# Estimating the Effective Support Size in Constant Query Complexity 

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#### Abstract

Estimating the support size of a distribution is a well-studied problem in statistics. Motivated by the fact that this problem is highly non-robust (as small perturbations in the distributions can drastically affect the support size) and thus hard to estimate, Goldreich [ECCC 2019] studied the query complexity of estimating the $\epsilon$-effective support size Ess $_{\epsilon}$ of a distribution $P$, which is equal to the smallest support size of a distribution that is $\epsilon$-far in total variation distance from $P$.

In his paper, he shows an algorithm in the dual access setting (where we may both receive random samples and query the sampling probability $p(x)$ for any $x$ ) for a bicriteria approximation, giving an answer in $\left[\operatorname{Ess}_{(1+\beta) \epsilon},(1+\gamma) \operatorname{Ess}_{\epsilon}\right]$ for some values $\beta, \gamma>0$. However, his algorithm has either super-constant query complexity in the support size or super-constant approximation ratio $1+\gamma=\omega(1)$. He then asked if this is necessary, or if it is possible to get a constant-factor approximation in a number of queries independent of the support size.

We answer his question by showing that not only is complexity independent of $n$ possible for $\gamma>0$, but also for $\gamma=0$, that is, that the bicriteria relaxation is not necessary. Specifically, we show an algorithm with query complexity $O\left(\frac{1}{\beta^{3} \epsilon^{3}}\right)$. That is, for any $0<\epsilon, \beta<1$, we output in this complexity a number $\tilde{n} \in\left[\operatorname{Ess}_{(1+\beta) \epsilon}, \operatorname{Ess}_{\epsilon}\right]$. We also show that it is possible to solve the approximate version with approximation ratio $1+\gamma$ in complexity $O\left(\frac{1}{\beta^{2} \epsilon}+\frac{1}{\beta \epsilon \gamma^{2}}\right)$. Our algorithm is very simple, and has 4 short lines of pseudocode.


[^0]
## 1 Introduction

Estimating the support size of a distribution is one of the most fundamental problems in statistics and has been studied over many decades, starting with the paper of Fisher, Corbet, and Williams [6] in 1943. Estimating the support size in full generality is, however, impossible. This is because any distribution is infinitesimally close to a distribution with arbitrarily large support. One common approach is to assume a lower bound on the elements' probabilities [14, 16, 17, 18]. This assumption is, however, not always reasonable in practice. This motivated Goldreich [8] to study algorithms for estimating a relaxed quantity known as the effective support size. The $\epsilon$-effective support size (abbreviated as $\mathrm{Ess}_{\epsilon}$ ) of a distribution $P$ is defined as the smallest $n$ such that there exists a distribution $P^{\prime}$ supported on $n$ elements that is $\epsilon$-far in total variation distance, that is $\left\|P-P^{\prime}\right\|_{T V}=\epsilon$. This problem is also too non-robust to be estimable in sublinear complexity. However, it leads to a natural bicriteria approximation: we may ask to find a value in $\left[\operatorname{Ess}_{(1+\beta) \epsilon},(1+\gamma) \operatorname{Ess}_{\epsilon}\right]$ for some $\gamma, \beta>0$.

Support size estimation fits in the general subfield of distribution testing, where the goal is to test or learn properties of a distribution from samples or queries to the distribution. Various settings have been considered in the context of distribution testing. One common setting is the dual access setting [1, 11, 4], where in addition to sampling access to the distribution, we may ask for the sampling probability $p(x)$ of any item $x$. Goldreich [8] studied the effective support size estimation in exactly this setting, and this is also the setting we use in this paper. In [8], Goldreich gave algorithms that either have super-constant approximation ratio of $O\left(\log ^{(k)}(n / \epsilon)\right)$ for any fixed constant $k$, or $\gamma=0$ and query complexity $O\left(\log ^{*}(n / \epsilon) / \operatorname{poly}(\epsilon \beta)\right)^{\text {冋 }}$. In this paper, we show that it is possible to get the best of both worlds: query complexity independent of the support size $n$ in complexity $O(\operatorname{poly}(1 /(\epsilon \beta)))$, and having $\gamma=0$. That is, we show that the bicriteria approximation is not necessary, and that the relaxation of the problem in terms of $\beta>0$ is sufficient for the problem to be efficiently solvable. This answers positively the following question posed by Goldreich. Specifically, all of the following questions have a positive answer:

> Open Problem 1.10 from [8], Open Problem 100 from (approximators of the effective support size with performance guarantees that are oblivious of the distribution): For a constant $\beta>0$, does there exist an algorithm that, on input $\epsilon>0$ and oracle access to $P$, uses $s(\epsilon)$ queries and outputs an $f(\epsilon$-factor approximation of the $[\epsilon,(1+\beta) \cdot \epsilon]$-effective support size of $P$, where $s$ and $f$ are functions of $\epsilon$ only? If so, can both functions be polynomials in $1 / \epsilon$ ? And, if so, can we have $s(\epsilon)=$ poly $(1 / \epsilon)$ and $f=1 ? 2$

Specifically, we give an algorithm for $1+\gamma$-approximate $[\epsilon,(1+\beta) \epsilon]$-effective support size in time $O\left(\frac{1}{\beta^{2} \epsilon}+\frac{1}{\beta \epsilon \gamma^{2}}\right)$. How do we decrease the $\gamma$ to 0 ? Goldreich proved that one may decrease the $\gamma$ to 0 , at the cost of an increase in $\beta$ by a factor of $\gamma / \epsilon$ (Observation (4). This means we may compute a $\beta \epsilon / 2$-approximate estimate of the $[\epsilon,(1+\beta / 2) \epsilon]$ effective support size, which will also be a 1 -approximation of the $[\epsilon,(1+\beta) \epsilon]$-effective support size, as desired. This results in an algorithm with complexity $O\left(\frac{1}{\beta^{3} \epsilon^{3}}\right)$ for the unicriterion approximation version of the problem.

### 1.1 Our techniques.

If our distribution is uniform, it would be natural to sample an item $y$ and return $1 / p(y)$, which would be equal to the universe size. It is easy to show that this in fact gives an unbiased estimate

[^1]for general distributions:
$$
\mathbb{E}\left[\frac{1}{p(y)}\right]=\sum_{y \in U} p(y) \frac{1}{p(y)}=\sum_{y \in U} 1=|U| .
$$

This simple estimator has in fact been used to estimate support size, such as in [4, 13]. Our estimator uses this observation as a starting point, and bears resemblance to [4, 13].

