# Sorting and Selection in Rounds with Adversarial Comparisons 

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#### Abstract

We continue the study of selection and sorting of $n$ numbers under the adversarial comparator model, where comparisons can be adversarially tampered with if the arguments are sufficiently close.

We derive a randomized sorting algorithm that does $O\left(n \log ^{2} n\right)$ comparisons and gives a correct answer with high probability, addressing an open problem of Ajtai, Feldman, Hassadim, and Nelson [AFHN15]. Our algorithm also implies a selection algorithm that does $O(n \log n)$ comparisons and gives a correct answer with high probability. Both of these results are a log factor away from the naive lower bound. [AFHN15] shows an $\Omega\left(n^{1+\varepsilon}\right)$ lower bound for both sorting and selection in the deterministic case, so our results also prove a discrepancy between what is possible with deterministic and randomized algorithms in this setting.

We also consider both sorting and selection in rounds, exploring the tradeoff between accuracy, number of comparisons, and number of rounds. Using results from sorting networks, we give general algorithms for sorting in $d$ rounds where the number of comparisons increases with $d$ and the accuracy decreases with $d$. Using these algorithms, we derive selection algorithms in $d+O(\log d)$ rounds that use the same number of comparisons as the corresponding sorting algorithm, but have a constant accuracy. Notably, this gives selection algorithms in $d$ rounds that use $n^{1+o(1)}$ comparisons and have constant accuracy for all $d=\omega(1)$, which still beats the deterministic lower bound of $\Omega\left(n^{1+\varepsilon}\right)$.


## 1 Introduction

Comparison-based sorting and selection are two of the most well-studied computational problems, with applications in all aspects of computing. Often, these problems are studied with the goal to minimize the number of comparisons needed. Classical results show that sorting takes $\Theta(n \log n)$ comparisons [GVN47] and selection takes $\Theta(n)$ comparisons [ $\left.\mathrm{BFP}^{+} 73\right]$.

Comparison-based sorting and selection have also been extensively studied in the parallel case. The round-based model of parallelism we consider was introduced by Valiant [Val75] for comparisonbased problems, where groups of comparisons are done in rounds of interaction. There followed a long line of research in parallel sorting and selection [BR81, HH81, HH82, BT83, Pip87, AA88b, Bol88, WZ93]. Particularly similar to the problems studied in this paper are parallel sorting with limited closure [BT83, Alo86] and sorting networks of arity $k$ with low depth [TL85, PP89, NHAT89, BG90, CS92, Chv92, LB95, GL97, ZLG98, SYW14, DKP22], the latter of which we will use to derive our general sorting algorithms.

However, often it is not possible to guarantee comparisons are completely precise. For example, when ranking chess players, or comparing job applicants. Due to this, the problems of sorting and selection with imprecise comparators have also been widely considered.

[^0]Depending on the model, the manner in which the comparator is imprecise differs. The adversarial comparison model we will study was introduced in [AFHN15]. If the values being compared differ by more than some threshold $\delta$, the comparison is correct, otherwise the result of the comparison can be chosen arbitrarily by an adversary. There are two adversary models that have been considered (as described in $\left[\mathrm{AFJ}^{+} 18\right]$ ): the non-adaptive model, where all of the comparisons must be predetermined by the adversary before the algorithm is run, and the adaptive model, where the comparisons can be chosen by the adversary at the time they are queried, possibly depending on the previous queries made by the algorithm. In this paper, we focus entirely on the adaptive model, and as our results are all upper bounds, all of our results imply equivalent results for the easier non-adaptive model. By scaling, we assume $\delta=1$.

Since the comparisons are imprecise, it is impossible to always determine the correct result, so algorithms in this setting instead strive to achieve a small approximation factor, which measures how far away the returned solution is from the correct one. More precisely, we say an ordering $Y$ of a set $X$ is a $k$-approximate sorting if all inversions differ in value by at most $k$. In their original paper, Ajtai, Feldman, Hassidim, and Nelson [AFHN15] give deterministic $k$-approximate algorithms for sorting and selection that use $O\left(4^{k} \cdot n^{1+1 / 2^{k-1}}\right)$ and $O\left(2^{k} \cdot n^{1+1 / 2^{k-1}}\right)$ comparisons respectively. They also give a lower bound of $\Omega\left(n^{1+1 / 2^{k-1}}\right)$ for deterministic $k$-approximate sorting and selection. Special consideration is given to the case of selecting the maximum element, for which they give a randomized algorithm that uses $O(n)$ comparisons and returns a 3-approximation with probability $1-n^{-r}$. This maximum selection result was then improved by Acharya, Falahatgar, Jafarpour, Orlitsky, and Suresh [AFJ $\left.{ }^{+} 18\right]$ who gave a randomized algorithm that uses $O\left(n \log \frac{1}{\varepsilon}\right)$ comparisons and returns a 2 -approximation with probability $1-\varepsilon$. The study of these problems in the parallel setting was introduced by Gopi, Kamath, Kulkarni, Nikolov, Wu, and Zhang [GKK+ 20], where they gave a randomized $d$-round algorithm that uses $O\left(n^{1+\frac{1}{2^{d}-1}} d\right)$ comparisons and returns a 3 -approximate maximum with probability 0.9 . This raises the following questions: can randomized algorithms yield an improvement in sorting and general selection? How many comparisons are required to do sorting and general selection in $d$ rounds?

### 1.1 Results, Techniques, and Discussion

To describe our results, we more formally define the model and the problems of approximate sorting and selection.

Definition 1.1. Suppose we are given $n$ items $x_{1}, \ldots, x_{n}$ with unknown real values. An adversarial comparator $C$ is a function that takes two items $x_{i}$ and $x_{j}$ and returns $\max \left\{x_{i}, x_{j}\right\}$ if $\left|x_{i}-x_{j}\right|>1$ and $x_{i}$ or $x_{j}$ adversarially otherwise.

Throughout this paper, we assume the adaptive adversary model [ $\mathrm{AFJ}^{+}$18], where the adversarial comparisons may depend on previous queries made by the algorithm.

We first define a notion of $k$-approximate sorting, in which inversions may only occur between values that differ by at most $k$.

Definition 1.2. We say $x_{i} \geq_{k} x_{j}$ if $x_{i} \geq x_{j}-k$. For sets of items $Y, Z$, we say $Y \geq_{k} Z$ if $x_{i} \geq_{k} x_{j}$ for all $x_{i} \in Y, x_{j} \in Z$. We say some ordering $x_{1}, \ldots, x_{n}$ of items in a set $X$ is a $k$-approximate sorting if $x_{j} \geq_{k} x_{i}$ for all $j>i$. Equivalently, for any pair $x_{i}, x_{j}$ in the wrong order, they must differ by at most $k$.

This leads to a notion of approximate $i$-selection, in which the result must be the $i$-th element of some approximate sorting.

Definition 1.3. We say an item $x^{*}$ in a set $X$ is a $k$-approximate $i$-selection if there exists $a$ $k$-approximate sorting $x_{1}, \ldots, x_{n}$ of $X$ such that $x_{i}=x^{*}$.

We show that this definition is equivalent to the result differing from the "actual" $i$-th smallest element by at most $k$.

Lemma 1.4. An item $x_{j}$ is a $k$-approximate $i$-selection if and only if $\left|x_{j}-x_{i}\right| \leq k$ where $x_{i}$ is the actual $i$-th smallest element of $X$.

We proceed with our results. We begin in the non-parallel setting (although our algorithms still have good round guarantees). We provide the following near-optimal approximate sorting algorithm.

Theorem 1.5. There exists a randomized algorithm that takes $O\left(n \log ^{2} n\right)$ comparisons, uses $O(\log n)$ parallel rounds, and returns a 4 -approximate sorting with probability $>1-\frac{1}{n^{2}}$.

Since any approximate sorting algorithm must be able to correctly sort any list of numbers after scaling, such an algorithm must take $\Omega(n \log n)$ comparisons by the well-known lower bound. Thus, this result is a log factor away from optimal. Note that no algorithm can give better than a 2-approximation, as the adversary can force $0>1>2>0$, which can make $0,1,2$ indistinguishable $\left[\mathrm{AFJ}^{+} 18\right]$. The best prior result is of $\left[\mathrm{AFJ}^{+} 18\right]$, where they show quicksort gives a 2 -approximate sorting in $O(n \log n)$ expected comparisons against the non-adaptive adversary. This approach falls apart against the adaptive adversary, however, as if all values are the same, the adversary can force all pivots to compare less than all elements, forcing the algorithm to do $\Omega\left(n^{2}\right)$ comparisons. Our result shows that it is possible to get a constant approximate sorting in near-optimal number of comparisons even against the adaptive adversary. Previously, this problem had also been studied in the deterministic case [AFHN15], where an upper bound of $O\left(4^{k} \cdot n^{1+1 / 2^{k-1}}\right)$ and a lower bound of $\Omega\left(n^{1+1 / 2^{k-1}}\right)$ comparisons were proven for $k$-approximate sorting. Taking this result with $k=4$, we get a lower bound of $\Omega\left(n^{9 / 8}\right)$ for 4 -approximate deterministic sorting. Thus, our algorithm shows a distinction between randomized and deterministic algorithms in this problem. To get an $\widetilde{O}(n)$ deterministic algorithm, one could at best provide a $\Omega(\log \log n)$-approximation.

Our algorithm uses the fact that randomized quicksort has good comparison complexity if there are not big groups of close elements, as the adversary cannot force the pivots too far away. Thus, if randomized quicksort does not work, there must be a large cluster of close elements that we can exploit. We then estimate the order of each element using a $O(\log n)$ size sample of items, using the existence of this cluster to guarantee our estimates are accurate. Finally, we use these approximate orders to find a partition of the input items, and recursively solve as in quicksort.

Our algorithm also implies a similar selection algorithm.
Corollary 1.6. For any $i$, there exists a randomized algorithm that takes $O(n \log n)$ comparisons and returns a 4-approximate $i$-selection with probability $>1-\frac{1}{n^{2}}$.

Similarly, this result is a $\log$ factor away from optimal. Again, the best prior result is of [AFJ ${ }^{+}$18], where their analysis also shows that quickselect gives a 2 -approximate selection in $O(n)$ expected comparisons against the non-adaptive adversary. Against the adaptive adversary, this approach fails in an identical way to quicksort. Our result shows that it is possible to get a constant approximate selection in near-optimal number of comparisons even against the adaptive adversary. This was also studied in the deterministic setting [AFHN15], where an equivalent $\Omega\left(n^{9 / 8}\right)$ lower bound was shown, so we also show a distinction between randomized and deterministic in this case. Similarly, any $\widetilde{O}(n)$ deterministic algorithm could at best return a $\Omega(\log \log n)$-approximation.

Our approach is identical to the sorting algorithm, except we only have to recursively solve on the relevant side of the partition, as in quickselect.

Next, we provide a family of algorithms that explore the tradeoff between number of rounds, number of comparisons, and approximation factor in the sorting case.

Theorem 1.7. For any integer $d>0$, there exists a deterministic algorithm that takes $d$ rounds, uses $n^{1+O(1 / d)} d$ comparisons, and returns a $2 d$-approximate sorting.

Again, any approximate sorting algorithm must be able to correctly sort any list of numbers, so such an algorithm must take $\Omega\left(n^{1+1 / d}\right)$ comparisons [BT83]. Thus, this algorithm is optimal up to a constant factor of $1 / d$ in the exponent. However, this constant factor is large, as it arises from the notoriously bad constant of the AKS sorting network [AKS83]. The best prior result is the aforementioned deterministic algorithms from [AFHN15]. Their $k$-approximate algorithm uses $\Omega\left(n^{1-\frac{1}{2^{k}-1}}\right)$ rounds and $O\left(4^{k} n^{1+1 / 2^{k-1}}\right)$ comparisons. Thus, their algorithm uses $\Omega\left(n^{2 / 3}\right)$ rounds at best. We drastically improve this by giving algorithms that can use an arbitrarily small number of rounds, which could not be done by any prior algorithm (except for the trivial 1 round round robin tournament). However, our comparison bound is worse than that of [AFHN15], as it is not possible to achieve their comparison complexity even for regular sorting in rounds.

