Approximation Algorithms for the Weighted Nash Social Welfare via Convex and Non-Convex Programs

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Abstract

In an instance of the weighted Nash Social Welfare problem, we are given a set of m indivisible items, \mathcal{G} , and n agents, \mathcal{A} , where each agent $i \in \mathcal{A}$ has a valuation $v_{ij} \geq 0$ for each item $j \in \mathcal{G}$. In addition, every agent i has a non-negative weight w_i such that the weights collectively sum up to 1. The goal is to find an assignment $\sigma : \mathcal{G} \to \mathcal{A}$ that maximizes $\prod_{i \in \mathcal{A}} \left(\sum_{j \in \sigma^{-1}(i)} v_{ij} \right)^{w_i}$, the product of the weighted valuations of the players. When all the weights equal $\frac{1}{n}$, the problem reduces to the classical Nash Social Welfare problem, which has recently received much attention. In this work, we present a $5 \cdot \exp\left(2 \cdot D_{\text{KL}}(\mathbf{w} \mid\mid \frac{1}{n})\right) =$ $5 \cdot \exp\left(2 \log n + 2\sum_{i=1}^{n} w_i \log w_i\right)$ -approximation algorithm for the weighted Nash Social Welfare problem, where $D_{\text{KL}}(\mathbf{w} \mid\mid \frac{1}{n})$ denotes the KL-divergence between the distribution induced by \mathbf{w} and the uniform distribution on [n].

We show a novel connection between the convex programming relaxations for the unweighted variant of Nash Social Welfare presented in [CDG⁺17, AGSS17], and generalize the programs to two different mathematical programs for the weighted case. The first program is convex and is necessary for computational efficiency, while the second program is a nonconvex relaxation that can be rounded efficiently. The approximation factor derives from the difference in the objective values of the convex and non-convex relaxation.

1 Introduction

In an instance of the weighted Nash Social Welfare problem, we are given a set of *m* indivisible items \mathcal{G} , and a set of *n* agents, \mathcal{A} . Every agent $i \in \mathcal{A}$ has a weight $w_i \ge 0$ such that $\sum_{i \in \mathcal{A}} w_i = 1$ and an additive valuation function $\mathbf{v}_i : 2^{\mathcal{G}} \to \mathbb{R}_{\ge 0}$. Let $v_{ij} := \mathbf{v}_i(\{j\})$. The goal is to find an assignment

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of items, $\sigma : \mathcal{G} \to \mathcal{A}$, to maximize the following welfare function:

$$\prod_{i \in \mathcal{A}} \left(\sum_{j \in \sigma^{-1}(i)} v_{ij} \right)^{w_i}.$$
 (1)

For ease of notation, we will work with the log objective and denote

$$NSW(\sigma) = \sum_{i \in \mathcal{A}} w_i \log \left(\sum_{j \in \sigma^{-1}(i)} v_{ij} \right).$$
(2)

Let $OPT = \max_{\sigma: \mathcal{G} \to \mathcal{A}} NSW(\sigma)$ denote the optimal log objective. The case where $w_i = \frac{1}{n}$ for each $i \in \mathcal{A}$ is the much-studied "symmetric" or unweighted Nash social welfare problem, where the objective is the geometric mean of agents' valuations.

Fair and efficient division of resources among agents is a fundamental problem arising in various fields [BT05, BT96, BCE⁺16, RW98, Rot15, You94]. While there are many social welfare functions which can be used to evaluate the efficacy of an assignment of goods to the agents, the Nash Social Welfare function is well-known to interpolate between fairness and overall utility. The unweighted Nash Social Welfare function first appeared as the solution to an arbitration scheme proposed by Nash for two-person bargaining games and was later generalized to multiple players [NJ50, KN79]. Since then, it has been widely used in numerous fields to model resource allocation problems. An attractive feature of the objective is that it is invariant under scaling by any of the agent's valuations, and therefore, each agent can specify its utility in its own units (see [CM04] for a detailed treatment). While the theory of Nash Social Welfare objective was initially developed for divisible items, more recently, it has been applied in the context of indivisible items. We refer the reader to [CKM⁺19] for a comprehensive overview of the problem in the latter setting. Indeed, optimizing the Nash Social Welfare objective also implies notions of fairness, such as *envy-free* allocation in an approximate sense [CKM⁺19, BKV18].

The Nash Social Welfare function with weights (also referred to as asymmetric or non-symmetric Nash Social Welfare) was first studied in the seventies [HS72, Kal77] in the context of two-person bargaining games. For example, in the bargaining context, it allows different agents to have different weights. Due to this flexibility, problems in many diverse domains can be modeled using the weighted objective, including bargaining theory [CM04, LV07], water resource allocation [FKL12, HLZ13], and climate agreements [YIWZ17]. From a context of indivisible goods, the study of this problem has been much more recent [GKK20, GHV21, GHL⁺23]. In this work, we aim to shed light on the weighted Nash Social Welfare problem, mainly focusing on mathematical programming relaxations for the problem.

1.1 Our Results and Contributions

Our main result is an exp $(2\log 2 + \frac{1}{2e} + 2D_{\text{KL}}(\mathbf{w} || \mathbf{u})) \approx 4.81 \cdot \exp(2\log n - 2\sum_{i=1}^{n} w_i \log \frac{1}{w_i})$ approximation algorithm for the weighted Nash Social Welfare problem with additive valuations. When all the weights are the same, this gives a constant factor approximation. Our algorithm
builds on and extends a convex programming relaxation for the unweighted variant of Nash Social
Welfare presented in [CG15, CDG⁺17, AGSS17]. In the following theorem, we state the guarantee
in terms of the log-objective, and therefore, the guarantee becomes an additive one. **Theorem 1.** Let $(\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w})$ be an instance of the weighted Nash Social Welfare problem with $\sum_{i \in \mathcal{A}} w_i = 1$ and $|\mathcal{A}| = n$ agents. There exists a polynomial time algorithm (Algorithm 1) that, given $(\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w})$, returns an assignment $\sigma : \mathcal{G} \to \mathcal{A}$ such that

$$\mathrm{NSW}(\sigma) \ge \mathrm{OPT} - 2\log 2 - \frac{1}{2e} - 2 \cdot D_{\mathrm{KL}}(\mathbf{w} \mid\mid \mathbf{u}),$$

where OPT is the optimal log-objective for the instance and $D_{KL}(\mathbf{w}||\mathbf{u}) = \log n - \sum_{i \in \mathcal{A}} w_i \log \frac{1}{w_i}$.

Observe that the KL-divergence term $D_{\text{KL}}(\mathbf{w} || \mathbf{u}) = \left(\log n - \sum_{i \in \mathcal{A}} w_i \log \frac{1}{w_i}\right)$ is always upper bounded by $\log(nw_{\text{max}})$, which is exactly the guarantee of previous work [GHL⁺23]. In many settings, the term $2 \cdot D_{\text{KL}}(\mathbf{w} || \mathbf{u})$ can be significantly smaller than nw_{max} . For example, consider the setting where $w_1 = \frac{1}{\log n}$ and $w_i = \frac{1}{n-1}(1 - \frac{1}{\log n})$ for i = 2, ..., n, i.e., one agent has a significantly higher weight than the others. Then

$$D_{\mathrm{KL}}(\mathbf{w} || \mathbf{u}) = \frac{1}{\log n} \log\left(\frac{n}{\log n}\right) + \left(1 - \frac{1}{\log n}\right) \log\left(\frac{n}{n-1}\left(1 - \frac{1}{\log n}\right)\right)$$
$$= 1 - \frac{\log\log n}{\log n} + \left(1 - \frac{1}{\log n}\right) \log\left(\frac{n}{n-1}\right) + \left(1 - \frac{1}{\log n}\right) \log\left(1 - \frac{1}{\log n}\right)$$
$$\leq 1 + \log\left(\frac{n}{n-1}\right) \leq 2.$$

In this case, our results imply an O(1)-approximation, while previous results imply an $O(\frac{n}{\log n})$ -approximation.

Our algorithm relies on two mathematical programming relaxations for the weighted Nash Social Welfare problem, both of which generalize the convex relaxation for the unweighted version [CG15, CDG⁺17, AGSS17]. The first relaxation, (NCVX-Weighted), is non-convex but retains a lot of structural insights obtained for the convex relaxation in the symmetric version. We show that the same rounding algorithm as in the symmetric version [CG15] gives an O(1)-approximation for the weighted version when applied to a fractional solution of the non-convex program. Although (NCVX-Weighted) can be rounded efficiently, unfortunately, we cannot solve this relaxation due to its non-convex nature. Now, the second mathematical programming relaxation, (CVX-Weighted), comes to the rescue. This relaxation is convex and thus can be solved efficiently, but is challenging to round. Our algorithm solves the convex relaxation, then uses the non-convex relaxation to measure the change in objective as it processes the solution and eventually rounds to an integral assignment. The approximation factor of $D_{\text{KL}}(\mathbf{w} || \mathbf{u})$ arises due to the difference in objective values of these two programs. Section 1.3 provides a technical overview of the properties of the two relaxations.

Before stating our second result, we describe the two previous convex programming relaxations for the unweighted Nash Social Welfare problem presented in [CDG⁺17] and [AGSS17].

Equivalence of Relaxations. Building on the algorithm of [CG15], [CDG⁺17] introduced the following relaxation for the unweighted Nash Social Welfare problem.

$$\max_{b,q} \quad \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} b_{ij} \log \left(v_{ij} \right) - \sum_{j \in \mathcal{G}} \left(\sum_{i \in \mathcal{A}} b_{ij} \right) \log \left(\sum_{i \in \mathcal{A}} b_{ij} \right)$$
(CVX-Unweighted)

s.t.
$$\sum_{j} b_{ij} = 1 \quad \forall i \in \mathcal{A}$$

 $\sum_{i} b_{ij} \leq 1 \quad \forall j \in \mathcal{G}$
 $b_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A} \times \mathcal{G}$

They showed that (CVX-Unweighted) is a convex relaxation of the Nash Social Welfare objective, and the prices used by the algorithm presented in [CG15] can be obtained as dual variables of (CVX-Unweighted). Interestingly, the convex relaxation is not in terms of the assignment variables. Indeed, given an optimal assignment $\sigma : \mathcal{G} \to \mathcal{A}$, the corresponding setting of the variables b_{ij} is

$$b_{ij} = \begin{cases} \frac{v_{ij}}{\sum_{k \in \sigma^{-1}(i)} v_{ik}} & \text{if } \sigma(j) = i\\ 0 & \text{otherwise.} \end{cases}$$
(3)

One can verify that **b** satisfies all the constraints in (CVX-Unweighted), and its objective value is equal to the log of the geometric means of the valuations.

[AGSS17] presented a different convex programming relaxation¹, (LogConcave-Unweighted), for unweighted NSW. They showed that the objective of (LogConcave-Unweighted) is a log-concave function in x and convex in log y to obtain an e-approximation for unweighted NSW.

$$\begin{split} \max_{\mathbf{x} \ge \mathbf{0}} \inf_{\mathbf{y} > \mathbf{0}} & \sum_{i \in \mathcal{A}} \log \left(\sum_{j \in \mathcal{G}} x_{ij} \, v_{ij} \, y_j \right) \\ \text{s.t.} & \sum_{i \in \mathcal{A}} x_{ij} = 1 \quad \forall j \in \mathcal{G} \\ & \prod_{j \in S} y_j \ge 1 \quad \forall S \in \binom{\mathcal{G}}{n}. \end{split}$$
 (LogConcave-Unweighted)

Here, $\binom{\mathcal{G}}{n}$ denotes the collection of subsets of \mathcal{G} of size *n*, where $n = |\mathcal{A}|$.

On the surface, the programs (LogConcave-Unweighted) and (CVX-Unweighted), and their corresponding rounding algorithm are quite different: [CDG⁺17] used intuition from economics and market equilibrium to both arrive at (CVX-Unweighted) and also to round it, while [AGSS17] uses the properties of log-concave polynomials to round (CVX-Unweighted).

However, our next result shows that these two convex programs indeed optimize the same objective.

Theorem 2. The optimal values of (LogConcave-Unweighted) and (CVX-Unweighted) are the same.

The proof of this theorem, presented in Appendix B.1, relies on a series of transformations using convex duality.

Besides providing a novel connection between two very different approaches to the unweighted problem, Theorem 2 is also vital to derive our main algorithm for weighted Nash Social Welfare. Independently generalizing either of these approaches to the weighted case is challenging: [CDG⁺17, CG15] use intuition from economics to arrive at (CVX-Unweighted), and these concepts

¹The program is concave in *x* and convex in $\log(y)$. A change of variable $y_j \mapsto \exp(-z_j)$ gives a concave-convex program in *x* and *z*.

do not generalize to the weighted case. On the other hand, there is a natural convex generalization of (LogConcave-Unweighted) for the weighted case², it is not log-concave, and therefore the machinery introduced in cannot be used to analyze it.

Our approach leverages the connection between [CDG⁺17] and [AGSS17] stated in Theorem 2 to derive a more natural convex relaxation of weighted Nash Social Welfare, given by (CVX-Weighted). We provide more concrete details on this relationship and how it leads to the two programs for weighted NSW in Appendix B.

1.2 Preliminaries

KL-Divergence. For two probability distributions \mathbf{p} , \mathbf{q} over the same discrete domain \mathcal{X} , the KL-divergence between \mathbf{p} and \mathbf{q} is defined as

$$D_{\mathrm{KL}}(\mathbf{p} \mid\mid \mathbf{q}) = \sum_{x \in \mathcal{X}} p(x) \log\left(\frac{p(x)}{q(x)}\right).$$

It is well-known, via Gibb's inequality, that the KL-divergence between two distributions is nonnegative and is zero if and only if **p** and **q** are identical.

We use this fact crucially in the following claim.

Claim 1.1. *Given positive reals* z_1, \ldots, z_d *, for any* $y_1, y_2, \ldots, y_d \ge 0$ *,*

$$\sum_{j=1}^d y_j \log\left(\sum_{j=1}^d z_j\right) - \sum_{j=1}^d y_j \log\left(\sum_{j=1}^d y_j\right) \ge \sum_{j=1}^d y_j \log z_j - \sum_{j=1}^d y_j \log y_j.$$

Proof. Define vectors $\mathbf{y} = (y_1, \ldots, y_d)$ and $\mathbf{z} = (z_1, \ldots, z_d)$. Then $\tilde{\mathbf{y}} = \frac{\mathbf{y}}{\|\mathbf{y}\|_1}$ and $\tilde{\mathbf{z}} = \frac{\mathbf{z}}{\|\mathbf{z}\|_1}$ define two probability distributions on [d]. The inequality is equivalent to $D_{\text{KL}}(\tilde{\mathbf{y}} \mid | \tilde{\mathbf{z}}) \ge 0$.

Moreover, if \mathbf{q} is the uniform distribution on \mathcal{X} and \mathbf{p} is an arbitrary distribution on the same domain, then

$$D_{\mathrm{KL}}(\mathbf{p} \mid\mid \mathbf{q}) = \log |\mathcal{X}| - \sum_{x \in \mathcal{X}} p(x) \log \frac{1}{p(x)}.$$

Feasibility Polytope. Consider a complete bipartite graph $G = (\mathcal{G} \cup \mathcal{A}, E)$ where E contains an edge (i, j) for each $i \in \mathcal{A}$ and $j \in \mathcal{G}$. Let $\mathcal{M}(\mathcal{A})$ denote the set of all matchings in G of size $|\mathcal{A}|$, i.e., matchings which have an edge incident to every vertex in \mathcal{A} . The convex hull of $\mathcal{M}(\mathcal{A})$, denoted by $\mathcal{P}(\mathcal{A}, \mathcal{G})$, is defined by the following polytope.

Definition 1 (Feasibility Polytope). For a set of *m* indivisible items, \mathcal{G} , and a set of *n* agents, \mathcal{A} , the *feasibility polytope, denoted by* $\mathcal{P}(\mathcal{A}, \mathcal{G})$, *is defined as*

$$\mathcal{P}(\mathcal{A},\mathcal{G}) := \left\{ \mathbf{b} \in \mathbb{R}_{\geq 0}^{|\mathcal{A}| \times |\mathcal{G}|} : \sum_{j \in \mathcal{G}} b_{ij} = 1 \ \forall i \in \mathcal{A} \ , \sum_{i \in \mathcal{A}} b_{ij} \leq 1 \ \forall j \in \mathcal{G} \right\}.$$

²We present this generalization in Appendix B.2

The constraint $\sum_{j \in \mathcal{G}} b_{ij} = 1$ is called the Agent constraint for agent *i*, and the constraint $\sum_{i \in \mathcal{A}} b_{ij} \leq 1$ is referred to as the Item constraint for item *j*.

We call $\mathcal{P}(\mathcal{A}, \mathcal{G})$ the feasibility polytope of $(\mathcal{A}, \mathcal{G})$ and will refer to points in $\mathcal{P}(\mathcal{A}, \mathcal{G})$ as either feasible points or solutions. In the next section, we use $\mathcal{P}(\mathcal{A}, \mathcal{G})$ to define the feasible regions for both mathematical programs.

1.3 Technical Overview

Programs for Weighted NSW. We introduce two mathematical programs, (CVX-Weighted) and (NCVX-Weighted) below as relaxations of the weighted Nash Social Welfare objective. By setting **b** to be the same value as (3), it is natural to see that both programs are indeed relaxations. The first program (CVX-Weighted) is a convex program for any non-negative weights **w**, whereas the second program is not a convex program when the weights are not identical.