The $\epsilon$-effective support size corresponds to ignoring the smallest probability items totaling $\epsilon$ probability mass. Let us therefore order the universe in order of increasing probabilities. We may then modify the above estimator as follows. We generate a sufficiently large random sample of items drawn from $P$ and compute the $(1+\beta / 2) \epsilon$-quantile $x$ of the samples with respect to the order, and define $p=P(x)$. If we sampled enough items, it should hold that $\mathbb{P}_{X \sim P}(X<x)=\epsilon^{*}$, for $\epsilon^{*} \approx(1+\beta / 2) \epsilon$. If all probabilities are distinct, we may then use $\mathbb{I}[p(X) \geq p(x)] / p(X)$, where $X \sim P$, as an unbiased estimat $\int^{3}$ of $\mathrm{Ess}_{\epsilon^{*}}$ :

$$
\mathbb{E}\left[\frac{\mathbb{I}[p(X) \geq p(x)]}{p(X)}\right]=\sum_{y \in U} p(y) \frac{\mathbb{I}[p(y) \geq p(x)]}{p(y)}=\sum_{y \in U} \mathbb{I}[p(y) \geq p(x)]=\operatorname{Ess}_{\epsilon^{*}}
$$

Here, we use $\mathbb{I}$ to denote the indicator random variable for an event. The final equality holds from a known observation (see Observation 3) that if the $n$ heaviest elements in $P$ have total probability $1-\epsilon$, then Ess $_{\epsilon}=n$.

We now bound the variance. In some of the cases, we may use a straightforward analysis that we will now describe; we briefly describe the final and most difficult case at the end of this subsection.

Specifically, we use the fact that for a random variable $X$ with $X \geq 0$ almost surely, $\operatorname{Var}[X] \leq \mathbb{E}[X] \sup [X]$, where $\sup [X]$ represents the maximum value $X$ may take. Because an indicator variable is at most 1 and $p(X) \geq p(x)=p$ whenever the indicator is true, this gives us

$$
\operatorname{Var}\left[\frac{\mathbb{I}[p(X) \geq p(x)]}{p(X)}\right] \leq \mathbb{E}\left[\frac{\mathbb{I}[p(X) \geq p(x)]}{p(X)}\right] \cdot \frac{1}{p}=\frac{\operatorname{Ess}_{\epsilon^{*}}}{p} .
$$

If it were the case that $\operatorname{Ess}_{\epsilon^{*}} \geq \frac{\epsilon^{*} \beta}{100 p}$, this would be sufficient, as we could use this to upper-bound the variance by $\leq \frac{100}{\epsilon^{*} \beta} \operatorname{Ess}_{\epsilon^{*}}^{2}$ which directly leads by Chebyshev's inequality to an algorithm with complexity independent of $n$.

The difficult case is thus when $\mathrm{Ess}_{\epsilon^{*}}<\frac{\epsilon^{*} \beta}{100 p}$. The basic idea is that in this case, we can show that $\operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}}$ is significantly larger than $\operatorname{Ess}_{\epsilon^{*}}$, meaning that the interval $\left[\operatorname{Ess}_{\epsilon^{*}}, \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}}\right]$ is large. As we are assuming $\epsilon^{*} \approx(1+\beta / 2) \epsilon$, any answer in this range is valid. Intuitively speaking, the fact that the range of valid outputs is large then makes the problem easier.

The items that are counted in $\operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}}$ but not in $\operatorname{Ess}_{\epsilon^{*}}$ have $\beta \epsilon^{*} / 4 \geq \beta \epsilon / 4$ probability mass, but each has a probability of being sampled at most $p$. There are therefore at least $\beta \epsilon /(4 p)$ of them. Hence, $\operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} \geq \beta \epsilon /(4 p)$ and $p \geq \beta \epsilon /\left(4 \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}}\right)$. This also implies that $\operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} \geq 2 \operatorname{Ess}_{\epsilon^{*}}$, since we know $\operatorname{Ess}_{\epsilon^{*}}<\frac{\epsilon^{*} \beta}{100 p}$, and $\epsilon^{*} \approx(1+\beta / 2) \epsilon$. We now argue that we return a value $\leq \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} \leq \operatorname{Ess}_{\epsilon}$. Since each estimate is unbiased with variance at most $\mathrm{Ess}_{\epsilon^{*}} / p$, by the Chebyshev inequality, if we average this estimate over $t$ samples, with high probability we return a value that is at most
$\operatorname{Ess}_{\epsilon^{*}}+O\left(\frac{1}{\sqrt{t}} \cdot \sqrt{\operatorname{Ess}_{\epsilon^{*}} / p}\right) \leq \operatorname{Ess}_{\epsilon^{*}}+O\left(\frac{1}{\sqrt{t}} \cdot \sqrt{\operatorname{Ess}_{\epsilon^{*}} \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} /(\epsilon \beta)}\right) \leq \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} \leq \operatorname{Ess}_{\epsilon}$.
Above, the first inequality holds by Chebyshev's inequality, the second inequality holds by our assumption $\operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} \geq \beta \epsilon /(4 p)$, and the last inequality holds by the assumption $\epsilon^{*} \approx$

[^2]$(1+\beta / 2) \epsilon$. The third inequality holds as long as $t \geq \frac{C}{\epsilon \beta}$ for some large constant $C$, since $\operatorname{Ess}_{\epsilon^{*}} \leq \frac{1}{2} \cdot \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}}$, Importantly, $t$ only needs to depend on $\beta$ and $\epsilon$, not on $n$.

It remains to prove that we return a value that is at least $\operatorname{Ess}_{(1+\beta) \epsilon}$. Unfortunately, the variance of our estimator can be arbitrarily large (if one of the probabilities is extremely small), and we thus cannot use Chebyshev's inequality to prove that the returned value will be close to the expectation, and thus not too small. We get around this issue by show a different random variable that is stochastically dominated $4^{4}$ by our estimator, and whose variance is small enough and expectation large enough for this argument to work. Since our estimator stochastically dominates this random variable, it is also not too small with good probability.

