mathcal{T}^{(\mathbf{n}, \mathbf{i})}(\mathbb{Z})$ defined by

$$E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * E_{\mathbf{b}} = \begin{cases} 1 & \text{if } \mathbf{b}_{\mathbf{i}} = \mathbf{a} \\ 0 & \text{otherwise} \end{cases} \quad \text{for each } \begin{matrix} \mathbf{a} \in [\mathbf{n}_{\mathbf{i}}], \\ \mathbf{b} \in [\mathbf{n}]. \end{matrix} \quad (1)$$

We will need a few technical lemmas on the tensors defined above,⁴ whose proofs can be found in Section 3. The first concerns the “limit case” of the empty tuple ϵ .

Lemma 3. $E_{\epsilon} = 1$. Moreover, given $q \in \mathbb{N}_0$ and $\mathbf{n} \in \mathbb{N}^q$, $\Pi_{\epsilon}^{\mathbf{n}}$ is the all-one tensor in $\mathcal{T}^{\mathbf{n}}(\mathbb{Z})$.

The following is a simple description of the entries of $\Pi_{\mathbf{i}}^{\mathbf{n}}$.

Lemma 4. Given $p, q \in \mathbb{N}_0$, $\mathbf{n} \in \mathbb{N}^q$, $\mathbf{i} \in [q]^p$, and $\mathbf{a} \in [\mathbf{n}_{\mathbf{i}}]$, we have $E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\mathbf{n}} = \sum_{\mathbf{b} \in [\mathbf{n}], \mathbf{b}_{\mathbf{i}} = \mathbf{a}} E_{\mathbf{b}}$.

The assignment $\mathbf{i} \mapsto \Pi_{\mathbf{i}}^{\mathbf{n}}$ creates a correspondence between tuples and tensors. More specifically, Lemma 5 shows that, under this assignment, the operation of tuple projection is translated into the operation of tensor contraction, while Lemma 6 shows that the tuple $\langle q \rangle$, that acts by projection as the identity on the set of tuples of appropriate length, corresponds to a tensor that acts by contraction as the identity on the space of tensors of appropriate size.

Lemma 5. Let $m, p, q \in \mathbb{N}_0$, and consider two tuples $\mathbf{i} \in [q]^p$ and $\mathbf{j} \in [p]^m$. Then, for any $\mathbf{n} \in \mathbb{N}^q$, $\Pi_{\mathbf{i}}^{\mathbf{n}} = \Pi_{\mathbf{j}}^{\mathbf{n}_{\mathbf{i}}} \overset{\circ}{*} \Pi_{\mathbf{i}}^{\mathbf{n}}$.

Lemma 6. Let $q, q' \in \mathbb{N}_0$, $\mathbf{n} \in \mathbb{N}^q$, $\mathbf{n}' \in \mathbb{N}^{q'}$, and $T \in \mathcal{T}^{(\mathbf{n}, \mathbf{n}')}(\mathbb{Z})$. Then $\Pi_{\langle q \rangle}^{\mathbf{n}} \overset{\circ}{*} T = T$.

In [29], the AIP hierarchy was characterised algebraically by using a multilinear construction. We state the characterisation in Theorem 11 below, after introducing the necessary terminology.

Definition 7 ([6, 16]). A *minion* \mathcal{M} is the disjoint union of nonempty sets $\mathcal{M}^{(p)}$ for $p \in \mathbb{N}$ equipped with operations $(\cdot)_{/\pi} : \mathcal{M}^{(p)} \rightarrow \mathcal{M}^{(q)}$ for all $\pi : [p] \rightarrow [q]$ that satisfy, for any $p, q, r \in \mathbb{N}$, $\pi : [p] \rightarrow [q]$, $\rho : [q] \rightarrow [r]$, $M \in \mathcal{M}^{(p)}$, the requirements (i) $(M_{/\pi})_{/\rho} = M_{/\rho \circ \pi}$, and (ii) $M_{/\text{id}} = M$.

Let \mathbf{H} be a k -uniform hypergraph having n vertices and m hyperedges. The *free hypergraph* $\mathbb{F}_{\mathcal{M}}(\mathbf{H})$ of a minion \mathcal{M} generated by \mathbf{H} is the (potentially infinite) k -uniform hypergraph on the vertex set $\mathcal{V}(\mathbb{F}_{\mathcal{M}}(\mathbf{H})) = \mathcal{M}^{(n)}$ whose hyperedges are defined as follows: Given $M_1, \dots, M_k \in \mathcal{M}^{(n)}$, the tuple (M_1, \dots, M_k) belongs to $\mathcal{E}(\mathbb{F}_{\mathcal{M}}(\mathbf{H}))$ if and only if there exists some $Q \in \mathcal{M}^{(m)}$ such that $M_i = Q_{/\pi_i}$ for any $i \in [k]$, where $\pi_i : \mathcal{E}(\mathbf{H}) \rightarrow \mathcal{V}(\mathbf{H})$ maps a hyperedge \mathbf{h} to its i -th entry h_i .

Example 8 ([6]). For any $p \in \mathbb{N}$, let $\mathcal{Z}_{\text{aff}}^{(p)}$ be the set of integer vectors of length p whose entries sum up to one. Given $\pi : [p] \rightarrow [q]$ and $\mathbf{v} \in \mathcal{Z}_{\text{aff}}^{(p)}$, let $\mathbf{v}_{/\pi}$ be the q -vector whose j -th entry is $\sum_{\ell \in \pi^{-1}(j)} v_{\ell}$ for each $j \in [q]$. One easily shows that $\mathcal{Z}_{\text{aff}} = \bigcup_{p \in \mathbb{N}} \mathcal{Z}_{\text{aff}}^{(p)}$ is a minion.

⁴The expression “ $x \stackrel{\text{L}\bullet}{=} y$ ” shall mean “ $x = y$ by Lemma \bullet ”. Similarly, “ $x \stackrel{\text{D}\bullet}{=} y$ ” and “ $x \stackrel{\circ}{=} y$ ” shall mean “ $x = y$ by Definition \bullet ” and “ $x = y$ by equation (\bullet) ”, respectively.

Definition 9 ([29]). Given $k \in \mathbb{N}$, the k -th tensor power⁵ of a digraph \mathbf{H} is the 2^k -uniform hypergraph $\mathbf{H}^{\otimes k}$ having vertex set $\mathcal{V}(\mathbf{H}^{\otimes k}) = \mathcal{V}(\mathbf{H})^k$ and hyperedge set $\mathcal{E}(\mathbf{H}^{\otimes k}) = \{\mathbf{h}^{\otimes k} : \mathbf{h} \in \mathcal{E}(\mathbf{H})\}$, where, for $\mathbf{h} \in \mathcal{E}(\mathbf{H})$, $\mathbf{h}^{\otimes k}$ is the tensor⁶ in $\mathcal{T}^{2 \cdot \mathbf{1}_k}(\mathcal{V}(\mathbf{H})^k)$ whose \mathbf{i} -th entry is $\mathbf{h}_{\mathbf{i}}$ for every $\mathbf{i} \in [2]^k$.

Example 10. Let us describe the free hypergraph of \mathcal{Z}_{aff} generated by $\mathbf{H}^{\otimes k}$, where \mathbf{H} is a digraph on n vertices. $\mathbb{F}_{\mathcal{Z}_{\text{aff}}}(\mathbf{H}^{\otimes k})$ is a (potentially infinite) 2^k -uniform hypergraph whose vertex set is $\mathcal{Z}_{\text{aff}}^{(n^k)}$, which we identify with the set of (cubical) tensors in $\mathcal{T}^{n \cdot \mathbf{1}_k}(\mathbb{Z})$ whose entries sum up to one. Each hyperedge of $\mathbb{F}_{\mathcal{Z}_{\text{aff}}}(\mathbf{H}^{\otimes k})$ consists of 2^k vertices, i.e., 2^k elements of $\mathcal{Z}_{\text{aff}}^{(n^k)}$. It is convenient to visualise it as a block tensor T belonging to $\mathcal{T}^{2 \cdot \mathbf{1}_k}(\mathcal{T}^{n \cdot \mathbf{1}_k}(\mathbb{Z})) = \mathcal{T}^{2n \cdot \mathbf{1}_k}(\mathbb{Z})$. Using Definition 7, we see that $T \in \mathcal{E}(\mathbb{F}_{\mathcal{Z}_{\text{aff}}}(\mathbf{H}^{\otimes k}))$ if and only if there exists some $Q \in \mathcal{Z}_{\text{aff}}^{(|\mathcal{E}(\mathbf{H}^{\otimes k})|)} = \mathcal{Z}_{\text{aff}}^{(|\mathcal{E}(\mathbf{H})|)}$ such that, for any $\mathbf{i} \in [2]^k$, the \mathbf{i} -th block of T is equal to $Q_{/\pi_{\mathbf{i}}}$, where $\pi_{\mathbf{i}} : \mathcal{E}(\mathbf{H}) \rightarrow \mathcal{V}(\mathbf{H})^k$ maps $\mathbf{h} \in \mathcal{E}(\mathbf{H})$ to $\mathbf{h}_{\mathbf{i}}$. It only remains to describe the entries of $Q_{/\pi_{\mathbf{i}}}$. According to Example 8, given any $\mathbf{h} \in \mathcal{V}(\mathbf{H})^k$, the \mathbf{h} -th entry of $Q_{/\pi_{\mathbf{i}}}$ is given by

$$E_{\mathbf{h}} * Q_{/\pi_{\mathbf{i}}} = \sum_{\ell \in \pi_{\mathbf{i}}^{-1}(\mathbf{h})} E_{\ell} * Q = \sum_{\substack{\ell \in \mathcal{E}(\mathbf{H}) \\ \ell_{\mathbf{i}} = \mathbf{h}}} E_{\ell} * Q. \quad (2)$$

The following result characterises acceptance for the AIP hierarchy.⁷

Theorem 11 ([29]). Let \mathbf{G}, \mathbf{H} be two digraphs and let $k \geq 2$. Then $\text{AIP}^k(\mathbf{G}, \mathbf{H}) = \text{YES}$ if and only if there exists a homomorphism $\xi : \mathbf{G}^{\otimes k} \rightarrow \mathbb{F}_{\mathcal{Z}_{\text{aff}}}(\mathbf{H}^{\otimes k})$ such that $\xi(\mathbf{g}_{\mathbf{i}}) = \Pi_{\mathbf{i}}^{n \cdot \mathbf{1}_k} * \xi(\mathbf{g})$ for any $\mathbf{g} \in \mathcal{V}(\mathbf{G})^k$, $\mathbf{i} \in [k]^k$.

2.2 The quest for crystals

Theorem 1 is established by proving the existence of certain highly symmetric tensors (Theorem 14, our second main result) and using them to fool the AIP hierarchy (Proposition 22). The tensors we will build enjoy the remarkable property of looking identical when observed from any angle, which is why we shall refer to them as to *crystals*.

Given $p, q \in \mathbb{N}$, we let $[q]_{\rightarrow}^p$ denote the set of increasing tuples in $[q]^p$; i.e., $[q]_{\rightarrow}^p = \{(i_1, \dots, i_p) \in [q]^p \text{ s.t. } i_1 < i_2 < \dots < i_p\}$. Moreover, we let $[q]_{\rightarrow}^0 = \{\epsilon\}$ for any $q \in \mathbb{N}$. Observe that $[q]_{\rightarrow}^p \neq \emptyset$ if and only if $p \leq q$.

Definition 12. For $q, n \in \mathbb{N}$, let M be an $n \times n$ integer matrix. A tensor $C \in \mathcal{T}^{n \cdot \mathbf{1}_q}(\mathbb{Z})$ is a q -dimensional M -crystal if $\Pi_{\mathbf{i}}^{n \cdot \mathbf{1}_q} * C = M$ for each $\mathbf{i} \in [q]_{\rightarrow}^2$.

⁵The expression ‘‘tensor product of digraphs’’ is sometimes used in the literature to indicate the direct or categorical product of digraphs. The tensor power used here is unrelated to that notion – in particular, as it is clear from Definition 9, the k -th tensor power of a digraph is not a digraph for $k > 1$.

⁶In particular, the number of hyperedges in $\mathbf{H}^{\otimes k}$ is equal to the number of edges in \mathbf{H} .