$\max_{\mathbf{b}} f_{\text{cvx}}(\mathbf{b}) := \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} w_i b_{ij} \log v_{ij}$	$\max_{\mathbf{b}} f_{\text{nevx}}(\mathbf{b}) := \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} w_i b_{ij} \log v_{ij}$
$-\sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{A}}w_ib_{ij}\log\left(\sum_{i\in\mathcal{A}}w_ib_{ij} ight)$	$-\sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{A}}w_ib_{ij}\log\left(\sum_{i\in\mathcal{A}}b_{ij} ight)$
$egin{aligned} &+\sum_{i\in\mathcal{A}}w_i\log w_i\ ext{s.t.} &\sum_{i=2}b_{ij}=1 orall i\in\mathcal{A} \end{aligned}$	s.t. $\sum_{i=2} b_{ij} = 1 \forall i \in \mathcal{A}$
$\sum_{i \in \mathcal{A}}^{j \in \mathcal{G}} b_{ij} \leq 1 orall j \in \mathcal{G}$ $h_{ii} \geq 0 orall (i, i) \in \mathcal{A} imes \mathcal{G}$	$\sum_{i \in \mathcal{A}}^{j \in \mathcal{G}} b_{ij} \leq 1 orall j \in \mathcal{G}$ $b_{ii} \geq 0 orall (i, i) \in \mathcal{A} imes \mathcal{G}$
$v_{ij} \geq 0$ $v(i,j) \in \mathcal{A} \times \mathcal{G}$	

(CVX-Weighted)

(NCVX-Weighted)

Lemma 3. (CVX-Weighted) and (NCVX-Weighted) are relaxations of the weighted Nash Social Welfare problem. Moreover, when the weights are symmetric, i.e., $w_i = 1/n$ for all $i \in A$, the programs (CVX-Weighted) and (NCVX-Weighted) are equivalent to the convex program (CVX-Unweighted).

We formally prove Lemma 3 in Appendix A.

Note that the constraints for both (CVX-Weighted) and (NCVX-Weighted) are identical to the feasibility polytope $\mathcal{P}(\mathcal{A}, \mathcal{G})$.

While analogous to (CVX-Unweighted), (CVX-Weighted) does not inherit a crucial property of (CVX-Unweighted), making (CVX-Weighted) challenging to round: optimal solutions of (CVX-Weighted) need not be acyclic. Furthermore, the integrality gap of (CVX-Weighted) is non-trivial even in the case when there are exactly *n* items. To circumvent these issues, we use (NCVX-Weighted) as an intermediate step in our rounding algorithm which has a desirable property: given a point $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$, one can efficiently find another point $\tilde{\mathbf{b}} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$ without decreasing the objective

 f_{ncvx} such that the graph formed by support of $\mathbf{\tilde{b}}$ is a forest, as stated in the following lemma. We formally define the support graphs in Definition 2.

The two relaxations. We observe that the objective of (CVX-Weighted) is a concave function, and thus, it is a polynomial time tractable convex program. However, the objective of (NCVX-Weighted) is not necessarily concave when the weights w_i are not uniform. Despite this, (NCVX-Weighted) still satisfies many desirable properties:

Lemma 4. Let $\overline{\mathbf{b}}$ be any feasible point in $\mathcal{P}(\mathcal{A}, \mathcal{G})$. Then there exists an acyclic solution, $\mathbf{b}^{\text{forest}}$, in the support of $\overline{\mathbf{b}}$ such that

$$f_{\text{ncvx}}(\mathbf{b}^{\text{forest}}) \geq f_{\text{ncvx}}(\overline{\mathbf{b}}).$$

Moreover, such a solution can be found in time polynomial in |A| *and* |G|*.*

Next, we establish that one can efficiently round any feasible point whose support graph is a forest to an integral assignment.

Theorem 5. For a Nash Social Welfare instance $(\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w})$, given a vector $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$ such that the support of \mathbf{b} is a forest, there exists a deterministic polynomial time algorithm (Algorithm 2) which returns an assignment $\sigma : \mathcal{G} \to \mathcal{A}$ such that

$$\mathrm{NSW}(\sigma) \ge f_{\mathrm{cvx}}(\mathbf{b}) - D_{\mathrm{KL}}(\mathbf{w} || \mathbf{u}) - 2\log 2 - \frac{1}{2e}.$$

We remark that our algorithm for rounding (NCVX-Weighted) (Algorithm 2) is the same as that in [CG15]. However, our analysis is quite different. Rather than using ideas from market interpretations of the problem, we utilize properties of (CVX-Weighted) and (NCVX-Weighted), which generalize to both the unweighted and the weighted versions of the problem.

Our analysis relies crucially on two facts: the relative stability of optimal points of (CVX-Weighted) and the interplay between the values of f_{cvx} and f_{ncvx} . First, we establish that any optimal point of (CVX-Weighted) is relatively stable; the difference between the objective values of an optimal solution and any feasible solution is independent of the valuations **v** and, therefore, can be bounded effectively. Second, we show that for any feasible solution, the difference between f_{cvx} and f_{ncvx} is, at most, the KL-Divergence between the weights and the constant vector.

Our analysis uses this stability property along with the structure of the feasibility polytope to iteratively sparsify an optimal solution and obtain a matching between the agents and bundles of items while only losing a constant factor in the objective. It is worth noting that the first term in the objective f_{ncvx} (and f_{cvx}) is linear in the variable **b**. As the constraint set on **b** is a matching polytope, the solution that optimizes a linear objective would be a matching in which all agents receive exactly one item. While such a matching would be very suboptimal compared to OPT, our algorithm constructs an augmented graph containing a matching with a value comparable to OPT. The crux of our algorithm is to find a feasible vector in the matching polytope for which f_{ncvx} is close to OPT, and the additional non-linear terms in f_{ncvx} are relatively small.

The remaining challenge to our approach is that (NCVX-Weighted) is not a convex program, and therefore, we cannot efficiently find a global optima that maximizes f_{ncvx} . However, we show that the objective of (NCVX-Weighted) and (CVX-Weighted) differ by at most the $D_{KL}(\mathbf{w} || \mathbf{u})$, as stated in the following lemma. We leverage this fact to initialize (NCVX-Weighted) with the globally optimal solution of (CVX-Weighted) to obtain the approximation guarantee.

Lemma 6. For any $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$ and weights $w_1, \ldots, w_n > 0$ with $\sum_{i \in \mathcal{A}} w_i = 1$,

$$0 \leq f_{\text{cvx}}(\mathbf{b}) - f_{\text{ncvx}}(\mathbf{b}) \leq D_{\text{KL}}(\mathbf{w} || \mathbf{u}) = \log n - \sum_{i \in \mathcal{A}} w_i \log \frac{1}{w_i}.$$

We obtain our main result in Theorem 1 by combining Lemma 4, Theorem 5, and Lemma 6.

1.4 Related Work

The problem of finding the allocation that maximizes the Nash Social Welfare objective is an NPhard problem, as was proven by [NNRR14]. Additionally, [Lee17] showed that finding such an allocation is also APX-hard. From an algorithmic perspective, the first constant factor approximation for the unweighted version was provided in [CG15] using analogies from market equilibrium. [CDG⁺17] provided an improved analysis of the algorithm from [CG15] and introduced a convex programming relaxation. Using an entirely different approach, [AGSS17] also provided a constant factor approximation for the unweighted variant, where their analysis employed the theory of log-concave polynomials. The best-known approximation factor with linear valuations of 1.45 is due to [BKV18], where they provide a pseudopolynomial-time algorithm that finds an allocation that is envy-free up to one good. Their algorithm is entirely combinatorial and runs in polynomial time when the valuations are bounded.

Another setting of interest is when the valuation of each agent is submodular instead of additive. For instance, [GHV21] gave a constant factor approximation algorithm for maximizing the unweighted Nash Social welfare function when the agents' valuations are Rado, a special subclass of submodular functions. In the weighted case, the approximation factor of this algorithm depends on the ratio of the maximum weight to the minimum weight. [LV22] provided a constant factor approximation algorithm for the unweighted case with submodular valuations. More recently, [GHL⁺23] gave a local search-based algorithm to obtain an $O(nw_{max})$ -approximation for the weighted case and a 4-approximation for the unweighted case with submodular valuations. Note that this $O(nw_{max})$ -approximation factor was also the previously best-known approximation for the weighted case, even when considering additive valuations.

2 Approximation Algorithm

Before describing our algorithm, we need the following definitions.

Definition 2 (Support Graph). For a vector $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$, the support graph of \mathbf{b} , denoted by $G_{\text{supp}}(\mathbf{b})$, *is a bipartite graph with vertex set* $\mathcal{A} \cup \mathcal{G}$. For any $i \in \mathcal{A}$ and $j \in \mathcal{G}$, the edge (i, j) belongs to the edge set of G if and only if $b_{ij} > 0$.

Definition 3 (Acyclic Solution). A vector $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$ is called an acyclic solution if the support graph of \mathbf{b} , $G_{supp}(\mathbf{b})$, does not contain any cycles.

For ease of notation, given any feasible point $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$, we use vector $\mathbf{q} \in \mathbb{R}^{|\mathcal{G}|}$ to denote the projection of **b** to \mathcal{G} , i.e.,

$$q_j := \sum_{i \in \mathcal{A}} b_{ij}$$

for each $j \in \mathcal{G}$. Since **q** is completely defined by **b**, with abuse of notation, we will interchangeably use $\mathcal{P}(\mathcal{A}, \mathcal{G})$ to denote feasible vectors **b** as well as (\mathbf{b}, \mathbf{q}) . Similarly, we will use $f_{ncvx}(\mathbf{b}, \mathbf{q})$ and $f_{cvx}(\mathbf{b}, \mathbf{q})$ to also denote the objective $f_{ncvx}(\mathbf{b})$ and $f_{cvx}(\mathbf{b})$, respectively. With a slight abuse of notation, we define

$$f_{\text{nevx}}(\mathbf{b},\mathbf{q}) := \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} w_i \, b_{ij} \log v_{ij} - \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} w_i \, b_{ij} \log q_j.$$

for any $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$ and its projection $\mathbf{q} \in \mathbb{R}^{|G|}$.

Our main algorithm, Algorithm 1, begins by finding the optimal solution $\overline{\mathbf{b}}$ to the convex program (CVX-Weighted). It then constructs another feasible point, $\mathbf{b}^{\text{forest}}$, in support of $\overline{\mathbf{b}}$ such that the support graph of $\mathbf{b}^{\text{forest}}$ is a forest and f_{ncvx} at $\mathbf{b}^{\text{forest}}$ is at least f_{ncvx} at $\overline{\mathbf{b}}$. In the final step, the algorithm rounds $\mathbf{b}^{\text{forest}}$ to an integral solution using Algorithm 2. Theorem 5 establishes a bound on the rounding error incurred during Algorithm 2.

Algorithm 1: A	pproximation	Algorithm for	Weighted Nas	sh Social Welfare
<i>(</i>)		()	()	

1 Input. NSW instance $(\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w})$

2 $\overline{\mathbf{b}}$ \leftarrow optimal solution of (CVX-Weighted)

- 3 $\overline{\mathbf{q}} \leftarrow \text{vector in } \mathbb{R}^{|\mathcal{G}|} \text{ with } \overline{q}_i = \sum_{i \in \mathcal{A}} \overline{b}_{ij}$
- 4 $(\mathbf{b}^{\text{forest}}, \mathbf{q}^{\text{forest}}) \leftarrow \text{acyclic solution in support of } \overline{\mathbf{b}} \text{ such that } f_{\text{ncvx}}(\mathbf{b}^{\text{forest}}) \geq f_{\text{ncvx}}(\overline{\mathbf{b}})$
- 5 $\sigma \leftarrow$ output of Algorithm 2 with input $(\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w}, \mathbf{b}^{\text{forest}}, \mathbf{q}^{\text{forest}})$
- 6 Output. σ

Lemma 4, which we re-state below for the reader's convenience, guarantees the existence of $\mathbf{b}^{\text{forest}}$, ensuring that the algorithm is well-defined. It is worth mentioning that for the unweighted case, the existence of an acyclic optimum was utilized by [CG15, CDG⁺17] for the convex program (CVX-Unweighted). In the weighted setting, this structural property is not inherited by the convex program (CVX-Weighted) but by the non-convex program (NCVX-Weighted).

Lemma 4. Let $\overline{\mathbf{b}}$ be any feasible point in $\mathcal{P}(\mathcal{A}, \mathcal{G})$. Then there exists an acyclic solution, $\mathbf{b}^{\text{forest}}$, in the support of $\overline{\mathbf{b}}$ such that

$$f_{\text{ncvx}}(\mathbf{b}^{\text{forest}}) \geq f_{\text{ncvx}}(\overline{\mathbf{b}}).$$

Moreover, such a solution can be found in time polynomial in |A| *and* |G|*.*

Proof. Let $G_{\text{supp}}(\bar{\mathbf{b}})$ contain a cycle $(i_0, j_0, i_1, \dots, j_{\ell-1}, i_\ell)$ with $i_0 = i_\ell$, where $i_x \in \mathcal{A}$ and $j_y \in \mathcal{G}$. The main idea is to modify the variables $\bar{\mathbf{b}}$ on this cycle while ensuring the value of $\bar{\mathbf{q}}$ does not change. If $\bar{\mathbf{q}}$ is fixed, then $f_{\text{ncvx}}(\cdot, \bar{\mathbf{q}})$ is linear in the input, and as a result, we can *cancel* the cycle by considering the following vector. Define $\boldsymbol{\delta} \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{G}|}$ with $\delta_{i_x j_x} := 1$ and $\delta_{i_{x+1} j_x} := -1$ for $x \in \{0, \dots, \ell-1\}$, and $\delta_{i_j} := 0$ otherwise.

Note that $\sum_{i \in A} \delta_{ij} = 0$ for any item *j*. As a result, for each $j \in G$,

$$\sum_{i\in\mathcal{A}}\bar{b}_{ij}+\varepsilon\delta_{ij}=\sum_{i\in\mathcal{A}}\bar{b}_{ij}=\bar{q}_j.$$

Therefore, the change in f_{ncvx} is given by

$$f_{\text{nevx}}(\bar{\mathbf{b}} + \varepsilon \boldsymbol{\delta}, \bar{\mathbf{q}}) - f_{\text{nevx}}(\bar{\mathbf{b}}, \bar{\mathbf{q}}) = \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} \varepsilon w_i \, \delta_{ij} \log v_{ij} - \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} \varepsilon w_i \, \delta_{ij} \log \bar{q}_j := \varepsilon \, h(\boldsymbol{\delta}, \bar{\mathbf{q}}).$$

Note that $h(\boldsymbol{\delta}, \bar{\mathbf{q}})$ is a linear function in $\boldsymbol{\delta}$. So, if $h(\boldsymbol{\delta}, \bar{\mathbf{q}}) > 0$, then setting $\varepsilon = \max_{x} b_{i_{x+1}j_x}$ ensures that $f_{\text{ncvx}}(\bar{\mathbf{b}} + \varepsilon \boldsymbol{\delta}, \bar{\mathbf{q}}) \ge f_{\text{ncvx}}(\bar{\mathbf{b}}, \bar{\mathbf{q}})$, and $\bar{\mathbf{b}} + \varepsilon \boldsymbol{\delta} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$. In addition, the number of cycles in $G_{\text{supp}}(\bar{\mathbf{b}} + \varepsilon \boldsymbol{\delta})$ is strictly less than the number of cycles in $G_{\text{supp}}(\bar{\mathbf{b}})$.

Similarly, if $h(\delta, \bar{\mathbf{q}}) \leq 0$, setting $\varepsilon = -\max_x b_{i_x j_x}$ gives the same guarantees. Iterating this cycle canceling process until the support contains no cycles leads to the required solution.

By combining Lemma 4 with Lemma 6, we obtain the following corollary.

Corollary 7. Let $\overline{\mathbf{b}}$ be any feasible point in $\mathcal{P}(\mathcal{A}, \mathcal{G})$. Then, there exists an acyclic solution, $\mathbf{b}^{\text{forest}}$, in the support of $\overline{\mathbf{b}}$ such that

$$f_{\text{cvx}}(\mathbf{b}^{\text{forest}}) \geq f_{\text{cvx}}(\overline{\mathbf{b}}) - D_{\text{KL}}(\mathbf{w} || \mathbf{u}).$$

Moreover, such a $\mathbf{b}^{\text{forest}}$ *can be found in time polynomial in* $|\mathcal{A}|$ *and* $|\mathcal{G}|$ *.*

Before presenting Algorithm 2, we give the proof of Theorem 1, which now follows directly from Theorem 5 and Corollary 7, as outlined below.

Proof of Theorem 1. Let $(\overline{\mathbf{b}}, \overline{\mathbf{q}})$ and $(\mathbf{b}^{\text{forest}}, \mathbf{q}^{\text{forest}})$ denote the feasible points defined in Step 1 and Step 3 of Algorithm 1, respectively. Let σ^* be the assignment returned by Algorithm 2 on input $(\mathbf{b}^{\text{forest}}, \mathbf{q}^{\text{forest}})$. By Theorem 5, we have

$$NSW(\sigma^{\star}) \geq f_{cvx}(\mathbf{b}^{\text{forest}}, \mathbf{q}^{\text{forest}}) - D_{KL}(\mathbf{w} || \mathbf{u}) - 2\log 2 - \frac{1}{2e}$$

$$\stackrel{(i)}{\geq} f_{cvx}(\overline{\mathbf{b}}, \overline{\mathbf{q}}) - 2 \cdot D_{KL}(\mathbf{w} || \mathbf{u}) - 2\log 2 - \frac{1}{2e}$$

$$\stackrel{(ii)}{\geq} OPT - 2 \cdot D_{KL}(\mathbf{w} || \mathbf{u}) - 2\log 2 - \frac{1}{2e}.$$

Here, (*i*) follows from Corollary 7 and (*ii*) follows from Lemma 3.

