

Breaking the Metric Voting Distortion Barrier

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Abstract

We consider the following well studied problem of metric distortion in social choice. Suppose we have an election with n voters and m candidates who lie in a shared metric space. We would like to design a voting rule that chooses a candidate whose average distance to the voters is small. However, instead of having direct access to the distances in the metric space, each voter gives us a ranked list of the candidates in order of distance. Can we design a rule that regardless of the election instance and underlying metric space, chooses a candidate whose cost differs from the true optimum by only a small factor (known as the *distortion*)?

A long line of work culminated in finding deterministic voting rules with metric distortion 3, which is the best possible for deterministic rules and many other classes of voting rules. However, without any restrictions, there is still a significant gap in our understanding: Even though the best lower bound is substantially lower at 2.112, the best upper bound is still 3, which is attained even by simple rules such as Random Dictatorship. Finding a rule that guarantees distortion $3 - \varepsilon$ for some constant ε has been a major challenge in computational social choice.

In this work, we give a rule that guarantees distortion less than 2.753. To do so we study a handful of voting rules that are new to the problem. One is *Maximal Lotteries*, a rule based on the Nash equilibrium of a natural zero-sum game which dates back to the 60's. The others are novel rules that can be thought of as hybrids of Random Dictatorship and the Copeland rule. Though none of these rules can beat distortion 3 alone, a careful randomization between Maximal Lotteries and any of the novel rules can.

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1 Introduction

Elections are a fundamental primitive in societal decision making. Through votes, people express their preferences and make collective decisions for social good. A common example of voting is single-winner elections, where voters select one winner from a pool of candidates. These candidates can be persons that represent the voters, or broader social options such as potential locations to build a public facility. Voting is also applicable to every-day situations, such as choosing one from many lunch options for a group of colleagues, or picking a game to play among a group of friends. A *voting rule* (or *social choice rule*) maps the voters’ preferences to a winning candidate.

A standard approach to evaluate the outcomes is to adopt the notion of utilitarian social efficiency. We assume that each voter has a cardinal utility function that maps each possible outcome to a real number quantitatively representing their preference for that outcome. From this utilitarian point of view, the optimal voting rule selects the outcome that optimizes the sum of the utilities.

In what has been classically studied and what has been practically implemented, voting rules are often based on ordinal rankings – these rules make decisions based only on each voter’s preference ordering, not the cardinal utilities, on the candidates. There are several considerations behind this. First, the restriction to ordinal rules simplifies the processes and the infrastructures needed for voting. Moreover, even though voters are assumed to have cardinal utilities, they may not be able to articulate them accurately, especially when these numbers represent differences in political stances in an abstract way. Finally, in a sense, this restriction to a common ordinal format gives each voter equal voting power.

Ordinal voting rules cannot always perfectly optimize social efficiency. Aiming to quantify the drawback of this format restriction – or this information loss from the perspective of social optimization – researchers have proposed the powerful notion of *distortion* [PR06, BCH⁺15, BR16, ABE⁺18]: It represents the worst-case ratio between the optimal efficiency and the efficiency of a particular ordinal voting rule (or in some contexts, the distortion-optimal ordinal voting rule). The worst-case distortion is generally not bounded by any constant, even after imposing normalization constraints on the cardinal utilities. This naturally calls for structural restrictions for the model.

In the seminal work of [ABP15, ABE⁺18], they proposed the influential framework of *metric distortion* – in particular, they imposed the natural assumption that the voters and the candidates lie in a shared (unknown) metric space, and a voter’s cardinal cost for a candidate is the distance between them in the metric space. This metric assumption is convincing if we think voters and candidates have positions in a political spectrum in the form of a metric space, or if the candidates are public facilities and voters’ costs are their travel costs to the selected facility. More formally, the metric distortion of a social choice rule is defined as the supremum of the ratio between the social cost (i.e., sum of costs of voters) of this rule and the optimal social cost, over all possible metric spaces and all induced ordinal preference profiles. The introduction of this metric constraint reduces the distortion of many social choice rules to constants, and the search for distortion-optimal social choice rules is very intriguing.

The journey to distortion 3 for deterministic rules. One fruitful line of work on metric distortion, including the original one of [ABP15, ABE⁺18], focuses on deterministic social choice rules. An immediate lower bound for deterministic rules is 3 (see Figure 1): There are two candidates $\{a, b\}$ and two voters where one voter prefers a to b and the other prefers b to a . Choosing either candidate gives distortion of 3. [ABP15, ABE⁺18] also showed that any voting rule selecting

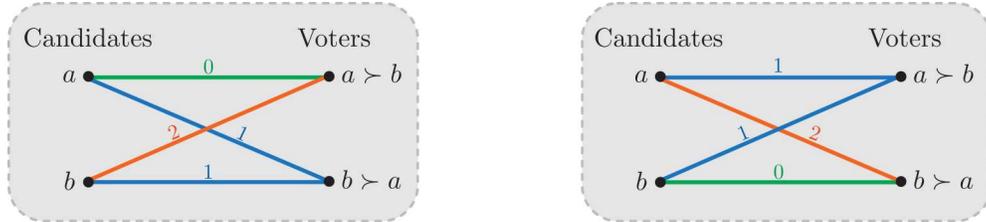


Figure 1: In an election with two candidates and two disagreeing voters, both of the above metric spaces are possible. No deterministic rule can have distortion less than 3, and no randomized rule can have distortion less than 2.

from the *uncovered set* (which we will introduce later along with its generalizations), such as the Copeland rule, guarantees an upper bound of 5. This gap between 3 and 5 was a tantalizing one and resisted researchers’ efforts (e.g. [GKM17]) for a few years. [MW19] were the first to reduce the gap: They proposed a novel weighted variant of the uncovered set to improve the upper bound to $2 + \sqrt{5} \approx 4.236$, and also showed that selecting from a novel *matching uncovered set* guarantees a distortion upper bound of 3. However, they did not manage to show that the set is always non-empty, and left it as a conjecture. This conjecture had been studied by many researchers (e.g. [Kem20a] who identified alternative and additional formulations) since then, until [GHS20], in their breakthrough result, proved it true. [GHS20] identified the crux of the conjecture and proved the existence of a candidate that satisfies their new simplified conditions, hence showing a distortion upper bound of 3 for both their novel rule of Plurality Matching and the one of [MW19]. This closes the gap for deterministic social choice rules. In the impressive work of [KK22], they proposed the novel, elegant, and practical voting rule Plurality Veto, which is also by itself a one-paragraph constructive proof of the conjecture of [MW19]. [KK23] further identified other related practical voting rules with distortion 3 that connects to the *proportional veto core* [Mou81], a classical notion in social choice.

The need for randomization and the barrier of 3. The above canonical lower bound of 3 only involves two candidates and two disagreeing voters, and one factor that induces this “large” distortion is the limit to *deterministic* social choice rules. This limit is traditionally motivated by people’s aversion to randomization for important social issues. However, nowadays, randomization is used in countless democratic (and non-democratic) processes, both those practically implemented and those theoretically studied. In that canonical example, it might even be more natural to randomize over the candidates, than to break the symmetry arbitrarily and pick one candidate deterministically. Moreover, there are situations where fractional solutions are acceptable, such as when allocating funding to the candidates.

In fact, soon after the first work on metric distortion, [FFG16] and [AP17] independently studied metric distortion without the restriction to deterministic rules. They both showed an upper bound of 3 for the simple rule of Random Dictatorship (which outputs the favorite candidate of a uniformly random voter), and gave a lower bound of 2 using that same two-voter-two-candidate example, leaving open the possibility that much better metric distortion could be achieved by randomized voting rules. This gap between 2 and 3 was a very (if not the most) intriguing question in the field of distortion.

The lack of progress on this question motivated researchers to look at fine-grained distortion

analysis within the instance classes of a fixed number of voters or candidates. For example, [AP17] showed that Random Dictatorship has distortion $3 - 2/n$ within the instance class of n voters, for any n . Several other works provided upper bounds for the instance class of m candidates, for any m : [FGMP19] proposed a rule called Random Oligarchy that can achieve an upper bound slightly worse than $3 - 2/m$;¹ [Kem20b] first showed an upper bound of $3 - 2/m$ by mixing Random Dictatorship with a rule named Proportional to Squares; [GHS20] proposed Smart Dictatorship, a variant of Random Dictatorship, which gives the same guarantee of $3 - 2/m$. These improvements over 3 vanish when we consider the supremum over all instances.

The first constant improvement for this gap is on the lower bound. [CR22] improved the lower bound to 2.1126, while [PS21] independently showed a lower bound of 2.0631. For the upper bound, there have been many different rules with distortion 3, but also many classes of rules which are known to be *unable* to beat 3: deterministic rules [ABE⁺18], truthful rules [FFG16], rules that only look at the top choices of the voters [GAX17], and rules that only looks at pairwise comparisons of candidates (i.e. a weighted tournament rule) [GKM17]. These results rule out a large swath of voting rules that have been studied in the metric distortion literature, which raises the following pressing question.

Question 1. Is there a voting rule with metric distortion better than 3? What might such a voting rule look like?

In this work, we answer this question by showing that a randomization over simple rules can achieve distortion less than 2.753.

1.1 Our Techniques and Voting Rules

The biased metric framework. Our work uses a new analysis framework that refines the linear programming approach introduced by [CR22]. Each metric can be viewed as a linear constraint on a potential voting rule, and their insight was to show that a relatively simple class of metrics, called the *biased metrics*, characterizes the most strict constraints. However, they were only able to analyze the biased metrics with some relaxations, and could only prove upper bounds for elections with three candidates. Our approach, on the other hand, allows us to precisely characterize the constraints imposed by the biased metrics. The resulting framework gives us more analysis power while retaining most of the simplicity, and gives us the intuition that leads to the break of the barrier of 3.

In the main body of the paper, we consider three randomized rules and analyze them and their mixtures using this framework. In [Appendix C](#), we also use this framework to revisit a variety of results proved in the metric distortion literature [ABP15, FFG16, AP17, MW19, GHS20, KK22] and show that they have short, simple proofs once the biased metric framework has been established. This suggests that the framework may be a helpful primitive for future work in the area.

A note on weighted tournament rules. *Weighted tournament rules* (or *C2 rules* [Fis77]) are a special class of voting rules that only consider pairwise comparisons (for each pair of candidates i, j , the proportion of voters that prefer i over j). Weighted tournament rules are desirable in many settings since they are often simple, interpretable, and efficiently implementable by sampling voters.

¹Their upper bound is $3 - 2 \cdot \min_{p \in [0,1]} (p^2(2-p) + (1-p)^3/(m-1))$, which is $3 - 2/m + O(1/m^2)$ as $m \rightarrow \infty$.