⁷The result in [29] is proved for arbitrary relational structures; for the purpose of this work, the less general version concerning digraphs is enough. Moreover, the definition of the AIP hierarchy and the other hierarchies characterised in [29] is formally different from the definition used here, in that it requires preprocessing the PCSP template and instance by ‘‘ k -enhancing’’ them, i.e., adding dummy constraints on k -tuples of variables. As proved in [29, Section A.1], that definition is equivalent to the more standard hierarchy definition used in [25], which we follow in this work.

Remark 13. For $\mathbf{n} \in \mathbb{N}^q$ and $\mathbf{i} \in [q]^p$, the tensor $\Pi_{\mathbf{i}}^{\mathbf{n}}$ introduced in Section 2.1 should be understood as a projection operator, that projects a given tensor T living in $\mathcal{T}^{\mathbf{n}}(\mathbb{Z})$ onto a new system of modes – namely, $\mathbf{n}_{\mathbf{i}}$. As an example, we have seen (cf. Lemma 6) that, if \mathbf{i} is the identity tuple (i.e., the tuple $\langle q \rangle$), contracting by $\Pi_{\mathbf{i}}^{\mathbf{n}}$ leaves T unaffected. More in general, if \mathbf{i} is a permutation (i.e., $\#(\mathbf{i}) = p = q$), $\Pi_{\mathbf{i}}^{\mathbf{n}}$ simply rotates the tensor by rearranging its modes. For instance, for $p = q = 2$, $\Pi_{(1,2)}^{\mathbf{n}}$ is the identity operator, while $\Pi_{(2,1)}^{\mathbf{n}}$ is the transpose operator. Indeed, letting $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$ and considering an $n_1 \times n_2$ matrix M , $\Pi_{(1,2)}^{\mathbf{n}} * M = M$ and $\Pi_{(2,1)}^{\mathbf{n}} * M = M^T$. If $p \leq q$, as it is the case for Definition 12, $\Pi_{\mathbf{i}}^{\mathbf{n}}$ projects a tensor T having q modes onto a smaller, p -dimensional space. In other words, $\Pi_{\mathbf{i}}^{\mathbf{n}} * T$ is a “ p -dimensional picture” of T .

Theorem 14. *Let $q, n \in \mathbb{N}$, and let M be an $n \times n$ integer matrix satisfying $M\mathbf{1}_n = M^T\mathbf{1}_n$. Then there exists a q -dimensional M -crystal.*

Our approach to prove Theorem 14 will be to show something slightly more general: Given a collection \mathcal{C} of pictures that is *realistic* – i.e., such that each pair of pictures is “locally compatible” with each other – one can always produce a tensor C such that photographing C from all angles results in the pictures in \mathcal{C} . After establishing this result (Proposition 20), Theorem 14 will easily follow, by letting all pictures be the same matrix M . We note that, even if the pictures in the definition of a crystal are two-dimensional objects (matrices), the results we shall prove are more conveniently phrased in terms of arbitrary-dimensional pictures.

Definition 15. For $p, q \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{N}^q$, a (p, \mathbf{n}) -album of pictures is a set $\mathcal{C} = \{C_{\mathbf{i}}\}_{\mathbf{i} \in [q]^p}$, such that $C_{\mathbf{i}} \in \mathcal{T}^{\mathbf{n}_{\mathbf{i}}}(\mathbb{Z})$ for each $\mathbf{i} \in [q]^p$. \mathcal{C} is a *realistic* album if

$$\Pi_{\mathbf{r}}^{\mathbf{n}_{\mathbf{i}}} * C_{\mathbf{i}} = \Pi_{\mathbf{s}}^{\mathbf{n}_{\mathbf{j}}} * C_{\mathbf{j}} \quad \text{for any} \quad \mathbf{i}, \mathbf{j} \in [q]^p, \mathbf{r}, \mathbf{s} \in [p]^{p-1} \text{ such that } \mathbf{i}_{\mathbf{r}} = \mathbf{j}_{\mathbf{s}}. \quad (3)$$

\mathcal{C} is a *realisable* album if there exists a tensor $C \in \mathcal{T}^{\mathbf{n}}(\mathbb{Z})$ such that $\Pi_{\mathbf{i}}^{\mathbf{n}} * C = C_{\mathbf{i}}$ for each $\mathbf{i} \in [q]^p$.

Remark 16. Crucially, the pictures in Definition 15 are *oriented*; this is enforced by taking $\mathbf{i} \in [q]^p_{\rightarrow}$ instead of $\mathbf{i} \in [q]^p$. Similarly, in Definition 12, we only require that “oriented pictures” of a crystal C should look identical. If we strengthened this requirement by asking that $\Pi_{\mathbf{i}}^{n \cdot \mathbf{1}_q} * C = M$ for *all* $\mathbf{i} \in [q]^2$, an M -crystal could only exist for a *symmetric* matrix M . Indeed, applying this strengthened requirement to the tuples $\mathbf{i} = (i_1, i_2)$ and $\mathbf{i}_{(2,1)} = (i_2, i_1)$, we would find

$$M = \Pi_{\mathbf{i}}^{n \cdot \mathbf{1}_q} * C = \Pi_{\mathbf{i}_{(2,1)}}^{n \cdot \mathbf{1}_q} * C \stackrel{L.5}{=} \Pi_{(2,1)}^{n \cdot \mathbf{1}_2} * \Pi_{\mathbf{i}}^{n \cdot \mathbf{1}_q} * C = \Pi_{(2,1)}^{n \cdot \mathbf{1}_2} * (\Pi_{\mathbf{i}}^{n \cdot \mathbf{1}_q} * C) = \Pi_{(2,1)}^{n \cdot \mathbf{1}_2} * M = M^T,$$

where the last equality follows from the discussion in Remark 13. This is not a sacrifice we are willing to make, as the crystal we shall need in Proposition 22 to fool AIP corresponds to an integer matrix having zero diagonal and whose entries sum up to one – which, as a consequence, cannot be symmetric, see (4).

It is not difficult to show that, if the pictures in an album are indeed photographs of some unique tensor, then they must be compatible. In other words, a realisable album must be realistic (cf. the beginning of the proof of Proposition 20). Proving that a realistic album is always realisable shall require some more work. We start by showing that the problem of checking if a realistic album is realisable does not change if we rotate the space where the tensors live.

Lemma 17. *Let $p, q \in \mathbb{N}$, let $\ell \in [q]^q$ be such that $\#(\ell) = q$, and let $\mathbf{n} \in \mathbb{N}^q$. If every realistic (p, \mathbf{n}_ℓ) -album of pictures is realisable then every realistic (p, \mathbf{n}) -album of pictures is realisable.*

Proposition 20 is proved through a nested induction, first on the dimension of the pictures (i.e., p), and second on the sum of the sizes of the modes of the tensor C that the pictures claim to depict (i.e., $\mathbf{n}^T \mathbf{1}_q$). Lemmas 18 and 19 contain the base cases for the second and the first inductions, respectively.

Lemma 18. *A realistic $(p, \mathbf{1}_q)$ -album of pictures is realisable for any $p, q \in \mathbb{N}$.*

Lemma 19. *A realistic $(1, \mathbf{n})$ -album of pictures is realisable for any $q \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{N}^q$.*

Proposition 20. *Let $p, q \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{N}^q$. A (p, \mathbf{n}) -album of pictures is realistic if and only if it is realisable.*

Proof of Theorem 14. Consider the $(2, n \cdot \mathbf{1}_q)$ -album of pictures $\mathcal{C} = \{C_{\mathbf{i}}\}_{\mathbf{i} \in [q]_{\rightarrow}^2}$, given by $C_{\mathbf{i}} = M$ for each $\mathbf{i} \in [q]_{\rightarrow}^2$. To check that \mathcal{C} is a realistic album, we only need to notice that $\Pi_1^{n \cdot \mathbf{1}_2} * M = M \mathbf{1}_n$ and $\Pi_2^{n \cdot \mathbf{1}_2} * M = M^T \mathbf{1}_n$ and use that, by hypothesis, $M \mathbf{1}_n = M^T \mathbf{1}_n$. It then follows from Proposition 20 that \mathcal{C} is a realisable album. Hence, there exists a tensor $C \in \mathcal{T}^{n \cdot \mathbf{1}_q}(\mathbb{Z})$ such that $\Pi_{\mathbf{i}}^{n \cdot \mathbf{1}_q} * C = M$ for each $\mathbf{i} \in [q]_{\rightarrow}^2$. By Definition 12, C is a q -dimensional M -crystal. \square

The results in this section are proved in Section 4. We point out that the proofs of Proposition 20 and of the lemmas needed to establish it are constructive, in that they allow to explicitly build a tensor whose projections are prescribed by a realistic album of pictures. As a consequence, the proof of Theorem 14 on the existence of crystals is constructive, too. We now give an example to illustrate the proof strategy.

Example 21. Throughout this example (and Example 23), we shall indicate the numbers -2 , -1 , 0 , 1 , 2 , and 3 by the colours blue, green, light grey, yellow, orange, and red, respectively. The goal is to build a 4-dimensional M -crystal, where

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{array}{|c|c|c|} \hline & & \color{yellow}{\square} \\ \hline \color{yellow}{\square} & & \color{green}{\square} \\ \hline & & \\ \hline \end{array}.$$

To this end, we consider the $(2, 3 \cdot \mathbf{1}_4)$ album of pictures \mathcal{C} such that all pictures are equal to $\begin{array}{|c|c|c|} \hline & & \color{yellow}{\square} \\ \hline \color{yellow}{\square} & & \color{green}{\square} \\ \hline & & \\ \hline \end{array}$. It is easy to check that \mathcal{C} is realistic (cf. the proof of Theorem 14); the goal is to show that \mathcal{C} is realisable, as the tensor $C \in \mathcal{T}^{3 \cdot \mathbf{1}_4}(\mathbb{Z})$ witnessing this fact would be the crystal we are looking for.

Following the proof of Proposition 20, we create two auxiliary albums $\hat{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ from \mathcal{C} . $\hat{\mathcal{C}}$ is a $(1, 3 \cdot \mathbf{1}_3)$ -album – i.e., both the pictures and the tensor that $\hat{\mathcal{C}}$ claims to depict have one fewer dimension than those for the original album \mathcal{C} . In particular, we see from the proof that all pictures in $\hat{\mathcal{C}}$ are the same vector $\begin{array}{|c|c|} \hline \color{yellow}{\square} & \color{green}{\square} \\ \hline \end{array}$. Again, it is not hard to check that $\hat{\mathcal{C}}$ is a realistic album. To check that it is realisable, we only need to find a 3-dimensional tensor such that summing its entries along all three modes yields $\begin{array}{|c|c|} \hline \color{yellow}{\square} & \color{green}{\square} \\ \hline \end{array}$. Either by inspection or using the proof

of Lemma 19, we find that $\hat{C} = \begin{array}{|c|c|c|} \hline \color{yellow}{\square} & & \\ \hline \color{green}{\square} & \color{yellow}{\square} & \\ \hline \color{yellow}{\square} & \color{green}{\square} & \color{yellow}{\square} \\ \hline \end{array} \in \mathcal{T}^{3 \cdot \mathbf{1}_3}(\mathbb{Z})$ satisfies these conditions. The second album we build is the $(2, (3, 3, 3, 2))$ -album $\tilde{\mathcal{C}}$ defined as follows: $\tilde{C}_{1,4} = \tilde{C}_{2,4} = \tilde{C}_{3,4} = \begin{array}{|c|c|} \hline \color{yellow}{\square} & \color{green}{\square} \\ \hline \end{array}$

(i.e., the matrix obtained by slicing off the rightmost column of $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$); each other picture in the album is obtained by taking the corresponding picture in \mathcal{C} and subtracting from it the corresponding projection of \hat{C} (i.e., $\tilde{C}_i = C_i - \Pi_i^{(3,1_3)} * \hat{C}$).