2.1 Rounding an Acyclic Solution

Given an acyclic solution **b**, Algorithm 2 returns an assignment, σ^* , such that NSW(σ^*) is comparable to $f_{\text{cvx}}(\mathbf{b})$, as stated in Theorem 5.

In the first step, Algorithm 2 finds an optimal solution, denoted by \mathbf{b}^* , to (CVX-Weighted) restricted to the support of \mathbf{b} , i.e., \mathbf{b}^* is the optimal solution to (CVX-Weighted) on input $(\mathcal{A}, \mathcal{G}, \tilde{\mathbf{v}}, \mathbf{w})$, where $\tilde{v}_{ij} = 0$ if $b_{ij} = 0$, and $\tilde{v}_{ij} = v_{ij}$ otherwise. This step is crucial as it allows us to utilize the stability properties of stationary points of (CVX-Weighted).

Next, the algorithm implements a "pruning" step to sparsify **b**^{*}: it removes edges between any item with $q_j^* < 1/2$ and its children in F^* . Here, F^* is the support graph of **b**^{*} with every tree rooted at agent nodes. This step is equivalent to assigning each item *j* with $q_j^* < 1/2$ to its parent agent in F^* . As a result, any item with $q_j^* < 1/2$ is a leaf in the pruned forest, \tilde{F} . Since removing edges will exclude certain items from being assigned to some agents, pruning can lead to a sub-optimal solution. We bound this loss in objective by showing the existence of a fractional solution

³If agent *i* in unmatched in *M*, we let $v_{iM(i)} = 0$

Algorithm 2: Algorithm for Rounding an Acyclic Solution

- **1 Input**. NSW instance $(\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w})$, acyclic solution $(\mathbf{b}, \mathbf{q}) \in \mathcal{P}(\mathcal{A}, \mathcal{G})$
- 2 ($\mathbf{b}^{\star}, \mathbf{q}^{\star}$) \leftarrow optimal solution of (CVX-Weighted) restricted to the support of (\mathbf{b}, \mathbf{q})
- **3** $F^{\star} \leftarrow G_{\text{supp}}(\mathbf{b}^{\star})$ with every tree rooted at an agent node
- 4 $\tilde{F} \leftarrow$ Forest obtained by removing edges between item *j* and its children in F^* whenever $q_i^* < 1/2$ /* pruning step */
- 5 $L_i^* \leftarrow$ set of leaf children of agent *i* in \widetilde{F} and let $L^* = \bigcup_i \{L_i^*\}$
- 6 $M^* \leftarrow$ matching between $\mathcal{A} \to \mathcal{G} \setminus L^*$ in \widetilde{F} which maximizes weight function

$$w_{\widetilde{F}}(M) := \sum_{i \in \mathcal{A}} w_i \log \left(v_{iM(i)} + \sum_{j \in L_i^*} v_{ij} \right)^3$$

7
$$\sigma^* \leftarrow \text{assignment of } \mathcal{G} \text{ to } \mathcal{A} \text{ with } \sigma^*(j) = i \text{ if } j \in \{L_i^* \cup M^*(i)\}$$
 /* matching step */
8 **Output.** σ^*

 $(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}})$ whose support graph is a subset of the pruned forest, \tilde{F} , and $f_{\text{cvx}}(\mathbf{b}^{\text{pruned}})$ is comparable to $f_{\text{cvx}}(\mathbf{b}^{\star})$. For concrete details, see Section 3.

It is important to emphasize that the algorithm does not need to find $(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}})$. The mere existence of $(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}})$ is enough to guarantee that the assignment returned by the algorithm will be good, as explained below.

After the pruning step, the algorithm assigns every leaf item in the pruned forest to its parent. We use L_i^* to denote the set of leaf items whose parent is agent *i* and $L^* = \bigcup_{i \in \mathcal{A}} L_i^*$ to denote the set of all leaf items in the pruned forest. So, each agent *i* receives all the items in the bundle L_i^* . In the matching step, the algorithm assigns at most one additional item to each agent by finding a maximum weight matching between agents \mathcal{A} and items $\mathcal{G} \setminus L^*$ (the set of non-leaf items in the pruned forest). This matching is determined using an augmented weight function, denoted by $w_{\tilde{F}}$. The weight of a matching M between \mathcal{A} and $\mathcal{G} \setminus L^*$ in the pruned forest is defined as follows:

$$w_{\widetilde{F}}(M) := \sum_{i \in \mathcal{A}} w_i \log \left(v_{iM(i)} + \sum_{j \in L_i^*} v_{ij} \right) ,$$

where $v_{iM(i)} = 0$ if *i* is not matched in *M*. Observe that this weight function exactly captures the weighted Nash Social Welfare objective when agent *i* is assigned the item set $S_i := \{M(i) \cup L_i^*\}$ for each $i \in A$. Moreover, finding the optimal matching *M* can be easily formulated as a maximum weight matching problem in a bipartite graph.

Since the standard linear programming relaxation for the bipartite matching problem is integral, it is enough to demonstrate the existence of a *fractional matching* with a large weight $w_{\tilde{F}}$ in the pruned forest. In Section 3.2, we show how to construct a fractional matching corresponding to $\mathbf{b}^{\text{pruned}}$, such that the weight of this matching is comparable to the objective $f_{\text{ncvx}}(\mathbf{b}^{\text{pruned}})$. We emphasize that this matching corresponding to $\mathbf{b}^{\text{pruned}}$ is only required for the sake of analysis: to lower bound the performance of the matching returned by the algorithm. We do not need to know $\mathbf{b}^{\text{pruned}}$ for the execution of the algorithm.

3 Rounding via the Non-Convex Relaxation

In this section, we prove Theorem 5 by establishing properties of support-restricted optimal solutions of (CVX-Weighted). First, in Lemma 8, we show that any optimum whose support is restricted to a forest can be "pruned" to a feasible solution while only losing a constant factor in the objective. Specifically, we show that given a support restricted optimum ($\mathbf{b}^*, \mathbf{q}^*$), we can construct a feasible solution ($\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}}$) such that any item with $q_j^{\text{pruned}} < 1/2$ is a leaf in support graph of $\mathbf{b}^{\text{pruned}}$, and $f_{\text{cvx}}(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}}) \ge f_{\text{cvx}}(\mathbf{b}^*, \mathbf{q}^*) - \log 2$.

Second, in Lemma 9, we demonstrate the existence of a matching in the support graph of $\mathbf{b}^{\text{pruned}}$ such that the augmented weight function of this matching differs from $f_{\text{ncvx}}(\mathbf{b}^{\text{pruned}})$ by a constant factor. After presenting these two lemmas, we provide the proof of Theorem 5.

Lemma 8. Let $(\mathbf{b}^*, \mathbf{q}^*)$ be the optimal solution of (CVX-Weighted) in the support of some acyclic feasible point $\mathbf{b}^{\text{forest}}$. Let F be a directed forest formed by $G_{\text{supp}}(\mathbf{b}^*)$ when every tree is rooted at an agent node. Then, there exists an acyclic feasible point $(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}})$ in $\mathcal{P}(\mathcal{A}, \mathcal{G})$ such that $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$ is a subgraph of $G_{\text{supp}}(\mathbf{b}^*)$ and

- $q_i^{\text{pruned}} \ge q_i^{\star}$ for any item *j* with $q_i^{\star} \ge 1/2$,
- each item with $q_i^* < 1/2$ is a leaf in $G_{supp}(\mathbf{b}^{pruned})$ connected to its parent in F, and
- $f_{\text{cvx}}(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}}) \ge f_{\text{cvx}}(\mathbf{b}^{\star}, \mathbf{q}^{\star}) \log 2.$

The proof of Lemma 8 relies on the stability properties of optimal solutions of (CVX-Weighted), as outlined in Section 3.1.

Lemma 9. Let (\mathbf{b}, \mathbf{q}) be an acyclic solution in $\mathcal{P}(\mathcal{A}, \mathcal{G})$ such that every item with $q_j < 1/2$ is a leaf in $G_{\text{support}}(\mathbf{b})$. Let $S : \mathcal{A} \to 2^{\mathcal{G}}$ be a function such that for each agent *i*, S(i) is a subset of the leaf items connected to agent *i* in $G_{\text{supp}}(\mathbf{b})$, and S(i) contains all children of agent *i* with $q_j < 1/2$. Then, there exists a matching *M* in $G_{\text{supp}}(\mathbf{b})$ between the vertices in \mathcal{A} and $\{\mathcal{G} \setminus \bigcup_i \{S(i)\}\}$ such that

$$\sum_{i\in\mathcal{A}} w_i \log\left(v_{iM(i)} + \sum_{j\in S(i)} v_{ij}\right) \ge f_{\mathrm{nevx}}(\mathbf{b}, \mathbf{q}) - \log 2 - \frac{1}{2e},$$

where $v_{iM(i)} = 0$ if agent *i* is not matched in M.

We prove this lemma in Section 3.2.

Proof of Theorem 5. Given (\mathbf{b}, \mathbf{q}) such that $G_{supp}(\mathbf{b})$ is a forest, let $(\mathbf{b}^*, \mathbf{q}^*)$ be the optimal solution of (CVX-Weighted) restricted to support of \mathbf{b} , let \tilde{F} denote the forest obtained after pruning $G_{supp}(\mathbf{b}^*)$. Let L_i^* denote the set of leaf children of agent *i* in \tilde{F} .

Let $(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}})$ be a feasible solution guaranteed by Lemma 8 on input $(\mathbf{b}^*, \mathbf{q}^*)$. Since Lemma 8 guarantees that $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$ is a subset of $G_{\text{supp}}(\mathbf{b}^*)$, and every item with $q_j^* < 1/2$ is a leaf in $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$, we conclude that $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$ is a subgraph of \widetilde{F} .

In addition, for any agent *i*, L_i^* is a subset of the leaf children of *i* in $G_{supp}(\mathbf{b}^{pruned})$ as $G_{supp}(\mathbf{b}^{pruned})$ is a subgraph of \widetilde{F} . Furthermore, if $q_j^{pruned} < 1/2$, then we claim that *j* is a leaf in $G_{supp}(\mathbf{b}^{pruned})$

with parent *i* such that $j \in L_i^*$ in \widetilde{F} . Since $q_j^{\text{pruned}} < 1/2$, by the first point of Lemma 8, we have $q_j^* < 1/2$. As a result, item *j* is a leaf in $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$ connected to its parent in \widetilde{F} . So, item *j* would be pruned in \widetilde{F} , and therefore, by definition, $j \in L_i^*$.

Therefore, for each agent *i*, the set L_i^{\star} is a subset of the set of leaves of agent *i* in $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$, and L_i^{\star} contains all the items with $q_j^{\text{pruned}} < 1/2$ in $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$. So, the function $S(i) = L_i^{\star}$ satisfies the constraints of Lemma 9 with input ($\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}}$).

Using Lemma 9 on $(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}})$ with function $S(i) = L_i^*$, we conclude that there exists a matching, M, in $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$ such that

$$\sum_{i \in \mathcal{A}} w_i \log \left(v_{iM(i)} + \sum_{j \in L_i^*} v_{ij} \right) = \sum_{i \in \mathcal{A}} w_i \log \left(v_{iM(i)} + \sum_{j \in S(i)} v_{ij} \right)$$
$$\geq f_{\text{nevx}}(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}}) - \log 2 - \frac{1}{2e}$$

Since $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$ is a subgraph of \tilde{F} , the matching M is also present in \tilde{F} . Therefore, the matching M^* (and corresponding assignment σ^*) returned by Algorithm 2 satisfies

$$NSW(\sigma^{\star}) = \sum_{i \in \mathcal{A}} w_i \log \left(v_{iM^{\star}(i)} + \sum_{j \in L_i^{\star}} v_{ij} \right) \stackrel{(i)}{\geq} \sum_{i \in \mathcal{A}} w_i \log \left(v_{iM(i)} + \sum_{j \in L_i^{\star}} v_{ij} \right)$$
$$\stackrel{(ii)}{\geq} f_{ncvx}(\mathbf{b}^{pruned}, \mathbf{q}^{pruned}) - \log 2 - \frac{1}{2e}$$
$$\stackrel{(iii)}{\geq} f_{cvx}(\mathbf{b}^{pruned}, \mathbf{q}^{pruned}) - D_{KL}(\mathbf{w} || \mathbf{u}) - \log 2 - \frac{1}{2e}$$
$$\stackrel{(iv)}{\geq} f_{cvx}(\mathbf{b}^{\star}, \mathbf{q}^{\star}) - D_{KL}(\mathbf{w} || \mathbf{u}) - 2\log 2 - \frac{1}{2e}$$
$$\stackrel{(v)}{\geq} f_{cvx}(\mathbf{b}, \mathbf{q}) - D_{KL}(\mathbf{w} || \mathbf{u}) - 2\log 2 - \frac{1}{2e}$$

Here, (*i*) follows from the optimality of M^* , (*ii*) follows from Lemma 9, (*iii*) follows from Lemma 6, (*iv*) follows from Lemma 9, and (*v*) follows from the optimality of \mathbf{b}^* .

3.1 Pruning Small Items

In this section, we prove Lemma 8 by establishing some properties of the set of (support restricted) optimal solutions of (CVX-Weighted) in Lemma 10 and Lemma 11.

First, we show that any optimal solution of (CVX-Weighted) is relatively stable, i.e., the change in function value when moving away from the optimal solution can be quantified in terms of how much we deviate from that solution. We formalize the stability property as follows.

Lemma 10. Let $(\mathbf{b}^*, \mathbf{q}^*)$ be the optimal solution of (CVX-Weighted) in the support of some acyclic feasible point $\mathbf{b}^{\text{forest}}$. Let (\mathbf{b}, \mathbf{q}) be a feasible point in $\mathcal{P}(\mathcal{A}, \mathcal{G})$ such that the support of \mathbf{b} is a subset of the support

of \mathbf{b}^* , and for any $j \in \mathcal{G}$, if $q_j^* = 1$, then $q_j = 1$. Then

$$f_{\text{cvx}}(\mathbf{b}^{\star},\mathbf{q}^{\star}) - f_{\text{cvx}}(\mathbf{b},\mathbf{q}) = \sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{A}}w_i b_{ij}\log\left(\frac{\sum_{i\in\mathcal{A}}w_i b_{ij}}{\sum_{i\in\mathcal{A}}w_i b_{ij}^{\star}}\right).$$

We provide the proof of this lemma in Appendix A.

Second, in Lemma 11, we show that any acyclic optimal solution of (CVX-Weighted) can be pruned to a feasible solution, denoted by $\mathbf{b}^{\text{pruned}}$, which is amenable to rounding. Specifically, we show that given a first-order stationary point $(\mathbf{b}^*, \mathbf{q}^*)$, we can construct a feasible solution $(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}})$ such that any item with $q_j^{\text{pruned}} < 1/2$ is a leaf in support of $\mathbf{b}^{\text{pruned}}$ and $b_{ij}^{\text{pruned}} \leq \min\{1, 2b_{ii}^*\}$ for any agent *i* and item *j*.

Lemma 11. Let $(\mathbf{b}^*, \mathbf{q}^*)$ be an acyclic feasible point in $\mathcal{P}(\mathcal{A}, \mathcal{G})$. Let F be a directed forest formed by $G_{supp}(\mathbf{b}^*)$ when every tree is rooted at an arbitrary agent node. Then, there exists a feasible solution $(\mathbf{b}^{pruned}, \mathbf{q}^{pruned})$ such that $G_{supp}(\mathbf{b}^{pruned})$ is a subgraph of $G_{supp}(\mathbf{b}^*)$,

- $q_j^{\star} \leq q_j^{\text{pruned}}$ for each item j with $q_j^{\star} \geq 1/2$,
- each item with $q_i^* < 1/2$ is a leaf in $G_{supp}(\mathbf{b}^{pruned})$ connected to its parent in F, and
- for any $(i, j) \in \mathcal{A} \times \mathcal{G}$, $b_{ij}^{\text{pruned}} \leq \min\{1, 2 \cdot b_{ij}^{\star}\}$.

Before proving Lemma 11, we use Lemma 11 along with Lemma 10 to prove Lemma 8.

Proof of Lemma 8. By Lemma 11, there exists a feasible solution ($\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}}$) such that the support graph, $G_{\text{supp}}(\mathbf{b}^{\text{pruned}})$, is a subgraph of $G_{\text{supp}}(\mathbf{b})$ and ($\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}}$) satisfies the first two items claimed in the lemma. Furthermore, for any $(i, j) \in \mathcal{A} \times \mathcal{G}$, $b_{ii}^{\text{pruned}} \leq \min\{1, 2 \cdot b_{ii}^{\star}\}$.