The Maximal Lotteries rule discussed in [Section 4](#) is a weighted tournament rule, and is in fact optimal among such rules. Our other rules, including Random Consensus Builder ([Section 5.1](#)), Random Dictatorship on the (Weighted) Uncovered set ([Section 5.2](#)) and Random Dictatorship on the Directed Maximal Independent Set ([Appendix B](#)), are “almost” weighted tournament rules: They only consider pairwise comparisons of the candidates and the ordering of a uniformly random voter. Note that this modification still allows the rules to be efficiently implemented with sampling.

Maximal Lotteries. The first rule we study is *Maximal Lotteries*. According to [\[Bra17\]](#), it (along with its variants) was first considered by [\[Kre65\]](#), independently rediscovered and studied in detail by [\[Fis84\]](#), and later also independently rediscovered by [\[LLL93, FR95, FM92, RS10\]](#). This rule has not been studied in the context of distortion.

Maximal Lotteries formulates the following zero-sum game: Two players 1 and 2 each proposes a distribution over the candidates, and then we independently draw a candidate c_1 from Player 1’s distribution, a candidate c_2 from Player 2’s distribution, and a uniformly random voter v . Player 1 wins if v prefers c_1 to c_2 and vice versa, breaking ties uniformly in case $c_1 = c_2$. Each player aims to maximize their winning probability. Maximal Lotteries outputs a Nash equilibrium of this zero-sum game, which can be computed in polynomial time.

We show that Maximal Lotteries has distortion 3. This on its own resolves an interesting question on the optimal distortion of weighted tournament rules. [\[GKM17\]](#) showed that no such rule can have distortion better than 3, and the best previously known upper bound was $2 + \sqrt{5}$ [\[MW19\]](#). Additionally, our framework admits a finer characterization on the worst-case instances: Intuitively, if Maximal Lotteries has distortion close to 3 in an instance, then for any candidate c_1 and c_2 where c_1 beats c_2 with a large margin in their pairwise comparison, c_1 cannot be much farther away to the true optimal candidate than c_2 . This motivates us to design complementary rules that can deal with these cases where Maximal Lotteries is “bad”.

Random Consensus Builder. Motivated by the discussion above, we conceptually build a directed graph where the vertices are the candidates. We draw an edge from c_1 to c_2 if c_1 beats c_2 with a large margin in their pairwise comparison. We are inspired by the following graph theory fact: In any directed graph, there exists an independent set, so that any vertex in the graph can be reached from a vertex in the independent set in at most two steps (see e.g. [\[BM76\]](#)). When Maximal Lotteries is “bad”, a candidate in this independent set must be close to the true optimal candidate. Additionally, candidates in this independent set must be relatively even in their pairwise comparisons due to our construction of the graph. These additional structures make Random Dictatorship on the independent set perform much better than distortion-3 in these cases.

Our Random Consensus Builder rule utilizes this intuition but only implicitly picks this independent set.² Random Consensus Builder picks a uniformly random voter and looks at remaining candidates from her least preferred one to her most preferred one. When we encounter a candidate c , we remove all candidates that c can pairwise beat with a large margin. In the end, we output the last candidate that we encounter. Conceptually, Random Consensus Builder naturally balances the opinion of a random voter with the general consensus.

Using our framework, we show that a randomization between Maximal Lotteries and Random Consensus Builder with proper parameters has distortion at most $2\sqrt{2} \approx 2.82843$.

²For the interested reader, another voting rule, Random Dictatorship on the Directed Maximal Independent Set, which more directly uses this idea, is discussed in [Appendix B](#).

RaDiUS: Random Dictatorship on the (Weighted) Uncovered Set. Our analysis of Random Consensus Builder uses properties that are reminiscent of the weighted uncovered set, proposed by [MW19] who showed that an arbitrary selection from this set (with a proper parameter) gives distortion $2 + \sqrt{5} \approx 4.23607$. This motivates us to propose RaDiUS (Random Dictatorship on the (Weighted) Uncovered Set) that outputs a uniformly random voter’s favorite candidate within the weighted uncovered set.

It turns out RaDiUS can give better guarantees than Random Consensus Builder. Using our framework, we show that a randomization between Maximal Lotteries and RaDiUS with proper parameters has distortion at most 2.75271.

1.2 Further Related Work

There has been a large body of work on distortion in social choice. We refer the reader to the survey of [AFSV21] for a more detailed overview of the field; below we briefly discuss some of them.

The first works on distortion did not impose the metric-space condition, assumed the utilities are non-negative, and defined distortion of a rule as the worst-case ratio between the optimal sum of utilities and the sum of utilities attained by the rule [PR06]. Many works made the unit-sum utility assumption, where for every voter, her sum of utilities on each candidate equals 1, to avoid uninteresting worst cases. Under this assumption, [CP11] showed that the Plurality rule has distortion $O(m^2)$ (for the class of instances with m candidates, same below). A matching $\Omega(m^2)$ lower bound for deterministic rules was later given by [CNPS17]. Under the same assumption, [BCH⁺15] proposed a randomized rule with distortion $O(\sqrt{m} \log^* m)$ and gave a lower bound of $\Omega(\sqrt{m})$ for any rule. This gap was closed by [EKPS22] who proposed a Stable Lottery (and Stable Committee) rule, which was inspired by fair committee selection literature, with distortion $O(\sqrt{m})$.

[GLS23] aimed to provide best-of-both-worlds guarantees for both the metric setting and the non-metric setting. They proposed novel deterministic and randomized social choice rules which guarantee constant metric distortion and almost optimal (for deterministic and randomized rules correspondingly) non-metric distortion.

Researchers have also looked beyond single-winner elections where we select one winner from the candidates. [CSV22] considered a model of multi-winner elections in the metric distortion setting, where they give complete characterizations for the optimal metric distortion. Graph problems such as selecting a perfect bipartite matching (e.g. [CFF⁺16, AS16, AZ21, ABFV22, ACR23]) in both metric and non-metric distortion settings have also received great attention.

Most works in this field aim to optimize the utilitarian aggregation of preferences, i.e., the sum or average of the utilities. Other works consider “fair” ways to aggregate preferences: [GKM17] proposed the *fairness ratio* in the metric setting, which is inspired by the mathematical idea of majorization and replaces the utilitarian aggregation by (loosely speaking) the worst-case symmetric and convex aggregation function, and showed a lower bound of 3 and an upper bound of 5. [GHS20] closed this gap by showing their Plurality Matching rule has a fairness ratio of 3. [EKPS22] studied proportional fairness, Nash welfare, and the core in the non-metric setting, and gave distortion bounds of $O(\log m)$ for all these objectives.

The distortion framework serves as a valuable tool to quantify the efficiency of voting rules, and therefore has been adopted in the study of various aspects of voting, such as the tradeoff between the amount of communication and the efficiency performance of voting rules [GAX17, FGMP19, MPSW19, Kem20b, MSW20]. The framework is also used to quantify the effect of certain social structures: [CDK17, CDK18] studied the representativeness of candidates on the population of

voters. In particular, they showed that when the candidates are drawn independently from the voter population, the metric distortion of social choice rules becomes much better. [FPW23] studied the effect of public spirit in the non-metric distortion framework. They showed that if every voter altruistically ranks the candidate according to a mixture of her own preference and the average preference of the voters, then the distortion of many social choice rules will drastically improve.

2 Preliminaries and Notation

Elections. An *election instance* is defined by a tuple $\mathcal{E} = (V, C, \succ_V)$, where V is a set of n voters, C is a set of m candidates, and \succ_V is a set of linear orders, one for each voter, where $i \succ_v j$ if voter v prefers candidate i over candidate j . Throughout the paper, we will use i, j, k, a, b, c to refer to candidates and u, v to refer to voters. We will have i^* denote the true best candidate when the metric space is fixed.

For a condition \mathcal{P} , we let $S_{\mathcal{P}}$ denote the subset of voters whose preferences satisfy \mathcal{P} . We also let $s_{\mathcal{P}} = |S_{\mathcal{P}}|/n$ be the proportion of these voters overall, or equivalently, the probability that a uniformly random voter's preference list satisfies property \mathcal{P} . For example, $S_{i \succ j}$ is the set of voters that prefer i over j , $S_{i, j \succ k}$ is the set of voters that prefer i and j over k , and $S_{I \succ j}$ is the set of voters that prefer all the candidates in I over j . Note that if $j \in I$ then $S_{I \succ j} = \emptyset$ and $s_{I \succ j} = 0$. We also let $\text{plu}(i) = s_{i \succ C \setminus \{i\}}$ be the proportion of voters whose first choice is i .

In [Section 4](#), we will also allow the property \mathcal{P} to be randomized, in which case $s_{\mathcal{P}}$ still makes sense but $S_{\mathcal{P}}$ does not. For example, if D is a distribution over candidates, then $s_{D \succ j}$ denotes the probability that a uniformly random voter prefers a candidate $i \sim D$ over j . That is,

$$s_{D \succ j} = \Pr_{i \sim D, v \sim V}[i \succ_v j] = \mathbb{E}_{i \sim D, v \sim V}[\mathbf{1}[i \succ_v j]].$$

Note that in the case that $i = j$, it will be natural to treat $\mathbf{1}[i \succ_v j]$ as $\frac{1}{2}$. To this end, in [Section 4](#) we will let $s_{i \succ i}$ be $\frac{1}{2}$ instead of 0 – this makes it so that the *Condorcet Game* is well defined, and it makes the proof of [Theorem 1](#) read much more smoothly.

We use this notation extensively and flexibly in the paper, and we may reiterate what certain instances mean in natural language to be clear.

Metric spaces. A metric space is a pair (\mathcal{M}, d) of a set \mathcal{M} and a distance metric $d : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}_{\geq 0}$ with the following three properties:

- (1) Positive definiteness: $d(x, y) \geq 0$ with equality if and only if $x = y$,
- (2) Symmetry: $d(x, y) = d(y, x)$,
- (3) Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

We will extend the notation of d to operate directly on the voters and candidates rather than the points they occupy in the metric space. Note that for simplicity we allow voters and candidates to be co-located in the space, so their distance may be zero. (That is to say, technically, we consider *pseudometric spaces*. This simplification does not change the distortion of any voting rule.)

When defining the biased metrics in [Section 3](#) and proving lower bounds in [Appendix A](#), we will specify metric spaces where we only explicitly define the distances between candidates and voters, and leave the other distances implicit. We make no use of the implicit distances, but to fully specify

the metric space one can use the *graph distance closure* of the explicitly defined distances. i.e., if the distance between two points is not explicitly defined, it should be taken to be the shortest path between those two points using the explicitly defined distances.