Our algorithm uses a connection between this problem and the problem of sorting networks that use a sorting oracle of arity $k$. We use a result based on the AKS sorting network [AKS83] that gives sorting networks with asymptotically optimal depth $O\left(\log _{k} n\right)$. We then show that these networks imply good algorithms for adversarial sorting, by showing that each round can incur at most 2 additional approximation error.

Since the constant factor in the exponent is large, we also provide an asymptotically worse algorithm (with respect to $d$ ) with smaller constant that is better for small constant $d$.

Theorem 1.8. For any integer $d>0$, there exists a deterministic algorithm that takes $d$ rounds, uses $n^{1+2 / \sqrt{d}} d$ comparisons, and returns a $2 d$-approximate sorting.

Our final result is an extension of these algorithms to selection algorithms that guarantee a constant approximation.

Theorem 1.9. For any integer $d>1$ and $i$, there exists a randomized algorithm that takes $d+$ $O(\log d)$ rounds, uses $n^{1+O(1 / d)} d \log n$ comparisons, and returns a 202-approximate $i$-selection with probability $>1-\frac{1}{n^{2}}$.

This result uses the previous sorting result, as well as the maximum selection in rounds result from $\left[\mathrm{GKK}^{+} 20\right]$. Similarly, such an algorithm must take $\Omega\left(n^{1+1 /(d+O(\log d))}\right)$ comparisons, so our algorithm is optimal up to a constant factor of $1 / d$ in the exponent. The best prior result is again the deterministic selection algorithms of [AFHN15], but their algorithms similarly use $\Omega\left(n^{1-\frac{1}{2^{k}-1}}\right)=\Omega\left(n^{2 / 3}\right)$ rounds. Thus, our algorithm is again a drastic improvement in terms of round complexity. On top of this, for $d=\omega(1)$, our algorithm uses $n^{1+o(1)}$ comparisons, which still beats the deterministic lower bound of [AFHN15], regardless of the number of rounds the deterministic algorithm uses. Thus, we show that randomized algorithms can beat the best deterministic algorithms even when restricted to an arbitrarily small number of rounds (as long as it increases with $n$ ).

Our algorithm repeatedly approximates the $k$-th element by taking $n^{2 / 3} \log n$ random subsets of size $n^{1 / 3}$, sorting them with depth $d$, and splitting around position $k / n^{2 / 3}$. This results in us reducing the problem to that of size $n^{5 / 6}$, which we can then solve with a constant approximation
using one of our sorting algorithms. The depth $d$ sorting does not guarantee a good approximation, so we instead use a gap-preserving property of all approximate sorting algorithms to show that this must give a good approximation if there are few elements close to the $k$-th smallest. If there are many close elements, we can instead sample a large subset and estimate the $k$-th smallest directly, which is likely to give us one of the close elements. We then show a method of combining these two algorithms to show it is possible to always get a good approximation.

The following tables summarize the previous results for the problems of approximate sorting and selection along with our contributions.

| Paper | Adversary | Randomized? | Approximation | Query Complexity | Round Complexity |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\left[\right.$ AFJ $\left.^{+} 18\right]$ | Non-Adaptive | Deterministic | 2 | $O(n \log n)$ | $O(\log n)$ |
| $[$ AFHN15 $]$ | Adaptive | Deterministic | $k$ | $O\left(4^{k} n^{1+1 / 2^{k-1}}\right)$ | $O\left(n^{1-1 /\left(2^{k}-1\right)}\right)$ |
| Our Paper | Adaptive | Randomized | 4 | $O\left(n \log ^{2} n\right)$ | $O(\log n)$ |
| Our Paper | Adaptive | Deterministic | $2 d$ | $n^{1+O(1 / d)} d \log n$ | $d$ |

Table 1: Sorting

| Paper | Adversary | Randomized? | Approximation | Query Complexity | Round Complexity |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\left[\right.$ AFJ $\left.^{+} 18\right]$ | Non-Adaptive | Deterministic | 2 | $O(n)$ | $O(\log n)$ |
| $[$ AFHN15 $]$ | Adaptive | Deterministic | $k$ | $O\left(2^{k} n^{1+1 / 2^{k-1}}\right)$ | $O\left(n^{1-1 /\left(2^{k}-1\right)}\right)$ |
| Our Paper | Adaptive | Randomized | 4 | $O(n \log n)$ | $O(\log n)$ |
| Our Paper | Adaptive | Deterministic | $2 d$ | $n^{1+O(1 / d)} d \log n$ | $d$ |
| Our Paper | Adaptive | Randomized | 202 | $n^{1+O(1 / d)} d \log n$ | $d+O(\log d)$ |

Table 2: Selection

### 1.2 Related Work

Imprecise comparisons were first considered by Rényi [Rén61] and Ulam [Ula91] in the setting of binary search. The model described allows for a bounded number of incorrect comparisons. An optimal algorithm for this problem was given by Rivest, Meyer, Kleitman, Winklmann, and Spencer $\left[\mathrm{RMK}^{+} 80\right]$ that uses $O(\log n)$ comparisons. This problem was considered in the parallel setting by Negro, Parlati, and Ritrovato [NPR95] where they give optimal algorithms for a fixed number of rounds and errors.

Binary search has also been considered in the setting where comparisons are incorrect with some probability $p<\frac{1}{2}$ [Hor63, BZ74, FRPU94]. Pelc [Pel89] gave an algorithm that uses $O(\log n)$ comparisons and gives the correct answer with probability $1-\varepsilon$ if $p<\frac{1}{3}$. For $\frac{1}{3} \leq p<\frac{1}{2}$, he gave an algorithm that uses $O\left(\log ^{2} n\right)$ comparisons. A later result from Borgstrom and Kosaraju [BK93] implies an optimal $O(\log n)$ algorithm for all $p<\frac{1}{2}$.

Sorting with imprecise comparisons was first considered by Lakshmanan, Ravikuman, and Ganesan [LRG91] in the model where the number of incorrect comparisons is bounded by a function $e(n)$. They gave a lower bound of $\Omega(n \log n+e n)$ comparisons and an upper bound of $O\left(n \log n+e n+e^{2}\right)$ comparisons. The upper bound was later improved to match the lower bound by Bagchi [Bag92] and Long [Lon92].

Sorting with comparisons that are incorrect with probability $p<\frac{1}{2}$ was considered by Feige, Peleg, Raghavan, and Upfal [FRPU94] where they gave an algorithm that uses $O(n \log (n / \varepsilon))$ queries and gives the correct answer with probability $1-\varepsilon$.

Another common model is that of sorting networks with faulty comparisons. In the model where $e$ gates may be faulty (i.e. do nothing), Yao and Yao [YY85] gave an algorithm that uses $O(n \log n+e n)$ gates. This model has been further studied in the cases where faulty gates may arbitrarily permute their inputs [AU90, LMP97] and where gates are faulty with some probability [Pio96].

Recently, a comparison model has been considered where some comparisons are not allowed to be made at all. This model was introduced by Huang, Kannan, and Khanna [HKK11] where they give a randomized algorithm that uses $\widetilde{O}\left(n^{3 / 2}\right)$ comparisons with high probability provided the input is sortable. They also give an algorithm that uses $\widetilde{O}\left(\min \left(n / p^{2}, n^{3 / 2} \sqrt{p}\right)\right)$ comparisons if the graph of forbidden comparisons is random with edge probability $1-p$. This was recently improved by Kuszmaul and Narayanan [KN22] who give corresponding algorithms using $\widetilde{O}(\sqrt{n m})$ and $O(n \log (n p))$ comparisons respectively.

None of these results apply to our comparison model, as the incorrect comparisons are either bounded or random. In our model, however, there can be any number of incorrect comparisons, which can be chosen adversarially. There is a notion of 'closeness' of elements that allows comparisons to be incorrect, for which there does not exist an analogue in other models. We note that if we were to 'disallow' all comparisons between items that differ by at most 1 , using the aforementioned algorithms would give a good approximation. However, this would require additional knowledge of which pairs of elements are sufficiently close, which the algorithm does not have in our model.

Solving comparison-based problems in rounds has also been widely considered [BR81, HH81, HH82, Kru83, Lei84, BH85, AAV86, AV87, AA88a, Bol88, BB90, AP90, WZ93, FRPU94, BMW16, BMP19, CAMTM20]. For sorting in $k$ rounds, Bollobás [Bol88] showed that $O\left(n^{1+1 / k} \frac{(\log n)^{2-2 / k}}{(\log \log n)^{1-1 / k}}\right)$ comparisons are sufficient, while Alon and Azar [AA88b] showed that $\Omega\left(n^{1+1 / k}(\log n)^{1 / k}\right)$ comparisons are necessary. For merging two sorted arrays in $k$ rounds, Haggkvist and Hell [HH82] showed that $\Theta\left(n^{1+1 /\left(2^{k}-1\right)}\right)$ comparisons is necessary and sufficient. For selecting an element of arbitrary order in $k$ rounds, Pippenger [Pip87] showed that $O\left(n^{1+1 /\left(2^{k}-1\right)}(\log n)^{2-2 /\left(2^{k}-1\right)}\right)$ comparisons are sufficient, while Alon, Azar, and Vishkin [AAV86] showed that $\Omega\left(n^{1+1 /\left(2^{k}-1\right)}(\log n)^{2 /\left(2^{k}-1\right)}\right)$ comparisons are necessary for a deterministic algorithm. Bollobás and Brightwell showed that, using a randomized algorithm, $O(n)$ comparisons is sufficient to select in $k \geq 4$ rounds with high probability. For sorted top- $m$ in $k$ rounds, Braverman, Mao, and Peres [BMP19] recently showed that $\widetilde{\Theta}\left(n^{2 / k} m^{(k-1) / k}+n\right)$ comparisons are both necessary and sufficient. Similar consideration of round complexity has also been done in the setting of best arm selection with multi-armed bandits [AAAK17, TZZ19].

These algorithms do not easily generalize to our setting, as they often heavily rely on the existence of a "correct" sorted order that is adhered to by the comparisons, as to derive information from the transitive closure of the comparison graph. We also note that the faulty comparison models considered in [BMP19, CAMTM20] suffer the same drawbacks as the previously described comparison models when extended to our model.

## 2 Techniques

Throughout, we use the fact that the naive round robin tournament algorithm that does all comparisons (denoted Tournament) guarantees a 2-approximate sorting [AFJ $\left.{ }^{+} 18\right]$. We also use the fact that any approximate sorting algorithm must preserve gaps in the input set of size at least 1 , as one
could move the values on each side of the gap arbitrarily far apart without affecting any comparisons. We say some item $x^{*}$ is a $k$-left-approximation of another item $x_{i}$ if $x^{*} \geq x_{i}-k$. Similarly, $x^{*}$ is a $k$-right-approximation of $x_{i}$ if $x^{*} \leq x_{i}+k$. Intuitively, $x^{*}$ is a left-approximation if it not "too far left" of $x_{i}$, and similarly for a right-approximation. If an element is both a $k$-left-approximation and a $k$-right-approximation of $x_{i}$ it must be a $k$-approximate $i$-selection.

### 2.1 Randomized Sorting

The main issue with sorting in the adversarial comparison setting is balancing worst-case approximation factor with worst-case number of comparisons. Standard sorting algorithms either always guarantee a good approximation, but can be forced to do many comparisons (i.e. quicksort), or always guarantee few comparisons, but can be forced to be give a bad approximation (i.e. mergesort). With deterministic algorithms, it was shown in [AFHN15] that it is not possible to get the best of both worlds, where they gave a tradeoff between approximation factor and comparisons. Our algorithm shows that in the randomized case, however, it is possible to be near-optimal in both aspects.