In this way, we obtain $\tilde{C}_{(1,2)} = \tilde{C}_{(1,3)} = \tilde{C}_{(2,3)} = \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$. This album is also realistic, and it is such that the sum of the dimensions is strictly smaller than the sum of the dimensions for the album \mathcal{C} . At this point, we simply iterate the process, by repeatedly “slicing” \tilde{C} into an album of 1-dimensional pictures (which we handle through Lemma 19) and a smaller album of 2-dimensional pictures, until we end up with an album such that all dimensions are shrunk to one, so that the tensor it depicts is a single number (see Lemma 18). Throughout this process, Lemma 17 guarantees that the tensors can be rotated in a way that we slice along the rightmost mode, thus avoiding complications with the orientations of the pictures. In this way, we find that the

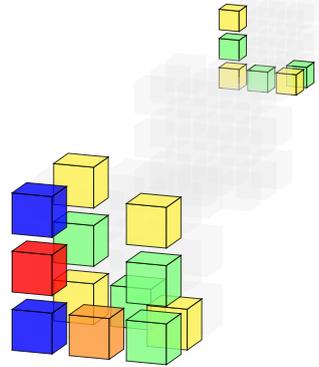


Figure 1: A 4-dimensional $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ -crystal.

album \tilde{C} depicts the tensor \tilde{C} whose two blocks are $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ and the all-zero $3 \times 3 \times 3$ tensor, respectively. Finally, to obtain a tensor depicted by the initial album \mathcal{C} (i.e., a 4-dimensional $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ -crystal), we glue together \tilde{C} and \hat{C} . The result is shown in Figure 1.

2.3 Approximate graph colouring

In this section, we prove the following result.

Theorem (Theorem 1 restated). *No level of the AIP hierarchy solves approximate graph colouring; i.e., for any fixed $3 \leq c \leq d$, there is no k such that the k -th level of AIP solves $\text{PCSP}(\mathbf{K}_c, \mathbf{K}_d)$.*

The next proposition shows that the crystals we mined in Section 2.2 are able to fool the affine integer programming hierarchy. After establishing this result, Theorem 1 will easily follow.

Proposition 22. *Let $k, n \in \mathbb{N}$ with $k \geq 2$, $n \geq 3$, and let \mathbf{G} be a loopless digraph. Then $\text{AIP}^k(\mathbf{G}, \mathbf{K}_n) = \text{YES}$.*

Example 23. We first illustrate Proposition 22 and its proof for the case $k = n = 3$ and $\mathbf{G} = \mathbf{K}_4$. Take the 4-dimensional $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ -crystal C in Figure 1, and consider the map $\xi : [4]^3 \rightarrow \mathcal{T}^{3 \cdot 1_3}(\mathbb{Z})$ defined by $\mathbf{g} \mapsto \Pi_{\mathbf{g}}^{3 \cdot 1_4} * C$; i.e., ξ applied to a triplet \mathbf{g} of modes is the projection of the 4-dimensional crystal onto the 3-dimensional hyperplane corresponding to \mathbf{g} . In particular, $\xi(\mathbf{g})$ is a $3 \times 3 \times 3$ cube. According to Theorem 11, to show that $\text{AIP}^3(\mathbf{K}_4, \mathbf{K}_3) = \text{YES}$, we need to prove that ξ is a homomorphism from $\mathbf{K}_4^{(3)}$ to $\mathbb{F}_{\mathcal{L}_{\text{aff}}}(\mathbf{K}_3^{(3)})$; i.e., that ξ maps hyperedges of $\mathbf{K}_4^{(3)}$ to hyperedges of $\mathbb{F}_{\mathcal{L}_{\text{aff}}}(\mathbf{K}_3^{(3)})$. (The extra condition $\xi(\mathbf{g}_i) = \Pi_i^{3 \cdot 1_3} * \xi(\mathbf{g})$ easily follows from the definition of ξ). Take, for example, the hyperedge $(1, 2)^{(3)} \in \mathcal{E}(\mathbf{K}_4^{(3)})$. Applying ξ entrywise to the $2^3 = 8$ entries of $(1, 2)^{(3)}$ yields the tensor $T \in \mathcal{T}^{2 \cdot 1_3}(\mathcal{T}^{3 \cdot 1_3}(\mathbb{Z})) = \mathcal{T}^{6 \cdot 1_3}(\mathbb{Z})$ in Figure 2. According to Example 10, to conclude that $T \in \mathcal{E}(\mathbb{F}_{\mathcal{L}_{\text{aff}}}(\mathbf{K}_3^{(3)}))$, we need to exhibit some $Q \in \mathcal{L}_{\text{aff}}^{(|\mathcal{E}(\mathbf{K}_3)|)} = \mathcal{L}_{\text{aff}}^{(6)}$ (i.e., some integer distribution over the edges of \mathbf{K}_3)

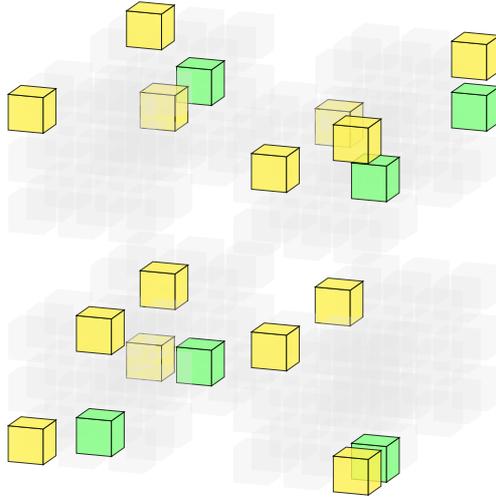


Figure 2: The tensor $\xi((1,2)^{\textcircled{3}})$. Each of the 8 blocks is obtained by projecting the 4-dimensional crystal from Figure 1 onto a 3-dimensional hyperplane.

such that, for any $\mathbf{i} \in [2]^3$, the \mathbf{i} -th block of T is $Q/\pi_{\mathbf{i}}$. Here it is where we use that the two-dimensional pictures of a crystal are all identical: The \mathbf{i} -th block of T is $\xi((1,2)_{\mathbf{i}}) = \Pi_{(1,2)_{\mathbf{i}}}^{3 \cdot \mathbf{1}_4} * C \stackrel{\text{L.5}}{=} \Pi_{\mathbf{i}}^{3 \cdot \mathbf{1}_2} * (\Pi_{(1,2)}^{3 \cdot \mathbf{1}_4} * C) = \Pi_{\mathbf{i}}^{3 \cdot \mathbf{1}_2} * \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$. As a consequence, we can let Q be the distribution encoded by the picture $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$; i.e., the distribution assigning weight 1 to the edges (1,3) and (2,1), and weight -1 to the edge (2,3).

Proof of Proposition 22. Suppose, without loss of generality, that $\mathcal{V}(\mathbf{G}) = [q]$ for some $q \in \mathbb{N}$. If $q = 1$, the proposition is trivially true, so we can assume $q \geq 2$. Consider the matrices

$$\hat{M} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}^{3 \cdot \mathbf{1}_2}(\mathbb{Z}) \quad \text{and} \quad M = \begin{bmatrix} \hat{M} & O \\ O & O \end{bmatrix} \in \mathcal{T}^{n \cdot \mathbf{1}_2}(\mathbb{Z}), \quad (4)$$

where O denotes the all-zero matrix of suitable size. Notice that $M\mathbf{1}_n = M^T\mathbf{1}_n = E_1$. (Recall that E_i is the i -th standard unit vector of length n for any $i \in [n]$.) Then, Theorem 14 provides us with a q -dimensional M -crystal $C \in \mathcal{T}^{n \cdot \mathbf{1}_q}(\mathbb{Z})$. Consider the map

$$\begin{aligned} \xi : \mathcal{V}(\mathbf{G})^k &\rightarrow \mathcal{T}^{n \cdot \mathbf{1}_k}(\mathbb{Z}) \\ \mathbf{g} &\mapsto \Pi_{\mathbf{g}}^{n \cdot \mathbf{1}_q} * C, \end{aligned} \quad (5)$$

which is well defined since $\Pi_{\mathbf{g}}^{n \cdot \mathbf{1}_q} \in \mathcal{T}^{(n \cdot \mathbf{1}_k, n \cdot \mathbf{1}_q)}(\mathbb{Z}) = \mathcal{T}^{n \cdot \mathbf{1}_{k+q}}(\mathbb{Z})$ for any $\mathbf{g} \in \mathcal{V}(\mathbf{G})^k$. We claim that ξ yields a homomorphism from $\mathbf{G}^{\textcircled{k}}$ to $\mathbb{F}_{\mathcal{Z}_{\text{aff}}}(\mathbf{K}_n^{\textcircled{k}})$. First, observe that, for any $\mathbf{g} \in \mathcal{V}(\mathbf{G})^k$,

$$\begin{aligned} \sum_{\mathbf{a} \in [n]^k} E_{\mathbf{a}} * \xi(\mathbf{g}) &\stackrel{(5)}{=} \sum_{\mathbf{a} \in [n]^k} E_{\mathbf{a}} * \Pi_{\mathbf{g}}^{n \cdot \mathbf{1}_q} * C \stackrel{\text{L.3}}{=} \Pi_{\epsilon}^{n \cdot \mathbf{1}_k} * \Pi_{\mathbf{g}}^{n \cdot \mathbf{1}_q} * C \stackrel{\text{L.5}}{=} \Pi_{\epsilon}^{n \cdot \mathbf{1}_q} * C \\ &\stackrel{\text{L.5}}{=} \Pi_{\epsilon}^{n \cdot \mathbf{1}_2} * \Pi_{\langle 2 \rangle}^{n \cdot \mathbf{1}_q} * C \stackrel{\text{D.12}}{=} \Pi_{\epsilon}^{n \cdot \mathbf{1}_2} * M \stackrel{\text{L.3}}{=} \sum_{\mathbf{b} \in [n]^2} E_{\mathbf{b}} * M = \mathbf{1}_n^T M \mathbf{1}_n = 1. \end{aligned}$$

Hence, $\xi(\mathbf{g}) \in \mathcal{L}_{\text{aff}}^{(n^k)} = \mathcal{V}(\mathbb{F}_{\mathcal{L}_{\text{aff}}}(\mathbf{K}_n^{\text{K}}))$, as required.

We now show that ξ preserves the hyperedges of \mathbf{G}^{K} . Recall from Definition 9 that $\mathcal{E}(\mathbf{G}^{\text{K}}) = \{\mathbf{g}^{\text{K}} : \mathbf{g} \in \mathcal{E}(\mathbf{G})\}$. Take $\mathbf{g} \in \mathcal{E}(\mathbf{G})$; we need to prove that $\xi(\mathbf{g}^{\text{K}}) \in \mathcal{E}(\mathbb{F}_{\mathcal{L}_{\text{aff}}}(\mathbf{K}_n^{\text{K}}))$. Observe first that $\xi(\mathbf{g}^{\text{K}}) = (\xi(\mathbf{g}_i))_{i \in [2]^k} \in \mathcal{T}^{2 \cdot 1^k}(\mathcal{T}^{n \cdot 1^k}(\mathbb{Z}))$. Let $\alpha \in [2]^2$ be such that $\mathbf{g}_\alpha \in [q]_{\rightarrow}^2$ (which is possible since $\#(\mathbf{g}) = 2$ as \mathbf{G} is loopless). Notice that $\alpha_\alpha = \langle 2 \rangle$. Consider the vector $Q \in \mathcal{T}^{n^2 - n}(\mathbb{Z})$ whose entries are indexed by the edges of \mathbf{K}_n and are defined as follows: For each $\mathbf{a} \in \mathcal{E}(\mathbf{K}_n)$, the \mathbf{a} -th entry of Q is $E_{\mathbf{a}} * \Pi_{\alpha}^{n \cdot 1_2} * M$. Observe that, for any $a \in [n]$, we have

$$E_{(a,a)} * \Pi_{\alpha}^{n \cdot 1_2} * M \stackrel{\text{L.4}}{=} \sum_{\substack{\mathbf{b} \in [n]^2 \\ \mathbf{b}_{\alpha} = (a,a)}} E_{\mathbf{b}} * M = \sum_{\substack{\mathbf{b} \in [n]^2 \\ \mathbf{b} = (a,a)_{\alpha}}} E_{\mathbf{b}} * M = E_{(a,a)} * M = 0, \quad (6)$$

where we have used that α is an involution and the diagonal entries of M are zero. We find