Given an election instance $\mathcal{E} = (V, C, \succ_V)$, we say that a distance metric d is *consistent* with \mathcal{E} (denoted $d \triangleright \mathcal{E}$) if for all $v \in V$, we have that $i \succ_v j$ implies $d(i, v) \leq d(j, v)$.

Voting rules, social cost, and distortion. For an election with underlying distance metric d , we denote the social cost of a candidate i to be their average distance to the voters. i.e.,

$$\text{SC}(i, d) := \frac{1}{n} \sum_{v \in V} d(i, v).$$

(In the literature, the social cost of a candidate is usually the *sum*, rather than the average, of distances. Since we are concerned about the ratio between costs, we can equivalently use the average-distance version, which we find easier to work with.) We will often just write $\text{SC}(i)$ when the relevant distance metric has been fixed, or is clear from context.

A *voting rule* (or *social choice rule*) f is a function that maps every election instance \mathcal{E} to a distribution over its candidates. Given this, the *distortion* of f is given by

$$\text{distortion}(f) = \sup_{\mathcal{E}} \sup_{d: d \triangleright \mathcal{E}} \frac{\mathbb{E}_{j \sim f(\mathcal{E})} [\text{SC}(j, d)]}{\min_{i \in C} \text{SC}(i, d)}.$$

We will also often refer to the distortion of f on a particular metric d , which is just the operand of the suprema above. When the election instance and voting rule are fixed, we will use p_j to denote the probability that the rule chooses candidate j on the instance.

The aforementioned *weighted tournament rules* are the class of voting rules that map $\langle s_{i \succ j} \rangle_{i, j \in C}$ to a distribution of candidates.

3 The Biased Metrics

The key tool that we use to understand the metric distortion of the social choice rules in [Section 5](#) is a refinement of the linear programming framework introduced by [\[CR22\]](#).

Suppose that we have an election instance. If we can design a rule such that for any metric consistent with the instance we have

$$\sum_{j \in C} (\text{SC}(j) - \text{SC}(i^*)) p_j \leq \lambda \cdot 2 \text{SC}(i^*), \tag{1}$$

then the rule has distortion at most $1 + 2\lambda$ on this instance (the reason for the unnecessary factor of 2 will be clear later). In this view, for a rule to have low distortion it has to satisfy a set of linear constraints, with one constraint imposed by each metric. As one might expect, some constraints may be redundant, so it is helpful to try and find a small set of metrics whose constraints imply those for all of the metrics. Then, one can show that a rule has low distortion just by showing that it has low distortion on the small set of metrics.

[\[CR22\]](#) defined the set of *biased metrics*, and showed that they satisfied this property.

Definition 1. Let (x_1, \dots, x_m) be a vector of nonnegative real numbers such that $x_{i^*} = 0$ for some i^* . Given an election instance, the *biased metric* for the vector (x_1, \dots, x_m) is defined as follows. For a voter v and candidate i , let

$$d(i^*, v) = \frac{1}{2} \max_{i, j: i \succeq_v j} (x_i - x_j),$$

$$d(j, v) - d(i^*, v) = \min_{k: j \succeq_v k} x_k.$$

The rough idea of the biased metrics is the following. Suppose we were given some fixed metric such that the distance from candidate j to the optimal candidate is x_j (so $x_{i^*} = 0$). Then we could imagine throwing out all of the other distances, and remaking them so that the distances from i^* to the voters is as small as possible, and the distances from the other candidates is as large as possible (compared to the distances from i^*). The former is to make the right side of Eq. (1) smaller and the latter is to make the left side larger, which will tighten the constraint. It turns out that to do this while respecting the triangle inequality and the preferences, one ends up with the above definition.

[CR22] gave proofs that the biased metrics are indeed valid distance metrics, and that they tighten the constraints in Eq. (1). For completeness, these proofs are included in Appendix D.

Now that it has been established that we only need to consider the constraints imposed by the biased metrics, let us see how to express these constraints. Suppose that we have a fixed biased metric given by a vector (x_1, \dots, x_m) . Let $I_t = \{k \in C : x_k \leq t\}$. Notice then that $d(j, v) - d(i^*, v) > t$ if and only if $v \in S_{I_t \succ j}$. To be clear, note that if $j \in I_t$ then $S_{I_t \succ j} = \emptyset$ and $s_{I_t \succ j} = 0$. It follows that

$$\text{SC}(j) - \text{SC}(i^*) = \mathbb{E}_{v \sim V} [d(j, v) - d(i^*, v)] = \int_0^\infty \Pr_{v \sim V} [d(j, v) - d(i^*, v) > t] dt = \int_0^\infty s_{I_t \succ j} dt$$

and so

$$\sum_{j \in C} (\text{SC}(j) - \text{SC}(i^*)) p_j = \int_0^\infty \sum_{j \notin I_t} s_{I_t \succ j} p_j dt.$$

We can use a similar approach to express $2\text{SC}(i^*)$. We have that $2d(i^*, v) \leq t$ if and only if $v \in S_{\forall i \succ_j, x_i - x_j \leq t}$. This is the set of voters v such that whenever $i \succ_v j$, we have $x_i - x_j \leq t$. It follows that

$$2\text{SC}(i^*) = \mathbb{E}_{v \sim V} [2d(i^*, v)] = \int_0^\infty (1 - \Pr_{v \sim V} [2d(i^*, v) \leq t]) dt = \int_0^\infty (1 - s_{\forall i \succ_j, x_i - x_j \leq t}) dt.$$

Therefore, the constraint imposed by the biased metric is

$$\int_0^\infty \sum_{j \notin I_t} s_{I_t \succ j} p_j dt \leq \lambda \int_0^\infty (1 - s_{\forall i \succ_j, x_i - x_j \leq t}) dt. \quad (2)$$

To get a sense of how the right side of this expression behaves, note that $s_{i^* \succ I_t^c} \geq s_{\forall i \succ_j, x_i - x_j \leq t}$. If a voter v satisfies the condition that $i \succ_v j$, we have that $x_i - x_j \leq t$, then for all k such that $x_k > t$ (the candidates in I_t^c), we must have $i^* \succ_v k$. In a lot of situations, using $s_{i^* \succ I_t^c}$ in place of the more complicated expression is sufficient. To this end, note that the following constraint implies Eq. (2).

$$\int_0^\infty \sum_{j \notin I_t} s_{I_t \succ j} p_j dt \leq \lambda \int_0^\infty (1 - s_{i^* \succ I_t^c}) dt. \quad (3)$$

[CR22] derived Eq. (3) (though written in a more discrete form), and then noted that one could consider an even stricter collection of constraints, which we will use to analyze Maximal Lotteries in Section 4.

$$\sum_{j \notin I} s_{I \succ j} p_j \leq \lambda (1 - s_{i^* \succ I^c}). \quad (4)$$

In particular, to show Eq. (2) for all possible metrics, it suffices to show the above inequality for all sets $I \neq \emptyset, C$, and all $i^* \in I$. The convenience of this is that there are finitely many possible constraints rather than the infinitely many constraints we would have if we used Eq. (2) or Eq. (3) (since each choice of the vector (x_1, \dots, x_m) may give a different constraint). Moreover, the metric space has been completely abstracted away – these constraints only involve terms that come from the election instance alone. However, it should be noted that Eq. (2) loses no generality (a rule has distortion $1 + 2\lambda$ if and only if it satisfies Eq. (2) for all biased metrics), but Eq. (3) and Eq. (4) may lose generality (the implication only goes one way).

To conclude this section, we introduce some notation that makes Eq. (2) easier to discuss.

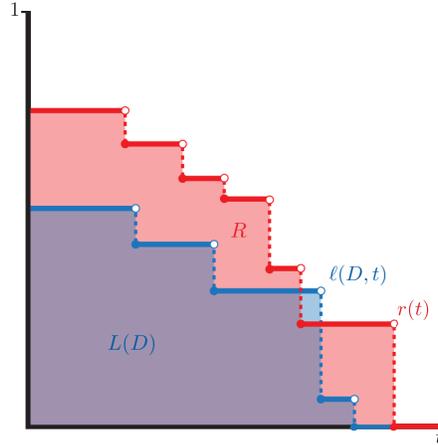


Figure 2: An example of the functions $r(t)$, $\ell(D, t)$, and the areas R and $L(D)$.

Once an election instance is fixed, we let

$$r(t) = 1 - s_{\forall i \succ j, x_i - x_j \leq t} \quad \text{and} \quad R = \int_0^\infty r(t) dt.$$

Given a distribution D over the candidates which chooses candidate j with probability p_j , we let

$$\ell(D, t) = \sum_{j \notin I_t} s_{I_t \succ j} p_j \quad \text{and} \quad L(D) = \int_0^\infty \ell(D, t) dt.$$

With this notation, we would like to design a rule which outputs a distribution D such that for all biased metrics, $L(D)/R \leq \lambda$ for a small fixed λ (to get distortion less than 3 we need $\lambda < 1$).

Though the expression for $r(t)$ may seem complicated and difficult to work with, ultimately we only use it in two simple ways. As mentioned, in [Section 4](#) we only use $r(t) \geq 1 - s_{i^* \succ_{I_t^c}}$. In [Section 5](#) it is only needed for [Proposition 3](#), which roughly says that if $r(t)$ is small, then the metric admits a nice structure that can be leveraged to get better distortion bounds.

4 Maximal Lotteries

In this section, we will study the distortion of Maximal Lotteries [\[Kre65\]](#). The voting rule is based on a zero-sum game (in our formulation, a constant-sum game) between two players Alice and Bob. The details of the game and the voting rule are below.

The Condorcet Game

- Simultaneously, Alice picks a distribution D_A and Bob picks a distribution D_B over the candidates.
- We sample $a \sim D_A$ and $b \sim D_B$.
- Alice and Bob's payoffs are $s_{a \succ b}$ and $s_{b \succ a}$ respectively. If $a = b$ then each player gets $\frac{1}{2}$.

Maximal Lotteries (ML)

- Choose a candidate from any Nash equilibrium distribution of the Condorcet game.

For the Condorcet game, note that under the notation we introduced in [Section 2](#), once D_A and D_B are picked the payoffs for Alice and Bob are $s_{D_A \succ D_B}$ and $s_{D_B \succ D_A}$ respectively. We remind the reader that in this section we treat $s_{i \succ i}$ as $\frac{1}{2}$ so that if both players choose the same candidate, their payoffs are equal.

Suppose that we fix an election instance. Let D be the distribution output by ML, which chooses candidate i with probability p_i . Let $P(I) = \sum_{i \in I} p_i$ be the probability that a candidate in I is chosen. Let $D(I)$ be the distribution conditioned on the chosen candidate coming from I . i.e., a candidate $i \in I$ is chosen with probability $p_i/P(I)$. Note that $D(I)$ is not well-defined if $P(I) = 0$, so we will deal with cases where this comes up separately.

