Near-Optimal Min-Sum Motion Planning for Two Square Robots in a Polygonal Environment^{*}

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Abstract

Let $W \subset \mathbb{R}^2$ be a planar polygonal environment (i.e., a polygon potentially with holes) with a total of n vertices, and let A, B be two robots, each modeled as an axis-aligned unit square, that can translate inside W. Given source and target placements $s_A, t_A, s_B, t_B \in W$ of A and B, respectively, the goal is to compute a *collision-free motion plan* π^* , i.e., a motion plan that continuously moves A from s_A to t_A and B from s_B to t_B so that A and B remain inside Wand do not collide with each other during the motion. Furthermore, if such a plan exists, then we wish to return a plan that minimizes the sum of the lengths of the paths traversed by the robots, $|\pi^*|$. Given W, s_A, t_A, s_B, t_B and a parameter $\varepsilon > 0$, we present an $n^2 \varepsilon^{-O(1)} \log n$ -time $(1 + \varepsilon)$ -approximation algorithm for this problem. We are not aware of any polynomial time algorithm for this problem, nor do we know whether the problem is NP-Hard. Our result is the first polynomial-time $(1 + \varepsilon)$ -approximation algorithm for an optimal motion planning problem involving two robots moving in a polygonal environment.

1 Introduction

The basic motion-planning problem is to decide whether a robot (i.e., a rigid or multi-link moving object) can move from a given start position to a given target position without colliding with obstacles on its way, and avoiding collision of different parts of the robot. If the answer is positive, we also want to plan such a motion. With the advancement of robotics, we witness the growing deployment of *teams* of robots in logistics, wildlife monitoring, buildings and bridges inspection and more. Motion planning for many robots requires that, in addition to not colliding with obstacles, the robots should not collide with one another, which in turn necessitates studying the problem in high-dimensional *configuration spaces*. Furthermore, we wish to ensure a good quality of the motion, such as being short or having a small makespan. Already for two simple robots, such as unit squares or discs, translating in a planar polygonal environment, little is known when it comes to optimizing

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the robot motion. Although polynomial-time algorithms are known for computing a collision-free motion plan of two simple robots [35], no polynomial-time algorithm is known for computing a plan such that the sum (or the maximum) of the path lengths of the two robots is minimized, nor is the problem known to be NP-hard. Even a polynomial-time constant-factor approximation algorithm is not known for this problem (without further restrictions).

Problem statement. Let $\Box = \{x \in \mathbb{R}^2 \mid ||x||_{\infty} \leq 1\}$ denote the unit-radius axis-aligned square centered at the origin, referred to as a *unit square* for short. For a point $p \in \mathbb{R}^2$ and a real value $\lambda \geq 0$, we use $p + \lambda \Box$ to denote the axis-parallel square of radius λ centered at p. Let A and B be two robots, each modeled as a unit square, that can translate inside the same closed planar polygonal environment (a connected polygon possibly with holes) \mathcal{W} with n vertices. A placement of A or B is represented by a point in \mathcal{W} — the position of its center. For such a placement to be free of collision with ∂W , the boundary of W, the representing point should be at L_{∞} -distance at least 1 from ∂W . We denote by \mathcal{F} , the *free space* of a single robot, the subset of W consisting of such points. Note that the robots may be at L_{∞} -distance 1 from ∂W and hence they are allowed to make contact with the obstacles. A (joint) configuration of A and B is represented as a pair $(p_A, p_B) \in \mathcal{W} \times \mathcal{W}$, where p_A (resp., p_B) is the placement of A (resp., B). We also represent a configuration as a point $p \in \mathbb{R}^4$, where the first (resp., last) pair of coordinates represent the placement of A (resp., B). The configuration space, called C-space for short, namely the set of all configurations, is thus represented as $\mathcal{W} \times \mathcal{W} \subset \mathbb{R}^4$. A configuration $\mathbf{p} = (p_A, p_B) \in \mathbb{R}^4$ is called free if $p_A, p_B \in \mathcal{F}$, that is, $p_A + \Box, p_B + \Box \subseteq \mathcal{W}$, and $||p_A - p_B||_{\infty} \ge 2$. Such a free configuration is called a kissing configuration if $||p_A - p_B||_{\infty} = 2$, i.e., the robots touch each other (but their interiors remain disjoint). Let $\mathbf{F} := \mathbf{F}(\mathcal{W})$ denote the (four-dimensional) free space, namely the set of all free configurations. Clearly, $\mathbf{F} \subset \mathcal{F} \times \mathcal{F}$.

Two free configurations $s, t \in \mathbf{F}$ are *reachable* if they lie in the same connected component of \mathbf{F} , i.e., there is a path contained in \mathbf{F} from s to t. For two reachable free configurations $s := (s_A, s_B), t := (t_A, t_B) \in \mathbf{F}$, a path $\pi \subseteq \mathbf{F}$ from s to t is called a *(feasible) plan* of A and B from s to t, or an (s, t)-plan for brevity. With a slight abuse of notation, we also use π as a (continuous) parameterization $\pi : [0, 1] \to \mathbf{F}$, with $\pi(0) = s$ and $\pi(1) = t$. For a path $\pi \subseteq \mathbf{F}$, let π_A (resp., π_B) be the projection of π onto the two-dimensional plane spanned by the first (resp., last) two coordinates, which specifies the path followed by A (resp., B) that π induces; we have $\pi_A, \pi_B \subset \mathcal{F}$. Let $\mathfrak{e}(\pi_A), \mathfrak{e}(\pi_B)$ denote the (Euclidean) arc length of the paths π_A, π_B , respectively, in \mathbb{R}^2 . We define $\mathfrak{e}(\pi)$, the *cost* of π , to be the sum of the lengths of π_A and π_B , i.e., $\mathfrak{e}(\pi) = \mathfrak{e}(\pi_A) + \mathfrak{e}(\pi_B)$. Let $\pi^*(s, t)$ denote an *optimal* (s, t)-plan, i.e., a plan that minimizes the sum of the lengths of the two paths.¹ If s and t are not reachable, i.e., they lie in different connected components of \mathbf{F} , then $\pi^*(s, t)$ does not exist. We refer to the problem of computing $\pi^*(s, t)$ as the *(optimal) min-sum motion-planning problem*. In this paper we study the min-sum motion-planning problem for two translating axis-aligned unit squares, and present a $(1 + \varepsilon)$ -approximation algorithm that runs in $n^2\varepsilon^{-O(1)}\log n$ time.

Related work. Algorithmic motion planning has been studied for well over fifty years in computer science and beyond. The rigorous study of algorithmic motion planning dates back to the work of Schwartz and Sharir [33] and Canny [9]. See [16, 17, 26, 28] for a review of key relevant results. We mention here only a small sample of these results—the ones that are most closely related to the problem at hand.

¹The existence of π^* can be proved using a simple compactness argument, since \mathcal{F} and \mathbf{F} are closed.

When only one square robot translates, or more generally when only one convex polygonal robot of a constant description complexity (that is, with a constant number of vertices) translates, the problem is equivalent—through C-space formulation—to moving a point robot amid polygonal obstacles with O(n) vertices, and it can be solved in $O(n \log n)$ time [10, 19, 43]. Interestingly, the analogous problem in 3D, namely finding the shortest path for a point robot amid polyhedral obstacles, is NP-hard [8] and fast $(1 + \varepsilon)$ -approximation algorithms are known [8, 34]. Note that this hard problem has only three degrees of freedom of motion, and there are other optimal motionplanning problems for robots with three degrees of freedom that are NP-hard [4, 5]. Our two-square problem has four degrees of freedom, which suggests it might be NP-hard as well, though, as we have remarked earlier, this is an open problem.

Computing a feasible (not necessarily optimal) plan for a team of translating unit square robots in a polygonal environment is PSPACE-hard [38] (see also [6, 7, 18, 20, 40, 46] for related intractibility results). Notwithstanding a rich literature on multi-robot motion planning in both continuous and discrete setting (robots moving on a graph in the latter setting), see, e.g., [11, 22, 23, 32, 36, 41, 42]. little is known about algorithms producing paths with provable quality guarantees. Approximation algorithms for minimizing the total path-length are given in [2, 37, 39] for a set of unit-disc robots assuming a certain separation between the start and goal positions, as well as from the obstacles. The separation assumption makes the problem considerably easier. A feasible plan always exists, and one can first compute an optimal path for each robot independently, ignoring other robots and then locally modify them so that the robots do not collide with each other during their motion. An O(1)-approximation algorithm was proposed in [13] for computing a plan that minimizes the makespan for a set of unit discs (or squares) in the plane without obstacles, again assuming some separation. Computing the min-sum motion plan for two unit squares/discs even in the absence of obstacles is non-trivial [14, 24]. We are unaware of any constant-factor approximation algorithms for the min-sum motion-planning problem even for two unit squares/discs in a planar polygonal environment without any assumptions on the work environment or on the start/final configurations.

Quite a few of the algorithmic results for teams of robots distinguish between the labeled and unlabeled versions: In the labeled version, like in the two-square problem studied here, each robot is designated its own unique target position. In the unlabeled case, each robot can finish at any of the (collective) target positions, as long as at the end of the motion all the target position are occupied by robots. For a team of unlabeled unit discs, an approximate solution for the minimum total path length is given in [39], assuming a certain separation between the start and goal positions of the robots, as well as from the obstacles. A similar result has also been obtained for a team of labeled unit discs in [37], using the slightly more relaxed requirement of the existence of *revolving areas* around the start and target positions. In both cases the approximation bounds are crude, and we omit them here. The latter result for labeled unit discs has recently been improved [2], to give an O(1)-approximation of the optimal total length of the paths, under exactly the same conditions as in [37].

The central and prevalent family of practical motion-planning techniques in robotics is based on sampling of the underlying C-space; see, e.g., [23, 26, 27], and [32] for a recent review. The original sampling-based motion-planning techniques aimed at finding a feasible solution by creating a roadmap of free configurations and connections between them in the C-space, while deferring the (necessarily suboptimal) optimization to a graph search on the resulting roadmap. This two-stage approach has detrimental effect on the quality of the approximation. For example, for a point robot moving amid polyhedra in 3-space, this approach could lead to paths that are hundredfold longer than optimal with high probability [29]. This shortcoming was rectified in a breakthrough paper by Karaman and Frazzoli [22], who presented a series of variants of the fundamental sampling-based techniques, that are guaranteed to be asymptotically optimal, namely converge to an optimal (e.g., shortest) path, when the number of samples tends to infinity. Most sampling-based planners come with only asymptotic guarantees of this type. Finite-time guarantees for sampling-based planners for a team of unit-disc robots are given in [11].

Another major line of work on optimizing multi-robot motion plans addresses a discrete version of the problem, where robots are moving on graphs. In this setting the robots are often referred to as *agents*, and the problem is called *Multi Agent Path Finding (MAPF)*. There is a rich literature on MAPF, and we refer the reader to the recent survey [41]. A commonly used optimization criterion (particularly in the study of MAPF, but elsewhere as well) is *makespan*, where we wish to minimize the time by which all the robots reach their destination, assuming they move in some prespecified maximum speed; see, e.g., [13, 46].

There are a variety of additional optimization criteria in robot motion planning. A common one, related to motion safety, is requiring high *clearance*, namely, requiring that the robot stays far from the obstacles in its environment—this can be obtained using Voronoi diagrams (e.g., [30]). In the context of multi-robot planning we may also require that the robots stay sufficiently far from one another (e.g., [11]). A natural requirement is to produce paths that are at once short and far away from obstacles, which is a more intricate task even for a single robot translating in the plane; see, e.g., [1, 44, 45].

Our contributions. We consider the following simple case of min-sum motion-planning for two unit-square robots. Let W be a polygonal environment, i.e., a polygon possibly with holes. As already stated, we assume that the two robots A and B are axis-parallel squares of side-length 2. Given a source and a target free configurations $(s_A, s_B), (t_A, t_B) \in W$, the goal is to compute a collision-free motion plan for A from s_A to t_A and B from s_B to t_B , such that the sum of the lengths of the two tours traversed by the robots is minimized, or otherwise report that there is no such collision-free motion plan. Our main result is the following theorem, which provides an efficient ε -approximation algorithm for this problem².

Theorem 1.1. Let W be a closed polygonal environment with n vertices, let A, B be two axis-parallel unit-square robots translating inside W, and let s, t be source and target configurations of A, B. For any $\varepsilon \in (0, 1)$, a motion plan π from s to t with $\mathfrak{e}(\pi) \leq (1 + \varepsilon)\mathfrak{e}(\pi^*)$, if there exists a such a motion, can be computed in $n^2 \varepsilon^{-O(1)} \log n$ time, where π^* is an optimal (s, t)-plan.

Although our result falls short of answering whether the min-sum problem for two robots is in P, it is a significant contribution to the theory of optimal multi-robot motion planning. First, as mentioned above, a polynomial-time algorithm was not known, even for constant-factor approximation, and we present an FPTAS for this problem. Second, we prove several structural properties of an optimal plan, which could lead to a polynomial-time algorithm in some special cases, e.g., when W is rectilinear and we consider the L_1 -length of a path. Note that our FPTAS does not rule out the possibility of the problem being NP-hard because, as in other NP-hard optimal motion-planning problems, the construction might use a polynomial number of bits. Finally, our algorithm is very simple and follows the widely-used sampling paradigm. More precisely, we sample a finite set $\mathcal{V} \subset \mathbf{F}$ of free configurations that contains s, t. We connect a pair of configurations

²In principle, our approach extends to two identical centrally-symmetric regular convex polygons, but the analysis becomes even more technical, so for simplicity we only focus on unit squares.

 $\mathbf{p} \coloneqq (p_A, p_B)$ and $\mathbf{q} \coloneqq (q_A, q_B)$ in \mathcal{V} by an edge if there is a *simple* (feasible) plan from \mathbf{p} to \mathbf{q} , namely, we can move A from p_A to q_A (not necessarily along a straight segment) while keeping B parked at p_B and then move B from p_B to q_B while A is parked at q_A , or vice-versa. The cost of the edge (\mathbf{p}, \mathbf{q}) is the minimum cost of such a plan. We then compute a shortest path in this graph. The question is, of course, how we (efficiently) choose a small number of free configurations (linear in n) so that the resulting graph is guaranteed to contain a path from \mathbf{s} to \mathbf{t} that corresponds to a near-optimal (\mathbf{s}, \mathbf{t}) -plan. Most of this paper is about answering this question. We note that the runtime of our algorithm nearly matches that of the best known algorithm for finding any (\mathbf{s}, \mathbf{t}) -plan for two unit squares in a planar polygonal environment, which takes $O(n^2)$ time [35].

There are four main technical contributions of this paper. First, we prove a few key properties of an optimal plan (Section 3). Concretely, we show that there is always an optimal plan in which only one robot moves at any given time while the other robot is *parked* (remains stationary). Thus an optimal plan can be represented as a sequence of *moves*, where each move is specified as a 3-tuple (R, π, p) , where $R \in \{A, B\}$ is the robot that is moving along a path $\pi \subseteq W$ and the other robot is parked at $p \in \mathcal{F}$, where $\pi \times \{p\}$ or $\{p\} \times \pi$ is in \mathbf{F} (π also encodes the starting and terminating placements of R in this move). We refer to such a plan as a *decoupled plan*.³

Second, we show that among all decoupled plans, there exists one in which for each move (R, π, p) , except possibly the first and the last moves, there is a point $q \in \pi$ such that (p, q) (or (q, p) as the case might be) is a kissing configuration. We refer to such a plan as a *kissing plan*. We use the kissing property to prove that there exists an optimal, kissing plan composed of $O(\mathfrak{c}(\pi) + 1)$ moves. Our usage of kissing configurations is different from earlier work (see, e.g., [3, 15, 21]) in a few ways. First, the focus of these works is on motion in contact. For example, Aronov *et al.* [3] use a continuum of kissing configurations to reduce the dimension of the underlying joint configuration space of a pair or of a triple of robots, under various extra conditions. In contrast, kissing configurations in this paper arise as part of individual robot moves, often a singular/discrete configuration, in a (possibly long) alternating sequence of moves. Second, earlier work deals with feasible motion, while we show that there exists optimal plans in which almost every move contains a kissing configuration.

Finally, we prove that there is always a kissing plan in which neither of the robots is ever parked deep inside *corridors*. A formal definitions of corridors is given in Section 2, but intuitively a corridor is a (narrow) region of \mathcal{F} bounded by two of its edges that is far from all vertices of \mathcal{F} and not wide enough to let one robot pass the other.

Next, using these three properties of an optimal plan, we show that we can deform an optimal kissing plan to a *tame plan*, at a slight increase of its cost, in which (roughly speaking) a robot is always parked near a vertex of W or of a corridor at each move. Furthermore, the deformed plan $\tilde{\pi}$ is composed of $O(\mathfrak{e}(\pi) + 1)$ moves and remains a kissing plan (Section 5). Ensuring the kissing property in this deformation is delicate and requires a rather involved argument, so we first prove the existence of a tame plan without ensuring the kissing property (Section 5). This weaker property already leads to an $n^3 \varepsilon^{-O(1)} \log n$ -time $(1 + \varepsilon)$ -approximation algorithm. A key ingredient in computing these deformations is the notion of *revolving areas* within \mathcal{F} , the two-dimensional free space with respect to one robot, roughly a unit square inside \mathcal{F} (again see below for a precise definition). We can show that if each of s_A, s_B, t_A, t_B lies in a revolving area, then there is an (s, t)-plan π composed of O(1) moves with cost $\mathfrak{e}(\pi) \leq \varrho(s_A, s_B) + \varrho(t_A, t_B) + O(1)$, where $\varrho(\cdot, \cdot)$ is

 $^{^{3}}$ We note that the notion of *decoupled* has been used in multiple ways in the context of multi-robot motion planning [25].



Figure 1. An (s, t)-plan π with $\langle \pi \rangle = (A, \pi_1, s_B), (B, \pi_2, p_2), (A, \pi_3, t_B). (s_A, s_B)$ is x-separated and not y-separated.

the geodesic distance between two points in \mathcal{F} . The notion of revolving areas was used in [2, 37] to make a strong separation assumption on each of the start and target configurations, which was exploited to compute a near-optimal plan. Here, we prove the existence of revolving areas in the neighborhood of a non-tame plan and use them for auxiliary parking spots to convert the plan into a near-optimal tame plan.

The existence of an kissing, tame, near-optimal (s, t)-plan π^* enables us to choose a set \mathcal{V} of $n\varepsilon^{-O(1)}$ (nearly) kissing configurations and to build a graph \mathcal{G} over them so that π^* can be retracted to a path in \mathcal{G} at a slight increase in its cost, thereby reducing the problem to computing a shortest path in \mathcal{G} . Ensuring that the two robots do not collide with each other in the retracted path requires care and thus the retraction map is somewhat involved. This retraction step introduces $O(\varepsilon)$ additive error, so we need a separate procedure to handle the case when $\mathfrak{c}(\pi^*)$ is small, say, at most 1/4. By exploiting the topology of \mathbf{F} , we describe an $O(n \log^2 n)$ -time O(1)-approximation algorithm for computing an optimal (s, t)-plan when $\mathfrak{c}(\pi^*) \leq 1/4$ (Section 8). We then plug it into the above algorithm to obtain a $(1 + \varepsilon)$ -approximation algorithm for all values of $\mathfrak{c}(\pi^*)$.

2 Preliminaries

Definitions. Let \mathcal{F} be the free space of one robot as defined above. For a point $p \in \mathcal{W}$, let $\mathcal{F}[p] := \{x \in \mathcal{F} \mid ||x - p||_{\infty} \ge 2\}$ be the set of all placements $x \in \mathcal{F}$ of A such that A does not collide with B if B is placed at p, i.e., $\operatorname{int}(x + \Box) \cap \operatorname{int}(p + \Box) = \emptyset$. It is well known that \mathcal{F} and $\mathcal{F}[p]$ are polygonal and have O(n) vertices, and that they can be computed in $O(n \log^2 n)$ time [12]. See Figure 1. Let X be the set of vertices of \mathcal{F} . We regard s_A, s_B, t_A, t_B as additional vertices of \mathcal{F} and add them to X. For $p, q \in \mathcal{F}$, let $\varrho(p, q)$ denote the geodesic distance between p and q in \mathcal{F} . We call a configuration $(a, b) \in \mathbf{F}$ x-separated if $|x(a) - x(b)| \ge 2$ and y-separated if $|y(a) - y(b)| \ge 2$. (a, b) is always x-separated or y-separated (or both) since $||a - b||_{\infty} \ge 2$.

Given source and target configurations $s, t \in \mathbf{F}$, we call an (s, t)-plan $\pi : [0, 1] \to \mathbf{F}$ decoupled if only one robot moves at any time while the other robot is *parked* at some point in \mathcal{F} , and if there is only a finite number of switches between the moving and parking robots. A decoupled plan can be represented as a finite sequence

$$(R_1, \pi_1, p_1), (R_2, \pi_2, p_2), \dots, (R_k, \pi_k, p_k),$$

where, for each i, (R_i, π_i, p_i) is called a *move*, with $R_i \in \{A, B\}$, $p_i \in \mathcal{F}$, and $\pi_i \subseteq \mathcal{F}[p_i]$. At such



Figure 2. The optimal (s, t)-plan moves A from s_A to p, then moves B from s_B to t_B , and then moves A from p to t_A . This example is adapted from [31].

a move, R_i moves along π_i and the other robot is parked at p_i . The plan π is the concatenation of Cartesian products of the form $\pi_i \times \{p_i\}$ or $\{p_i\} \times \pi_i$, depending on which robot is moving and which is parked. If R_1 is A (resp., B), then we set, for completeness, $p_0 \coloneqq s_A$ (resp., $p_0 \coloneqq s_B$). If $R_i \neq R_{i-1}$, then the initial point of π_i is p_{i-1} and p_i is the final point of π_{i-1} . Otherwise $R_i = R_{i-1}$ and the initial point of π_i is the final point of π_{i-1} and $p_i = p_{i-1}$. We call a move-sequence minimal if $R_i \neq R_{i-1}$ for all $1 < i \leq k$. If $R_i = R_{i-1}$, we can replace $(R_{i-1}, \pi_{i-1}, p_{i-1}) \circ (R_i, \pi_i, p_i)$ with $(R_i, \pi_{i-1} || \pi_i, p_i)$, and obtain a shorter sequence (recall that in this case $p_{i-1} = p_i$). Most of the time we will be working with a minimal sequence, but sometimes, when we deform a plan, it will be convenient to describe a non-minimal move sequence into which π can be compressed as above. For a given plan π , there is a unique minimal move sequence into which π can be compressed, which we represent as $\langle \pi \rangle$, and we define $\alpha(\pi) \coloneqq |\langle \pi \rangle|$ to be the number of moves in π .

For a path $\pi \subset \mathcal{F}$ and two values $\lambda, \lambda' \in [0, 1], \lambda < \lambda'$, we denote by $\pi(\lambda, \lambda')$ the *pathlet* of π between times λ and λ' , which itself is a path (with a suitable reparameterization). It will be convenient to specify the portion of a path π between two points $p, p' \in \pi$ using the notation $\pi[p, p']$. We define the distance between closest points in a pair of sets using either the L_2 -distance or the L_{∞} -distance. For any pair of subsets $X, Y \subset \mathbb{R}^2$, set

$$d_{\ell}(\mathsf{X},\mathsf{Y}) \coloneqq \min_{x \in \mathsf{X}, y \in \mathsf{Y}} ||x - y||_{\ell}, \quad \text{for } \ell \in \{2, \infty\}.$$

Lastly, throughout the paper, we refer to the robots A and B by their centers: we say that a robot is "in" a region R (at some time λ) if its center lies in R. Similarly, we say that a robot "enters" (resp., "exits") a region R (at some time λ) its center point is crossing into (resp., out of) R. To describe that the entire robot is contained in R, we say $p + \Box \subseteq R$ where p is the placement of its center.

Optimal plan for $\mathcal{W} = \mathbb{R}^2$. Suppose the work environment is the entire plane \mathbb{R}^2 , i.e., there are no obstacles. In this case, Esteban *et al.* [14] proved that an optimal plan is a piecewise-linear decoupled plan consisting of at most three moves, and each move consists of at most three line segments. See Figure 2 for an example. Note that the parking position in some cases (such as the one in Figure 2) is not necessarily near the initial/final placements, which is one of the challenges in developing an efficient algorithm for computing an optimal plan.

Revolving area. A revolving area is a unit(-radius) square $p + \Box$, for some $p \in \mathcal{F}$, that is contained in \mathcal{F} ; we denote it by $\operatorname{RA}(p)$ (Figure 3). For $p_A, p_B \in \partial \operatorname{RA}(p)$ with $||p_A - p_B||_{\infty} = 2$, (p_A, p_B) is a kissing configuration, and we say that this kissing configuration lies in the revolving area $\operatorname{RA}(p)$. In Section 4 we give useful lemmas regarding revolving areas, which play a key role in deforming an optimal path into a near-optimal *tame* plan (defined later in Section 5) that is easier to compute.



Figure 3. Example of a kissing configuration $(q, r) \in \mathbf{F}$ with $q, r \in \partial RA(p)$.