So using Lemma 10, the difference in objective between $(\mathbf{b}^{\star}, \mathbf{q}^{\star})$ to $(\mathbf{b}^{pruned}, \mathbf{q}^{pruned})$ is bounded as follows

$$f_{\text{cvx}}(\mathbf{b}^{\star}, \mathbf{q}^{\star}) - f_{\text{cvx}}(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}}) = \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} w_i b_{ij}^{\text{pruned}} \log\left(\frac{\sum_{i \in \mathcal{A}} w_i b_{ij}^{\text{pruned}}}{\sum_{i \in \mathcal{A}} w_i b_{ij}^{\star}}\right).$$

Since $b_{ij}^{\text{pruned}} \leq \min\{1, 2b_{ij}^{\star}\}$ for each (i, j), we have $\sum_i w_i b_{ij}^{\text{pruned}} \leq 2\sum_i w_i b_{ij}^{\star}$.

$$f_{\rm cvx}(\mathbf{b}^{\star}, \mathbf{q}^{\star}) - f_{\rm cvx}(\mathbf{b}^{\rm pruned}, \mathbf{q}^{\rm pruned}) \le \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} w_i b_{ij}^{\rm pruned} \log 2.$$
(4)

The feasibility of **b**^{pruned} implies

$$\sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} w_i b_{ij}^{\text{pruned}} = \sum_{i \in \mathcal{A}} w_i \sum_{j \in \mathcal{G}} b_{ij}^{\text{pruned}} = \sum_{i \in \mathcal{A}} w_i = 1.$$

Plugging this bound in equation (4) completes the proof.

Proof of Lemma 10. If **b**^{*} is an optimal solution of (CVX-Weighted), then using the KKT conditions, there exist real numbers λ_i for each $i \in A$, $\eta_j \ge 0$ for each $j \in G$, and $\alpha_{ij} \ge 0$ for every $(i, j) \in A \times G$ such that

$$\frac{\partial L}{\partial b_{ij}^{\star}} = w_i \log v_{ij} - w_i - w_i \log \left(\sum_{i \in \mathcal{A}} w_i b_{ij}^{\star} \right) - \lambda_i - \eta_j + \alpha_{ij} = 0.$$

In addition, by complementary slackness, we have $\eta_j(1 - \sum_{i \in \mathcal{A}} b_{ij}^*) = 0$ for each item *j* and $\alpha_{ij}b_{ij}^* = 0$ for each $(i, j) \in \mathcal{A} \times \mathcal{G}$. Using these complementary slackness conditions, if $b_{ij}^* > 0$, then

$$w_i \log v_{ij} = w_i + w_i \log \left(\sum_{i \in \mathcal{A}} w_i b_{ij}^{\star} \right) + \lambda_i + \eta_j.$$
(5)

Now, expanding the difference between the two function values, we get

$$f_{\text{cvx}}(\mathbf{b}^{\star}, \mathbf{q}^{\star}) - f_{\text{cvx}}(\mathbf{b}, \mathbf{q}) = \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} \left(b_{ij}^{\star} - b_{ij} \right) \cdot w_i \log v_{ij} - \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} w_i b_{ij}^{\star} \log \left(\sum_{i \in \mathcal{A}} w_i b_{ij}^{\star} \right) + \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} w_i b_{ij} \log \left(\sum_{i \in \mathcal{A}} w_i b_{ij} \right).$$
(6)

Substituting the value of v_{ij} from equation (5) in equation (6) gives

$$\begin{split} f_{\text{cvx}}(\mathbf{b}^{\star},\mathbf{q}^{\star}) - f_{\text{cvx}}(\mathbf{b},\mathbf{q}) &= \sum_{i\in\mathcal{A}}\sum_{j\in\mathcal{G}} (b_{ij}^{\star} - b_{ij}) \left(w_i \log\left(\sum_{i\in\mathcal{A}} w_i b_{ij}^{\star}\right) + \lambda_i + w_i + \eta_j \right) \\ &- \sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{A}} w_i b_{ij}^{\star} \log\left(\sum_{i\in\mathcal{A}} w_i b_{ij}^{\star}\right) + \sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{A}} w_i b_{ij} \log\left(\sum_{i\in\mathcal{A}} w_i b_{ij}\right) \\ &= \sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{A}} w_i b_{ij} \log\left(\frac{\sum_{i\in\mathcal{A}} w_i b_{ij}}{\sum_{i\in\mathcal{A}} w_i b_{ij}}\right) + \sum_{i\in\mathcal{A}} (\lambda_i + w_i) \left(\sum_{j\in\mathcal{G}} b_{ij}^{\star} - \sum_{j\in\mathcal{G}} b_{ij}\right) \\ &+ \sum_{j\in\mathcal{G}} \eta_j \left(\sum_{i\in\mathcal{A}} b_{ij}^{\star} - \sum_{i\in\mathcal{A}} b_{ij}\right). \end{split}$$

Using $\sum_{j \in \mathcal{G}} b_{ij} = \sum_{j \in \mathcal{G}} b_{ij}^{\star} = 1$ for every $i \in \mathcal{A}$, we get

$$f_{\text{cvx}}(\mathbf{b}^{\star},\mathbf{q}^{\star}) - f_{\text{cvx}}(\mathbf{b},\mathbf{q}) = \sum_{j\in\mathcal{G}}\sum_{i\in\mathcal{A}}w_i b_{ij}\log\left(\frac{\sum_{i\in\mathcal{A}}w_i b_{ij}}{\sum_{i\in\mathcal{A}}w_i b_{ij}^{\star}}\right) + \sum_{j\in\mathcal{G}}\eta_j \left(q_j^{\star} - q_j\right),$$

where the last equation follows from the definitions of q_j and q_j^* .

Note that by complementary slackness, $\eta_j(1 - q_j^*) = 0$ for any $j \in \mathcal{G}$. So if $q_j^* < 1$, then $\eta_j = 0$ and therefore $\eta_j(q_j^* - q_j) = 0$. If $q_j^* = 1$, then by the hypothesis of the Lemma, $q_j = 1$, and again we

obtain that $\eta_i(q_i^* - q_j) = 0$. Using this bound in the above equation gives

$$f_{\text{cvx}}(\mathbf{b}^{\star}, \mathbf{q}^{\star}) - f_{\text{cvx}}(\mathbf{b}, \mathbf{q}) = \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} w_i b_{ij} \log\left(\frac{\sum_{i \in \mathcal{A}} w_i b_{ij}}{\sum_{i \in \mathcal{A}} w_i b_{ij}^{\star}}\right).$$

Before proving Lemma 11, we need the following lemma about the feasibility of a solution when we decrease the b_{ij} for some edge (j, i) in the support forest of **b**.

Lemma 12. Let (\mathbf{b}, \mathbf{q}) be an acyclic feasible point in $\mathcal{P}(\mathcal{A}, \mathcal{G})$, and let F be a directed forest formed by $G_{\text{supp}}(\mathbf{b})$ when every tree is rooted at an arbitrary agent node. For a non-root agent i in F, let item j be its parent. Then, for any $0 \le \delta \le \min\{b_{ij}, 1 - b_{ij}\}$, there exists a feasible solution, $(\mathbf{b}^{\delta}, \mathbf{q}^{\delta})$ such that $b_{ij}^{\delta} = b_{ij} - \delta$, $q_i^{\delta} = q_j - \delta$, $q_{i'}^{\delta} \ge q_{j'}$ for all $j' \in \mathcal{G} \setminus \{j\}$, and

$$b_{i'j'}^{\delta} \begin{cases} \leq \min\{1, 2b_{i'j'}\} & \text{if } i', j' \in T(i) \\ = b_{i'j'} & \text{otherwise} , \end{cases}$$

where T(x) denotes the sub-tree rooted at x in F.

Proof of Lemma 11. We will iteratively build the solution ($\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}}$) satisfying these properties while ensuring it remains feasible. For a vertex $x \in \mathcal{A} \cup \mathcal{G}$, let par(x) denote its parent in $G_{\text{supp}}(\mathbf{b}^*)$, let C(x) denote the set of its children in $G_{\text{supp}}(\mathbf{b}^*)$, and let T(x) denote the sub-tree rooted at vertex x in $G_{\text{supp}}(\mathbf{b}^*)$.

Consider an item *j* with $q_j^* < 1/2$. To make the vertex corresponding to *j* a leaf, the algorithm removes all the edges between item *j* and its children C(j). To reflect this change, we will update the solution ($\mathbf{b}^*, \mathbf{q}^*$) to an intermediate solution ($\tilde{\mathbf{b}}, \tilde{\mathbf{q}}$) such that the support of $\tilde{\mathbf{b}}$ does not contain any edges between item *j* and its children. To maintain feasibility, we require:

$$\widetilde{q}_{j} = \widetilde{b}_{\text{par}(j)j} = b^{\star}_{\text{par}(j)j}$$
$$\widetilde{b}_{ij} = 0 \text{ for all } i \in C(j)$$
(7)

Note that $q_i^* < 1/2$ implies $b_{ij}^* < 1/2$ for each $i \in C(j)$. As a result, the decrease in b_{ij} satisfies

$$b_{ij}^{\star} - \widetilde{b}_{ij} \le \min\{b_{ij}^{\star}, 1 - b_{ij}^{\star}\}$$

for each $i \in C(j)$. So, we update $(\mathbf{b}, \mathbf{\tilde{q}})$ by iteratively applying Lemma 12 to edge $(j \to i)$ with $\delta = b_{ij}$ for each $i \in C(j)$. The updated solution satisfies $\tilde{b}_{ij} = 0$ for each $i \in C(j)$ and $\tilde{q}_j = q_j - \sum_{i \in C(j)} b_{ij} = b_{\text{par}(j)j} < 1/2$. Note that T(j) is the disjoint union of the sub-trees rooted at nodes in C(j). So for distinct $i_1, i_2 \in C(j)$, updating the edge $(j \to i_1)$ (and the sub-tree for i_1) does not affect the *b* values for any edge in $T(i_2)$ and vice versa. Therefore, by Lemma 12, we have $q_{j'}^* \leq \tilde{q}_{j'}$ for any item $j' \in T(j)$ and $\tilde{b}_{i'j'} \leq \min\{1, 2b_{i'j'}^*\}$ for any $i', j' \in T(j)$.

Since every item with $q_j^* < 1/2$ must become a leaf, we repeat the above process for any such item. The following fact is crucial to bound the values after multiple pruning processes: Pruning

item *j* only changes *b* values for edges in T(j), and item *j* becomes a leaf after that. So, if we prune ancestors of *j* after pruning *j*, the *b* values of edges in T(j) do not change further.

Let $(\mathbf{b}^{\text{pruned}}, \mathbf{q}^{\text{pruned}})$ be the solution obtained by pruning the set of items $J = \{j \in \mathcal{G} : q_j^* < 1/2\}$ in decreasing order of their height⁴. Pruning item *j* does not decrease the *q* value of any item other than *j*. Therefore, if $q_j^{\text{pruned}} < 1/2$, then $q_j^* < 1/2$, so item *j* has been pruned and is a leaf. For any item *j* with $q_j^* \ge 1/2$, its *q* value only increases when its nearest ancestor is pruned, and this is the only time its *q*-value changes. So we conclude that $q_j^{\text{pruned}} \ge q_j^*$ for each $j \in \mathcal{G}$.

To establish the third claim of the lemma, observe that the *b*-value of any edge in $G_{\text{supp}}(\mathbf{b}^*)$ changes at most twice during the pruning process: If $q_j^* \ge 1/2$, then item *j* itself is not pruned, and the *b* values of edges incident to *j* may change only when the nearest ancestor of *j* is pruned. By Lemma 12, $b_{ij}^{\text{pruned}} \le \min\{1, 2b_{ij}^*\}$ for each $i \in \mathcal{A}$. If $q_j^* < 1/2$, the *b* value of any edge from *j* to its children becomes zero when *j* is pruned, satisfying the claim. The *b* value of the edge $(\text{par}(j) \rightarrow j)$ does not change when we prune *j*, and it may increase when the nearest ancestor of *j* in *J* is pruned. If so, we have $b_{\text{par}(j)j}^{\text{pruned}} \le \min\{1, 2b_{\text{par}(j)j}^*\}$.

3.2 Fractional Matching and Analysis

In this section, we prove Lemma 9, which completes the proof of Theorem 5.

We establish Lemma 9 by proving two inequalities (in Lemmas 13 and 14) about the properties of f_{ncvx} at any feasible point whose support is a forest. Lemma 13 shows that f_{ncvx} can be upper bounded by a linear function in **b** while only losing a constant factor.

Lemma 13. Let (\mathbf{b}, \mathbf{q}) be an acyclic solution in $\mathcal{P}(\mathcal{A}, \mathcal{G})$ such that every item with $q_j < 1/2$ is a leaf in $G_{\text{support}}(\mathbf{b})$. Let $S : \mathcal{A} \to 2^{\mathcal{G}}$ be a function such that for each agent i, S(i) is a subset of leaf items connected to agent i in $G_{\text{supp}}(\mathbf{b})$, and S(i) contains all children of agent i with $q_j < 1/2$. Then

$$\sum_{i \in \mathcal{A}} w_i \left(\sum_{j \notin S(i)} b_{ij} \log v_{ij} + \sum_{j \in S(i)} b_{ij} \log \left(\sum_{j \in S(i)} v_{ij} \right) \right) \ge f_{\text{nevx}}(\mathbf{b}, \mathbf{q}) - \log 2 - \frac{1}{2e}$$

Lemma 14 demonstrates how the linear function obtained in Lemma 13 can be used as a lower bound for the maximum weight matching with the augmented weight function. A crucial component of the proof of this lemma is the fact that any feasible **b** in $\mathcal{P}(\mathcal{A}, \mathcal{G})$ corresponds to a point in the matching polytope where all agents are matched.

Lemma 14. Let (\mathbf{b}, \mathbf{q}) be an acyclic solution in $\mathcal{P}(\mathcal{A}, \mathcal{G})$ such that every item with $q_j < 1/2$ is a leaf in $G_{\text{support}}(\mathbf{b})$. Let $S : \mathcal{A} \to 2^{\mathcal{G}}$ be a function such that for each agent *i*, S(i) is a subset of leaf items connected to agent *i* in $G_{\text{supp}}(\mathbf{b})$, and S(i) contains all children of agent *i* with $q_j < 1/2$. Then, there exists a matching *M* in $G_{\text{supp}}(\mathbf{b})$ between vertices in \mathcal{A} and $\{\mathcal{G} \setminus \cup_i \{S(i)\}\}$ such that

$$\sum_{i \in \mathcal{A}} w_i \log \left(v_{iM(i)} + \sum_{j \in S(i)} v_{ij} \right) \ge \sum_{i \in \mathcal{A}} w_i \left(\sum_{j \notin S(i)} b_{ij} \log v_{ij} + \sum_{j \in S(i)} b_{ij} \log \left(\sum_{j \in S(i)} v_{ij} \right) \right) , \quad (8)$$

⁴Note that pruning items in decreasing order of their height is only an artifact of the analysis. The algorithm can prune items with $q_i^* < 1/2$ in any order.

where $v_{iM(i)} = 0$ if agent *i* is not matched in M.

Lemma 13 and Lemma 14 together establish Lemma 9. In the rest of this section, we provide the proofs of Lemma 13 and Lemma 14.

Proof of Lemma 13. Let $S := \bigcup_i \{S(i)\}$. Recall that

$$f_{\text{nevx}}(\mathbf{b}, \mathbf{q}) = \sum_{i \in \mathcal{A}} w_i \sum_{j \in \mathcal{G}} b_{ij} \log v_{ij} - \sum_{i \in \mathcal{A}} w_i \sum_{j \notin \mathcal{G}} b_{ij} \log q_j$$
$$= \sum_{i \in \mathcal{A}} w_i \sum_{j \notin S(i)} b_{ij} \log v_{ij} - \sum_{i \in \mathcal{A}} w_i \sum_{j \notin S(i)} b_{ij} \log q_j + \sum_{i \in \mathcal{A}} w_i \left(\sum_{j \in S(i)} b_{ij} \log v_{ij} - b_{ij} \log b_{ij} \right),$$
(9)

where the last equation follows from the fact that every item in S(i) is a leaf, i.e., if $j \in S(i)$, then $b_{i'j} = 0$ for every $i' \neq i$.

For an item $j \notin S$, we have $q_j \ge 1/2$. As a result,

$$-\sum_{i\in\mathcal{A}}w_i b_{ij}\log q_j \leq \log 2 \sum_{i\in\mathcal{A}}w_i b_{ij}.$$
(10)

Plugging this bound into equation (9) gives

$$f_{\text{ncvx}}(\mathbf{b}, \mathbf{q}) \leq \sum_{i \in \mathcal{A}} w_i \sum_{j \notin S(i)} b_{ij} \log v_{ij} - \sum_{i \in \mathcal{A}} w_i \sum_{j \notin S(i)} b_{ij} \log 2 + \sum_{i \in \mathcal{A}} w_i \left(\sum_{j \in S(i)} b_{ij} \log v_{ij} - b_{ij} \log b_{ij} \right).$$
(11)

As $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$, we have $\sum_{j \notin S(i)} b_{ij} = 1 - \sum_{j \in S(i)} b_{ij}$ for every agent *i*. Substituting this in equation (11) yields

$$f_{\text{nevx}}(\mathbf{b}, \mathbf{q}) \leq \sum_{i \in \mathcal{A}} w_i \sum_{j \notin S(i)} b_{ij} \log v_{ij} + \sum_{i \in \mathcal{A}} w_i \log 2 + \sum_{i \in \mathcal{A}} w_i \left(\sum_{j \in S(i)} b_{ij} \log v_{ij} - b_{ij} \log b_{ij} - b_{ij} \log 2 \right)$$
$$= \sum_{i \in \mathcal{A}} w_i \sum_{j \notin S(i)} b_{ij} \log v_{ij} + \log 2 + \sum_{i \in \mathcal{A}} w_i \left(\sum_{j \in S(i)} b_{ij} \log v_{ij} - b_{ij} \log b_{ij} - b_{ij} \log 2 \right), \quad (12)$$

where the last equation follows from $\sum_i w_i = 1$. For each agent $i \in A$, Claim 1.1 implies that

$$\sum_{j\in S(i)} b_{ij} \log v_{ij} - b_{ij} \log b_{ij} \leq \sum_{j\in S(i)} b_{ij} \log \left(\sum_{j\in S(i)} v_{ij}\right) - \sum_{j\in S(i)} b_{ij} \log \left(\sum_{j\in S(i)} b_{ij}\right).$$

So, for any agent *i*,

$$\sum_{j \in S(i)} b_{ij} \log v_{ij} - b_{ij} \log b_{ij} - b_{ij} \log 2$$

$$\leq \sum_{j \in S(i)} b_{ij} \log \left(\sum_{j \in S(i)} v_{ij} \right) - \sum_{j \in S(i)} b_{ij} \log \left(\sum_{j \in S(i)} b_{ij} \right) - \sum_{j \in S(i)} b_{ij} \log 2$$

$$\leq \sum_{j \in S(i)} b_{ij} \log \left(\sum_{j \in S(i)} v_{ij} \right) + \frac{1}{2e}, \qquad (13)$$

where the last inequality follows from $-x \log(x) - x \log 2 \le 1/(2e)$ for all $x \ge 0$ applied to $x = \sum_{j \in S(i)} b_{ij}$.