We will prove the following theorem.

Theorem 1. *For any fixed biased metric and any $t \geq 0$, we have*

$$\ell(D, t) \leq \frac{P(I_t^c)}{2} \leq r(t).$$

In particular, this implies that ML has distortion at most 3.

By a theorem due to [\[GKM17\]](#), no randomized or deterministic weighted tournament rule can have distortion better than 3, so ML is optimal among weighted tournament rules.

Proof of [Theorem 1](#). For ease of notation, let us fix t and set $I = I_t$. Let us first prove the theorem in the cases where $P(I)$ or $P(I^c)$ are zero, so that afterwards we can assume that $D(I)$ and $D(I^c)$ are well-defined.

If $P(I^c) = 0$, then we simply have that $\ell(D, t) = \frac{P(I^c)}{2} = 0$, and the theorem easily follows. If $P(I) = 0$, we need to show that $\ell(D, t) \leq \frac{1}{2} \leq r(t)$. Then note that

$$\ell(D, t) = \sum_{j \notin I} s_{I \succ j} p_j \leq \sum_{j \notin I} s_{i^* \succ j} p_j = s_{i^* \succ D}.$$

On the other hand, we have

$$r(t) \geq 1 - s_{i^* \succ I^c} \geq 1 - \min_{j \notin I} s_{i^* \succ j} \geq 1 - \sum_{j \in I^c} s_{i^* \succ j} p_j = 1 - s_{i^* \succ D}.$$

We have that $s_{i^* \succ D} \leq \frac{1}{2}$ since D weakly beats the strategy of deterministically picking i^* , so $\ell(D, t) \leq \frac{1}{2} \leq r(t)$. As desired.

Henceforth, let's assume that $D(I)$ and $D(I^c)$ are well-defined. First, using the fact that $s_{I \succ j} \leq \min_{i \in I} s_{i \succ j} \leq s_{D(I) \succ j}$, we have,

$$\ell(D, t) = \sum_{j \notin I} s_{I \succ j} p_j \leq \sum_{j \notin I} s_{D(I) \succ j} p_j = s_{D(I) \succ D(I^c)} \cdot P(I^c).$$

Similarly, using $s_{i^* \succ I^c} \leq \min_{j \notin I} s_{i^* \succ j} \leq s_{i^* \succ D(I^c)}$, we have

$$r(t) \geq 1 - s_{i^* \succ I^c} \geq 1 - s_{i^* \succ D(I^c)}.$$

Therefore, it suffices to show that

$$s_{D(I) \succ D(I^c)} \cdot P(I^c) \leq \frac{P(I^c)}{2} \leq 1 - s_{i^* \succ D(I^c)}. \quad (5)$$

To prove this, we will rely on two somewhat general properties of any equilibrium. The first claim, below, is equivalent to the first inequality in [Eq. \(5\)](#).

Claim 1. $s_{D(I) \succ D(I^c)} \leq \frac{1}{2}$.

Proof. Intuitively, this is true because if it were the case that $s_{D(I) \succ D(I^c)} > \frac{1}{2}$ then $D(I)$ could strictly beat D , contradicting the fact that it is an equilibrium.

The proof of the claim relies on two facts. For any distribution X over the candidates, we have

1. $s_{X \succ D} \leq \frac{1}{2}$, and
2. $s_{X \succ X} = \frac{1}{2}$.

The first follows by definition of D being an equilibrium, and the second follows by symmetry. Then we have that by the law of conditional expectation,

$$\begin{aligned} \frac{1}{2} &\geq s_{D(I) \succ D} = P(I^c) s_{D(I) \succ D(I^c)} + P(I) s_{D(I) \succ D(I)} \\ &= P(I^c) s_{D(I) \succ D(I^c)} + (1 - P(I^c)) \cdot \frac{1}{2} \end{aligned}$$

which means that $s_{D(I) \succ D(I^c)} \leq \frac{1}{2}$ as claimed. \square

Note that, applying the claim with I replaced with I^c we can in fact conclude that $s_{D(I) \succ D(I^c)} = s_{D(I^c) \succ D(I)} = \frac{1}{2}$.

Now it remains to show that

$$\frac{P(I^c)}{2} \leq 1 - s_{i^* \succ D(I^c)}.$$

For brevity, let $p = P(I^c)$ and $q = s_{i^* \succ D(I^c)}$. We want to show that $\frac{p}{2} \leq 1 - q$. We will also introduce $r = s_{D(I) \succ i^*}$. Consider three different strategies for the game: $D(I)$, $D(I^c)$, and just

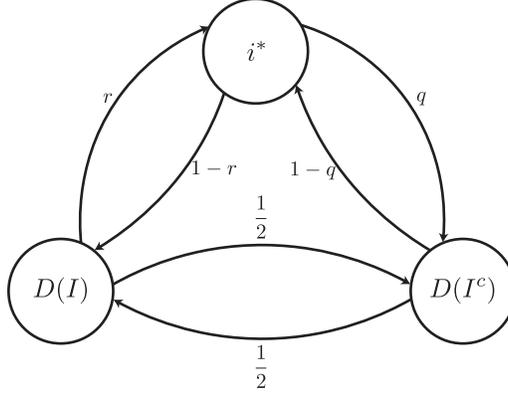


Figure 3: Three strategies $D(I), D(I^c), i^*$. The edge (A, B) is labeled with $s_{A \succ B}$.

deterministically choosing i^* . Then by conditional expectation and the assumption that D is an equilibrium, we have that

$$\frac{1}{2} \leq s_{D \succ i^*} = P(I^c)s_{D(I^c) \succ i^*} + P(I)s_{D(I) \succ i^*} = p(1 - q) + (1 - p)r.$$

On the other hand, the payoffs for these strategies satisfy a kind of triangle inequality, by the following claim.

Claim 2. For three strategies A, B, C , we have $s_{A \succ B} \leq s_{A \succ C} + s_{C \succ B}$.

Proof. We have that $s_{A \succ B} = \mathbb{E}_{i \sim A, j \sim B, v \sim V} [\mathbf{1}[i \succ_v j]]$, so

$$-s_{A \succ B} + s_{A \succ C} + s_{C \succ B} = \mathbb{E}_{i \sim A, j \sim B, k \sim C, v \sim V} [-\mathbf{1}[i \succ_v j] + \mathbf{1}[i \succ_v k] + \mathbf{1}[k \succ_v j]].$$

We claim that the term in the expectation is always nonnegative. If i, j, k are all different then this is clear because it is not possible that $i \succ_v j$ but then $k \succ_v i$ and $j \succ_v k$. If $j = k$ or $i = k$ then the first term cancels with the second or third terms. If $i = j$ then this term is

$$-\mathbf{1}[i \succ_v i] + \mathbf{1}[i \succ_v k] + \mathbf{1}[k \succ_v i] = \frac{1}{2}.$$

This means that the term in the expectation is always nonnegative, which proves the claim. \square

Now, applying the claim to our three strategies, we have that

$$\begin{aligned} s_{D(I) \succ i^*} &\leq s_{D(I) \succ D(I^c)} + s_{D(I^c) \succ i^*} \\ \implies r &\leq \frac{1}{2} + (1 - q). \end{aligned}$$

Combining this with our previous inequality relating p, q, r we have

$$\frac{1}{2} \leq p(1-q) + (1-p)r \leq p(1-q) + (1-p)\left(\frac{3}{2} - q\right) = (1-q) + \frac{1-p}{2}$$

which implies that $\frac{q}{2} \leq 1-q$ as desired. This completes the proof of [Theorem 1](#). \square

5 Two Rules that Complement Maximal Lotteries

In this section, we introduce two novel social choice rules, each of which can be mixed with Maximal Lotteries to get distortion better than 3: Random Consensus Builder (RCB) and Random Dictatorship on the Uncovered Set (RaDiUS). The two rules are very much cousins of one another, and their analyses have many commonalities. RCB has a slightly worse distortion guarantee, but we include it for several reasons: First, its analysis mirrors that of RaDiUS but is simpler, making it a natural warm-up for the tighter bound given by RaDiUS. Moreover, the final specification of the mixed rule using RCB is easier to state, and the distortion bound ends up being a clean algebraic number, $2\sqrt{2}$. These nice properties, along with the fact that RCB itself has a clean interpretation, make us believe that RCB, as a voting rule, can be of independent interest.

We will analyze the rules using our biased metric framework. Intuitively, the constraints will be harder to satisfy when $r(t)$ is often small, which makes R small. However, when $r(t)$ is small, we can show that the election instance and the metric admit a certain structure which can be leveraged in the analysis. This structure is characterized by the following definition, and it interplays with the function $r(t)$ via the subsequent proposition. This is actually all we need from $r(t)$ in this section.

Definition 2. Given an election instance, a biased metric is (α, β) -consistent if whenever $s_{k \succ i^*} \geq \beta$, we have $x_k \leq \alpha R$.

Proposition 3. *If $r(\alpha R) < \beta$ then the metric is (α, β) -consistent.*

Proof. $r(\alpha R) < \beta$ means $s_{\forall i \succ j, x_i - x_j \leq \alpha R} > 1 - \beta$. Then, whenever $s_{k \succ i^*} \geq \beta$, we have $s_{k \succ i^*} + s_{\forall i \succ j, x_i - x_j \leq \alpha R} > 1$, which means that there exists a voter v such that $k \succ_v i^*$ and whenever $i \succ_v j$, we have $x_i - x_j \leq \alpha R$. This exactly implies that $x_k \leq \alpha R$. \square

Once the election instance and metric space are fixed, we will analyze both rules under the assumption that the metric is (α, β) -consistent. This assumption will crucially come into play in [Section 6](#) where we want these rules to perform particularly well when α is close to 0 and β is close to $\frac{1}{2}$. The results in this section will still extend to the general case by the following proposition.

Proposition 4. *All biased metrics are $(\frac{1}{\beta}, \beta)$ -consistent for all $\beta \in (0, 1)$.*

Proof. If $s_{k \succ i^*} \geq \beta$ we have

$$R = 2 \text{SC}(i^*) \geq (x_k - x_{i^*})s_{k \succ i^*} \geq x_k \cdot \beta.$$

This means that $x_k \leq \frac{1}{\beta}R$, and the proposition follows. \square

Both of our rules, RCB and RaDiUS, are parameterized by a tunable value $\beta \in (\frac{1}{2}, 1)$. Very roughly speaking, they both construct the graph on candidates where (i, j) is an edge whenever $s_{i \succ j} \geq \beta$, with the goal of choosing candidates that can always reach the low-distortion candidates in few hops. Note that the rules have natural interpretations when $\beta = \frac{1}{2}$ and $\beta = 1$ which we will briefly discuss, but to avoid dealing with these (ultimately irrelevant) edge cases in the proofs, we will restrict β to the open interval.