$$\begin{aligned} \sum_{\mathbf{a} \in \mathcal{E}(\mathbf{K}_n)} E_{\mathbf{a}} * Q &= \sum_{\mathbf{a} \in \mathcal{E}(\mathbf{K}_n)} E_{\mathbf{a}} * \Pi_{\alpha}^{n \cdot 1_2} * M \stackrel{(6)}{=} \sum_{\mathbf{a} \in [n]^2} E_{\mathbf{a}} * \Pi_{\alpha}^{n \cdot 1_2} * M \stackrel{\text{L.3}}{=} \Pi_{\epsilon}^{n \cdot 1_2} * \Pi_{\alpha}^{n \cdot 1_2} * M \\ &\stackrel{\text{L.5}}{=} \Pi_{\epsilon}^{n \cdot 1_2} * M \stackrel{\text{L.3}}{=} \sum_{\mathbf{b} \in [n]^2} E_{\mathbf{b}} * M = \mathbf{1}_n^T M \mathbf{1}_n = 1, \end{aligned}$$

which means that $Q \in \mathcal{L}_{\text{aff}}^{(|\mathcal{E}(\mathbf{K}_n)|)}$. We now aim to show that $\xi(\mathbf{g}_i) = Q / \pi_i$ for any $i \in [2]^k$. We obtain

$$\begin{aligned} \xi(\mathbf{g}_i) &\stackrel{(5)}{=} \Pi_{\mathbf{g}_i}^{n \cdot 1_q} * C = \Pi_{\mathbf{g}_{\alpha \alpha_i}}^{n \cdot 1_q} * C \stackrel{\text{L.5}}{=} \Pi_i^{n \cdot 1_2} * \Pi_{\mathbf{g}_{\alpha \alpha}}^{n \cdot 1_q} * C \stackrel{\text{L.5}}{=} \Pi_i^{n \cdot 1_2} * (\Pi_{\alpha}^{n \cdot 1_2} * \Pi_{\mathbf{g}_{\alpha}}^{n \cdot 1_q}) * C \\ &= \Pi_i^{n \cdot 1_2} * (\Pi_{\alpha}^{n \cdot 1_2} * (\Pi_{\mathbf{g}_{\alpha}}^{n \cdot 1_q} * C)) \stackrel{\text{D.12}}{=} \Pi_i^{n \cdot 1_2} * (\Pi_{\alpha}^{n \cdot 1_2} * M) = \Pi_i^{n \cdot 1_2} * \Pi_{\alpha}^{n \cdot 1_2} * M. \end{aligned}$$

Hence, for any $\mathbf{a} \in [n]^k$,

$$\begin{aligned} E_{\mathbf{a}} * \xi(\mathbf{g}_i) &= E_{\mathbf{a}} * \Pi_i^{n \cdot 1_2} * \Pi_{\alpha}^{n \cdot 1_2} * M \stackrel{\text{L.4}}{=} \sum_{\substack{\mathbf{b} \in [n]^2 \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}} * \Pi_{\alpha}^{n \cdot 1_2} * M \stackrel{(6)}{=} \sum_{\substack{\mathbf{b} \in \mathcal{E}(\mathbf{K}_n) \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}} * \Pi_{\alpha}^{n \cdot 1_2} * M \\ &= \sum_{\substack{\mathbf{b} \in \mathcal{E}(\mathbf{K}_n) \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}} * Q \stackrel{(2)}{=} E_{\mathbf{a}} * Q / \pi_i. \end{aligned}$$

It follows that $\xi(\mathbf{g}_i) = Q / \pi_i$ for any $i \in [2]^k$, as wanted, so $\xi(\mathbf{g}^{\text{K}}) \in \mathcal{E}(\mathbb{F}_{\mathcal{L}_{\text{aff}}}(\mathbf{K}_n^{\text{K}}))$, which means that ξ is indeed a homomorphism.

To be able to apply Theorem 11 and conclude that $\text{AIP}^k(\mathbf{G}, \mathbf{K}_n) = \text{YES}$, we are only left to observe that, for any $\mathbf{g} \in \mathcal{V}(\mathbf{G})^k$ and any $\mathbf{i} \in [k]^k$,

$$\xi(\mathbf{g}_i) \stackrel{\text{D.5}}{=} \Pi_{\mathbf{g}_i}^{n \cdot 1_q} * C \stackrel{\text{L.5}}{=} \Pi_i^{n \cdot 1_k} * \Pi_{\mathbf{g}}^{n \cdot 1_q} * C = \Pi_i^{n \cdot 1_k} * (\Pi_{\mathbf{g}}^{n \cdot 1_q} * C) \stackrel{(5)}{=} \Pi_i^{n \cdot 1_k} * \xi(\mathbf{g}),$$

as desired. \square

We remark that Proposition 22 does *not* hold for $n = 2$, cf. the discussion in Appendix A.

Proof of Theorem 1. Consider three integers c, d, k such that $3 \leq c \leq d$ and $2 \leq k$. Taking \mathbf{K}_{d+1} as \mathbf{G} in Proposition 22, we find that $\text{AIP}^k(\mathbf{K}_{d+1}, \mathbf{K}_c) = \text{YES}$; however, clearly, $\mathbf{K}_{d+1} \not\rightarrow \mathbf{K}_d$. Hence, the k -th level of AIP does not solve $\text{PCSP}(\mathbf{K}_c, \mathbf{K}_d)$. \square

3 Proofs of results from Section 2.1

Lemma (Lemma 3 restated). $E_\epsilon = 1$. Moreover, given $q \in \mathbb{N}_0$ and $\mathbf{n} \in \mathbb{N}^q$, $\Pi_\epsilon^{\mathbf{n}}$ is the all-one tensor in $\mathcal{T}^{\mathbf{n}}(\mathbb{Z})$.

Proof. Since $\mathbb{N}^0 = \{\epsilon\}$ and $[\epsilon] = \{\epsilon\}$, E_ϵ is well defined and lives in $\mathcal{T}^\epsilon(\mathbb{Z}) = \mathbb{Z}$. Its unique entry – i.e., its ϵ -th entry – is 1 by definition.

Setting $p = 0$ yields $[q]^p = \{\epsilon\}$, so $\Pi_\epsilon^{\mathbf{n}}$ is well defined and lives in $\mathcal{T}^{(\mathbf{n}, \mathbf{n})}(\mathbb{Z}) = \mathcal{T}^{\mathbf{n}}(\mathbb{Z})$. Using that $E_\epsilon = 1$, as shown above, and applying the definition (1), we find that, for any $\mathbf{b} \in [\mathbf{n}]$,

$$\Pi_\epsilon^{\mathbf{n}} * E_{\mathbf{b}} = E_\epsilon * \Pi_\epsilon^{\mathbf{n}} * E_{\mathbf{b}} = 1,$$

as required. □

Lemma (Lemma 4 restated). Given $p, q \in \mathbb{N}_0$, $\mathbf{n} \in \mathbb{N}^q$, $\mathbf{i} \in [q]^p$, and $\mathbf{a} \in [\mathbf{n}_i]$, we have

$$E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\mathbf{n}} = \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}}.$$

Proof. If $p = 0$, we have $\mathbf{i} = \mathbf{a} = \epsilon$. Using both claims in Lemma 3, we get

$$E_\epsilon * \Pi_\epsilon^{\mathbf{n}} = \Pi_\epsilon^{\mathbf{n}} = \sum_{\mathbf{b} \in [\mathbf{n}]} E_{\mathbf{b}} = \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_\epsilon = \epsilon}} E_{\mathbf{b}},$$

as required. Suppose now that $p \in \mathbb{N}$. In this case, we can assume that $q \in \mathbb{N}$ as $[0]^p = \emptyset^p = \emptyset$. For any $\mathbf{a}' \in [\mathbf{n}]$, we have

$$\sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}} * E_{\mathbf{a}'} = \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a} \\ \mathbf{b} = \mathbf{a}'}} 1 = \begin{cases} 1 & \text{if } \mathbf{a}' = \mathbf{a} \\ 0 & \text{otherwise} \end{cases} = E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * E_{\mathbf{a}'},$$

thus proving the result. □

Lemma (Lemma 5 restated). Let $m, p, q \in \mathbb{N}_0$, and consider two tuples $\mathbf{i} \in [q]^p$ and $\mathbf{j} \in [p]^m$. Then, for any $\mathbf{n} \in \mathbb{N}^q$,

$$\Pi_{\mathbf{j}}^{\mathbf{n}} = \Pi_{\mathbf{j}}^{\mathbf{n}_i} * \Pi_{\mathbf{i}}^{\mathbf{n}}.$$

Proof. Take $\mathbf{a} \in [\mathbf{n}_i]$ and $\mathbf{a}' \in [\mathbf{n}]$, and observe that

$$\begin{aligned} E_{\mathbf{a}} * (\Pi_{\mathbf{j}}^{\mathbf{n}_i} * \Pi_{\mathbf{i}}^{\mathbf{n}}) * E_{\mathbf{a}'} &= E_{\mathbf{a}} * \Pi_{\mathbf{j}}^{\mathbf{n}_i} * \Pi_{\mathbf{i}}^{\mathbf{n}} * E_{\mathbf{a}'} \stackrel{\text{L.4}}{=} \sum_{\substack{\mathbf{b} \in [\mathbf{n}_i] \\ \mathbf{b}_j = \mathbf{a}}} E_{\mathbf{b}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * E_{\mathbf{a}'} = \sum_{\substack{\mathbf{b} \in [\mathbf{n}_i] \\ \mathbf{b}_j = \mathbf{a} \\ \mathbf{a}' = \mathbf{b}}} 1 = \begin{cases} 1 & \text{if } \mathbf{a}' = \mathbf{a} \\ 0 & \text{otherwise} \end{cases} \\ &= E_{\mathbf{a}} * \Pi_{\mathbf{j}}^{\mathbf{n}} * E_{\mathbf{a}'}, \end{aligned}$$

whence the result follows. □

Lemma (Lemma 6 restated). *Let $q, q' \in \mathbb{N}_0$, $\mathbf{n} \in \mathbb{N}^q$, $\mathbf{n}' \in \mathbb{N}^{q'}$, and $T \in \mathcal{T}^{(\mathbf{n}, \mathbf{n}')}(\mathbb{Z})$. Then*

$$\Pi_{\langle q \rangle}^{\mathbf{n}} * T = T.$$

Proof. For any $\mathbf{a} \in [\mathbf{n}]$, we find

$$E_{\mathbf{a}} * (\Pi_{\langle q \rangle}^{\mathbf{n}} * T) = E_{\mathbf{a}} * \Pi_{\langle q \rangle}^{\mathbf{n}} * T \stackrel{\text{L.4}}{=} \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_{\langle q \rangle} = \mathbf{a}}} E_{\mathbf{b}} * T = \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b} = \mathbf{a}}} E_{\mathbf{b}} * T = E_{\mathbf{a}} * T,$$

as required. \square

4 Proofs of results from Section 2.2

Lemma (Lemma 17 restated). *Let $p, q \in \mathbb{N}$, let $\ell \in [q]^q$ be such that $\#(\ell) = q$, and let $\mathbf{n} \in \mathbb{N}^q$. If every realistic $(p, \mathbf{n}\ell)$ -album of pictures is realisable then every realistic (p, \mathbf{n}) -album of pictures is realisable.*

Proof. Since every permutation can be expressed as the composition of inversions, it is enough to consider the case that ℓ is an inversion; in particular, $\ell = \langle q \rangle$.

