

# Tight Bounds for Vertex Connectivity in Dynamic Streams

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## Abstract

We present a streaming algorithm for the vertex connectivity problem in dynamic streams with a (nearly) optimal space bound: for any  $n$ -vertex graph  $G$  and any integer  $k \geq 1$ , our algorithm with high probability outputs whether or not  $G$  is  $k$ -vertex-connected in a single pass using  $\tilde{O}(kn)$  space<sup>1</sup>.

Our upper bound matches the known  $\Omega(kn)$  lower bound for this problem even in insertion-only streams—which we extend to multi-pass algorithms in this paper—and closes one of the last remaining gaps in our understanding of dynamic versus insertion-only streams. Our result is obtained via a novel analysis of the previous best dynamic streaming algorithm of Guha, McGregor, and Tench [PODS 2015] who obtained an  $\tilde{O}(k^2n)$  space algorithm for this problem. This also gives a model-independent algorithm for computing a “certificate” of  $k$ -vertex-connectivity as a union of  $O(k^2 \log n)$  spanning forests, each on a random subset of  $O(n/k)$  vertices, which may be of independent interest.

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<sup>1</sup>Throughout the paper, we use  $\tilde{O}(f) := O(f \cdot \text{poly} \log f)$  to hide poly-logarithmic factors.

# 1 Introduction

The *vertex connectivity* of an undirected graph  $G = (V, E)$ , with  $n$  vertices and  $m$  edges, is the size of the smallest vertex cut in  $G$ , defined as the minimum number of vertices whose removal disconnects the graph (or turns it into a singleton vertex). Finding the vertex connectivity of a graph is a fundamental problem in combinatorial optimization and has been extensively studied in the literature; see, e.g., [Kle69, Pod73, ET75, BDD<sup>+</sup>82, LLW88, HRG00, LNP<sup>+</sup>21]. This problem can be solved in “polylogarithmic max-flow time” via a result of [LNP<sup>+</sup>21], which combined with the recent breakthrough improvement for max-flow computation in [CKL<sup>+</sup>22], leads to an  $m^{1+o(1)}$  time algorithm for vertex connectivity.

We study the vertex connectivity problem in *dynamic streams*. In this model, the edges of the input graph  $G$  are presented to the algorithm as a sequence of both edge insertions and deletions. The goal is to, given an integer  $k$  at the start of the stream, process the stream with limited space and at the end output whether or not the graph is *k-vertex-connected*, namely, its vertex connectivity is at least  $k$ . In fact, in our algorithm, we focus on not only deciding if the graph is *k-vertex-connected* or not, but also outputting a *certificate* of *k-vertex-connectivity* defined as follows:

**Definition 1.1.** *For any graph  $G = (V, E)$ , a **certificate** of *k-vertex-connectivity* for  $G$  is a subgraph on the same vertex set  $H = (V, E_H)$  such that  $G$  is *k-vertex-connected* if and only if  $H$  is *k-vertex-connected*.*

A certificate of *k-vertex-connectivity* needs  $\Omega(kn)$  edges since any *k-vertex-connected* graph has at least  $kn/2$  edges, as it needs to have a minimum degree of at least  $k$ . Mader’s theorem (see [Proposition A.2](#)) implies that there is also always a certificate with  $O(kn)$  edges although we will not use this result directly in our dynamic streaming algorithm.

In insertion-only streams with no edge deletions, it has been known since the introduction of the model in [FKM<sup>+</sup>05], that one can find a certificate of *k-vertex connectivity* in  $\tilde{O}(kn)$  space using the sparsification techniques of [CKT93] or [EGIN97]<sup>2</sup>, which is nearly optimal. Moreover, [SW15] proved that  $\Omega(kn)$  space is needed even for the original (decision) problem, thus settling the space complexity of the problem in insertion-only streams, up to logarithmic factors. But when it comes to dynamic streams, the best upper bound achieves  $\tilde{O}(k^2n)$  space [GMT15] with no better known lower bounds. We close this gap in this paper.

Our main result is a general (model-independent) approach for computing a vertex connectivity certificate.

**Result 1** (Formalized in [Theorem 1](#)). *For any graph  $G$  and integer  $k \geq 1$ , let  $H$  be a subgraph of  $G$  with  $O(kn \log n)$  edges obtained as a union of  $O(k^2 \log n)$  spanning forests, each on a random subset of  $O(n/k)$  vertices chosen independently of the others. Then, with high probability,  $H$  is a certificate of *k-vertex-connectivity* for  $G$ . This certificate also preserves all vertex cuts of size up to  $k$  and can determine if any two given vertices are *k-vertex-connected* or not.*

[Result 1](#) reduces the problem of finding a certificate of *k-vertex-connectivity* to computing spanning forests on random subsets of vertices. This gives a general approach for solving *k-vertex-connectivity* across different models that can be of its own independent interest. In particular, [Result 1](#) immediately leads to a dynamic streaming algorithms for *k-vertex-connectivity* when combined with the dynamic streaming algorithms of [AGM12a] for computing spanning forests.

**Result 2** (Formalized in [Theorem 2](#)). *There is a randomized single-pass  $\tilde{O}(kn)$  space algorithm that solves *k-vertex-connectivity* with high probability in dynamic streams.*

[Result 2](#) also closes one of the last remaining gaps in understanding of dynamic versus insertion-only streams. Starting with the breakthrough of [AGM12a] that initiated the study of dynamic graph streams, various graph problems such as cut sparsifiers [AGM12b], spectral sparsifiers [KLM<sup>+</sup>17], densest subgraph [MTVV15], subgraph counting [AGM12b], and  $(\Delta + 1)$ -vertex coloring [ACK19] were shown to admit algorithms in dynamic streams with similar guarantees as those of insertion-only streams. In particular, for the closely related problem of *k-edge-connectivity*, it was shown already by [AGM12a] how to obtain similar

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<sup>2</sup>While this connection has been observed in several places, e.g., in [SW15, GMT15], we are not aware of a concrete reference that includes this proof in the streaming model and thus provide a self-contained proof in [Appendix A](#) for completeness.

bounds in dynamic streams as insertion-only streams. For a few other problems such as maximum matchings and minimum vertex cover, strong separations between the two models were proven in [Kon15, AKLY16] (see also [DK20, AS22, NS22]); a conjectured separation for the shortest path problem also appears in [FKN21]. Before our [Result 2](#) however, it was not clear vertex connectivity belongs to which family of these problems.

As a secondary result, we also extend the previous  $\Omega(kn)$  lower bound of [SW15] to multiple passes.

**Result 3** (Formalized in [Theorem 3](#)). *Any randomized  $p$ -pass streaming algorithm that solves  $k$ -vertex-connectivity with constant probability even in insertion-only streams, needs  $\Omega(kn/p)$  space.*

[Result 3](#) is proven by lower bounding the communication complexity of  $k$ -vertex-connectivity problem with  $\Omega(kn)$  bits. This also answers an open problem of [BBE<sup>+</sup>22] in negative that asks whether recent equivalences between vertex connectivity and vertex-capacitated max-flow in [LNP<sup>+</sup>21] in classical setting also extends to communication complexity model (a recent result of [BBE<sup>+</sup>22] shows that vertex-capacitated max-flow (unit capacity) can be solved with  $\tilde{O}(n)$  communication, while our communication lower bound in general implies an  $\Omega(n^2)$  lower bound for determining the vertex connectivity of a graph, thus ruling out such an equivalence). We remark that our [Result 3](#), similar to the previous lower bound for streaming vertex connectivity in [SW15], holds on multi-graphs (with at most two parallel edges per pairs of vertices); our algorithm in [Result 2](#) also works for multi-graphs. It remains an interesting open question to prove any streaming lower bound for vertex connectivity on simple graphs as well (even for single-pass algorithms).