5.1 Random Consensus Builder

Below is the description of our first rule.

β -Random Consensus Builder (RCB)

- Choose a uniformly random voter v . Initially the candidate under consideration, i , is v 's least favorite candidate.
- Until a candidate is chosen:
 - Eliminate all of the candidates j such that $j \succ_v i$ and $s_{i \succ j} \geq \beta$.
 - If there are no uneliminated candidates that v prefers over i then we choose i . Otherwise, update i to be v 's least favorite uneliminated candidate that v prefers over i .

We will say that i *eliminates* j if j is eliminated in the iteration where i is the candidate under consideration.

One interpretation of this rule is that it chooses a random voter v (the consensus builder) whose preferences are the main guide of the rule, but if the voters as a whole have consensus disagreements with v 's preferences, then their view overrules. In particular, in each stage of the rule v considers a candidate that is undesirable according to v 's preferences, but if a large fraction of voters views this candidate as preferable over some other candidates, then those candidates are removed from contention. The threshold for how strong the consensus needs to be is tuned by this parameter β .

We can find another interesting interpretation of this rule by considering how it would operate at the extremes where $\beta = \frac{1}{2}$ and $\beta = 1$. If $\beta = 1$, the chosen candidate is always v 's top choice, and so the rule is exactly Random Dictatorship. On the other hand, if $\beta = \frac{1}{2}$, then it is not hard to see that if the chosen candidate is a , for every other candidate b , either a defeats b (i.e. $s_{a \succ b} \geq \frac{1}{2}$) or a defeats a candidate who defeats b . The candidates that satisfy this property are called the *uncovered set*. Some well known voting rules always output a candidate in the uncovered set, including the well known Copeland rule. In this sense, β also is a measure of interpolation between these kinds of rules and Random Dictatorship.

We will show the following theorem.

Theorem 2. *Suppose that we have an election instance with an (α, β) -consistent underlying metric. Then if D is the distribution output by β -RCB, we have*

$$L(D) \leq (\alpha + \beta)R.$$

Corollary 5. *β -RCB guarantees distortion at most $1 + 2(\beta + \frac{1}{\beta})$.*

Proof of Theorem 2. Suppose v is the consensus builder. Let j_v be the candidate that β -RCB picks. Note that each candidate is either at some point the candidate under consideration (candidate i), or it is eliminated by some other candidate.

If i^* is not eliminated during the rule then let $k_v = i^*$, and otherwise let k_v be the candidate that eliminates i^* . In order to prove the theorem, we will use the following three critical properties of k_v :

- (I) $x_{k_v} \leq \alpha R$,
- (II) $j_v \succeq_v k_v$,
- (III) $s_{k_v \succ j_v} < \beta$.

(I) follows because either $k_v = i^*$ in which case $x_{k_v} = 0$, or k_v eliminates i which means that $s_{k_v \succ i^*} \geq \beta$ and so by the fact that the metric is (α, β) -consistent, we have $x_{k_v} \leq \alpha R$. (II) follows because both k_v and j_v are at some point under consideration, and we consider candidates from lowest to highest on v 's preference list. (III) follows because at some point k_v is under consideration, and so either $k_v = j_v$ in which case $s_{k_v \succ j_v} = 0 < \beta$, or k_v did not eliminate j_v which means $s_{k_v \succ j_v} < \beta$.

We will use (I) and (III) to get a good upper bound on $L(D)$, and (I) and (II) to get a good lower bound on R .

We have that

$$\begin{aligned} \text{SC}(j_v) - \text{SC}(i^*) &\leq s_{j_v \succeq k_v} \min(x_{k_v}, x_{j_v}) + s_{k_v \succ j_v} x_{j_v} \\ &\leq (1 - \beta) \min(x_{k_v}, x_{j_v}) + \beta x_{j_v} \\ &\leq (1 - \beta) \alpha R + \beta x_{j_v}. \end{aligned}$$

The second line follows from the first because $x_{j_v} \geq \min(x_{k_v}, x_{j_v})$ and so the expression is maximized when $s_{k_v \succ j_v}$ (which is bounded above by β) is as large as possible. It follows that

$$L(D) = \frac{1}{n} \sum_{v \in V} (\text{SC}(j_v) - \text{SC}(i^*)) \leq \alpha(1 - \beta)R + \beta \cdot \frac{1}{n} \sum_{v \in V} x_{j_v}.$$

On the other hand, since $j_v \succeq_v k_v$, we have $2d(v, i^*) \geq x_{j_v} - x_{k_v} \geq x_{j_v} - \alpha R$. It follows that

$$R = 2 \text{SC}(i^*) \geq -\alpha R + \frac{1}{n} \sum_{v \in V} x_{j_v} \implies \frac{1}{n} \sum_{v \in V} x_{j_v} \leq (1 + \alpha)R.$$

Plugging this into our upper bound on $L(D)$, we get

$$L(D) \leq \alpha(1 - \beta)R + \beta(1 + \alpha)R = (\alpha + \beta)R$$

as desired. □

5.2 Random Dictatorship on the (Weighted) Uncovered Set

Next, we consider a rule similar in spirit to RCB, but with better distortion guarantees.

β-Random Dictatorship on the (Weighted) Uncovered Set (RaDiUS)

- Say that a covers b if $s_{a \succ b} \geq \beta$ and for any c , if $s_{c \succ a} \geq \beta$ then $s_{c \succ b} \geq \beta$.
- Let U be the set of candidates that are not covered by any other candidate.
- Choose a uniformly random voter and output their favorite candidate in U .

The set U was previously considered by [MW19] in the context of deterministic rules. They showed that there exists a β such that any candidate from the set U (which they called the *weighted uncovered set*) has distortion at most $2 + \sqrt{5}$.

To see how this rule is similar to β -RCB, consider the following proposition. It also conveniently gives a proof that the set U is always non-empty (which was proved in a different way in [MW19]).

Proposition 6. *Suppose that U is the weighted uncovered set constructed by β -RaDiUS. Then β -RCB always outputs a member from U .*

Proof. Let's suppose towards a contradiction that for some voter v , the candidate j_v chosen by RCB is covered by some other candidate a .

For j_v to be chosen, it must have been the case that a was eliminated by some candidate c such that $j_v \succ_v c$. Otherwise, if $j_v \succ_v a$ then a would have eliminated j_v and if $a \succ_v j_v$ then a would not have been eliminated when j_v is the candidate under consideration (or before by c) and so the rule would not have terminated in that iteration.

But then $s_{c \succ a} \geq \beta$, and so by the definition of a covering j_v , we must have $s_{c \succ j_v} \geq \beta$. But then c would have eliminated j_v , which is a contradiction. \square

Before we get into the distortion guarantee for β -RaDiUS, we prove two more facts that will be helpful to us.

Proposition 7. *The covering relation is transitive.*

Proof. Suppose a covers b and b covers c . We claim that a covers c . Since b covers c and $s_{a \succ b} \geq \beta$ we have $s_{a \succ c} \geq \beta$. Now suppose that for some d , $s_{d \succ a} \geq \beta$. Then since a covers b we have $s_{d \succ b} \geq \beta$ but then since b covers c we have $s_{d \succ c} \geq \beta$. So indeed, a covers c . \square

Proposition 8. *If a candidate is not in U then it is covered by a candidate in U .*

Proof. Suppose that we build a graph on the candidates where (a, b) is an edge if a covers b . We cannot have a cycle in this graph, because by transitivity this would imply that some candidate i covers itself, which would imply the impossible $s_{i \succ i} \geq \beta > \frac{1}{2} > 0$.

If a candidate i is not in U , it must have positive in-degree. Since the graph is acyclic, by arbitrarily following edges backwards from i , we must eventually reach a candidate j with in-degree zero. This means that $j \in U$ and there is a path from j to i , which by transitivity means that j covers i . \square

Now we prove the following distortion guarantee.

Theorem 3. *Suppose that we have an election instance with an (α, β) -consistent underlying metric. Then if D is the distribution output by β -RaDiUS, we have*

$$L(D) \leq (\alpha(1 - \beta^2) + \beta)R.$$

Corollary 9. *β -RaDiUS guarantees distortion at most $1 + 2/\beta$.*

The proof is similar in structure to the proof of [Theorem 2](#). The key difference is that rather than using the same candidate k_v which satisfies the properties (I), (II), (III), we will have one candidate k_v which satisfies properties (I) and (III) and another candidate k^* that satisfies properties (I) and (II). The advantage is that in the later case, we have the same candidate k^* for *all* voters v , which will allow us to get a stronger lower bound on R . However, having different candidates for the two cases makes the argument a little more complicated.

Proof of Theorem 3. Once again, let j_v be the candidate that is output when v is the randomly chosen voter. Let's assume that $j_v \neq i^*$, otherwise the rule picks the best candidate and all of the bounds will only be improved. Then since $j_v \in U$ it must be the case that i^* *does not* cover j_v . Unpacking the definition, this means that either

- (a) $s_{i^* \succ j_v} < \beta$, or
- (b) there exists some k such that $s_{k \succ i^*} \geq \beta$ but $s_{k \succ j_v} < \beta$.

Define k_v so that $k_v = i^*$ if (a) occurs and $k_v = k$ if (b) occurs. In either case we have once again that $x_{k_v} \leq \alpha R$ and $s_{k_v \succ j_v} < \beta$. These are the properties (I) and (III) used in the proof of Theorem 2, and so by an identical argument we can show that

$$L(D) = \frac{1}{n} \sum_{v \in V} (\text{SC}(j_v) - \text{SC}(i^*)) \leq \alpha(1 - \beta)R + \beta \cdot \frac{1}{n} \sum_{v \in V} x_{j_v}.$$

Now define k^* so that $k^* = i^*$ if $i^* \in U$ and otherwise, k^* is some member of U which covers i^* (which exists by Proposition 8). Once again, we have that either $k^* = i^*$ and so $x_{k^*} = 0$, or $s_{k^* \succ i^*} \geq \beta$ and since the metric is (α, β) -consistent we have $x_{k^*} \leq \alpha R$. In addition, we have that $j_v \preceq_v k^*$, since $k^* \in U$ and j_v is v 's favorite candidate in U . Thus, k^* satisfies properties (I) and (II) used in the proof of Theorem 2.