### 1.2 Related work.

The problem of support size estimation has been studied over many decades. To the best of our knowledge, the problem was first considered in 1943 under parametric assumptions by Fisher et al. [6]. Under slightly different assumptions, the problem was then considered in 1953 by Good [10]. A large number of works have since followed (see [7] for a survey). However, no approach with formal guarantees without parametric assumptions was known until the study of this problem in the context of distribution testing.

Distribution testing has also enjoyed a long line of research over the past few decades (see Canonne [3] for a survey). The study of the support size estimation problem in the context of distribution testing started more recently, with [14] in 2009. Perhaps the most common parametrization of this problem in distribution testing is by the smallest probability of any item. That is, one assumes that $1 / n \leq p(x)$ for any item $x$ in the universe, and $n$ is now no longer the universe size. In this setting, a line of research [14, 16, 17, 18, lead to an algorithm with complexity $O\left(\frac{n}{\log n} \log ^{2}(1 / \epsilon)\right)$ to estimate the support up to additive error $\epsilon \cdot n$ in the setting when we have sampling access to the distribution.

There are several settings that are commonly studied in distribution testing. Among them are the dual setting, notably systematically studied by Canonne and Rubinfeld [4], and probabilityrevealing samples defined by Onak and Sun [13]. The dual setting has also been considered prior to [4] in [1, 11]. The dual setting assumes that we may ask for the sampling probability of an item. In the probability-revealing samples setting, we get with each sampled item, its sampling probability. The difference is that we may ask the dual oracle for probabilities even of items that have not been sampled. A related setting is the "learning-based" distribution framework [5], which is similar to the probability-revealing samples setting except with each sampled item, we only receive an $O(1)$-approximation to the sampling probability rather than the exact sampling probability. In the dual and probability-revealing samples settings, it is possible to get in time $O\left(1 / \epsilon^{2}\right)$ an additive $\pm \epsilon n$ approximation, and in the learning-based setting, it is possible to get the same approximation in time $O\left(n^{1-\Theta\left(1 / \log \epsilon^{-1}\right)} \cdot \log \epsilon^{-1}\right)$ [5]. In all of these settings, we again choose $n$ such that $1 / n \leq p(x)$ for all $x$ in the universe, which means $n$ can be much larger than the universe size [4, 13, 5]. This may, however, be a very poor approximation, if some sampling probabilities are very small. This motivates the notion of effective support size, as this is known to be optimal [4, 13] and relative approximation is thus not possible in complexity independent of $n$.

While effective support size was first defined by Blais, Canonne, and Gur [2] and also studied in Stewart, Diakonikolas, and Canonne [15], the specific problem of estimating effective support size was first studied later by Goldreich [8]. The main motivation for this relaxation of the problem is that it is possible to get a relative approximation to the effective support size, even if there are no promises on the minimum probability. Specifically, Goldreich shows that for any $\epsilon>0$ and any fixed $\beta>0$, it is possible to get a $(1+\gamma)$-approximation to the $[\epsilon,(1+\beta) \epsilon]$-effective support size, in complexity $s$, for:

[^3]1. $s=O(1 / \epsilon)$ and $1+\gamma=O\left(\epsilon^{-1} \log (n / \epsilon)\right)$,
2. $s=\widetilde{O}(1 / \epsilon)$ and $1+\gamma=O(\log (n / \epsilon))$.
3. For any constants $t, k \in \mathbb{N}$, it holds that $s=\widetilde{O}\left(t / \epsilon^{1+\frac{1}{k}}\right)$ and $1+\gamma=\widetilde{O}\left(\log ^{(t)}(n / \epsilon)\right)$.
4. For any constant $k \in \mathbb{N}$, it holds that $s=\widetilde{O}\left(\log ^{*}(n / \epsilon) / \epsilon^{1+\frac{1}{k}}\right)$ in expectation and $\gamma=\beta$.

Simplicity: To our knowledge, the only work to study estimating effective support size is that of Goldreich [8]. We note that our algorithm not only achieves a better query complexity but is also substantially simpler and shorter, both in terms of algorithm description and analysis.

## 2 Preliminaries

### 2.1 Effective support size and its properties.

The effective support size of a distribution $P$ is defined as follows.
Definition 1 (Definition 1.1 from [8]). The $\epsilon$-effective support size $\operatorname{Ess}_{\epsilon}(P)$ is defined as the smallest $n$ such that $P$ is $\epsilon$-close in total variation distance to some distribution $P^{\prime}$ whose support has size $n$.

As Goldreich [8, Proposition 1.6] proves, it is not possible to efficiently estimate the $\epsilon$-effective support size. Instead, one has to use a relaxation of this notion.

Definition 2 (Definition 1.2 from [8]). A value $\tilde{n}$ is a $(1+\gamma)$-approximate effective $\left[\epsilon_{1}, \epsilon_{2}\right]$-support size if $n \in\left[\operatorname{Ess}_{\epsilon_{2}},(1+\gamma) \operatorname{Ess}_{\epsilon_{1}}\right]$.

We prove that, in fact, one does need the error parameter $\gamma$ in the sense that the problem is efficiently solvable even for $\gamma=0$.

We now state two observations of Goldreich [8] that we will need. The first says that $\epsilon$-effective support size is equal to the support size after removing the least likely elements with a total mass of $\epsilon$. The second one says that we may decrease $\gamma$ to 0 at the cost of an increase in $\beta$ by a factor of $\gamma / \epsilon$.

Observation 3 (Observation 1.4 in [8]). If $P$ has $\epsilon$-effective support size $n$, then $P$ is $\epsilon$-close to a distribution that has support that consists of the $n$ heaviest elements in $P$, with ties broken arbitrarily.

Observation 4 (Observation 1.5 in [8]). If a random variable $X$ is a $(1+\gamma)$-factor approximation of the $\left[\epsilon_{1}, \epsilon_{2}\right]$-effective support size of $P$, then $X /(1+\gamma)$ is an $\left[\epsilon_{1}, \epsilon_{2}+\gamma /(1+\gamma)\right]$-effective support size of $P$. In particular, for $\gamma=\beta \epsilon$, we have $\gamma /(1+\gamma)<\beta \epsilon$. Therefore, if a random variable $X$ is a $(1+\beta \epsilon)$-factor approximation of the $[\epsilon,(1+\beta) \epsilon]$-effective support size of $P$, then $X /(1+\gamma)$ is an $[\epsilon,(1+2 \beta) \epsilon]$-effective support size of $P$.