**Our techniques.** [Result 1](#) (and by extension [Result 2](#)) is obtained via a novel and improved *analysis* of the dynamic streaming algorithm of [GMT15] (with minor modifications), that leads to an improved bound on the size of the certificate<sup>3</sup>. [GMT15] presented algorithms for two *relaxations* of the vertex connectivity problem in dynamic streams:

- **$k$ -vertex-query-connectivity** problem: the algorithm is additionally given a set  $X$  of size at most  $k$  after the stream and the goal is to determine whether removing  $X$  from  $G$  disconnects the graph or not (this is similar to the vertex-failure connectivity oracle problem studied in [LS22]). [GMT15] showed that this problem can be solved in  $\tilde{O}(kn)$  space (and that this is also nearly optimal for this problem).
- **promised-gap  $k$ -vertex-connectivity** problem: the algorithm is given a parameter  $\varepsilon \in (0, 1)$  at the start of the stream and the goal is to determine whether the vertex connectivity of the input graph  $G$  is at least  $k$  or at most  $(1 - \varepsilon) \cdot k$ . [GMT15] showed that this problem could be solved in  $\tilde{O}(kn/\varepsilon)$  space (using an algorithm quite similar to the one for the previous case).

Both these algorithms work roughly as follows (think of  $\varepsilon = \Theta(1)$  for the second one in this context): for  $\tilde{O}(k^2)$  times in parallel, sample  $\tilde{O}(n/k)$  vertices from the input graph uniformly at random and maintain a spanning forest on these vertices using the dynamic streaming algorithm of [AGM12a] (this is basically the same approach taken in our [Result 1](#)). Then, solve the problem on these stored set of edges at the end of the stream. This approach can be implemented in  $\tilde{O}(k^2) \cdot \tilde{O}(n/k) = \tilde{O}(kn)$  space and [GMT15] proves, using a somewhat different analysis, that this solves the problem in each case with high probability.

One can solve the original  $k$ -vertex-connectivity problem using this approach as follows. For the query problem, boost the probability of success of the algorithm to  $1 - n^{-k}$  by running the algorithm in parallel  $\Theta(k \log n)$  times; this allows for taking a union bound over at most  $\binom{n}{k}$  possible choices for the query set  $X$  and testing whether removal of *any* of them can disconnect the graph. For the promised-gap problem, we can set  $\varepsilon = 1/k$  which allows us to distinguish between graphs with vertex connectivity  $k$  versus  $k - 1$  and thus solve the  $k$ -vertex-connectivity problem. Nevertheless, as is apparent, either of these solutions leads to an algorithm with  $\tilde{O}(k^2n)$  space which is sub-optimal.

The key novelty in our work is another analysis of essentially the same vertex-sampling plus spanning forest computation approach of [GMT15]. This analysis allows us to “beat the union bound” over all possible  $k$ -subsets of vertices as candidate choices for the vertex cut, that was the source of the additional factor  $k$  in space in the algorithm of [GMT15]. In particular, our analysis consists of two parts. We first show that all pairs of vertices with sufficiently “high” vertex connectivity, say, at least  $2k$ , remain at least  $k$ -vertex-connected even over the stored edges of the sampling approach (this part is quite similar to the guarantee

<sup>3</sup>The algorithm in [GMT15] is presented as a streaming algorithm but it implicitly gives a certificate with  $O(k^2n \log n)$  edges.

of promised-gap algorithm of [GMT15]). We then prove that *all* edges in the input graph with “low” vertex connectivity between their endpoints, say, less than  $2k$ , are recovered by this sampling approach entirely<sup>4</sup>. Finally, we combine these two parts to argue that the sampled set of edges is a certificate for  $k$ -vertex-connectivity of the input graph and conclude the proof. This proof more generally shows that *all* minimum (global) vertex cuts as well as all  $s$ - $t$  vertex cuts of size up to  $k$  are preserved in this sampling process.

## 2 Preliminaries

**Notation.** For a graph  $G = (V, E)$ , we use  $\deg(v)$  and  $N(v)$  for each vertex  $v \in V$  to denote the degree and neighborhood of  $v$ , respectively. For a subset  $F$  of edges in  $E$ , we use  $V(F)$  to denote the vertices incident on  $F$ ; similarly, for a set  $U$  of vertices,  $E(U)$  denotes the edges incident on  $U$ . We further use  $G[U]$  for any set  $U$  of vertices to denote the induced subgraph of  $G$  on  $U$ . For any two vertices  $s, t \in V$ , we say that a collection of  $s$ - $t$  paths are vertex-disjoint if they do not share any vertices other than  $s$  and  $t$ .

We use the following standard forms of Chernoff bounds.

**Proposition 2.1** (Chernoff bound; c.f. [DP09]). *Suppose  $X_1, \dots, X_m$  are  $m$  independent random variables with range  $[0, b]$  each for some  $b \geq 1$ . Let  $X := \sum_{i=1}^m X_i$  and  $\mu_L \leq \mathbb{E}[X] \leq \mu_H$ . Then, for any  $\varepsilon > 0$ ,*

$$\Pr(X > (1 + \varepsilon) \cdot \mu_H) \leq \exp\left(-\frac{\varepsilon^2 \cdot \mu_H}{(3 + \varepsilon) \cdot b}\right) \quad \text{and} \quad \Pr(X < (1 - \varepsilon) \cdot \mu_L) \leq \exp\left(-\frac{\varepsilon^2 \cdot \mu_L}{(2 + \varepsilon) \cdot b}\right).$$

We use the term “with high probability” to mean with probability at least  $1 - 1/n^c$  for some large constant  $c > 0$ , which can be made arbitrarily large by increasing the space of our algorithms with a constant factor.

**Dynamic graph streams.** The dynamic graph streaming model is defined formally as follows.

**Definition 2.2.** *A dynamic stream  $\sigma = (\sigma_1, \dots, \sigma_N)$  defines a multi-graph  $G = (V, E)$  on  $n$  vertices. Each entry of the stream is a tuple  $\sigma_k = (i_k, j_k, \Delta_k)$  for  $i_k, j_k \in [n]$  and  $\Delta_k \in \{-1, +1\}$ . The multiplicity of an edge  $(u, v)$  is defined as:*

$$A(u, v) = \sum_{\sigma_k: i_k=u \wedge j_k=v} \Delta_k.$$

*The multiplicity of every edge is required to be always non-negative.*

The goal in this model is to design algorithms that can process a dynamic stream using limited space and at the end of the stream, output a solution to the underlying problem for the (multi-)graph defined by the stream. Throughout the paper, we measure the space of the algorithms in *bits*.

We use the algorithm of [AGM12a] that can find a spanning forest of a graph in a dynamic stream.

**Proposition 2.3** ([AGM12a]). *There is an algorithm that given any  $N$ -vertex graph  $G$  in a dynamic stream and  $\delta \in (0, 1)$ , computes a spanning forest  $T$  of  $G$  with probability at least  $1 - \delta$  in  $O(N \log^3(N/\delta))$  space.*

## 3 A Certificate of Vertex Connectivity

We present a certificate of  $k$ -vertex-connectivity in this section, formalizing [Result 1](#). Our algorithm is virtually identical (up to changing constants and ignoring implementation details in dynamic streams) to the algorithm in [GMT15] for the  $k$ -vertex-**query**-connectivity problem mentioned in the introduction. However, we provide an improved analysis showing that it also works for the  $k$ -vertex-connectivity problem.