It follows that for every voter v ,

$$2d(v, i^*) \geq x_{j_v} - x_{k^*}.$$

Moreover, if v satisfies $k^* \preceq_v i^*$, the inequality can be stronger. In this case, $j_v \preceq_v k^* \preceq_v i^*$ and so

$$2d(v, i^*) \geq x_{j_v} = x_{k^*} + (x_{j_v} - x_{k^*}).$$

Since $k^* \preceq_v i^*$ for at least a β fraction of voters v , we have

$$R = 2 \text{SC}(i^*) \geq \beta x_{k^*} + \frac{1}{n} \sum_{v \in V} (x_{j_v} - x_{k^*}) = -(1 - \beta)x_{k^*} + \frac{1}{n} \sum_{v \in V} x_{j_v},$$

where we crucially use the fact that all voters share the same k^* . It follows that

$$\frac{1}{n} \sum_{v \in V} x_{j_v} \leq R + (1 - \beta)x_{k^*} \leq (1 + (1 - \beta)\alpha)R.$$

Plugging this into our upper bound on $L(D)$, we have

$$\begin{aligned} L(D) &\leq \alpha(1 - \beta)R + \beta(1 + (1 - \beta)\alpha)R \\ &= (\alpha(1 - \beta^2) + \beta)R \end{aligned}$$

as claimed. □

6 Mixing Rules

Even though none of the three social choice rules we introduced can beat distortion 3 (see [Appendix A](#)), it turns out that mixing them in a careful way can.

Let us introduce the general technique for analyzing the mixture of these rules. Suppose that we have a given election instance and a biased metric. We would like to show that a particular rule achieves low distortion on this instance and metric. Consider the graph of $r(t)$ that is fixed by the metric. For each $\beta \in (\frac{1}{2}, 1)$, let $\alpha(\cdot)$ be the function such that $\alpha(\beta) \cdot R = \min\{t : r(t) < \beta\}$. Informally, if we draw the horizontal line $y = \beta$, then this line intersects the graph of $r(t)$ at the point $(\alpha(\beta)R, \beta)$. (If the intersection is a line segment, we take the rightmost point on the segment.)

Unsurprisingly, the function $\alpha(\cdot)$ is directly related to the α we were considering in [Section 5](#): [Proposition 3](#) tells us that if we have $\alpha(\cdot)$ corresponding to a biased metric, then the metric is $(\alpha(\beta), \beta)$ -consistent for all $\beta \in (\frac{1}{2}, 1)$.

Moreover, we can use this function $\alpha(\beta)$ to get a tighter bound on the distortion of ML. Let D_{ML} be the distribution output by ML. Then [Theorem 1](#) tells us that

$$\ell(D_{\text{ML}}, t) \leq \frac{P(I_t^c)}{2} \leq r(t)$$

and since $P(I_t^c) \leq 1$, we have $\ell(D_{\text{ML}}, t) \leq \min(\frac{1}{2}, r(t))$. On the other hand, the area that is below $r(t)$ but above the horizontal line $\frac{1}{2}$ is exactly $R \int_{\frac{1}{2}}^1 \alpha(\beta) d\beta$, and so it follows that

$$L(D_{\text{ML}}) + R \int_{\frac{1}{2}}^1 \alpha(\beta) d\beta \leq R \implies L(D_{\text{ML}}) \leq \left(1 - \int_{\frac{1}{2}}^1 \alpha(\beta) d\beta\right) R. \quad (6)$$

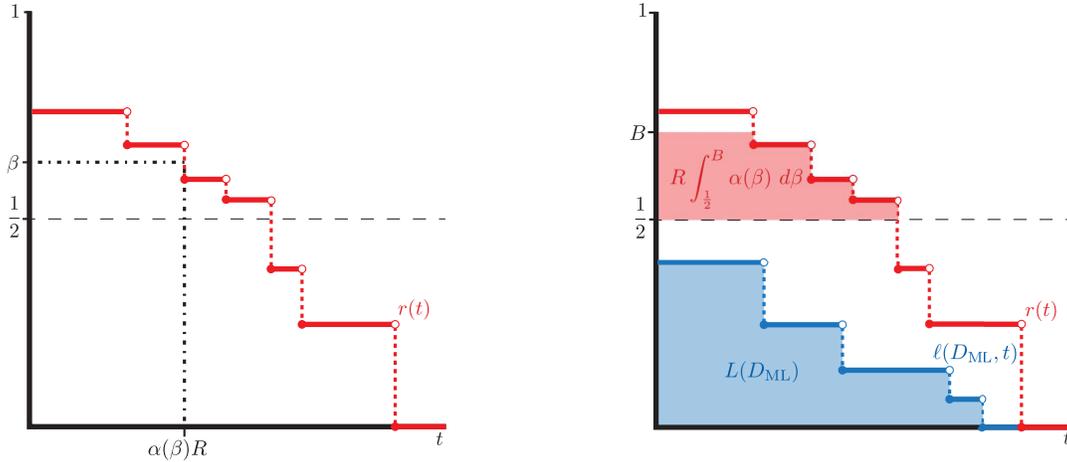


Figure 4: Left: For each β , the horizontal line at β intersects $r(t)$ at $(\alpha(\beta)R, \beta)$. Right: If the area above $\frac{1}{2}$ and below $r(t)$ is large, we can get a better bound on $L(D_{\text{ML}})/R$.

This mixture of rules does well because there is a sense in which ML and β -RCB/ β -RaDiUS are complementary. For the analysis of ML not to have much wiggle room, the curve $r(t)$ should be above the line $\frac{1}{2}$ very little. But then it means that $\alpha(\beta)$ is small for values of β that are slightly larger than $\frac{1}{2}$, and with smaller α and β , we get much better guarantees in [Theorem 2](#) and [Theorem 3](#).

The rules will have three parameters: p , B , and $\rho(\cdot)$. Both rules run ML with probability p , and otherwise run β -RCB or β -RaDiUS where $\beta \in (\frac{1}{2}, B)$ is drawn from a distribution with probability density function ρ . It will turn out that in the analysis, we will fix p and ρ as a function of B , and then to get the best distortion we just need to optimize a single variable function. In the warm-up, the right ρ is just uniform, so to simplify we will omit that parameter.

6.1 Warm-up: ML and RCB Get Distortion $2\sqrt{2}$

We note that the rule and proof might read more smoothly with the right values for p and B baked in, but we will keep these as variables to illustrate the mechanics of the proof technique.

ML mixed with RCB

- With probability p , run Maximal Lotteries.
- With probability $1 - p$, choose a *uniformly* random $\beta \in (\frac{1}{2}, B)$ and run β -Random Consensus Builder.

Theorem 4. *With $p = \frac{1}{\sqrt{2}}$ and $B = \sqrt{2} - \frac{1}{2}$, the rule ML mixed with RCB has distortion at most $2\sqrt{2} \approx 2.828$.*

Proof. Suppose that we have a fixed election instance and metric space. Let D_{ML} be the distribution output by ML, let D_β be the distribution output by β -RCB, and let D be the overall distribution of the rule. Note that β is chosen according to the probability density function $\frac{1}{B-\frac{1}{2}}$, so

$$\begin{aligned} L(D) &= pL(D_{\text{ML}}) + (1-p) \int_{\frac{1}{2}}^B \frac{1}{B-\frac{1}{2}} \cdot L(D_\beta) \, d\beta \\ &= pL(D_{\text{ML}}) + \frac{1-p}{B-\frac{1}{2}} \int_{\frac{1}{2}}^B L(D_\beta) \, d\beta. \end{aligned}$$

So then applying [Eq. \(6\)](#) and [Theorem 2](#), we get

$$\begin{aligned} \frac{L(D)}{R} &\leq p \left(1 - \int_{\frac{1}{2}}^B \alpha(\beta) \, d\beta \right) + \frac{1-p}{B-\frac{1}{2}} \int_{\frac{1}{2}}^B (\alpha(\beta) + \beta) \, d\beta \\ &= p + \frac{1-p}{B-\frac{1}{2}} \int_{\frac{1}{2}}^B \beta \, d\beta + \left(-p + \frac{1-p}{B-\frac{1}{2}} \right) \int_{\frac{1}{2}}^B \alpha(\beta) \, d\beta \\ &= p + \frac{1}{2}(1-p)(B + \frac{1}{2}) + \left(\frac{1-p(B + \frac{1}{2})}{B-\frac{1}{2}} \right) \int_{\frac{1}{2}}^B \alpha(\beta) \, d\beta. \end{aligned}$$

Now, $\alpha(\beta)$ is a function which depends on the metric, which could be chosen adversarially. However, we can completely eliminate this “dangerous” term by choosing p and B such that its coefficient is 0. This is perhaps where the magic of the proof happens – by carefully balancing the two rules, we can get a kind of destructive interference that eliminates any dangerous terms.

Choosing $p = \frac{1}{B+\frac{1}{2}}$, we get

$$\frac{L(D)}{R} \leq \frac{1}{B+\frac{1}{2}} + \frac{1}{2}(B - \frac{1}{2}).$$

It is not hard to check that choosing $B = \sqrt{2} - \frac{1}{2}$ minimizes the above expression, at which point it is also $\sqrt{2} - \frac{1}{2}$. This gives us distortion $1 + 2(\sqrt{2} - \frac{1}{2}) = 2\sqrt{2}$. \square

6.2 ML and RaDiUS Get Distortion 2.753

ML mixed with RaDiUS

- With probability p , run Maximal Lotteries.
- With probability $1 - p$, sample $\beta \in (\frac{1}{2}, B)$ according to the probability density function $\rho(\beta)$ and run β -RaDiUS.

Theorem 5. *With appropriate choices for p, B , and $\rho(\cdot)$ the rule ML mixed with RaDiUS has distortion at most 2.753.*

Proof. Let D_{ML} and D_{β} be defined as in the proof of [Theorem 4](#), but with β -RaDiUS in place of β -RCB. Then we get

$$\begin{aligned} \frac{L(D)}{R} &\leq p \left(1 - \int_{\frac{1}{2}}^B \alpha(\beta) \, d\beta \right) + (1-p) \int_{\frac{1}{2}}^B \rho(\beta) (\alpha(\beta)(1-\beta^2) + \beta) \, d\beta \\ &= p + (1-p) \int_{\frac{1}{2}}^B \rho(\beta) \beta \, d\beta + \int_{\frac{1}{2}}^B \alpha(\beta) (-p + \rho(\beta)(1-p)(1-\beta^2)) \, d\beta \\ &= 1 - (1-p) \int_{\frac{1}{2}}^B \rho(\beta)(1-\beta) \, d\beta + \int_{\frac{1}{2}}^B \alpha(\beta) (-p + \rho(\beta)(1-p)(1-\beta^2)) \, d\beta. \end{aligned}$$

The last line uses the fact that $\int_{\frac{1}{2}}^B \rho(\beta) \, d\beta = 1$. In order to make the coefficient of $\alpha(\beta)$ equal to 0, we set

$$\rho(\beta) = \frac{p}{(1-p)(1-\beta^2)}$$

which means that

$$1 = \int_{\frac{1}{2}}^B \rho(\beta) \, d\beta = \frac{p}{1-p} \int_{\frac{1}{2}}^B \frac{d\beta}{1-\beta^2} \implies p = \frac{1}{1 + \int_{\frac{1}{2}}^B \frac{d\beta}{1-\beta^2}}.$$

With these choices, we have

$$\begin{aligned} \frac{L(D)}{R} &\leq 1 - p \int_{\frac{1}{2}}^B \frac{d\beta}{1+\beta} \\ &= 1 - \frac{\int_{\frac{1}{2}}^B \frac{d\beta}{1+\beta}}{1 + \int_{\frac{1}{2}}^B \frac{d\beta}{1-\beta^2}} \\ &= 1 - \frac{\ln \frac{2}{3} + \ln(1+B)}{1 - \frac{1}{2} \ln 3 + \frac{1}{2}(\ln(1+B) - \ln(1-B))}. \end{aligned}$$

Using numerical optimization methods, we find that the best choice is $B \approx 0.876353$, which gives distortion 2.75271. \square

7 Discussion

In this work, we studied Maximal Lotteries in the distortion setting and proposed novel simple rules of Random Consensus Builder and RaDiUS. Using our biased metric framework, we show that a mix between ML and RCB has metric distortion at most $2\sqrt{2}$, and a mix between ML and RaDiUS has distortion at most 2.753.