Let $\mathcal{C} = \{C_{\mathbf{i}}\}_{\mathbf{i} \in [q]_{\rightarrow}^p}$ be a realistic (p, \mathbf{n}) -album of pictures. For any $\mathbf{i} \in [q]_{\rightarrow}^p$, let \mathbf{i}^+ be the (unique) tuple in $[p]^p$ such that $\ell_{\mathbf{i}^+} \in [q]_{\rightarrow}^p$. Let also \mathbf{i}^- be the (unique) tuple in $[p]^p$ such that $\mathbf{i}_-^+ = \mathbf{i}_-^- = \langle p \rangle$. For each $\mathbf{i} \in [q]_{\rightarrow}^p$, define the tensor

$$\tilde{C}_{\mathbf{i}} = \Pi_{\mathbf{i}^-}^{\mathbf{n}\ell_{\mathbf{i}^+}} * C_{\ell_{\mathbf{i}^+}}. \quad (7)$$

Observe that $\tilde{C}_{\mathbf{i}} \in \mathcal{T}^{\mathbf{n}\ell_{\mathbf{i}^+}}(\mathbb{Z})$, so $\tilde{\mathcal{C}} = \{\tilde{C}_{\mathbf{i}}\}_{\mathbf{i} \in [q]_{\rightarrow}^p}$ is a $(p, \mathbf{n}\ell)$ -album of pictures. We claim that $\tilde{\mathcal{C}}$ is a realistic album. To prove the claim, take $\mathbf{i}, \mathbf{j} \in [q]_{\rightarrow}^p$ and $\mathbf{r}, \mathbf{s} \in [p]_{\rightarrow}^{p-1}$ such that $\mathbf{i}_{\mathbf{r}} = \mathbf{j}_{\mathbf{s}}$. We need to show that

$$\Pi_{\mathbf{r}}^{\mathbf{n}\ell_{\mathbf{i}}} * \tilde{C}_{\mathbf{i}} = \Pi_{\mathbf{s}}^{\mathbf{n}\ell_{\mathbf{j}}} * \tilde{C}_{\mathbf{j}}. \quad (8)$$

Let $\alpha, \beta \in [p-1]^{p-1}$ be the (unique) tuples such that $\mathbf{i}_{\mathbf{r}\alpha}^- \in [p]_{\rightarrow}^{p-1}$ and $\alpha\beta = \beta\alpha = \langle p-1 \rangle$. We claim that $\mathbf{j}_{\mathbf{s}\alpha}^- \in [p]_{\rightarrow}^{p-1}$. Indeed, for any $x, y \in [p-1]$ such that $x < y$ we have

$$\begin{aligned} i_{\mathbf{r}\alpha x}^- < i_{\mathbf{r}\alpha y}^- &\Rightarrow \ell_{i_{\mathbf{r}\alpha x}^-} < \ell_{i_{\mathbf{r}\alpha y}^-} \Rightarrow \ell_{i_{\mathbf{r}\alpha x}} < \ell_{i_{\mathbf{r}\alpha y}} \Rightarrow \ell_{j_{\mathbf{s}\alpha x}} < \ell_{j_{\mathbf{s}\alpha y}} \\ &\Rightarrow \ell_{j_{\mathbf{s}\alpha x}^+} < \ell_{j_{\mathbf{s}\alpha y}^+} \Rightarrow j_{\mathbf{s}\alpha x}^- < j_{\mathbf{s}\alpha y}^-, \end{aligned}$$

thus proving the claim. Therefore,

$$\begin{aligned} \Pi_{\mathbf{r}}^{\mathbf{n}\ell_{\mathbf{i}}} * \tilde{C}_{\mathbf{i}} &\stackrel{(7)}{=} \Pi_{\mathbf{r}}^{\mathbf{n}\ell_{\mathbf{i}}} * \left(\Pi_{\mathbf{i}^-}^{\mathbf{n}\ell_{\mathbf{i}^+}} * C_{\ell_{\mathbf{i}^+}} \right) = \Pi_{\mathbf{r}}^{\mathbf{n}\ell_{\mathbf{i}}} * \Pi_{\mathbf{i}^-}^{\mathbf{n}\ell_{\mathbf{i}^+}} * C_{\ell_{\mathbf{i}^+}} \stackrel{\text{L.5}}{=} \Pi_{\mathbf{i}_{\mathbf{r}}^-}^{\mathbf{n}\ell_{\mathbf{i}^+}} * C_{\ell_{\mathbf{i}^+}} = \Pi_{\mathbf{i}_{\mathbf{r}\alpha\beta}^-}^{\mathbf{n}\ell_{\mathbf{i}^+}} * C_{\ell_{\mathbf{i}^+}} \\ &\stackrel{\text{L.5}}{=} \Pi_{\beta}^{\mathbf{n}\ell_{\mathbf{i}_{\mathbf{r}\alpha}^-}} * \Pi_{\mathbf{i}_{\mathbf{r}\alpha}^-}^{\mathbf{n}\ell_{\mathbf{i}^+}} * C_{\ell_{\mathbf{i}^+}} = \Pi_{\beta}^{\mathbf{n}\ell_{\mathbf{i}_{\mathbf{r}\alpha}^-}} * \left(\Pi_{\mathbf{i}_{\mathbf{r}\alpha}^-}^{\mathbf{n}\ell_{\mathbf{i}^+}} * C_{\ell_{\mathbf{i}^+}} \right) \end{aligned} \quad (9)$$

and, similarly,

$$\Pi_{\mathbf{s}}^{\mathbf{n}\ell_{\mathbf{j}}} * \tilde{C}_{\mathbf{j}} = \Pi_{\beta}^{\mathbf{n}\ell_{\mathbf{j}_{\mathbf{s}\alpha}^-}} * \left(\Pi_{\mathbf{j}_{\mathbf{s}\alpha}^-}^{\mathbf{n}\ell_{\mathbf{j}^+}} * C_{\ell_{\mathbf{j}^+}} \right). \quad (10)$$

Let us now focus on the tuples $\ell_{\mathbf{i}_+}, \ell_{\mathbf{j}_+} \in [q]_{\rightarrow}^p$ and $\mathbf{i}_\alpha^-, \mathbf{j}_\alpha^- \in [p]_{\rightarrow}^{p-1}$. Observe that

$$\ell_{\mathbf{i}_+} = \ell_{\mathbf{i}_\alpha^-} = \ell_{\mathbf{j}_\alpha^-} = \ell_{\mathbf{j}_+}.$$

Using that \mathcal{C} is a realistic album, we deduce

$$\Pi_{\mathbf{i}_\alpha^-}^{\mathbf{n}_{\ell_{\mathbf{i}_+}}} * C_{\ell_{\mathbf{i}_+}} = \Pi_{\mathbf{j}_\alpha^-}^{\mathbf{n}_{\ell_{\mathbf{j}_+}}} * C_{\ell_{\mathbf{j}_+}}. \quad (11)$$

Combining (9), (10), and (11), and recalling that $\mathbf{i}_r = \mathbf{j}_s$, yields (8), thus proving that $\tilde{\mathcal{C}}$ is a realistic (p, \mathbf{n}_ℓ) -album of pictures, as claimed. From the hypothesis of the lemma, we deduce that $\tilde{\mathcal{C}}$ is realisable, so there exists a tensor $\tilde{C} \in \mathcal{T}^{\mathbf{n}_\ell}(\mathbb{Z})$ such that $\Pi_{\mathbf{i}}^{\mathbf{n}_\ell} * \tilde{C} = \tilde{C}_{\mathbf{i}}$ for each $\mathbf{i} \in [q]_{\rightarrow}^p$. Define $C = \Pi_{\ell}^{\mathbf{n}_\ell} * \tilde{C} \in \mathcal{T}^{\mathbf{n}}(\mathbb{Z})$ (where we are using that $\ell_\ell = \langle q \rangle$). Given $\mathbf{i} \in [q]_{\rightarrow}^p$, we find

$$\begin{aligned} \Pi_{\mathbf{i}}^{\mathbf{n}} * C &= \Pi_{\mathbf{i}}^{\mathbf{n}} * (\Pi_{\ell}^{\mathbf{n}_\ell} * \tilde{C}) = \Pi_{\mathbf{i}_+}^{\mathbf{n}} * (\Pi_{\ell}^{\mathbf{n}_\ell} * \tilde{C}) \stackrel{\text{L.5}}{=} \Pi_{\mathbf{i}_-}^{\mathbf{n}_{\mathbf{i}_+}} * \Pi_{\mathbf{i}_+}^{\mathbf{n}_{\mathbf{i}_+}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * \Pi_{\ell}^{\mathbf{n}_\ell} * \tilde{C} \\ &\stackrel{\text{L.5}}{=} \Pi_{\mathbf{i}_-}^{\mathbf{n}_{\mathbf{i}_+}} * \Pi_{\ell_{\mathbf{i}_+}}^{\mathbf{n}_\ell} * \tilde{C} = \Pi_{\mathbf{i}_-}^{\mathbf{n}_{\mathbf{i}_+}} * (\Pi_{\ell_{\mathbf{i}_+}}^{\mathbf{n}_\ell} * \tilde{C}) = \Pi_{\mathbf{i}_-}^{\mathbf{n}_{\mathbf{i}_+}} * \tilde{C}_{\ell_{\mathbf{i}_+}}. \end{aligned} \quad (12)$$

Notice that $\ell_{\ell_{\mathbf{i}_+}} = \mathbf{i}$, which is an increasing tuple. Hence, $(\ell_{\mathbf{i}_+})^+ = \mathbf{i}^-$ and, consequently, $(\ell_{\mathbf{i}_+})^- = \mathbf{i}^+$. It follows from (7) that

$$\tilde{C}_{\ell_{\mathbf{i}_+}} = \Pi_{\mathbf{i}_+}^{\mathbf{n}_{\ell_{\mathbf{i}_+}}} * C_{\ell_{\ell_{\mathbf{i}_+}}} = \Pi_{\mathbf{i}_+}^{\mathbf{n}_{\mathbf{i}}} * C_{\mathbf{i}}. \quad (13)$$

Combining (12) and (13) yields

$$\Pi_{\mathbf{i}}^{\mathbf{n}} * C = \Pi_{\mathbf{i}_-}^{\mathbf{n}_{\mathbf{i}_+}} * (\Pi_{\mathbf{i}_+}^{\mathbf{n}_{\mathbf{i}}} * C_{\mathbf{i}}) = \Pi_{\mathbf{i}_-}^{\mathbf{n}_{\mathbf{i}_+}} * \Pi_{\mathbf{i}_+}^{\mathbf{n}_{\mathbf{i}}} * C_{\mathbf{i}} \stackrel{\text{L.5}}{=} \Pi_{\langle p \rangle}^{\mathbf{n}_{\mathbf{i}}} * C_{\mathbf{i}} \stackrel{\text{L.6}}{=} C_{\mathbf{i}},$$

which concludes the proof that \mathcal{C} is a realisable album of pictures. \square

Lemma (Lemma 18 restated). *A realistic $(p, \mathbf{1}_q)$ -album of pictures is realisable for any $p, q \in \mathbb{N}$.*

Proof. Let $\mathcal{C} = \{C_{\mathbf{i}}\}_{\mathbf{i} \in [q]_{\rightarrow}^p}$ be a realistic $(p, \mathbf{1}_q)$ -album of pictures. For any $\mathbf{i} \in [q]_{\rightarrow}^p$, $C_{\mathbf{i}} \in \mathcal{T}^{(\mathbf{1}_q)\mathbf{i}}(\mathbb{Z}) = \mathcal{T}^{\mathbf{1}_p}(\mathbb{Z})$. We claim that $C_{\mathbf{i}} = C_{\mathbf{j}}$ for any $\mathbf{i}, \mathbf{j} \in [q]_{\rightarrow}^p$. Define, for each pair $\mathbf{i}, \mathbf{j} \in [q]_{\rightarrow}^p$, their *distance* $d(\mathbf{i}, \mathbf{j})$ as the cardinality of the set $\{t \in [p] : i_t \neq j_t\}$. Suppose, for the sake of contradiction, that the claim is false, and let $\mathbf{i}, \mathbf{j} \in [q]_{\rightarrow}^p$ attain the minimum distance among all pairs \mathbf{i}', \mathbf{j}' for which $C_{\mathbf{i}'} \neq C_{\mathbf{j}'}$. Let $\alpha = \max\{t \in [p] : i_t \neq j_t\}$. Assume, without loss of generality, that $i_\alpha < j_\alpha$, and define a new tuple $\ell \in [q]_{\rightarrow}^p$ obtained from \mathbf{i} by replacing i_α with j_α . Observe that $i_1 < i_2 < \dots < i_{\alpha-1} < i_\alpha < j_\alpha < j_{\alpha+1} = i_{\alpha+1} < i_{\alpha+2} < \dots < i_p$, so $\ell \in [q]_{\rightarrow}^p$. Letting $\mathbf{r} \in [p]_{\rightarrow}^{p-1}$ be obtained from $\langle p \rangle$ by deleting its α -th entry, observe that $\mathbf{i}_r = \ell_r$. Using that \mathcal{C} is a realistic album, we obtain $\Pi_{\mathbf{r}}^{\mathbf{1}_p} * C_{\mathbf{i}} = \Pi_{\mathbf{r}}^{\mathbf{1}_p} * C_{\ell}$. Therefore,

$$E_{\mathbf{1}_p} * C_{\mathbf{i}} \stackrel{\text{L.4}}{=} E_{\mathbf{1}_{p-1}} * \Pi_{\mathbf{r}}^{\mathbf{1}_p} * C_{\mathbf{i}} = E_{\mathbf{1}_{p-1}} * \Pi_{\mathbf{r}}^{\mathbf{1}_p} * C_{\ell} \stackrel{\text{L.4}}{=} E_{\mathbf{1}_p} * C_{\ell},$$

so $C_\ell = C_i \neq C_j$. But this contradicts the choice of the pair (\mathbf{i}, \mathbf{j}) , as $d(\ell, \mathbf{j}) = d(\mathbf{i}, \mathbf{j}) - 1$. Hence, the claim is true. We can then define a tensor $C \in \mathcal{T}^{1_q}(\mathbb{Z})$ by setting $E_{1_q} * C = E_{1_p} * C_i$ for any $\mathbf{i} \in [q]_{\rightarrow}^p$. In this way, we get

$$E_{1_p} * \Pi_{\mathbf{i}}^{1_q} * C \stackrel{\text{L.4}}{=} E_{1_q} * C = E_{1_p} * C_i.$$