Substituting equation (13) in equation (12), we get

$$\begin{split} f_{\mathrm{ncvx}}(\mathbf{b},\mathbf{q}) &\leq \sum_{i \in \mathcal{A}} w_i \sum_{j \notin S(i)} b_{ij} \log v_{ij} + \log 2 + \sum_{i \in \mathcal{A}} w_i \left(\sum_{j \in S(i)} b_{ij} \log \left(\sum_{j \in S(i)} v_{ij} \right) + \frac{1}{2e} \right) \\ &= \sum_{i \in \mathcal{A}} w_i \left(\sum_{j \notin S(i)} b_{ij} \log v_{ij} + \sum_{j \in S(i)} b_{ij} \log \left(\sum_{j \in S(i)} v_{ij} \right) \right) + \log 2 + \frac{1}{2e}, \end{split}$$

where the last inequality again follows from $\sum_{i \in A} w_i = 1$.

Proof of Lemma 14. In this proof, we will analyze a matching that either assigns the bundle S(i) to an agent or a single item $j \notin \bigcup_i \{S(i)\}$. Observe that the algorithm clearly finds an assignment with a larger objective as

$$\log\left(v_{iM(i)} + \sum_{j \in S(i)} v_{ij}\right) \ge \max\left\{\log v_{iM(i)}, \log\left(\sum_{j \in S(i)} v_{ij}\right)\right\}.$$

So, for each agent $i \in A$, we create a new leaf item ℓ_i with $v_{i\ell_i} = \sum_{j \in S(i)} v_{ij}$ corresponding to the set of items in S(i). Define $S := \bigcup_i \{S(i)\}$ and $\tilde{\mathcal{G}} := \{\mathcal{G} \setminus S\} \cup \{\ell_i\}_{i \in A}$. We show that the maximum weight matching in the bipartite graph $(\mathcal{A}, \tilde{\mathcal{G}})$ suffices to prove the lemma. As the matching polytope is integral, it is enough to demonstrate the existence of a fractional matching of a large value.

Using **b**, we define fractional assignment variables **x** as follows:

$$\begin{aligned} x_{ij} &:= b_{ij} \quad \forall i \in \mathcal{A}, j \in \{\mathcal{G} \setminus L\} \\ x_{i\ell_i} &:= \sum_{j \in S(i)} b_{ij} \quad \forall i \in \mathcal{A}. \end{aligned}$$

The L.H.S. of equation (8) can be stated in terms of x as

$$\sum_{i \in \mathcal{A}} w_i \left(\sum_{j \notin S(i)} b_{ij} \log v_{ij} + \sum_{j \in S(i)} b_{ij} \log \left(\sum_{j \in S(i)} v_{ij} \right) \right) = \sum_{i \in \mathcal{A}} \sum_{j \in \widetilde{\mathcal{G}}} x_{ij} w_i \log v_{ij}.$$
(14)

Observe that **x** lies in the convex hull of matchings between agents \mathcal{A} and items $\tilde{\mathcal{G}}$ in which every

agent is matched as x satisfies the following properties:

$$\sum_{j \in \widetilde{\mathcal{G}}} x_{ij} = \sum_{j \notin S(i)} b_{ij} + \sum_{j \in S(i)} b_{ij} = 1 \quad \forall i \in \mathcal{A}$$
$$\sum_{i \in \mathcal{A}} x_{ij} \le 1 \quad \forall j \in \widetilde{\mathcal{G}}.$$

Here, for item $j \notin S$, the second inequality is inherited from the feasibility of **b**. The constraint for $\ell_{i'}$ for some $i' \in A$ is implied by the constraint $\sum_{i \in A} x_{ij} = x_{i'j} = \sum_{j \in S(i)} b_{ij} \leq \sum_{j \in G} b_{ij} \leq 1$, where the last constraint again follows from the feasibility of **b**.

Using the integrality of the matching polytope, there exists a matching $\widetilde{M} : \mathcal{A} \to \widetilde{\mathcal{G}}$ such that

$$\sum_{i \in \mathcal{A}} \sum_{j \in \widetilde{\mathcal{G}}} x_{ij} w_i \log v_{ij} \le \sum_{i \in \mathcal{A}} w_i \log v_{i\widetilde{M}(i)}.$$
(15)

Now consider a matching $M : \mathcal{A} \to \mathcal{G}$ with $M(i) = \emptyset$ if $\widetilde{M}(i) = \ell_i$, and $M(i) = \widetilde{M}(i)$ otherwise. Then

$$\sum_{i \in \mathcal{A}} w_i \log v_{i\widetilde{M}(i)} \le \sum_{i \in \mathcal{A}} w_i \log \left(v_{iM(i)} + \sum_{j \in S(i)} v_{ij} \right).$$
(16)

Then equations (14), (15), and (16) together imply

$$\sum_{i \in \mathcal{A}} w_i \log \left(v_{iM(i)} + \sum_{j \in S(i)} v_{ij} \right) \ge \sum_{i \in \mathcal{A}} w_i \left(\sum_{j \notin S(i)} b_{ij} \log v_{ij} + \sum_{j \in S(i)} b_{ij} \log \left(\sum_{j \in S(i)} v_{ij} \right) \right).$$

4 Conclusion and Open Questions

This paper introduces a convex and a non-convex relaxation for the weighted (asymmetric) Nash Social Welfare problem to give an $O(\exp(2D_{KL}(\mathbf{w} || \mathbf{u})))$ -approximation. Both of these relaxations play a crucial role in obtaining the approximation algorithm for the problem. There are two natural open questions. First, is the factor $\exp(2D_{KL}(\mathbf{w} || \mathbf{u}))$ necessary in the approximation guarantee? Equivalently, is it possible to obtain a constant factor approximation for the weighted Nash Social Welfare problem? It is important to emphasize that we lose the $\exp(2D_{KL}(\mathbf{w} || \mathbf{u}))$ when relating the objectives of the two relaxations; we only lose a constant factor when rounding the non-convex relaxation. A direct approach may exist to solve the non-convex formulation that gives an improved approximation guarantee.

The second question is whether the techniques introduced in this work generalize to more general valuation functions, particularly submodular valuations for the weighted Nash Social Welfare problem. While there are constant factor approximation algorithms for unweighted Nash Social Welfare with submodular valuations, obtaining anything better than $O(nw_{max})$ -approximation for the weighted variant of the problem remains an open question.

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A Omitted Proofs and Lemmas

Proof of Lemma 3. Let $\sigma : \mathcal{G} \to \mathcal{A}$ be the optimal assignment of an instance of NSW, $(\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w})$. For each agent $i \in \mathcal{A}$, define $V_i = \sum_{j \in \sigma^{-1}(i)} v_{ij}$. Using σ , we define a vector $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$ as

$$b_{ij} := \begin{cases} \frac{v_{ij}}{V_i} & \text{if } \sigma(j) = i\\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $\sum_{i \in A} b_{ij} \leq 1$ for each $j \in G$ and $\sum_{i \in G} b_{ij} = 1$ for each $i \in A$. We will now

show that $f_{\text{cvx}}(\mathbf{b})$ and $f_{\text{ncvx}}(\mathbf{b})$ are both equal to NSW(σ).

$$\begin{split} f_{\text{cvx}}(\mathbf{b}) &= \sum_{j \in \mathcal{G}} \frac{w_{\sigma(j)} v_{\sigma(j)j}}{V_{\sigma(j)}} \log v_{\sigma(j)j} - \sum_{j \in \mathcal{G}} \frac{w_{\sigma(j)} v_{\sigma(j)j}}{V_{\sigma(j)}} \log \left(\frac{w_{\sigma(j)} v_{\sigma(j)j}}{V_{\sigma(j)}}\right) + \sum_{i \in \mathcal{A}} w_i \log w_i \\ &= \sum_{j \in \mathcal{G}} \frac{w_{\sigma(j)} v_{\sigma(j)j}}{V_{\sigma(j)}} \log \left(\frac{V_{\sigma(j)}}{w_{\sigma(j)}}\right) + \sum_{i \in \mathcal{A}} w_i \log w_i \\ &= \sum_{i \in \mathcal{A}} w_i \sum_{j \in \sigma^{-1}(i)} \frac{v_{ij}}{V_i} \log \left(\frac{V_i}{w_i}\right) + \sum_{i \in \mathcal{A}} w_i \log w_i \\ &\stackrel{(i)}{=} \sum_{i \in \mathcal{A}} w_i \log \left(\frac{V_i}{w_i}\right) + \sum_{i \in \mathcal{A}} w_i \log w_i = \sum_{i \in \mathcal{A}} w_i \log V_i = \text{NSW}(\sigma) \,, \end{split}$$

where (i) follows from definition of V_i .

Similarly, we have

$$f_{\text{nevx}}(\mathbf{b}) = \sum_{j \in \mathcal{G}} \frac{w_{\sigma(j)} v_{\sigma(j)j}}{V_{\sigma(j)}} \log v_{\sigma(j)j} - \sum_{j \in \mathcal{G}} \frac{w_{\sigma(j)} v_{\sigma(j)j}}{V_{\sigma(j)}} \log \left(\frac{v_{\sigma(j)j}}{V_{\sigma(j)}}\right)$$
$$= \sum_{j \in \mathcal{G}} \frac{w_{\sigma(j)} v_{\sigma(j)j}}{V_{\sigma(j)}} \log V_{\sigma(j)} = \sum_{i \in \mathcal{A}} w_i \sum_{j \in \sigma^{-1}(i)} \frac{v_{ij}}{V_i} \log V_i$$
$$= \sum_{i \in \mathcal{A}} w_i \log V_i = \text{NSW}(\sigma).$$

For the second claim in the lemma, when $w_i = 1/n$ for each *i*, for any $\mathbf{b} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$, we have

$$\begin{split} f_{\text{cvx}}(\mathbf{b}) &= \frac{1}{n} \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} b_{ij} \log v_{ij} - \frac{1}{n} \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} b_{ij} \log \left(\frac{\sum_{i \in \mathcal{A}} b_{ij}}{n}\right) - \log n \\ &= \frac{1}{n} \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} b_{ij} \log v_{ij} - \frac{1}{n} \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} b_{ij} \log \left(\sum_{i \in \mathcal{A}} b_{ij}\right) + \frac{1}{n} \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} b_{ij} \log n - \log n \\ &= \frac{1}{n} \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} b_{ij} \log v_{ij} - \frac{1}{n} \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} b_{ij} \log \left(\sum_{i \in \mathcal{A}} b_{ij}\right), \end{split}$$

where we used $\sum_{j \in \mathcal{G}} b_{ij} = 1$ for every *i* in the last inequality. Similarly, substituting $w_i = 1/n$ for each *i* in f_{ncvx} completes the proof.

Proof of Lemma 6. We will show that

$$f_{\text{cvx}}(\mathbf{b}) - f_{\text{ncvx}}(\mathbf{b}) = D_{\text{KL}}(\mathbf{w} || \mathbf{u}) - D_{\text{KL}}(\mu || \theta)$$
,

where μ , θ are two probability distributions on \mathcal{G} given by

$$\mu(j) = \sum_{i \in \mathcal{A}} w_i b_{ij}$$
 and $\theta(j) = \frac{\sum_{i \in \mathcal{A}} b_{ij}}{n}$.

Using $\sum_{i \in A} w_i = 1$ and $\sum_{j \in G} b_{ij} = 1$ for each $i \in A$, one can verify that $\sum_{j \in G} \mu(j) = 1 = \sum_{j \in G} \theta(j)$. Expanding the difference between the functions gives

$$\begin{split} f_{\text{cvx}}(\mathbf{b}) - f_{\text{ncvx}}(\mathbf{b}) &= \sum_{i \in \mathcal{A}} w_i \log w_i - \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} w_i b_{ij} \log \left(\sum_{i \in \mathcal{A}} w_i b_{ij}\right) + + \sum_{i,j} w_i b_{ij} \log \left(\sum_{i \in \mathcal{A}} b_{ij}\right) \\ &= \sum_{i \in \mathcal{A}} w_i \log w_i - \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} w_i b_{ij} \log \left(\frac{\sum_{i \in \mathcal{A}} w_i b_{ij}}{\sum_{i \in \mathcal{A}} b_{ij}}\right) \\ &= \sum_{i \in \mathcal{A}} w_i \log w_i + \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} w_i b_{ij} \log n - \sum_{j \in \mathcal{G}} \mu(j) \log \left(\frac{\mu(j)}{\theta(j)}\right) \\ &= \sum_{i \in \mathcal{A}} w_i \log(nw_i) - \sum_{j \in \mathcal{G}} \mu(j) \log \left(\frac{\mu(j)}{\theta(j)}\right) \qquad (\text{using } \sum_j b_{ij} = 1) \\ &= D_{\text{KL}}(\mathbf{w} \mid \mid \mathbf{u}) - D_{\text{KL}}(\mu \mid \mid \theta). \end{split}$$

As $D_{KL}(\mu, \theta) \ge 0$, the above equation implies

$$f_{\text{cvx}}(\mathbf{b}) - f_{\text{ncvx}}(\mathbf{b}) \leq D_{\text{KL}}(\mathbf{w} || \mathbf{u}).$$

For the lower bound, it suffices to show that $D_{KL}(\mu || \theta) \leq D_{KL}(\mathbf{w} || \mathbf{u})$. To see this, we expand the definition:

$$\begin{split} D_{\mathrm{KL}}(\mu \mid\mid \theta) &= \sum_{j \in \mathcal{G}} \left(\sum_{i \in \mathcal{A}} w_i b_{ij} \right) \log \left(\frac{n \sum_{i \in \mathcal{A}} w_i b_{ij}}{\sum_{i \in \mathcal{A}} b_{ij}} \right) \\ &= \log(n) + \sum_{j \in \mathcal{G}} \left(\sum_{i \in \mathcal{A}} w_i b_{ij} \right) \log \left(\frac{\sum_{i \in \mathcal{A}} w_i b_{ij}}{\sum_{i \in \mathcal{A}} b_{ij}} \right) \\ &= \log(n) + \sum_{j \in \mathcal{G}} \left(\sum_{i \in \mathcal{A}} b_{ij} \right) \left(\sum_{i \in \mathcal{A}} \frac{b_{ij}}{\sum_{i \in \mathcal{A}} b_{ij}} \cdot w_i \right) \log \left(\sum_{i \in \mathcal{A}} \frac{b_{ij}}{\sum_{i \in \mathcal{A}} b_{ij}} \cdot w_i \right) \\ &\leq \log(n) + \sum_{j \in \mathcal{G}} \left(\sum_{i \in \mathcal{A}} b_{ij} \right) \left(\sum_{i \in \mathcal{A}} \frac{b_{ij}}{\sum_{i \in \mathcal{A}} b_{ij}} \right) w_i \log(w_i) \\ &= \log(n) + \sum_{j \in \mathcal{A}} w_i \log(w_i) \sum_{j \in \mathcal{G}} b_{ij} \\ &= \log(n) + \sum_{i \in \mathcal{A}} w_i \log(w_i) = D_{\mathrm{KL}}(\mathbf{w} \mid\mid \mathbf{u}). \end{split}$$

Here, the only inequality uses the convexity of $x \log(x)$, and the last equality follows from the feasibility of **b**.

Proof of Lemma 12. For $x \in A \cup G$, let C(x) denote the children of node x in F and let T(x) denote the sub-tree rooted at node x. We will prove this lemma by induction on the height of agent i, building $(\mathbf{b}^{\delta}, \mathbf{q}^{\delta}) \in \mathcal{P}(A, G)$ in the process.

For the base case, assume agent *i* has height 1, i.e., T(i) consists of only leaf item nodes that are the children of node *i*. We define a new vector \mathbf{b}^{δ} with $b_{i'j'}^{\delta} = b_{i'j'}$ for any $i' \neq i$ and $j' \in \mathcal{G}$. Note

that setting $b_{ij}^{\delta} = b_{ij} - \delta$ and $q_j^{\delta} = q_j - \delta$ only violates the Agent constraint for agent *i*. So we will update the values of **b** in T(i) to make the solution feasible.