In the style of quicksort, our algorithm aims to partition the input set $X$ into two sets $Y$ and $\bar{Y}$ such that $\bar{Y} \geq_{4} Y$. If we can guarantee this in every recursive call, we must return a 4 -approximate sorting, as no pair differing by more than 4 can ever be put in the wrong order [AFJ $\left.{ }^{+} 18\right]$. To ensure a good bound on the number of comparisons, we also require $|Y|,|\bar{Y}| \geq n / 8$.

The first phase of our algorithm attempts to partition using random pivots $O(\log n)$ times. Let $x_{L}$ and $x_{R}$ be the $n / 8$-th smallest and $n / 8$-th largest elements of $X$ respectively. For a fixed pivot $x_{p}$, if $x_{p}>x_{L}+1$, at least $n / 8$ items must compare less than $x_{p}$. Thus, if $\left|X \cap\left[x_{L}, x_{L}+1\right]\right| \leq n / 4$, by a Chernoff bound, less than half of the pivots we try will have less than $n / 8$ items compare less with high probability. Similarly, if $\left|X \cap\left[x_{R}-1, x_{R}\right]\right| \leq n / 4$, less than half of the pivots we try will have less than $n / 8$ items compare greater with high probability. If both of these inequalities are satisfied, it follows that we will find a "good" partition from one of our pivots with high probability. Otherwise, without loss of generality we assume more than half of the pivots had less than $n / 8$ compare less. In this case, we have $\left|X \cap\left[x_{L}, x_{L}+1\right]\right|>n / 4$ with high probability. We will exploit this property in the other phases of the algorithm.

The second phase of our algorithm estimates the order of each element $x_{i}$ by comparing it to a small subset of $X$. Then, we create a partition by taking $Y$ to be the elements with estimated order less than $n / 4$, and $\bar{Y}$ the elements with estimated order at least $n / 4$. For some fixed item $x_{i}<x_{L}-1$, there must be $7 n / 8$ items that compare greater than it, so by a Chernoff bound, $x_{i}$ ends up in $Y$ with high probability. Similarly, if $x_{i}>x_{L}+2$, there must be $3 n / 8$ items that compare less than it (since $\left|X \cap\left[x_{L}, x_{L}+1\right]\right|>n / 4$ ), so by a Chernoff bound it ends up in $\bar{Y}$ with high probability. Thus, by a union bound, $\bar{Y} \geq_{3} Y$ with high probability. We note that if we did not have such high density in $\left[x_{L}, x_{L}+1\right]$, the elements of order around $n / 4$ (which could be arbitrarily far apart value-wise) would be impossible to differentiate with a small sample.

The final phase of our algorithm ensures $|Y|,|\bar{Y}| \geq n / 8$. Without loss of generality we assume $|Y| \leq|\bar{Y}|$. If $|Y| \geq n / 8$, nothing needs to be done as both sets are sufficiently large. Otherwise, we repeatedly sample $m=O(\log n)$ elements of $\bar{Y}$, sort them, and move the $m / 8$ smallest elements to $Y$ until $|Y| \geq n / 8$. Since $|Y|<n / 8$ and $\left|X \cap\left[x_{L}, x_{L}+1\right]\right|>n / 4$ before any iteration, there must be at least $n / 4$ items $\leq x_{L}$ that are not in $Y$. Thus, by a Chernoff bound, the $m / 8$ smallest elements are all $\leq x_{L}$ with high probability. Since Tournament incurs error at most 2 , it follows that $|Y| \geq_{4} Y$ at the end with high probability. Note that by choosing a random permutation to guarantee disjoint subsets, we can do this sampling in parallel.

### 2.2 Sorting in Rounds

The main issue with sorting in rounds is guaranteeing good comparison complexity. Many of the state of the art algorithms for sorting in rounds heavily rely on the existence of a "correct" sorting order to guarantee a low number of comparisons.

We use low-depth sorting networks of arity $m$ using Tournament to implement the sorting oracle. Since each call to Tournament incurs error at most 2, each round incurs error at most 2, so we can show the total error is bounded by 2 times the number of rounds.

### 2.3 Selection in Rounds

Our algorithm improves the approximation factor from the previous sections algorithm with a small $O(\log d)$ additional round overhead. We use a similar approach to the maximum selection algorithm given in $\left[\mathrm{GKK}^{+} 20\right]$, where we first describe an algorithm that gives a good approximation if there are few elements around the actual answer, then describe an algorithm that gives a good approximation if there are many elements around the actual answer, then show a way to combine them to guarantee a good approximation always. Since we are looking for an element in the middle of the sorted order, however, there is some additional complexity with considering close elements on each side of the desired element.

Let $x_{i}$ be the actual $i$-th smallest element of $X$. The first part of our algorithm guarantees a good left-side approximation if $\left|X \cap\left[x_{i}-1, x_{i}\right]\right| \leq \frac{1}{10} n^{2 / 3}$. Similarly, it guarantees a good right-side approximation if $\left|X \cap\left[x_{i}, x_{i}+1\right]\right| \leq \frac{1}{10} n^{2 / 3}$. We aim to partition $X$ into three sets $Z, Y, \Gamma$ such that elements of $Z$ are "less than" $x_{i}$, elements of $\Gamma$ are "greater than" $x_{i}$, and $Y$ is the set of "candidate" elements to be $x_{i}$. We sample $c n^{2 / 3} \log n$ subsets of $X$ of size $n^{1 / 3}$, each time sorting using the depth $d$ algorithm from the previous subsection. We then take the elements within $n^{1 / 6}$ of the $k / n^{2 / 3}$-th element of each subset and add them to $Y$ (the set of candidates). We say that elements in positions to the left of $k / n^{2 / 3}-n^{1 / 6}$ are on the left side, and the rest of the elements are on the right side. After sampling all subsets, elements which are not in the set of candidates are partitioned into $Z$ and $\Gamma$ based on how frequently they are on the left side. Assume $\left|X \cap\left[x_{i}-1, x_{i}\right]\right| \leq \frac{1}{10} n^{2 / 3}$, the other case is symmetric. In this case, roughly $90 \%$ of the sampled subsets will contain no elements in $\left[x_{i}-1, x_{i}\right]$. For each of these subsets, since our sorting algorithm must be gap-preserving, the sorting must be correct with respect to $\left[x_{i}-1, x_{i}\right]$. It thus follows by a tail bound of the Hypergeometric distribution and a union bound that for all $x_{j}<x_{i}-1, x_{j}$ will end up in $Z \cup Y$ with high probability. Similarly, for $x_{j}>x_{i}, x_{j}$ will end up in $Y \cup \Gamma$ with high probability. Finally, if $|Z| \geq k$, we return the maximum element of $Z$ computed with a depth $O(\log d)$ maximum finding algorithm from [GKK $\left.{ }^{+} 20\right]$. If $|Z|+|Y|<k$, we return the minimum element of $\Gamma$ similarly. Otherwise, we return the $(i-|Z|)$-th element of $Y$ as determined by a constant depth, constant approximate sorting algorithm from the previous section (since $|Y|=O\left(n^{5 / 6}\right)$ ). If $|Z| \geq i$, there must be some element of $Z$ that is $\geq x_{i}$, so since the maximum finding algorithm returns a constant approximation, we return a constant left-approximation. If $|Z|+|Y|<i$, we return some element of $\Gamma$, and all elements of $\Gamma$ are $\geq x_{i}-1$. Otherwise, there can be at most $i-|Z|-1$ elements of $Y$ that are $<x_{i}$, so the actual $(k-|Z|)$-th element is $\geq x_{i}$, so since we use a constant approximate sorting, the element we return is a constant left-approximation as desired.

The second part of our algorithm guarantees a good left side approximation given $\mid X \cap\left[x_{i}-\right.$ $\left.1, x_{i}\right] \left\lvert\,>\frac{1}{10} n^{2 / 3}\right.$ (and there is a symmetric algorithm that guarantees a good right side approximation). The idea is simple: sample $O\left(n^{5 / 6}\right)$ elements of $x$, sort them with a constant approximation algorithm in constant rounds, and then take the $\left(k / n^{1 / 6}-n^{5 / 12}\right)$-th element. By another Hypergeometric tail bound, it follows that we always get a good right-approximation, and also get a good
left-approximation if $\left|X \cap\left[x_{i}-1, x_{i}\right]\right|>\frac{1}{10} n^{2 / 3}$ as desired.
Finally, we describe how to combine these algorithms. First, we run the sparse algorithm and get the result $x^{*}$. Then, we count the number of elements that compare less than the result. If this value is $\leq i-1$, we must have $x^{*} \leq x_{i}+1$. We then call the left-side dense algorithm and return the greater of the two results (according to the comparator). Since one of the two algorithms must return a constant left-approximation, the latter algorithm always returns a constant right approximation, and we already know that $x^{*}$ is a constant right approximation, it follows that in the end we return a constant approximation. The case where the number of elements that compare less is $\geq i$ is handled symmetrically.

## 3 Preliminaries

In this section, we give some basic definitions and results in the adversarial comparison setting which will serve as the basis for many of our algorithms.

Definition 3.1. Element $x_{j}$ in the set $X=\left\{x_{1}, \ldots, x_{n}\right\}$ is of $k$-order $i$ if there exists a partition $S_{1}, S_{2}$ of $X \backslash\left\{x_{j}\right\}$ such that $\left|S_{1}\right|=i-1$, and $S_{2} \cup\left\{x_{j}\right\} \geq_{k} S_{1} \cup\left\{x_{j}\right\}$.

This the notion of approximate selection that was originally introduced in [AFHN15]. We show that this is equivalent to the intuitive notion we previously described, and then show that it is equivalent to something that is easier to work with.

Lemma 3.2. An item $x_{j}$ in $X$ is of $k$-order $i$ if and only if $x_{j}$ is a $k$-approximate $i$-selection.
Proof. Assume $S_{1}, S_{2}$ exist as in the definition of $x_{j}$ being $k$-order $i$. Then, since $S_{2} \cup\left\{x_{j}\right\} \geq_{k}$ $S_{1} \cup\left\{x_{j}\right\}$, the concatenation of the sorted order of $S_{1}, x_{j}$, and the sorted order of $S_{2}$ in that order is a $k$-approximate sorting by definition. Thus, $x_{j}$ is a $k$-approximate $i$-selection as desired.

Similarly, assume there exists a $k$-approximate sorting $Y$ of $X$ where $y_{i}=x_{j}$. Then, taking $S_{1}$ to be the first $i-1$ elements of $Y$, and $S_{2}$ to be the final $n-i$ elements, it follows by definition that $S_{2} \cup\left\{x_{j}\right\} \geq_{k} S_{1} \cup\left\{x_{j}\right\}$. Thus, $x_{j}$ is of $k$-order $i$ as desired.

Lemma 3.3. An item $x_{j}$ in $X$ is a $k$-approximate $i$-selection if and only if $\left|x_{j}-x_{i}\right| \leq k$ where $x_{i}$ is the actual $i$-th smallest element of $X$.

Proof. Consider some $k$-approximate sorting $Y$ of $X$ where $y_{i}=x_{j}$. Without loss of generality, assume $x_{j} \leq x_{i}$. Since $x_{i}$ is the $i^{\text {th }}$ smallest element of $X$, there must be at least $n-i+1$ elements of $X$ that are $\geq x_{i}$. Thus, there must exist some element $x_{\ell} \geq x_{i}$ that is to the left of $x_{j}$ in $Y$ since there are only $n-i$ places to the right that they can be. It follows that $x_{j} \geq_{k} x_{\ell}$, which implies $x_{j} \geq_{k} x_{i}$ since $x_{\ell} \geq x_{i}$. Thus, $x_{i} \geq x_{j} \geq x_{i}-k$, so $\left|x_{j}-x_{i}\right| \leq k$.