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<sup>4</sup>The fact that the number of these edges itself is sufficiently small is a direct corollary of Mader’s theorem ([Proposition A.2](#)) that states that every graph with  $O(kn)$  edges has a  $(2k)$ -vertex-connected subgraph. This theorem is at the heart of existing (near) optimal algorithms for vertex connectivity in insertion-only streams (see [Appendix A](#)). Nevertheless, since our proof requires additionally recovering these edges via a particular sampling method, it does *not* rely on Mader’s theorem, and instead, as a corollary, implies a weaker variant of Mader’s theorem (with  $O(kn \log n)$  edges instead) via a probabilistic argument quite different from the standard proofs of this theorem ([Appendix B](#)).

**Algorithm 1.** An algorithm for computing a certificate of  $k$ -vertex-connectivity.

**Input:** A graph  $G = (V, E)$  and an integer  $k$ .

**Output:** A certificate  $H$  for  $k$ -vertex-connectivity of  $G$ .

1. For  $i = 1, 2, \dots, r := (200k^2 \ln n)$  do the following:
  - Let  $V_i$  be a subset of  $V$  where each vertex is sampled independently with probability  $1/k$ .
  - Let  $G_i = G[V_i]$  be the induced subgraph of  $G$  on  $V_i$ .
  - Compute a spanning forest  $T_i$  of  $G_i$ .
2. Output  $H := T_1 \cup T_2 \cup \dots \cup T_r$  as a certificate for  $k$ -vertex-connectivity of  $G$ .

The following theorem proves the main guarantee of this algorithm.

**Theorem 1.** *Algorithm 1, given any graph  $G = (V, E)$  and any integer  $k \geq 1$ , outputs a certificate  $H$  of  $k$ -vertex-connectivity of  $G$  with  $O(kn \cdot \log n)$  edges with high probability.*

The analysis in the proof of [Theorem 1](#) is twofold. We first show that pairs of vertices that are at least  $2k$ -vertex-connected in  $G$  stay  $k$ -vertex-connected in  $H$ . Secondly, we show that edges whose endpoints are not  $2k$ -vertex-connected in  $G$  will be preserved in  $H$ . Putting these together, we then show that  $H$  is a certificate for  $k$ -vertex-connectivity of  $G$  and has at most  $\tilde{O}(kn)$  edges.

We start by bounding the number of edges of the certificate  $H$ . We first show that the sum of sizes of  $V_i$  is  $O(kn \log n)$  with high probability.

**Claim 3.1.**  $\sum_{i=1}^r |V_i| = O(kn \cdot \log n)$  with high probability.

*Proof.* For any iteration  $i \in [r]$ , the graph  $G_i$  has  $n/k$  vertices in expectation. We have  $r = O(k^2 \ln n)$  iterations so  $\sum_{i=1}^r |V_i| = O(kn \cdot \log n)$  in expectation. We prove this is the case with high probability as well.

For  $i \in [r]$ , let  $X_i$  be the random variable denoting the number of vertices in  $V_i$ . We know that  $0 \leq X_i \leq n$  (the inequalities are tight when  $V_i = \emptyset$  and  $V_i = V$ ). Let  $X = \sum_i X_i$  be the random variable governing the sum of sizes of  $V_i$ 's. We have  $\mathbb{E}[X_i] = n/k$  implying  $\mathbb{E}[X] = \mu = r \cdot (n/k)$ . Using a Chernoff bound ([Proposition 2.1](#)) with parameters  $b = n$  and  $\varepsilon = 1$  we get:

$$\Pr(X > 2\mu) \leq \exp\left(\frac{-\mu}{4b}\right) = \exp\left(\frac{-r \cdot (n/k)}{4n}\right) = \exp(-50k \ln n) \leq n^{-50}.$$

Thus, we get that the  $\sum_{i=1}^r |V_i| \leq 2\mu_H = O(kn \cdot \log n)$  with high probability as well. ■

**Lemma 3.2.** *The certificate  $H$  in [Algorithm 1](#) has  $O(kn \cdot \log n)$  edges with high probability.*

*Proof.* Each spanning forest  $T_i$  has at most  $|V_i|$  edges. Thus, the total number of edges in  $H$  can be bounded by  $\sum_{i=1}^r |V_i| = O(kn \cdot \log n)$  with high probability (by [Claim 3.1](#)). ■

We now prove the correctness of this algorithm in the following lemma.

**Lemma 3.3.** *Subgraph  $H$  of [Algorithm 1](#) is a certificate of  $k$ -vertex-connectivity for  $G$  with high probability.*

[Lemma 3.3](#) will be proven in two steps. We first show that every pair of vertices that have at least  $2k$  vertex-disjoint paths between them in  $G$  have at least  $k$  vertex-disjoint paths in  $H$  with high probability<sup>5</sup>.

**Lemma 3.4.** *Every pair of vertices  $s, t$  in  $G$  that have at least  $2k$  vertex-disjoint paths between them in  $G$  have at least  $k$  vertex-disjoint paths in  $H$  with high probability.*

<sup>5</sup>By Menger's theorem ([Proposition A.1](#)), this is equivalent to saying any pair of vertices that are  $(2k)$ -vertex-connected in  $G$  remain at least  $k$ -vertex-connected in  $H$ . However, we do not need to explicitly use Menger's theorem in our proofs.

We then show that every edge whose endpoints have less than  $2k$  vertex-disjoint paths between them in  $G$  will belong to  $H$  as well.

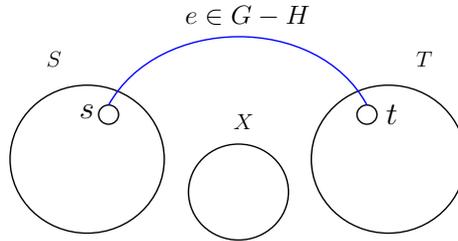
**Lemma 3.5.** *Every edge  $(s, t) \in G$  that has less than  $2k$  vertex-disjoint paths between its endpoints in  $G$  belongs to  $H$  also with high probability.*

The proofs of these lemmas appear in the next two subsections. We first use these lemmas to prove Lemma 3.3 and conclude the proof of Theorem 1.

*Proof of Lemma 3.3.* We first condition on the events in Lemma 3.4 and Lemma 3.5 both of which happen with high probability. We also condition on the event that  $H$  has the same set of vertices as  $G$ . This event also happens with high probability because the probability that a given vertex is not in  $H$  is  $(1 - 1/k)^r \leq \exp(-200k \cdot \ln n) = n^{-200k}$  and hence by a union bound, all vertices in  $G$  are also in  $H$  with high probability. All in all, by a union bound, all the above events happen together with high probability.

We need to show that  $H$  is  $k$ -vertex-connected iff  $G$  is  $k$ -vertex-connected. If  $H$  is  $k$ -vertex-connected then  $G$  is also  $k$ -vertex-connected simply because  $H$  is a subgraph of  $G$  (and crucially on the same set of vertices, namely, it is a spanning subgraph).

We now assume towards a contradiction that  $G$  is  $k$ -vertex-connected, but  $H$  is not. This means that there is a vertex cut  $X$  of size at most  $k - 1$  such that there is a partition  $(S, X, T)$  of  $V$  with no edges between  $S$  and  $T$  in  $H$  (so that removing  $X$  disconnects  $H$ ). Since  $G$  is  $k$ -vertex-connected,  $X$  cannot be a vertex cut in  $G$  and thus  $G$  has an edge  $e = (s, t)$  between  $S$  and  $T$  (see Figure 1).