By the feasibility of **b**, $b_{ij} + \sum_{k \in C(i)} b_{ik} = 1$, and for every item node $k \in C(i)$, $q_k = b_{ik} < 1$. Using Lemma 15 with $\alpha = b_{ij}$ and $\beta_k = b_{ik}$, there exist δ_k for each $k \in C(i)$ such that

$$\begin{split} b_{ij} - \delta + \sum_{k \in C(i)} b_{ik} (1 + \delta_k) &= 1 \\ b_{ik} (1 + \delta_k) &\leq 1 \quad \forall k \in C(i) \\ 0 &\leq \delta_k \leq 1 \quad \forall k \in C(i). \end{split}$$

So, for each $k \in C(i)$, we set $b_{ik}^{\delta} = b_{ik}(1 + \delta_k)$. Note that $b_{ik}^{\delta} \leq 1$, and as $\delta_k \leq 1$, we have

$$b_{ik}^{\delta} = b_{ik}(1+\delta_k) \le 2b_{ik}.$$

As every item in C(i) is a leaf, we also have

$$q_k \leq q_k^\delta = b_{ik}^\delta = b_{ik}(1+\delta_k) \leq 1$$

for each item $k \in C(i)$. The Agent constraint for agent *i* satisfies

$$\sum_{k \in C(i)} b_{ik}^{\delta} = b_{ij} - \delta + \sum_{k \in C(i)} b_{ik}(1 + \delta_k) = 1.$$

Therefore, $\mathbf{b}^{\delta} \in \mathcal{P}(\mathcal{A}, \mathcal{G})$ and $b_{i'j'}^{\delta} \leq \min\{1, 2b_{ij'}\}$ for each $j' \in T(i)$.

For the induction hypothesis, assume that the lemma is true whenever the height of agent *i* is at most $\ell - 1$ for some integer $\ell > 1$. We now show that the statement also holds when the height of agent *i* is ℓ .

Again, setting $b_{ij}^{\delta} = b_{ij} - \delta$ and $q_j^{\delta} = q_j - \delta$ violates the Agent constraint for agent *i*. Similar to the base case, we can find $\delta_k \in (0, 1)$ for each $k \in C(i)$ such that $b_{ik}(1 + \delta_k) \leq 1$ and

$$b_{ij} - \delta + \sum_{k \in C(i)} b_{ik} (1 + \delta_k) = 1.$$

Setting $b_{ik}^{\delta} = b_{ik}(1 + \delta_k)$ for each $k \in C(i)$ will ensure that \mathbf{b}^{δ} satisfies the Agent constraint for agent *i*. However, this can violate the Item constraint for some item $k \in C(i)$, as $q_k^{\delta} = q_k + \delta_k b_{ik}$. So, we inductively update the values of \mathbf{b}^{δ} and \mathbf{q}^{δ} for the sub-tree rooted at item *k* for which such a violation occurs.

Consider an item $k \in C(i)$ such that $q_k^{\delta} = q_k + \delta_k b_{ik} > 1$. So we decrease $b_{i'k}$ for each $i' \in C(k)$ to ensure that q_k^{δ} is at most 1 as follows. Define $\gamma := q_k + \delta_k b_{ik} - 1$. Using the fact that $q_k = \sum_{i' \in C(k)} b_{i'k} + b_{ik}$, we bound γ as follows.

$$\begin{split} \gamma &= q_k + \delta_k b_{ik} - 1 = \sum_{i' \in C(k)} b_{i'k} + b_{ik} + \delta_k b_{ik} - 1 \\ &\leq \sum_{i' \in C(k)} b_{i'k}. \end{split} \qquad (using \ b_{ik}(1 + \delta_k) \leq 1) \end{split}$$

Therefore, there exist numbers $\gamma_{i'} \ge 0$ for each $i' \in C(k)$ such that $\gamma_{i'} \le b_{i'k}$ and $\sum_{i' \in C(k)} \gamma_{i'} = \gamma$. We would like to update $b_{i'k}^{\delta} = b_{i'k} - \gamma_{i'}$ for each $i' \in C(k)$, but this violates the Agent constraint for agent i' when $\gamma_{i'} > 0$. We inductively update the solution for subtree T(i') as follows. First, note that $q_k^{\delta} = 1 \ge q_k$ after this update, as shown below.

$$\begin{aligned} q_k^{\delta} &= b_{ik}^{\delta} + \sum_{i' \in C(k)} b_{i'k}^{\delta} = b_{ik}(1+\delta_k) + \sum_{i' \in C(k)} (b_{i'k} - \gamma_{i'}) = q_k + \delta_i b_{ik} - \sum_{i' \in C(k)} \gamma_{i'} \\ &= 1 - \gamma + \sum_{i' \in C(k)} \gamma_{i'} \\ &= 1. \end{aligned}$$
 (by definition of γ)
$$&= 1. \end{aligned}$$

So now, $(\mathbf{b}^{\delta}, \mathbf{q}^{\delta})$ only violates Agent constraints for agents in C(k).

We claim that for each agent $i' \in C(k)$

$$\gamma_{i'} \le \min\{b_{i'k}, \ 1 - b_{i'k}\}. \tag{17}$$

Before proving this inequality, we use it to complete the proof.

Using the induction hypothesis, for each $i' \in C(k)$, there exists feasible $(\mathbf{b}^{\gamma_{i'}}, \mathbf{q}^{\gamma_{i'}})$ which differs from $(\mathbf{b}^{\delta}, \mathbf{q}^{\delta})$ only in the sub-tree rooted at i' such that for any $\hat{j} \in T(i')$,

$$q_{\hat{j}}^{\gamma_{i'}} \ge q_{\hat{j}}^{\delta} = q_{\hat{j}}$$

and for any $\hat{i}, \hat{j} \in T(i')$,

$$b_{\hat{i}\hat{j}}^{\gamma_{i'}} \leq \min\{1, 2 \cdot b_{\hat{i},\hat{j}}^{\delta}\} = \min\{1, 2 \cdot b_{\hat{i},\hat{j}}\}.$$

So for each $i' \in C(k)$ with $\gamma_{i'} > 0$, we set $b_{\hat{i}\hat{j}}^{\delta} = b_{\hat{i}\hat{j}}^{\gamma_{i'}}$ for every $\hat{i}, \hat{j} \in T(i')$ to get the required solution.

We now only need to establish equation (17). By definition, $\gamma_{i'} \leq b_{i'k}$ for each $i' \in C(k)$. Additionally, $\gamma_{i'} \leq \gamma$, so it suffices to show that $\gamma \leq 1 - b_{i'k}$ for every $i' \in C(k)$. Recall that

$$\gamma = q_k + \delta_k b_{ik} - 1$$

$$\stackrel{(i)}{\leq} \delta_k b_{ik} \stackrel{(ii)}{\leq} b_{ik} \stackrel{(iii)}{\leq} q_k - b_{i'k} \stackrel{(iv)}{\leq} b_{i'k}.$$

Here, (*i*) and (*iv*) follow from $q_k \leq 1$, (*ii*) follows from $\delta_k \leq 1$, and (*iii*) holds as $b_{ik} + \sum_{i' \in C(k)} b_{i'k} = q_k$. This completes the proof of (17).

Lemma 15. Let $\alpha > 0$ and $\beta_1, \ldots, \beta_k > 0$ with $\alpha + \sum_{j=1}^k \beta_i = 1$. For any $0 < \delta \le \min\{\alpha, 1-\alpha\}$, there exist real numbers $\delta_1, \ldots, \delta_k$ such that

$$\alpha - \delta + \sum_{j \in [k]} \beta_j (1 + \delta_j) = 1$$

$$\beta_j (1 + \delta_j) \le 1 \quad \forall j \in [k]$$

$$0 \le \delta_j \le 1 \quad \forall j \in [k].$$
(18)

Proof. As the above system contains only linear constraints in δ , we use Farkas' Lemma to show

the existence of $\{\delta_j\}_{j=1}^k$. Re-arranging the constraints gives

$$\sum_{j \in [k]} \beta_j \delta_j = \delta$$

$$\beta_j \delta_j \leq 1 - \beta_j \quad \forall j \in [k]$$

$$0 \leq \delta_j \leq 1 \quad \forall j \in [k]$$
(19)

If there do not exist real numbers $\{\delta_j\}_{j=1}^k$ satisfying (19), then by Farkas' Lemma, there exist real numbers η , $\{\gamma_j\}_{j=1}^k$, $\{\lambda_j\}_{j=1}^k$ such that

$$\beta_{j}\eta + \beta_{j}\gamma_{j} + \lambda_{j} \ge 0 \quad \forall j \in [k]$$

$$\gamma_{i}, \lambda_{i} \ge 0$$
(20)

$$\delta\eta + \sum_{j \in [k]} (1 - \beta_j)\gamma_j + \sum_{j \in [k]} \lambda_j < 0$$
(21)

Adding equation (20) for all $j \in [k]$, we get

$$\eta \sum_{j \in [k]} \beta_j + \sum_{j \in [k]} \beta_j \gamma_j + \sum_{j \in [k]} \lambda_j \ge 0.$$

Since $\alpha + \sum_{j \in [k]} \beta_j = 1$, this implies $\eta(1 - \alpha) + \sum_{j \in [k]} \beta_j \gamma_j + \sum_{j \in [k]} \lambda_j \ge 0$. In addition, since $\beta_i > 0$, we also have $\alpha < 1$. Therefore, dividing by $1 - \alpha$ and re-arranging gives

$$\sum_{j \in [k]} \frac{\beta_j \gamma_j}{1 - \alpha} + \sum_{j \in [k]} \frac{\lambda_j}{1 - \alpha} \ge -\eta.$$
(22)

On the other hand, equation (21) implies

$$-\eta > \sum_{j \in [k]} \frac{(1 - \beta_j)\gamma_j}{\delta} + \sum_{j \in [k]} \frac{\lambda_j}{\delta}.$$
(23)

On comparing equations (22) and (23), we obtain

$$\sum_{j \in [k]} \frac{\beta_j \gamma_j}{1 - \alpha} + \sum_{j \in [k]} \frac{\lambda_j}{1 - \alpha} > \sum_{j \in [k]} \frac{(1 - \beta_j) \gamma_j}{\delta} + \sum_{j \in [k]} \frac{\lambda_j}{\delta}.$$
(24)

We will now derive a contradiction to (24).

As $\delta \leq 1 - \alpha$, we have $1/(1 - \alpha) \leq 1/\delta$, and therefore,

$$\sum_{j \in [k]} \frac{\lambda_j}{1 - \alpha} \le \sum_{j \in [k]} \frac{\lambda_j}{\delta},$$
(25)

where we use the fact that $\lambda_j > 0$ for all $j \in [k]$.

In addition, for any $j \in [k]$

$$\frac{\beta_j}{1-\alpha} - \frac{(1-\beta_j)}{\delta} \le \frac{\beta_j}{1-\alpha} - \frac{(1-\beta_j)}{\alpha} = \frac{\alpha+\beta_j-1}{\alpha(1-\alpha)} \le 0.$$
(26)

Here, the first inequality follows from $\delta \leq \alpha$, and the last inequality follows from the facts that $\alpha + \sum_{j \in [k]} \beta_j = 1$ and $\alpha, \beta_j > 0$.

On adding equation (25) and equation (26) for all $j \in [k]$, we obtain

$$\sum_{j \in [k]} \frac{\beta_j \gamma_j}{1 - \alpha} + \sum_{j \in [k]} \frac{\lambda_j}{1 - \alpha} \le \sum_{j \in [k]} \frac{(1 - \beta_j) \gamma_j}{\delta} + \sum_{j \in [k]} \frac{\lambda_j}{\delta},$$

which contradicts equation (24). Therefore, there exist real numbers $\{\delta_j\}_{j=1}^k$ satisfying (18).

B Relationships Between the Mathematical Programs

This section provides the proof of Theorem 2 by establishing a relationship between two natural convex programming relaxations for the unweighted Nash Social Welfare problem. We then build upon this relationship to derive (CVX-Weighted) for the weighted Nash Social Welfare problem.

To ensure that the optimum values of all the convex programs mentioned below are bounded, we assume that the instance of Nash Social Welfare ($\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w}$) satisfies the following assumption.

Assumption B.1. Let $G[\mathcal{G}, \mathcal{A}, \mathbf{v}]$ denote the support graph of the valuation function. The support graph is the bipartite graph between agents and items with an edge between agent *i* and item *j* iff $v_{ij} > 0$. We assume that there exists a matching of size $|\mathcal{A}|$ in $G[\mathcal{G}, \mathcal{A}, \mathbf{v}]$. In other words, the objective of the Nash Social Welfare problem is not zero for $(\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w})$.

It is straightforward to verify this assumption given an instance of Nash Social Welfare.

The proof of Theorem 2 uses the following two results. The first result is the classical Sion's Minimax Theorem, which can be found as Corollary 3.3 from [Sio58].

Theorem 16 (Sion's Minimax Theorem). *Let* M *and* N *be convex spaces, one of which is compact, and* f(x, y) *a function on* $M \times N$ *that is quasi-concave-convex and (upper semicontinuous)-(lower semicontinuous). Then*

$$\sup_{\mathbf{x}\in M} \inf_{y\in N} f(x,y) = \inf_{y\in N} \sup_{\mathbf{x}\in M} f(x,y).$$

The second result was proved in [AGSS17].

Lemma 17 (Lemma 4.3 in [AGSS17]). Let $p : \mathbb{R}_{\geq 0}^m \to \mathbb{R}_{\geq 0}$ be a positive function satisfying the following properties:

- $p(\alpha \mathbf{y}) = \alpha^n p(\mathbf{y})$ for all $\mathbf{y} \ge 0$,
- $\log p(\mathbf{y})$ is convex in $\log \mathbf{y}$.

Then the following inequality holds

$$\inf_{\mathbf{y}>0:y^{S}\geq 0,\forall S\in \binom{[m]}{n}}\log p(\mathbf{y})=\sup_{\boldsymbol{\alpha}\in[0,1]^{m},\sum_{j}\alpha_{j}=n}\inf_{\mathbf{y}>0}\quad \log p(\mathbf{y})-\sum_{j=1}^{m}\alpha_{j}\log(y_{j}).$$

While the original result in [AGSS17] assumed *p* to be a homogeneous polynomial with positive coefficients, their proof only relies on the two properties presented in Lemma 17.

B.1 Proof of Theorem 2

To prove Theorem 2, we start with the (LogConcave-Unweighted) and derive the convex program (CVX-Unweighted) via a sequence of duals presented in Lemmas 19, 20, and 21.

Let \mathcal{P} and \mathcal{Q} denote the feasible regions for **x** and **y** in (LogConcave-Unweighted), respectively.

$$\mathcal{P} := \left\{ \mathbf{x} \in \mathbb{R}_{\geq 0}^{\mathcal{A} \times \mathcal{G}} : \sum_{i \in \mathcal{A}} x_{ij} = 1 \quad \forall j \in \mathcal{G} \right\}$$
$$\mathcal{Q} := \left\{ \mathbf{y} \in \mathbb{R}_{\geq 0}^{\mathcal{G}} : \prod_{j \in S} y_j \ge 1 \quad \forall S \in \binom{\mathcal{G}}{n} \right\}.$$

Note that the inner function in the objective

$$f(x) = \inf_{\mathbf{y} \in \mathcal{Q}} \sum_{i \in \mathcal{A}} \log \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} y_j \right),$$

is bounded above (y = 1 belongs to Q), and the domain of x, P, is compact (Bounded and Closed sets in Euclidean space are compact using Heine-Borel Theorem).

Lemma 18 shows that the inner infimum of (LogConcave-Unweighted) is $> -\infty$ for any integral allocation **x** that assigns at least one item to each agent in the support of **v**. We know such an allocation exists by Assumption B.1.

Lemma 18. For any integral allocation $\mathbf{x} \in \mathcal{P} \cap \{0,1\}^{|\mathcal{A}| \times |\mathcal{G}|}$,

$$\inf_{\mathbf{y}\in\mathcal{Q}} \quad \sum_{i\in\mathcal{A}} \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} y_j\right) = \sum_{i\in\mathcal{A}} \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij}\right).$$

Proof. Let $\sigma : \mathcal{G} \to \mathcal{A}$ be the allocation corresponding to \mathbf{x} , i.e., $\sigma(j) = i$ iff $x_{ij} = 1$ and let $\mathcal{S} = \{S \in \binom{\mathcal{G}}{n} : \forall i \in \mathcal{A}, \exists j \in S \text{ such that } x_{ij} = 1\}$. For any $\mathbf{y} \in \mathcal{Q}$,

$$\begin{split} \sum_{i \in \mathcal{A}} \log \left(\sum_{j \in \mathcal{G}} x_{ij} \, v_{ij} \, y_j \right) &= \log \left(\sum_{S \in \mathcal{S}} y^S \prod_{j \in S} x_{\sigma(j)j} v_{\sigma(j)j} \right) \\ &\geq \log \left(\sum_{S \in \mathcal{S}} \prod_{j \in S} x_{\sigma(j)j} v_{\sigma(j)j} \right) = \sum_{i \in \mathcal{A}} \log \left(\sum_{j \in \mathcal{G}} x_{ij} \, v_{ij} \right). \end{split}$$

Here, the only inequality holds because $y^S \ge 1$ for each $S \in S$. Setting $y_j = 1$ for each $j \in G$ gives the equality.