We conclude that $\Pi_{\mathbf{i}}^{1_q} * C = C_i$ for any $\mathbf{i} \in [q]_{\rightarrow}^p$, which means that \mathcal{C} is a realisable album. \square

Lemma (Lemma 19 restated). *A realistic $(1, \mathbf{n})$ -album of pictures is realisable for any $q \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{N}^q$.*

Proof. If $q = 1$, the result is trivially true, so we assume $q \geq 2$. Notice that $[q]_{\rightarrow}^1 = [q]$, so each element of $[q]_{\rightarrow}^1$ is a single number. We prove the statement by induction on $\mathbf{n}^T \mathbf{1}_q$. If $\mathbf{n}^T \mathbf{1}_q = q$, then $\mathbf{n} = \mathbf{1}_q$, and the result follows from Lemma 18. Suppose that $\mathbf{n}^T \mathbf{1}_q \geq q + 1$. Using Lemma 17, we can assume $n_q \geq 2$ without losing generality. Let $\mathcal{C} = \{C_i\}_{i \in [q]}$ be a realistic $(1, \mathbf{n})$ -album of pictures; observe that C_i is a vector in $\mathcal{T}^{n_i}(\mathbb{Z})$ for each $i \in [q]$. Set $\ell = E_{n_q} * C_q$ (i.e., ℓ is the last entry of C_q), and consider a new family of tensors $\tilde{\mathcal{C}} = \{\tilde{C}_i\}_{i \in [q]}$ defined by

$$\tilde{C}_i = \begin{cases} C_i - \ell E_{n_i} & \text{if } i \in [q-1] \\ (E_1 * C_q, \dots, E_{n_q-1} * C_q) & \text{if } i = q. \end{cases}$$

Let $\tilde{\mathbf{n}} = \mathbf{n} - E_q$ and notice that $\tilde{\mathbf{n}} \in \mathbb{N}^q$ since $n_q \geq 2$. We have that $C_i \in \mathcal{T}^{\tilde{n}_i}(\mathbb{Z})$ for each $i \in [q]$, so $\tilde{\mathcal{C}}$ is a $(1, \tilde{\mathbf{n}})$ -album of pictures.

We now show that $\tilde{\mathcal{C}}$ is realistic. By definition, $[1]_{\rightarrow}^0 = \{\epsilon\}$, so we only need to show that

$$\Pi_{\epsilon}^{\tilde{n}_i} * \tilde{C}_i = \Pi_{\epsilon}^{\tilde{n}_j} * \tilde{C}_j \quad \forall i, j \in [q]. \quad (14)$$

We claim that

$$\Pi_{\epsilon}^{\tilde{n}_i} * \tilde{C}_i = \Pi_{\epsilon}^{n_i} * C_i - \ell \quad \forall i \in [q]. \quad (15)$$

Then, (14) would follow from the fact that \mathcal{C} is a realistic album. If $i \in [q-1]$,

$$\Pi_{\epsilon}^{\tilde{n}_i} * \tilde{C}_i = \Pi_{\epsilon}^{n_i} * (C_i - \ell E_{n_i}) \stackrel{\text{L.3}}{=} \Pi_{\epsilon}^{n_i} * C_i - \ell,$$

so (15) holds in this case. Moreover,

$$\Pi_{\epsilon}^{\tilde{n}_q} * \tilde{C}_q = \Pi_{\epsilon}^{n_q-1} * (E_1 * C_q, \dots, E_{n_q-1} * C_q) \stackrel{\text{L.3}}{=} \sum_{b \in [n_q-1]} E_b * C_q = \mathbf{1}_{n_q} * C_q - \ell \stackrel{\text{L.3}}{=} \Pi_{\epsilon}^{n_q} * C_q - \ell,$$

so (15) holds in this case as well. We conclude that $\tilde{\mathcal{C}}$ is indeed a realistic album.

Since $\tilde{\mathbf{n}}^T \mathbf{1}_q = \mathbf{n}^T \mathbf{1}_q - 1$, we have from the inductive hypothesis that $\tilde{\mathcal{C}}$ is realisable, so there exists a tensor $\tilde{C} \in \mathcal{T}^{\tilde{\mathbf{n}}}(\mathbb{Z})$ such that $\Pi_{\tilde{\mathbf{i}}}^{\tilde{\mathbf{n}}} * \tilde{C} = \tilde{C}_i$ for each $i \in [q]$. Define a tensor $C \in \mathcal{T}^{\mathbf{n}}(\mathbb{Z})$ by setting, for each $\mathbf{b} \in [\mathbf{n}]$,

$$E_{\mathbf{b}} * C = \begin{cases} \ell & \text{if } \mathbf{b} = \mathbf{n} \\ 0 & \text{if } \mathbf{b} \neq \mathbf{n} \text{ and } b_q = n_q \\ E_{\mathbf{b}} * \tilde{C} & \text{if } b_q \neq n_q. \end{cases} \quad (16)$$

(Notice that the last line of the right-hand side of the above expression is well defined as, if $b_q \neq n_q$, then $\mathbf{b} \in [\tilde{\mathbf{n}}]$.) Take $i \in [q]$; we claim that $\Pi_i^{\mathbf{n}} * C = C_i$. For $a \in [n_i]$, we find

$$E_a * \Pi_i^{\mathbf{n}} * C \stackrel{\text{L.4}}{=} \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ b_i = a}} E_{\mathbf{b}} * C.$$

For $i \neq q$, this yields

$$E_a * \Pi_i^{\mathbf{n}} * C = \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ b_i = a \\ b_q = n_q}} E_{\mathbf{b}} * C + \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ b_i = a \\ b_q \neq n_q}} E_{\mathbf{b}} * C \stackrel{(16)}{=} \ell \cdot \delta_{a, n_i} + \sum_{\substack{\mathbf{b} \in [\tilde{\mathbf{n}}] \\ b_i = a}} E_{\mathbf{b}} * \tilde{C}$$

(where δ_{a, n_i} is 1 if $a = n_i$, 0 otherwise)

$$\stackrel{\text{L.4}}{=} \ell \cdot \delta_{a, n_i} + E_a * \Pi_i^{\tilde{\mathbf{n}}} * \tilde{C} = \ell \cdot \delta_{a, n_i} + E_a * \tilde{C}_i = \ell \cdot \delta_{a, n_i} + E_a * (C_i - \ell E_{n_i}) = E_a * C_i.$$

For $i = q$, if $a = n_q$ we get

$$E_a * \Pi_q^{\mathbf{n}} * C = \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ b_q = n_q}} E_{\mathbf{b}} * C \stackrel{(16)}{=} \ell = E_a * C_q,$$

while if $a \neq n_q$ we get

$$\begin{aligned} E_a * \Pi_q^{\mathbf{n}} * C &= \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ b_q = a}} E_{\mathbf{b}} * C \stackrel{(16)}{=} \sum_{\substack{\mathbf{b} \in [\tilde{\mathbf{n}}] \\ b_q = a}} E_{\mathbf{b}} * \tilde{C} \stackrel{\text{L.4}}{=} E_a * \Pi_q^{\tilde{\mathbf{n}}} * \tilde{C} = E_a * \tilde{C}_q \\ &= E_a * (E_1 * C_q, \dots, E_{n_q-1} * C_q) = E_a * C_q. \end{aligned}$$

It follows that $\Pi_i^{\mathbf{n}} * C = C_i$, as claimed. Therefore, \mathcal{C} is a realisable album. \square

Proposition (Proposition 20 restated). *Let $p, q \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{N}^q$. A (p, \mathbf{n}) -album of pictures is realistic if and only if it is realisable.*

Proof. Let $\mathcal{C} = \{C_{\mathbf{i}}\}_{\mathbf{i} \in [q]_{\rightarrow}^p}$ be a realisable album of pictures; i.e., there exists $C \in \mathcal{T}^{\mathbf{n}}(\mathbb{Z})$ such that $\Pi_{\mathbf{i}}^{\mathbf{n}} * C = C_{\mathbf{i}}$ for each $\mathbf{i} \in [q]_{\rightarrow}^p$. For any $\mathbf{i}, \mathbf{j} \in [q]_{\rightarrow}^p$ and $\mathbf{r}, \mathbf{s} \in [p]_{\rightarrow}^{p-1}$ such that $\mathbf{i}_{\mathbf{r}} = \mathbf{j}_{\mathbf{s}}$, we find

$$\Pi_{\mathbf{r}}^{\mathbf{n}_{\mathbf{i}}} * C_{\mathbf{i}} = \Pi_{\mathbf{r}}^{\mathbf{n}_{\mathbf{i}}} * (\Pi_{\mathbf{i}}^{\mathbf{n}} * C) = \Pi_{\mathbf{r}}^{\mathbf{n}_{\mathbf{i}}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * C \stackrel{\text{L.5}}{=} \Pi_{\mathbf{r}}^{\mathbf{n}} * C = \Pi_{\mathbf{j}_{\mathbf{s}}}^{\mathbf{n}} * C \stackrel{\text{L.5}}{=} \Pi_{\mathbf{s}}^{\mathbf{n}_{\mathbf{j}}} * \Pi_{\mathbf{j}}^{\mathbf{n}} * C = \Pi_{\mathbf{s}}^{\mathbf{n}_{\mathbf{j}}} * C_{\mathbf{j}},$$

which shows that \mathcal{C} is a realistic album. Hence, the “if” part of the statement is true. Next, we focus on the “only if” part.

We prove the result by nested induction, first on p and second on $\mathbf{n}^T \mathbf{1}_q$. For $p = 1$, the result follows from Lemma 19. Suppose that $p \geq 2$. For $\mathbf{n}^T \mathbf{1}_q = q$ (which implies $\mathbf{n} = \mathbf{1}_q$), the result follows from Lemma 18. Suppose that $\mathbf{n}^T \mathbf{1}_q \geq q + 1$. Using Lemma 17, we can safely assume $n_q \geq 2$. If $q = 1$, then $[q]_{\rightarrow}^p = \emptyset$ and the statement is trivially true, so we can assume $q \geq 2$. Let $\mathcal{C} = \{C_{\mathbf{i}}\}_{\mathbf{i} \in [q]_{\rightarrow}^p}$ be a realistic (p, \mathbf{n}) -album of pictures; we need to show that \mathcal{C} is realisable.