Figure 4. Two examples of corridors K (shaded) with blockers e_i, e_j that contains squares $S \subset \mathcal{F}$ with radii strictly less than 2 since their centers lie in the interiors of the corridors. The left (resp., right) corridor has direction vector u_{ij} with angle $\pi/4$ (resp., 0). Both examples are maximal since the portals σ_L have L_{∞} -length 2 and σ_R contain a vertex g of \mathcal{F} . from portals σ^L, σ^R , respectively (not drawn to scale).

Corridor and sanctum. Intuitively, a corridor K is a (narrow) trapezoid in \mathcal{F} bounded by two edges of \mathcal{F} , so that if one robot is parked inside K, the other one cannot pass around it (within K). This implies that when both robots are in the same corridor, their motions are constrained in ways that we will later explore. We now give a formal definition. Let e_i, e_j be a pair of edges of \mathcal{F} that support an axis-aligned square (of any size) contained in \mathcal{F} , i.e., there exists an axis-aligned square $S \subset \mathcal{F}$ such that e_i (resp., e_j) touches a vertex of S, say v_i (resp., v_j), but does not intersect int(S). Let $u_{ij} \in [0, \pi)$ be a direction normal to the segment $v_i v_j$; $u_{ij} = k\pi/4$ for some $0 \le k \le 3$. ⁴ A corridor K bounded by e_i, e_j is a trapezoid such that (i) two of the edges of K are portions of e_i, e_j , called blockers; (ii) the other edges of K, called portals, are normal to the direction u_{ij} ; (iii) the L_{∞} -length of each portal (i.e., the L_{∞} -distance between its endpoints) is at most 2; and (iv) no vertex of \mathcal{F} lies in the interior of K. See Figure 4. We refer to u_{ij} as the direction of the corridor. The following lemma directly follows from condition (iii).

Lemma 2.1. Let K be a corridor with direction vector u. For any segment $vw \subset int(K)$ normal to $u, ||v - w||_{\infty} < 2$. Furthermore, $||v - w||_2 < 2$ if u is axis-parallel, otherwise $||v - w||_2 < 2\sqrt{2}$.

A corridor K is maximal if there is no other corridor that contains K. If K is maximal, condition (iv) is "tight" for at least one portal σ of K in the sense that there is a vertex of \mathcal{F} (not necessarily an endpoint of e_i or e_j) on σ . In particular, there is a convex vertex of \mathcal{F} on the shorter portal of K; if both portals have the same length, both contain such vertices.

Let \mathcal{K} be the set of all maximal corridors in \mathcal{F} . We charge each corridor $K \in \mathcal{K}$ to a vertex of \mathcal{F} on its shorter portal. The conditions are easily seen to imply that the corridors of \mathcal{K} are pairwise disjoint, from which it follows that any vertex of \mathcal{F} is charged at most O(1) times. There are O(n)vertices of \mathcal{F} , which implies $|\mathcal{K}| = O(n)$.

⁴If v_i or v_j is not unique, i.e., when e_i and e_j are axis-aligned, we can choose v_i or v_j (or both) so that $u_{ij} \in \{0, \pi/2\}$.



Figure 5. Illustrations of various portal-parallel lines supporting segments in K, and the sanctum K^S of K.



Figure 6. Illustration of the proof of Lemma 2.2.

Let ℓ_L, ℓ_R be the lines supporting the portals σ_L, σ_R of K, and let $\operatorname{len}(K)$ be the L_{∞} -distance between ℓ_L, ℓ_R . Let u_L (resp., u_R) be the inner normal of σ_L (resp., σ_R), i.e., pointing toward the interior of K; $u_L = -u_R$. For D = L, R and any value $\tau \ge 0$, let $\ell_D^{(\tau)}$ be the line ℓ_D shifted in direction u_D at L_{∞} -distance τ from ℓ_D , let $\sigma_D^{(\tau)}$ be the segment $K \cap \ell_D^{(\tau)}$, and let $K^{(\tau)} \subseteq K$ be the (possibly empty) trapezoid bounded by the blockers of K and segments $\sigma_L^{(\tau)}, \sigma_R^{(\tau)}$. (We assume here that τ is sufficiently small so as to guarantee the shifts from ℓ_L to $\ell^{(\tau)}$ and from ℓ_R to $\ell_R^{(\tau)}$ do not collide.) Note that $K = K^{(0)}$. Similarly, we define portal-parallel lines and segments by points that they contain: For any point $p \in K$, let ℓ_p be the line normal to u_L (and u_R) containing p, and let $\sigma_p := K \cap \ell_p$. For any corridor $K \in \mathcal{K}$ with len $(K) \ge 20$, we define its *sanctum* to be $K^S := K^{(10)} \subset K$. See Figure 5. A corridor $K \in \mathcal{K}$ with len(K) < 20 has an empty sanctum. The following two lemmas capture the essence of a corridor.

Lemma 2.2. Let $K \in \mathcal{K}$ be a maximal corridor, and let u its direction, i.e. one of the unit vectors normal to the portals of K. Let I be a time interval in a plan π of A and B, during which both robots are in K, i.e., $\pi_A(\lambda), \pi_B(\lambda) \in K$ for all $\lambda \in I$. Then the sign of $g(\lambda) \coloneqq \langle \pi_A(\lambda) - \pi_B(\lambda), u \rangle$ is the same for all $\lambda \in I$, where $\langle \cdot \rangle$ is the inner product.

Proof. Suppose to the contrary that there exist two time instances $\lambda_1, \lambda_2 \in I$, with $\lambda_1 < \lambda_2$, such that $g(\lambda_1) < 0$ and $g(\lambda_2) > 0$ (or the other way around). Since π_A, π_B are continuous functions, there exists $\lambda_0 \in (\lambda_1, \lambda_2)$ with $g(\lambda_0) = 0$. But then $\pi_A(\lambda_0)$ and $\pi_B(\lambda_0)$ lie on a segment parallel to the portals of K and thus $||\pi_A(\lambda_0) - \pi_B(\lambda_0)||_{\infty} < 2$, which means that the robots intersect at these placements, contradicting the assumption that π is a feasible plan. Hence, $g(\cdot)$ has the same sign over the entire interval I. See Figure 6.

The following lemma describes a crucial relationship between revolving areas and corridors.

Lemma 2.3. Suppose $p \in \mathcal{F}$ is a point such that p does not lie in any corridor of \mathcal{K} and $d_{\infty}(p, \mathsf{X}) \geq 1$, where X denotes the set of vertices of \mathcal{F} plus $\{s_A, s_B, t_A, t_B\}$ and the vertices of all maximal corridors

in \mathcal{K} , i.e., the endpoints of their portals. Then there is a revolving area $q + \Box \subseteq \mathcal{F}$, for some $q \in \mathcal{F}$, that contains p.

Proof. Let $S := p + r \square$ be the largest axis-aligned square in \mathcal{F} centered at p, where $r \ge 0$. If $r \ge 1$, then $r + \square \subset \mathcal{F}$ and the claim holds, so suppose r < 1. S is supported by at least two edges e_i, e_j of \mathcal{F} , otherwise it could be expanded. Let σ be the segment connecting the vertices v_i, v_j of S on edges e_i, e_j , respectively. The L_{∞} -distance between v_i, v_j is 2r < 2 by definition, and we have, for any point $q \in \sigma$,

$$d_{\infty}(q, \mathsf{X}) \ge d_{\infty}(p, \mathsf{X}) - r > 0.$$

Then it is easy to verify that σ is itself a corridor, so there is a maximal corridor $K \in \mathcal{K}$ such that $\sigma \subseteq K$. So $p \in K$, which is a contradiction.

3 Well-structured Optimal Plans

We present a sequence of transformations for optimal plans, which leads to the existence of an optimal plan with certain desirable properties. Using the easily established fact that \mathbf{F} is polyhedral, it can be shown that an optimal plan is piecewise linear with its breakpoints lying on 2-faces of $\partial \mathbf{F}$. We show that there always exists a piecewise-linear, decoupled plan such that a robot is never parked in the sanctum of a corridor, and the moving robot *kisses* the parked robot in each move, except possibly in the first and the last moves.

We first observe that each facet (three-dimensional face) of $\partial \mathbf{F}$ corresponds to a maximal connected set of placements at which some vertex (resp., edge) of one of the robots touches some edge (resp., vertex) of ∂W or of the other robot. This implies that each connected component of $\partial \mathbf{F}$ is a polyhedral region in \mathbb{R}^4 . The distance between two points $a, b \in \mathbf{F}$ is the sum of the Euclidean length of the projections of b - a onto the 2-planes formed by the first and the last pairs of coordinates, so it is the L_1 -distance of two L_2 -distances. Still, we claim that an optimal path (in \mathbf{F}) must be piecewise linear, with bends only at 2-faces (or faces of lower dimension) of $\partial \mathbf{F}$. This follows since both the L_2 and L_1 -distances satisfy the triangle inequality, and since paths that bend at the relative interior of some 3-face of \mathbf{F} can be shortened. Hence, from now on we only consider piecewise-linear plans.

3.1 Decoupled optimal plans

We begin by proving that there always exists an optimal (piecewise-linear) plan that is decoupled, i.e., only one robot moves at any given time. Such *decoupled* plans are desirable, as during the motion of the moving robot, the parked robot can be treated as an additional obstacle that is part of the environment. Thus, given the start and target placements, s and t, of the moving robot, at some single move in the plan, and the position p of the parked robot, the optimal motion for the moving robot is the shortest path from s to t in $\mathcal{F}[p]$.

Lemma 3.1. Given reachable configurations $s, t \in \mathbf{F}$, there is always a piecewise-linear, decoupled, optimal (s, t)-plan.

Proof. Let $\boldsymbol{\pi} = \langle s = x^0, x^1, \dots, x^R = t \rangle$ be a piecewise-linear optimal $(\boldsymbol{s}, \boldsymbol{t})$ -plan in \mathbf{F} , where $\boldsymbol{\pi}^i = x^{i-1}x^i$, for $1 \leq i \leq k$, is a line segment in \mathbf{F} . Let π^i_A (resp., π^i_B) be the line segment $x^{i-1}_A x^i_A$ (resp., $x^{i-1}_B x^i_B$) in \mathcal{F} , along which A (resp., B) moves during plan $\boldsymbol{\pi}^i$, i.e., it is the projection of $\boldsymbol{\pi}^i$



Figure 7. Illustrations of the proof of Lemma 3.1. The top illustrates an example where predicate (P1) holds, and the bottom illustrates an example where predicate (P2) holds, with $\varphi(\lambda_B) < \lambda_B$. The blue and red lines are the respective projected paths π_B and π_A , and the dotted blue and red squares are axis-parallel squares of radius 2, i.e., copies of $2\Box$, centered at the four endpoints of the two projected paths.

onto the A-plane (resp., B-plane). We also use $\pi^i : [0,1] \to \mathbf{F}$ to denote the (linear) parameterization of the segment $x^{i-1}x^i$, and similarly define the projected parameterizations $\pi^i_A, \pi^i_B : [0,1] \to \mathcal{F}$, i.e., $\pi^i(t) = (\pi^i_A(t), \pi^i_B(t))$, for $t \in [0,1]$. We claim that we can either move A first along π^i_A while B is parked at x^{i-1}_B followed by moving B along π^i_B while A is parked at x^i_A or vice-versa. We note that A can be moved first followed by B if $\pi^i_A \subseteq \mathcal{F}[x^{i-1}_B]$ and $\pi^i_B \subseteq \mathcal{F}[x^i_A]$. Similarly, B can be moved first followed by A if $\pi^i_A \subseteq \mathcal{F}[x^i_B]$ and $\pi^i_B \subseteq \mathcal{F}[x^{i-1}_A]$.

Suppose to the contrary that a decoupled plan does not exist for π^i , i.e., the predicate

$$\left(\pi_A^i \subseteq \mathcal{F}[x_B^{i-1}] \land \pi_B^i \subseteq \mathcal{F}[x_A^i]\right) \lor \left(\pi_A^i \subseteq \mathcal{F}[x_B^i] \land \pi_B^i \subseteq \mathcal{F}[x_A^{i-1}]\right)$$

is not true. Then at least one of the following four predicates must hold:

- $\begin{array}{ll} (\mathrm{P1}) & \pi_A^i \not\subseteq \mathcal{F}[x_B^{i-1}] \wedge \pi_A^i \not\subseteq \mathcal{F}[x_B^i]. \\ (\mathrm{P2}) & \pi_A^i \not\subseteq \mathcal{F}[x_B^{i-1}] \wedge \pi_B^i \not\subseteq \mathcal{F}[x_A^{i-1}]. \\ (\mathrm{P3}) & \pi_B^i \not\subseteq \mathcal{F}[x_A^i] \wedge \pi_B^i \not\subseteq \mathcal{F}[x_A^{i-1}]. \end{array}$
- $(\mathbf{P4}) \ \pi_B^i \not\subseteq \mathfrak{F}[x_A^i] \wedge \pi_A^i \not\subseteq \mathfrak{F}[x_B^i].$

In each case we show the existence of a time $\lambda^* \in [0,1]$ for which

$$||\pi_A^i(\lambda^*) - \pi_B^i(\lambda^*)||_{\infty} < 2,$$

which would imply that π^i is not a feasible plan, and thereby yield the desired contradiction. First, consider (P1). Since $\pi^i \subseteq \mathbf{F}, \pi^i_A, \pi^i_B \subseteq \mathcal{F}$. Therefore (P1) implies that

$$\pi^i_A \cap \operatorname{int}(x^{i-1}_B + 2\Box) \neq \varnothing \quad \text{and} \quad \pi^i_A \cap \operatorname{int}(x^i_B + 2\Box) \neq \varnothing.$$

Then there exist $\lambda_0, \lambda_1 \in [0, 1]$ such that

$$||\pi_A^i(\lambda_0) - \pi_B^i(0)||_{\infty}, \ ||\pi_A^i(\lambda_1) - \pi_B^i(1)||_{\infty} < 2$$

(recall that $x_B^{i-1} = \pi_B^i(0), x_B^i = \pi_B^{i-1}(1)$). For a value $\lambda \in [0,1]$, let $\varphi(\lambda) \in [0,1]$ be such that $\pi_A^i(\varphi(\lambda))$ is the point closest to $\pi_B^i(\lambda)$ on the segment $x_B^{i-1}x_B^i$. We have

$$||\pi_A^i(\varphi(0)) - \pi_B^i(0)||_{\infty}, \ ||\pi_A^i(\varphi(1)) - \pi_B^i(1)||_{\infty} < 2,$$

which holds because of the existence of λ_0 , λ_1 above. This means that

$$\begin{split} |x(\pi_A^i(\varphi(0))) - x(\pi_B^i(0))| &< 2, \\ |y(\pi_A^i(\varphi(0))) - y(\pi_B^i(0))| &< 2, \\ |x(\pi_A^i(\varphi(1))) - x(\pi_B^i(1))| &< 2, \\ |y(\pi_A^i(\varphi(1))) - y(\pi_B^i(1))| &< 2. \end{split}$$

Recalling that π_A , π_B are line segments, this implies that, for any $\alpha \in [0, 1]$, we also have

$$\begin{aligned} |x(\pi_A^i((1-\alpha)\varphi(0)+\alpha\varphi(1)))-x(\pi_B^i(\alpha))| &< 2, \\ |y(\pi_A^i((1-\alpha)\varphi(0)+\alpha\varphi(1)))-y(\pi_B^i(\alpha))| &< 2. \end{aligned}$$

That is, $||\pi_A^i((1-\alpha)\varphi(0) + \alpha\varphi(1)) - \pi_B^i(\alpha)|| < 2$. This in turn implies that $||\pi_A^i(\varphi(\lambda)) - \pi_B^i(\lambda)||_{\infty} < 2$ for all $\lambda \in [0, 1]$. Since φ is a continuous function, there exists a $\lambda^* \in [0, 1]$ such that $\varphi(\lambda^*) = \lambda^*$. But then

$$||\pi_A^i(\lambda^*) - \pi_B^i(\lambda^*)||_{\infty} = ||\pi_A^i(\varphi(\lambda^*)) - \pi_B^i(\lambda^*)||_{\infty} < 2,$$

which contradicts the assumption that $\pi^i \subseteq \mathbf{F}$. Hence (P1) does not hold.

Next, suppose that (P2) holds. Then there exist $\lambda_A, \lambda_B \in (0, 1)$ such that

$$||\pi_A^i(\lambda_A) - \pi_B^i(0)||_{\infty}, ||\pi_B^i(\lambda_B) - \pi_A^i(0)||_{\infty} < 2.$$

Without loss of generality, assume that $\lambda_A \geq \lambda_B$. For a value $\lambda \in [0, \lambda_B]$, let $\varphi(\lambda) \in [0, \lambda_A]$ be the parameter of the closest point to $\pi_B^i(\lambda)$ on the segment $\pi_A^i(0)\pi_A^i(\lambda_A)$. As above,

$$||\pi_A^i(\varphi(\lambda) - \pi_B^i(\lambda))||_{\infty} < 2 \text{ for all } \lambda \in [0, \lambda_B].$$

If $\varphi(\lambda_B) \geq \lambda_B$, then

$$||\pi_{A}^{i}(\lambda_{B}) - \pi_{B}^{i}(\lambda_{B})||_{\infty} \le \max\{||\pi_{A}^{i-1}(0) - \pi_{B}^{i}(\lambda_{B})||_{\infty}, ||\pi_{A}^{i}(\varphi(\lambda_{B})) - \pi_{B}^{i}(\lambda_{B})||_{\infty}\} < 2,$$

which contradicts the fact that $\pi^i(\lambda_B) = (\pi^i_A(\lambda_B), \pi^i_B(\lambda_B)) \in \mathbf{F}$. Hence, assume that $\varphi(\lambda_B) < \lambda_B$. Since $\varphi(0) > 0$, there exists a value $\lambda^* \in (0, \lambda_B]$ such that $\varphi(\lambda^*) = \lambda^*$, and we obtain the same contradiction as above. Hence (P2) does not hold.

Predicates (P3) and (P4) are analogous to (P1) and (P2), respectively, by either switching the roles of A and B or by reversing the time direction. We thus conclude that none of (P1)–(P4) holds, implying that there is a decoupled plan for π^i . Repeating this argument for all $i \leq k$, and observing that the endpoints of each π do not change by the transformation, we conclude that there is a decoupled, piecewise-linear optimal plan.



Figure 8. Example of three moves, π_i and π_{i+2} of robot A and π_{i+1} of robot B, in a plan π . By modifying π_i, π_{i+2} so that A parks at p'_{i+1} instead of p_{i+1} , B kisses A during move π_i .

3.2 Kissing plans

We call a piecewise-linear decoupled plan π a kissing plan if the robots kiss on all but possibly the first and the last moves. Formally, let $\langle \pi \rangle = (R_1, \pi_1, p_1), \ldots, (R_k, \pi_k, p_k)$ be the move sequence of π . Then π is a kissing plan if, for all 1 < i < k, there exists a point $q_i \in \pi_i$ such that (p_i, q_i) is a kissing configuration. We show that a decoupled plan can be converted into a kissing plan, without changing the images of the paths traveled by A and B in the plan, by reducing the number of moves and adjusting the parking places (Figure 8). We obtain the following:

Lemma 3.2. Let π be a piecewise-linear, decoupled, optimal plan with the minimum number of moves. There exists a piecewise-linear, decoupled, kissing, optimal plan π' with the same number of moves, such that the first move is made by the same robot as in π , and the pathlet of the first move in π' contains that of π .

Proof. The proof is by induction on k. Let $\langle \boldsymbol{\pi} \rangle = (R_1, \pi_1, p_1), \ldots, (R_k, \pi_k, p_k)$. If k = 2, the claim holds trivially, that is, vacuously, so assume k > 2. Without loss of generality, A moves first, i.e., $R_1 = A$. Then $(p_1 = s_B), p_3, p_5, \ldots$ are the parking placements of B; $(p_0 = s_A), p_2, p_4, \ldots$ are the parking placements of A; $p_k = t_A, p_{k+1} = t_B$ if k is odd, and $p_k = t_B, p_{k+1} = t_A$ if k is even; $\pi_1 \coloneqq \pi_A[s_A, p_2]$ is the motion of A in the first move and $\pi_2 \coloneqq \pi_B[s_B, p_3]$ is the motion of B in its first move. There are two cases to consider.

1. If $(\pi_3 \oplus \Box) \cap (\pi_2 \oplus \Box) = \emptyset$ then

$$\boldsymbol{\pi}' \coloneqq \begin{cases} (A, \pi_1 \| \pi_3, p_1), (B, \pi_2 \| \pi_4, p_4), (R_5, \pi_5, p_5), \dots, (R_k, \pi_k, p_k) & \text{if } k > 3\\ (A, \pi_1 \| \pi_3, s_B), (B, \pi_2, t_A) & \text{if } k = 3 \end{cases}$$

is a decoupled, optimal plan with fewer than k moves, which contradicts the assumption that π has the fewest moves among all decoupled, optimal plans.

2. If $(\pi_3 \oplus \Box) \cap (\pi_2 \oplus \Box) \neq \emptyset$, let p' be the first point reached on π_3 such that $(p' + \Box) \cap (\pi_2 \oplus \Box) \neq \emptyset$; note that p' may be p_2 . By the choice of p', the interior of $p' + \Box$ is disjoint from $\pi_2 \oplus \Box$, so Bkisses A at that placement when moving along π_2 . Define $\pi_{3<} \coloneqq \pi_3[p_2, p']$ and $\pi_{3>} \coloneqq \pi_3[p', p_4]$. Again, the choice of p' also implies that $\pi_{3<} \oplus \Box$ is interior disjoint from $\pi_2 \oplus \Box$. Then

$$\boldsymbol{\pi}' \coloneqq (A, \pi_1 \| \pi_{3<}, s_B), (B, \pi_2, p'), (A, \pi_{3>}, p_3), (R_4, \pi_4, p_4), \dots, (R_k, \pi_k, p_k)$$

is a decoupled, optimal (s, t)-plan in which B kisses A, parked at p', as it moves along π_2 .

Set $s' \coloneqq (p', s_B)$. Let π'_0 be the decoupled (s', t)-plan composed of all but the first move of π' . Then $\alpha(\pi'_0) = \alpha(\pi') - 1 = \alpha(\pi) - 1$. Furthermore, π'_0 is a decoupled, optimal (s', t)-plan. We apply the induction hypothesis to π'_0 to obtain a decoupled, kissing, optimal (s', t)-plan π''_0 satisfying the lemma, with B making the first move (B, π'_2, p') . Since the lemma guarantees that $\pi_2 \subset \pi'_2$, B kisses A (parked at p') during the first move of π''_0 . Set $\pi'' \coloneqq (A, \pi_1 || \pi_{3<}, s_A) || \pi''_0$. Then the robots kiss on all moves of π'' except possibly in the first and the last moves. Furthermore

$$\alpha(\boldsymbol{\pi}'') = \alpha(\boldsymbol{\pi}''_0) + 1 = \alpha(\boldsymbol{\pi}'_0) + 1 = \alpha(\boldsymbol{\pi}),$$

and $\pi_1 \subseteq \pi_1 || \pi_{3<}$. Hence π'' satisfies the lemma, which establishes the induction step and thus completes the proof of the lemma.

3.3 Bounding alternations

In a sequence of lemmas, we show that for any $s, t \in \mathbf{F}$, there exists a decoupled, kissing, optimal (s, t)-plan π with $\alpha(\pi) = O(\mathfrak{q}(\pi) + 1)$. We begin with a simple observation whose proof is omitted.

Lemma 3.3. Let e be a horizontal or vertical segment of length at most 2. Then $e \cap \mathcal{F}$ is a connected (possibly empty) interval.

For any region $\nabla \subseteq \mathcal{F}$ and any two points $p, q \in \nabla$, let $\varrho_{\nabla}(p, q)$ be the length of the shortest (p, q)-path in $\nabla \cap \mathcal{F}$. Note that $\varrho(p, q) = \varrho_{\mathcal{F}}(p, q)$.

Lemma 3.4. Let S be any axis-aligned unit-radius square. (i) $S \cap \mathcal{F}$ is composed of xy-monotone components (without holes). (ii) At most two components intersect ∂S . (iii) For any p, q that lie in the interior of a common component of $S \cap \mathcal{F}$, there exists an xy-monotone (p,q)-path P such that $|P| = \varrho_S(p,q) = \varrho(p,q)$.

Proof.

- (i) The claim is immediate from Lemma 3.3.
- (ii) By Lemma 3.3, at most one connected component of $S \cap \mathcal{F}$ intersects each edge of S. So if three connected components of $S \cap \mathcal{F}$ intersect ∂S , two opposite edges, say, horizontal edges of S intersect different connected components of $S \cap \mathcal{F}$. Let C_1 (resp., C_2) with $C_1 \neq C_2$ be the connected component of $S \cap \mathcal{F}$ that intersects the bottom (resp., top) edge of S, and let a_1, b_1 (resp., a_2, b_2) be the left and right endpoints of C_1 (resp., C_2) with the bottom (resp., top) edge. Without loss of generality assume that $x(a_1) < x(a_2)$. Then by Lemma 3.3, $x(a_1) < x(b_1) < x(a_2) < x(b_2)$. Let C_3 be the third component of $S \cap \mathcal{F}$ that intersects, say, the left edge of S, and let a_3, b_3 be the intersection segment of $C_3 \cap \partial S$, with $y(a_3) < y(b_3)$. Since C_1 does not intersect the top edge of S, the highest point of C_1 , denoted by q, lies inside S, i.e., $y(a_1) = y(b_1) < y(q) < y(a_2) = y(b_2)$. Furthermore, by Lemma 3.3, $x(a_3) < x(q) < x(a_2)$ and $y(a_3) > y(q)$. Finally, since $q \in \partial \mathcal{F}$, there is a point $\hat{q} \in \partial \mathcal{W}$ such that $||q - \hat{q}||_{\infty} = 1$ and $y(\hat{q}) = y(q) + 1$. Note that $a_2 \notin \operatorname{int}(\hat{q} + \Box)$. Since $y(a_2) - y(q) < 2$,



Figure 9. Examples of x-separated configurations s, t and squares Q_A, Q_B that satisfy Lemma 3.5, s_A is left of s_B . (left) s_A, s_B are both left of their respective target placements, t_A, t_B . B moves first from s_B to t_B and then A moves from s_A to t_A . (right) s_A is left of t_A but s_B is right of t_B , so both 2-move plans are feasible.