An immediate future direction is to further close the gap (2.112, 2.753) of optimal metric distortion. Towards this, we propose the following ideas:

- Our RaDiUS rule is a hybrid of Random Dictatorship and a deterministic weighted tournament rule of [MW19]. Is there a deterministic weighted tournament rule with better distortion than $2 + \sqrt{5} \approx 4.236$? Such a result is very interesting on its own, and can potentially serve as an ingredient for a rule with better distortion than 2.753. (Note that our analysis for ML pinned down the optimal metric distortion for weighted tournament rules at 3. For deterministic weighted tournament rules, the gap is $[3, 2 + \sqrt{5}]$.)
- Our RaDiUS rule uses the notion of weighted uncovered set, which was designed to show a deterministic rule with good metric distortion. Would ideas that lead to distortion-optimal deterministic rules be useful, such as the matching uncovered set [MW19] and related ideas [Kem20a], Plurality Matching [GHS20], Plurality Veto [KK22] and its variants [KK22, KK23]?
- The biased metric framework potentially has more power than we have utilized in our proofs. Theorems 2 and 3 show that for some function $f(\cdot, \cdot)$, their respective rules have distortion $1 + 2f(\alpha, \beta)$ under this assumption. If one can show a similar theorem for a new rule, but with a smaller function $f(\cdot, \cdot)$, then this would improve the distortion upper bound. One could also attempt to go beyond our proof structure. For example, Eq. (3) is a set of simpler and stricter constraints (derived by [CR22]) than what we use, but we do not know what distortion bounds we can achieve after this simplification. Further understanding the structures of biased metrics can be helpful in improving the metric distortion bounds.

Another intriguing direction is to find “simpler” rules that have good metric distortion:

- We managed to break the barrier of 3 by mixing simple rules. Can we do this using an even simpler rule, e.g., one which does not look like a randomization between simple rules? A similar question can be asked for some non-metric distortion settings, where the Stable Lottery (or Stable Committee) rule, which looks like a randomization between two simple rules, gives optimal $\Theta(\sqrt{m})$ distortion [EKPS22].
- Can we break the barrier of 3 by using a minimal amount of randomness – for example, randomizing between at most two candidates, or only using randomness to sample a single voter (as RCB, RaDiUS, and Random Dictatorship do)?

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A Lower Bounds for RCB and RaDiUS

In this section, we will show that β -RCB and β -RaDiUS always have worst-case distortion at least 3. In particular, we will show that [Corollary 9](#) is tight and [Corollary 5](#) is almost tight. The reason for the gap between the upper and lower bounds for β -RCB is that [Theorem 2](#) can be improved in some parameter regimes (though not in a way that improves the final distortion). Details on this are included in [Appendix C.1](#).

Theorem 6. β -RaDiUS has distortion at least $1 + 2/\beta$.

Proof. Consider an instance where the candidates are i^* , k^* , and a large set U . A $1 - \beta$ fraction of voters ranks $i^* \succ U \succ k^*$. When we write the set U in this fashion, it means that these voters order the candidates of U every way in equal proportion. The remaining β fraction of voters has rankings of the form $j \succ k^* \succ i^* \succ U \setminus j$ for some $j \in U$. Each $j \in U$ is equally likely to be j .

Now we have that $s_{k^* \succ i^*} = \beta$, and for each $j \in U$, $s_{i^* \succ j} = 1 - \beta/|U|$ and $s_{k^* \succ j} = \beta(1 - \frac{1}{|U|})$. So in the graph where (a, b) is an edge if $s_{a \succ b} \geq \beta$, the only edges are (k^*, i^*) and (i^*, j) for each $j \in U$ (eventually we will take $|U| \rightarrow \infty$, so we should treat $|U|$ as large enough that $1 - \beta/|U| > \beta$). Then k^* covers i^* , since there is an edge from k^* to i^* and k^* has no in-edges. On the other hand,

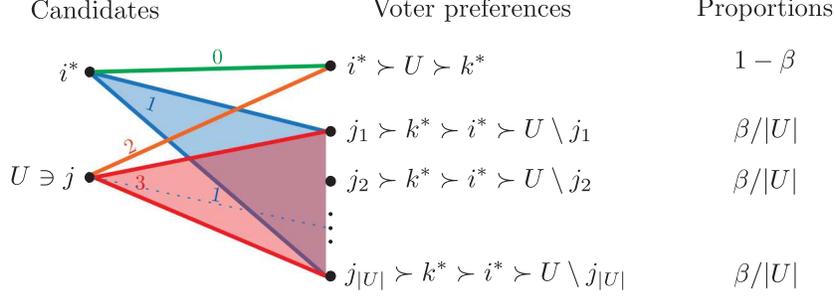


Figure 5: The lower bound instance for β -RCB.

i^* does not cover any $j \in U$, because there is no edge from k^* to j . Thus, the weighted uncovered set is $U \cup \{k^*\}$.

But then β -RaDiUS will always choose a candidate in U . Considering the biased metric where $x_i = 2$ for all $i \neq i^*$ we have

$$\begin{aligned} \text{SC}(i^*) &= \beta, \\ \text{SC}(j) - \text{SC}(i^*) &= 2 \left(1 - \frac{\beta}{|U|} \right). \end{aligned}$$

Taking $|U| \rightarrow \infty$, we get that

$$\frac{\text{SC}(j) - \text{SC}(i^*)}{2\text{SC}(i^*)} = \frac{1}{\beta}$$

which exactly corresponds to distortion $1 + 2/\beta$. \square

Theorem 7. β -RCB has distortion at least $1 + 2/\beta$.

Proof. Suppose that the candidates are i^* and several large sets C_1, C_2, \dots, C_{T+1} , where T is also large. We will eventually let T and the size of these sets go to infinity, so for ease, we make the sizes equal to T as well.

We will treat $\{i^*\}$ as C_0 . The idea is to have: Firstly, $s_{c_{i+1} \succ c_i} \geq \beta$ for each $c_i \in C_i$ and $c_{i+1} \in C_{i+1}$; and secondly, as i increases, $s_{c_i \succ i^*}$ decreases until $s_{c_{T+1} \succ i^*} \approx 0$. We will want that when any given voter is the chosen voter, some candidate in C_1 will eliminate i^* , and then some candidate in C_2 will eliminate everyone in C_1 above i^* and so on, so that all the candidates above and including i^* will get eliminated. In the end, we will get stuck with a candidate in C_{T+1} with the highest cost.

Consider the following voter preferences. A $1 - \beta - 1/T$ fraction of voters has a ranking of the form

$$i^* \succ C_{T+1} \succ C_T \succ \dots \succ C_1.$$

(These sets should always be thought of as representing candidates ordered in every way equally often.) Another $1/T$ fraction of voters will have rankings of the form

$$C_{T+1} \succ C_T \succ \dots \succ C_1 \succ i^*.$$

Then for each $1 \leq t \leq T$, a β/T fraction of voters has a ranking of the form

$$(C_t \setminus c_t) \succ \dots \succ (C_1 \setminus c_1) \succ i^* \succ C_{T+1} \succ \dots \succ C_{t+1} \succ c_t \succ \dots \succ c_1.$$

For each $i \leq t$, each member of C_i is the candidate c_i in equal proportion. For brevity, we call the set of voters with rankings of the above form S_i .

First, we will show that for $c_i \in C_i$ and $c_{i+1} \in C_{i+1}$, we have $s_{c_{i+1} \succ c_i} \geq \beta$. First let us see this for $i = 0$. We have that $c_1 \succ i^*$ for the $1/T$ fraction of voters in $s_{C_1 \succ i^*}$, and for all but $1/T$ fraction of voters in $S_1 \cup \dots \cup S_T$. This means that

$$s_{c_1 \succ i^*} = \frac{1}{T} + \beta \left(1 - \frac{1}{T}\right) = \beta + \frac{1-\beta}{T} > \beta.$$

Now for $i > 0$, we have that $c_{i+1} \succ c_i$ for the $1 - \beta$ fraction of voters outside of $S_1 \cup \dots \cup S_T$. For the remaining voters, if $c_i \succ_v c_{i+1}$, then one of two things must be true of v : Either $v \in S_i$, or $v \in S_{i'}$ for some $i' > i$ and the candidate c_{i+1} is the candidate from C_{i+1} which is below i^* . Each of these two events is true for at most $1/T$ of the voters in $S_1 \cup \dots \cup S_T$, so it follows that

$$s_{c_{i+1} \succ c_i} \geq 1 - \beta + \beta(1 - \frac{2}{T}) = 1 - \frac{2}{T}\beta$$

and taking T sufficiently large, this is at least β .

For $1 \leq i < j$ we have $s_{c_i \succ c_j} < \beta$ since $c_j \succ c_i$ for the $1 - \beta$ fraction of voters outside of $S_1 \cup \dots \cup S_T$, at at least one voter in S_1 . These conclusions imply that when a voter outside of $s_{C_1 \succ i^*}$ is chosen by the rule, the candidates in C_{T+1} cannot be eliminated, and the candidates above and including i^* will all get eliminated. This means that with probability at least $1 - \frac{1}{T}$, the rule chooses a candidate in C_{T+1} .