Consider some element $x_{j}$ in $X$ such that $\left|x_{j}-x_{i}\right| \leq k$. Let $Y$ be the sorted order of $X$, but with $x_{j}$ and $x_{i}$ swapped. All pairs that do not contain $x_{i}$ or $x_{j}$ must still be in the right order. Pairs $\left(x_{i}, x_{\ell}\right)$ that are in the wrong order must have $x_{\ell}$ between $x_{i}$ and $x_{j}$ (or equal to $x_{j}$ ), so $x_{\ell}$ and $x_{i}$ must differ by at most $k$. An identical argument applies to pairs $\left(x_{j}, x_{\ell}\right)$, so it follows that $Y$ is a $k$-approximate sorting as desired.

Corollary 3.4. If $Y$ is a $k$-approximate sorting of $X$, and $S$ is the actual sorting of $X,\left|y_{i}-s_{i}\right| \leq k$ for all $i$.

We define the notion of a gap-preserving algorithm, where elements must be correctly sorted with respect to a gap of size 1 . This will be useful in proving the correctness of our later algorithms.

Definition 3.5. We say a sorting algorithm is gap-preserving if, given there exists a gap of length 1 in the input, the sorting algorithm returns all elements before the gap before all elements after the gap. Formally, given input $X$ such that there exists a gap $[y, y+1)$ where $X \cap[y, y+1)=\emptyset$, the sorting algorithm must return all elements of $X$ less than $y$ before all elements of $X$ greater than $y$.

Lemma 3.6. All approximate sorting algorithms are gap-preserving.
Proof. We can shift the elements on one side of the gap arbitrarily far away without affecting any comparison results. Thus, if there exists an input for which a $\tau(n)$-approximate algorithm is not gap-preserving, then we can make the approximation factor larger than $\tau(n)$ by shifting one side by more than that, a contradiction.

Recall that the Tournament algorithm for sorting a set $X$ of items, as originally defined in [AFHN15], does all pairwise comparisons, and orders the items by the number of "wins" they have (i.e. the number of elements that compare less).

Lemma 3.7. Tournament is a 2-approximate sorting algorithm.
Proof. If $x_{i}>x_{j}+2, x_{i}$ must compare greater than all elements $\leq x_{j}+1$ including $x_{j}$. However, $x_{j}$ can at most compare greater than all elements $\leq x_{j}+1$ excluding itself, so it must come before $x_{i}$ as desired.

We show that partitioning as in quicksort guarantees a good approximation factor, which will be the basis of our randomized sorting algorithm. This was originally shown in $[A F J+18]$.

Lemma 3.8. Let $x_{i}$ be some item in a set $X$. Let $S=\left\{x_{j} \mid x_{j}<_{c} x_{i}\right\}$ and $T=X \backslash S$. We must have $T \geq_{2} S$.

Proof. All elements of $S$ must be $\leq x_{i}+1$, and all elements of $T$ must be $\geq x_{i}-1$. Thus, for $x_{j} \in S, x_{k} \in T, x_{j}-x_{k} \leq x_{i}+1-\left(x_{j}-1\right)=2$ as desired.

Lemma 3.9. If a sorting algorithm repeatedly partitions the input set $X$ into two sets $S, T$ such that $T \geq_{k} S$, recursively sorts $S$ and $T$ and then concatenates them, it is guaranteed to result in a $k$-approximate sorting.

Proof. Assume for the sake of contradiction that there exist $x_{i}, x_{j}$ for $i>j$ in the final order such that $x_{i} \not ¥_{k} x_{j}$. We must have put $x_{i}$ in $T$ and $x_{j}$ in $S$ in some recursive call, but this contradicts $T \geq_{k} S$, as desired.

Throughout, when referring to a sorted order, we assume a fixed sorted order with ties broken arbitrarily. Unless otherwise stated, all logarithms are base $e$. We often ignore rounding errors that vanish for large $n$.

## 4 A Randomized Sorting (and Selection) Algorithm

In this section we prove Theorem 1.5 by describing an algorithm RSort. This algorithm is similar to quicksort in the sense that we aim to partition the original set of items $X$ into two sets $S, T$, and then recursively sort $S$ and $T$ and concatenate them. Recall by Lemma 3.9 that it is sufficient to have $T \geq_{4} S$ in every call. Thus, our algorithm aims to find a partition $S, T$ of $X$ such that $T \geq_{4} S$. To
ensure the recursion depth is $O(\log |X|)$, we also aim to have $|S|,|T| \geq \frac{|X|}{8}$. Our algorithm consists of three phases, which we will analyze independently. Throughout the algorithm, we let $n$ be the size of the current set $X$ the function is being called on, and we let $N$ be the size of the initial set $X$ that RSort was called on. This distinction is important, as we want our probability guarantees to be with respect to the size of the original caller.

### 4.1 The Pivot Phase

```
Algorithm 1 Pivot Phase
    \(R \leftarrow 0\)
    \(L \leftarrow 0\)
    loop \(8 c_{1} \log N\) times
        pick a pivot \(x_{p}\) at random
        \(Y \leftarrow\left\{x \in X: x<_{c} x_{p}\right\}\)
        \(\bar{Y} \leftarrow X \backslash Y\)
        if \(\min (|Y|,|\bar{Y}|) \geq \frac{n}{8}\) then
            return \((Y, \bar{Y})\)
        else if \(|Y|<\frac{n}{8}\) then
            \(L \leftarrow L+1\)
        else
            \(R \leftarrow R+1\)
        end if
    end loop
```

In this phase, we aim to use a pivot as in quicksort to find the desired partition, in which case we return early. If we do not find such a pivot, the input set has additional structure with high probability, which we will use in the rest of the algorithm.
$8 c_{1} \log N$ elements $x_{p}$ are randomly chosen and used as pivots. This splits $X$ into two sets $Y$ and $\bar{Y}$ such that $\bar{Y} \geq_{2} Y$ by Lemma 3.8. If both of these sets are sufficiently large, then we have found our desired partition, and we return early. Otherwise, if $|Y|<\frac{n}{8}$, we say $x_{p}$ goes left and if $|\bar{Y}|<\frac{n}{8}$, we say $x_{p}$ goes right. Let $x_{L}$ be the $n / 8$-th smallest element of $X$ and $x_{R}$ the $n / 8$-th largest. If $x_{p}>x_{L}+1$, then $x_{p}$ cannot go left, and symmetrically, if $x_{p}<x_{R}-1$, then $x_{p}$ cannot go right. Let $X_{L}=\left\{x_{p} \in X: x_{p} \leq x_{L}+1\right\}$ and $X_{R}=\left\{x_{p} \in X: x_{p} \geq x_{R}-1\right\}$. The elements in $X \backslash\left(X_{L} \cup X_{R}\right)$ are thus guaranteed to neither go left or go right. Intuitively, if this set is sufficiently big, we expect to find such a pivot. Otherwise, either $X_{L}$ or $X_{R}$ must be large. The variables $L$ and $R$ in the code count how many pivots go left and right respectively. Again intuitively, we expect $L>R$ if $X_{R}$ is small and vice versa. These intuitive statements are captured in the following lemmas:

Lemma 4.1. If $\left|X_{L}\right|<\frac{3 n}{8}$, for any constant $r>0$, we can choose $c_{1}$ sufficiently large such that $\operatorname{Pr}\left[L \geq 4 c_{1} \log N\right.$ after pivot phase $]<\frac{1}{N^{r}}$.

Proof. Let $A_{i}$ be a random variable that takes value 1 if the $i^{\text {th }}$ pivot $x_{p}$ is in $X_{L}$, and 0 otherwise. Note that if $A_{i}$ is 0 , we cannot increment $L$ in the $i^{\text {th }}$ iteration, so we have $L \leq A=\sum_{i} A_{i}$. Let $\mu=\mathbb{E}[A]=8 c_{1} \log N \frac{\left|X_{L}\right|}{n} \geq c_{1} \log N$. By a Chernoff bound, we have:

$$
\begin{aligned}
\operatorname{Pr}\left[L \geq 4 c_{1} \log N \text { at line } 20\right] & \leq \operatorname{Pr}\left[A \geq 4 c_{1} \log N\right] \\
& =\operatorname{Pr}[A \geq(1+\delta) \mu]
\end{aligned}
$$

Where $\delta=\frac{n}{2\left|X_{L}\right|}-1>\frac{1}{3}$

$$
\begin{aligned}
& \leq e^{-\frac{\delta^{2} \mu}{2+\delta}} \\
& <e^{-\frac{\mu}{21}} \\
& \leq e^{-\log N \frac{c_{1}}{21}} \\
& =N^{-\frac{c_{1}}{21}}
\end{aligned}
$$

Thus, choosing $c_{1} \geq 21 r$, we get

$$
\operatorname{Pr}\left[L \geq 4 c_{1} \log N \text { at line } 20\right]<\frac{1}{N^{r}}
$$

Corollary 4.2. If $\left|X_{R}\right|<\frac{3 n}{8}$, for any constant $r>0$, we can choose $c_{1}$ sufficiently large such that $\operatorname{Pr}\left[R \geq 4 c_{1} \log N\right.$ after pivot phase $]<\frac{1}{N^{r}}$.

## Proof. Symmetric.

Throughout the rest of the analysis, we will assume $L \geq R$. The other case is handled symmetrically.

### 4.2 The Sample Phase

```
Algorithm 2 Sample Phase
    \(Y \leftarrow \emptyset\)
    for \(x_{i} \in X\) do
        \(C \leftarrow 0\)
        loop \(8 c_{2} \log N\) times
            Choose \(z \in X\) at random
            if \(z<_{c} x_{i}\) then
                        \(C \leftarrow C+1\)
            end if
        end loop
        if \(C<2 c_{2} \log N\) then
            \(Y \leftarrow Y \cup\left\{x_{i}\right\}\)
        end if
    end for
```

In this phase, for each element $x_{i}$ we estimate its position in the sorted array by comparing it to a small subset of $X$. All elements with estimated position less than $\frac{n}{4}$ are put in set $Y$ and the remaining elements are put in $\bar{Y}$. Since $\left|X_{L}\right| \geq \frac{3 n}{8}$, all elements with positions between $\frac{n}{8}$ and $\frac{3 n}{8}$ are in $\left[x_{L}, x_{L}+1\right]$. Thus, since all elements $<x_{L}-1$ compare less than all of these elements, we intuitively expect them to have estimated position less than $\frac{n}{4}$ even on a small subset. Similarly, since all elements $>x_{L}+2$ compare greater than all of those elements, we intuitively expect them to have estimated position greater than $\frac{n}{4}$. These statements are captured in the following lemmas:
Lemma 4.3. If $\left|X_{L}\right| \geq \frac{3 n}{8}$, for any constant $r>0$ we can choose $c_{2}$ sufficiently large such that

$$
\operatorname{Pr}\left[\exists x_{i}<x_{L}-1: C \geq 2 c_{2} \log N\right]<\frac{1}{N^{r}}
$$

Proof. Let $U$ be the set of the smallest $n / 8$ elements of $X$. Consider some iteration of the loop on line 25 where $x_{i}<x_{L}-1$. Let $A_{i}$ be a random variable that takes value 1 if the $i^{\text {th }}$ random element is $\in U$ and 0 otherwise. Note that if $A_{i}$ is 0 , we cannot increment $C$ in the $i^{\text {th }}$ iteration. Thus, $C \leq A=\sum_{i} A_{i}$. Let $\mu=\mathbb{E}[A]=8 c_{2} \log N \frac{|U|}{n}=c_{2} \log N$, we have:

$$
\begin{aligned}
\operatorname{Pr}\left[C \geq 2 c_{2} \log N\right] & \leq \operatorname{Pr}\left[A \geq 2 c_{2} \log N\right] \\
& =\operatorname{Pr}[A \geq(1+\delta) \mu]
\end{aligned}
$$

Where $\delta=1$

$$
\begin{aligned}
& \leq e^{-\frac{\delta^{2} \mu}{2+\delta}} \\
& =e^{-\frac{\mu}{3}} \\
& =e^{-\log N \frac{c_{2}}{3}} \\
& =N^{-\frac{c_{2}}{3}}
\end{aligned}
$$

Thus, choosing $c_{2} \geq 3(r+1)$, we get

$$
\operatorname{Pr}\left[C \geq 2 c_{2} \log N\right] \leq \frac{1}{N^{r+1}}
$$

By a union bound:

$$
\begin{aligned}
\operatorname{Pr}\left[\exists x_{i}<x_{L}-1: C \geq 2 c_{2} \log N\right] & \leq \#\left\{x_{i}<x_{L}-1\right\} \frac{1}{N^{r+1}} \\
& <\frac{1}{N^{r}} .
\end{aligned}
$$

Lemma 4.4. If $\left|X_{L}\right| \geq \frac{3 n}{8}$, for any constant $r>0$ we can choose $c_{2}$ sufficiently large such that

$$
\operatorname{Pr}\left[\exists x_{i}>x_{L}+2: C<2 c_{2} \log N\right]<\frac{1}{N^{r}} .
$$

Proof. Similar to the previous Lemma, using the fact that $\left|X_{L}\right| \geq \frac{3 n}{8} \Longrightarrow \#\left\{x_{i} \leq x_{L}+1\right\} \geq \frac{3 n}{8}$.
Corollary 4.5. If $\left|X_{L}\right| \geq \frac{3 n}{8}$, for any constant $r>0$ we can choose $c_{2}$ sufficiently large such that after the sample phase, $\operatorname{Pr}\left[\max (Y)>x_{L}+2\right]<\frac{1}{N^{r}}$.