**Figure 1:** An illustration of the partition  $(S, X, T)$  of vertices of  $G$  and  $H$ . There are no edges between  $S$  and  $T$  in  $H$ , while  $G$  has at least one edge  $e = (s, t)$  between  $S$  and  $T$ , to ensure its  $k$ -vertex-connectivity as  $|X| < k$ .

We now consider two cases.

- Case 1:  $s$  and  $t$  have at least  $2k$  vertex-disjoint paths between them in  $G$ .  
By conditioning on the event of Lemma 3.4, we can say that  $s$  and  $t$  have at least  $k$  vertex-disjoint paths in  $H$ . Deleting  $X$  can remove at most  $|X| \leq k - 1$  of these paths in  $H - X$ . This implies that there is still an  $s$ - $t$  path in  $H - X$  and thus there is an edge between  $S$  and  $T$  in  $H - X$ , a contradiction.
- Case 2:  $s$  and  $t$  have less than  $2k$  vertex-disjoint paths between them in  $G$ .  
Since there are fewer than  $2k$  vertex-disjoint paths between  $s$  and  $t$  in  $G$ , by conditioning on the event of Lemma 3.5,  $e$  would be preserved in  $H$ , a contradiction with  $H$  having no edge between  $S$  and  $T$ .

In conclusion, we get that  $H$  is a certificate of  $k$ -vertex-connectivity for  $G$  with high probability. ■

Theorem 1 now follows immediately from Lemma 3.2 and Lemma 3.3.

Before moving on from this section, we present the following corollary of Algorithm 1 that allows for using this algorithm for some other related problems in dynamic streams as well.

**Corollary 3.6.** *The subgraph  $H$  output by Algorithm 1 with high probability satisfies the following guarantees:*

- (i) *For any pair of vertices  $s, t$  in  $G$ , there are at least  $k$  vertex-disjoint  $s$ - $t$  paths in  $G$  iff there at least  $k$  vertex-disjoint  $s$ - $t$  paths in  $H$  (this holds even if  $G$  is not  $k$ -vertex-connected).*

- (ii) Every vertex cut of  $H$  with size less than  $k$  is a vertex cut in  $G$  and vice versa (this means all vertex cuts of  $G$  are preserved in  $H$  as long as their size is less than  $k$ ).

The proof of this corollary is identical to that of [Lemma 3.3](#) and is thus omitted.

### 3.1 Proof of [Lemma 3.4](#)

We prove [Lemma 3.4](#) in this part following the same approach as in [\[GMT15\]](#). For this proof, without loss of generality, we can assume that  $k > 1$ : for  $k = 1$ , each graph  $G_i$  is the same as  $G$  and thus the algorithm in [Proposition 2.3](#) computes an  $s$ - $t$  path which will be added to  $H$ , trivially implying the proof.

Fix any pair of vertices  $s, t$  with at least  $2k$  vertex-disjoint paths between them. We choose an arbitrary set  $X$  of vertices with size  $k - 1$ , and the goal is to show that  $s$  and  $t$  remain connected in the graph  $H - X$  with very high probability. We do so by showing that out of the at least  $k$  vertex-disjoint paths between  $s$  and  $t$  in  $G - X$ , with probability  $1 - n^{-\Theta(k)}$ , at least one of them is entirely sampled as part of the *subset* of  $G_i$ 's for  $i \in [r]$  that do not contain any vertex from  $X$ . This will be sufficient to prove existence of a  $s$ - $t$  path in  $H - X$ . A union bound over the  $\binom{n}{k-1}$  choices of  $X$  and  $\binom{n}{2}$  pairs  $s, t$  concludes the proof.

Fix  $X$  as a set of  $k - 1$  vertices that contains neither  $s$  nor  $t$ . Define:

$$I(X) := \{i \in [r] : V_i \cap X = \emptyset\}; \tag{1}$$

that is, the indices of sampled graphs in  $G_1, \dots, G_r$  that contain no vertex from  $X$ . We first argue that  $|I(X)|$  is large with high probability.

**Claim 3.7.**  $\Pr(|I(X)| \leq r/8) \leq n^{-5k}$ .

*Proof.* Fix any index  $i \in [r]$  and a vertex  $v \in X$ . The probability that  $v$  is not sampled in  $V_i$  is  $(1 - 1/k)$  by definition and thus,

$$\Pr(V_i \cap X = \emptyset) = (1 - 1/k)^{k-1} \geq 1/4,$$

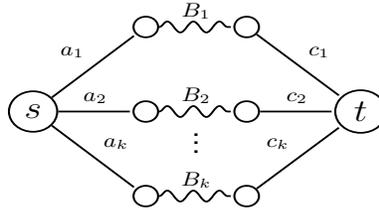
given that  $k > 1$  (as argued earlier) and the choice of vertices is independent in  $V_i$ . Therefore, we have,

$$\mathbb{E}|I(X)| = r \cdot (1 - 1/k)^{k-1} \geq r/4.$$

By an application of the Chernoff bound ([Proposition 2.1](#)) with  $\mu_L = r/4$  and  $\varepsilon = 1/2$ , we have,

$$\Pr(|I(X)| \leq r/8) \leq \exp(-r/4 \cdot 1/10) < n^{-5k}. \quad \blacksquare$$

In the rest of the proof we condition on the event that  $|I(X)| \geq r/8$ . To continue, we need some definitions. There are more than  $k$  vertex-disjoint paths between  $s$  and  $t$  in  $G - X$  since there were  $2k$  of them in  $G$  and only  $k - 1$  vertices (set  $X$ ) are deleted. Choose  $k$  of them arbitrarily denoted by  $P_1(X), \dots, P_k(X)$ . For each path  $P_j(X)$ , let  $a_j$  be the edge incident to  $s$ ,  $B_j$  be the remaining path until the final edge  $c_j$  which is incident to  $t$  - it is possible for  $a_i$  and  $c_i$  to be the same and  $B_i$  be empty (see [Figure 2](#) for an illustration).



**Figure 2:** An illustration of the  $s$ - $t$  paths  $P_1(X), P_2(X), \dots, P_k(X)$ . Each  $P_j(X)$  consists of an edge  $a_j$  from  $s$ , a path  $B_j$  until the last edge  $c_j$  to  $t$ .

We define the notion of “preserving” a path.

**Definition 3.8.** Let  $G_X := \cup_{i \in I(X)} G_i$  be the union of graphs indexed in [Eq \(1\)](#). We say that a path  $P$  in  $G - X$  is **preserved** in  $G_X$  iff for every edge  $e \in P$ , there exists at least one  $i \in I(X)$  such that  $e \in G_i$ ; in other words, the entire path  $P$  belongs to  $G_X$ .

We are going to show that with high probability, at least one path  $P_j(X)$  for  $j \in [k]$  is preserved by  $G_X$ . Before that, we have the following claim that allows us to use this property to conclude the proof.