Lemma 19. The optimal value of (LogConcave-Unweighted) is the same as

$$\inf_{\boldsymbol{\delta}} \max_{\mathbf{x} \in \mathcal{P}} \sum_{i \in \mathcal{A}} \log \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} e^{-\delta_j} \right) + \sum_{j \in \mathcal{G}} \max(0, \delta_j).$$
(Unweighted-Primal)

Proof. For a fixed $\mathbf{x} \in \mathcal{P}$, using Lemma 17 with $p_{\mathbf{x}}(\mathbf{y}) = \prod_{i \in \mathcal{A}} \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} y_j \right)$, we get

$$\inf_{\mathbf{y}>0:y^{S}\geq 0,\forall S\in\binom{\mathcal{G}}{n}}\log p_{\mathbf{x}}(\mathbf{y}) = \inf_{\mathbf{y}>0:y^{S}\geq 0,\forall S\in\binom{\mathcal{G}}{n}}\sum_{i\in\mathcal{A}}\log\left(\sum_{j\in\mathcal{G}}x_{ij}\,v_{ij}\,y_{j}\right) \\
= \sup_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|},\sum_{j}\alpha_{j}=n}\inf_{\mathbf{y}>0}\sum_{i\in\mathcal{A}}\log\left(\sum_{j\in\mathcal{G}}x_{ij}\,v_{ij}\,y_{j}\right) - \sum_{j\in\mathcal{G}}\alpha_{j}\log(y_{j}).$$

Substituting $\delta_j = -\log(y_j)$ and taking a maximum over **x**, we get

$$\max_{\mathbf{x}\in\mathcal{P}} \inf_{\mathbf{y}>0: \mathbf{y}^{S}\geq 0, \forall S\in\binom{\mathcal{G}}{n}} \log p_{\mathbf{x}}(\mathbf{y}) = \sup_{\mathbf{x}\in\mathcal{P}, \boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|}, \sum_{j}\alpha_{j}=n} \inf_{\boldsymbol{\delta}} \sum_{i\in\mathcal{A}} \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} e^{-\delta_{j}}\right) + \sum_{j\in\mathcal{G}} \alpha_{j}\delta_{j}.$$

As the domains of both x and α are compact, using Theorem 16 on the previous equation, we get

$$\max_{\mathbf{x}\in\mathcal{P}}\inf_{\mathbf{y}>0:y^{S}\geq 0,\forall S\in\binom{\mathcal{G}}{n}}\log p_{\mathbf{x}}(\mathbf{y})=\inf_{\boldsymbol{\delta}}\max_{\mathbf{x}\in\mathcal{P}}\max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|},\sum_{j}\alpha_{j}=n}\sum_{i\in\mathcal{A}}\log\left(\sum_{j\in\mathcal{G}}x_{ij}v_{ij}e^{-\delta_{j}}\right)+\sum_{j\in\mathcal{G}}\alpha_{j}\delta_{j}.$$

Finally, the following claim completes the proof.

$$\inf_{\boldsymbol{\delta}} \max_{\mathbf{x}\in\mathcal{P}} \max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|},\sum_{j}\alpha_{j}=n} \sum_{i\in\mathcal{A}} \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} e^{-\delta_{j}}\right) + \sum_{j\in\mathcal{G}} \alpha_{j}\delta_{j}$$
$$= \inf_{\boldsymbol{\delta}} \max_{\mathbf{x}\in\mathcal{P}} \sum_{i\in\mathcal{A}} \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} e^{-\delta_{j}}\right) + \sum_{j\in\mathcal{G}} \max(0,\delta_{j}).$$
(27)

For proving the claim, we define functions

$$f_1(\boldsymbol{\delta}, \mathbf{x}, \boldsymbol{\alpha}) = \sum_{i \in \mathcal{A}} \log \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} e^{-\delta_j} \right) + \sum_{j \in \mathcal{G}} \alpha_j \delta_j, \text{ and}$$
$$f_2(\boldsymbol{\delta}, \mathbf{x}) = \sum_{i \in \mathcal{A}} \log \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} e^{-\delta_j} \right) + \sum_{j \in \mathcal{G}} \max(0, \delta_j).$$

Observe that for any δ and $\alpha \in [0, 1]^{|G|}$, $\alpha_j \delta_j \leq \max(0, \delta_j)$. Therefore, for any δ , \mathbf{x} and $\alpha \in [0, 1]^{|G|}$, we have $f_1(\delta, \mathbf{x}, \alpha) \leq f_2(\delta, \mathbf{x})$. As a result,

$$\inf_{\boldsymbol{\delta}} \max_{\mathbf{x}\in\mathcal{P}} \max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|}, \sum_{j}\alpha_{j}=n} f_{1}(\boldsymbol{\delta}, \mathbf{x}, \boldsymbol{\alpha}) \leq \inf_{\boldsymbol{\delta}} \max_{\mathbf{x}\in\mathcal{P}} f_{2}(\boldsymbol{\delta}, \mathbf{x}).$$
(28)

To establish an inequality in the other direction, first note that $f_1(\delta, \mathbf{x}, \alpha) = f_1(\delta + t \cdot \mathbf{1}, \mathbf{x}, \alpha)$ for any $t \in \mathbb{R}$. So, for a fixed δ , let t_{δ} denote a value of t for which the n largest values of $\delta + t_{\delta} \cdot \mathbf{1}$ are non-negative and the m - n smallest values of δ are non-positive. Then

$$\max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|},\sum_{j}\alpha_{j}=n} f_{1}(\boldsymbol{\delta}, \mathbf{x}, \boldsymbol{\alpha}) = \max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|},\sum_{j}\alpha_{j}=n} f_{1}(\boldsymbol{\delta}+t_{\delta}\cdot\mathbf{1}, \mathbf{x}, \boldsymbol{\alpha})$$
$$= \sum_{i\in\mathcal{A}} \log\left(\sum_{j\in\mathcal{G}} x_{ij}v_{ij}e^{-\delta_{j}-t_{\delta}}\right) + \max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|},\sum_{j}\alpha_{j}=n} \sum_{j\in\mathcal{G}} \alpha_{j}(\delta_{j}+t_{\delta}).$$
(29)

The term $\sum_{j \in \mathcal{G}} \alpha_j (\delta_j + t_{\delta})$ is maximized when $\alpha_j = 1$ for the largest *n* coordinates of $\delta + t \cdot \mathbf{1}$. As a result, we get

$$\sum_{i \in \mathcal{A}} \log \left(\sum_{j \in G} x_{ij} v_{ij} e^{-\delta_j - t_\delta} \right) + \sum_{j \in \mathcal{G}} \max(0, \delta_j + t_\delta) = f_2(\boldsymbol{\delta} + t_\delta \cdot \mathbf{1}, \mathbf{x}).$$
(30)

Combining equations (29) and (30), and taking max over x, we have

$$\max_{\mathbf{x}\in\mathcal{P}} \max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|}, \sum_{j}\alpha_{j}=n} f_{1}(\boldsymbol{\delta}, \mathbf{x}, \boldsymbol{\alpha}) = \max_{\mathbf{x}\in\mathcal{P}} f_{2}(\boldsymbol{\delta}+t_{\delta}\cdot\mathbf{1}, \mathbf{x}) \geq \inf_{\boldsymbol{\gamma}} \max_{\mathbf{x}\in\mathcal{P}} f_{2}(\boldsymbol{\gamma}, \mathbf{x}).$$

Taking an infimum over δ , we obtain

$$\inf_{\boldsymbol{\delta}} \max_{\mathbf{x}\in\mathcal{P}} \max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|}, \sum_{j}\alpha_{j}=n} f_{1}(\boldsymbol{\delta}, \mathbf{x}, \boldsymbol{\alpha}) \geq \inf_{\boldsymbol{\delta}} \inf_{\boldsymbol{\gamma}} \max_{\mathbf{x}\in\mathcal{P}} f_{2}(\boldsymbol{\gamma}, \mathbf{x}) \\
= \inf_{\boldsymbol{\gamma}} \max_{\mathbf{x}\in\mathcal{P}} f_{2}(\boldsymbol{\gamma}, \mathbf{x}).$$
(31)

Here, the last equality follows as the function being optimized does not depend on δ .

Combining equations (28) and (31) completes the proof of equation (27).

Lemma 20. The optimal values of (Unweighted-Primal) is the same as that of the following program.

$$\begin{split} \inf_{\delta, \mathbf{r}, \gamma} & \sum_{j \in \mathcal{G}} e^{r_j} + \sum_{i \in \mathcal{A}} \gamma_i + \sum_{j \in \mathcal{G}} \delta_j - n \\ & r_j + \gamma_i + \delta_j \geq \log v_{ij} \quad \forall (i, j) \in \mathcal{A} \times \mathcal{G} \\ & \delta \geq \mathbf{0}. \end{split}$$
 (Unweighted-Dual)

Proof. For a fixed δ , let us first re-write the internal maximum of (Unweighted-Primal) as

$$\max_{\mathbf{x},\mathbf{u}} \sum_{i \in \mathcal{A}} \log u_i + f(\boldsymbol{\delta})$$

$$u_i \leq \sum_{j \in \mathcal{G}} x_{ij} v_{ij} e^{-\delta_j} \quad \forall i \in \mathcal{A}$$

$$\sum_{i \in \mathcal{A}} x_{ij} \leq 1 \quad \forall j \in \mathcal{G}$$

$$\mathbf{x} \geq \mathbf{0},$$

$$(32)$$

where $f(\boldsymbol{\delta}) = \sum_{j \in \mathcal{G}} \max(0, \delta_j)$.

Let β_i , p_j , and θ_{ij} be the Lagrange dual variables associated with the constraints corresponding to agent *i*, item *j*, and agent-item pair(*i*, *j*), respectively. The Lagrangian of the above convex program is defined as follows

$$L(\mathbf{x}, \mathbf{u}, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{p}) = f(\boldsymbol{\delta}) + \left[\sum_{i \in \mathcal{A}} \log u_i + \sum_{i \in \mathcal{A}} \beta_i \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} e^{-\delta_j} - u_i \right) + \sum_{j \in \mathcal{G}} p_j (1 - \sum_{i \in \mathcal{A}} x_{ij}) + \sum_{i,j} \theta_{ij} x_{ij} \right]$$
$$= f(\boldsymbol{\delta}) + \left[\sum_{i \in \mathcal{A}} (\log u_i - \beta_i u_i) + \sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{G}} x_{ij} \left(\beta_i v_{ij} e^{-\delta_j} + \theta_{ij} - p_j \right) + \sum_{j \in \mathcal{G}} p_j \right].$$

The Lagrange dual of (32) is given by

$$g(\boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{p}) = \max_{\mathbf{x} \in \mathcal{P}, \mathbf{u} \ge 0} L(\mathbf{x}, \mathbf{u}, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{p}).$$
(33)

Observe that solution $x_{ij} = 1/n$ for each $(i, j) \in A \times G$ lies in the relative interior of \mathcal{P} . Since all the constraints are affine, Slater's condition is satisfied for (Unweighted-Primal). Thus, the optimal value of the infimum of Lagrange dual over β , θ , $p \ge 0$ is exactly equal to the optimum of (32).

The KKT conditions imply that the optimal solutions must satisfy

$$\frac{1}{u_i} - \beta_i = 0 \quad \forall i \in \mathcal{A}$$
$$\beta_i v_{ij} e^{-\delta_j} - \sum_{j \in \mathcal{G}} p_j + \theta_{ij} = 0 \quad \forall (i, j) \in \mathcal{A} \times \mathcal{G}.$$

The KKT conditions imply that $u_i = 1/\beta_i$ for each $i \in A$ maximizes the Lagrangian. For the supremum over **x**, **u** in (33) to stay finite, the second KKT condition is necessary and sufficient. Substituting these conditions in the Langrangian gives the following convex program.

$$\inf_{\mathbf{p},\boldsymbol{\beta},\boldsymbol{\theta}} f(\boldsymbol{\delta}) + \sum_{j \in \mathcal{G}} p_j - \sum_{i \in \mathcal{A}} \log \beta_i - n$$
$$p_j = \beta_i v_{ij} e^{-\delta_j} + \theta_{ij} \quad \forall (i,j) \in \mathcal{A} \times \mathcal{G}$$
$$\mathbf{p}, \boldsymbol{\beta}, \boldsymbol{\theta} \ge \mathbf{0}.$$

Observe that we can remove θ from the above program while making the first constraint an inequality. By substituting $r_i = \log p_i$, $\gamma_i = -\log \beta_i$, the above program is equivalent to

$$\inf_{\mathbf{r}, \gamma} \quad f(\boldsymbol{\delta}) + \sum_{j \in \mathcal{G}} e^{r_j} + \sum_{i \in \mathcal{A}} \gamma_i + \sum_{j \in \mathcal{G}} -n r_j + \gamma_i + \delta_j \ge \log v_{ij} \quad \forall (i, j) \in \mathcal{A} \times \mathcal{G}.$$

As (Unweighted-Primal) involves an infimum over δ , whenever $\delta_j < 0$, we can increase it to $\delta_j = 0$ without increasing the value of $f(\delta)$ and maintaining feasibility. Using this observation and taking an infimum over δ , the above program gives (Unweighted-Dual).

Lemma 21. The optimal values of (Unweighted-Dual) and (CVX-Unweighted) are the same.

Proof. Let b_{ij} be the Lagrange dual variable associated with constraint $r_j + \gamma_i + \delta_j \ge \log v_{ij}$ of (Unweighted-Dual) and let τ_{ij} be the Lagrange dual variable associated with constraint $\delta_{ij} \ge 0$.

The Lagrangian of (Unweighted-Dual) is defined as follows

$$L(\mathbf{r}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \mathbf{b}, \boldsymbol{\tau}) = \sum_{j \in \mathcal{G}} e^{r_j} + \sum_{i \in \mathcal{A}} \gamma_i + \sum_{j \in \mathcal{G}} \delta_j - n + \sum_{i,j} b_{ij} (\log v_{ij} - r_j - \gamma_i - \delta_j) - \sum_{j \in \mathcal{G}} \delta_j \tau_j$$

=
$$\sum_{j \in \mathcal{G}} (e^{r_j} - (\sum_{i \in \mathcal{A}} b_{ij})r_j) + \sum_{i \in \mathcal{A}} \gamma_i (1 - \sum_{j \in \mathcal{G}} b_{ij}) \sum_{j \in \mathcal{G}} \delta_j (1 - \tau_j - \sum_{i \in \mathcal{A}} b_{ij}) + \sum_{i,j} b_{ij} \log v_{ij} - n.$$

The Lagrange dual of (Unweighted-Dual) is given by

$$g(\mathbf{b}, \boldsymbol{\tau}) = \inf_{\boldsymbol{\delta} \ge 0, \mathbf{r}, \boldsymbol{\gamma}} L(\mathbf{r}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \mathbf{b}, \boldsymbol{\tau}).$$
(34)

One can verify that Slater's condition is satisfied by (Unweighted-Dual). So, the supremum of (34) with **b**, $\tau \ge 0$ is equal to the optimum of (Unweighted-Dual).

The KKT conditions for the Langrangian give

$$e^{r_j}-\sum_{i\in\mathcal{A}}b_{ij}=0$$
 $1-\sum_{j\in\mathcal{G}}b_{ij}=0$ $1-\tau_j-\sum_{j\in\mathcal{A}}b_{ij}=0.$

The KKT conditions imply $r_j = \log \left(\sum_{i \in A} b_{ij}\right)$ for each $j \in \mathcal{G}$ minimizes the Lagrangian. For the infimum over γ , δ in (34) to stay finite, the conditions $1 = \sum_{j \in \mathcal{G}} b_{ij}$ and $1 - \tau_j = \sum_{i \in \mathcal{A}} b_{ij}$ are necessary and sufficient. Substituting these conditions in the Lagrangian, we get

$$\begin{split} \sup_{\mathbf{b}, \boldsymbol{\tau}} & \sum_{i,j} b_{ij} \log v_{ij} - \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} b_{ij} \log \left(\sum_{i \in \mathcal{A}} b_{ij} \right) + \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} b_{ij} - n \\ & \sum_{j \in \mathcal{G}} b_{ij} = 1 \\ & \sum_{i \in \mathcal{A}} b_{ij} = 1 - \tau_j \\ & \mathbf{b}, \boldsymbol{\tau} \ge \mathbf{0}. \end{split}$$

Observe that the supremum in the above program can be switched to maximum as the feasible region is compact and the objective is bounded. Also note that $\sum_{i,j} b_{ij} = n$ for any **b** in the feasible region. As a result, the last two terms in the objective cancel each other. Finally, on substituting $q_j = \sum_{i \in A} b_{ij}$ in the above program, we obtain (CVX-Unweighted).

B.2 Generalization to Weighted Nash Social Welfare

Given an instance of weighted Nash Social Welfare $(\mathcal{A}, \mathcal{G}, \mathbf{v}, \mathbf{w})$ where $\sum_{i \in \mathcal{A}} w_i = 1$ and $\mathbf{w} \ge \mathbf{0}$, we introduce the following program as a generalization of (LogConcave-Unweighted) program.

$$\begin{array}{ll} \max_{\mathbf{x} \ge 0} \min_{\mathbf{y} > 0} & \sum_{i \in \mathcal{A}} w_i \log \left(\sum_{j \in \mathcal{G}} x_{ij} \ v_{ij} \ y_j^{1/w_i} \right) \\ \text{s.t.} & \sum_{i \in \mathcal{A}} x_{ij} = 1 \quad \forall j \in \mathcal{G} \end{array}$$
 (LogConcave-Weighted)

$$\prod_{j\in S} y_j \ge 1 \quad \forall S \in \binom{\mathcal{G}}{n}.$$

Observe that the feasible region of (LogConcave-Weighted) is given by $\mathbf{x} \in \mathcal{P}$ and $\mathbf{y} \in \mathcal{Q}$, which is identical to that of (LogConcave-Unweighted).