 $y(a_2) - y(\widehat{q}) < 1$ implying that $x(a_2) - x(\widehat{q}) \ge 1$. On the other hand, $x(\widehat{q}) \ge x(q) - 1$. Putting these together, we obtain that $y(\widehat{q}) - y(a_3) < y(\widehat{q}) - y(q) = 1$, $y(a_3) - y(\widehat{q}) < y(a_2) - y(\widehat{q}) < 1$, $x(a_3) - x(\widehat{q}) < x(q) - x(\widehat{q}) \le 1$. $x(\widehat{q}) - x(a_3) \le x(a_3) - 1 - x(a_3) < 1$ since $x(a_2) - x(a_3) < 2$. In other words, $|x(a_3) - x(\widehat{q})|, |y(a_3) - y(\widehat{q})| < 1$, so $||a_2 - q||_{\infty} < 1$ and hence $a_3 \notin \mathcal{F}$, contradicting the assumption that $a_3 \in \partial \mathcal{F}$. Hence, c_3 does not exist. A similar argument shows that a third component of $S \cap \mathcal{F}$ cannot intersect the right edge of S.

(iii) Let C be a connected component of S∩𝔅. Since C is xy-monotone, for any two points a, b ∈ C, the shortest path from a to b within C is xy-monotone, implying that ρ_S(a, b) ≤ ||a − b||₁. Let p, q ∈ C be two points such that ρ(p,q) < ρ_S(p,q). Then the shortest path, denoted by ψ, from p to q in 𝔅 leaves S. Let a, b be two consecutive intersection points of ψ with ∂S where ψ crosses ∂S, i.e., a, b ∈ ∂S and ψ(a, b) ∩ S = {a, b}. But ψ(a, b) is longer than following ∂S from a to b (along the shorter of the two portions of ∂S), which implies that ¢(ψ(a, b)) > ||a − b||₁. On the other hand, ρ_S(a, b) ≤ ||a − b||₁, contradicting that ψ is the shortest path from p to q in 𝔅. Hence, ρ_S(p,q) = ρ(p,q).

The next lemma shows that there is a simple optimal motion between configurations as long as they are sufficiently close and both x-separated or both y-separated.

Lemma 3.5. Let Q_A, Q_B be axis-aligned unit-radius squares. For $\mathbf{s} = (s_A, s_B), \mathbf{t} = (t_A, t_B) \in \mathbf{F}$ such that s_A, t_A (resp., s_B, t_B) lie in a common component of $\operatorname{int}(Q_A) \cap \mathcal{F}$ (resp., $\operatorname{int}(Q_B) \cap \mathcal{F}$) and \mathbf{s} and \mathbf{t} are both x-separated or both y-separated, there exists a (trivially kissing) plan π with $\mathfrak{e}(\pi) = \varrho(s_A, t_A) + \varrho(s_B, t_B)$ and $\alpha(\pi) \leq 2$.

Proof. Without loss of generality, the configurations are x-separated. Using standard transformations as necessary, we can assume $x(s_A) - x(s_B) \ge 2$. Then $x(s_A) - 2 < x(t_A) < x(s_A) + 2$ (resp., $x(s_B) - 2 < x(t_B) < x(s_B) + 2$) since s_A, t_A (resp., s_B, t_B) lie in the interior of Q_A (resp., Q_B). (t_A, t_B) is x-separated so $|x(t_A) - x(t_B)| \ge 2$. If $x(t_A) - x(t_B) \le -2$ then $x(t_A) < x(s_B) \le x(s_A) - 2$, which is a contradiction. Hence $x(t_A) - x(t_B) \ge 2$. Let P_A be the xy-monotone (s_A, t_A) -path in $Q_A \cap \mathcal{F}$ and let P_B be the xy-monotone (s_B, t_B) -path in $Q_B \cap \mathcal{F}$ from Lemma 3.4. There are two cases.

First, suppose $x(s_A) - x(t_A)$ and $x(s_B) - x(t_B)$ are zero or their signs are the same, say, nonnegative for concreteness. See Figure 9(left). Then P_B lies to the right of line $x = s_A + 2$ and hence $P_B \subset \mathcal{F}[s_A]$. Similarly, P_A lies to the left of line $x = t_B - 2$ and hence $P_A \subset \mathcal{F}[t_B]$. Otherwise, $x(s_A) - x(t_A)$ and $x(s_B) - x(t_B)$ are non-zero and their signs are different; for concreteness, suppose $x(s_A) - x(t_A) < 0 < x(s_B) - x(t_B)$. See Figure 9(right). Then $x(s_A) < x(t_A) \le x(t_B) - 2 < x(t_A) - 2$. See Figure 9. Then P_B lies to the right of line $x = x(t_A) + 2$, and hence right of line $x = x(s_A) + 2$, so $P_B \subset \mathcal{F}[s_A]$. Similarly, P_A lies to the left of line $x = x(t_B) - 2$ so $P_A \subset \mathcal{F}[t_B]$.

Thus, in either case, the desired plan π is to first move B along P_B while A is parked at s_A , then move A along P_A while B is parked at t_B , which is trivially kissing since it has at most two moves. The other cases are symmetric.

The previous lemma allows us to shortcut kissing plans and to use a packing argument to establish a useful upper bound on the number of moves in an optimal plan.

Lemma 3.6. Given reachable configurations $s, t \in \mathbf{F}$, there exists a decoupled, kissing, optimal (s, t)-plan $\boldsymbol{\pi} = (\pi_A, \pi_B)$ with $\alpha(\boldsymbol{\pi}) \leq c(\min\{\mathfrak{c}(\pi_A), \mathfrak{c}(\pi_B)\} + 1)$, for some global constant $c \geq 1$.

Proof. Without loss of generality, assume $\mathfrak{c}(\pi_A) \leq \mathfrak{c}(\pi_B)$. Let \mathbb{G} be the axis-aligned uniform grid with square cells of radius 1 such that all parking places lie in the interior of grid cells and π does not pass through a vertex of \mathbb{G} . Let $\mathcal{G} \subset \mathbb{G}$ be the set of grid cells that contain at least one parking place of A. It is easily seen that $|\mathcal{G}| \leq 4\mathfrak{c}(\pi_A)$. We will show that we can shortcut π to obtain a new plan π' if necessary so that $\mathfrak{c}(\pi') \leq \mathfrak{c}(\pi)$, A is parked only O(1) times in each cell of \mathcal{G} , the parking places of A in π' are a subset of those in π and π' is also a kissing plan. For a cell $g \in \mathbb{G}$, let $N(g) \subset \mathbb{G}$ be the set of cells $g' \in \mathbb{G}$ such that there exists a pair of points $p \in g, q \in g'$ with $||p-q||_{\infty} = 2$, i.e., (p,q) is a kissing configuration. Note that $|N(g)| \leq 25$.

Fix a cell $g \in \mathcal{G}$. Let C be a connected component of $g \cap \mathcal{F}$ that contains a parking place of A. Recall that π is a kissing plan so B kisses A at each parking place of A. For each parking place ξ of A in C, we label it with cell $\tau \in \mathbb{G}$ if B was in cell τ when it kissed A at ξ . If there are more than one such cell, we arbitrarily choose one of them. If C contains more than two parking places of A with the same label τ such that all of them are x-separated or all of them are y-separated, then we shortcut π as follows. Let λ^- (resp., λ^+) be the first (resp., last) time instance such that $\pi(\lambda^-)$ (resp., $\pi(\lambda^+)$) is a x-separated kissing configuration with $\pi_A(\lambda^-) \in \xi$, $\pi_B(\lambda^-) \in \tau$ (resp., $\pi_A(\lambda^+) \in \xi$, $\pi_B(\lambda^+) \in \tau$). We replace $\pi(\lambda^-, \lambda^+)$ with the $(\pi(\lambda^-), \pi(\lambda^+))$ -plan described in Lemma 3.5 of cost $\varrho(\pi_A(\lambda^-), \pi_A(\lambda^+)) + \varrho(\pi_B(\lambda^-), \pi_B(\lambda^+))$. We repeat this procedure in C until there are no such parking places of A in g. We repeat this step for all cells $g \in \mathcal{G}$. Let π' be the resulting plan. By construction, $\mathfrak{c}(\pi') \leq \mathfrak{c}(\pi)$ and π' is a kissing plan.

We now bound $\alpha(\pi')$. First note that π intersects at most two components of $g \cap \mathcal{F}$ for each cell $g \in \mathcal{G}$. For each such connected component, the plan π' has at most $4|N(g)| \leq 100$ parking places. Therefore g contains at most 200 parkings of A in the plan π' . Summing over all cells of \mathcal{G} , we obtain that A is parked $O(\lceil |\pi_A| \rceil) = O(\mathfrak{e}(\pi_A) + 1)$ times in the plan π' . Since A and B park alternately, $\alpha(\pi') = O(\mathfrak{e}(\pi_A) + 1)$.

4 Paths Inside a Corridor

In this section we prove the existence of a decoupled, kissing, optimal plan in which neither of the two robots is ever parked in the sanctum of a corridor. We prove this result by introducing some convenient notations and establishing a few properties of a decoupled path inside a corridor.

Suppose π is an decoupled, kissing, optimal (s, t)-plan, and let $K \in \mathcal{K}$ be a corridor such that one of the robots, say, A, enters K and parks inside the sanctum K^S of K. We have that

 $s_A, s_B, t_A, t_B \notin \operatorname{int}(K)$ since no point in X lies in the the interior any corridor by definition. Let $I_A := [\lambda_A^-, \lambda_A^+]$ be a maximal time interval during which (the center of) A is inside $K^{(2)}$, which contains the time λ at which $\pi_A(\lambda) \in K^S$. Let σ_0, σ_1 be the (not necessarily distinct) portals of K last crossed in $\pi_A(0, \lambda_A^-)$ and first crossed in $\pi_A(\lambda_A^+, 1)$, respectively. Then $\pi_A(\lambda_A^-), \pi_A(\lambda_A^+)$ lie on the edges of $\sigma_0^{(2)}$ and $\sigma_1^{(2)}$ of $K^{(2)}$, respectively, which again are not necessarily distinct.

We show that π can be transformed to another decoupled, kissing, optimal (s, t)-plan without increasing the cost, so that A does not park inside the sanctum K^S during the interval I_A . We accomplish this in stages. First, we show that B enters K for an interval I_B , with $I_A \cap I_B \neq \emptyset$, and it also enters (resp., exits) K through the portal σ_0 (resp., σ_1) (Lemma 4.1). Next, we show (in Lemma 4.3) that if $\sigma_0 = \sigma_1$, i.e., A (and B) enters and exits K at the same portal then A does not enter the sanctum K^S during the interval I_A and B also does not enter the sanctum during the interval I_B . If $\sigma_0 \neq \sigma_1$, then both A and B cross the sanctum K^S . In this case, we first argue that π can be deformed so that $\pi_A(I_A \cap I_B)$ consists of at most one breakpoint and the breakpoint lies near the portals of K, so it is outside $K^{(6)}$ and A is not parked inside K^S during I_A . A similar claim holds for B (see Lemma 4.4). This deformation may result in losing the kissing property of π during the interval $I_A \cap I_B$. Finally, we show that we can reparameterize the paths π_A and π_B without changing their images, i.e., merging two or more moves into one or adjusting the parking places, so that the resulting (s, t)-plan is decoupled, kissing, and optimal, and neither A nor B is parked inside K^S during the interval I_A (Lemma 4.5).

Lemma 4.1. There is a maximal interval I_B such that $I_A \cap I_B \neq \emptyset$ and B is in K during I_B , i.e., $\pi_B(\lambda) \in K$ for all $\lambda \in I_B$. Furthermore B enters and exits K during I_B through the same portals as A.

Proof. If $\pi_B(\lambda) \notin K$ for all $\lambda \in I_A$, then $K^S \subseteq \bigcap_{\lambda \in I_A} \mathcal{F}[\pi_B(\lambda)]$ and there is no need to park A inside K^S . That is, we can first move A along $\pi_A(I_A)$ while B is parked at $\pi_B(\lambda_A^-)$, then park A at the portal σ_1 and move B along $\pi_B(I_A)$, and then follow the rest of the plan, $\pi(\lambda_A^-, 1)$. So we assume that $\pi_B(\lambda) \in K$ for some $\lambda \in I_A$.

Let $I_B := [\lambda_B^-, \lambda_B^+]$ be a maximal interval with $I_A \cap I_B \neq \emptyset$ during which B is in K. Let u be a vector normal to σ_0 . By Lemma 2.2, the sign of $\langle \pi_A(\lambda) - \pi_B(\lambda), u \rangle$ is the same for all $\lambda \in I_A \cap I_B$. If A and B enter through different portals of K, then we claim that π is not an optimal plan. Indeed, if A enters and exits at the same portal, (i.e., $\sigma_0 = \sigma_1$), then we can shortcut $\pi_A(I_A)$ along $\sigma_0^{(2)}$ to obtain a cheaper (s, t)-plan, and if A exits at the other portal (i.e., $\sigma_0 \neq \sigma_1$), we can shortcut $\pi_B(I_B)$ along that portal (at which B entered) to obtain a cheaper (s, t)-plan. A similar short-cutting argument holds if B does not exit through the same portal as A. This completes the proof of the lemma.

For simplicity, we make some assumptions about the orientation of the features of K without loss of generality. Recall that if at least one blocker of K is vertical (resp., horizontal) its portals are horizontal (resp., vertical). First, we assume that the portals of K are vertical or have slope -1 by rotating the setting by $\pi/2$ as necessary. Then the slopes of the blockers of K are strictly positive if its portals have slope -1, otherwise one blocker has non-negative slope and the other blocker has non-positive slope. Second, we assume that A and B enter K from its "left portal," which formally is the portal whose endpoint is the leftmost vertex of K (if the portals are vertical this is obvious); let σ_L be this portal and σ_R be the other. See Figure 10.

In the following, we prove that neither A nor B is parked inside the sanctum of K during the interval $I_A \cap I_B$. Without loss of generality, assume that $\lambda_A^- \leq \lambda_B^-$ because otherwise we can swap



Figure 10. Illustrations of corridors with the left portals σ_L shown as thick. The left and top-right examples have portals with slope -1 and the bottom-right example has vertical portals.



Figure 11. Illustrations of $\boldsymbol{\xi}, \boldsymbol{\zeta}$ when $\sigma_0 = \sigma_1$ (top) and $\sigma_0 \neq \sigma_1$ (bottom), not to scale.

A and B. We modify the plan π , without changing the images of π_A, π_B , so that B is "near" A when A enters or exits K: Let Δ_0 be the trapezoid formed by the blockers of K and segments $\sigma_0, \sigma_0^{(2)}$. If $\sigma_0 = \sigma_1$, let $\Delta_1 = \Delta_0$, otherwise let Δ_1 be the trapezoid formed by the blockers of K and segments $\sigma_1^{(4)}, \sigma_1^{(2)}$. See Figure 11. Set $\xi_A \coloneqq \pi_A(\lambda_A^-), \zeta_A \coloneqq \pi_A(\lambda_A^+)$. If $\pi_B(\lambda_A^-) \in \Delta_0$, we set $\xi_B \coloneqq \pi_B(\lambda_A^-)$. If $\pi_B(\lambda_A^-) \notin \Delta_0$, we park A at ξ_A and move B from $\pi_B(\lambda_A^-)$ until B enters Δ_0 through σ_0 and park B at this point, which we denote by ξ_B . Then we follow the plan π as before. We use λ_A^- to denote the time instance at which A is at ξ_A and B is at ξ_B , and we continue to use π to denote the unmodified plan. Similarly, if $\pi_B(\lambda_A^+) \in \Delta_1$, then set $\zeta_B \coloneqq \pi_B(\lambda_A^+)$. If $\pi_B(\lambda_A^+) \notin \Delta_1$ then we park A at $\zeta_A \coloneqq \pi_A(\lambda_A^+) \in \sigma_1^{(2)}$ and move B along π_B until it enters Δ_1 through a point $\xi_B \in \sigma_0$ if $\sigma_0 = \sigma_1$ or through a point $\xi_B \in \sigma_1^{(4)}$ otherwise. See Figure 11 again. We use $\widehat{\lambda}_A^+$ to denote the time instance at which A (resp., B) is at point ξ_A (resp., ζ_B). We continue to use π to denote the modified plan. It can be verified that the modified plan is feasible. Since the images of π_A and π_B remained the same, the cost also remains the same. Set $\boldsymbol{\xi} = (\xi_A, \xi_B) = \boldsymbol{\pi}(\lambda_A)$ and $\boldsymbol{\zeta} = (\zeta_A, \zeta_B) = \boldsymbol{\pi}(\widehat{\lambda}_A^+)$. We note that the resulting plan may not be kissing during the interval $[\lambda_A^-, \lambda_A^+]$, e.g., A may not kiss B on its move to ζ_A and B may not kiss A on its move to ζ_B . We will convert it into a kissing plan after we are done modifying the plan inside K (see Lemma 4.5).

We extend our notation for portal-parallel lines and segments to define them by points that they contain: For any point $p \in K$, let ℓ_p the line normal to u_L (and u_R) containing p, and let $\sigma_p := K \cap \ell_p$. We next prove the following technical lemma, which shows that if A is sufficiently deep inside and ahead of B in K, then B does not obstruct A from reaching further inside the corridor; moreover, the shortest path for A inside the corridor is the shortest path in the plane that avoids B. By swapping the roles of A and B and reflecting the setting over the y = -x line, it can be used more generally.

Lemma 4.2. Let K be a corridor with direction vector $u \in \{0, \pi/4\}$. For any configurations $(s_A, p_B), (t_A, p_B) \in \mathcal{F}$ such that $s_A \in K^{(2)}, p_B, t_A \in K, \langle p_B, u \rangle < \langle s_A, u \rangle < \langle t_A, u \rangle$, and $d_{\infty}(\ell_{s_A}, \ell_{t_A}) \geq 2$, the shortest path P_A from s_A to t_A in $cl(\mathbb{R}^2 \setminus (p_B + 2\Box))$ is such that:

- (i) P_A is contained in $cl(K \setminus (p_B + 2\Box)) \subset K \cap \mathcal{F}[p_B]$,
- (ii) if the portals of K are vertical then P_A is segment $s_A t_A$ and $\mathfrak{e}(P_A) = ||s_A t_A||_2$, and
- (iii) if the portals of K have slope -1 then P_A is segment $s_A t_A$ if $s_A t_A \cap \operatorname{int}(p_B w) = \emptyset$ and $P_A = s_A w || w t_A$ otherwise, where w is the top-right vertex w of $p_B + 2\Box$, and $\mathfrak{e}(P_A) < 2\sqrt{5} + || s_A t_A ||_2$.

Proof. See Figure 12. Let P_A be the shortest path from s_A to t_A in $cl(\mathbb{R}^2 \setminus (p_B + 2\Box))$. If P_A has no breakpoints, P_A is segment $s_A t_A$ so $P_A \subset K \setminus (p_B + 2\Box)$ since $s_A, t_A \in K$ and K is convex. Then (i) holds when P_A is a segment. We will prove (i) when P_A is not a segment later.

We next prove (ii). Assume the portals are vertical. Since the portals are vertical, we have

$$x(\sigma_L) \le x(p_B) \le x(s_A) \le x(t_A) - 2 \le x(\sigma_R) - 4.$$

If $x(s_A) \ge x(p_B) + 2$ then s_A, t_A lie to the right of (the right vertical edge of) $p_B + 2\Box$ and hence $P_A = s_A t_A$. So assume $|x(s_A) - x(p_B)| < 2$ for sake of contradiction. By definition, $(s_A, p_B) \in \mathcal{F}$, so $|y(s_A) - y(p_B)| \ge 2$. $s_A, p_B \in K$ so the segment $p_B s_A \subset K$ because K is convex. For concreteness, suppose $y(p_B) \le y(s_A) - 2$ so the segment $p_B s_A$ has positive slope; the other case is symmetric.



Figure 12. Illustrations of the proof of Lemma 4.2. (top) K has vertical portals so P_A is a segment. (bottom) K has portals with slope -1 and segment $s_A t_A$ (not shown) intersects $int(p_B + 2\Box)$ and hence segment $p_B w$. The region Δ contains vertex w of $p_B + 2\Box$.

Recall that because K has vertical portals, K has one blocker with non-positive slope and the other has non-negative slope and $|\sigma_{\eta}| < 2$ for all points $\eta \in int(K)$ by Lemma 2.1. Let Q be the rectangle with s_A (resp., p_B) as its bottom-left (resp., top-right) vertex and let e_1 (resp., e_2) be the left (resp. right) vertical edge of Q; $|e_1|, |e_2| \ge 2$. The blocker with non-positive slope must not intersect int(Q) and the blocker with non-negative slope must intersect at most one of $int(e_1)$ or $int(e_2)$ (otherwise a blocker would intersect $int(p_Bs_A)$ which is impossible since $p_Bs_A \subset K$). Then either $e_1 \subset K$ or $e_2 \subset K$. If $e_1 \subset K$ (resp. $e_2 \subset K$) then $|\sigma_{p_B}| \ge |e_1| \ge 2$ (resp., $|\sigma_{s_A}| \ge |e_2| \ge 2$), which is a contradiction. So $P_A = s_A t_A$ as claimed. This proves (ii).

We next prove (i) when $P_A \neq s_A t_A$. Assume the portals have slope -1. If $P_A = s_A t_A$ then (iii) holds by the discussion above, so assume otherwise. Let h be the halfspace containing t_A defined by the line supporting σ_{s_A} . $\langle p_B, u \rangle < \langle s_A, u \rangle \leq \langle t_A, u \rangle$ implies $p_B \notin h$ since $t_A \in h$ and u is normal to σ_{s_A} . Then the top-right vertex w of $p_B + 2\Box$ is the only vertex that lies in int(h). It is easy to see that the shortest path $P_A \subset h$ since $t_A \in h$ (otherwise P_A must leave h, wrap around $p_B + 2\Box \setminus h$, and then re-enter h, which can only be longer). Since $P_A \neq s_A t_A$, $s_A t_A$ must intersect $p_B + 2\Box) \cap h$, and hence it intersects segment $p_B w$. Then $P_A = s_A w || w t_A$, as wis the only vertex of $cl(\mathbb{R}^2 \setminus (p_B + 2\Box))$ in h. It follows that either $x(s_A) < x(w) < x(t_A)$ and $y(s_A) > y(w) > y(t_A)$ or all of the inequalities are reversed. For concreteness, assume the former; the other case is symmetric. We have $\langle p_B, u \rangle < \langle s_A, u \rangle < \langle t_A, u \rangle$, and $d_{\infty}(\ell_{s_A}, \ell_{t_A}) \ge 2$ by definition. The latter implies $d_2(\ell_{s_A}, \ell_{t_A}) \ge 2\sqrt{2}$. Then

$$\langle w, u \rangle = \langle p_B, u \rangle + 2\sqrt{2} < \langle s_A, u \rangle + 2\sqrt{2} \le \langle s_A, u \rangle + d_2(\ell^{(s_A)}, \ell^{(t_A)}) = \langle t_A, u \rangle + d_2(\ell^{(t_A)}, \ell^{(t_A)}) = \langle t_A, u \rangle + d_2(\ell^{(t_A)}, \ell^{(t_A)}) = \langle t_A, u \rangle + d_2(\ell^{(t_A)}, \ell^{(t_A)}) = \langle t_A, u \rangle + d_2(\ell^{(t_$$

Then we have

$$\langle s_A, u \rangle < \langle w, u \rangle < \langle t_A, u \rangle,$$

where the first inequality follows from the fact $w \in int(h)$. Now let Δ be the trapezoid with edges

on $\sigma_{s_A}, \sigma_{t_A}$ and horizontal lines $y = y(s_A), y = y(t_A); s_A$ (resp., t_A) is top-left (resp., bottom-right) vertex of Δ . It follows that $w \in \Delta$ since $y(s_A) \ge x(w) \ge y(t_A)$ and the previous inequalities. Recall that the blockers of K have positive slope. Then the top (resp., bottom) blocker of K lies above the top (resp. bottom) horizontal edge of Δ and hence $\Delta \subset K^{(2)}$. Then the segments $s_A w, w t_A$ lie in $K^{(2)}$ because it is convex. It follows that $P_A = s_A w || w t_A \subset \operatorname{cl}(K^{(2)} \setminus (p_B + 2\Box))$ as desired.