### 2.2 Distribution testing settings.

The model for a distribution $P$ on a universe $U$ is defined as a pair of oracles $\left(\mathrm{SAMP}_{P}, \mathrm{EVAL}_{P}\right)$ which are in turn defined as follows. Upon being queried, $\mathrm{SAMP}_{P}$ returns a sample from $P$, independent from all previously returned samples. $\operatorname{EVAL}_{P}(x)$ for $x \in U$ returns the probability $p(x)$ of $x$ being sampled from $P$.

The probability-revealing samples model was defined by Onak and Sun [13]. For a distribution $P$, we define a probability-revealing oracle $R E V_{P}$ as an oracle that returns $(x, p(x))$ for $x \sim P$, independently of all previous calls of the oracle. The difference between these two settings that one may also use the EVAL oracle on items that have not been sampled in the dual access model,
but not in the probability-revealing samples model. Hence, the dual access model in general is more powerful than the probability-revealing samples model. Our algorithm in fact will only query probabilities for elements that have already been sampled, so it works both in the dual access model and the probability-revealing samples model.

We also briefly remark that, while not actually relevant for our upper bounds, for effective support size estimation, the query complexities in these two models are in fact equivalent. For symmetric properties, one may assume that we use EVAL either on sampled items, or on items selected uniformly at random from the not-yet-seen part of the support ${ }^{5}$ In addition, if we want the query complexity to have no dependence on the universe size $|U|$, sampling uniformly from the not-yet-seen part of the support is useless because one can make $|U|$ arbitrarily large by adding elements of probability 0 , and sampling uniformly means that with overwhelming probability we will only see elements with probability 0 .

### 2.3 Notions from probability theory.

First, we note that we use $P$ to denote a distribution, and $p(x)$ to denote the probability of sampling $x$ from $P$. We also use $\mathbb{I}$ to denote an indicator random variable. In other words, for an event $E, \mathbb{}[E]=1$ if $E$ occurs and $\mathbb{I}[E]=0$ otherwise.

For a real-valued random variable $X$, we define $\sup [X]=\sup _{t \in \mathbb{R}}\{t: \mathbb{P}(X \geq t)>0\}$ : $\sup [X]$ roughly represents the largest real number that $X$ may take. If $\mathbb{P}(X \geq t)>0$ for all $t \in \mathbb{R}$, then $\sup [X]=\infty$. In addition, if $X$ is conditioned on some variable $Y$, we can define $\sup [X \mid Y=y]=\sup _{t \in \mathbb{R}}\{t: \mathbb{P}(X \geq t \mid Y=y)>0\}$.

In this paper, we need some common notions from probability theory. Two note-worthy ones are that of total variation distance and stochastic domination. The total variation distance of two distributions $P_{1}, P_{2}$ supported on $U$ can be for finite $U$ written as

$$
\left\|P_{1}-P_{2}\right\|_{T V}=\frac{1}{2} \sum_{x \in U}\left|p_{1}(x)-p_{2}(x)\right|
$$

We say that a real random variable $X_{1}$ stochastically dominates a real random variable $X_{2}$ if there exists a coupling $\left(X_{1}^{\prime}, X_{2}^{\prime}\right)$ such that $X_{1}^{\prime} \geq X_{2}^{\prime}$ almost surely. We also use an equivalent definition, which states $X_{1}$ stochastically dominates $X_{2}$, iff for any value $\phi$, we have $\mathbb{P}\left[X_{1} \geq \phi\right] \geq \mathbb{P}\left[X_{2} \geq \phi\right]$.

## 3 Effective support size estimation

We assume an arbitrary total ordering $\leq$ on the support. This may be assumed WLOG and without seeing any samples; for instance, each element will have some number or categorical label associated with it, so we may set $\leq$ as the natural lexicographic order on the labels.

We then define a total ordering $\prec$ such that $x_{1} \prec x_{2}$ if $p\left(x_{1}\right)<p\left(x_{2}\right)$ or if $p\left(x_{1}\right)=p\left(x_{2}\right)$ and $x_{1}<x_{2}$. (We also define $x_{1} \preccurlyeq x_{2}$ to mean $x_{1} \prec x_{2}$ or $x_{1}=x_{2}$ ). We define the $\epsilon$-quantile of a distribution $P$ over a universe $U$ to be the $x_{\epsilon} \in U$ that is smallest w.r.t. $\prec$ such that $\mathbb{P}_{X \sim P}\left(X \preccurlyeq x_{\epsilon}\right)>\epsilon$. One can verify (using Observation (3) that Ess $_{\epsilon}$ equals the number of elements that are $\succcurlyeq x_{\epsilon}$ if $x_{\epsilon}$ is the $\epsilon$-quantile. In addition, if given a sample $R$ from the distribution, we define the $\epsilon$-quantile of $R$ w.r.t. $\prec$ to be the smallest $x \in U$ (under the $\prec$ ordering) such that $\#\{X \in R: X \preccurlyeq x\}>\epsilon \cdot|R|$.

Finally, for any $0<\epsilon<1$, define $x_{\epsilon}$ as the $\epsilon$-quantile of $P$, and $p_{\epsilon}:=p\left(x_{\epsilon}\right)$.

[^4]Given this, we can now describe our algorithm, described in Algorithm 1. Indeed, our algorithm is very simple and only requires 4 lines of pseudocode description. We will assume WLOG that $\beta \leq 0.2$ and $\gamma \leq 0.2$ throughout the analysis. In addition, we assume WLOG that $(1+\beta) \cdot \epsilon<1$, as if $(1+\beta) \cdot \epsilon \geq 1$, then any distribution with support 1 (i.e., a point mass on any element) has total variation distance at most $1 \leq(1+\beta) \cdot \epsilon$ from $P$, so we may output 1 as our estimate of the effective support size.

```
Algorithm 1: Get a \((\gamma, \beta)\)-approximate estimate of the \(\epsilon\)-effective support size.
    \(R \leftarrow\) sample of size \(\frac{180}{\beta^{2} \epsilon}\)
    \(x \leftarrow(1+\beta / 2) \epsilon\)-quantile of \(R\) w.r.t. \(\prec\)
    \(y_{1}, \ldots, y_{t} \leftarrow\) sample of size \(t=\frac{500}{\epsilon \beta \gamma^{2}}\)
    return \((1+\gamma / 2) S\), where \(S=\frac{1}{t} \sum_{i=1}^{t} \frac{\mathbb{I}\left[y_{i} \succcurlyeq x\right]}{p\left(y_{i}\right)}\)
```

First, we show that the $x$ created in Algorithm 1 is an approximate $\epsilon$ quantile of $P$.
Lemma 5. Let $x$ represent the output of the second line of Algorithm 1. With probability at least $\frac{9}{10}$, there exists $\epsilon^{*} \in[(1+\beta / 4) \epsilon,(1+3 \beta / 4) \epsilon]$ such that $x$ is the $\epsilon^{*}$ quantile of $P$.