The main result of this section is the following.

Theorem 22. The optimal values of (LogConcave-Weighted) and (CVX-Weighted) are the same.

We prove Theorem 22 analogously to Theorem 2, starting with (LogConcave-Weighted) and deriving (CVX-Weighted) via a sequence of duals presented in Lemmas 24, 25, and 26.

We start by establishing that LogConcave-Weighted is indeed a relaxation of the weighted Nash Social Welfare, and the inner infimum is bounded in the following lemma.

Lemma 23. For any integral allocation $\mathbf{x} \in \mathcal{P} \cap \{0, 1\}^{|\mathcal{A}| \times |\mathcal{G}|}$,

$$\inf_{\mathbf{y}\in\mathcal{Q}} \quad \sum_{i\in\mathcal{A}} w_i \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} y_j^{1/w_i}\right) = \sum_{i\in\mathcal{A}} w_i \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij}\right)$$

Proof. For each *i*, let $S_i = \{j \in \mathcal{G} : x_{ij} = 1\}$ be the allocation corresponding to **x**. Then for any $\mathbf{y} \in \mathcal{Q}$,

$$\sum_{i \in \mathcal{A}} w_i \log\left(\sum_{j \in \mathcal{G}} x_{ij} \, v_{ij} \, y_j^{1/w_i}\right) - \sum_{i \in \mathcal{A}} w_i \log\left(\sum_{j \in \mathcal{G}} x_{ij} \, v_{ij}\right) = \sum_{i \in \mathcal{A}} w_i \log\left(\frac{\sum_{j \in S_i} v_{ij} \, y_j^{1/w_i}}{\sum_{j \in S_i} v_{ij}}\right).$$
(35)

Now for positive reals c_1, \ldots, c_m with $\sum_{j=1}^m c_j = 1$, and $0 \le p \le q$, the weighted power mean inequality states that for any $\mathbf{z} \in \mathbb{R}_{\ge 0}^m$,

$$\left(\sum_{j=1}^{m} c_j z_j^p\right)^{1/p} \le \left(\sum_{j=1}^{m} c_j z_j^q\right)^{1/q}.$$
(36)

This inequality follows from Jensen's inequality.

For each $i \in A$, define $q_i = \frac{1}{w_i}$ and $c_j^{(i)} = \frac{v_{ij}}{\sum\limits_{j \in S_i} v_{ij}}$ for every $j \in S_i$. Since $q_i = \frac{1}{w_i} \ge 1$, using equation 36, we get

$$w_i \log \left(rac{\sum_{j \in S_i} v_{ij} \ y_j^{1/w_i}}{\sum\limits_{j \in S_i} v_{ij}}
ight) \ge \log \left(rac{\sum_{j \in S_i} v_{ij} \ y_j}{\sum\limits_{j \in S_i} v_{ij}}
ight) \ge 0$$

for each agent *i*. Summing this inequality over all agents and substituting in (35) gives

$$\sum_{i \in \mathcal{A}} w_i \log \left(\sum_{j \in \mathcal{G}} x_{ij} \ v_{ij} \ y_j^{1/w_i} \right) \geq \sum_{i \in \mathcal{A}} w_i \log \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} \right).$$

Observe that equality holds when $y_j = 1$ for all $j \in \mathcal{G}$.

Lemma 24. The optimal value of (LogConcave-Weighted) is the same as

$$\inf_{\boldsymbol{\delta}} \max_{\mathbf{x} \in \mathcal{P}} \sum_{i \in \mathcal{A}} w_i \log \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} e^{-\delta_j / w_i} \right) + \sum_{j \in \mathcal{G}} \max(0, \delta_j).$$
(Weighted-Primal)

The following fact is crucial to the proof of this lemma.

Fact B.1. Let $p(\mathbf{y}) = w \log \left(\sum_{j=1}^{m} c_j y_j^{1/w} \right)$ with w > 0 and $c_j \ge 0$ for each j. Then $\log p(\mathbf{y})$ is a convex function in $\log(\mathbf{y})$.

Proof. For a fixed $\mathbf{x} \in \mathcal{P}$, the function

$$p_{\mathbf{x}}(\mathbf{y}) = \prod_{i \in \mathcal{A}} \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} y_j^{1/w_i} \right)^{w_i}$$

satisfies all the prerequisites of Lemma 17. The first property is easy to verify and the second property follows from Fact B.1. Therefore, by Lemma 17, we get

$$\inf_{\mathbf{y}>0:y^{S}\geq 0,\forall S\in\binom{\mathcal{G}}{n}}\log p_{\mathbf{x}}(\mathbf{y}) = \inf_{\mathbf{y}>0:y^{S}\geq 0,\forall S\in\binom{\mathcal{G}}{n}}\sum_{i\in\mathcal{A}}w_{i}\log\left(\sum_{j\in\mathcal{G}}x_{ij}v_{ij}y_{j}^{1/w_{i}}\right) \\
= \sup_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|},\sum_{j}\alpha_{j}=n}\inf_{\mathbf{y}>0}\sum_{i\in\mathcal{A}}w_{i}\log\left(\sum_{j\in\mathcal{G}}x_{ij}v_{ij}y_{j}^{1/w_{i}}\right) - \sum_{j\in\mathcal{G}}\alpha_{j}\log(y_{j}).$$

Substituting $\delta_i = -\log(y_i)$, and taking the supremum over **x**, we get

$$\max_{\mathbf{x}\in\mathcal{P}} \inf_{\mathbf{y}>0:y^{S}\geq 0,\forall S\in\binom{\mathcal{G}}{n}} \log p_{\mathbf{x}}(\mathbf{y}) = \sup_{\mathbf{x}\in\mathcal{P},\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|},\sum_{j}\alpha_{j}=n} \inf_{\boldsymbol{\delta}} \sum_{i\in\mathcal{A}} w_{i} \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} e^{-\delta_{j}/w_{i}}\right) + \sum_{j\in\mathcal{G}} \alpha_{j}\delta_{j}.$$

As the domains of both x and α are compact, using Theorem 16, we get

$$\begin{aligned} \max_{\mathbf{x}\in\mathcal{P}} \inf_{\mathbf{y}>0:y^{S}\geq 0,\forall S\in\binom{\mathcal{G}}{n}} \sum_{i\in\mathcal{A}} w_{i} \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} y_{j}^{1/w_{i}}\right) \\ &= \inf_{\boldsymbol{\delta}} \max_{\mathbf{x}\in\mathcal{P}} \max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|}, \sum_{j}\alpha_{j}=n} \sum_{i\in\mathcal{A}} w_{i} \log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} e^{-\delta_{j}/w_{i}}\right) + \sum_{j\in\mathcal{G}} \alpha_{j}\delta_{j}. \end{aligned}$$

Finally, we claim that

$$\begin{split} &\inf_{\boldsymbol{\delta}} \max_{\mathbf{x}\in\mathcal{P}} \max_{\boldsymbol{\alpha}\in[0,1]^{|\mathcal{G}|},\sum_{j}\alpha_{j}=n} \sum_{i\in\mathcal{A}} w_{i}\log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} e^{-\delta_{j}/w_{i}}\right) + \sum_{j\in\mathcal{G}} \alpha_{j}\delta_{j} \\ &= \inf_{\boldsymbol{\delta}} \max_{\mathbf{x}\in\mathcal{P}} \sum_{i\in\mathcal{A}} w_{i}\log\left(\sum_{j\in\mathcal{G}} x_{ij} v_{ij} e^{-\delta_{j}/w_{i}}\right) + \sum_{j\in\mathcal{G}} \max(0,\delta_{j}). \end{split}$$

The proof of this claim is identical to the proof of the unweighted case in equation (27). **Lemma 25.** *The optimal value of* (Weighted-Primal) *is the same as that of the following program.*

$$\inf_{\boldsymbol{\delta}, \mathbf{r}, \boldsymbol{\gamma}} \quad \sum_{j \in \mathcal{G}} e^{r_j} + \sum_{i \in \mathcal{A}} w_i \gamma_i + \sum_{j \in \mathcal{G}} \delta_j + \sum_{i \in \mathcal{A}} (w_i \log w_i - w_i)$$
(Weighted-Dual)
$$r_j + \gamma_i + \frac{\delta_j}{w_i} \ge \log v_{ij} \quad \forall (i, j) \in \mathcal{A} \times \mathcal{G}.$$

Proof. For a fixed δ , let us first re-write the internal maximum of (Weighted-Primal) as

$$\max_{\mathbf{x},\mathbf{u}} \sum_{i \in \mathcal{A}} w_i \log u_i + f(\boldsymbol{\delta})$$

$$u_i \leq \sum_{j \in \mathcal{G}} x_{ij} v_{ij} e^{-\delta_j / w_i} \quad \forall i \in \mathcal{A}$$

$$\sum_{i \in \mathcal{A}} x_{ij} \leq 1 \quad \forall j \in \mathcal{G}$$

$$\mathbf{x} \geq \mathbf{0},$$

$$(37)$$

where $f(\boldsymbol{\delta}) = \sum_{j \in \mathcal{G}} \max(0, \delta_j)$.

Let β_i , p_j , and θ_{ij} be the Lagrange dual variables associated with the constraints corresponding to agent *i*, item *j*, and agent-item pair(*i*, *j*), respectively. The Lagrangian of the above convex program is defined as follows

$$L(\mathbf{x}, \mathbf{u}, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{p})$$

$$= f(\boldsymbol{\delta}) + \sum_{i \in \mathcal{A}} w_i \log u_i + \sum_{i \in \mathcal{A}} \beta_i \left(\sum_{j \in \mathcal{G}} x_{ij} v_{ij} e^{-\delta_j / w_i} - u_i \right) + \sum_{j \in \mathcal{G}} p_j \left(1 - \sum_{i \in \mathcal{A}} x_{ij} \right) + \sum_{i,j} \theta_{ij} x_{ij}$$

$$= f(\boldsymbol{\delta}) + \left[\sum_{i \in \mathcal{A}} (w_i \log u_i - \beta_i u_i) + \sum_{i,j} x_{ij} \left(\beta_i v_{ij} e^{-\delta_j / w_i} + \theta_{ij} - p_j \right) + \sum_{j \in \mathcal{G}} p_j \right].$$

The Lagrange dual of (37) is given by

$$g(\boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{p}) = \max_{\mathbf{x} \in \mathcal{P}, \mathbf{u} \ge 0} L(\mathbf{x}, \mathbf{u}, \boldsymbol{\beta}, \boldsymbol{\theta}, \mathbf{p}).$$

Observe that solution $x_{ij} = 1/n$ is in the relative interior of \mathcal{P} . Since all the constraints are affine, Slater's condition is satisfied. Thus the optimum value of the infimum of Lagrange dual over β , θ , $p \ge 0$ is exactly equal to the optimum of (37).

The KKT conditions for the Lagrangian imply

$$rac{w_i}{u_i} - eta_i = 0 \quad orall i \in \mathcal{A} \ eta_i v_{ij} e^{-\delta_j/w_i} - \sum_{j \in \mathcal{G}} p_j + heta_{ij} = 0 \quad orall (i,j) \in \mathcal{A} imes \mathcal{G}.$$

The KKT conditions imply that $u_i = w_i/\beta_i$ for each $i \in A$ maximizes the Lagrangian. For the supremum over **x**, **u** in (34) to stay finite, the second KKT condition is necessary and sufficient.

Substituting these conditions in the Langrangian gives the following convex program.

$$\inf_{\mathbf{p},\boldsymbol{\beta},\boldsymbol{\theta}} f(\boldsymbol{\delta}) + \sum_{j \in \mathcal{G}} p_j + \sum_{i \in \mathcal{A}} (w_i \log w_i - w_i) - \sum_{i \in \mathcal{A}} w_i \log \beta_i$$
$$p_j = \beta_i v_{ij} e^{-\delta_j / w_i} + \theta_{ij} \quad \forall (i,j) \in \mathcal{A} \times \mathcal{G}$$
$$\mathbf{p}, \boldsymbol{\beta}, \boldsymbol{\theta} \ge 0$$

Observe that we can remove θ from the above program while making the first constraint an inequality. By substituting $r_i = \log p_i$, $\gamma_i = -\log \beta_i$, the above program is equivalently to

$$\inf_{\mathbf{r},\boldsymbol{\gamma}} \quad f(\boldsymbol{\delta}) + \sum_{j \in \mathcal{G}} e^{r_j} + \sum_{i \in \mathcal{A}} w_i \, \gamma_i + \sum_{i \in \mathcal{A}} (w_i \log w_i - w_i)$$
$$r_j + \gamma_i + \frac{\delta_j}{w_i} \ge \log v_{ij} \quad \forall (i,j) \in \mathcal{A} \times \mathcal{G}.$$

As (Weighted-Primal) involves an infimum over δ , whenever $\delta_j < 0$, we can increase it to $\delta_j = 0$ without increasing the value of $f(\delta)$ and maintaining feasibility in the above program. Using this observation and taking an infimum over δ gives (Weighted-Dual).

Lemma 26. The optimal value of (Weighted-Dual) is the same as that of (CVX-Weighted).

Proof. Let \hat{b}_{ij} be the Lagrange dual variable associated with constraint $r_j + \gamma_i + \delta_j \ge \log v_{ij}$ of (Unweighted-Dual) and let \mathbf{y}_{ij} be the Lagrange dual variable associated with constraint $\delta_{ij} \ge 0$. The Lagrangian of (Weighted-Dual) is defined as follows

$$\begin{split} L(\mathbf{r}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \hat{\mathbf{b}}, \boldsymbol{\tau}) &= \sum_{j \in \mathcal{G}} e^{r_j} + \sum_{i \in \mathcal{A}} w_i \gamma_i + \sum_{j \in \mathcal{G}} \delta_j + \sum_{i,j} \hat{b}_{ij} (\log v_{ij} - r_j - \gamma_i - \frac{\delta_j}{w_i}) \\ &- \sum_{j \in \mathcal{G}} \delta_j \tau_j + \sum_{i \in \mathcal{A}} (w_i \log w_i - w_i) \\ &= \sum_{j \in \mathcal{G}} (e^{r_j} - (\sum_{i \in \mathcal{A}} \hat{b}_{ij}) r_j) + \sum_{i \in \mathcal{A}} \gamma_i (w_i - \sum_{j \in \mathcal{G}} \hat{b}_{ij}) + \sum_{j \in \mathcal{G}} \delta_j (1 - \tau_j - \sum_{i \in \mathcal{A}} \frac{\hat{b}_{ij}}{w_i}) \\ &+ \sum_{i,j} \hat{b}_{ij} \log v_{ij} + \sum_{i \in \mathcal{A}} (w_i \log w_i - w_i). \end{split}$$

The Lagrange dual of (Weighted-Dual) is given by

$$g(\hat{\mathbf{b}}, \boldsymbol{\tau}) = \inf_{\boldsymbol{\delta} \ge 0, \mathbf{r}, \boldsymbol{\gamma}} L(\mathbf{r}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \hat{\mathbf{b}}, \boldsymbol{\tau}).$$
(38)

One can verify that Slater's condition is satisfied by (Weighted-Dual). So, the supremum of (38) with **b**, $\tau \ge 0$ is equal to the optimum of (Weighted-Dual).

The KKT conditions for the Langrangian imply

$$e^{r_j} - \sum_{i \in \mathcal{A}} \hat{b}_{ij} = 0$$

 $w_i - \sum_{j \in \mathcal{G}} \hat{b}_{ij} = 0$

$$1-\tau_j-\sum_{j\in A}\frac{\hat{b}_{ij}}{w_i}=0$$

The KKT conditions imply that the minimizer for r_j is given by $r_j = \log\left(\sum_{i \in \mathcal{A}} \hat{b}_{ij}\right)$. For the infimum over γ , δ to stay finite, the conditions $w_i = \sum_{j \in \mathcal{G}} \hat{b}_{ij}$ for each $i \in \mathcal{A}$ and $1 - \tau_j = \sum_{i \in \mathcal{A}} \hat{b}_{ij}$ for each $j \in \mathcal{G}$ are necessary and sufficient. Substituting these conditions in the Lagrange dual, we get

$$\begin{split} \sup_{\hat{\mathbf{b}},\boldsymbol{\tau}} & \sum_{i,j} \hat{b}_{ij} \log v_{ij} - \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} \hat{b}_{ij} \log \left(\sum_{i \in \mathcal{A}} \hat{b}_{ij} \right) + \sum_{j \in \mathcal{G}} \sum_{i \in \mathcal{A}} \hat{b}_{ij} + \sum_{i \in \mathcal{A}} (w_i \log w_i - w_i) \\ & \sum_{j \in \mathcal{G}} \hat{b}_{ij} = w_i \\ & \sum_{i \in \mathcal{A}} \frac{\hat{b}_{ij}}{w_i} = 1 - \tau_j \\ & \hat{\mathbf{b}}_{i}, \boldsymbol{\tau} \ge \mathbf{0}. \end{split}$$

Observe that the supremum in the above program can be switched to maximum because the feasible region is compact and the objective is bounded. Also, $\sum_{i,j} \hat{b}_{ij} = \sum_{i \in \mathcal{A}} w_i$ for any feasible $\hat{\mathbf{b}}$. Finally, substituting $b_{ij} = \frac{\hat{b}_{ij}}{w_i}$ and $q_j = \sum_{i \in \mathcal{A}} b_{ij}$, we obtain (CVX-Weighted).