It remains to prove $\mathfrak{c}(P_A) \leq 2\sqrt{5} + ||s_A - t_A||_2$ in (iii) when P_A has w as a breakpoint. By assumption that $\langle p_B, u \rangle \leq \langle s_A, u \rangle$, we have $x(s_A) \geq x(p_B) - 2 \geq x(w) - 4$, where the second inequality follows from the fact w is the top-right vertex of $p_B + 2\Box$. The L_{∞} -distance between the endpoints of $\sigma_{s_A}, \sigma_{t_A}$ are less than 2 by Lemma 2.1 because $s_A, t_A \in \operatorname{int}(K)$, so $y(s_A) - y(w) \leq y(s_A) - y(t_A) < 2$. Putting everything together, we have $|x(s_A) - x(w)| \leq 4$ and $|y(s_A) - y(w)| < 2$, and hence $||s_A - w||_2 < \sqrt{4^2 + 2^2} = 2\sqrt{5}$. Clearly $||w - t_A||_2 \leq ||s_A - t_A||_2$, so

$$\mathfrak{e}(P_A) = ||s_A - w||_2 + ||w - t_A||_2 \le 2\sqrt{5} + ||s_A - t_A||_2.$$

Having shown (i)–(iii), this concludes the proof.

Now we are ready to prove the first main lemma of this section.

Lemma 4.3. If $\sigma_0 = \sigma_1$, i.e., A enters and exits K from the same portal during interval I_A then A does not enter the sanctum K^S during I_A . Similarly B does not enter K^S during the interval $I_A \cap I_B$.

Proof. For sake of contradiction, suppose A enters K^S during interval I_A . Then $\pi_A(I_A)$ intersects segments $\sigma_0^{(2)}$ and $\sigma_0^{(10)}$. Let $\gamma_A^0 \in \sigma_0^{(4)}$ (resp., $\gamma_A^1 \in \sigma_0^{(4)}$) be the first (resp., last) point of $\pi_A(I_A)$ on $\sigma_0^{(4)}$. See Figure 13. We now deform π by replacing $\pi(I_A)$ with another plan $\hat{\pi} := (\hat{\pi}_A, \hat{\pi}_B)$:

- (1) Move A from ξ_A to γ_A^0 while B is parked at ξ_B ,
- (2) move B from ξ_B to ζ_B while A is parked at γ_A^0 ,
- (3) move A from γ_A^0 to γ_A^1 while B is parked at ζ_B , and
- (4) continue moving A from γ_A^1 to ζ_A while B is parked at ζ_B .

In each move (i), i = 1, ..., 4, the moving robot follows the shortest feasible path $\hat{\pi}_i$ between its start and target placements, while the other robot is parked. Note that ξ_B, ξ_A, γ_A^0 and $\zeta_B, \zeta_A, \gamma_A^1$ satisfy the conditions as p_B, s_A, t_A in the statement of Lemma 4.2, respectively, by definition. Then $\mathfrak{c}(\hat{\pi}_1) \leq 2\sqrt{5} + ||\xi_A - \gamma_A^0||_2$ and $\mathfrak{c}(\hat{\pi}_4) \leq 2\sqrt{5} + ||\zeta_A - \gamma_A^1||_2$. Furthermore, $d_{\infty}(\Delta_0, \sigma_0^{(4)}) \geq 2$ by definition. It follows that segment $\xi_B\zeta_B \subset \mathcal{F}[\gamma_0^1]$ and segment $\gamma_A^0\gamma_B^1 \subset \mathcal{F}[\zeta_B]$, so $\hat{\pi}_2 = \xi_B\zeta_B$ and $\hat{\pi}_3 = \gamma_A^0\gamma_A^1$. Then

$$\begin{aligned} \mathfrak{c}(\widehat{\boldsymbol{\pi}}) &\leq (2\sqrt{5} + ||\xi_A - \gamma_A^0||_2) + ||\xi_B - \zeta_B||_2 + ||\gamma_A^0 - \gamma_A^1||_2 + (2\sqrt{5} + ||\zeta_A - \gamma_A^1||_2) \\ &\leq ||\xi_B - \zeta_B||_2 + ||\xi_A - \gamma_A^0||_2 + ||\zeta_A - \gamma_A^1||_2 + 4\sqrt{5} + 2\sqrt{2}, \end{aligned}$$

where the last inequality follows from the fact $|\sigma_0^{(4)}| < 2\sqrt{2}$ by Lemma 2.1. On the other hand, $\mathfrak{e}(\pi_B) \geq ||\xi_B - \zeta_B||_2$ and

$$\mathfrak{e}(\pi_A) \ge ||\xi_A - \gamma_A^0||_2 + ||\zeta_A - \gamma_A^1||_2 + \mathfrak{e}(\pi_A[\gamma_A^0, \gamma_A^1]) \ge ||\xi_A - \gamma_A^0||_2 + ||\zeta_A - \gamma_A^1||_2 + 12$$



Figure 13. Illustration of Lemma 4.3. $\pi_A[\gamma_A^0, \gamma_A^1]$ is depicted as dashed and $\hat{\pi}_i$ are thick for i = 1, ..., 4. Note that this example is not drawn to scale.

where the last inequality follows from the fact $c(\pi_A[\gamma_A^0, \gamma_A^1]) \ge 2d_{\infty}(\sigma_0^{(4)}, \sigma_0^{(10)}) \ge 2(6) = 12$ by definition. Putting everything together, we have

$$\mathfrak{e}(\boldsymbol{\pi}) - \mathfrak{e}(\widehat{\boldsymbol{\pi}}) \ge 12 - 4\sqrt{5} - 2\sqrt{2} > 0.$$

But then π is not optimal, which is a contradiction. We conclude that our assumption that $\pi_A(I_A)$ enters the sanctum K^S is false. By Lemma 2.2, $\pi_B(I_A \cap I_B)$ also does not enter the sanctum K^S . \Box

In the rest of this section we further assume that A and B exit K through its right portal in addition to the assumption that they enter K through its left portal, i.e., $\sigma_L = \sigma_0$ and $\sigma_R = \sigma_1$.

Lemma 4.4. There exists a decoupled, optimal $(\boldsymbol{\xi}, \boldsymbol{\zeta})$ -plan $\boldsymbol{\psi} = (\psi_A, \psi_B) \subset K \times K$ that consists of two moves: first move A along the shortest (ξ_A, ζ_A) -path in $\mathcal{F}[\xi_B]$ while B is being parked at ξ_B , and then move B along the shortest (ξ_B, ζ_B) -path in $\mathcal{F}[\zeta_A]$ while A is being parked at ζ_A .

Proof. Let u be the direction of K. We have $\xi_A \in \sigma_1^{(2)}$, $\xi_B \in \Delta_0$, $\zeta_A \in \sigma_1^{(4)}$, $\zeta_B \in \Delta_1$ and $\langle \xi_B, u \rangle < \langle \xi_A, u \rangle < \langle \zeta_B, u \rangle < \langle \zeta_A, u \rangle$ by definition.

Let ψ_A be the shortest path from ξ_A to ζ_A in $\mathbb{R}^2 \setminus (\xi_B + 2\Box)$, and let ψ_B be the shortest path from ξ_B to ζ_B in $\mathbb{R}^2 \setminus (\zeta_B + 2\Box)$. Then we have $\psi_A \subset \mathcal{F}[\xi_B]$ and $\psi_B \subset \mathcal{F}[\zeta_A]$ by Lemma 4.2, where the latter is obtained by swapping the roles of A and B and reflecting the setting over the y = -xline to apply the lemma. Let $\psi = (\psi_A, \psi_B)$ be the (ξ, ζ) -plan. It remains to show that ψ is an optimal plan, which does not follow immediately from the fact each of ψ_A, ψ_B are "locally" optimal in the sense they are shortest paths between specific placements while the other robot is at another specific placement. Recall that some optimal plans have more than two (but at most three) moves even for the case $\mathcal{W} = \mathcal{F} = \mathbb{R}^2$ [14, 31], as mentioned in Section 2 (see the example in Figure 2). So we have to rely on the structure of ξ, ζ .

We first rule out the easy case, which is when ψ_A, ψ_B are both segments. See Figure 14(topleft,bottom). Then $\mathfrak{c}(\boldsymbol{\psi}) = ||\xi_A - \zeta_A||_2 + ||\xi_B - \zeta_B||_2$, which is optimal since any plan must have at least this cost. By Lemma 4.2, ψ_A, ψ_B are segments when the portals are vertical, in which case we are done. So suppose one of them, say, ψ_A is not the segment $\xi_A\zeta_A$; the case where $\psi_B \neq \xi_B\zeta_B$ is symmetric. Then $\xi_A\zeta_A \cap \operatorname{int}(\xi_B + 2\Box) \neq \emptyset$, and by Lemma 4.2, $\psi_A = \xi_A w ||w\zeta_A$, where w is the top-right vertex of $\xi_B + 2\Box$. It follows that either $x(\xi_A) < x(w) < x(\zeta_A)$ and $y(\xi_A) > y(w) > y(\zeta_A)$, or all inequalities are reversed. We assume the former; the latter case is symmetric. See Figure 14(top-right).

The current $\boldsymbol{\xi}, \boldsymbol{\zeta}$ satisfy a case in [31], specifically case "Zone III(2)" in their Section 4.3.1, which implies that $\boldsymbol{\psi}$ is optimal in the plane without obstacles, and hence in our setting, since we have



Figure 14. Illustrations of cases from the proof of Lemma 4.4. Top-left: The portals have slope -1, and ψ_A, ψ_B are segments. Top-right: The portals have slope -1 and ψ_A has a breakpoint w which is a vertex of $\xi_B + 2\Box$; ψ_B is a segment. Bottom: The portals are vertical, so ψ_A, ψ_B are segments. Only in the bottom example are $\boldsymbol{\xi}, \boldsymbol{\zeta}$ kissing configurations, which is necessary when portals are axis-aligned since the lines containing the placements are at Euclidean distance 2 in that case.

already shown ψ is feasible in our setting (i.e., inside K). To verify that ξ, ζ satisfies their case "Zone III(2)," it suffices⁵ to have the following properties in addition to the inequalities on the x-and y-coordinates above:

- (P1) $\xi_B \in \operatorname{int}(\xi_A \zeta_A \oplus 2\Box),$
- (P2) $\xi_A \in \operatorname{int}(\xi_B \zeta_B \oplus 2\Box),$
- $(P3) ||\xi_B \zeta_B||_{\infty} \ge 2,$
- (P4) $|x(\xi_B) x(\xi_A)| < 2$ and $y(\xi_B) \le y(\xi_A) 2$, i.e., ξ_B lies below $\xi_A + 2\Box$.
- (P5) $|y(\zeta_B) y(\zeta_A)| < 2$ and $x(\zeta_B) \le x(\zeta_A) 2$, i.e., ζ_B lies left of $\zeta_A + 2\Box$.

See Figure 14 (top-right) again. (P1) follows from the fact $\xi_A \zeta_A \cap \operatorname{int}(\xi_B + 2\Box) \neq \emptyset$. Since $\langle \xi_B, u \rangle < \langle \xi_B, u \rangle < \langle \zeta_B, u \rangle, \xi_B \zeta_B$ crosses σ_{ξ_A} . Then we have

$$\sigma_{\xi_A} \subset \operatorname{int}(\xi_A + 2\Box) \subset \operatorname{int}(\xi_A \zeta_A \oplus 2\Box),$$

where the first containment follows from Lemma 2.1, and hence (P2) holds. We next prove (P3). We have that

$$||\xi_B - \zeta_B||_{\infty} \ge d_{\infty}(\Delta_0, \Delta_1) \ge d_{\infty}(\ell_0^{(2)}, \ell_1^{(4)}) \ge \operatorname{len}(K) - 2 - 4 \ge 14,$$

where the second inequality follows from the definitions of Δ_0, Δ_1 being trapezoids bounded between $\sigma_0, \ell_0^{(2)}$ and $\ell_0^{(4)}, \ell_0^{(2)}$, respectively, and the last inequality follows from the fact $\operatorname{len}(K) \geq 20$ since K has a non-empty sanctum $K^S = K^{(10)} \subset K$. So (P3) holds.

Next, we have that the segment $\sigma_0^{(2)} = \sigma_{\xi_A}$ (resp., $\sigma_1^{(2)} = \sigma_{\zeta_A}$) has slope -1, and the L_{∞} -distance between its endpoints is than 2 by Lemma 2.1. We have $||\xi_A - \xi_B||_{\infty}, ||\zeta_A, \zeta_B - \geq ||_{\infty} 2$ since $\xi, \zeta \in \mathbf{F}$ are configurations. It follows that, with $\langle \xi_B, u \rangle < \langle \xi_A, u \rangle$ (resp., $\langle \zeta_B, u \rangle < \langle \xi_A, u \rangle$), $x(\xi_B) < x(\xi_A) + 2$ (resp., $x(\zeta_B) < x(\zeta_A) + 2$). We have $x(\xi_B) + 2 = x(w) > x(\xi_A)$, so together we have

⁵We note that their case "Zone III(2)" is more general in the sense that properties (P1)-(P5) are only sufficient conditions, which is easy to verify from their paper.

 $|x(\xi_B) - x(\xi_A)| < 2$, the first part of (P4). The second part follows from $y(\xi_B) - 2 = y(w) < y(\xi_A)$, so we have (P4).

It remains to prove (P5). We have $y(\zeta_A) < y(\xi_A)$. Then $y(\zeta_A) \leq y(\xi_A)$. For sake of contradiction, suppose we have $y(\zeta_B) \leq y(\zeta_A) - 2$. The bottom endpoint q of $\ell_0^{(2)}$ has $y(q) > y(\xi_A) - 2$ since ξ_A lies on the segment and its L_{∞} -length is less than 2 as described above. But then

$$y(\zeta_B) \le y(\zeta_A) - 2 < y(\xi_A) - 2 \le y(q),$$

which is a contradiction since the blockers of K have positive slope, i.e., the bottom endpoint of segment σ_{ζ_B} , which contains ζ_B , lies on or above y = y(q). So $y(\zeta_B) > y(\zeta_A) - 2$. A similar argument implies $y(\zeta_B) < y(\zeta_A) + 2$. So we have $|y(\zeta_B) - y(\zeta_A)| < 2$, the first part of (P5). Since ζ is a configuration, so we have $||\zeta_A - \zeta_B||_{\infty} \ge 2$. Since $|y(\zeta_B) - y(\zeta_A)| < 2$, it must be that $|x(\zeta_B) - x(\zeta_A)| \ge 2$. It cannot be the case that $x(\zeta_B) \ge x(\zeta_A) + 2$ since $\langle \zeta_B, u \rangle < \langle \zeta_A, u \rangle$. So $x(\zeta_B) \le x(\zeta_A) - 2$, and we have (P5). This concludes the proof.

Lemma 4.5. Suppose a robot, say A, is parked inside the sanctum of a corridor K at time $\lambda \in [0,1]$ in a decoupled, kissing, optimal (\mathbf{s}, \mathbf{t}) -plan π , and let I_A be a maximal time interval with $\lambda \in I_A$ during which A is inside $K^{(2)}$. Let I_B be the maximal time interval with $I_A \cap I_B \neq \emptyset$ as given by Lemma 4.1. Then there exists a decoupled, kissing, optimal (\mathbf{s}, \mathbf{t}) -plan π' and an interval $I \supseteq I_A \cup I_B$ such that neither A nor B parks inside the sanctum K^S of K during $\pi'(I)$ and $\pi(\lambda) = \pi'(\lambda)$ for all $\lambda \notin I$.

Proof. Following the notation above, by Lemma 4.3, $\sigma_0 \neq \sigma_1$. We then modify the plan during the interval I_A as described above (preceding Lemma 4.1). Note that $\pi(\widehat{\lambda}_A^-, \widehat{\lambda}_A^+)$ is a $(\boldsymbol{\xi}, \boldsymbol{\zeta})$ -plan. By Lemma 4.4, we can replace $[\widehat{\lambda}_A^-, \widehat{\lambda}_A^+]$ with the $(\boldsymbol{\xi}, \boldsymbol{\zeta})$ -plan $\boldsymbol{\psi}$ without increasing the cost of the overall plan. We thus obtain a decoupled, optimal $(\boldsymbol{s}, \boldsymbol{t})$ -plan π' such that $\pi_A(I_A) \cap K^{(4)}$ and $\pi_B(I_A) \cap K^{(4)}$ are line segments, $\pi'(I_A)$ consists of two moves, and no robot is parked inside $K^{(4)}$. However, the resulting plan $\pi' = (\pi'_A, \pi'_B)$ may not be kissing. We convert it into a kissing plan without changing the images of π'_A and π'_B by applying the construction described in the proof of Lemma 3.2 repeatedly, as follows.

Let ψ_A (resp., ψ_B) be the path followed by A (resp., B) in the $(\boldsymbol{\xi}, \boldsymbol{\zeta})$ -plan. Let ψ_A^- (resp., ψ_B^-) be the move of A (resp., B) that brought it to ξ_A (resp., ζ_A), and let p_A^- (resp., p_B^-) be the initial point of ψ_A^- (resp., ψ_B^-), i.e., where A (resp., B) was parked before ξ_A (resp., ξ_B). Similarly, let ψ_A^+ (resp., ψ_B^+) be the move of A (resp., B) that took it from ζ_A (resp., ζ_B) to its next parking position denoted by p_A^+ (resp., p_B^+). These six moves are the only moves which might not be kissing. For simplicity, assume that none of them is the first or last move of π' . First consider the case where $\widehat{\lambda}_A^- = \widehat{\lambda}_A^-$, i.e., B has already parked at ξ_B when A reached ξ_A , in which case the sequence of moves is

$$\dots, (B, \psi_B^-, p_A^-), (A, \psi_A^-, \xi_B), (A, \psi_A, \xi_B), (B, \psi_B, \zeta_A), \dots$$

We note that A moves in both the second and third move, so we can transform the sequence as

$$\dots, (B, \psi_B^-, p_A^-), (A, \psi_A^- \| \psi_A, \xi_B), (B, \psi_B, \zeta_A), \dots$$

Since the original plan was kissing, B kisses A while moving along the path ψ_B^- . If A kisses B during $\psi_A^- \|\psi_A$, we do not need to modify the moves $\psi_B, \psi_A^- \|\psi_A$. So assume A does not kiss B in this move. Consider the first point of ψ_B , denoted by η , that intersects $\psi_A^- \|\psi_A \oplus 2\Box$. Since both ψ_A

and $\psi_B \operatorname{cross} \sigma_0^{(2)}$ and the L_{∞} -distance between its endpoints is less than 2 by Lemma 2.1, η lies to the left of $\sigma_0^{(2)}$. Let $\psi_B^{\leq} := \psi_B[\eta, \zeta_B]$. We now park B at η instead of ξ_B , i.e., the plan becomes

...,
$$(B, \psi_B^- || \psi_B^<, p_A^-), (A, \psi_A^- || \psi_A, \eta), (B, \psi_B^>, \zeta_A), \dots$$

Now both $(B, \psi_B^- || \psi_B^<, p_A^-)$ and $(A, \psi_A^- || \psi_A, \eta)$ are kissing moves and the plan remains feasible.

Next, suppose $\lambda_A < \lambda_A^-$, i.e., A is parked at ξ_A while B moves from p_B^- to ξ_B , i.e., the plan π' is of the form

$$\dots, (A, \psi_A^-, p_B^-), (A, \psi_B^-, \xi_A), (A, \psi_A, \xi_B), (B, \psi_B, \zeta_A), \dots$$

Since $\xi_B \in \sigma_0$ and $\xi_A \in \sigma_0^{(2)}$ in this case and the original plan was a kissing plan, it is easily seen that A kisses B during the ψ_A^- move. Next, the path ψ_B^- lies to the left of σ_0 while $\xi_A \in \sigma_0^{(2)}$, so if B kisses A while moving along ψ_B^- , it happens only at ξ_B in which case A also kisses B during the move ψ_A . Suppose B does not kiss A during ψ_B^- . Since ψ_A lies to the right of $\sigma_0^{(2)}$, $\operatorname{int}(\psi_A + 2\Box) \cap \psi_B^- = \varnothing$. Therefore, we can combine ψ_A^- and ψ_A , i.e., the plan becomes

$$\dots, (A, \psi_A^- \| \psi_A, p_B^-), (B, \psi_B^-, \zeta_A), (B, \psi_B, \zeta_A), \dots$$

Of course, A kisses B during $\psi_A^- || \psi_A$. B moves in both the second and third move, so we can transform the sequence as

$$\ldots, (A, \psi_A^- \| \psi_A, p_B^-), (B, \psi_B^- \| \psi_B, \zeta_A), \ldots$$

In summary, we convert π' into another plan without changing the images of the paths so that it is kissing until the move that contains ψ_A . Furthermore, the new parking place η we added (only in the first case) lies outside $K^{(2)}$. We continue this process until the moves containing paths ψ_B, ψ_A^+ , and ψ_B^+ also become kissing. However, they push the parkings of A and B only later, i.e., beyond the time at which B leaves ζ_B . Hence, we conclude that the transformation converts π' into a kissing plan without adding a parking in K^S during the interval I_A and also without changing the images of the paths π'_A, π'_B . This concludes the proof of the lemma.

By applying Lemma 4.5 repeatedly, we obtain the following corollary.

Corollary 4.6. For any reachable configurations $s, t \in \mathbf{F}$, there exists a decoupled, kissing, optimal (s, t)-plan in which no robot parks inside the sanctum of a corridor of \mathcal{K} .

5 Near-Optimal Tame Plans

Let X be the set of vertices of \mathcal{F} plus $\{s_A, s_B, t_A, t_B\}$ and the vertices of all maximal corridors in \mathcal{K} , i.e., the endpoints of their portals. In this section, we show that a kissing, decoupled, optimal plan can be deformed by paying a fixed (constant) cost so that all robots are parked near a point of X. For two parameters Δ^-, Δ^+ with $0 \leq \Delta^- \leq \Delta^+$, we say that a point $p \in \mathcal{F}$ is (Δ^-, Δ^+) -close (to X) if $d_{\infty}(p, \mathsf{X}) \in [\Delta^-, \Delta^+]$. Often we will be interested in only one of Δ^- and Δ^+ , so we say is Δ -close (resp., Δ -far, Δ -tight) if $d_{\infty}(p, \mathsf{X}) \leq \Delta$ (resp., $d_{\infty}(p, \mathsf{X}) \geq \Delta, d_{\infty}(p, \mathsf{X}) = \Delta$). A decoupled (s, t)-plan $\pi = (\pi_A, \pi_B)$ is called Δ -tame (or tame if the value of Δ is clear from the context) if every parking place on π_A, π_B is Δ -close. The following lemma is the main result of this section and one of the crucial properties on which our algorithm relies. Throughout this section, we set $\Delta_0 := 30$, which is simply a constant that is sufficiently large for our needs.

Lemma 5.1. Given reachable configurations $s, t \in \mathbf{F}$, let π be a decoupled, kissing (s, t)-plan. For any parameter $\Delta \geq \Delta_0$, there exists a decoupled, kissing, Δ -tame (s, t)-plan π' such that $\pi' = \pi$ if $\mathfrak{c}(\pi) \leq \Delta$, and $\mathfrak{c}(\pi') \leq \mathfrak{c}(\pi) + c_1$ and $\alpha(\pi') \leq \alpha(\pi) + c_2$ otherwise, where $c_1 \geq \Delta_0$ and $c_2 > 0$ are absolute constants that do not depend on Δ .

For any $\varepsilon \in (0,1]$ and optimal plan π^* , if $\mathfrak{c}(\pi^*) \leq c_1/\varepsilon$, then π^* is obviously (c_1/ε) -tame (recalling that s_A , s_B , t_A , t_B are in X). Otherwise, by Lemma 5.1, there exists a (c_1/ε) -tame (s, t)-plan of cost at most $\mathfrak{c}(\pi^*) + c_1 \leq (1 + \varepsilon)\mathfrak{c}(\pi^*)$. Hence, using Lemma 3.6 to bound the number of moves, we obtain:

Corollary 5.2. Given reachable configurations $s, t \in \mathbf{F}$ and $\varepsilon \in (0, 1]$, there exists a decoupled, kissing, (c_1/ε) -tame (s, t)-plan π with $\mathfrak{e}(\pi) \leq (1+\varepsilon)\mathfrak{e}(\pi^*)$ and $\alpha(\pi) \leq c_2(\mathfrak{e}(\pi^*)+1)$, where $c_1 \geq \Delta_0$ and $c_2 > 0$ are absolute constants that do not depend on ε .

Let π be an optimal, decoupled, kissing (s, t)-plan. By Corollary 4.6, we can assume that no robot is parked inside the sanctum of a corridor. Let $\ell := \alpha(\pi)$ and let $(R_1, \pi_1, p_1), \ldots, (R_\ell, \pi_\ell, p_\ell)$ be the sequence of moves of π . Let i (resp., j), $1 < i \leq j < \ell$, be the smallest (resp., largest) index such that p_i, p_j are $(\Delta - 4)$ -far, i.e., p_i (resp. p_j) is the first (resp. last) $(\Delta - 4)$ -far parking place in π . If there are no such indices, then π is Δ -tame and we are done. So suppose i, j exist. Note that it can be that i = j. By the definitions of corridors and sanctums, p_i and p_j do not lie inside a corridor $K \in \mathcal{K}$ because any point in $K \setminus K^S$ is $(\Delta_0 - 4)$ -close, p_i, p_j are $(\Delta - 4)$ -far, and $\Delta \geq \Delta_0$. Therefore there is a revolving area around each of p_i and p_j by Lemma 2.3.