Proof. If $\max (Y)>x_{L}+2$ then we must have had $C<2 c_{2} \log N$ for some $x_{i}>x_{L}+2$, which happens with probability $<\frac{1}{N^{r}}$ by the previous Lemma.
Corollary 4.6. If $\left|X_{L}\right| \geq \frac{3 n}{8}$, for any constant $r>0$ we can choose $c_{2}$ sufficiently large such that after the sample phase, $\operatorname{Pr}\left[\min (X \backslash Y)<x_{L}-1\right]<\frac{1}{N^{r}}$.

### 4.3 The Shifting Phase

```
Algorithm 3 Shifting Phase
    Let \(P\) be a random permutation of \(X \backslash Y\)
    \(i \leftarrow 0\)
    \(B \leftarrow 4 c_{3} \log N\)
    while \(|Y|<\frac{n}{8}\) do
        \(Z \leftarrow P[i . . i+7 B)\)
        Tournament \((Z)\)
        \(Y \leftarrow Y \cup Z[0 . . B)\)
        \(i \leftarrow i+7 B\)
    end while
    Let \(P\) be a random permutation of \(Y\)
    \(i \leftarrow 0\)
    while \(|Y|>\frac{7 n}{8}\) do
        \(Z \leftarrow P[i . . i+7 B)\)
        \(Z \leftarrow \operatorname{Tournament}(Z)\)
        \(Y \leftarrow Y \backslash Z[6 B . .7 B)\)
        \(i \leftarrow i+7 B\)
    end while
    Let \(\bar{Y}=X \backslash Y\)
    return \((Y, \bar{Y})\)
```

In this phase, if either $Y$ or $\bar{Y}$ is too big, we move some elements to the other set to ensure they both have size $\geq n / 8$. Since the two cases are symmetric, without loss of generality, we assume $|Y| \leq|\bar{Y}|$. We partition $\bar{Y}$ into small subsets, and move the minimum $1 / 8$-th of each subset into $Y$ until $|Y| \geq n / 8$. Since at least $3 n / 8$ elements of $X$ are $\leq x_{L}+1$, at least $1 / 4$-th of the elements in $\bar{Y}$ are $\leq x_{L}+1$, so even for small subsets we expect the smallest $1 / 8$-th to be all $\leq x_{L}+1$. Thus, since Tournament returns a 2-approximate sorting by Lemma 3.7, we expect the elements we add to $Y$ to be $\leq x_{L}+3$. These intuitive statements are captured in the following lemmas:

Lemma 4.7. If $\left|X_{L}\right| \geq \frac{3 n}{8}$ and $\max (Y) \leq x_{L}+2$ after the sample phase, for any constant $r>0$ we can choose $c_{3}$ sufficiently large such that after the shifting phase, $\operatorname{Pr}\left[\max (Y)>x_{L}+3\right]<\frac{1}{N^{r}}$

Proof. Consider some iteration of the loop on line 40. Recall that Tournament returns a 2 approximate sorting. Thus, if in each iteration of the loop, $Z$ has at least $B$ elements $\leq x_{L}+1$, then we are guaranteed to only add elements $\leq x_{L}+3$ to $Y$. Since $\left|X_{L}\right| \geq \frac{3 n}{8}$ and $|Y|<\frac{n}{8}$, there must be at least $\frac{n}{4}$ elements $x \in X \backslash Y$ such that $x \leq x_{L}+1$. Let $U$ be the set of the $\frac{n}{4}$ smallest elements of $X \backslash Y$, breaking ties arbitrarily. Let $A_{i}$ be a random variable that takes value 1 if $Z_{i} \notin U$ (before sorting), and 0 otherwise. Note that for any subset $S$ of $\left\{A_{i}\right\}$, $\operatorname{Pr}\left[\bigwedge_{i \in S} A_{i}\right]=\operatorname{Pr}\left[A_{S_{0}}\right] \operatorname{Pr}\left[A_{S_{1}} \mid A_{S_{0}}\right] \ldots \operatorname{Pr}\left[A_{S_{|S|} \mid} \mid A_{S_{0}}, \ldots, A_{S_{|S|-1}}\right]=\frac{3}{4} \frac{\frac{3 n}{4}-1}{n-1} \ldots \frac{\frac{3 n}{4}-|S|}{n-|S|} \leq\left(\frac{3}{4}\right)^{|S|}$. Let
$C=\#\left\{x_{i} \in Z \mid x_{i}>x_{L}+1\right\}$. Clearly, $x_{i}>x_{L}+1 \Longrightarrow x_{i} \notin U$, so $C \leq A=\sum_{i} A_{i}$. We have:

$$
\begin{aligned}
\operatorname{Pr}[C \geq 6 B] & \leq \operatorname{Pr}[A \geq 6 B] \\
& \leq e^{-7 B\left(2\left(\frac{6}{7}-\frac{3}{4}\right)^{2}\right)} \\
& =e^{-\frac{9 B}{56}} \\
& =e^{-\log N \frac{9 c_{3}}{14}} \\
& =N^{-\frac{9 c_{3}}{14}}
\end{aligned}
$$

Thus, choosing $c_{3} \geq \frac{14(r+1)}{9}$

$$
\leq \frac{1}{N^{r+1}}
$$

By a union bound:

$$
\begin{aligned}
\operatorname{Pr}[C \geq 5 B \text { on some iteration }] & \leq \frac{n}{B} \frac{1}{N^{r+1}} \\
& <\frac{1}{N^{r}} .
\end{aligned}
$$

Here we use the generalized Chernoff bound from Theorem 1.1 of [IK10].
Lemma 4.8. If $\left|X_{L}\right| \geq \frac{3 n}{8}$ and $\min (X \backslash Y) \geq x_{L}-1$ after the sample phase, for any constant $r>0$ we can choose $c_{3}$ sufficiently large such that after the shifting phase, $\operatorname{Pr}\left[\min (X \backslash Y)<x_{L}-2\right]<\frac{1}{N^{r}}$.
Proof. Similar to Lemma 9, noting that since $x_{L}$ is the $\frac{n}{8}$ th smallest element of $X$ and $|X \backslash Y|<\frac{n}{8}$, there must be at least $\frac{3 n}{4}$ elements $x \in Y$ such that $x \geq x_{L}$.

### 4.4 Tying it together

We conclude the probability bounds for the algorithm and describe the comparison and round complexity.

Lemma 4.9. For any constant $r>0$, we can choose $c_{1}, c_{2}, c_{3}$ sufficiently large such that the probability that we split $X$ into sets $Y, \bar{Y}$ such that $\bar{Y} \geq_{4} Y$ is $>1-\frac{1}{N^{r}}$.

Proof. As described in the previous sections, there are 6 failure points at which something may go wrong and we may end up with $\bar{Y} \not \unlhd_{4} Y$. By a union bound, it follows that the probability that $\bar{Y} \geq_{4} Y$ is at least $1-\frac{6}{N^{r+1}}>1-\frac{1}{N^{r}}$ for sufficiently large $c_{1}, c_{2}, c_{3}$ as desired.

Theorem 4.10. For any constant $r>0$, we can choose $c_{1}, c_{2}, c_{3}$ sufficiently large such that RSort returns a 4 -approximate sorting with probability $>1-\frac{1}{N^{r}}$.
Proof. Recall that it is sufficient for every recursive call to satisfy $\bar{Y} \geq_{4} Y$. Since we reduce the size of the input by a constant factor in each recursive call, there must be $O(N)$ total recursive calls. Thus, by a union bound, we return a 4 -approximate sorting with probability at least $1-O(N) \frac{1}{N^{r+2}}>$ $1-\frac{1}{N^{r}}$ as desired.

Theorem 4.11. RSort uses $O\left(N \log ^{2} N\right)$ comparisons.
Proof. It is clear that any recursive call takes $O(n \log N)$ comparisons. It follows by a well known recurrence that $O\left(N \log ^{2} N\right)$ comparisons are thus required in total.

Theorem 4.12. RSort uses in $O(\log N)$ rounds.

Proof. The different iterations of each loop in RSort are clearly independent, so we can do them in parallel. Thus each call to RSort takes $O(1)$ rounds. Additionally, each layer of the recursion can also be done in parallel. Since we reduce the size of the input by a constant factor in each call, the recursion depth is $O(\log N)$ and thus the algorithm works in $O(\log N)$ rounds.

Theorem 1.5 thus follows from the previous three Theorems.
By only recursively solving on the relevant side, this sorting algorithm implies a selection algorithm that returns a 4 -approximation with probability $>1-\frac{1}{N^{r}}$ that uses $O(N \log N)$ comparisons and $O(\log N)$ rounds. Corollary 1.6 thus follows.

## 5 A General Sorting Algorithm In Rounds

In this section, we use a connection to sorting networks to give a general sorting algorithm in rounds. We consider sorting networks of arity $k$ : rather than being able to compare and swap two elements, we can sort any $k$ elements.

Theorem 5.1. [Chv92] For all $m \geq 2$, there exists an arity $m$ sorting network of depth $O\left(\log _{m} n\right)$.
Corollary 5.2. For any integer $d>0$, there exists a sorting network of arity $n^{O(1 / d)}$ and depth $d$.
This result comes from the AKS sorting network construction [AKS83], which has a notoriously big constant factor. Thus, we also consider asymptotically worse (with respect to $d$ ) networks with smaller constant factors, which are better for small $d$.

Theorem 5.3. [PP89] For all $m \geq 2$, there exists an arity $m$ sorting network of depth $4 \log _{m}^{2} n$.
Corollary 5.4. For any integer $d>0$, there exists a sorting network of arity $n^{2 / \sqrt{d}}$ and depth $d$.
We connect this result to the adversarial comparison setting by showing that these sorting networks imply approximate sorting algorithms. Since Tournament gives a 2 -approximate sorting, by implementing the sorting oracle with Tournament, we in some sense guarantee that the total approximation error only accumulates by 2 on each level of the network. Thus, for a depth $d$ network, we get a $2 d$-approximate algorithm.

Lemma 5.5. Let $a$ and $b$ be arrays of length $n$. If $\left|a_{i}-b_{i}\right| \leq k$ for all $i$, then $\mid \operatorname{sorted}(a)[i]-$ $\operatorname{sorted}(b)[i] \mid \leq k$ for all $i$.