Set $\hat{\mathbf{n}} = (n_1, \dots, n_{q-1}) \in \mathbb{N}^{q-1}$. For any $\mathbf{i} \in [q-1]_{\rightarrow}^{p-1}$, we define $\hat{C}_{\mathbf{i}} \in \mathcal{T}^{\hat{\mathbf{n}}_i}(\mathbb{Z})$ by $E_{\mathbf{a}} * \hat{C}_{\mathbf{i}} = E_{(\mathbf{a}, n_q)} * C_{(\mathbf{i}, q)}$ for each $\mathbf{a} \in [\hat{\mathbf{n}}_i]$. Observe that the last expression is well defined, as $\mathbf{i} \in [q-1]_{\rightarrow}^{p-1}$ implies that $(\mathbf{i}, q) \in [q]_{\rightarrow}^p$. We claim that the family $\hat{\mathcal{C}} = \{\hat{C}_{\mathbf{i}}\}_{\mathbf{i} \in [q-1]_{\rightarrow}^{p-1}}$ is a realistic $(p-1, \hat{\mathbf{n}})$ -album of pictures. Take $\mathbf{i}, \mathbf{j} \in [q-1]_{\rightarrow}^{p-1}$ and $\mathbf{r}, \mathbf{s} \in [p-1]_{\rightarrow}^{p-2}$ such that $\mathbf{i}_{\mathbf{r}} = \mathbf{j}_{\mathbf{s}}$. For any $\mathbf{a} \in [\hat{\mathbf{n}}_{\mathbf{i}_{\mathbf{r}}}]$, we find

$$\begin{aligned} E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\hat{\mathbf{n}}_i} * \hat{C}_{\mathbf{i}} &\stackrel{\text{L.4}}{=} \sum_{\substack{\mathbf{b} \in [\hat{\mathbf{n}}_i] \\ \mathbf{b}_{\mathbf{r}} = \mathbf{a}}} E_{\mathbf{b}} * \hat{C}_{\mathbf{i}} = \sum_{\substack{\mathbf{b} \in [\hat{\mathbf{n}}_i] \\ \mathbf{b}_{\mathbf{r}} = \mathbf{a}}} E_{(\mathbf{b}, n_q)} * C_{(\mathbf{i}, q)} = \sum_{\substack{\mathbf{c} \in [\mathbf{n}_{(\mathbf{i}, q)}] \\ \mathbf{c}_{(\mathbf{r}, p)} = (\mathbf{a}, n_q)}} E_{\mathbf{c}} * C_{(\mathbf{i}, q)} \\ &\stackrel{\text{L.4}}{=} E_{(\mathbf{a}, n_q)} * \Pi_{(\mathbf{r}, p)}^{\mathbf{n}_{(\mathbf{i}, q)}} * C_{(\mathbf{i}, q)} \end{aligned} \quad (17)$$

and, similarly,

$$E_{\mathbf{a}} * \Pi_{\mathbf{s}}^{\hat{\mathbf{n}}_j} * \hat{C}_{\mathbf{j}} = E_{(\mathbf{a}, n_q)} * \Pi_{(\mathbf{s}, p)}^{\mathbf{n}_{(\mathbf{j}, q)}} * C_{(\mathbf{j}, q)}. \quad (18)$$

We now use the fact that \mathcal{C} is a realistic album. In particular, we apply (3) to the tuples $(\mathbf{i}, q), (\mathbf{j}, q) \in [q]_{\rightarrow}^p$ and $(\mathbf{r}, p), (\mathbf{s}, p) \in [p]_{\rightarrow}^{p-1}$ (note that $(\mathbf{i}, q)_{(\mathbf{r}, p)} = (\mathbf{i}_{\mathbf{r}}, q) = (\mathbf{j}_{\mathbf{s}}, q) = (\mathbf{j}, q)_{(\mathbf{s}, p)}$). Since $(\mathbf{a}, n_q) \in [\mathbf{n}_{(\mathbf{i}, q)_{(\mathbf{r}, p)}}]$, we obtain

$$E_{(\mathbf{a}, n_q)} * \Pi_{(\mathbf{r}, p)}^{\mathbf{n}_{(\mathbf{i}, q)}} * C_{(\mathbf{i}, q)} = E_{(\mathbf{a}, n_q)} * \Pi_{(\mathbf{s}, p)}^{\mathbf{n}_{(\mathbf{j}, q)}} * C_{(\mathbf{j}, q)}.$$

Combining this with (17) and (18) yields

$$E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\hat{\mathbf{n}}_i} * \hat{C}_{\mathbf{i}} = E_{\mathbf{a}} * \Pi_{\mathbf{s}}^{\hat{\mathbf{n}}_j} * \hat{C}_{\mathbf{j}}.$$

We conclude that $\hat{\mathcal{C}}$ is a realistic album, as claimed. It follows from the inductive hypothesis that $\hat{\mathcal{C}}$ is realisable, so we can find a tensor $\hat{C} \in \mathcal{T}^{\hat{\mathbf{n}}}(\mathbb{Z})$ such that $\Pi_{\mathbf{i}}^{\hat{\mathbf{n}}} * \hat{C} = \hat{C}_{\mathbf{i}}$ for each $\mathbf{i} \in [q-1]_{\rightarrow}^{p-1}$. Let now $\tilde{\mathbf{n}} = \mathbf{n} - E_q \in \mathbb{N}^q$. For any $\mathbf{i} \in [q]_{\rightarrow}^p$, define a tensor $\tilde{C}_{\mathbf{i}} \in \mathcal{T}^{\tilde{\mathbf{n}}_i}(\mathbb{Z})$ as follows: If $i_p \neq q$ (in which case $\mathbf{i} \in [q-1]_{\rightarrow}^p$) we set $\tilde{C}_{\mathbf{i}} = C_{\mathbf{i}} - \Pi_{\mathbf{i}}^{\hat{\mathbf{n}}} * \hat{C}$; if $i_p = q$, for $\mathbf{b} \in [\tilde{\mathbf{n}}_i]$, we set $E_{\mathbf{b}} * \tilde{C}_{\mathbf{i}} = E_{\mathbf{b}} * C_{\mathbf{i}}$ (where the last expression is well defined as $[\tilde{\mathbf{n}}] \subseteq [\mathbf{n}]$, so $[\tilde{\mathbf{n}}_i] \subseteq [\mathbf{n}_i]$). We claim that the family $\tilde{\mathcal{C}} = \{\tilde{C}_{\mathbf{i}}\}_{\mathbf{i} \in [q]_{\rightarrow}^p}$ is a realistic $(p, \tilde{\mathbf{n}})$ -album of pictures. To that end, we shall first prove that the equation

$$E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\tilde{\mathbf{n}}_i} * \tilde{C}_{\mathbf{i}} = \begin{cases} E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\mathbf{n}_i} * C_{\mathbf{i}} & \text{if } i_{r_{p-1}} = q \\ E_{\mathbf{a}} * (\Pi_{\mathbf{r}}^{\mathbf{n}_i} * C_{\mathbf{i}} - \hat{C}_{\mathbf{i}_{\mathbf{r}}}) & \text{otherwise} \end{cases} \quad (19)$$

is satisfied for any $\mathbf{i} \in [q]_{\rightarrow}^p$, any $\mathbf{r} \in [p]_{\rightarrow}^{p-1}$, and any $\mathbf{a} \in [\tilde{\mathbf{n}}_{\mathbf{i}_{\mathbf{r}}}]$. First, notice that, if $i_p = q$,

$$[\tilde{\mathbf{n}}_i] = [\tilde{n}_{i_1}] \times \dots \times [\tilde{n}_{i_{p-1}}] \times [\tilde{n}_{i_p}] = [n_{i_1}] \times \dots \times [n_{i_{p-1}}] \times [n_q - 1] = \{\mathbf{b} \in [\mathbf{n}_i] : b_p \neq n_q\}$$

while, if $i_p \neq q$, $\tilde{\mathbf{n}}_i = \hat{\mathbf{n}}_i = \mathbf{n}_i$, so $[\tilde{\mathbf{n}}_i] = [\hat{\mathbf{n}}_i] = [\mathbf{n}_i]$. Suppose that $i_{r_{p-1}} = q$. In this case, we have $r_{p-1} = p$ and $i_p = q$. Hence,

$$E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\tilde{\mathbf{n}}_i} * \tilde{C}_{\mathbf{i}} \stackrel{\text{L.4}}{=} \sum_{\substack{\mathbf{b} \in [\tilde{\mathbf{n}}_i] \\ \mathbf{b}_{\mathbf{r}} = \mathbf{a}}} E_{\mathbf{b}} * \tilde{C}_{\mathbf{i}} = \sum_{\substack{\mathbf{b} \in [\mathbf{n}_i] \\ \mathbf{b}_{\mathbf{r}} = \mathbf{a} \\ b_p \neq n_q}} E_{\mathbf{b}} * C_{\mathbf{i}} = \sum_{\substack{\mathbf{b} \in [\mathbf{n}_i] \\ \mathbf{b}_{\mathbf{r}} = \mathbf{a}}} E_{\mathbf{b}} * C_{\mathbf{i}} \stackrel{\text{L.4}}{=} E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\mathbf{n}_i} * C_{\mathbf{i}},$$

so (19) holds in this case. Suppose now that $i_{r_{p-1}} \neq q$. This can happen either if $i_p \neq q$ (case a), or if $i_p = q$ and $r_{p-1} \neq p$ (case b), and it implies that $\mathbf{i}_r \in [q-1]_{\rightarrow}^{p-1}$. In case a,

$$\Pi_{\mathbf{r}}^{\tilde{\mathbf{n}}_i} * \tilde{C}_i = \Pi_{\mathbf{r}}^{\mathbf{n}_i} * (C_i - \Pi_{\mathbf{i}}^{\hat{\mathbf{n}}} * \hat{C}) = \Pi_{\mathbf{r}}^{\mathbf{n}_i} * C_i - \Pi_{\mathbf{r}}^{\mathbf{n}_i} * \Pi_{\mathbf{i}}^{\hat{\mathbf{n}}} * \hat{C} \stackrel{\text{L.5}}{=} \Pi_{\mathbf{r}}^{\mathbf{n}_i} * C_i - \Pi_{\mathbf{r}}^{\hat{\mathbf{n}}} * \hat{C} = \Pi_{\mathbf{r}}^{\mathbf{n}_i} * C_i - \hat{C}_{\mathbf{i}_r},$$

where the last equality follows from the property of \hat{C} . So, (19) holds in this case. In case b, we must have $\mathbf{r} = \langle p-1 \rangle$. Hence,

$$\begin{aligned} E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\tilde{\mathbf{n}}_i} * \tilde{C}_i &\stackrel{\text{L.4}}{=} \sum_{\substack{\mathbf{b} \in [\tilde{\mathbf{n}}_i] \\ \mathbf{b}_{\langle p-1 \rangle} = \mathbf{a}}} E_{\mathbf{b}} * \tilde{C}_i = \sum_{\substack{\mathbf{b} \in [\mathbf{n}_i] \\ \mathbf{b}_{\langle p-1 \rangle} = \mathbf{a} \\ b_p \neq n_q}} E_{\mathbf{b}} * C_i = \sum_{\substack{\mathbf{b} \in [\mathbf{n}_i] \\ \mathbf{b}_{\langle p-1 \rangle} = \mathbf{a}}} E_{\mathbf{b}} * C_i - E_{(\mathbf{a}, n_q)} * C_i \\ &\stackrel{\text{L.4}}{=} E_{\mathbf{a}} * \Pi_{\langle p-1 \rangle}^{\mathbf{n}_i} * C_i - E_{(\mathbf{a}, n_q)} * C_i = E_{\mathbf{a}} * \Pi_{\langle p-1 \rangle}^{\mathbf{n}_i} * C_i - E_{(\mathbf{a}, n_q)} * C_{(\mathbf{i}_{\langle p-1 \rangle}, q)} \\ &= E_{\mathbf{a}} * \Pi_{\langle p-1 \rangle}^{\mathbf{n}_i} * C_i - E_{\mathbf{a}} * \hat{C}_{\mathbf{i}_{\langle p-1 \rangle}} = E_{\mathbf{a}} * (\Pi_{\mathbf{r}}^{\mathbf{n}_i} * C_i - \hat{C}_{\mathbf{i}_r}), \end{aligned}$$

where the penultimate equality comes from the definition of \hat{C} and from the fact that, in this case, $\tilde{\mathbf{n}}_{\mathbf{i}_r} = \hat{\mathbf{n}}_{\mathbf{i}_r}$, so $\mathbf{a} \in [\hat{\mathbf{n}}_{\mathbf{i}_r}]$. We conclude that (19) also holds in case b. Using (19) and the fact that \tilde{C} is a realistic album, we easily conclude that \tilde{C} is a realistic album, too. Indeed, take $\mathbf{i}, \mathbf{j} \in [q]_{\rightarrow}^p$ and $\mathbf{r}, \mathbf{s} \in [p]_{\rightarrow}^{p-1}$ such that $\mathbf{i}_r = \mathbf{j}_s$, and choose $\mathbf{a} \in [\tilde{\mathbf{n}}_{\mathbf{i}_r}]$. Observe that $i_{r_{p-1}} = j_{s_{p-1}}$. If $i_{r_{p-1}} = q$, we find

$$E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\tilde{\mathbf{n}}_i} * \tilde{C}_i = E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\mathbf{n}_i} * C_i = E_{\mathbf{a}} * \Pi_{\mathbf{s}}^{\mathbf{n}_j} * C_j = E_{\mathbf{a}} * \Pi_{\mathbf{s}}^{\tilde{\mathbf{n}}_j} * \tilde{C}_j;$$

otherwise,

$$E_{\mathbf{a}} * \Pi_{\mathbf{r}}^{\tilde{\mathbf{n}}_i} * \tilde{C}_i = E_{\mathbf{a}} * (\Pi_{\mathbf{r}}^{\mathbf{n}_i} * C_i - \hat{C}_{\mathbf{i}_r}) = E_{\mathbf{a}} * (\Pi_{\mathbf{s}}^{\mathbf{n}_j} * C_j - \hat{C}_{\mathbf{j}_s}) = E_{\mathbf{a}} * \Pi_{\mathbf{s}}^{\tilde{\mathbf{n}}_j} * \tilde{C}_j.$$

It follows that \tilde{C} is indeed a realistic album, as claimed. Since $\tilde{\mathbf{n}}^T \mathbf{1}_q = \mathbf{n}^T \mathbf{1}_q - 1$, we can then apply the inductive hypothesis to deduce that \tilde{C} is realisable, so there exists a tensor $\tilde{C} \in \mathcal{T}^{\tilde{\mathbf{n}}}(\mathbb{Z})$ such that $\Pi_{\mathbf{i}}^{\tilde{\mathbf{n}}} * \tilde{C} = \tilde{C}_i$ for each $\mathbf{i} \in [q]_{\rightarrow}^p$.