The proof of Lemma 5.1 is based on the following observation, which is proved in Lemma 5.8. Let $\mathbf{s} = (s_A, s_B), \mathbf{t} = (t_A, t_B) \in \mathbf{F}$ be reachable kissing configurations with the property that there exist $r^-, r^+ \in \mathcal{F}$ such that $s_A, s_B \in \operatorname{RA}(r^-), t_A, t_B \in \operatorname{RA}(r^+)$, and r^-, r^+ are 3-far. Then there exists a decoupled, kissing (\mathbf{s}, \mathbf{t}) -plan $\tilde{\pi}$ with $\mathfrak{e}(\tilde{\pi}) \leq \varrho(s_A, t_A) + \varrho(s_B, t_B) + O(1)$ and $\alpha(\tilde{\pi}) = O(1)$, and all parking places in $\tilde{\pi}$ lie in $\operatorname{RA}(r^-)$ or $\operatorname{RA}(r^+)$. Since π is a kissing plan, there are kissing configurations $q = (q_A, q_B)$ and $q' = (q'_A, q'_B)$ on moves i and j. If q_A, q'_A, q_B, q'_B each is $(\Delta - 2)$ -close and lies in a revolving area then Lemma 5.1 follows from Lemma 5.8 but we may not be so lucky— q_A or q'_A may not be $(\Delta - 2)$ -close or may not lie in revolving areas, so the proof is much more involved.

The proof of Lemma 5.8, however, can be slightly adapted to prove the following variant:

Lemma 5.3. Let $\boldsymbol{u} = (u_A, u_B), \boldsymbol{v} = (v_A, v_B) \in \mathbf{F}$ be two configurations such that there exist four points $\overline{u}_A, \overline{u}_B, \overline{v}_A, \overline{v}_B \in \mathcal{F}$ with $u_A \in \operatorname{RA}(\overline{u}_A), u_B \in \operatorname{RA}(\overline{u}_B), v_A \in \operatorname{RA}(\overline{v}_A), v_B \in \operatorname{RA}(\overline{v}_B),$ and $\overline{u}_A, \overline{u}_B, \overline{v}_A, \overline{v}_B$ are 3-far, then there exists a decoupled $(\boldsymbol{u}, \boldsymbol{v})$ -plan $\widetilde{\boldsymbol{\pi}}$ with $\boldsymbol{e}(\widetilde{\boldsymbol{\pi}}) \leq \varrho(u_A, u_B) + \varrho(v_A, v_B) + 78$, $\alpha(\widetilde{\boldsymbol{\pi}}) \leq 40$, and all parking places of $\widetilde{\boldsymbol{\pi}}$ lie in the four revolving areas.

Using Lemma 5.3, we can prove the following weaker version of Lemma 5.1, which guarantees that $\tilde{\pi}$ is Δ -tame but does not guarantee the kissing property.

Lemma 5.4. Given reachable configurations $s, t \in \mathbf{F}$, let π be a decoupled, kissing (s, t)-plan. For any parameter $\Delta \geq \Delta_0$, there exists a decoupled Δ -tame (s, t)-plan $\tilde{\pi}$ such that $\tilde{\pi} = \pi$ if $\mathfrak{e}(\pi) \leq \Delta$ and $\mathfrak{e}(\tilde{\pi}) \leq \mathfrak{e}(\pi) + c_1$ and $\alpha(\tilde{\pi}) \leq \alpha(\pi) + c_2$ otherwise, for some absolute constants $c_1, c_2 > 0$ independent of Δ .

Proof. Let p_i, p_j be as defined above. Suppose $R_i = B$, i.e., A moves from p_{i-2} to p_i in the (i-1)-st move along π_{i-1} and is parked at p_i , then B moves from p_{i-1} to p_{i+1} along π_i in the *i*-th move. Let u_A be the last point along π_{i-1} that is $(\Delta - 4)$ -close, i.e., $d_{\infty}(\pi_{i-1}[u_A, p_i], X) \ge \Delta - 4$. Recall

that p_{i-2} is $(\Delta - 4)$ -close. Note that u_A may be p_{i-2} or p_i , and u_A is $(\Delta - 4)$ -tight. Since p_i does not lie in a corridor, we claim that u_A also does not lie inside a corridor. Indeed if $u_A \in K$ for some $K \in \mathcal{K}$, then A exits K at some point $\xi \in \pi_{i-1}[u_A, p_i]$ but then $d_{\infty}(\xi, \mathsf{X}) < \Delta_0 - 4 \leq \Delta - 4$, contradicting that u_A is the last $(\Delta - 4)$ -close point on π_{i-1} . Since u_A does not lie in a corridor, by Lemma 2.3, there is a $(\Delta - 5, \Delta - 3)$ -close point $\overline{u}_A \in \mathcal{F}$ such that $u_A \in \operatorname{RA}(\overline{u}_A)$.

Next, *B* kisses *A* parked at p_i during the *i*-th move. Since p_i is $(\Delta - 4)$ -far, π_i contains a $(\Delta - 6)$ -far point. If $\pi_i \cap (u_A + 2\Box) = \emptyset$, let u_B be the last $(\Delta - 6)$ -close point on π_i if there exists one and $u_B = p_{i-1}$ otherwise (i.e., all points on π_i are $(\Delta - 6)$ -far). Then u_B is $(\Delta - 6)$ -tight. On the other hand, if $\pi_i \cap (u_A + 2\Box) \neq \emptyset$, let u_B be the first intersection point of π_i with $u_A + 2\Box$, i.e., $\pi_i[p_{i-1}, u_B] \cap \operatorname{int}(u_A + 2\Box) = \emptyset$. Since u_A is $(\Delta - 4)$ -tight, u_B is $(\Delta - 6, \Delta - 2)$ -close.

Since p_i and u_A are not inside a corridor, a similar argument as above implies that u_B is also not in a corridor. Therefore there exists a $(\Delta - 7, \Delta - 1)$ -close point \overline{u}_B such that $u_B \in \operatorname{RA}(\overline{u}_B)$. Set $\boldsymbol{u} = (u_A, u_B)$.

Without loss of generality, assume that $R_j = B$. Then using a symmetric argument, we find points $v_A \in \pi_{j+1}$ such that $v_A \in \operatorname{RA}(\overline{v}_A)$ and v_A is $(\Delta - 4)$ -close, and $v_B \in \pi_j$ such that v_B is $(\Delta - 6, \Delta - 4)$ -close and $v_B \in \operatorname{RA}(\overline{v}_B)$, for some $(\Delta - 7, \Delta - 1)$ -close points $\overline{v}_A, \overline{v}_B \in \mathcal{F}$. Set $\boldsymbol{v} = (v_A, v_B)$.

Since $\overline{u}_A, \overline{u}_B, \overline{v}_A, \overline{v}_B$ each is $(\Delta - 7)$ -far and $\Delta - 7 \ge \Delta_0 - 7 \ge 3$, each is $(3, \Delta - 1)$ -close. Let $\boldsymbol{\psi} = (\psi_A, \psi_B)$ be the decoupled $(\boldsymbol{u}, \boldsymbol{v})$ -plan according to Lemma 5.3, with $\langle \boldsymbol{\psi} \rangle = (S_1, \psi_1, q_1), \ldots, (S_h, \psi_h, q_h)$. We obtain a new $(\boldsymbol{s}, \boldsymbol{t})$ -plan $\widetilde{\boldsymbol{\pi}}$ by replacing $\pi_A[u_A, v_A]$ and $\pi_B[u_B, v_B]$ with ψ_A and ψ_B , respectively. More precisely,

$$\langle \widetilde{\boldsymbol{\pi}} \rangle = (R_1, \pi_1, p_1), \dots, (R_{i-2}, \pi_{i-2}, p_{i-2}), (A, \pi_{i-1}[p_{i-2}, u_A], p_{i-1}), (B, \pi_i[p_{i-1}, u_B], u_A) \circ \langle \boldsymbol{\psi} \rangle \circ (B, \pi_j[v_B, p_{j+1}], v_A), (A, \pi_{j+1}[v_A, p_{j+2}], p_{j+1}), (R_{j+2}, \pi_{j+2}, p_{j+2}), \dots, (R_\ell, \pi_\ell, p_\ell).$$

It is easily seen that $\widetilde{\pi}$ is a (feasible) (s, t)-plan. By Lemma 5.3, all parking places in ψ and thus in $\widetilde{\pi}$ are Δ -close, $\mathfrak{e}(\widetilde{\pi}) \leq \mathfrak{e}(\pi) + 78$, and $\alpha(\widetilde{\pi}) \leq \alpha(\pi) + 40$.

A similar argument as for Corollary 5.2, but using Lemma 5.4, implies the following corollary.

Corollary 5.5. Given reachable configurations $s, t \in \mathbf{F}$ and $\varepsilon \in (0, 1]$, there exists a decoupled (c_1/ε) -tame (s, t)-plan π with $\mathfrak{e}(\pi) \leq (1 + \varepsilon)\mathfrak{e}(\pi^*)$ and $\alpha(\pi) \leq c_2(\mathfrak{e}(\pi^*) + 1)$, where $c_1, c_2 > 0$ are absolute constants that do not depend on ε .

Returning to the proof of Lemma 5.1, we first briefly sketch the idea. Let $\lambda_i \in [0, 1]$ (resp., $\lambda_j \in [0, 1]$) be the earliest (resp., latest) time during the move i (resp., j) such that $\pi(\lambda_i)$ (resp., $\pi(\lambda_j)$) is a kissing configuration; there exists such a value since π is kissing. If $R_i = B$ then $\pi_A(\lambda_i) = p_i$ and $\pi_B(\lambda_i) \in \pi_i$, and $\pi_A(\lambda_i) \in \pi_i$ and $\pi_B(\lambda_i) = p_i$ otherwise; the same holds for λ_j . We similarly define λ_{i-1} (resp., λ_{j+1}) to be the latest (resp., earliest) time during the move i - 1 (resp., j + 1) such that $\pi(\lambda_{i-1})$ (resp., $\pi(\lambda_{j+1})$) is a kissing configuration; if no such configuration exists, then i - 1 = 1 (resp., $j + 1 = \ell$) and we set $\lambda_{i-1} = 0$ (resp., $\lambda_{j+1} = 1$). Then $0 \leq \lambda_{i-1} \leq \lambda_i \leq \lambda_j \leq \lambda_{j+1} \leq 1$. If i = j then $\pi_A(\lambda_i, \lambda_j)$ and $\pi_B(\lambda_i, \lambda_j)$ are points. For $0 \leq r \leq 3$, let $a_r \coloneqq \pi_A(\lambda_{i-1+r})$ and $b_r \coloneqq (\lambda_{i-1+r})$. Without loss of generality, $R_{i-1} = A$ and $R_i = B$, so A moves first from $a_0 = p_{i-2}$ to $a_1 = p_i$ then B moves from b_0 to b_1 in the given motion plan $\pi(\lambda_i, \lambda_j)$. The proof of Lemma 5.1 is divided into two cases:



Figure 15. Illustration of the plan described in the proof of Lemma 5.6, where s_A, t_B lie on the left edge of RA(q) and t_A, s_B lie on the right edge of RA(q).

- (i) There exists a $(\Delta 6)$ -close point on $\pi_A[\lambda_i, \lambda_j]$ or $\pi_B[\lambda_i, \lambda_j]$, say, on $\pi_A[\lambda_i, \lambda_j]$. In this case, we find two $(\Delta 6)$ -close points q^-, q^+ on $\pi_A[\lambda_i, \lambda_j]$ and modify $\pi_A[\lambda_{i-1}, \lambda_{j+1}]$ and $\pi_B[\lambda_{i-1}, \lambda_{j+1}]$, using the above observation, so that A and B are parked at Δ -close points near a_0, b_0, a_3, b_3, q^- , or q^+ and they lie in revolving areas. The surgery on π_A, π_B increases their lengths by O(1) and adds O(1) new alternations (Section 5.2).
- (ii) There is no $(\Delta 6)$ -close point on $\pi_A[\lambda_i, \lambda_j]$ or $\pi_B[\lambda_i, \lambda_j]$. In this case, we find Δ -close parking places in the vicinity of $\pi_A[\lambda_{i-1}, \lambda_i]$, $\pi_B[\lambda_{i-1}, \lambda_i]$, $\pi_A[\lambda_j, \lambda_{j+1}]$, and $\pi_B[\lambda_j, \lambda_{j+1}]$ and again modify $\pi_A[\lambda_{i-1}, \lambda_{j+1}]$ and $\pi_B[\lambda_{i-1}, \lambda_{j+1}]$. We cannot always guarantee the existence of revolving areas that contain parking places. Therefore the surgery as well as the analysis is more involved. Nevertheless, we are able to argue that the increase in the cost of the plan and in the number of alternations is O(1) (Section 5.3).

5.1 Auxiliary lemmas

We next prove a sequence of lemmas for the that show the existence of near-optimal plans so that parking places lie in revolving areas containing initial or final placements. For simplicity, we do not try to minimize the error terms. They are used heavily when proving Lemma 5.1. For a point $q \in \partial \text{RA}(p)$, we define $\operatorname{anti}(q, p) \coloneqq (2x(p) - x(q), 2y(p) - y(q))$ as the point of intersection of $\partial \text{RA}(p)$ with the open ray emanating from q towards p. Note that $||q - \operatorname{anti}(q, p)||_{\infty} = 2$. We may omit the second parameter of $\operatorname{anti}(\cdot, \cdot)$ when it is clear from context.

Lemma 5.6. For any revolving area RA := $q \oplus \Box \subseteq \mathcal{F}$ and configurations $s = (s_A, s_B), t = (t_A, t_B) \in \mathbf{F}$ with $s_A, s_B, t_A, t_B \in \mathbf{RA}$, there exists a kissing (s, t)-plan π such that all parking places lie in RA, $\pi_A, \pi_B \subset \mathbf{RA}, \ \epsilon(\pi) \leq 12$, and $\alpha(\pi) \leq 8$.

Proof. Let π_A be the shortest (s_A, t_A) -path on ∂RA ; $\mathfrak{c}(\pi_A) \leq 4$. Without loss of generality, assume s_A lies on the left edge of RA, s_B lies on the right edge of RA, and π_A traces ∂RA from s_A to t_A



Figure 16. Illustration of the plan described in the proof of Lemma 5.7. Only the first part, from (s_A, s_B) to $(v^-, \operatorname{anti}(v^-))$, is shown. A, centered at v^- and depicted as blue, kisses B, centered at $\operatorname{anti}(v^-)$ and depicted as red.

in clockwise direction. See Figure 15. If t_A lies on the same edge as s_A , then t_B lies on the same edge as t_B and we move A to t_A and B to t_B with total length at most 4. Otherwise, we make A, B y-separated by moving A up to the top-left vertex v_0 of RA (above s_A) and moving B down to the bottom-right vertex anti (v_0) of RA (below s_B). This consists of two moves with total length at most 4.

If t_A lies on the top edge of RA, t_B lies on the bottom edge of RA and we move A right to t_A and B left to t_B . This consists of two more moves with total length of at most 4, so the overall length is at most 8. Otherwise, t_A, t_B lie on the right and left edges of RA, respectively, and we move A right to the top-right vertex v_1 of RA then move B to the bottom-left vertex anti (v_1) of RA for total length 4. Finally, we move A down to t_A and move B up to t_B with total length 4. The overall length is 12 in this case there are at most six moves.

The following lemma proves that if one robot lies in a revolving area far enough from the set of vertices X, there is a simple near-optimal kissing plan that moves the other robot between any two points that do not lie in the interior of the revolving area.

Lemma 5.7. Let $r \in \mathcal{F}$ be a 3-far point such that $\operatorname{RA} := \operatorname{RA}(r) \subseteq \mathcal{F}$. Let $s_B, t_B \in \operatorname{RA}$ and $(s_A, s_B), (t_A, t_B) \in \mathbf{F}$ such that s_A, t_A lie in the same component of \mathcal{F} . Then there exists a kissing $((s_A, s_B), (t_A, t_B))$ -plan π such that all parking places lie in $\operatorname{RA}, \mathfrak{e}(\pi) \leq \varrho(s_A, t_A) + 24$, and $\alpha(\pi) \leq 14$.

Proof. Let P be a shortest (s_A, t_A) -path in \mathcal{F} . If $P \subset \mathcal{F}[s_B]$, then define π as the simple plan where A moves directly from s_A to t_A along P, and then moves B from s_B to t_B along the shortest path in $(r + 2\Box) \cap \mathcal{F}[t_A]$ (such a path must exist). In this case $\mathfrak{c}(\pi) \leq \varrho(s_A, s_B) + 4$ and $\alpha(\pi) \leq 2$ and we are done. So suppose otherwise.

Let C be the component of $(r + 3\Box) \cap \mathcal{F}$ containing r (and s_B, t_B). Since $s_B, t_B \in \operatorname{RA}(r)$, $s_B + 2\Box, t_B + 2\Box \subset \mathcal{C}$. By assumption, P intersects $\operatorname{int}(s_B + 2\Box)$, so P intersects $\operatorname{int}(C)$. Let $c^$ and c^+ be the first and last points on P such that $c^-, c^+ \in C$. Note that c^-, c^+ may be s_A, t_A , respectively. Let v^-, v^+ be the closest vertices of ∂RA to c^-, c^+ , respectively. See Figure 16. Note that $s_A, t_A \notin int(RA)$ since $(s_A, s_B), (t_A, t_B) \in \mathbf{F}$ and $s_B, t_B \in RA$.

Let π_0 be the kissing $((s_A, s_B), (v^-, \operatorname{anti}(v^-)))$ -plan where we move B to $\operatorname{anti}(v^-)$ along segment $s_B\operatorname{anti}(v^-)$ then move A to v^- along $P[s_A, c^-] \| c^- v^-$. Then $\mathfrak{e}(\pi_0) \leq \mathfrak{e}(P[s_A, c^-]) + 4\sqrt{2}$ and $\alpha(\pi_0) \leq 3$. A similar construction gives a kissing $((v^+, \operatorname{anti}(v^+)), (t_A, t_B))$ -plan π_2 with $\mathfrak{e}(\pi_2) \leq \mathfrak{e}(P[c^+, t_A]) + 4\sqrt{2}$ and $\alpha(\pi_2) \leq 3$. Next, let π_1 be the kissing $((v^-, \operatorname{anti}(v^-)), (v^+, \operatorname{anti}(v^+)))$ -plan from Lemma 5.6 with $\alpha(\pi_1) \leq 8$ and $\mathfrak{e}(\pi_1) \leq 12$. Putting everything together, $\pi \coloneqq \pi_1 \circ \pi_2 \circ \pi_3$ is a kissing $((s_A, s_B), (t_A, t_B))$ -plan with

$$\mathfrak{e}(\pi) \le \mathfrak{e}(P[s_A, c^-]) + \mathfrak{e}(P[c^+, t_A]) + 8\sqrt{2} + 12 \le \varrho(s_A, t_A) + 24$$

3 + 8 \le 14.

and $\alpha(\pi) \le 3 + 3 + 8 \le 14$.

With the previous lemma in hand, we describe simple near-optimal kissing plans that move both robots from one revolving area to another, provided they are far enough from vertices of \mathcal{F} .

Lemma 5.8. Let $r_1, r_2 \in \mathcal{F}$ be 3-far points such that $\operatorname{RA}_1 \coloneqq \operatorname{RA}(r_1), \operatorname{RA}_2 \coloneqq \operatorname{RA}(r_2) \subset \mathcal{F}$. Let $s = (s_A, s_B), t = (t_A, t_B) \in \mathbf{F}$ be two kissing configurations such that $s_A, s_B \in \operatorname{RA}_1$ and $t_A, t_B \in \operatorname{RA}_2$. Then there exists a kissing (s, t)-plan π such that all parking places lie in RA_1 or RA_2 ,

$$\mathfrak{e}(\boldsymbol{\pi}) \leq \varrho(s_A, t_A) + \varrho(s_B, t_B) + 78,$$

and $\alpha(\boldsymbol{\pi}) \leq 40$.

Proof. Let $v_1 \in \partial \operatorname{RA}_1$ and $v_2 \in \partial \operatorname{RA}_2$ be any two points such that $||v_1 - v_2||_{\infty} \geq 2$. (Note that v_1, v_2 must exist.) Then $(v_1, v_2) \in \mathbf{F}$. Let π_0, π_3 be the kissing $((s_A, s_B), (v_1, \operatorname{anti}(v_1, r_1)))$ -plan and $((\operatorname{anti}(v_2, r_2), v_2), (t_A, t_B))$ -plan by Lemma 5.6, respectively. Next, let π_1, π_2 be the kissing $((v_1, \operatorname{anti}(v_1, r_1)), (v_1, v_2))$ -plan and $((v_1, v_2), (\operatorname{anti}(v_1, r_2), v_2)$ -plan by Lemma 5.7, respectively. Putting everything together, $\pi \coloneqq \pi_0 \circ \pi_1 \circ \pi_2 \circ \pi_3$ is a kissing $((s_A, s_B), (t_A, t_B))$ -plan with

$$\begin{aligned} \mathfrak{e}(\boldsymbol{\pi}) &= \sum_{i=0}^{3} \mathfrak{e}(\boldsymbol{\pi}_{i}) \leq 12 + (\varrho(\operatorname{anti}(v_{1}, r_{1}), v_{2}) + 24) + (\varrho(v_{1}, \operatorname{anti}(v_{1}, r_{2})) + 24) + 12 \\ &\leq \varrho(\operatorname{anti}(v_{1}, r_{1}), v_{2}) + \varrho(v_{1}, \operatorname{anti}(v_{1}, r_{2})) + 72 \\ &\leq \varrho(s_{A}, t_{A}) + \varrho(s_{B}, t_{B}) + 78, \end{aligned}$$

where the last inequality follows because $\rho(p,q) = \sqrt{2} < 3/2$ for any two points p,q on the boundary of a revolving area. Finally,

$$\alpha(\pi) = \sum_{i=0}^{3} \alpha(\pi_i) \le 6 + 14 + 14 + 6 = 40.$$

The following lemma shows that if the initial and final configurations are kissing configurations, the paths traversed by the robots in any optimal plan have similar lengths.

Lemma 5.9. Let $s, t \in \mathbf{F}$ be two kissing configurations such that s_A, s_B, t_A, t_B lie in revolving areas $\operatorname{RA}(s_A), \operatorname{RA}(s_B), \operatorname{RA}(t_A), \operatorname{RA}(t_B)$, respectively, where $r_{s_A}, r_{s_B}, r_{t_A}, r_{t_B} \in \mathcal{F}$ each is 3-far, and let π be an optimal (s, t)-plan. Then $|\mathfrak{q}(\pi_A) - \mathfrak{q}(\pi_B)| \leq 150$.



Figure 17. Abstract diagram of the paths $\pi_A[a_0, a_3], \pi_B[b_0, b_3]$ and various points as defined in Section 5.2. The thick pathlets are $(\Delta - 6)$ -far, the black dashed lines represent segments in \mathcal{F} between kissing configurations, and the grey dashed squares are revolving areas. The dotted red path represents the path of B followed during the new plan.

Proof. Suppose not. Without loss of generality, $\mathfrak{c}(\pi_A) < \mathfrak{c}(\pi_B) - 150$. Since $r_{s_A}, r_{s_B}, r_{t_A}, r_{t_B}$ are 3-far, we have that segments $r_{s_A}r_{s_B}$ and $r_{t_A}r_{t_B}$ lie in \mathcal{F} . Let π_0 be the $((s_A, s_B), (s_A, \operatorname{anti}(s_A, r_{s_A}))$ -plan from Lemma 5.7, let π_1 be the $((s_A, \operatorname{anti}(s_A, r_{s_A})), (t_A, \operatorname{anti}(t_A, r_{t_A})))$ -plan from Lemma 5.8, and let π_2 be the $((t_A, \operatorname{anti}(t_A, r_{t_A})), (t_A, t_B))$ -plan from Lemma 5.7. Then $\pi' \coloneqq \pi_0 \circ \pi_1 \circ \pi_2$ is a (s, t)-plan with

$$\begin{aligned} \mathbf{e}(\mathbf{\pi}') &= \sum_{i=0}^{2} \mathbf{e}(\mathbf{\pi}_{i}) \\ &= (\varrho(s_{B}, \operatorname{anti}(s_{A}, r_{s_{A}}))) + 24) \\ &+ (\varrho(s_{A}, t_{A}) + \varrho(\operatorname{anti}(s_{A}, r_{s_{A}}), \operatorname{anti}(t_{A}, r_{t_{A}})) + 78) \\ &+ (\varrho(\operatorname{anti}(t_{A}, r_{t_{A}}), t_{B}) + 24) \\ &\leq (|s_{B}\operatorname{anti}(s_{A}, r_{s_{A}})| + |s_{A}s_{B}|) \\ &+ 2\varrho(s_{A}, t_{A}) + |s_{A}\operatorname{anti}(s_{A}, r_{s_{A}})| + |t_{A}\operatorname{anti}(t_{A}, r_{t_{A}})| \\ &+ (|t_{A}t_{B}| + |\operatorname{anti}(t_{A}, r_{t_{A}})t_{A}|) + 126 \\ &\leq 9 + 2(\mathbf{e}(\mathbf{\pi}_{A})) + 3 + 3 + 9 + 126 < 2\mathbf{e}(\mathbf{\pi}_{A}) + 150 \end{aligned}$$

Then, by assumption, we have $\mathfrak{c}(\pi') < \mathfrak{c}(\pi_A) + \mathfrak{c}(\pi_B) = \mathfrak{c}(\pi)$, which contradicts the optimality of π .