Proof. We proceed by induction over $n$. When $n=1$, the result is trivial. Otherwise, let $i=\operatorname{argmin}(a), j=\operatorname{argmin}(b)$. Without loss of generality, assume $a[i] \leq b[j]$. If $i=j$, then $|\operatorname{sorted}(a)[0]-\operatorname{sorted}(b)[0]| \leq k$ and the result follows by the induction hypothesis. Otherwise, we claim that $|b[i]-a[j]| \leq k$. If $b[i] \geq a[j]$, then $|b[i]-a[j]|=b[i]-a[j] \leq b[i]-a[i] \leq k$. Otherwise, $|b[i]-a[j]|=a[j]-b[i] \leq a[j]-b[j] \leq k$. Thus, we can swap $a[i]$ and $a[j]$ and the assumption still holds. We thus reduce to the already solved $i=j$ case as desired.

Lemma 5.6. If there exists a sorting network with arity $k$ and depth $d$, then there exists $a 2 d$ approximate sorting algorithm in d rounds that takes $O(n k d)$ comparisons.

Proof. Consider directly running the sorting network, using Tournament to sort. Clearly, $O(d n / k)$ groups are sorted, and each takes $O\left(k^{2}\right)$ time, so the total time taken is $O(n k d)$. We claim that after the $r$-th round, the current element at position $i$ differs by the "correct" element at position $i$ (the element that would be there if all comparisons were correct) by at most $2 r$. We prove this
by induction. When $r=0$, the result is trivial. Otherwise, after $r-1$ rounds, each element must differ by at most $2 r-2$ from the "correct" element. By the previous lemma, it follows that in each group that is being sorted, the elements of the correct sorting of the current elements differ by the elements of the correct sorting of the correct elements by at most $2 r-2$. Since Tournament gives a 2 -approximate sorting, it follows by Corollary 3.4 that after sorting the elements differ by the "correct" elements by at most $2 r$ by the triangle inequality as desired.

Theorem 1.7 and Theorem 1.8 follow. By letting $d$ be an arbitrarily large constant, we can get a constant round, constant approximate algorithm that uses $O\left(n^{1+\varepsilon}\right)$ comparisons for any $\varepsilon>0$.

## 6 A General Selection Algorithm In Rounds

In this section, we extend the sorting algorithms in the previous section to selection algorithms that return a constant approximation regardless of $d$. We first provide an algorithm that gives a good approximation if there are few elements close to the answer. Then, we provide an algorithm that gives a good approximation if there are many elements close to the answer. We then show that it is possible to combine these to always achieve a constant approximation.

### 6.1 Sparse Selection

Let $L_{x}=\left\{x_{i} \mid x_{k}-1 \leq x_{i} \leq x_{k}\right\}$ and $R_{x}=\left\{x_{i} \mid x_{k} \leq x_{i} \leq x_{k}+1\right\}$. This part of the algorithm returns a 200 -approximation on the left side if $\left|L_{x}\right|$ is sufficiently small, and a 200-approximation on the right side if $\left|R_{x}\right|$ is sufficiently small. Specifically, if $\left|L_{x}\right| \leq \frac{1}{10} n^{2 / 3}$, then $x^{*} \geq_{200} x_{i}$ where $x^{*}$ is the returned item. Similarly, if $\left|R_{x}\right| \leq \frac{1}{10} n^{2 / 3}$, then $x_{i} \geq_{200} x^{*}$.

We aim to partition $X$ into three sets: $Z, Y, \Gamma$ where $Z$ is the set of elements definitely to the left of $x_{k}, Y$ is the set of candidate elements to be $x_{k}$, and $\Gamma$ is the set of elements definitely to the right of $x_{k}$. We also want $|Y|=O\left(n^{1-\varepsilon}\right)$, so we can sort $Y$ with a constant approximate algorithm. We sample $c n^{2 / 3} \log n$ subsets of $X$ of size $n^{1 / 3}$, each time sorting with the $d$ round algorithm from the previous section. We then take the elements of each subset close to the $k / n^{1 / 3}$-th position and add them to the set of candidates. The elements that are not candidates at the end are partitioned into left and right depending whether they were to the left or the right of the $k / n^{1 / 3}$-th position more frequently. If $\left|L_{x}\right|$ is sufficiently small, we expect most of the subsets to not contain any elements of $L_{x}$, and thus since Sort must be gap-preserving, the subsets must be roughly correctly sorted around position $k / n^{1 / 3}-\frac{\left|L_{x}\right|}{2}$. Thus, we expect our candidates to be $\geq x_{k}-1$. Similarly, when $\left|R_{x}\right|$ is sufficiently small, we expect our candidates to be $\leq x_{k}+1$. This gives us our desired result.

Let $d>1$ be arbitrary. Let Sort be the $d$ round sorting algorithm from Theorem 1.7 and let 100 -Sort be the sorting algorithm obtained by taking $d=100$ in Theorem 1.8.

Theorem 6.1. [GKK+ 20] For any $r>0$, there exists a $\log _{2} d$ round maximum/minimum finding algorithm GetMax/GetMin that uses $O\left(n^{1+\frac{1}{d-1}} \log d \log n\right)$ comparisons and returns a 5 -approximate maximum/minimum with probability $>1-\frac{1}{n^{r}}$.
${ }^{*}$ Their algorithm guarantees a 3 -approximation, but only probability $\geq 0.9$. Running it $O(\log n)$ times and taking the tournament maximum/minimum of the results boosts the probability to $>1-\frac{1}{n^{r}}$, but increases the approximation factor to 5 .

```
Algorithm 4 Sparse Selection
    function \(\operatorname{SelectSparse}(X, k)\)
        \(Y \leftarrow \emptyset\)
        \(L \leftarrow[0] * n\)
        loop \(c n^{2 / 3} \log n\) times
            Generate a subset \(S\) of \(X\) of size \(n^{1 / 3}\)
            \(S \leftarrow \operatorname{Sort}(S)\)
            \(T \leftarrow S\left[k / n^{2 / 3}-n^{1 / 6}: k / n^{2 / 3}+n^{1 / 6}\right]\)
            for \(x_{i} \in S\left[: k / n^{2 / 3}-n^{1 / 6}\right]\) do
                    \(L[i] \leftarrow L[i]+1\)
            end for
            \(Y \leftarrow Y \cup T\)
        end loop
        \(Z \leftarrow \emptyset\)
        for \(i=0 . . n-1\) do
            if \(x_{i} \notin Y\) and \(L[i]>\frac{c}{2} \log n\) then
                \(Z \leftarrow Z \cup\left\{x_{i}\right\}\)
            end if
        end for
        \(\Gamma \leftarrow X \backslash(Y \cup Z)\)
        if \(k \leq|Z|\) then
            return \(\operatorname{GetMax}(Z)\)
        else if \(k \leq|Z|+|Y|\) then
            \(Y \leftarrow 100\)-Sort \((Y)\)
            return \(Y[k-|Z|-1]\)
        else
            return \(\operatorname{GetMin}(\Gamma)\)
        end if
    end function
```

Lemma 6.2. If $\left|L_{x}\right| \leq \frac{1}{10} n^{2 / 3}$, for $r>0$ and $x_{i}>x_{L}$ there exists c large enough that $\operatorname{Pr}\left[L[i]>\frac{c}{2} \log n\right]<$ $\frac{1}{n^{r}}$.
Proof. Consider the iterations in which $x_{i}$ is chosen. By a union bound,

$$
\operatorname{Pr}\left[L_{x} \cap S \neq \emptyset \mid x_{i} \in S\right] \leq\left|L_{x}\right| \frac{n^{1 / 3}-1}{n-1} \leq \frac{1}{10} n^{2 / 3} \frac{n^{1 / 3}-1}{n-1} \leq \frac{2}{10}
$$

for $n$ sufficiently large. Conditioning on $x_{i}$ being $\in S$, the number of elements of $S$ that are $\leq x_{k}$ (call this $V$ ) follows a $\operatorname{Hypergeometric}\left(n, k, n^{1 / 3}\right)^{\dagger}$ distribution. By a tail bound:

$$
\begin{aligned}
\operatorname{Pr}\left[V \leq k / n^{2 / 3}-n^{1 / 6} \mid x_{i} \in S\right] & =\operatorname{Pr}\left[V \leq\left(k / n-n^{-1 / 6}\right) n^{1 / 3}\right] \\
& \leq e^{-2\left(n^{-1 / 6}\right)^{2} n^{1 / 3}} \\
& =e^{-2} .
\end{aligned}
$$

If $L_{x} \cap S=\emptyset$ and $V>k / n^{2 / 3}-n^{1 / 6}$, since Sort is gap-preserving by Lemma 3.6, $L[i]$ cannot increase in this iteration. It thus follows by a union bound that $L[i]$ increases with probability at

[^1]most $\frac{2}{10}+e^{-2}<0.4$. Thus, $\operatorname{Pr}\left[x_{i} \in S\right.$ and $L[i]$ increases $]<\frac{0.4}{n^{2 / 3}}$. Let $\mu=\mathbb{E}[L[i]] \leq c n^{2 / 3} \log n \frac{0.4}{n^{2 / 3}}=$ $0.4 c \log n$. By a Chernoff bound:
\[

$$
\begin{aligned}
\operatorname{Pr}\left[L[i]>\frac{c}{2} \log n\right] & \leq \operatorname{Pr}[L[i]>(1+1 / 4) \mu] \\
& \leq e^{-(1 / 4)^{2} \mu /(2+1 / 4)} \\
& \leq e^{-0.4 c \log n / 36}
\end{aligned}
$$
\]

Choosing $c>90 r$ :

$$
<\frac{1}{n^{r}}
$$

as desired.

Corollary 6.3. If $\left|L_{x}\right| \leq \frac{1}{10} n^{2 / 3}$, for any $r>0$ there exists $c$ large enough that $\operatorname{Pr}\left[Z \cap\left(x_{L}, \infty\right)=\emptyset\right]>$ $1-\frac{1}{n^{r}}$.

Proof. This follows by a union bound and the previous lemma.
Corollary 6.4. If $\left|L_{x}\right| \leq \frac{1}{10} n^{2 / 3}$, for any $r>0$ there exists c large enough that $\operatorname{Pr}\left[\Gamma \cap\left(-\infty, x_{L}-1\right)=\emptyset\right]>$ $1-\frac{1}{n^{r}}$.

Proof. Symmetric.
Let $x^{*}$ be the value returned by SelectSparse.
Lemma 6.5. If $\left|L_{x}\right| \leq \frac{1}{10} n^{2 / 3}$, for any $r>0$ there exists c large enough that $\operatorname{Pr}\left[x^{*} \geq_{200} x_{k}\right]>1-\frac{1}{n^{r}}$.
Proof. We claim it suffices that $Z \cap\left(x_{L}, \infty\right)=\emptyset$ and $\Gamma \cap\left(-\infty, x_{L}-1\right)=\emptyset$. If $|Z| \geq k$, then there must be an element of $Z$ that is $\geq x_{k}$. Thus, since GetMax returns a 5 -approximation with sufficiently large probability, $x^{*} \geq_{5} x_{k}$ with sufficiently large probability. If $|Z|+|Y|<k$, then we return some element of $\Gamma$ which is $\geq_{1} x_{k}$ if $\Gamma \cap\left(-\infty, x_{L}-1\right)=\emptyset$. Otherwise, if $|Z|<k$ and $|Z|+|Y| \geq k$, there must be at most $k-|Z|-1$ elements of $Y$ that are $<x_{k}$. Thus, the $(k-|Z|)$-th element of $Y$ is $\geq x_{k}$. Since 100 -Sort returns a 200 -approximation, it follows that $x^{*} \geq 200 x_{k}$ with sufficiently large probability as desired.

Corollary 6.6. If $\left|R_{x}\right| \leq \frac{1}{10} n^{2 / 3}$, for any $r>0$ there exists $c$ large enough that $\operatorname{Pr}\left[x_{k} \geq_{200} x^{*}\right]>$ $1-\frac{1}{n^{r}}$.