Remark. We say "there exists $\epsilon^{*}$ " as the choice of $\epsilon^{*}$ may not be unique. For instance, if $P$ were a point mass on a single element $x$, then $x$ is the $\epsilon$ quantile for all $0<\epsilon<1$.

Proof. Let $x_{(1+\beta / 4) \epsilon}$ represent the $(1+\beta / 4) \epsilon$ quantile of $P$. If $x \prec x_{(1+\beta / 4) \epsilon}$, then $\#\{X \in R$ : $X \preccurlyeq x\}>(1+\beta / 2) \epsilon \cdot|R|$ by definition, so $k_{1}:=\#\left\{X \in R: X \prec x_{(1+\beta / 4) \epsilon}\right\}>(1+\beta / 2) \epsilon \cdot|R|$. However, $k_{1} \sim \operatorname{Bin}(|R|, \eta)$ where $\eta \leq(1+\beta / 4) \epsilon$ by definition. Hence, the probability that $x \prec x_{(1+\beta / 4) \epsilon}$ is at $\operatorname{most} \mathbb{P}(\operatorname{Bin}(|R|,(1+\beta / 4) \epsilon)>(1+\beta / 2) \cdot|R|)$, which by the Chernoff bound is at most

$$
\exp \left(-((\beta / 4) /(1+\beta / 4))^{2} \cdot|R| \cdot(1+\beta / 4) \epsilon / 3\right) \leq \exp \left(-\beta^{2} \cdot|R| \cdot \epsilon / 60\right) \leq 1 / 20
$$

where the first inequality follows since we assumed $\beta \leq 0.2$ and the last inequality follows since $|R|=\frac{180}{\beta^{2} \epsilon}$.

Similarly, we let $x_{(1+3 \beta / 4) \epsilon}$ represent the $(1+3 \beta / 4) \epsilon$ quantile of $P$. If $x \succ x_{(1+3 \beta / 4) \epsilon}$, then $k_{2}:=\#\left\{X \in R: X \preccurlyeq x_{(1+3 \beta / 4) \epsilon}\right\} \leq \#\{X \in R: X \prec x\} \leq(1+\beta / 2) \epsilon \cdot|R|$. As $k_{2} \sim \operatorname{Bin}(|R|, \eta)$ for some $\eta>(1+3 \beta / 4) \epsilon$, the probability that $x \succ x_{(1+3 \beta / 4) \epsilon}$ is at most $\mathbb{P}(\operatorname{Bin}(|R|,(1+3 \beta / 4) \epsilon) \leq(1+\beta / 2) \cdot|R|)$, which by the Chernoff bound is at most

$$
\exp \left(-((\beta / 4) /(1+3 \beta / 4))^{2} \cdot|R| \cdot(1+3 \beta / 4) \epsilon / 3\right) \leq \exp \left(-\beta^{2} \cdot|R| \cdot \epsilon / 60\right) \leq 1 / 20
$$

So, with probability at least $9 / 10, x_{(1+\beta / 4) \epsilon} \preccurlyeq x \preccurlyeq x_{(1+3 \beta / 4) \epsilon}$. In this case, there must exist $\epsilon^{*} \in[(1+\beta / 4) \epsilon,(1+3 \beta / 4) \epsilon]$ such that $x$ is the $\epsilon^{*}$ quantile of $P$.

We next prove the following auxiliary lemma.
Lemma 6. For any $\epsilon<1$, recall that $x_{\epsilon}$ represents the $\epsilon$ quantile of $P$ and $p_{\epsilon}=p\left(x_{\epsilon}\right)$. Then, for any $0<\epsilon, \alpha<1$, it holds that $\operatorname{Ess}_{(1-\alpha) \epsilon} \geq \frac{\epsilon \alpha}{p_{\epsilon}}$.
Proof. Assume without loss of generality that the elements are sorted in increasing order of probability, i.e., $p\left(x_{1}\right) \leq p\left(x_{2}\right) \leq \cdots \leq p\left(x_{n}\right)$. We may also assume all elements have nonzero probability by removing all elements with 0 probability. (Indeed, this does not affect $\operatorname{Ess}_{(1-\alpha) \epsilon}$ or $p_{\epsilon}$.) For simplicity, we define $a:=x_{(1-\alpha) \epsilon}$ and $b:=x_{\epsilon}$. Then, for $X \sim P$, $\mathbb{P}(X \prec a) \leq(1-\alpha) \epsilon<\mathbb{P}(X \preccurlyeq a)$, and $\mathbb{P}(X \prec b) \leq \epsilon<\mathbb{P}(X \preccurlyeq b)$. Importantly, this means $\mathbb{P}(a \preccurlyeq X \preccurlyeq b)=\mathbb{P}(X \preccurlyeq b)-\mathbb{P}(X \prec a)>\alpha \cdot \epsilon$. However, $p(b)=p_{\epsilon}$, and $p(c) \leq p_{\epsilon}$ for any $a \preccurlyeq c \preccurlyeq b$. Thus, $(b-a+1) \cdot p_{\epsilon} \geq \mathbb{P}(a \preccurlyeq X \preccurlyeq b)>\alpha \cdot \epsilon$, which means that $b-a+1>\frac{\alpha \cdot \epsilon}{p_{\epsilon}}$.

Next, we remark that since $a$ is the $(1-\alpha) \epsilon$ quantile of $P$, the $(1-\alpha) \epsilon$ effective support size is precisely the number of elements which are $\succcurlyeq a$. Since all elements between $a$ and $b$ in the order fall in this category, we have that $\operatorname{Ess}_{(1-\alpha) \epsilon} \geq b-a+1$.