5.2 Case (i): Existence of a $(\Delta - 6)$ -close point on $\pi_A(\lambda_i, \lambda_j)$ or $\pi_B(\lambda_i, \lambda_j)$.

We next prove case (i) of Lemma 5.1. For concreteness, suppose there is a $(\Delta - 6)$ -close point on $\pi_A(\lambda_i, \lambda_j)$; the other case is similar. Let q^- (resp., q^+) be the first (resp., last) point on $\pi_A(\lambda_i, \lambda_j)$ which is $(\Delta - 6)$ -far and thus $(\Delta - 6)$ -tight. Then there are revolving areas RA⁻ := $r^- + \Box$ and RA⁺ := $r^+ + \Box$ containing q^-, q^+ respectively, for points $r^-, r^+ \in \mathcal{F}$. The high-level idea for the plan is to move A, B from (a_0, b_0) into RA⁻, then move both to RA⁺, then finally move both to

 (a_3, b_3) . All parking places during the plan are near $a_0, b_0, a_3, b_3, q^-, q^+$, so they are Δ -close, as desired.

Let a'_1 be the first point on $\pi_A[a_1, q^-] \cap RA^-$, and set $b'_1 \coloneqq \operatorname{anti}(a'_1, r^-)$, and define $a'_2, b'_2 \in RA^+$ similarly. r^-, r^+ are $(\Delta - 7, \Delta - 5)$ -close, so a'_1, b'_1, a'_2, b'_2 are $(\Delta - 8, \Delta - 4)$ -close. We next describe a decoupled, kissing $((a_0, b_0), (a'_1, b'_1))$ -plan π_0 . There are two cases.

- 1. Suppose $\pi_A[a_1, q^-] \subset \mathcal{F}[b_0]$. Since $R_{i-1} = A$, $\pi_A[a_0, a_1] \subset \mathcal{F}[b_0]$, and hence $\pi_A[a_1, q^-] \subset \mathcal{F}[b_0]$. Let a'_1 be the first point on $\pi_A[a_1, q^-] \cap \operatorname{RA}^-$, and set $b'_1 \coloneqq \operatorname{anti}(a'_1, r^-)$. See Figure 17. In this case, we first move A from a_0 directly to a'_1 in a single move. Then we move B from b_0 to b'_1 using the decoupled, kissing plan from Lemma 5.7. Let π_0 be the resulting decoupled, kissing $((a_0, b_0), (a'_1, b'_1))$ -plan.
- 2. Otherwise, $\pi_A[a_1, q^-] \not\subset \mathcal{F}[b_0]$. Then $\pi_A[a_1, q^-]$ intersects the interior of $b_0 + 2\Box$. Since all points on $\pi_A[a_1, q^-]$ are $(\Delta 6)$ -far, b_0 is $(\Delta 4)$ -far. Furthermore, $b_0 = p_{i-1}$ is $(\Delta 6)$ -close by definition of p_i ; b_0 is $(\Delta 8, \Delta 4)$ -close. Let (a^*, b^*) be a decoupled, kissing configuration on the revolving area containing b_0 . Then a^*, b^* are $(\Delta 10, \Delta 2)$ -close. Let π_0 be the decoupled, kissing $((a_0, b_0), (a'_1, b'_1))$ -plan obtained by first applying the kissing $((a_0, b_0), (a^*, b^*))$ -plan from Lemma 5.7 then applying the $((a^*, b^*), (a'_1, b'_1))$ -plan from Lemma 5.8.

It is easy to verify that, in either case, all parking places in π_0 except a_0, b_0, a'_1, a'_1 are $(\Delta - 10, \Delta - 2)$ -close, $\mathfrak{c}(\pi_0) \leq \mathfrak{c}(\pi_A[a_0, a_1]) + \mathfrak{c}(\pi_B[b_0, b_1]) + 2\mathfrak{c}(\pi_A[a_1, a'_1]) + 200$ and $\alpha(\pi_0) \leq 40$. Similarly, we construct a decoupled, kissing $((a'_2, b'_2), (a_3, b_3))$ -plan π_2 where all parking places except a'_2, b'_2, a_3, b_3 are $(\Delta - 10, \Delta - 2)$ -close, $\mathfrak{c}(\pi_2) \leq \mathfrak{c}(\pi_A[a_2, a_3]) + \mathfrak{c}(\pi_B[b_2, b_3]) + 2\mathfrak{c}(\pi_A[a'_2, a_2]) + 200$ and $\alpha(\pi_0) \leq 40$. Let π_1 be the decoupled, kissing $((a'_1, b'_1), (a'_2, b'_2))$ -plan from Lemma 5.8 with

$$\mathfrak{e}(\boldsymbol{\pi}_1) \le \varrho(a_1', a_2') + \varrho(b_1', b_2') + 78 \le 2\mathfrak{e}(\boldsymbol{\pi}_A[a_1', a_2']) + 150$$

and $\alpha(\pi_1) \leq 40$. Then $\pi' \coloneqq \pi_0 \circ \pi_1 \circ \pi_2$ is a decoupled, kissing $((a_0, b_0), (a_3, b_3))$ -plan with

$$\mathfrak{e}(\boldsymbol{\pi}') \leq \mathfrak{e}(\pi_A[a_0, a_3]) + \mathfrak{e}(\pi_B[b_0, b_1]) + \mathfrak{e}(\pi_A[a_1, a_2]) + \mathfrak{e}(\pi_B[b_2, b_3]) + 550 \\ \leq \mathfrak{e}(\pi_A[a_0, a_3]) + \mathfrak{e}(\pi_B[b_0, b_3]) + 750,$$

where the last inequality follows the fact $\mathfrak{c}(\pi_A[a_1, a_2]) \leq \mathfrak{c}(\pi_B[b_1, b_2]) + 150$ by Lemma 5.9. Furthermore, all parking places besides a_0, b_0, a_3, b_3 are $(\Delta - 2)$ -close. We replace $\pi(\lambda_i, \lambda_j)$ with π' in π , which completes the proof.

5.3 Case (ii): No $(\Delta - 6)$ -close point on $\pi_A(t_i, t_j)$ or $\pi_B(t_i, t_j)$.

Without loss of generality, $R_{i-1} = A$ and $R_i = B$. For concreteness, we assume $R_j = A$ and $R_{j+1} = B$; the other case is similar. Then $\pi_A[a_0, a_1] \subset \mathcal{F}[b_0]$ and $\pi_B[b_2, b_3] \subset \mathcal{F}[a_3]$.

The high-level idea is to try to follow the approach taken in the previous proof and find revolving areas centered at $(\Delta - c)$ -close points near $\pi_A[a_0, a_1]$ and $\pi_B[b_2, b_3]$ that B and A can reach, respectively, "without straying too far" from their original paths in π , for a sufficiently large constant c. If we find such revolving areas, we first move A then B to the former revolving area, then move them both to the latter revolving area, then finally move A then B to (a_3, b_3) . Otherwise, if we are unable to find such revolving areas, we instead find and use a sequence of kissing configurations whose points may not be in revolving areas. We need to be more careful when choosing such configurations, as not only must they be Δ -close and kissing, we must be able to



Figure 18. Abstract diagram of the paths $\pi_A[a_0, a_3], \pi_B[b_0, b_3]$ and various points as defined in Section 5.3 when p_B^+ is *distant*. The thick pathlets are $(\Delta - 6)$ -far, the black dashed lines represent segments in \mathcal{F} between kissing configurations, and the grey dashed square is a revolving area containing q_B^+ . The dotted blue path represents the path of A followed during the new plan; B mainly follows π_B .

move the robots to one from (a_0, b_0) , between them, and from one to (a_3, b_3) but without relying on the auxiliary lemmas for plans between configurations with at least one robot in a revolving area as done in the previous case.

Let $q_{\bar{A}}^-$ (resp., $q_{\bar{A}}^+$) be the last (resp., first) $(\Delta - 6)$ -close point on $\pi_A[a_0, a_1]$ (resp., on $\pi_A[a_2, a_3]$). Let $q_{\bar{B}}^-, q_{\bar{B}}^+$ be defined similarly. If $\pi_A[a_0, q_{\bar{A}}^-] \cap \operatorname{int}(\pi_B[b_0, q_{\bar{B}}^-] \oplus 2\Box) \neq \emptyset$, let $p_{\bar{A}}^-$ be the last point on $\pi_A[a_0, q_{\bar{A}}^-] \cap (\pi_B[b_0, q_{\bar{B}}^-] \oplus 2\Box)$ and let $k_{\bar{A}}^-$ be the last point on $\pi_B[b_0, q_{\bar{B}}^-]$ contained in $p_{\bar{A}}^- + 2\Box$; otherwise, $p_{\bar{A}}^-, k_{\bar{A}}^-$ are NIL. Define $p_{\bar{B}}^+, k_{\bar{B}}^+$ similarly: If $\pi_B[q_{\bar{B}}^-, b_3] \cap \operatorname{int}(\pi_A[q_{\bar{A}}^+, a_3] \oplus 2\Box) \neq \emptyset$, let $p_{\bar{B}}^+$ be the first point on $\pi_B[q_{\bar{B}}^+, b_3] \cap (\pi_A[q_{\bar{A}}^+, a_3] \oplus 2\Box)$ and let $k_{\bar{B}}^+$ be the last point on $\pi_A[q_{\bar{A}}^-, a_3]$ contained in $p_{\bar{B}}^+ + 2\Box$; otherwise, $p_{\bar{B}}^+, k_{\bar{B}}^+$ are NIL. Note that none of these points lie in sanctums of corridors by definition and the fact that none of a_i, b_i lie in sanctums, for i = 0, 1, 2, 3. Indeed, if $q_{\bar{A}}^-$ lies in the sanctum of a corridor K, then $\pi_A[q_{\bar{A}}^-, a_1]$ crosses the portals of K, and such crossings are within L_1 -distance 1 of the portal endpoints in X; the same holds for $q_{\bar{A}}^+, q_{\bar{B}}^-, q_{\bar{B}}^+$. If $p_{\bar{A}}^-$ lies in a sanctum K^S of a corridor K, then $k_{\bar{A}}^- \in K$ and $\pi_A[a_0, q_{\bar{A}}^-]$ and $\pi_B[b_0, q_{\bar{B}}^-]$ span K^S . Then $\pi_A[a_0, q_{\bar{A}}^-] \cap K^S \subset \operatorname{int}(\pi_B[b_0, q_{\bar{B}}^-] \oplus 2\Box)$, by definition of sanctum, so $p_{\bar{A}}^-$ lies outside K^S . A similar argument shows $k_{\bar{A}}^-, p_{\bar{B}}^+, k_{\bar{B}}^+$ do not lie in sanctums. Note that when $p_{\bar{A}}^-$ is NIL, a_0 may be s_A , and when $p_{\bar{B}}^+$ is NIL, b_3 may be t_B . See Figure 18. We say $p_{\bar{A}}^-$ or $p_{\bar{B}}^+$ is *distant* if it is not NIL and it is ($\Delta - 8$)-far. There are two main cases.

At least one of p_A^- , p_B^+ is distant. (This is the case that most closely resembles that of case (i).) Without loss of generality, p_B^+ is distant. We set (a'_3, b'_3) to be any kissing configuration where the points lie in (the boundary of) the revolving area centered at a $(\Delta - 7, \Delta - 5)$ -close point that contains q_B^+ . a'_3, b'_3 are $(\Delta - 8, \Delta - 4)$ -close. Let

$$P_A \coloneqq \pi_A[a_0, a_2] \| a_2 b_2 \| \pi_B[b_2, q_B^+] \| q_B^+ a_3'.$$

Note that $a_2b_2 \subset \mathcal{F}$ since a_2, b_2 are $(\Delta - 6)$ -far, and hence $P_A \subset \mathcal{F}$. See Figure 18 again. There are two cases.

1. Suppose $P_A \subset \mathcal{F}[b_0]$. Then we let π_0 be the decoupled, kissing $((a_0, b_0), (a'_3, b'_3)$ -plan by



Figure 19. Abstract diagram of the paths $\pi_A[a_0, a_3]$, $\pi_B[b_0, b_3]$ and various points as defined in Section 5.3 when neither p_A^-, p_B^+ are *distant*. In this example, p_A^- is not NIL and p_B^+ is NIL.

first moving A from a_0 to a'_3 along P_A , then applying the kissing $((a'_3, b_0), (a'_3, b'_3)$ -plan from Lemma 5.7 to move B from b_0 to b'_3 .

2. Otherwise, $P_A \not\subset \mathcal{F}[b_0]$. We have that the prefix $P[a_0, q_A^-] \subseteq \mathcal{F}[b_0]$ and that all points on the suffix $P_A[q_A^-, a_3']$ are $(\Delta - 6)$ -far, so if P_A intersects the interior of $b_0 + 2\Box$, it must intersect on $P_A[q_A^-, a_3']$. Then b_0 is $(\Delta - 8)$ -far, and hence is contained in a revolving area. We have that b_0 is $(\Delta - 6)$ -close by definition of p_i . Let (a^*, b^*) be a kissing configuration on the revolving area centered at a $(\Delta - 9, \Delta - 5)$ -close point that contains b_0 . a^*, b^* are $(\Delta - 10, \Delta - 4)$ -close. Let π_0 be the decoupled, kissing $((a_0, b_0), (a_3', b_3'))$ -plan obtained by first applying the decoupled, kissing $((a_0, b_0), (a^*, b^*))$ -plan from Lemma 5.7 then applying the $((a^*, b^*), (a_3', b_3'))$ -plan from Lemma 5.8.

In either case, we have a kissing $((a_0, b_0), (a'_3, b'_3))$ -plan π_0 where all parking places except a_0, b_0, a'_3, b'_3 are $(\Delta - 10, \Delta - 4)$ -close. Then let π_1 be the kissing $((a'_3, b'_3), (a_3, b_3))$ -plan where we first apply the kissing $((a'_3, b'_3), (a_3, q^+_B))$ -plan from Lemma 5.7 then move B from q^+_B to b_3 along $\pi_B[q^+_B, b_3]$. Then $\pi' := \pi_0 \circ \pi_1$ is a kissing $((a_0, b_0), (a_3, b_3))$ -plan with all parking places except a_0, b_0, a_3, b_3 being $(\Delta - 2)$ -close.

Let $(\pi'_A, \pi'_B) = \pi'$. An argument similar to that in the proof of Lemma 5.9 implies that

$$|\mathfrak{e}(\pi_A[a_2,k_B^+]) - \mathfrak{e}(\pi_B[b_2,p_B^+])| = O(1).$$

Similar to the proof of case (i), a tedious but straightforward analysis that incorporates each upper bound on the costs and number of moves from the O(1) applications of Lemmas 5.7 and 5.8 implies that

$$\begin{aligned} \mathbf{c}(\boldsymbol{\pi}') &= \mathbf{c}(\pi'_A) + \mathbf{c}(\pi'_B) \\ &\leq \left(\mathbf{c}(\pi_A[a_0, a_2]) + \mathbf{c}(\pi_B[b_2, p_B^+]) + \mathbf{c}(\pi_A[k_B^+, a_2])\right) + \\ &\left(\mathbf{c}(\pi_B[b_0, b_2]) + \mathbf{c}(\pi_B[b_2, p_B^+]) + \mathbf{c}(\pi_B[p_B^+, b_3])\right) + O(1) \\ &\leq \left(\mathbf{c}(\pi_A[a_0, a_2]) + \mathbf{c}(\pi_A[a_2, k_B^+]) + \mathbf{c}(\pi_A[k_B^+, a_2])\right) + \\ &\left(\mathbf{c}(\pi_B[b_0, b_2]) + \mathbf{c}(\pi_B[b_2, p_B^+]) + \mathbf{c}(\pi_B[p_B^+, b_3])\right) + O(1) \\ &\leq \mathbf{c}(\pi_A[a_0, a_3]) + \mathbf{c}(\pi_B[b_0, b_3]) + O(1) \\ &\leq \mathbf{c}(\boldsymbol{\pi}(\lambda_i, \lambda_j)) \end{aligned}$$

for a constant $c_1 > c_0$. A similar bound $\alpha(\pi') \leq \alpha(\pi) + c_2$ for a constant $c_2 > 0$ follows. We replace $\pi(\lambda_i, \lambda_j)$ with π' in π , which completes the proof for this subcase.

Neither of p_A^-, p_B^+ are distant. Since p_A^- is not distant, either (i) p_A^- is NIL or (ii) p_A^- is $(\Delta - 8)$ -close. If p_A^- is NIL, let $(a'_0, b'_0) \coloneqq (a_0, b_0)$ and let π_0 be the trivial $((a_0, b_0), (a_0, b_0))$ -plan where neither robot moves. Otherwise, let $(a'_0, b'_0) \coloneqq (p_A^-, k_A^-)$ and let π_0 be the $((a_0, b_0), (p_A^-, k_A^-))$ -plan obtained by moving A from a_0 to p_A^- along $\pi_A[a_0, p_A^-]$ then moving B from b_0 to k_A^- along $\pi_B[b_0, k_A^-]$. In either case, π_0 is a kissing $((a_0, b_0), (a'_0, b'_0))$ -plan and a'_0, b'_0 are $(\Delta - 6)$ -close. We similarly define (a'_3, b'_3) and a kissing $((a'_3, b'_3), (a_3, b_3))$ -plan π_2 . See Figure 19. It remains to define a kissing $((a'_0, b'_0), (a'_3, b'_3))$ -plan, π_1 .

By the choice of a'_0, b'_0, a'_3, b'_3 , we have

- $\pi_A[a'_0, q_A^-] \subset \mathcal{F}[z]$ for any point $z \in \pi_B[b'_0, q_B^-]$,
- $\pi_A[q_A^+, a_3'] \subset \mathcal{F}[z]$ for any point $z \in \pi_B[q_B^+, b_3']$,
- $\pi_B[b'_0, q_B^-] \subset \mathcal{F}[z]$ for any point $z \in \pi_A[a'_0, q_A^-]$, and
- $\pi_B[q_B^+, b'_3] \subset \mathcal{F}[z]$ for any point $z \in \pi_A[q_A^+, a'_3]$.

Furthermore, we have that all points on $\pi_A[q_A^-, q_A^+]$ and $\pi_B[q_B^-, q_B^+]$ are $(\Delta - 6)$ -far. There are two cases.

- 1. Suppose $\pi_A[q_A^+, a_3'] \cap \operatorname{int}(\pi_B[b_0', q_B^-] \oplus 2\Box) = \varnothing$. Then $\pi_B[b_0', q_B^-] \subset \mathcal{F}[a_3']$. Furthermore, if $\pi_A[q_A^-, q_A^+]$ intersects the interior of $\pi_B[b_0', q_B^-]$, then b_0' is $(\Delta 6)$ -far, and hence is contained in a revolving area centered at a $(\Delta 7)$ -far point. Since b_0' is $(\Delta 6)$ -close, the center of the revolving area is $(\Delta 5)$ -close. So we either move A directly to a_3' from a_0' on $\pi_A[a_0', a_3']$ or we apply Lemma 5.7 if that path is not in $\mathcal{F}[b_0']$. Similarly, we either move B directly to b_3' from b_0' on $\pi_B[b_0', b_3']$ or we apply Lemma 5.7 if that path is not in $\mathcal{F}[a_3']$. Let π_1 be the resulting kissing $((a_0', b_0'), (a_3', b_3'))$ -plan. All parking places in π_1 except a_0', b_0', a_3', b_3' are $(\Delta 4)$ -close.
- 2. Otherwise, $\pi_A[q_A^+, a_3'] \cap \operatorname{int}(\pi_B[b_0', q_B^-] \oplus 2\Box) \neq \emptyset$. Let p_A^* be the first point on $\pi_A[q_A^+, a_3']$ such that $p_A^* \in \pi_B[b_0', q_B^-] \oplus 2\Box$, and let k_B^* be the last point on $\pi_B[b_0', q_B^-]$ such that $k_B^* \in p_A^* + 2\Box$. Then (p_A^*, k_B^*) is a kissing configuration, $\pi_A[q_A^+, p_A^*] \subset \mathcal{F}[b_0']$, and $\pi_B[k_B^-, q_B^-] \subset \mathcal{F}[p_A^*]$ Then we proceed similar to the earlier case where p_B^+ is distant: If p_A^* is $(\Delta - 8)$ -close we construct a kissing $((a_0', b_0'), (a_3', b_3'))$ -plan π_1 by applying Lemma 5.7 at most twice as necessary. Specifically, we move A to p_A^* on $\pi_A[a_0', p_A^*]$, possibly moving B within a revolving area containing b_0' if A collides with B, then move B to k_A^* on $\pi_B[b_0', b_3']$, possibly moving



Figure 20. Abstract diagrams of the paths $\pi_A[a'_0, a'_3], \pi_B[b'_0, b'_3]$ and various points as defined in Section 5.3 when q_A^-, q_B^+ are not distant and $\pi_A[q_A^+, a'_3]$ intersects the interior of $\pi_B[b'_0, q_B^-] \oplus 2\Box$. The thick pathlets are $(\Delta - 6)$ -far, the black dashed lines represent segments in \mathcal{F} between kissing configurations, and the grey dashed square is a revolving area containing q_A^+ . The dotted blue and red paths represent the paths of A and B followed during the new plan, respectively. (top) p^* is $(\Delta - 8)$ -close. (bottom) p^* is $(\Delta - 8)$ -far.

A within a revolving area containing a'_3 if B collides with A, and then finally move A to a'_3 on $\pi_A[p^*_A, a'_3]$. See Figure 20(top). Otherwise, p^*_A is $(\Delta - 8)$ -far, so we cannot park A there. Instead, we construct a kissing $((a'_0, b'_0), (a'_3, b'_3))$ -plan π_1 by first moving A then B to the revolving area containing q^+_A using a plan similar to that in case (1). See Figure 20(bottom). Then we move A to a'_3 using the plan from Lemma 5.7, move B to b'_3 along the reversal of $\pi_A[a_2, q^*_A]$ followed by $a_2b_2||\pi_B[b_2, b'_3]$, possibly moving A within a revolving area containing a'_3 if B collides with A. In either case, it can be verified that all parking places in π_1 except a'_0, b'_0, a'_3, b'_3 are $(\Delta - 2)$ -close.

In either case we have a $((a'_0, b'_0), (a'_1, b'_1))$ -plan π_1 where all parking places are $(\Delta - 4)$ -close. Let $(\pi^1_A, \pi^1_B) = \pi_1$. Suppose case (2) occurs. An argument similar to that in the proof of Lemma 5.9 implies that

$$|\mathfrak{e}(\pi_A[a_2, p_A^*]) - \mathfrak{e}(\pi_B[k_A^*, b_1])| = O(1).$$

Then again, in either case (1) or (2), a tedious but straightforward analysis that incorporates each upper bound on the costs and number of moves from the O(1) applications of Lemmas 5.7 and 5.8 implies that

$$\begin{aligned} \mathfrak{e}(\boldsymbol{\pi}') &= \mathfrak{e}(\pi_A^1) + \mathfrak{e}(\pi_B^1) \\ &\leq \left(\mathfrak{e}(\pi_A[a'_0, a_2]) + \mathfrak{e}(\pi_A[a_2, p_A^*]) + \mathfrak{e}(\pi_A[p_A^*, a'_3])\right) + \\ &\left(\mathfrak{e}(\pi_B[b'_0, k_B^*]) + \mathfrak{e}(\pi_A[a_2, p_A^*]) + \mathfrak{e}(\pi_B[b_2, b'_3])\right) + O(1) \\ &\leq \left(\mathfrak{e}(\pi_A[a'_0, a_2]) + \mathfrak{e}(\pi_A[a_2, p_A^*]) + \mathfrak{e}(\pi_A[p_A^*, a'_3])\right) + \\ &\left(\mathfrak{e}(\pi_B[b'_0, k_B^*]) + \mathfrak{e}(\pi_B[k_B^*, b_2]) + \mathfrak{e}(\pi_B[b_2, b'_3])\right) + O(1) \\ &\leq \mathfrak{e}(\pi_A[a'_0, a'_3]) + \mathfrak{e}(\pi_B[b'_0, b'_3]) + O(1). \end{aligned}$$

It is easy to verify that

$$\mathfrak{c}(\boldsymbol{\pi}_0) \le \mathfrak{c}(\pi_A[a_0, a_0']) + \mathfrak{c}(\pi_B[b_0, b_0']) + O(1) \text{ and } \alpha(\boldsymbol{\pi}_0) = O(1)$$

and

$$\mathfrak{e}(\pi_2) \le \mathfrak{e}(\pi_A[a'_3, a_3]) + \mathfrak{e}(\pi_B[b'_3, b_3]) + O(1) \text{ and } \alpha(\pi_2) = O(1).$$

Set $\pi' \coloneqq \pi_0 \circ \pi_1 \circ \pi_2$. It follows that $\mathfrak{e}(\pi') = \mathfrak{e}(\pi_A[a_0, a_3]) + \mathfrak{e}(\pi_B[b_0, b_3]) + O(1)$ and $\alpha(\pi) = O(1)$. We replace $\pi(\lambda_i, \lambda_j)$ with π' in π , which completes the proof for this subcase.