Proof. Symmetric.
Theorem 6.7. SelectSparse uses $n^{1+O(1 / d)} d \log n$ comparisons and $d+\max \left(100, \log _{2} d\right)$ rounds.
Proof. All of the comparisons come from Sort, 100-Sort, and GetMax/GetMin. The number of comparisons is thus bounded by $c n^{2 / 3} \log n\left(n^{1 / 3}\right)^{1+O(1 / d)}+n^{1+\frac{1}{d-1}} \log d \log n+\left(c n^{5 / 6} \log n\right)^{6 / 5}=$ $n^{1+O(1 / d)} d \log n$ as desired. All iterations of the loop can be done in parallel, so the number of rounds is bounded by $d+\log _{2} d$ if GetMax is called, and by $d+100$ if 100 -Sort is called.

### 6.2 Dense Selection

Here we give two algorithms: Select+ and Select-, the former of which will return a good approximation if $\left|L_{x}\right|>\frac{1}{10} n^{2 / 3}$, and the latter if $\left|R_{x}\right|>\frac{1}{10} n^{2 / 3}$.

The idea is simple: take a large sample ( $\operatorname{size} c n^{5 / 6} \log n$ ), sort it with a constant approximate algorithm, and return the element in roughly the $k$-th position.

```
Algorithm 5 Dense Selection
    function Select \(\pm(X, k)\)
        \(n \leftarrow|X|\)
        Generate a subset \(S\) of \(X\) of size \(c n^{5 / 6} \log n\)
        \(S \leftarrow 100\)-Sort \((S)\)
        return \(S\left[c k \log n / n^{1 / 6} \pm c n^{5 / 12} \log n\right]\)
    end function
```

Let $x^{*}$ be the item returned by Select $\pm$.
Lemma 6.8. If $\left|L_{x}\right|>\frac{1}{10} n^{2 / 3}$, for any $r>0$ there exists clarge enough such that Select-returns a 201-approximation with probability $>1-\frac{1}{n^{r}}$.

Proof. It suffices to prove the actual $\left(k / n^{1 / 6}-n^{5 / 12}\right)$-th smallest element of $S$ is in $L_{x}$, since 100 -Sort returns a 200-approximation. The number of elements of $S$ that are $\leq x_{k}$ (call this $V$ ) follows a Hypergeometric $\left(n, k, c n^{5 / 6} \log n\right)^{\ddagger}$ distribution. Thus, by a tail bound:

$$
\begin{aligned}
\operatorname{Pr}\left[V \leq c k \log n / n^{1 / 6}-c n^{5 / 12} \log n\right] & =\operatorname{Pr}\left[V \leq\left(k / n-n^{-5 / 12}\right) c n^{5 / 6} \log n\right] \\
& \leq e^{-2\left(n^{-5 / 12}\right)^{2} c n^{5 / 6} \log n} \\
& \leq n^{-2 c}
\end{aligned}
$$

Similarly, the number of elements of $S$ that are $<x_{k}-1$ (call this $U$ ) follows a Hypergeometric ( $n, k-$ $\left.\frac{1}{10} n^{2 / 3}, c n^{5 / 6} \log n\right)^{\S}$ distribution. Thus, by a tail bound:

$$
\begin{aligned}
\operatorname{Pr}\left[U \geq k / n^{1 / 6}-n^{5 / 12}\right] & =\operatorname{Pr}\left[\left(\left(k-n^{2 / 3} / 10\right) / n+n^{-1 / 3} / 10-n^{-5 / 12}\right) c n^{5 / 6} \log n\right] \\
& \leq e^{-2\left(n^{-1 / 3} / 10-n^{-5 / 12}\right)^{2} c n^{5 / 6} \log n} \\
& <n^{-2 c}
\end{aligned}
$$

for $n$ sufficiently large. Thus, the probability of the actual $\left(k / n^{1 / 6}-n^{5 / 12}\right)$-th smallest element is in $L_{x}$ is at least $1-2 n^{-2 c}>1-\frac{1}{n^{r}}$ for $c$ sufficiently large by a union bound.

Corollary 6.9. If $\left|R_{x}\right|>\frac{1}{10} n^{2 / 3}$, for any $r>0$ there exists $c$ large enough such that Select+ returns a 201-approximation with probability $>1-\frac{1}{n^{r}}$.

Proof. Symmetric.
Lemma 6.10. If $x^{*}$ is the item returned by Select-, for any $r>0$ there exists $c$ sufficiently large that $x_{k} \geq_{200} x^{*}$ with probability $>1-\frac{1}{n^{r}}$.

[^2]Proof. This is implicitly proven in the previous lemma, where we prove the position in the original array of $x^{*}$ is less than $k$.

Corollary 6.11. If $x^{*}$ is the item returned by Select + , for any $r>0$ there exists $c$ sufficiently large that $x^{*} \geq_{200} x_{k}$ with probability $>1-\frac{1}{n^{r}}$

Proof. Symmetric.
Theorem 6.12. Select $\pm$ uses $O\left(n \log ^{6 / 5} n\right)$ comparisons and 100 rounds.
Proof. All comparisons are done in 100-Sort, so the number of comparisons is $O\left(\left(n^{5 / 6} \log n\right)^{6 / 5}\right)=$ $O\left(n \log ^{6 / 5} n\right)$. Since 100-Sort takes 100 rounds, so does Select $\pm$.

### 6.3 Combining

```
Algorithm 6 Pivot
    function \(\operatorname{Count}\left(X, x_{i}\right)\)
        \(c \leftarrow 0\)
        for \(x \in X\) do
            if \(x<_{c} x_{i}\) then
                \(c \leftarrow c+1\)
            end if
        end for
        return \(c\)
    end function
```

```
Algorithm 7 Selection
    function \(\operatorname{Select}(X, k)\)
        \(x_{i} \leftarrow \operatorname{SelectSparse}(X, k)\)
        \(c_{i} \leftarrow \operatorname{Count}\left(X, x_{i}\right)\)
        if \(c_{i}<k\) then
            \(x_{j} \leftarrow\) Select \(-(X, k)\)
            if \(x_{j}>_{c} x_{i}\) then return \(x_{j}\)
            else return \(x_{i}\)
            end if
        else
            \(x_{j} \leftarrow\) Select \(+(X, k)\)
            if \(x_{j}<_{c} x_{i}\) then return \(x_{j}\)
            else return \(x_{i}\)
            end if
        end if
    end function
```

Lemma 6.13. For any $r>0$, we can choose $c$ sufficiently large that $\left|x_{k}-x^{*}\right| \leq \max (200,1+$ $\left.\min \left(\left|x_{k}-x_{i}\right|,\left|x_{k}-x_{j}\right|\right)\right)$ with probability $>1-\frac{1}{n^{r}}$.

Proof. By symmetry, we may assume without loss of generality that $c_{i}<k$. In this case, we return the item with the larger Count. Since we return the maximum of $x_{i}$ and $x_{j}$ (according to the
comparator), we must have $\max \left(x_{i}, x_{j}\right)-1 \leq x^{*} \leq \max \left(x_{i}, x_{j}\right)$. By a result from the previous section, $x_{j} \leq x_{k}+200$ with probability $>1-\frac{1}{n^{r}}$. If $x_{i}$ was $>x_{k}+1$, then it would compare greater than $x_{k}$ and everything before it, contradicting $c_{i}<k$. Thus, $x_{i} \leq x_{k}+1$. It follows that with probability $>1-\frac{1}{n^{r}}, \max \left(x_{i}, x_{j}\right) \leq x_{k}+200$. Thus, if either $x_{i}>x_{k}$ or $x_{j}>x_{k},\left|x^{*}-x_{k}\right| \leq 200$ as desired. Otherwise, if both $x_{i} \leq x_{k}$ and $x_{j} \leq x_{k}$, then $\left|x_{k}-x^{*}\right|=x_{k}-x^{*} \leq x_{k}-\left(\max \left(x_{i}, x_{j}\right)-1\right)=$ $\min \left(\left|x_{k}-x_{i}\right|,\left|x_{k}-x_{j}\right|\right)+1$ as desired.

Theorem 6.14. For $r>0$ there exists $c$ sufficiently large that Select returns a 202-approximate $k$-selection with probability $>1-\frac{1}{n^{r}}$.

Proof. We consider cases based on the sizes of $L_{x}$ and $R_{x}$ :
If $\left|L_{x}\right| \leq \frac{1}{10} n^{2 / 3}$ and $\left|R_{x}\right| \leq \frac{1}{10} n^{2 / 3}$, then we have $x_{i} \geq_{200} x_{k}$ and $x_{k} \geq_{200} x_{i}$ with probability $>1-\frac{2}{n^{r+1}}$, in which case we have $\left|x_{k}-x_{i}\right| \leq 200$. By the previous lemma it follows that $\left|x^{*}-x_{k}\right| \leq 201$ with probability $>1-\frac{3}{n^{r+1}}$, so we return a 201-approximate $k$-selection with probability $>1-\frac{3}{n^{r+1}}>1-\frac{1}{n^{r}}$ as desired.

If $\left|L_{x}\right|>\frac{1}{10} n^{2 / 3}$ and $\left|R_{x}\right|>\frac{1}{10} n^{2 / 3}$, then $\left|x_{j}-x_{k}\right| \leq 201$ with probability $>1-\frac{1}{n^{r+1}}$. Thus, by the previous lemma, $\left|x^{*}-x_{k}\right| \leq 202$ with probability $>1-\frac{2}{n^{r+1}}$. It follows that $x^{*}$ is a 202-approximate $k$-selection with probability $>1-\frac{2}{n^{r+1}}>1-\frac{1}{n^{r}}$ as desired.

If $\left|L_{x}\right| \leq \frac{1}{10} n^{2 / 3}$ and $\left|R_{x}\right|>\frac{1}{10} n^{2 / 3}$, we have $x_{i} \geq_{200} x_{k}$ with probability $>\frac{1}{n^{r+1}}$. Thus, either $x_{i} \leq x_{k}+1$ in which case we have $\left|x_{i}-x_{k}\right| \leq 200$, or $x_{i}>x_{k}+1$ in which case $x_{j}$ must come from Select+ and thus $\left|x_{j}-x_{k}\right| \leq 201$ with probability $>\frac{1}{n^{r+1}}$. By the previous lemma, it thus follows that $\left|x^{*}-x_{k}\right| \leq 202$ with probability $>\frac{3}{n^{r+1}}$. It follows that $x^{*}$ is a 202-approximate $k$-selection with probability $>1-\frac{3}{n^{r+1}}>1-\frac{1}{n^{r}}$ as desired.

The case where $\left|L_{x}\right|>\frac{1}{10} n^{2 / 3}$ and $\left|R_{x}\right| \leq \frac{1}{10} n^{2 / 3}$ is symmetric.
Theorem 6.15. Select takes $n^{1+O(1 / d)} d \log n$ comparisons and $d+102+\min \left(100, \log _{2} d\right)$ rounds .
Proof. All comparisons are done in SelectSparse, Select土 and Count. The number of comparisons done by the two calls to Count is bounded by $2 n$. Thus, the total number of comparisons is bounded by $n^{1+O(1 / d)} d \log n+n \log ^{6 / 5} n+2 n=n^{1+O(1 / d)} d \log n$. Each call to Count takes one round, so the total number of rounds is bounded by $d+\min \left(100, \log _{2} d\right)+100+2=$ $d+102+\min \left(100, \log _{2} d\right)$.

Theorem 1.9 follows.

## 7 Open Problems

- Is there an algorithm to find a 3-approximate sorting or selection with high probability in $\widetilde{O}(n)$ time?
- Is there an algorithm to find a constant-approximate sorting with high probability in $O(n \log n)$ time?
- Is there an algorithm to find a constant-approximate selection with high probability in $O(n)$ time?
- Can we improve the lower or upper bounds for $k$-approximate sorting and selection in $d$ rounds?