We now define a tensor $C \in \mathcal{T}^{\mathbf{n}}(\mathbb{Z})$ by setting, for each $\mathbf{b} \in [\mathbf{n}]$,

$$E_{\mathbf{b}} * C = \begin{cases} E_{\mathbf{b}_{\langle q-1 \rangle}} * \hat{C} & \text{if } b_q = n_q \\ E_{\mathbf{b}} * \tilde{C} & \text{if } b_q \neq n_q. \end{cases} \quad (20)$$

Take $\mathbf{i} \in [q]_{\rightarrow}^p$ and $\mathbf{a} \in [\mathbf{n}_i]$. To conclude the proof, we need to show that

$$E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * C = E_{\mathbf{a}} * C_i. \quad (21)$$

Observe that

$$E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * C \stackrel{\text{L.4}}{=} \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}} * C = \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a} \\ b_q = n_q}} E_{\mathbf{b}} * C + \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a} \\ b_q \neq n_q}} E_{\mathbf{b}} * C \stackrel{(20)}{=} \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a} \\ b_q = n_q}} E_{\mathbf{b}_{\langle q-1 \rangle}} * \hat{C} + \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}} * \tilde{C}. \quad (22)$$

Let us denote the first and the second summand of the rightmost expression in (22) by α and β , respectively. Suppose first that $i_p = q$. If $a_p \neq n_q$, we see that $\alpha = 0$, so

$$E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * C \stackrel{(22)}{=} \sum_{\substack{\mathbf{b} \in [\tilde{\mathbf{n}}] \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}} * \tilde{C} \stackrel{L.4}{=} E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\tilde{\mathbf{n}}} * \tilde{C} = E_{\mathbf{a}} * \tilde{C}_{\mathbf{i}} = E_{\mathbf{a}} * C_{\mathbf{i}};$$

if $a_p = n_q$, we get $\beta = 0$, so

$$\begin{aligned} E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * C &\stackrel{(22)}{=} \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a} \\ b_q = n_q}} E_{\mathbf{b}_{\langle q-1 \rangle}} * \hat{C} = \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}_{\langle q-1 \rangle}} * \hat{C} = \sum_{\substack{\mathbf{c} \in [\tilde{\mathbf{n}}] \\ \mathbf{c}_{i_{\langle p-1 \rangle} = \mathbf{a}_{\langle p-1 \rangle}}} E_{\mathbf{c}} * \hat{C} \stackrel{L.4}{=} E_{\mathbf{a}_{\langle p-1 \rangle}} * \Pi_{\mathbf{i}_{\langle p-1 \rangle}}^{\tilde{\mathbf{n}}} * \hat{C} \\ &= E_{\mathbf{a}_{\langle p-1 \rangle}} * \hat{C}_{\mathbf{i}_{\langle p-1 \rangle}} = E_{(\mathbf{a}_{\langle p-1 \rangle}, n_q)} * C_{(\mathbf{i}_{\langle p-1 \rangle}, q)} = E_{\mathbf{a}} * C_{\mathbf{i}}. \end{aligned}$$

Suppose now that $i_p \neq q$, in which case $\mathbf{i} \in [q-1]_{\rightarrow}^p$. We obtain

$$\begin{aligned} \alpha &= \sum_{\substack{\mathbf{b} \in [\mathbf{n}] \\ \mathbf{b}_i = \mathbf{a} \\ b_q = n_q}} E_{\mathbf{b}_{\langle q-1 \rangle}} * \hat{C} = \sum_{\substack{\mathbf{c} \in [\tilde{\mathbf{n}}] \\ \mathbf{c}_i = \mathbf{a}}} E_{\mathbf{c}} * \hat{C} \stackrel{L.4}{=} E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\tilde{\mathbf{n}}} * \hat{C}, \\ \beta &= \sum_{\substack{\mathbf{b} \in [\tilde{\mathbf{n}}] \\ \mathbf{b}_i = \mathbf{a}}} E_{\mathbf{b}} * \tilde{C} \stackrel{L.4}{=} E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\tilde{\mathbf{n}}} * \tilde{C} = E_{\mathbf{a}} * \tilde{C}_{\mathbf{i}} = E_{\mathbf{a}} * (C_{\mathbf{i}} - \Pi_{\mathbf{i}}^{\tilde{\mathbf{n}}} * \hat{C}) = E_{\mathbf{a}} * C_{\mathbf{i}} - E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\tilde{\mathbf{n}}} * \hat{C}, \end{aligned}$$

and it follows that

$$E_{\mathbf{a}} * \Pi_{\mathbf{i}}^{\mathbf{n}} * C \stackrel{(22)}{=} \alpha + \beta = E_{\mathbf{a}} * C_{\mathbf{i}}.$$

Therefore, (21) holds, \mathcal{C} is realisable, and the proof is concluded. \square

A Affine integer programming and 2-colouring

Recall the statement of Proposition 22.

Proposition (Proposition 22 restated). *Let $k, n \in \mathbb{N}$ with $k \geq 2$, $n \geq 3$, and let \mathbf{G} be a loopless digraph. Then $\text{AIP}^k(\mathbf{G}, \mathbf{K}_n) = \text{YES}$.*

Proposition 22 does not hold for $n = 2$; i.e., for the 2-clique \mathbf{K}_2 . The reason why the proof does not work in this case is that the argument relies on the existence of an $n \times n$ “picture-matrix” M that (i) is entrywise integer and affine (i.e., its entries sum up to one), (ii) has zero diagonal, and (iii) satisfies $M\mathbf{1}_n = M^T\mathbf{1}_n$. The purpose of (i) is to make M the 2-dimensional projection of a proper solution to the k -th level of AIP, the purpose of (ii) is to make the solution compatible with the edges of \mathbf{K}_n , and the purpose of (iii) is to make sure that the album of pictures consisting of copies of M is realistic, so that an M -crystal exists. For $n \geq 3$, (4) provides an example of such matrices; for $n \leq 2$, however, one readily checks that there is no matrix satisfying all three requirements.

In fact, one can show using the algebraic machinery developed in [6] that the base level of the AIP hierarchy is already powerful enough to solve $\text{CSP}(\mathbf{K}_2) = \text{PCSP}(\mathbf{K}_2, \mathbf{K}_2)$. Let the symbol “ \equiv_2 ” mean “equal mod 2”.

Definition 24 ([6]). Let A, B be sets and let $L \geq 3$ be an odd integer. A function $f : A^L \rightarrow B$ is said to be *alternating* if

- $f(\mathbf{a}) = f(\mathbf{a}_i)$ for any $\mathbf{a} \in A^L$ and any $\mathbf{i} \in [L]^L$ such that $\#(\mathbf{i}) = L$ and \mathbf{i} preserves the parity (i.e., $i_j =_2 j \ \forall j \in [L]$), and
- $f(\mathbf{a}, b, b) = f(\mathbf{a}, c, c)$ for any $\mathbf{a} \in A^{L-2}$ and any $b, c \in A$.

Given $L \in \mathbb{N}$ and two digraphs $\mathbf{H}, \tilde{\mathbf{H}}$ such that $\mathbf{H} \rightarrow \tilde{\mathbf{H}}$, an L -ary *polymorphism* from \mathbf{H} to $\tilde{\mathbf{H}}$ is a homomorphism from \mathbf{H}^L to $\tilde{\mathbf{H}}$, where \mathbf{H}^L is the L -th *direct power* of \mathbf{H} – i.e., the digraph whose vertex set is $\mathcal{V}(\mathbf{H}^L) = \mathcal{V}(\mathbf{H})^L$ and whose edge set is $\mathcal{E}(\mathbf{H}^L) = \{(\mathbf{h}, \ell) : \mathbf{h}, \ell \in \mathcal{V}(\mathbf{H})^L \text{ and } (h_i, \ell_i) \in \mathcal{E}(\mathbf{H}) \ \forall i \in [L]\}$.

The next result characterises the power of the affine integer programming relaxation (denoted by AIP) as defined in [6], which essentially corresponds to the case $k = 1$ of the hierarchy considered in this work. (In particular, AIP is weaker than any level of the AIP hierarchy, in the sense that any PCSP template solved by AIP is also solved by any level of the hierarchy.)

Theorem 25 ([6]). AIP solves $\text{PCSP}(\mathbf{H}, \tilde{\mathbf{H}})$ if and only if there exist L -ary alternating polymorphisms from \mathbf{H} to $\tilde{\mathbf{H}}$ for any odd $L \geq 3$.

Let the vertex set of \mathbf{K}_2 be $\mathcal{V}(\mathbf{K}_2) = \{0, 1\}$. Given an odd integer $L \geq 3$, consider the function $f : \{0, 1\}^L \rightarrow \{0, 1\}$ defined by $\mathbf{h} \mapsto \sum_{i=1}^L (-1)^{i+1} h_i \pmod 2$, and notice that it is alternating. We claim that f is a polymorphism from \mathbf{K}_2 to itself – i.e., a homomorphism from \mathbf{K}_2^L to \mathbf{K}_2 . Take $\mathbf{h}, \ell \in \{0, 1\}^L$ and suppose that $(\mathbf{h}, \ell) \in \mathcal{E}(\mathbf{K}_2^L)$; i.e., $h_i + \ell_i = 1$ for each $i \in [L]$. If $f(\mathbf{h}) = f(\ell)$, we find

$$\begin{aligned} f(\mathbf{h}) &=_{\mathbb{Z}_2} \sum_{i=1}^L (-1)^{i+1} h_i =_{\mathbb{Z}_2} \sum_{i=1}^L (-1)^{i+1} \ell_i =_{\mathbb{Z}_2} \sum_{i=1}^L (-1)^{i+1} (1 - h_i) =_{\mathbb{Z}_2} \sum_{i=1}^L (-1)^{i+1} - \sum_{i=1}^L (-1)^{i+1} h_i \\ &=_{\mathbb{Z}_2} 1 - f(\mathbf{h}), \end{aligned}$$

so $2f(\mathbf{h}) =_{\mathbb{Z}_2} 1$, a contradiction. As a consequence, $f(\mathbf{h}) \neq f(\ell)$, so $f((\mathbf{h}, \ell)) \in \mathcal{E}(\mathbf{K}_2)$. Hence, f is a polymorphism, as claimed. Using Theorem 25, it follows that AIP solves $\text{CSP}(\mathbf{K}_2)$, which means that $\text{AIP}(\mathbf{G}, \mathbf{K}_2)$ outputs YES exactly when \mathbf{G} is 2-colourable, i.e., bipartite.

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