6 Discretizing the Free Space

In this section, we further transform the optimal tame plans described in the previous section by "retracting" all parking places to a discrete set of points. For any $\varepsilon \in (0, 1)$, let \mathbb{G} be the axis-aligned grid centered at the origin whose cells are ε -radius squares.

Let $\varepsilon \in (0,1)$ be a parameter. We show how to choose a set $\mathcal{V} \subset \mathbb{R}^2$ of $O(n(\Delta/\varepsilon)^2)$ points so that a decoupled, Δ -tame (s, t)-plan π can be deformed into another decoupled, $(\Delta + 2\varepsilon)$ -tame (s, t)-plan $\hat{\pi}$ such that (i) the robots are parked at points of \mathcal{V} , (ii) $\mathfrak{e}(\hat{\pi}) \leq \mathfrak{e}(\pi) + \varepsilon \alpha(\pi)$, and (iii) $\alpha(\hat{\pi}) = c\alpha(\pi)$, where c is an absolute constant that does not depend on ε .

Let π be a decoupled, Δ -tame (s, t)-plan. We can assume that $\alpha(\pi) = O(\mathfrak{c}(\pi) + 1)$. Let \mathbb{G} be the axis-aligned uniform grid with square cells of radius ε such that all parking places lie in the

interior of grid cells and π does not pass through a vertex of \mathbb{G} . Let $\mathcal{F}^{\#}$ be the overlay of \mathbb{G} and \mathcal{F} , restricted to \mathcal{F} . Each face of $\mathcal{F}^{\#}$ is a connected component of $\mathcal{F} \cap g$ for some grid cell g of \mathbb{G} . Let \mathcal{V} be the set of vertices of $\mathcal{F}^{\#}$. Our goal is to "retract" the parking places of π to the points of \mathcal{V} , i.e., the robots are parked at the points of \mathcal{V} instead of their original parking places. Furthermore, since π is kissing, we want to ensure that the retracted path is ε -nearly-kissing, i.e., whenever a robot moves, it comes within L_{∞} -distance 4ε of the boundary of the other robot (parked at a vertex of \mathcal{V}). However, if for a parking place q, say, of A, we pick only one point in \mathcal{V} to park A at instead of q, B may collide with A during its next move, especially since B kisses A at q during the next move of π . Hence, we may have to choose multiple points of \mathcal{V} (in the neighborhood of q) and move A between them during the next move of B, to ensure that A and B do not collide. Each such move of A increases the cost of the plan by $O(\varepsilon)$, so we cannot move A too many times. Furthermore, we want to maintain the property of being decoupled (i.e., only one robot moves at a time), which means that when we move A between nearby points of \mathcal{V} to make way for B, we must first park B somewhere, also in \mathcal{V} . These technical constraints make the retraction rather involved. We now describe the retraction in detail, but first state a lemma which follows easily from Lemma 3.3.

Lemma 6.1. Let π be a decoupled (s, t)-plan, and let ℓ be a horizontal or vertical line. During a single move of π , if the path P followed by a robot intersects ℓ at two points that are less than two distance apart, then P can be shortened without affecting the rest of the plan.

We assume that π satisfies Lemma 6.1. Assume that $\langle \pi \rangle = (R_1, \pi_1, p_1), \ldots, (R_k, \pi_k, p_k)$. Assume $p_0 = s_A, p_1 = s_B$, i.e., $R_1 = A$. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_{k-1}$ be the time instances at which π switches from move i to i + 1. Set $\lambda_0 \coloneqq 0$ and $\lambda_{k+1} \coloneqq 1$. Then for $0 \le i \le k$, $\pi(\lambda_i) = (p_i, p_{i+1})$ if i is even and $\pi(\lambda_i) = (p_{i+1}, p_i)$ if i is odd. We describe the retraction of each move one by one. For each move, the retracted plan consists of O(1) moves and increases the cost by $O(\varepsilon)$.

Let $e_0 \coloneqq (-1, 0), e_1 \coloneqq (0, -1), e_2 \coloneqq (1, 0)$, and $e_3 \coloneqq (0, 1)$ be the four standard directions; $e_i = -e_{i+2 \pmod{4}}$. Set $S = \{e_i \mid 0 \le i \le 3\}$. For an edge γ of an axis-aligned square, the inner normal of γ is one of the e_i 's, namely if γ is the left edge of the square then the inner normal is $e_2 = (1, 0)$, and so on. For a connected component C of $g \cap \mathcal{F}$ of a cell $g \in \mathbb{G}$ and for $0 \le i \le 3$, let $\xi_i(C)$ be an extremal vertex of C in direction e_i . (Note that the four vertices may not be distinct.) For a point $q \in \mathbb{R}^2$, let C(q) be the face of $\mathcal{F}^{\#}$ that contains q. With a slight abuse of notation, we use $\xi_i(q)$ to denote $\xi_i(C(q))$. Set $\Xi(q) \coloneqq \{\xi_i(q) \mid 0 \le i \le 3\}$.

The retracted plan $\widehat{\pi}$ that we construct maintains the following invariant: for $0 \leq i \leq k$, $||\widehat{\pi}_A(\lambda_i) - \widehat{\pi}_B(\lambda_i)||_{\infty} \geq 2$, $\widehat{\pi}_A(\lambda_i) \in \mathcal{F}[\widehat{\pi}_B(\lambda_i)]$, and $\widehat{\pi}_B(\lambda_i) \in \mathcal{F}[\widehat{\pi}_A(\lambda_i)]$. We set $\widehat{\pi}_A(\lambda_0) \coloneqq s_A$ and $\widehat{\pi}_B(\lambda_0) \coloneqq s_B$. The invariant obviously holds for λ_0 . Assume that we have retracted the first i-1 moves of π . We now describe the retraction of the *i*-th move. Without loss of generality, assume that *i* is even, so $R_i = B$, *A* is parked at p_i , *B* moves along π_i , $\pi(\lambda_{i-1}) = (p_i, p_{i-1})$ and $\pi(\lambda_i) = (p_i, p_{i+1})$. Let *C* be the face of $\mathcal{F}^{\#}$ containing p_i , $\Box_i \coloneqq p_i + 2\Box$, and $\Box_i \coloneqq p_i + 2(1 + \varepsilon)\Box$. The intersection of $\Box_i \setminus \Box_i$ with the line supporting an edge of \Box_i consists of two segments, each connecting the edges of \Box_i and \Box_i . We refer to these eight segments, over the four edges of \Box_i , as extension chords. These extension chords partition $\Box_i \setminus \Box_i$ into four corner squares and four side rectangles. Since π is a feasible plan, $\pi_i \cap \operatorname{int}(\Box_i) = \emptyset$. For a point $q \in \mathbb{R}^2$, let $S(q) \coloneqq \{e_j \in S \mid |(q - p_i) \cdot e_j| \ge 2\}$ and $C(q) \coloneqq \{\xi_j(q) \mid e_j \in S(q)\}$. Since π is feasible, $S(q) \neq \emptyset$ for all $q \in \pi_i$. To define the retraction of the *i*-th move, we define events during the interval $[\lambda_{i-1}, \lambda_i]$ at which *A* is (possibly) moved from one point of $\Xi(p_i)$ to another while *B* is parked at a point of \mathcal{V} . There are two types of events:

(i) Boundary event. A time instance λ is a boundary event if $\pi_i(\lambda) \in \partial \Box_i$ and B enters \Box_i

immediately after λ_i .

(ii) Separation event. A time instance λ is a separation event if $\pi_i(\lambda) \in \Box_i$ and $p_i + \Box$ and $\pi_i(\lambda) + \Box$ stop being x-separated or y-separated, i.e., $|x(p_i) - x(\pi_i(\lambda))| = 2$ or $|y(p_i) - y(\pi_i(\lambda))| = 2$, and hence $\pi_i(\lambda)$ lies on one of the eight extension chords of \Box_i and π_i leaves a corner square of \Box_i and enters a side rectangle.

By Lemma 6.1, π_i crosses each edge of \Box_i at most three times and each extension chord at most once, so there are at most six boundary events and eight separation events during the *i*-th move. Hence, there are at most 14 events. We also add λ_i as an event to ensure that at the end of the *i*-th move *B* is parked at a point of \mathcal{V} , so that the invariant is maintained. Let $\theta_1 < \theta_2 < \ldots < \theta_{i-1} < (\theta_i = \lambda_i)$ be the sequence of events along π_i . For any time θ between two consecutive events (θ_{i-1}, θ_i) , $S(\pi_i(\theta))$ does not change, so *B* move along $\pi_i(\theta_{i-1}, \theta_i)$ and *A* remains parked at a vertex in $C(\pi_i(\theta))$. By the above invariant, *A* is parked at a vertex of $C(\pi_i(\lambda_{i-1}))$ at time λ_{i-1} . So assume that we have processed events $\theta_1, \ldots, \theta_{j-1}$ and retracted the plan until θ_j . *B* crosses an edge of \Box_i or one of its extension chords. Let e_k be the inner normal of the extension chord. If *A* is currently parked at the vertex $\xi_k(p_i)$ then $\xi_k(p_i) \in C(\pi_i(\theta))$ for all $[\theta_j, \theta_{j+1})$ and hence no action is needed. Otherwise, we first move *B* to the vertex $\xi_{k+2}(\pi_i(\theta_j))$ from $\pi_i(\theta_j)$ along an *xy*-monotone path in $C(\pi_i(\theta_j))$ and park it there, then move *A* from its current position to the vertex $\xi_k(p_i)$ within $C(p_i)$ by first moving it along an *xy*-monotone path to p_i and then from p_i to $\xi_k(p_i)$ again along an *xy*-monotone path. After *A* is parked at $\xi_k(p_i)$, we move *B* back to $\pi_i(\theta_j)$ then move it along π_i from $\pi_i(\theta_j)$ toward $\pi_i(\theta_{j+1})$. It can be verified that these paths are feasible.

If $\theta_j = \lambda_i$, then we have to find an appropriate parking place for *B*. By construction, *A* is parked at a vertex $\xi_k(p_i) \in C(p_{i+1})$. Then we move *B* from p_{i+1} to $\xi_{k+2}(p_{i+1})$, i.e., the farthest vertex of $C(p_{i+1})$ in direction $-e_k$, as above. This step ensures that the invariant is satisfied after move *i*.

Let π' be the plan obtained by retracting the given plan π . Overall, each move of π involves at most 15 events, and each event involves three new moves along xy-monotone paths in faces of $\mathcal{F}^{\#}$. By definition of $\mathcal{F}^{\#}$, each new move has length at most $2\varepsilon\sqrt{2} < 3\varepsilon$. Hence the total increase in cost is less than $135\varepsilon\alpha(\pi)$. Furthermore, all parking places of π' are at vertices of \mathcal{V} within L_{∞} -distance 2ε of parking places in π . By re-picking $\varepsilon := \varepsilon/135$, we have the following lemma.

Lemma 6.2. Let $\varepsilon \in (0,1)$ be a parameter, and let π be a decoupled, Δ -tame (s,t)-plan. There exists a decoupled, $(\Delta + 2\varepsilon)$ -tame, (s,t)-plan π' such that $\mathfrak{e}(\pi') \leq \mathfrak{e}(\pi) + \varepsilon \alpha(\pi)$ and $\alpha(\pi') = c\alpha(\pi)$, and every parking place of π' is in \mathcal{V} , for some constant c > 0 that does not depend on ε , Δ . If π is kissing, then π' is ε -nearly-kissing.

7 Algorithm

We are now ready to describe our algorithm to compute an (s, t)-plan π with $\mathfrak{c}(\pi) \leq (1+\varepsilon)\mathfrak{c}(\pi^*)$ for any $\varepsilon \in (0, 1]$. We first describe an $n^3 \varepsilon^{-O(1)} \log n$ -time algorithm (Lemma 7.1) under the assumption that $\mathfrak{c}(\pi^*) > 1/4$. With further efforts, we present a near-quadratic time algorithm (Lemma 7.2) and how to remove the assumption (Section 7.2).

The algorithm consists of three stages. First, we choose a set $\widetilde{\mathcal{V}}$ of $O(n/\varepsilon^4)$ points so that a robot is always parked at one of the points in $\widetilde{\mathcal{V}}$. Next, we construct a graph $\mathcal{G} = (\mathcal{C}, \mathcal{E})$ where $\mathcal{C} \subseteq \widetilde{\mathcal{V}} \times \widetilde{\mathcal{V}}$ is a set of (feasible) configurations and each edge is a (decoupled) plan between a pair

of configurations of C with one move. We compute a shortest path in \mathcal{G} , which corresponds to an (s, t)-plan $\widehat{\pi}$ with $\mathfrak{c}(\widehat{\pi}) \leq (1 + \varepsilon)\mathfrak{c}(\pi^*) + O(\varepsilon) \leq \mathfrak{c}(\pi) \leq (1 + O(\varepsilon))\mathfrak{c}(\pi^*)$ for $\mathfrak{c}(\pi^*) \geq 1/4$.

Set $\overline{\varepsilon} := \varepsilon/c_0$ and $\Delta := c_1/\overline{\varepsilon}$ where $c_0, c_1 > 0$ are sufficiently large constants (independent of ε) to be chosen later. Let \mathbb{G} , $\mathcal{F}^{\#}$, and \mathcal{V} be the same as in Section 6 but using $\overline{\varepsilon}$ for ε . Let F_{ε} be the set of faces of $\mathcal{F}^{\#}$ that contain a Δ -close point; any point in a face $C \in \mathsf{F}_{\varepsilon}$ is $(\Delta + 2\overline{\varepsilon})$ -close. Let $\widetilde{\mathcal{V}}$ be the set of vertices of F_{ε} ; $|\widetilde{\mathcal{V}}| = O(n\Delta^2/\overline{\varepsilon}^2) = O(n/\varepsilon^4)$. We now describe the weighted graph $\mathcal{G} = (\mathcal{C}, \mathcal{E})$. We set $\mathcal{C} := \{(a, b) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{V}} \mid ||a - b||_{\infty} \geq 2\}$. Note that $s, t \in \mathcal{C}$ and $\mathcal{C} \subset \mathsf{F}$. We construct \mathcal{E} as follows: For every ordered triple $(u, v, p) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{V}} \times \widetilde{\mathcal{V}}$ with $u \neq v$ and $||p - u||_2, ||p - v||_2 \geq 2$, we set $\omega((u, p) \to (v, p)) = \omega((p, u) \to (p, v)) := \varrho_{\mathcal{F}[p]}(u, v)$, and if this value is not ∞ we add edges $(u, p) \to (v, p)$ and $(p, u) \to (p, v)$ to \mathcal{E} with $\omega((u, p) \to (v, p)) = \omega((p, u) \to (p, v))$ as their weight, which corresponds to moving A (resp., B) from u to v along a shortest path in $\mathcal{F}[p]$ while B (resp., A) is parked at p. Then $|\mathcal{E}| = |\widetilde{\mathcal{V}}|^3 = O(n^3/\varepsilon^{12})$.

Finally, we compute a shortest path (by weight) Φ in \mathcal{G} from s to t. After having computed Φ , the (s, t)-plan corresponding to Φ can be retrieved in a straightforward manner, and the cost of the resulting plan is the same as the weight of the path. We conclude by stating the following lemma:

Lemma 7.1. Given $s, t \in \mathbf{F}$, and $\varepsilon \in (0, 1)$, there exists a path Φ from s to t in \mathcal{G} , if s, t are reachable, whose weight is at most $(1 + \varepsilon) \mathfrak{c}(\pi^*) + O(\varepsilon)$, which is $(1 + O(\varepsilon)) \mathfrak{c}(\pi^*)$ if $\mathfrak{c}(\pi^*) > 1/4$, where π^* is a decoupled, optimal (s, t)-plan. Conversely, a path Φ from s to t in \mathcal{G} corresponds to an (s, t)-plan $\hat{\pi}$ of cost $\omega(\Phi)$. Furthermore, a shortest path from s to t in \mathcal{G} can be computed in $O(n^3 \varepsilon^{-12} \log n)$ time.

Proof. By Corollary 5.5, there exists a decoupled $(\Delta = c_1/\overline{\epsilon})$ -tame plan π with $\mathfrak{c}(\pi) \leq (1 + \overline{\epsilon})\mathfrak{c}(\pi^*)$ and $\alpha(\pi) \leq c_2(\alpha(\pi^*) + 1)$ for some constants $c_1, c_2 > 0$. (We make use of the stronger Corollary 5.2 that guarantees π is kissing when we improve the algorithm in the next subsection.) Then, by Lemma 6.2 with $\overline{\epsilon}$ as parameter ϵ , there exists a $(\Delta + 2\overline{\epsilon})$ -tame, decoupled plan π' such that all parking places belong to $\widetilde{\mathcal{V}}$ and

$$\mathfrak{e}(\boldsymbol{\pi}') \leq \mathfrak{e}(\boldsymbol{\pi}) + \overline{\varepsilon}\alpha(\boldsymbol{\pi}) \leq (1+\overline{\varepsilon})\mathfrak{e}(\boldsymbol{\pi}^*) + c_2\overline{\varepsilon}(\mathfrak{e}(\boldsymbol{\pi}^*)+1) \leq (1+\overline{\varepsilon}(1+c_2))\mathfrak{e}(\boldsymbol{\pi}^*) + c_2\overline{\varepsilon}.$$

At this point, we have additive error $O(\varepsilon)$. Here we make use of our assumption that $\mathfrak{c}(\pi^*) > 1/4$ and have

$$\mathfrak{e}(\boldsymbol{\pi}') \leq (1 + \overline{\varepsilon}(1 + 5c_2))\mathfrak{e}(\boldsymbol{\pi}^*).$$

Then by choosing $c_0 \coloneqq 1 + 5c_2$, we have $\overline{\varepsilon} = \varepsilon/(1 + 5c_2)$ and hence

$$\mathfrak{e}(\boldsymbol{\pi}') \leq (1+\varepsilon)\mathfrak{e}(\boldsymbol{\pi}^*).$$

Let $\langle \boldsymbol{\pi}' \rangle = (R_1, \pi_1, p_1), \ldots, (R_\ell, \pi_\ell, p_\ell)$. Without loss of generality, assume that $R_1 = A$. Then we map $\widehat{\boldsymbol{\pi}}$ to a path from \boldsymbol{s} to \boldsymbol{t} in \mathcal{G} as follows. For each $1 \leq i \leq \ell, \pi_i$ is a path followed by one of the robots from p_{i-1} to p_{i+1} while the other is parked at p_i , so $||p_{i+1} - p_i||_{\infty} \geq 2$ and $\varrho_{\mathcal{F}[p_i]}(p_{i-1}, p_{i+1}) \leq \mathfrak{c}(\pi_i)$. Therefore $(p_{i-1}, p_i) \rightarrow (p_{i+1}, p_i), (p_i, p_{i-1}) \rightarrow (p_i, p_{i+1}) \in \mathcal{E}$ with their weights being at most $\mathfrak{c}(\pi_i)$. Hence $\boldsymbol{s} = (p_0, p_1) \rightarrow (p_2, p_1) \rightarrow (p_2, p_3) \rightarrow \ldots \rightarrow \boldsymbol{t}$ is a path in \mathcal{G} of weight at most $\mathfrak{c}(\boldsymbol{\pi})$.

Converting Φ to a decoupled (s, t)-plan of cost at most $\omega(\Phi)$ is straightforward and omitted from here. It remains to analyze the runtime of the algorithm. \mathcal{F} and $\widetilde{\mathcal{V}}$ can be computed in $O(n \log^2 n + |\widetilde{\mathcal{V}}|) = O(n(\log^2 n + 1/\varepsilon^4))$ time [12]. For any ordered pair $(u, p) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{V}}, \mathcal{F}[p]$ can be computed from \mathcal{F} in $O(n \log n)$ time and processed [19] in $O(n \log n)$ time into a data structure that answers $O(\log n)$ -time shortest-path queries from u to any query point $v \in \mathcal{F}[p]$. So we can compute $\omega((u, p) \to (v, p)) = \omega((p, u) \to (p, v))$ in $O(n \log n + |\widetilde{\mathcal{V}}| \log n) = O((n/\varepsilon^4) \log n)$ time, for all $v \in \widetilde{\mathcal{V}}$. Repeating this process for all $O((n/\varepsilon^4)^2)$ pairs $(u, p) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{V}}$, we compute \mathcal{G} and its edge weights in $O(|\mathcal{E}| \log n) = O((n/\varepsilon^4)^3 \log n)$ time. Finally, computing the shortest path Φ in \mathcal{G} and reporting its corresponding plan takes $O(|\mathcal{E}| + |\mathcal{C}| \log |\mathcal{C}|)$ time using Dijkstra's algorithm, which is dominated by the $O(|\mathcal{E}| \log n)$ time to build \mathcal{G} . Therefore the overall running time is $O(n^3 \varepsilon^{-12} \log n)$. \Box

7.1 Reducing the runtime

Now we describe how to reduce the runtime to $O(n^2 \varepsilon^{-O(1)} \log n)$ using Corollary 5.2 (instead of Corollary 5.5). The high-level idea is to reduce the number of vertices, $|\mathcal{C}|$, from $O(n^3 \operatorname{poly}(\log n, 1/\varepsilon))$ to $O(n^2 \operatorname{poly}(\log n, 1/\varepsilon))$ while maintaining the $O(|\widetilde{\mathcal{V}}|)$ degree of each node. The effect is that the size of each of $|\mathcal{C}|, |\mathcal{E}|$ reduces by a factor of n, which reduces the overall runtime by a factor of n.

We first describe the graph $\mathcal{G} = (\mathcal{C}, \mathcal{E})$. We set $\mathcal{C} := \{(a, b) \in \tilde{\mathcal{V}} \times \tilde{\mathcal{V}} \mid 2 \leq ||a - b||_{\infty} \leq 2(1 + \bar{\varepsilon})\}$. Note the new condition that $||a - b||_{\infty} \leq 2(1 + \bar{\varepsilon})$. For a pair of nearby configurations $u = (u_A, u_B), v = (v_A, v_B) \in \mathcal{C}$, we consider two possible (u, v)-plans: (i) keep A parked at u_A while B moves from u_B to v_B along a shortest path in $\mathcal{F}[u_A]$, then park B at v_B and move A from u_A to v_A along a shortest path in $\mathcal{F}[v_B]$, and (ii) keep B parked at u_B while A moves from u_A to v_A along a shortest path in $\mathcal{F}[u_B]$, then park A at v_A and move B from u_B to v_B along a shortest path in $\mathcal{F}[v_A]$. Set

$$\omega(\boldsymbol{u},\boldsymbol{v}) = \min\{\varrho_{\mathcal{F}[\boldsymbol{u}_B]}(\boldsymbol{u}_A,\boldsymbol{v}_A) + \varrho_{\mathcal{F}[\boldsymbol{v}_A]}(\boldsymbol{u}_B,\boldsymbol{v}_B), \varrho_{\mathcal{F}[\boldsymbol{u}_A]}(\boldsymbol{u}_B,\boldsymbol{v}_B) + \varrho_{\mathcal{F}[\boldsymbol{v}_B]}(\boldsymbol{u}_A,\boldsymbol{v}_A)\}.$$

If $\omega(\boldsymbol{u}, \boldsymbol{v}) < \infty$, we add $\boldsymbol{u} \to \boldsymbol{v}$ to \mathcal{E} with $\omega(\boldsymbol{u}, \boldsymbol{v})$ as its weight. Then $|\mathcal{E}| = |\widetilde{\mathcal{V}}|^2 = O(n^2/\varepsilon^8)$. For a fixed configuration $\boldsymbol{u} \coloneqq (u_A, u_B) \in \mathcal{C}$, we compute the shortest path from u_A to all points of $\widetilde{\mathcal{V}}$ within $\mathcal{F}[u_B]$, using the same data structure as before [19], and do the same for u_B to all points of $\widetilde{\mathcal{V}}$ in $\mathcal{F}[u_A]$. After repeating this step for all configurations in \mathcal{C} , we have all the information to compute $\omega(\boldsymbol{u}, \boldsymbol{v})$ for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{C} \times \mathcal{C}$. The overall runtime can be shown to be $O(|\mathcal{E}| \log n)$ as before, which is $O(n^2 \varepsilon^{-8} \log n)$ here.

A similar argument for Lemma 7.1 that uses Corollary 5.2 instead of Corollary 5.5 proves the following lemma, which is the same as Lemma 7.1 except that the plan $\hat{\pi}$ is ε -nearly-kissing.

Lemma 7.2. Given $s, t \in \mathbf{F}$, and $\varepsilon \in (0, 1)$, there exists a path Φ from s to t in \mathcal{G} , if s, t are reachable, whose weight is at most $(1 + \varepsilon) \diamond (\pi^*) + O(\varepsilon)$, which is bounded by $(1 + O(\varepsilon)) \diamond (\pi^*)$ if $\diamond (\pi^*) > 1/4$, where π^* is a decoupled, kissing, optimal (s, t)-plan. Conversely, a path Φ from s to t in \mathcal{G} corresponds to a decoupled, ε -nearly-kissing (s, t)-plan $\widehat{\pi}$ of cost $\omega(\Phi)$. Furthermore, a shortest path from s to t in \mathcal{G} can be computed in $O(n^2 \varepsilon^{-8} \log n)$ time.