To summarize, we have $\operatorname{Ess}_{(1-\alpha) \epsilon} \geq b-a+1 \geq \frac{\alpha \cdot \epsilon}{p_{\epsilon}}$, which completes the proof.
We are now ready to prove our main result.
Theorem 7. Suppose that $0<\epsilon<1$ and $0<\beta, \gamma \leq 0.2$. Then, with probability at least $2 / 3$, Algorithm 1 returns a $(1+\gamma)$-factor approximation to the $[\epsilon,(1+\beta) \epsilon]$ effective support size. Its sample complexity is $O\left(\frac{1}{\beta^{2} \epsilon}+\frac{1}{\epsilon \beta \gamma^{2}}\right)$.
Proof. The sample complexity is clearly as claimed. We thus focus on correctness. Recall that we may assume WLOG that $(1+\beta) \cdot \epsilon<1$. We will show that $(1-0.4 \gamma) \operatorname{Ess}_{(1+\beta) \epsilon} \leq S \leq$ $(1+0.4 \gamma) \operatorname{Ess}_{\epsilon}$, where $S$ is defined in Line 4 of Algorithm 1 . Since our final estimate is $(1+0.5 \gamma) S$, and since $1 \leq(1-0.4 \gamma) \cdot(1+0.5 \gamma)$ and $(1+0.4 \gamma) \cdot(1+0.5 \gamma) \leq 1+\gamma$ for $\gamma \leq 0.2$, this implies our final estimate is in the range $\left[\operatorname{Ess}_{(1+\beta) \epsilon},(1+\gamma) \operatorname{Ess}_{\epsilon}\right]$, as desired.

Recall that $x$ is the element generated in Line 2 of Algorithm 1. Define $p:=p(x)$, and let $\epsilon^{*}$ be such that $x$ is the $\epsilon^{*}$ quantile of $P$. Let $\mathcal{E}_{1}$ denote the event that we can choose $\epsilon^{*} \in[(1+\beta / 4) \epsilon,(1+3 \beta / 4) \epsilon]$. (By Lemma 5, $\mathcal{E}_{1}$ holds with at least $9 / 10$ probability.) Consider the random variable $Y=\mathbb{T}[X \succcurlyeq x] / p(X)$ for $X \sim P$. We have that

$$
\mathbb{E}\left[Y \mid \epsilon^{*}\right]=\sum_{y \in U} p(y) \frac{\mathbb{I}[y \succcurlyeq x]}{p(y)}=\#\{y \in U: y \succcurlyeq x\}=\operatorname{Ess}_{\epsilon^{*}} .
$$

At the same time, note that $\sup \left[Y \mid \epsilon^{*}\right] \leq \frac{1}{p}$ (where we recall $p:=p(x)$ and $x=x_{\epsilon^{*}}$ ) since $\mathbb{I}[X \succcurlyeq x]=1$ implies $p(X) \geq p$. Therefore,

$$
\operatorname{Var}\left[Y \mid \epsilon^{*}\right] \leq \mathbb{E}\left[Y^{2} \mid \epsilon^{*}\right] \leq \mathbb{E}\left[Y \mid \epsilon^{*}\right] \sup \left[Y \mid \epsilon^{*}\right] \leq \operatorname{Ess}_{\epsilon^{*}} / p .
$$

We thus have

$$
\mathbb{E}\left[S \mid \epsilon^{*}\right]=\mathbb{E}\left[Y \mid \epsilon^{*}\right]=\mathrm{Ess}_{\epsilon^{*}},
$$

recalling that $S$ is an average of $t$ copies of the random variable $Y$. It also holds that

$$
\operatorname{Var}\left[S \mid \epsilon^{*}\right]=\operatorname{Var}\left[Y \mid \epsilon^{*}\right] / t \leq \operatorname{Ess}_{\epsilon^{*}} /(t p),
$$

where we note that $t=\frac{500}{\epsilon \beta \gamma^{2}}$ depends on $\epsilon$ but not on $\epsilon^{*}$. Conditioning on $\epsilon^{*} \geq \epsilon$ (equivalently, $t \geq \frac{500}{\epsilon^{*} \beta \gamma^{2}}$, which holds on $\mathcal{E}_{1}$ ), we have that

$$
\operatorname{Var}\left[S \mid \epsilon^{*}, \epsilon^{*} \geq \epsilon\right] \leq \epsilon^{*} \beta \gamma^{2} \operatorname{Ess}_{\epsilon^{*}} /(500 p) .
$$

Therefore, by the (conditional) Chebyshev inequality, assuming $\mathcal{E}_{1}$ and conditioning on $\epsilon^{*} \in$ $[(1+\beta / 4) \epsilon,(1+3 \beta / 4) \epsilon]$, it holds with probability at least $9 / 10$ that

$$
\left|S-\operatorname{Ess}_{\epsilon^{*}}\right| \leq \sqrt{\epsilon^{*} \cdot \beta \cdot \gamma^{2} / 50} \cdot \sqrt{\operatorname{Ess}_{\epsilon^{*}} / p}
$$

We call the event when this is the case $\mathcal{E}_{2}$.
We split the rest of the analysis into two main cases. The first case is when $\operatorname{Ess}_{\epsilon^{*}} \geq \frac{\epsilon^{*} \beta}{8 p}$, and the second case is when $\operatorname{Ess}_{\epsilon^{*}} \leq \frac{\epsilon^{*} \beta}{8 p}$.