**Claim 3.9.** *If any  $s$ - $t$  path  $P_j(X)$  for  $j \in [k]$  is preserved in  $G_X$  then  $s$  and  $t$  are connected in  $H - X$ .*

*Proof.* Given that  $P := P_j(X)$  is preserved, we have that for any edge  $e = (u, v) \in P$ , there is some graph  $G_i$  for  $i \in I(X)$  that contains  $e$ . This means that  $u, v$  are connected in  $G_i$  which in turn implies that the spanning forest  $T_i$  of  $G_i$  contains a path between  $u$  and  $v$ . Moreover, since  $i \in I(X)$ , we know that  $G_i$  and hence  $T_i$  contain no vertices of  $X$  and thus  $u$  and  $v$  are connected in  $T_i - X$  as well. Stitching together these  $u$ - $v$  paths for every edge  $(u, v) \in P$  then gives us a walk between  $s$  and  $t$  in  $H - X$ , implying that  $s$  and  $t$  are connected in  $H - X$ . ■

We will now prove that some path  $P_j(X)$  for  $j \in [k]$  is preserved with very high probability.

**Claim 3.10.** *Conditioned on  $|I(X)| \geq r/8$ , the followings three probabilities are each at most  $n^{-2k}$ :*

1.  $\Pr(a_j \notin G_X \text{ for at least } k/3 \text{ values of } j \in [k]);$
2.  $\Pr(B_j \not\subseteq G_X \text{ for at least } k/3 \text{ values of } j \in [k]);$
3.  $\Pr(c_j \notin G_X \text{ for at least } k/3 \text{ values of } j \in [k]).$

*Proof.* To start the proof, note that even conditioned on a choice of  $I(X)$ , the vertices in each path  $P_j(X)$  appear independently in each graph  $G_i$  for  $i \in I(X)$ . This is because these paths do not intersect with  $X$  and by the independence in sampling of each graph  $G_i$  for  $i \in [r]$ . Moreover, given that these paths are vertex-disjoint (although share  $s$  and  $t$ ), the choices of their *inner* vertices across each graph  $G_i$  for  $i \in [r]$ , are independent. We crucially use these properties in this proof.

An edge is present in  $G_i$  if both of its endpoints are sampled which happens with probability  $1/k^2$ . Thus, each edge in  $B_j$  is not present in  $G_i$  with probability  $(1 - 1/k^2)$  and hence is not present in  $G_X$  with probability  $(1 - 1/k^2)^{|I(X)|}$ . Hence, by the union bound,

$$\Pr(B_j \not\subseteq G_X) \leq |B_j| \cdot (1 - 1/k^2)^{|I(X)|} \leq n \cdot (1 - 1/k^2)^{r/8} \leq n \cdot \exp(-200k^2 \ln n / 8k^2) = n^{-24}.$$

Finally, note that since the paths  $B_j$  for  $j \in [k]$  are vertex-disjoint, the probability of the above event is independent for each one. Thus,

$$\Pr(B_j \not\subseteq G_X \text{ for at least } k/3 \text{ values of } j \in [k]) \leq \binom{k}{k/3} (n^{-24})^{k/3} \leq 2^k \cdot n^{-8k} \leq n^{-7k}.$$

As such, the entire path  $B_j$  will lie inside  $G_X$  for at least  $2k/3$  values of  $j \in [k]$  with very high probability.

The analysis for  $a_j$ 's and  $c_j$ 's is slightly different since one of their endpoints, namely,  $s$  and  $t$ , respectively, is shared across all of them. But the proofs for  $a_j$ 's and  $c_j$ 's are entirely symmetric, so we just consider  $a_j$ 's. Consider the set of indices

$$I_s(X) := I(X) \cap \{i \in [r] : s \in G_i\};$$

that is the graphs in  $I(X)$  which additionally contain the vertex  $s$ . For  $i \in I_s(X)$ , the graph  $G_i$  contains the vertex  $s$  and but no vertex from  $X$ . We know that  $\mathbb{E}|I_s(X)| = |I(X)|/k$  since probability of sampling vertex  $s$  in any  $G_i$  is  $1/k$ . By an application of the Chernoff bound (Proposition 2.1) with  $\varepsilon = 0.5$ , we have,

$$\Pr(|I_s(X)| \leq |I(X)|/2k) \leq \exp(-|I(X)|/10k) \leq \exp(-200k^2 \ln n / 80k) = n^{-2.5k}.$$

Moreover, for any  $i \in I_s(X)$ , the probability that  $a_j$  is in  $G_i$  is  $1/k$ . Thus, for any fixed  $j \in [k]$ ,

$$\Pr(a_j \notin G_X) = (1 - 1/k)^{|I_s(X)|}.$$

Combining the above two equations, we have,

$$\Pr(a_j \notin G_X \text{ for at least } k/3 \text{ values of } j \in [k])$$

$$\begin{aligned}
&\leq \Pr\left(|I_s(X)| \leq |I(X)|/2k\right) + \Pr\left(a_j \notin G_X \text{ for at least } k/3 \text{ values of } j \mid |I_s(X)| > |I(X)|/2k\right) \\
&\hspace{15em} \text{(by the law of total probability)} \\
&\leq n^{-2.5k} + \binom{k}{k/3} \left(\left(1 - 1/k\right)^{|I(X)|/2k}\right)^{k/3} \leq n^{-2.5k} + 2^k \cdot \left(\exp(-200k^2 \ln n/16k^2)\right)^{k/3} < n^{-2k}.
\end{aligned}$$

The same property also holds for  $c_j$ 's by symmetry, concluding the proof.  $\blacksquare$

By union bound over the events of [Claims 3.7](#) and [3.10](#), we have that there exists an index  $j \in [k]$  such that the path  $P_j(X)$  is preserved in  $G_X$ . Thus, by [Claim 3.9](#), for a fixed choice of  $X$ , and  $s, t$ , the probability that  $s$  and  $t$  are not connected in  $H - X$  is at most  $4n^{-2k}$ . A union bound over the choices of  $X$  and  $s, t$ , then implies that the probability that even one such choice of  $X$  and  $s, t$  exists is at most

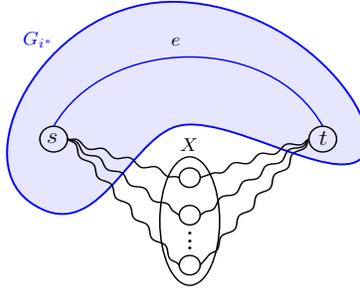
$$\binom{n}{k-1} \cdot \binom{n}{2} \cdot 4n^{-2k} \leq n^{k+1} \cdot 4n^{-2k} = 4n^{-k+1} < 4n^{-1},$$

since  $k \geq 2$ . This completes the proof of [Lemma 3.4](#).

### 3.2 Proof of [Lemma 3.5](#)

We now prove [Lemma 3.5](#). For this proof also, without loss of generality, we can assume that  $k > 1$ : for  $k = 1$ , each graph  $G_i$  is the same as  $G$  and thus the algorithm in [Proposition 2.3](#) computes the only  $s$ - $t$  path, namely, the edge  $(s, t)$  (as  $s$  and  $t$  can only be 1-connected through the edge  $(s, t)$ ) which will be added to  $H$ , thus trivially implying the proof. We now consider the main case.

Fix any pair of vertices  $s, t \in G$  which have less than  $2k$  vertex-disjoint paths between them. We know that deleting the edge  $(s, t)$  and some set of vertices  $X$  of size less than  $2k$  should disconnect  $s$  and  $t$ . For any  $i \in [r]$ , we call the graph  $G_i$  **good** if it samples both  $s$  and  $t$  and does not sample any vertex from  $X$ . See [Figure 3](#) for an illustration.