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## References

[AA88a] Noga Alon and Yossi Azar. The average complexity of deterministic and randomized parallel comparison-sorting algorithms. SIAM Journal on Computing, 17(6):11781192, 1988.
[AA88b] Noga Alon and Yossi Azar. Sorting, approximate sorting, and searching in rounds. SIAM Journal on Discrete Mathematics, 1(3):269-280, 1988.
[AAAK17] Arpit Agarwal, Shivani Agarwal, Sepehr Assadi, and Sanjeev Khanna. Learning with limited rounds of adaptivity: Coin tossing, multi-armed bandits, and ranking from pairwise comparisons. In Conference on Learning Theory, pages 39-75. PMLR, 2017.
[AAV86] Noga Alon, Yossi Azar, and Uzi Vishkin. Tight complexity bounds for parallel comparison sorting. In 27th Annual Symposium on Foundations of Computer Science (sfcs 1986), pages 502-510. IEEE, 1986.
[AFHN15] Miklós Ajtai, Vitaly Feldman, Avinatan Hassidim, and Jelani Nelson. Sorting and selection with imprecise comparisons. ACM Transactions on Algorithms (TALG), 12(2):1-19, 2015.
[AFJ ${ }^{+}$18] Jayadev Acharya, Moein Falahatgar, Ashkan Jafarpour, Alon Orlitsky, and Ananda Theertha Suresh. Maximum selection and sorting with adversarial comparators. The Journal of Machine Learning Research, 19(1):2427-2457, 2018.
[AKS83] Miklós Ajtai, János Komlós, and Endre Szemerédi. An 0 ( $\mathrm{n} \log \mathrm{n}$ ) sorting network. In Proceedings of the fifteenth annual ACM symposium on Theory of computing, pages 1-9, 1983.
[Alo86] Noga Alon. Eigenvalues, geometric expanders, sorting in rounds, and ramsey theory. Combinatorica, 6(3):207-219, 1986.
[AP90] Yossi Azar and Nicholas Pippenger. Parallel selection. Discrete Applied Mathematics, 27(1-2):49-58, 1990.
[AU90] Shay Assaf and Eli Upfal. Fault tolerant sorting network. In Proceedings [1990] 31st Annual Symposium on Foundations of Computer Science, pages 275-284. IEEE, 1990.
[AV87] Yossi Azar and Uzi Vishkin. Tight comparison bounds on the complexity of parallel sorting. SIAM Journal on Computing, 16(3):458-464, 1987.
[Bag92] Amitava Bagchi. On sorting in the presence of erroneous information. Information Processing Letters, 43(4):213-215, 1992.
[BB90] Béla Bollobás and Graham Brightwell. Parallel selection with high probability. SIAM Journal on Discrete Mathematics, 3(1):21-31, 1990.
[ $\left.\mathrm{BFP}^{+} 73\right]$ Manuel Blum, Robert W. Floyd, Vaughan R. Pratt, Ronald L. Rivest, Robert Endre Tarjan, et al. Time bounds for selection. J. Comput. Syst. Sci., 7(4):448-461, 1973.
[BG90] Richard Beigel and John Gill. Sorting n objects with a k-sorter. IEEE Transactions on Computers, 39(5):714-716, 1990.
[BH85] Béla Bollobás and Pavol Hell. Sorting and graphs. In Ivan Rival, editor, Graphs and Order: The Role of Graphs in the Theory of Ordered Sets and Its Applications, pages 169-184. Springer, 1985.
[BK93] Ryan S Borgstrom and S Rao Kosaraju. Comparison-based search in the presence of errors. In Proceedings of the twenty-fifth annual ACM symposium on Theory of computing, pages 130-136, 1993.
[BMP19] Mark Braverman, Jieming Mao, and Yuval Peres. Sorted top-k in rounds. In Conference on Learning Theory, pages 342-382. PMLR, 2019.
[BMW16] Mark Braverman, Jieming Mao, and S. Matthew Weinberg. Parallel algorithms for select and partition with noisy comparisons. In Proceedings of the 48 th Annual ACM Symposium on the Theory of Computing, STOC '16, pages 851-862, New York, NY, USA, 2016. ACM.
[Bol88] Béla Bollobás. Sorting in rounds. Discrete Mathematics, 72(1-3):21-28, 1988.
[BR81] Béla Bollobás and Moshe Rosenfeld. Sorting in one round. Israel Journal of Mathematics, 38:154-160, 1981.
[BT83] Béla Bollobás and Andrew Thomason. Parallel sorting. Discrete Applied Mathematics, 6(1):1-11, 1983.
[BZ74] Marat Valievich Burnashev and Kamil'Shamil'evich Zigangirov. An interval estimation problem for controlled observations. Problemy Peredachi Informatsii, 10(3):5161, 1974.
[CAMTM20] Vincent Cohen-Addad, Frederik Mallmann-Trenn, and Claire Mathieu. Instanceoptimality in the noisy value-and comparison-model* accept, accept, strong accept: Which papers get in? In Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms, SODA '20, pages 2124-2143, Philadelphia, PA, USA, 2020. SIAM.
[Chv92] Vasek Chvátal. Lecture notes on the new aks sorting network. 1992.
[CS92] Robert Cypher and Jorge LC Sanz. Cubesort: A parallel algorithm for sorting n data items with s-sorters. Journal of Algorithms, 13(2):211-234, 1992.
[DKP22] Natalia Dobrokhotova-Maikova, Alexander Kozachinskiy, and Vladimir V. Podolskii. Constant-depth sorting networks. Electron. Colloquium Comput. Complex., TR22116, 2022.
[FRPU94] Uriel Feige, Prabhakar Raghavan, David Peleg, and Eli Upfal. Computing with noisy information. SIAM Journal on Computing, 23(5):1001-1018, 1994.
[GKK ${ }^{+}$20] Sivakanth Gopi, Gautam Kamath, Janardhan Kulkarni, Aleksandar Nikolov, Zhiwei Steven Wu, and Huanyu Zhang. Locally private hypothesis selection. In Conference on Learning Theory, pages 1785-1816. PMLR, 2020.
[GL97] Qingshi Gao and Zhiyong Liu. Sloping-and-shaking: Multiway merging and sorting. Science in China Series E: Technological Sciences, 40:225-234, 1997.
[GVN47] Herman Heine Goldstine and John Von Neumann. Planning and coding of problems for an electronic computing instrument. 1947.
[HH81] Roland Häggkvist and Pavol Hell. Parallel sorting with constant time for comparisons. SIAM Journal on Computing, 10(3):465-472, 1981.
[HH82] Roland Häggkvist and Pavol Hell. Sorting and merging in rounds. SIAM Journal on Algebraic Discrete Methods, 3(4):465-473, 1982.
[HKK11] Zhiyi Huang, Sampath Kannan, and Sanjeev Khanna. Algorithms for the generalized sorting problem. In 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science, pages 738-747. IEEE, 2011.
[Hor63] Michael Horstein. Sequential transmission using noiseless feedback. IEEE Transactions on Information Theory, 9(3):136-143, 1963.
[IK10] Russell Impagliazzo and Valentine Kabanets. Constructive proofs of concentration bounds. In International Workshop on Randomization and Approximation Techniques in Computer Science, pages 617-631. Springer, 2010.
[KN22] William Kuszmaul and Shyam Narayanan. Stochastic and worst-case generalized sorting revisited. In 2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS), pages 1056-1067. IEEE, 2022.
[Kru83] Clyde P. Kruskal. Searching, merging, and sorting in parallel computation. IEEE Transactions on Computers, C-32(10):942-946, 1983.
[LB95] De-Lei Lee and Kenneth E. Batcher. A multiway merge sorting network. IEEE transactions on parallel and distributed systems, 6(2):211-215, 1995.
[Lei84] Tom Leighton. Tight bounds on the complexity of parallel sorting. In Proceedings of the 16th Annual ACM Symposium on the Theory of Computing, STOC '84, pages 71-80, New York, NY, USA, 1984. ACM.
[LMP97] Tom Leighton, Yuan Ma, and C Greg Plaxton. Breaking the ( $\mathrm{n} \log 2 \mathrm{n}$ ) barrier for sorting with faults. Journal of Computer and System Sciences, 54(2):265-304, 1997.
[Lon92] Philip M Long. Sorting and searching with a faulty comparison oracle. Citeseer, 1992.
[LRG91] KB Lakshmanan, Bala Ravikumar, and K Ganesan. Coping with erroneous information while sorting. IEEE Transactions on Computers, 40(09):1081-1084, 1991.
[NHAT89] Toshio Nakatani, S-T Huang, Bruce W. Arden, and Satish K. Tripathi. K-way bitonic sort. IEEE Transactions on Computers, 38(2):283-288, 1989.
[NPR95] Alberto Negro, Giuseppe Parlati, and P Ritrovato. Optimal adaptive search: reliable and unreliable models. In Proc. 5th Italian Conf. on Theoretical Computer Science, pages 211-231. World Scientific, 1995.
[Pel89] Andrzej Pelc. Searching with known error probability. Theoretical Computer Science, 63(2):185-202, 1989.
[Pio96] Marek Piotrów. Depth optimal sorting networks resistant to k passive faults. In Proceedings of the seventh annual ACM-SIAM symposium on Discrete algorithms, pages 242-251, 1996.
[Pip87] Nicholas Pippenger. Sorting and selecting in rounds. SIAM Journal on Computing, 16(6):1032-1038, 1987.
[PP89] Bruce Parker and Ian Parberry. Constructing sorting networks from k-sorters. Information Processing Letters, 33(3):157-162, 1989.
[Rén61] Alfréd Rényi. On a problem of information theory. MTA Mat. Kut. Int. Kozl. B, 6(MR143666):505-516, 1961.
[RMK $\left.{ }^{+} 80\right]$ Ronald L. Rivest, Albert R. Meyer, Daniel J. Kleitman, Karl Winklmann, and Joel Spencer. Coping with errors in binary search procedures. Journal of Computer and System Sciences, 20(3):396-404, 1980.
[SYW14] Feng Shi, Zhiyuan Yan, and Meghanad Wagh. An enhanced multiway sorting network based on n-sorters. In 2014 IEEE Global Conference on Signal and Information Processing (GlobalSIP), pages 60-64. IEEE, 2014.
[TL85] SS Tseng and Richard CT Lee. A parallel sorting scheme whose basic operation sorts n elements. International journal of computer $\mathcal{B}^{\text {information sciences, 14(6):455-467, }}$ 1985.
[TZZ19] Chao Tao, Qin Zhang, and Yuan Zhou. Collaborative learning with limited interaction: Tight bounds for distributed exploration in multi-armed bandits. In 2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS), pages 126-146. IEEE, 2019.
[Ula91] Stanislaw M Ulam. Adventures of a Mathematician. Univ of California Press, 1991.
[Val75] Leslie G Valiant. Parallelism in comparison problems. SIAM Journal on Computing, 4(3):348-355, 1975.
[WZ93] Avi Wigderson and David Zuckerman. Expanders that beat the eigenvalue bound: Explicit construction and applications. In Proceedings of the twenty-fifth annual ACM symposium on Theory of computing, pages 245-251, 1993.
[YY85] Andrew C Yao and F Frances Yao. On fault-tolerant networks for sorting. SIAM Journal on Computing, 14(1):120-128, 1985.
[ZLG98] Lijun Zhao, Zhiyong Liu, and Qingshi Gao. An efficient multiway merging algorithm. Science in China Series E: Technological Sciences, 41:543-551, 1998.


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[^1]:    ${ }^{\dagger}$ If there are multiple items with value $x_{k}$, the second argument can be larger than $k$, but that can only make the bounds better.

[^2]:    ${ }^{\ddagger}$ Similarly to before, the second argument can be $>k$, but it only makes the bounds better.
    ${ }^{\S}$ Again, the second argument could be larger.