We start by the simple case when $\mathrm{Ess}_{\epsilon^{*}} \geq \frac{\epsilon^{*} \beta}{8 p}$. In this case, the value $\mathrm{Ess}_{\epsilon^{*}}$ is relatively large, and this already guarantees a good approximation. Since $\operatorname{Ess}_{\epsilon^{*}} \geq \frac{\epsilon^{*} \beta}{8 p}$, we have that $1 / p \leq \frac{8 \operatorname{Ess}_{\epsilon^{*}}}{\epsilon^{*} \beta}$. It therefore holds on $\mathcal{E}_{2}$ that

$$
\left|S-\operatorname{Ess}_{\epsilon^{*}}\right| \leq \sqrt{\epsilon^{*} \cdot \beta \cdot \gamma^{2} / 50} \cdot \sqrt{\operatorname{Ess}_{\epsilon^{*}} / p} \leq 0.4 \gamma \cdot \operatorname{Ess}_{\epsilon^{*}} .
$$

We now consider the case when $\operatorname{Ess}_{\epsilon^{*}} \leq \frac{\epsilon^{*} \beta}{8 p}$. It holds, by Lemma 6 , that $\operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} \geq \frac{\epsilon^{*} \beta}{4 p}$, and it therefore holds $\operatorname{Ess}_{\epsilon^{*}} \leq \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} / 2$. We may thus bound

$$
\begin{aligned}
\operatorname{Ess}_{\epsilon^{*}}+\sqrt{\epsilon^{*} \cdot \beta \cdot \gamma^{2} / 50} \cdot \sqrt{\operatorname{Ess}_{\epsilon^{*}} / p} & \leq \operatorname{Ess}_{\epsilon^{*}}+\sqrt{\epsilon^{*} \beta / 8} \cdot \sqrt{\operatorname{Ess}_{\epsilon^{*}} / p} \\
& \leq \operatorname{Ess}_{\epsilon^{*}}+\sqrt{\operatorname{Ess}_{\epsilon^{*}} \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} / 2} \\
& \leq \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} / 2+\sqrt{\operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} \cdot \operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} / 4} \\
& =\operatorname{Ess}_{(1-\beta / 4) \epsilon^{*}} \leq \operatorname{Ess}_{\epsilon}
\end{aligned}
$$

where the last inequality holds on $\mathcal{E}_{1}$.
Next, we need to argue that $S \geq(1-0.4 \gamma) \operatorname{Ess}_{(1+\beta) \epsilon}$. We do this by defining a random variable $S^{\prime}$ that is stochastically dominated by $S$, and at the same time it has low enough variance that we may use the Chebyshev inequality to show that, with high constant probability, $S^{\prime} \geq \operatorname{Ess}_{(1+\beta) \epsilon}$. Specifically, we define $S^{\prime}=\frac{1}{t} \sum_{i=1}^{t} Y_{i}^{\prime}$, where

$$
Y_{i}^{\prime}:= \begin{cases}1 / p\left(y_{i}\right) & y_{i} \succcurlyeq x_{(1+\beta) \epsilon} \\ 1 / p_{(1+\beta) \epsilon} & x \preccurlyeq y_{i} \prec x_{(1+\beta) \epsilon} \\ 0 & y_{i} \prec x\end{cases}
$$

where we recall that each $y_{i} \stackrel{\text { i.i.d. }}{\sim} P$. (Recall that $x_{(1+\beta) \epsilon}$ is the $(1+\beta) \epsilon$ quantile of $P$, and $x$ is the $\epsilon^{*}$ quantile of $P$.) Note that this also implies each $Y_{i}^{\prime}$ is i.i.d.

We now prove that $S^{\prime}$ is stochastically dominated by $S$. The random variable $S$ is average of $t$ independent copies of $Y$ while $S^{\prime}$ is an average of $t$ random variables $Y_{i}^{\prime}$. It is thus sufficient to prove that $Y$ stochastically dominates $Y_{i}^{\prime}$, since the $Y_{i}^{\prime}$ variables are i.i.d. We do this by demonstrating a coupling between $Y$ and $Y_{i}^{\prime}$ in which it always holds $Y \geq Y_{i}^{\prime}$. Specifically, consider $Y$ and $Y_{i}^{\prime}$ with the same sample $y$. We have the following three cases.

1. If $y \succcurlyeq x_{(1+\beta) \epsilon}$, then $Y_{i}^{\prime}=1 / p(y)=Y$, since the indicator of $y \succcurlyeq x$ is 1 .
2. If $x \preccurlyeq y \prec x_{(1+\beta) \epsilon}$, then $Y=1 / p(y)$ and $Y_{i}^{\prime}=1 / p_{(1+\beta) \epsilon}$. However, $p(y) \leq p\left(x_{(1+\beta) \epsilon}\right)=$ $p_{(1+\beta) \epsilon}$, so $1 / p(y) \geq 1 / p_{(1+\beta) \epsilon}$.
3. If $y \prec x$, then $Y=Y_{i}^{\prime}=0$, where $Y=0$ since the indicator of $y \succcurlyeq x$ is 0 .

In all cases, $Y \geq Y_{i}^{\prime}$, so $Y$ stochastically dominates $Y_{i}^{\prime}$. Thus, $S$ stochastically dominates $S^{\prime}$.
At the same time, assuming $\epsilon^{*}$ is such that $x=x_{\epsilon^{*}} \preccurlyeq x_{(1+\beta) \epsilon}$, it holds that

$$
\begin{aligned}
\mathbb{E}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right] & =\sum_{\substack{y \in U \\
y \succcurlyeq x_{(1+\beta) \epsilon}}} p(y) \cdot \frac{1}{p(y)}+\sum_{\substack{y \in U \\
x \preccurlyeq y \prec x_{(1+\beta) \epsilon}}} p(y) \cdot \frac{1}{p_{(1+\beta) \epsilon}} \\
& =\#\left\{y \in U: y \succeq x_{(1+\beta) \epsilon}\right\}+\frac{1}{p_{(1+\beta) \epsilon}} \cdot \mathbb{P}_{X \sim P}\left(x \preccurlyeq X \prec x_{(1+\beta) \epsilon}\right) \\
& =\operatorname{Ess}(1+\beta) \epsilon+\frac{1}{p_{(1+\beta) \epsilon}} \cdot\left(\mathbb{P}_{X \sim P}\left(X \prec x_{(1+\beta) \epsilon}\right)-\mathbb{P}_{X \sim P}(X \prec x)\right) .
\end{aligned}
$$