**Figure 3:** An illustration of a good graph  $G_{i^*}$  wherein both vertices  $s$  and  $t$  are sampled and all the vertices in set  $X$  are not. Thus, none of the  $s$ - $t$  paths, except for the edge  $e$ , exist in  $G_{i^*}$  since they all pass through  $X$ . Therefore, the spanning forest  $T_{i^*}$  necessarily contains the edge  $e = (s, t)$ .

We have,

$$\Pr(G_i \text{ is good}) = 1/k^2 \cdot (1 - 1/k)^{2k-1} \geq 1/8k^2. \quad (\text{as } k \geq 2 \text{ so } (1 - 1/k)^{2k-1} \geq (1/2)^3)$$

Given the independence of choices of  $G_i$  for  $i \in [r]$ , we have,

$$\Pr(\text{No } G_i \text{ is good}) \leq (1 - 1/8k^2)^r \leq \exp(-200k^2 \ln n/8k^2) = n^{-25}.$$

Therefore, there is a graph  $G_{i^*}$  for  $i^* \in [r]$  where  $s$  and  $t$  are sampled but  $X$  is not (see [Figure 3](#)). This means that the spanning forest  $T_{i^*}$  has to contain the edge  $(s, t)$  as there is no other path between  $s$  and  $t$  (we have effectively “deleted”  $X$  by not sampling it). Thus, the edge  $(s, t)$  belongs to  $H$  with probability at least  $1 - n^{-25}$ . A union bound over all possible pairs  $s, t \in G$  concludes the proof.

## 4 The Dynamic Streaming Algorithm

We present our single pass dynamic streaming algorithm for  $k$ -vertex-connectivity in this section. The algorithm outputs a certificate of  $k$ -vertex-connectivity for the input graph at the end of the stream. Thus, by the definition of the certificate, to know whether or not the input graph is  $k$ -vertex-connected, it suffices to test if the certificate is  $k$ -vertex-connected, which can be done at the end of the stream using any offline algorithm. The following theorem formalizes [Result 2](#).

**Theorem 2.** *There is a randomized dynamic streaming algorithm that given an integer  $k \geq 1$  before the stream and a graph  $G = (V, E)$  in the stream, outputs a certificate  $H$  of  $k$ -vertex-connectivity of  $G$  with high probability using  $O(kn \cdot \log^4 n)$  bits of space.*

This algorithm is just an implementation of [Algorithm 1](#) in dynamic streams. We fix the vertex sets  $V_i$  in [Algorithm 1](#) before the stream so the only thing we need to specify is how we compute the spanning forests during the stream. We compute a spanning forest  $T_i$  of  $G_i$  in the stream using the dynamic streaming algorithm in [Proposition 2.3](#) with parameters  $N = |V_i|$  and  $\delta = n^{-4}$ . After the stream, we output the certificate  $H$ . This completes the description of the streaming algorithm.

We start by bounding the space of this algorithm.

**Lemma 4.1.** *This algorithm uses  $O(kn \cdot \log^4 n)$  bits of space with high probability.*

*Proof.* During the stream, we run a spanning forest algorithm for each graph  $G_i$  for  $i \in [r]$ . The algorithm of [Proposition 2.3](#) with parameters  $N = |V_i|$  and  $\delta = n^{-4}$  takes at most  $c|V_i| \cdot \log^3(n^4 |V_i|)$  bits of space for some absolute constant  $c$ . We store  $r = O(k^2 \ln n)$  spanning forests so the total space taken is

$$\sum_{i=1}^r c|V_i| \cdot \log^3(n^4 |V_i|) \leq c \log^3(n^5) \sum_{i=1}^r |V_i| = O(kn \log^4 n),$$

where the first inequality uses  $|V_i| \leq n$  and the second one uses  $\sum_{i=1}^r |V_i| = O(kn \log n)$  (by [Claim 3.1](#)). ■

We are now ready to prove [Theorem 2](#).

*Proof of Theorem 2.* By [Lemma 4.1](#), this algorithm uses  $O(kn \cdot \log^4 n)$  bits of space with high probability. The high probability guarantee can be even moved from the space bound to the correctness in the following way: if the space of the algorithms at any point increases beyond this high-probability bound guarantee, we simply terminate the algorithm and output “fail”. This happens with negligible probability by [Lemma 4.1](#). The algorithm is also correct with high probability by [Theorem 1](#). Moreover, none of the spanning forest algorithms of [Proposition 2.3](#) fail with high probability (by a union bound over the failure probabilities of the  $r$  spanning forest algorithms). Thus, the streaming algorithm (deterministically) uses  $O(kn \cdot \log^4 n)$  bits of space and, by union bound, with high probability outputs a certificate of  $k$ -vertex-connectivity. ■

## 5 The Lower Bound

In this section, we extend the prior single-pass lower bound of [\[SW15\]](#) for vertex connectivity to multi-pass algorithms. The following theorem formalizes [Result 3](#).

**Theorem 3.** *For any integer  $p \geq 1$ , any randomized  $p$ -pass insertion only streaming algorithm that given an integer  $1 \leq k \leq n/2$  before the stream and an  $n$ -vertex (multi-)graph  $G = (V, E)$  in the stream, outputs whether  $G$  is  $k$ -vertex connected with probability at least  $2/3$ , needs  $\Omega(kn/p)$  bits of space.*

We use the standard approach of proving lower bounds on the space of streaming algorithms via communication complexity (this is further spelled out in the proof of [Theorem 3](#)). The communication lower bound itself is proven using a reduction from the well-known set disjointness problem defined as follows.

**Definition 5.1 (Set Disjointness (DISJ<sub>N</sub>)).** For any integer  $N \geq 1$ , DISJ<sub>N</sub> is defined as follows: Alice and Bob are given length  $N$  binary strings  $x \in \{0, 1\}^N$  and  $y \in \{0, 1\}^N$ , respectively. They can communicate back and forth and need to output “No” if there exists an index  $i \in [N]$  such that  $x_i = y_i = 1$  and “Yes” otherwise. We assume both players have access to a shared source of randomness.

We use the following standard lower bound on the communication complexity of this problem.

**Proposition 5.2** ([KS92, Raz90, BYJKS04]). For any integer  $N \geq 1$ , any two-way randomized protocol for DISJ<sub>N</sub> that errs with probability at most  $1/3$  needs  $\Omega(N)$  bits of communication.

We use this result to prove a communication complexity lower bound for vertex connectivity.

**Proposition 5.3.** For any integers  $n, k \geq 1$  such that  $1 \leq k \leq n/2$  the following is true. Any randomized communication protocol wherein Alice and Bob receive edges of an  $n$ -vertex (multi-)graph  $G = (V, E)$  partitioned between the two, and can output whether or not  $G$  is  $k$ -vertex-connected with probability at least  $2/3$  requires  $\Omega(kn)$  bits of communication.

*Proof.* We start with a high level sketch of the proof. We use a reduction from the DISJ<sub>N</sub> communication problem for  $N = \Theta(kn)$ . Alice and Bob construct a bipartite graph  $G$  on  $n$  fixed vertices and pick their edges based on the values in their input strings  $x$  and  $y$  in DISJ<sub>N</sub>.  $G$  will be constructed in a way that if  $x$  and  $y$  are disjoint, then  $G$  will contain a complete bipartite graph and has vertex connectivity  $k$ ; otherwise, at least one edge is missing and the graph has vertex connectivity strictly less than  $k$ . Thus, solving  $k$ -vertex connectivity also solves DISJ<sub>N</sub> implying the space lower bound. We now formalize this idea.