7.2 Handling nearby configurations

We now describe how we compute an (s, t)-plan of cost at most $(1 + \varepsilon) \mathfrak{c}(\pi^*)$ even when $\mathfrak{c}(\pi^*) \leq 1/4$. The algorithm described in the following Section 8 (cf. Lemma 8.1) either reports an 8-approximation $\gamma \leq 2$ of $\mathfrak{c}(\pi^*)$, i.e., $\mathfrak{c}(\pi^*) \leq \gamma \leq 8\mathfrak{c}(\pi^*)$, or it reports that $\mathfrak{c}(\pi^*) > 1/4$. So we first run this algorithm. If it reports $\mathfrak{c}(\pi^*) > 1/4$, we run the algorithm above (with improved runtime). Otherwise, we have $\gamma \leq 2$ and $\mathfrak{c}(\pi^*) \leq \gamma \leq 8\mathfrak{c}(\pi^*)$. Then $\gamma/8 \leq \mathfrak{c}(\pi^*) \leq \gamma \leq 2$. In this case, we simply run the above algorithm except we set $\overline{\varepsilon} := \gamma \varepsilon / c_0$ for a parameter $c_0 > 0$ to be chosen later and set $\Delta := \gamma$. Then $\widetilde{\mathcal{V}}$ contains $(\gamma + 2\overline{\varepsilon})$ -close points and $|\widetilde{\mathcal{V}}| = O(n\Delta^2/\overline{\varepsilon}^2) = O(n\gamma^2/(\gamma\varepsilon)^2) = O(n/\varepsilon^2)$. Following the same argument as in the proof of Lemma 7.1, we claim that c_0 can be chosen so that there exists a $(\Delta + 2\overline{\varepsilon})$ -tame plan π' with $\mathfrak{c}(\pi') \leq (1+\varepsilon)\mathfrak{c}(\pi^*)$ and all parking places of π' are in $\widetilde{\mathcal{V}}$.

To prove the claim, note that π^* is trivially $(\Delta = \gamma)$ -tame since $\mathfrak{c}(\pi^*) \leq \gamma$. By Lemma 3.6, we have

$$\alpha(\boldsymbol{\pi}^*) \le c_2(\mathfrak{e}(\boldsymbol{\pi}^*) + 1) \le 3c_2$$

for a constant $c_2 > 0$. Then, by Lemma 6.2 with $\overline{\varepsilon}$ as parameter ε , there exists a decoupled, $(\Delta + 2\overline{\varepsilon})$ -tame, $\overline{\varepsilon}$ -nearly-kissing plan π' with all parking places of π' in $\widetilde{\mathcal{V}}$ and

$$\mathfrak{e}(\boldsymbol{\pi}') \leq \mathfrak{e}(\boldsymbol{\pi}^*) + \overline{\varepsilon}\alpha(\boldsymbol{\pi}^*) \leq \mathfrak{e}(\boldsymbol{\pi}^*) + 3c_2\overline{\varepsilon} = \mathfrak{e}(\boldsymbol{\pi}^*) + 3c_2\gamma\varepsilon/c_0 \leq (1 + 24c_2\varepsilon/c_0)\mathfrak{e}(\boldsymbol{\pi}^*),$$

where the last inequality follows by $c(\pi^*) \ge \gamma/8$. So we choose $c_0 \coloneqq 1/(24c_2)$. This proves the claim. The rest of the analysis follows from the previous algorithm, including the runtime analysis, since the algorithm from Lemma 8.1 only takes $O(n \log^2 n)$ additional time.

8 O(1)-Approximate Plans for Close Configurations

In this section we prove Lemma 8.1, which states we can compute either an 8-approximation γ of π^* or detect that $\mathfrak{e}(\pi^*) > 1/4$ in $O(n \log^2 n)$ time. First, we introduce some notations.

If all moves of a plan π are xy-monotone, we say π is xy-monotone. For a (piecewise-linear) xy-monotone (s, t)-plan π , $s, t \in \mathbf{F}$, let $\$(\pi)$ be the L_1 -cost of π , i.e., if $\langle u_1, u_2, \ldots, u_g \rangle$ (resp., $\langle v_1, v_2, \ldots, v_h \rangle$) is the sequence of vertices of π_A (resp., π_B), then

$$\$(\boldsymbol{\pi}) = \sum_{i=1}^{g-1} ||u_i - u_{i+1}||_1 + \sum_{i=1}^{h-1} ||v_i - v_{i+1}||_1.$$

Recall that we say a configuration $(a, b) \in \mathbf{F}$ is *x*-separated if $|x(a) - x(b)| \ge 2$ and is *y*-separated if $|y(a) - y(b)| \ge 2$. We now describe the algorithm given in the following main lemma of this section.

Lemma 8.1. Let W be a polygonal environment with n vertices, let A, B be two robots each modeled as a unit square, and let $\mathbf{s} = (s_A, s_B), \mathbf{t} = (t_A, t_B) \in \mathbf{F}$. There is an algorithm that in $O(n \log^2 n)$ time either reports a value $\gamma \leq 2$ with $\mathfrak{c}(\boldsymbol{\pi}^*) \leq \gamma \leq 8\mathfrak{c}(\boldsymbol{\pi}^*)$ or reports that $\mathfrak{c}(\boldsymbol{\pi}^*) > 1/4$; when both such a value γ exists and $\mathfrak{c}(\boldsymbol{\pi}^*) > 1/4$, it reports either outcome arbitrarily.

Algorithm. Let $\Box_A := (s_A + (1/4)\Box) \cap (t_A + (1/4)\Box)$ and $\Box_B := (s_B + (1/4)\Box) \cap (t_B + (1/4)\Box)$. The algorithm searches for a (s, t)-plan $\pi = (\pi_A, \pi_B)$ contained in $\Box_A \times \Box_B$ with $\alpha(\pi) \le 4$ and minimum L_1 -cost, $\$(\pi)$. As we will prove, the search only needs to be successful at finding such a plan when $\mathfrak{e}(\pi^*) \le 1/4$, so the algorithm is described assuming that is true.

We now describe the rest of the algorithm assuming that A moves first; by repeating the subroutines with the roles of A and B swapped we cover both cases. There are three main steps.

Step (I). We first do a simple check. Let C_A (resp., C_B) be the component of $\Box_A \cap \mathcal{F}$ (resp., $\Box_B \cap \mathcal{F}$) containing s_A (resp., s_B). If $t_A \notin C_A$ (resp., $t_B \notin C_B$) then s_A, t_A (resp., s_B, t_B) lie in different components of $\mathcal{F} \cap \Box_A$ (resp., $\mathcal{F} \cap \Box_B$) and we report that $\mathfrak{c}(\pi^*) > 1/4$. Otherwise, we proceed to Step (II).



Figure 21. Illustration of s_A, t_A, s_B, t_B positioned as assumed in Step (III) of the algorithm in Lemma 8.1, where $s_A + \Box, s_B + \Box$ are solid and $t_A + \Box, t_B + \Box$ are dashed.

Step (II). Now let $C'_A \subseteq C_A$ (resp., $C'_B \subseteq C_B$) be the component of $C_A \cap \mathcal{F}[s_B]$ (resp., $C_B \cap \mathcal{F}[t_A]$) containing s_A (resp., t_B). It is possible that $C_A = C'_A$ or $C_B = C'_B$. We next check if there exists a plan with at most two moves: We first check if $t_A \in C'_A$ and $s_B \in C'_B$. If so, there exists an xy-monotone path π_A from s_A to t_A in C'_A , i.e., while B is parked at s_B , and an xy-monotone path from s_B to t_B in C'_B , i.e., while A is parked at t_A , by Lemma 3.4. Then we report the cost $\mathfrak{c}(\pi)$ of the corresponding xy-monotone plan π . Otherwise, we proceed to the next step, Step (III).

We will later prove that if $\mathfrak{c}(\pi^*) \leq 1/4$ and s, t are both x-separated or both y-separated, then Step (II) must find and report a plan π . Hence s is only x-separated and t is only y-separated, or vice-versa. So, we continue our search for a plan π assuming without loss of generality that s is x-separated and t is y-separated in Step (III).

Step (III). For concreteness, assume that

$$x(s_B) \le x(s_A) - 2$$
 and $y(t_B) \le y(t_A) - 2$.

Under the assumption $\mathfrak{e}(\pi^*) \leq 1/4$ it can be shown that

$$x(s_B) \le x(s_A) - 2$$
 and $y(s_A) - 2 < y(s_B) \le y(s_A) - 7/4$,

and

$$y(t_B) \le y(t_A) - 2$$
 and $x(t_A) - 2 < x(t_B) \le x(t_A) - 7/4$.

See Figure 21. We search for a plan π with at most four moves, i.e., π is of the following form, for two points $p_A \in \Box_A \cap \mathcal{F}$ and $p_B \in \Box_B \cap \mathcal{F}$:

$$\langle \boldsymbol{\pi} \rangle = (A, \pi_1, s_B), (B, \pi_2, p_A), (A, \pi_3, p_B), (B, \pi_4, t_A).$$

If p_A (resp., p_B) is s_A or t_A (resp., s_B or t_B), then the first or last move by A (resp., B) is the trivial path, respectively. For any configuration $(p_A, p_B) \in \mathbf{F}$, let $\mathbf{\Pi}(p_A, p_B)$ be the plan where A moves from s_A then parks at p_A on the first move, B moves from s_B then parks at p_B on the second move, A moves from p_A then parks at t_A on the third move, and finally B moves from p_B to t_B on the fourth move (if possible). We define a set of candidate configurations in \mathbf{F} , then we choose and report the plan $\mathbf{\Pi}(p_A, p_B)$ which is feasible and minimizes its L_1 -cost $(\mathbf{\Pi}(p_A, p_B))$ over all candidate configurations (p_A, p_B) . The details are as follows.

Let \mathcal{L}_A be the set of axis-parallel lines that contain s_A, t_A , i.e.,

$$\mathcal{L}_A \coloneqq \{x = x(s_A), y = y(s_A), x = x(t_A), y = y(t_A)\}$$

and let \mathcal{L}_B be the set of axis-parallel lines that contain s_B, t_B . Let ϕ be the vector (1, 1). Let \widetilde{Q} be the overlay of $(C'_A \cup \mathcal{L}_A) - \phi$ and $(C'_B \cup \mathcal{L}_B) + \phi$. Finally, let $\widetilde{Q}^{||}$ be the overlay of \widetilde{Q} with a set of vertical lines through every vertex of \widetilde{Q} . Then every vertex of $\widetilde{Q}^{||}$ lies on a vertical line that contains at least one real vertex of \widetilde{Q} . Let \widetilde{V} be the vertices of $\widetilde{Q}^{||}$. Let $\widetilde{\mathcal{P}} \subseteq \widetilde{V} \times \widetilde{V}$ be the subset of pairs $(\widetilde{p}_A, \widetilde{p}_B)$ such that:

(i)
$$\widetilde{p}_A + \phi \in C'_A$$
 and $\widetilde{p}_B - \phi \in C'_B$, and

(ii) $x(\tilde{p}_A) = x(\tilde{p}_B)$ and $y(\tilde{p}_A) \ge y(\tilde{p}_B)$, i.e., \tilde{p}_B lies below \tilde{p}_A on the same vertical line,

(iii) and the plan $\mathbf{\Pi}(\widetilde{p}_A + \phi, \widetilde{p}_B - \phi)$ is feasible.

If $|\widetilde{\mathbf{P}}| = \emptyset$, we report that $\mathfrak{e}(\pi^*) > 1/4$. Otherwise, we report the L_2 -cost $\mathfrak{e}(\mathbf{\Pi}(\widetilde{p}_A + \phi, \widetilde{p}_B - \phi))$ of the corresponding plan $\mathbf{\Pi}(\widetilde{p}_A + \phi, \widetilde{p}_B - \phi)$ for the pair $(\widetilde{p}_A, \widetilde{p}_B) \in \widetilde{\mathbf{P}}$ that minimizes the L_1 -cost $\mathfrak{I}(\mathbf{\Pi}(\widetilde{p}_A + \phi, \widetilde{p}_B - \phi))$.

This concludes the algorithm.

Correctness. It is easy to verify that if the algorithm succeeds to find a plan π and reports its L_2 -cost $\mathfrak{e}(\pi)$ in Step (II) or Step (III) that $\pi \subset \Box_A \times \Box_B$, $\alpha(\pi) \leq 4$, and π is feasible. If the algorithm reports $\mathfrak{e}(\pi^*) > 1/4$ in Step (I), then s_A, t_A (resp., s_B, t_B) lie in different components of C_A (resp., C_B) and hence the path π_A (resp., π_B) in any feasible (s, t)-plan (π_A, π_B) must exit \Box_A (resp., \Box_B). So the algorithm behaves correctly in this case. If the algorithm reports a plan π in Steps (II) or (III), all parking places of A (resp., B) are contained in \Box_A (resp., \Box_B) and hence the L_2 -cost of each (xy-monotone) move is at most 1/2. It follows that $\mathfrak{e}(\pi) \leq (1/2)\alpha(\pi) \leq 2$.

First suppose $\mathfrak{c}(\pi^*) > 1/4$. If the algorithm fails in both Step (II) and Step (III) to find any path and report its cost, it correctly reports $\mathfrak{c}(\pi^*) > 1/4$. Otherwise, the algorithm reports the cost $\mathfrak{c}(\pi)$ of a plan π , where $\mathfrak{c}(\pi) \leq 2$ by the discussion above. Then

$$\mathfrak{e}(\boldsymbol{\pi}^*) \leq \mathfrak{e}(\boldsymbol{\pi}) \leq 2 \leq 8\mathfrak{e}(\boldsymbol{\pi}^*).$$

In either case, the algorithm behaves as claimed.

Next, suppose $\mathfrak{c}(\pi^*) \leq 1/4$. If the algorithm succeeds in Step (II), the cost reported is $\mathfrak{c}(\pi^*)$, and the algorithm behaves as claimed. So suppose Step (II) fails. As claimed in the description of the algorithm, it must be that s is only x-separated and t is only y-separated, or vice-versa. Indeed, for sake of contradiction, suppose s, t are both, say, x-separated. Then Lemma 3.5 implies there exists an (optimal xy-monotone) (s, t)-plan with at most two moves since there is a unit square that contains \Box_A and one that contains \Box_B . Step (II) checks for such plans, so it must succeed in this case, which is a contradiction. Henceforth, we assume A moves first in π^* and s, t are oriented as assumed in the algorithm, i.e., s is only x-separated and t is only y-separated,

$$x(s_B) \le x(s_A) - 2$$
 and $y(s_A) - 2 < y(s_B) \le y(s_A) - 7/4$,

and

$$y(t_B) \le y(t_A) - 2$$
 and $x(t_A) - 2 < x(t_B) \le x(t_A) - 7/4$

To finish the proof, we prove that Step (III) succeeds to find a plan $\boldsymbol{\pi} \subset \Box_A \times \Box_B$, under the assumption that $\boldsymbol{\mathfrak{e}}(\boldsymbol{\pi}^*) \leq 1/4$, with $\boldsymbol{\mathfrak{e}}(\boldsymbol{\pi}) \leq 8\boldsymbol{\mathfrak{e}}(\boldsymbol{\pi}^*)$. Let $(\pi_A^*, \pi_B^*) = \boldsymbol{\pi}^*$. Since π_A^*, π_B^* are continuous, there is a time instance $\lambda \in (0, 1)$ such that $(q_A, q_B) = \boldsymbol{\pi}^*(\lambda)$ is both x-separated and y-separated, in particular, $|x(q_A) - x(q_B)| = 2$. Then $q_A \in \Box_A$, $q_B \in \Box_B$, and $x(q_B) = x(q_A) - 2$ since $\boldsymbol{\mathfrak{e}}(\boldsymbol{\pi}^*) \leq 1/4$

and $x(s_B) < x(s_A) - 2$. By Lemma 3.4, there exists an *xy*-monotone optimal (s, q)-plan π_0 with at most two moves, since s, q are both *x*-separated, and there exists an *xy*-monotone optimal (q, t)-plan π_1 with at most two moves, since q, t are both *y*-separated. Then $\langle \pi_0 \rangle \circ \langle \pi_1 \rangle$ is an *xy*-monotone optimal (s, t)-plan. So assume $\langle \pi^* \rangle = \langle \pi_0 \rangle \circ \langle \pi_1 \rangle$. In particular, π^* is of the form

$$\langle \boldsymbol{\pi}^* \rangle = (A, \pi_1, s_B), (B, \pi_2, q_A), (A, \pi_3, q_B), (B, \pi_4, t_A).$$

Then $\pi_1 \subset C'_A$ (resp., $\pi_4 \subset C'_B$) and hence $q_A \in C'_A$ (resp., $q_B \in C'_B$). Let $\tilde{q}_A \coloneqq q_A - \phi$ (resp., $\tilde{q}_B \coloneqq q_B + \phi$), and let \tilde{g}_A (resp., \tilde{g}_B) be the cell of $\widetilde{Q}^{||}$ containing \widetilde{q}_A (resp., \widetilde{q}_B). By definition of $\widetilde{Q}^{||}$ and the fact that $q_A \in C'_A$ (resp., $q_B \in C'_B$), we have $\widetilde{g}_A \subseteq C'_A - \phi$ (resp., $\widetilde{g}_B \subseteq C'_B + \phi$). See that $x(\widetilde{q}_B) = x(\widetilde{q}_A)$ and $y(\widetilde{q}_B) \leq y(\widetilde{q}_A)$ since $x(q_B) = x(q_A) - 2$ and $y(q_B) \leq y(q_A) - 2$. That is, $\widetilde{q}_A, \widetilde{q}_B$ lie on the same vertical line with \widetilde{q}_A above \widetilde{q}_B .

Let $\tilde{p}_A \coloneqq \tilde{q}_A$ and $\tilde{p}_B \coloneqq \tilde{q}_B$ initially. Then $(\Pi(\tilde{p}_A + \phi, \tilde{p}_B - \phi)) = (\pi^*)$. Using the fact that \tilde{Q} includes the lines of $\mathcal{L}_A - \phi, \mathcal{L}_B + \phi$ and the convexity of \tilde{g}_A, \tilde{g}_B , it can be shown that \tilde{p}_A, \tilde{p}_B can be shifted to vertices of \tilde{g}_A, \tilde{g}_B , respectively, while maintaining that $x(\tilde{p}_B) = x(\tilde{p}_A), y(\tilde{p}_B) \leq y(\tilde{p}_A)$, and $(\Pi(\tilde{p}_A + \phi, \tilde{p}_B - \phi)) = (\pi^*)$. Then there exists a pair $(\tilde{p}_A, \tilde{p}_B) \in \tilde{V} \times \tilde{V}$ where \tilde{p}_A (resp., \tilde{p}_B) is a vertex of \tilde{g}_A (resp. \tilde{g}_B) with $(\Pi(\tilde{p}_A + \phi, \tilde{p}_B + \phi)) \leq (\pi^*)$. That is, $(\tilde{p}_A, \tilde{p}_B)$ satisfies conditions (i) and (ii) in the definition of $\tilde{\mathcal{P}}$.

To finish the proof, it suffices to prove $(\tilde{p}_A, \tilde{p}_B)$ satisfies condition (iii) so that $(\tilde{p}_A, \tilde{p}_B) \in \widetilde{\mathbf{\mathcal{P}}}$. To this end, we show that conditions (i) and (ii) imply (iii); i.e., (iii) is only included to make it by definition that the algorithm only reports costs of feasible plans in Step (III). Let $(\tilde{p}_A, \tilde{p}_B) \in V \times V$ be a pair that satisfies conditions (i) and (ii). Let $p_A := \tilde{p}_A + \phi$ and $p_B := \tilde{p}_B - \phi$. By definition of C'_A and C'_B and conditions (i) and (ii), we have $p_A \in C'_A, p_B \in C'_B$, and $(p_A, p_B) \in \mathbf{F}$. Then there is a xy-monotone path $\pi_1 \subset \mathcal{F}[s_B]$ from s_A to p_A and an xy-monotone path $\pi_4 \subset \mathcal{F}[t_A]$ from p_B to t_B by Lemma 3.4. We next show there is an xy-monotone path $\pi_3 \subset \mathcal{F}[p_B]$ from p_A to t_A . Since $p_A, t_A \in C_A$, there exists an xy-monotone path $\pi_3 \subset C_A$ from p_A to t_A by Lemma 3.4. It remains to show $\pi_3 \cap \operatorname{int}(p_B + 2\Box) = \emptyset$ so that we may conclude $\pi_3 \subset C_A \setminus (p_B + 2\Box) \subset$ $\mathcal{F}[p_B]$. By definition, $C'_B \subset \mathcal{F}[t_A]$, and $p_B \in C'_B$, so $(t_A, p_B) \in \mathbf{F}$. Then $x(p_B) + 2 \leq x(t_A)$ or $y(p_B) + 2 \le y(t_A)$. By condition (ii) and the fact π_3 is xy-monotone, in the former case we have $x(p_B) + 2 = x(p_A) \le x(t_A)$ so π_3 does not cross left of the line $x = x(p_B) + 2$, and in the latter case we have $y(p_B) + 2 \leq y(p_A), y(t_A)$ so π_3 does not cross below the line $y = y(p_B) + 2$. Hence, in either case, $\pi_3 \cap \operatorname{int}(p_B + 2\Box) = \emptyset$, i.e., $\pi_3 \subset \mathcal{F}[p_B]$, as desired. A symmetric argument implies there is an xy-monotone path $\pi_2 \subset \mathcal{F}[p_A]$ from s_B to p_B . Putting everything together, the plan π with $\langle \boldsymbol{\pi} \rangle = (A, \pi_1, s_B), (B, \pi_2, p_A), (A, \pi_3, p_B), (B, \pi_4, t_A)$ is a $(\boldsymbol{s}, \boldsymbol{t})$ -plan with $\$(\boldsymbol{\pi}) = \$(\boldsymbol{\pi}^*)$. Then $\mathfrak{e}(\boldsymbol{\pi}) \leq \sqrt{2} \mathfrak{s}(\boldsymbol{\pi}^*) \leq \mathfrak{s}\mathfrak{e}(\boldsymbol{\pi}^*)$, which completes the proof.

Runtime analysis. We first compute the components C_A, C'_A, C_B, C'_B in $O(n \log^2 n)$ time [12]. Then Step (I) and Step (II) take O(n) time. Consider Step (III). Since C'_A and C'_B are xy-monotone by Lemma 3.4, the O(1) lines in $\mathcal{L}_A - \phi, \mathcal{L}_B + \phi$ each intersect O(1) segments of $C'_A - \phi, C'_B + \phi$, so the overlay \widetilde{Q} has O(n) vertices and is computed in $O(n \log n)$ time. Furthermore, the vertical lines overlayed with Q to define $\widetilde{Q}^{||}$ each intersects O(1) segments of \widetilde{Q} , so $\widetilde{Q}^{||}$ and its set of vertices \widetilde{V} is also computed in $O(n \log n)$ time. Then there are O(1) vertices in \widetilde{V} that lie on any vertical line. It follows that, by condition (ii) in the definition of $\widetilde{\mathcal{P}}, |\widetilde{\mathcal{P}}| = O(n)$; in particular, O(1) pairs in $\widetilde{\mathcal{P}}$ lie on any common vertical line. For a given pair $(\widetilde{p}_A, \widetilde{p}_B) \in \widetilde{V} \times \widetilde{V}$, condition (i) is checked in O(1)time by marking the faces g of $\widetilde{Q}^{||}$ which of $C'_A - \phi, C'_B + \phi$ (possibly both) that contain g when $\widetilde{Q}^{||}$ is computed. Condition (ii) is implied by (i) and (ii), as argued above, so it does not need to be checked directly (it is only included in the definition so that the algorithm obviously only reports costs of feasible plans). It follows that $\tilde{\mathcal{P}}$ can be computed in O(n) time. If $|\tilde{\mathcal{P}}| = 0$ we report $\mathfrak{c}(\pi^*) > 1/4$, otherwise we find the pair $(\tilde{p}_A, \tilde{p}_B)$ for which $\mathfrak{c}(\mathbf{\Pi}(\tilde{p}_A + \phi, \tilde{p}_B - \phi))$ is minimized in O(1) time per pair, using the fact that

 $\mathfrak{c}(\Pi(\widetilde{p}_{A}+\phi,\widetilde{p}_{B}-\phi)) = ||s_{A}-(\widetilde{p}_{A}+\phi)||_{1} + ||(\widetilde{p}_{A}+\phi)-t_{A}||_{1} + ||s_{B}-(\widetilde{p}_{B}-\phi)||_{1} + ||(\widetilde{p}_{B}-\phi)-t_{B}||_{1}.$

Overall, the algorithm takes $O(n \log^2 n)$ time.

9 Conclusion

We have described a $(1 + \varepsilon)$ -approximation algorithm for the min-sum motion planning problem for two congruent square robots in a planar polygonal environment with running time $n^2 \varepsilon^{-O(1)} \log n$, i.e., our algorithm is an FPTAS. We also describe an $O(n \log^2 n)$ -time 8-approximation algorithm for the problem when the cost of the optimal plan is less than 1/4, which is used as a subroutine in our FPTAS. We conclude with some questions for future work. Can our techniques be extended

- (i) to obtain a $(1 + \varepsilon)$ -approximation algorithm for min-sum motion planning for k > 2 robots with running time $(n/\varepsilon)^{O(k)}$?
- (ii) to work for translating robots with congruent shapes other than squares, such as other centrally-symmetric regular polygons, disks, or convex polygons?
- (iii) to optimize both *clearance* and the total lengths of the paths in some fashion, where clearance is the minimum distance from any robot to any other robot or obstacle during the plan?

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