Since $x$ is the $\epsilon^{*}$ quantile, $\mathbb{P}(X \prec x) \leq \epsilon^{*}$, and since $x_{(1+\beta) \epsilon}$ is the $(1+\beta) \epsilon$ quantile, $\mathbb{P}(X \preccurlyeq$ $\left.x_{(1+\beta) \epsilon}\right)>(1+\beta) \epsilon$, which means $\mathbb{P}\left(X \prec x_{(1+\beta) \epsilon}\right)>(1+\beta) \epsilon-\mathbb{P}\left(X=x_{(1+\beta) \epsilon}\right)=(1+\beta) \epsilon-p_{(1+\beta) \epsilon}$. In addition, we also have that $\mathbb{P}\left(X \prec x_{(1+\beta) \epsilon}\right)-\mathbb{P}(X \prec x) \geq 0$. Therefore, we have

$$
\begin{aligned}
\mathbb{E}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right] & \geq \operatorname{Ess}_{(1+\beta) \epsilon}+\frac{1}{p_{(1+\beta) \epsilon}} \max \left((1+\beta) \epsilon-p_{(1+\beta) \epsilon}-\epsilon^{*}, 0\right) \\
& =\operatorname{Ess}_{(1+\beta) \epsilon}+\max \left(\frac{\beta \epsilon}{4 p_{(1+\beta) \epsilon}}-1,0\right)
\end{aligned}
$$

where the last inequality holds on $\mathcal{E}_{1}$, since that implies $\epsilon^{*} \leq(1+3 \beta / 4) \epsilon$. Since $\operatorname{Ess}_{(1+\beta) \epsilon} \geq 1$, this means that assuming $\mathcal{E}_{1}$,

$$
\mathbb{E}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right] \geq \max \left(\operatorname{Ess}_{(1+\beta) \epsilon}, \frac{\beta \epsilon}{4 p_{(1+\beta) \epsilon}}\right)
$$

Next, assuming $\mathcal{E}_{1}$, we have that

$$
\operatorname{Var}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right] \leq \mathbb{E}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right] \sup \left[Y_{i}^{\prime} \mid \epsilon^{*}\right]=\mathbb{E}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right] / p_{(1+\beta) \epsilon} \leq 4 \mathbb{E}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right]^{2} /(\beta \epsilon),
$$

where the last inequality holds because $\mathbb{E}\left[Y^{\prime} \mid \epsilon^{*}\right] \geq \beta \epsilon /\left(4 p_{(1+\beta) \epsilon}\right)$. Therefore,

$$
\operatorname{Var}\left[S^{\prime} \mid \epsilon^{*}\right] \leq 4 \mathbb{E}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right]^{2} /(\beta \epsilon t) \leq \gamma^{2} \mathbb{E}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right]^{2} / 125=\gamma^{2} \mathbb{E}\left[S^{\prime} \mid \epsilon^{*}\right]^{2} / 125 .
$$

By the Chebyshev inequality, we then have with probability at least $9 / 10$, that $S^{\prime} \geq(1-$ $0.4 \gamma) \mathbb{E}\left[S^{\prime} \mid \epsilon^{*}\right]=(1-0.4 \gamma) \mathbb{E}\left[Y_{i}^{\prime} \mid \epsilon^{*}\right] \geq(1-0.4 \gamma) \operatorname{Ess}_{(1+\beta) \epsilon}$. We call the event when this happens $\mathcal{E}_{3}$.

We have shown that the algorithm gives a correct output on $\mathcal{E}_{1} \cap \mathcal{E}_{2} \cap \mathcal{E}_{3}$. It holds that

$$
\begin{aligned}
\mathbb{P}\left[\mathcal{E}_{1} \cap \mathcal{E}_{2} \cap \mathcal{E}_{3}\right] & =1-\mathbb{P}\left[\neg \mathcal{E}_{1} \cup \neg \mathcal{E}_{2} \cup \neg \mathcal{E}_{3}\right] \\
& \geq 1-\mathbb{P}\left[\neg \mathcal{E}_{1}\right]-\mathbb{P}\left[\neg \mathcal{E}_{2} \mid \mathcal{E}_{1}\right]-\mathbb{P}\left[\neg \mathcal{E}_{3} \mid \mathcal{E}_{1}\right]>2 / 3,
\end{aligned}
$$

where the last inequality holds because we bounded above each of the three probabilities by 1/10.

As a direct corollary of combining Theorem 7 with Observation 4, we have the following.
Corollary 8. By setting $\gamma=\epsilon \cdot \beta$ in Algorithm 1 and outputting $(1+\gamma / 2) /(1+\gamma) \cdot S$ instead of $(1+\gamma / 2) \cdot S$ in the final line of Algorithm 1, we return an $[\epsilon,(1+2 \beta) \epsilon]$-effective support size. The sample complexity is $O\left(\frac{1}{\epsilon^{3} \beta^{3}}\right)$.

## Acknowledgments

Shyam Narayanan would like to thank Prof. Mikkel Thorup and BARC for allowing him to visit BARC, which facilitated this research.

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[^0]:    ${ }^{*}$ Shyam Narayanan is supported by the NSF GRFP Fellowship, and the NSF TRIPODS Program (award DMS-2022448)
    ${ }^{\dagger}$ Jakub Tětek is supported by the VILLUM Foundation grant 16582.

[^1]:    ${ }^{1}$ We use $\log { }^{(k)}(x)$ to denote the $k$ th iterated $\operatorname{logarithm}$ of $x$, i.e., $\log ^{(1)}(x):=\log x$ and for $k \geq 2, \log ^{(k)}(x):=$ $\log \left(\log ^{(k-1)}(x)\right)$. We use $\log ^{*}(x)$ to denote the smallest nonnegative integer $t$ such that $\log ^{(t)}(x) \leq 1$.
    ${ }^{2}$ We remark that we have slightly rephrased their problem: in this paper, we set $1+\beta$ to be what they have set as $\beta$.

[^2]:    ${ }^{3}$ The unbiasedness also follows from the fact that this is a special case of the Hansen-Hurwitz estimator 12 .

[^3]:    ${ }^{4}$ We define stochastic domination in Section 2 .

[^4]:    ${ }^{5}$ This may be argued roughly as follows: we take a uniformly random permutation $\pi$ of the universe. By symmetry, this does not affect correctness. At the same time, no matter the distribution of $x$ that the algorithm queries, we have $\operatorname{EVAL}_{\pi^{-1}(P)}(x)=\operatorname{EVAL}_{P}\left(\pi^{-1}(x)\right)$, but $\pi^{-1}(x)$ has conditional distribution of being uniform on the not-yet-sampled items. This holds even for adaptive queries, as after each adaptive query we have no information on $\pi$ outside of the elements we sampled/queried.