To prove the bound for parameters  $n$  and  $k$ , we start with an instance of DISJ<sub>N</sub> such that  $N = k \cdot (n - k)$ . Alice and Bob construct an  $n$ -vertex bipartite graph  $G = (L \sqcup R, E)$  with  $k$  vertices on  $L$  and  $n - k$  vertices on  $R$  as follows:

- *Vertices:* the vertices in  $L$  are  $u_1, u_2, \dots, u_k$  and the vertices in  $R$  are  $v_1, v_2, \dots, v_{n-k}$ .
- *Edges:* the indices of Alice’s string  $x$  and Bob’s string  $y$  can be expressed using coordinates  $i \in [k]$  and  $j \in [n - k]$  (since  $N = k \cdot (n - k)$ ). If  $x_{i,j} = 0$  then Alice has an edge  $(u_i, v_j)$  and if  $y_{i,j} = 0$  then Bob has an edge  $(u_i, v_j)$  (this way, there can be up to two edges between any pairs of vertices).

The following claim is the key part for establishing the correctness of our reduction.

**Claim 5.4.**  $G$  is  $k$ -vertex connected iff  $x$  and  $y$  are disjoint.

*Proof.* If  $x$  and  $y$  are disjoint, then for every  $i \in [k], j \in [n - k]$  either  $x_{i,j} = 0$  or  $y_{i,j} = 0$  and thus edge  $(u_i, v_j)$  exists in  $G$ . Thus,  $G$  contains a complete bipartite graph. Deleting any set  $X$  of  $k - 1$  vertices leave at least one vertex  $u_i \in L$  and one vertex  $v_j \in R$ . Since  $u_i$  and  $v_j$  are connected, and any vertex in  $L$  is connected to  $v_j$  and any vertex in  $R$  is connected to  $u_i$ , we have that  $G - X$  is connected. Therefore,  $G$  is  $k$ -vertex-connected.

If  $x$  and  $y$  are not disjoint, then there are indices  $i^*$  and  $j^*$  such that  $x_{i^*,j^*} = 1$  and  $y_{i^*,j^*} = 1$  implying that edge  $(u_{i^*}, v_{j^*})$  does not exist in  $G$ . Deleting all vertices in  $L$  except  $u_{i^*}$  disconnects  $v_{j^*}$  from the rest of the graph. Thus, there is a vertex cut of size  $k - 1$  implying that  $G$  is not  $k$ -vertex connected. ■ **Claim 5.4**

The proof of **Proposition 5.3** now follows from **Proposition 5.2**: Alice and Bob, given any instance  $(x, y)$  of DISJ<sub>N</sub>, can construct the graph  $G$  in the reduction without any communication and run the protocol for  $k$ -vertex-connectivity on  $G$ . If the protocol returns  $G$  is  $k$ -vertex-connected, they return “Yes” and otherwise they return “No”. The correctness follows from **Claim 5.4**. This implies that the  $k$ -vertex-connectivity protocol needs  $\Omega(N) = \Omega(kn)$  communication by **Proposition 5.2**, concluding the proof. ■

We can now obtain **Theorem 3** as a standard corollary of **Proposition 5.3**.

*Proof of Theorem 3.* Given a  $p$ -pass streaming algorithm for the  $k$ -vertex-connectivity problem, Alice and Bob can use the algorithm to solve the communication problem as follows. Alice treats her edges in the communication problem as the first part of the stream and Bob treats his edges as the second part of the stream. The players run the streaming algorithm on this stream by communicating the memory content whenever they finish running that pass of the algorithm on their input. This requires sending the memory content for  $2p - 1$  times until Bob can compute the answer of the streaming algorithm.

Assuming we start with a  $p$ -pass streaming algorithm that uses only  $o(kn/p)$  bits of space, the above approach gives us a communication protocol with  $o(kn)$  communication for  $k$ -vertex-connectivity, with the same probability of success as the streaming algorithm. This contradicts [Proposition 5.3](#), and concludes the proof of [Theorem 3](#). ■

## Acknowledgements

We are grateful to Zachary Langley and Michael Saks for helpful conversations about certificates of vertex connectivity and to the anonymous reviewers of SOSA 2023 for their useful comments on the presentation of the paper.

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## A An Insertion-Only Streaming Algorithm

We describe the insertion-only streaming algorithm for  $k$ -vertex-connectivity here. This algorithm has been folklore in the literature already since the introduction of graph streaming model in [FKM<sup>+</sup>05] (but we are not aware of any explicit reference for this result). Several references in the past attribute the algorithm to [EGIN97] but in fact, it appears that [CKT93] also has almost the complete proof which they present as an online algorithm that outputs a certificate of vertex-connectivity (as in Definition 1.1)<sup>6</sup>. We provide the algorithm and its analysis in the insertion-only streaming model in this appendix for completeness.

**Preliminaries.** Before we present the algorithm we mention two important propositions which will be useful in the analysis of the algorithm. The first is Menger’s theorem which gives an equivalent definition of  $k$ -vertex-connectivity via vertex-disjoint paths.

**Proposition A.1** (Menger’s Theorem; c.f. [Wes01, Theorem 17]). *Let  $G$  be an undirected graph and  $s$  and  $t$  be two non-adjacent vertices. Then the size of the minimum vertex cut for  $s$  and  $t$  is equal to the maximum number of vertex-disjoint paths between  $s$  and  $t$ . Moreover, a graph is  $k$ -vertex-connected if and only if every pair of vertices has at least  $k$  vertex-disjoint paths in between.*

The next is Mader’s theorem on existence of  $k$ -vertex-connected subgraphs on sufficiently dense graphs.

**Proposition A.2** (Mader’s Theorem; c.f. [Die05, Theorem 1.4.3]). *For any  $k > 1$ , if an undirected graph has at least  $2k - 1$  vertices and at least  $(2k - 3)(n - k + 1) + 1$  edges, it contains a  $k$ -vertex-connected subgraph.*

Unlike Menger’s theorem, there are not many sources that contain a complete proof of Mader’s theorem in the above formulation and with the given parameters (despite being a well-known result mentioned in various sources, e.g., with a different formulation in [Die05, Theorem 1.4.3]). Thus, we also present a simple proof of this theorem in Appendix B for interested readers.

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<sup>6</sup>To the best of our knowledge, the first version of [CKT93] is a technical report in 1991 which predates the conference version of [EGIN97] from 1992.

**The insertion-only streaming algorithm.** We will reprove the following folklore theorem.

**Theorem 4** (cf. [CKT93, EGIN97, SW15, GMT15]). *There is a deterministic insertion-only streaming algorithm that given an integer  $k \geq 1$  before the stream and a graph  $G = (V, E)$  in the stream, outputs a certificate  $H$  of  $k$ -vertex-connectivity of  $G$  using  $O(kn \log n)$  bits of space.*

The algorithm is very simple: when an edge  $(u, v)$  arrives in the stream, store the edge if and only if the number of vertex-disjoint paths between  $u$  and  $v$  is less than  $k$ .

**Algorithm 2.** An insertion-only streaming algorithm for  $k$ -vertex connectivity.

**Input:** A graph  $G = (V, E)$  specified in a stream and an integer  $k$  specified at the beginning of the stream.

**Output:** A certificate  $H$  for  $k$ -vertex connectivity of  $G$ .

1. Let  $F = \emptyset$ . When any edge  $e = (u, v)$  arrives, if the maximum number of vertex-disjoint paths between  $u$  and  $v$  in  $(V, F)$  is less than  $k$ , update  $F \leftarrow F \cup \{e\}$  (otherwise,  $F$  remains unchanged).
2. When the stream ends output  $H := (V, F)$  as a certificate for  $k$ -vertex connectivity of  $G$ .

We start by proving the correctness of the algorithm.

**Lemma A.3.** *Subgraph  $H$  output by Algorithm 2 is a certificate for  $k$ -vertex-connectivity of  $G$ .*

*Proof.* We need to show that  $H$  is  $k$ -vertex-connected iff  $G$  is  $k$ -vertex-connected. If  $H$  is  $k$ -vertex-connected then  $G$  is  $k$ -vertex-connected since  $H$  is a subgraph of  $G$  on the same set of vertices.

Suppose now towards a contradiction that  $G$  is  $k$ -vertex connected, but  $H$  is not. This means that there is a vertex cut  $X$  of size at most  $k - 1$  such that  $S, X, T$  is a partition of  $V$  and there are no edges between  $S$  and  $T$ . Since  $G$  is  $k$ -vertex-connected, it has an edge  $e = (u, v)$  between  $S$  and  $T$ . Edge  $e$  is not stored in  $H$  by Algorithm 2 and so  $u$  and  $v$  have at least  $k$  vertex disjoint paths between them. But this means that deleting  $X$ , a set of at most  $k - 1$  vertices, cannot disconnect  $S$  and  $T$ , leading to a contradiction. ■

We now prove the following claim which will be helpful in proving the space bound.

**Claim A.4.** *The certificate  $H$  of Algorithm 2 does not contain any subgraph that is  $(k + 1)$ -vertex connected.*

*Proof.* Assume for contradiction that  $H$  contains a subgraph  $J$  that is  $(k + 1)$ -vertex connected. Let  $e = (u, v)$  be the last edge added to  $J$  by Algorithm 2. By Proposition A.1, this means  $u$  and  $v$  have at least  $k + 1$  vertex-disjoint paths between them in  $J$  and thus have at least  $k$  vertex disjoint paths between them in  $J - \{e\}$ . Therefore, when  $e$  arrives in the stream, it is not stored since  $u$  and  $v$  already have  $k$  vertex disjoint paths between them in  $H - \{e\}$ . But this is a contradiction with  $e$  being in  $H$ . ■

Finally, we prove that  $H$  contains at most  $2kn$  edges.

**Lemma A.5.** *The certificate  $H$  of Algorithm 2 contains at most  $2kn$  edges.*

*Proof.* If  $n < 2k - 1$  then  $H$  contains at most  $n(n - 1)/2 \leq 2kn$  edges proving the claim. Thus, consider  $n \geq 2k - 1$ . If  $H$  has more than  $2kn$  edges then by Proposition A.2 it contains a  $(k + 1)$ -vertex connected subgraph. But  $H$  cannot contain any subgraph that is  $(k + 1)$ -vertex connected by Claim A.4. ■

We can now conclude the proof of Theorem 4.

*Proof of Theorem 4.* Lemma A.3 proves that  $H$  is a certificate for  $k$ -vertex-connectivity of  $G$ . Lemma A.5 proves that  $H$  contains at most  $2kn$  edges implying that Algorithm 2 uses  $O(kn \log n)$  bits of space. ■

## B Mader's Theorem

We present a self-contained proof of Mader's theorem in this section for the interested reader. Consider the following restatement of the proposition.

**Proposition A.2** (Mader's Theorem; c.f. [Die05, Theorem 1.4.3]). *For any  $k > 1$ , if an undirected graph has at least  $2k - 1$  vertices and at least  $(2k - 3)(n - k + 1) + 1$  edges, it contains a  $k$ -vertex-connected subgraph.*

*Proof.* We fix a value of  $k$  and prove the proposition by induction on  $n$ , the number of vertices. Our induction hypothesis is as follows: For any  $t \geq 2k$ , if an undirected graph has  $t - 1$  vertices and at least  $(2k - 3)(t - k) + 1$  edges then it contains a  $k$ -vertex connected subgraph.

**Base case:** when  $t = 2k$ .

We have  $m \geq (2k - 3)(k) + 1 = 2k^2 - 3k + 1$ . A clique on  $2k - 1$  vertices has  $(2k - 1)(2k - 2)/2 = 2k^2 - 3k + 1$  edges. Thus, the only graph on  $2k - 1$  vertices that satisfies the edge lower bound is a clique that is  $k$ -vertex-connected and thus has subsets that are  $k$ -vertex-connected.

**Induction step:** We assume the hypothesis for integers up to  $t$  and prove it for  $t + 1$ , that is if an undirected graph has  $t$  vertices and at least  $(2k - 3)(t - k + 1) + 1$  edges then it contains a  $k$ -vertex-connected subgraph.

Assume towards a contradiction that there is a graph  $G$  with  $t$  vertices and at least  $(2k - 3)(t - k + 1) + 1$  edges which contains no  $k$ -vertex connected subgraph. We first show that  $G$  has a large minimum degree.

**Claim B.1.**  *$G$  has minimum degree  $\delta \geq 2k - 2$ .*

*Proof.* Consider a vertex  $v$  with minimum degree  $\delta$ . Removing  $v$  leaves the graph with  $t - 1 \geq 2k - 1$  vertices and  $m' \geq (2k - 3)(t - k + 1) + 1 - \delta$  edges. If  $m' \geq (2k - 3)(t - k) + 1$  then  $G$  contains a  $k$ -vertex-connected subgraph by induction; thus, we need to have  $\delta \geq 2k - 2$ . ■

We know that  $G$  is not  $k$ -vertex-connected which implies there is a vertex cut  $X$  with at most  $k - 1$  vertices which when deleted disconnects  $G$  into components  $S$  and  $T := V - X - S$ . By Claim B.1, for any vertex  $u \in S$ ,  $\deg(u) \geq 2k - 2$ . Moreover, since there are no edges between  $S$  and  $T$ , any vertex  $u \in S$  has neighbors only in  $X$  and  $S$ . Thus, since  $|X| < k$ , we need  $S$  to have at least  $k - 1$  vertices other than  $u$  to satisfy the degree requirement of  $u$ , which implies  $|S| \geq k$ . By symmetry, we also have  $|T| \geq k$ .

Let  $G_1$  be the induced subgraph of  $G$  on  $S \cup X$  with  $n_1$  vertices and let  $G_2$  be the induced subgraph of  $G$  on  $T \cup X$  with  $n_2$  vertices. Both  $G_1$  and  $G_2$  do not contain any  $k$ -vertex-connected subgraphs and have at least  $2k - 1$  vertices, so they have strictly fewer than  $(2k - 3)(n_1 - k + 1) + 1$  and  $(2k - 3)(n_2 - k + 1) + 1$  edges, respectively. We now sum the number of edges  $m_1$  of  $G_1$  and  $m_2$  of  $G_2$ :

$$\begin{aligned} m_1 + m_2 &\leq (2k - 3)(n_1 - k + 1) + (2k - 3)(n_2 - k + 1) \\ &= (2k - 3)(n_1 + n_2 - 2k + 2) \\ &\leq (2k - 3)(t - k + 1) \\ &< m \end{aligned} \quad (\text{Since } m \geq (2k - 3)(t - k + 1) + 1)$$

But we know  $m_1 + m_2 \geq m$  because  $G_1$  and  $G_2$  cover all edges of  $G$  (and can even over count some edges, namely, those with both endpoints in  $X$ ). Thus, we arrive at a contradiction and such a graph  $G$  cannot exist. Therefore, we have shown the induction step and proved the proposition. ■