The biased metric we consider is the same as before, where $x_i = 2$ for $i \neq i^*$. With this we have:

$$\begin{aligned} \text{SC}(i^*) &= \beta, \\ \text{SC}(j) - \text{SC}(i^*) &= 2 \left(1 - \frac{1}{T}\right), \end{aligned}$$

for $j \in C_{T+1}$. Since the rule chooses a candidate in C_{T+1} with probability at least $1 - \frac{1}{T}$, we get

$$\frac{\mathbb{E}[\text{SC}(j) - \text{SC}(i^*)]}{2 \text{SC}(i^*)} \geq \frac{(1 - \frac{1}{T})^2}{\beta}$$

and taking $T \rightarrow \infty$, this corresponds to distortion $1 + 2/\beta$. □

B A Third Complementary Rule

In this section, we will give a brief discussion on another rule which is similar in flavor to β -RCB and β -RaDiUS. We chose not to include it in the main body of the paper because the guarantees are worse and the analysis is more complicated, but the rule itself is interesting and could be of independent interest. It may also give some insight into what led to β -RCB and β -RaDiUS.

The rule is based on the following well known combinatorial fact (see for instance, [BM76, Theorem 10.2]).

Lemma 10. *For any directed graph $G = (V, E)$, there exists an independent set U such that for all $v \in V$ there exists $u \in U$ such that there is a path of length at most 2 from u to v .*

The set U is a directed analogue of a maximal independent set (MIS). In an undirected graph, an MIS will satisfy the property that every vertex is either in the set, or is distance 1 from a vertex in the set.

Proof. [BM76] give a proof by induction, which can be reworked into a nice algorithm to construct the set U :

- Arbitrarily order the vertices v_1, v_2, \dots, v_n .
- For $i = 1, \dots, n$:
 - If v_i is not eliminated, eliminate all v_j such that $j > i$ and $(v_i, v_j) \in E$.
- For $i = n, \dots, 1$:
 - If v_i is not eliminated, eliminate all v_j such that $j < i$ and $(v_i, v_j) \in E$.
- The vertices that are not eliminated form the set U .

First, let us see why U is an independent set. Suppose that there existed $v_i, v_j \in U$ such that there is an edge from v_i to v_j . Since both are never eliminated, v_i would have eliminated v_j in the first loop if $i < j$ and in the second loop if $i > j$. This is a contradiction.

Next, suppose we have some $v \notin U$. Then either v was eliminated by some $u \in U$, or by some $v' \notin U$ who was eliminated by some $u \in U$ (v' eliminated v in the first loop, then u eliminated v' in the second loop). In the first case, there is a path of length 1 from u to v , and in the second case there is a path of length 2. Thus, U indeed satisfies the requirements. \square

The key idea of the rule is to apply this lemma to the graph on candidates where there is an edge from a to b if $s_{a \succ b} \geq \beta$, for some fixed $\beta \in (\frac{1}{2}, 1)$. Then, even though we do not know i^* , we know there is some $j^* \in U$ such that there exists some k such that $s_{j^* \succ k} \geq \beta$ and $s_{k \succ i^*} \geq \beta$. On the other hand, for any $j \in U$, we have that $s_{j \succeq j^*} \geq 1 - \beta$. This is somewhat similar to the condition we used in the proof of [Theorem 3](#) (there it is the same condition as if $j^* = k$) and so we can reason about it in a similar way. However, the extra distance between j and i^* (having j^* and k rather than just k) makes the analysis weaker.

Ultimately the rule is as follows.

β -Random Dictatorship on the Directed MIS (β -RDDMIS)

- Fix an arbitrary ordering of the candidates c_1, c_2, \dots, c_m .
- For $i = 1, \dots, m$:
 - If c_i is not eliminated, eliminate all c_j such that $j > i$ and $s_{c_i \succ c_j} \geq \beta$.
- For $i = m, \dots, 1$:
 - If c_i is not eliminated, eliminate all c_j such that $j < i$ and $s_{c_i \succ c_j} \geq \beta$.

- Let U be the set of candidates that were never eliminated. Pick a uniformly random voter and choose their favorite candidate in U .

For the distortion guarantees, we need to make a slight modification to the definition of (α, β) -consistent. Instead of the definition being whenever $s_{k \succ i^*} \geq \beta$ we have $x_k \leq \alpha R$, it will be whenever $s_{k \succ i} \geq \beta$ we have $x_k - x_i \leq \alpha R$.

The point of this change is so that for j^* we have that $x_{j^*} \leq 2\alpha R$. Using the same approach as in the proofs of [Theorem 2](#) and [Theorem 3](#), we can establish the following theorem.

Theorem 8. *Suppose that we have an election instance with an (α, β) -consistent underlying metric. Then if D is the distribution output by β -RDDMIS, we have*

$$L(D) \leq (\alpha(1 - \beta)(2 + 3\beta) + \beta)R.$$

Note that this is always worse than [Theorem 3](#), but when β is close to 1, it is actually better than [Theorem 2](#). We will omit the proof of the theorem, since the mechanics are the same as the proofs we have seen before.

Now, the fact that β -RDDMIS uses an arbitrary ordering of the candidates leaves open the possibility that by cleverly choosing the order of the candidates, one can get a better bound on the distortion. Indeed, all of the following modifications improve the guarantee, at least for some values of β :

- (1) Choose a random voter, and use their preference list as the candidate order.
- (2) Do the above, but then use the same random voter in the last step to select a candidate.
- (3) Either of the above, but use the preference list in *reverse* for the candidate order.

In fact, (3) with the same voter in the last step is exactly β -RCB (since we consider candidates in reverse, the voter’s favorite candidate in the set U will be decided after the first loop). As one might expect, figuring out the optimal way to randomize between all of these different options makes the rule and its analysis quite complicated. One can at least improve the upper bound on $L(D)/R$ to $\alpha(1 - \beta)(\frac{1}{2} + \frac{9}{2}\beta) + \beta$ by randomizing between (1) and (2), and randomizing between this, β -RCB, and ML one can get distortion a little less than $2\sqrt{2}$. It turns out that β -RaDiUS gets a better guarantee than all of these for all values of α and β so these do not lead to any improvement.

C Re-deriving Known Bounds via Biased Metrics

In this section, we revisit some of the various rules that have been studied in the metric voting distortion literature, and show how upper bounds on their distortions can be proved using the biased metric framework introduced by [\[CR22\]](#) and refined by our work. The fact that this can be done is unsurprising – since the biased metrics the hardest metrics, any distortion upper bound for a rule must somehow be arguing about them under-the-hood. However, many of these upper bounds also essentially re-do the work of defining the biased metrics (at least in some partial sense), and so having the biased metrics as a primitive can simplify some of the proofs. We will demonstrate this for a handful of results, suggesting that the biased metric may be a useful primitive for future work.

C.1 Distortion for (Weighted) Uncovered Set Rules

In [Section 5.2](#), we showed an upper bound on the distortion of running Random Dictatorship restricted to the candidates in the weighted uncovered set U . In this section, we will show the following distortion bound on choosing *any* candidate in U .

Theorem 9. *Let U be the β -weighted uncovered set for some $\beta \in [\frac{1}{2}, 1)$. For any (α, β) -consistent metric, any candidate $j \in U$ will satisfy*

$$\frac{\text{SC}(j) - \text{SC}(i^*)}{2\text{SC}(i^*)} \leq \alpha \min\left(0, 1 - \frac{\beta^2}{1 - \beta}\right) + \frac{\beta}{1 - \beta}.$$

Note that the coefficient of α is nonzero as long as $\beta \leq \varphi^{-1}$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Using $\alpha = \frac{1}{\beta}$, we get the following corollary.

Corollary 11. *For all metrics, a candidate in U has distortion at most $1 + 2/\beta$ for $\beta \in [\frac{1}{2}, \varphi^{-1}]$, and at most $1 + 2\frac{\beta}{1-\beta}$ for $\beta \in [\varphi^{-1}, 1)$.*

Taking $\beta = \frac{1}{2}$, this recovers the result of [\[ABP15, ABE⁺18\]](#) that any rule that outputs a candidate in the (unweighted) uncovered set has distortion at most 5. This includes the well known Copeland rule. Taking $\beta = \varphi^{-1}$ this also recovers the distortion $2 + \sqrt{5}$ rule due to [\[MW19\]](#).

We also note that by [Proposition 6](#), these bounds will also apply to β -RCB. In particular, [Corollary 11](#) improves [Corollary 5](#) for $\beta \in [\frac{1}{2}, 0.682]$, which explains why there is a gap between [Corollary 5](#) and [Theorem 7](#). In fact, [Theorem 7](#) is tight for $\beta \in [\frac{1}{2}, \varphi^{-1}]$.

These bounds of course also apply to β -RaDiUS, but it turns out that it does not improve on [Theorem 3](#). In particular, for $\alpha \leq \frac{1}{\beta}$, $\alpha(1 - \beta^2) + \beta$ is smaller than the expression in [Theorem 9](#).

Proof of [Theorem 9](#). Let j be any candidate in the β -weighted uncovered set. The fact that j is not covered by i^* implies that either $s_{j \succ i^*} \geq 1 - \beta$, or there exists k such that $s_{k \succ i^*} \geq \beta$ and $s_{j \succ k} \geq 1 - \beta$. If the former occurs, we just let $k = i^*$ so that we have the single condition that $s_{k \succ i^*} \geq \beta$ and $s_{j \succ k} \geq 1 - \beta$.

By a similar argument as in the proofs of [Theorem 2](#) and [Theorem 3](#), we can show that

$$\text{SC}(j) - \text{SC}(i^*) \leq x_k + \beta \max(x_j - x_k, 0)$$

and

$$2\text{SC}(i^*) \geq \beta x_k + (1 - \beta) \max(x_j - x_k, 0) \implies \max(x_j - x_k, 0) \leq \frac{1}{1 - \beta} (2\text{SC}(i^*) - \beta x_k).$$

It follows that

$$\text{SC}(j) - \text{SC}(i^*) \leq x_k \left(1 - \frac{\beta^2}{1 - \beta}\right) + \frac{\beta}{1 - \beta} \cdot 2\text{SC}(i^*).$$

If $1 - \frac{\beta^2}{1 - \beta} \geq 0$, then we can use $x_k \leq \alpha \cdot 2\text{SC}(i^*)$. Otherwise, we just use $x_k \geq 0$. Putting these two cases together gives us the desired result. \square

C.2 Matchings in Domination Graphs

In the pursuit of a deterministic distortion 3 rule, [MW19] gave the following definition, and proved the subsequent theorem.

Definition 3. Given an election instance and a pair of candidates a, b , the bipartite graph $G(a, b)$ is defined as follows. Each side of the graph is a copy of the set of voters V . There is an edge from v to v' if there exists a candidate c such that $a \succeq_v c$ and $c \succeq_{v'} b$.

The *matching uncovered set* is the set of candidates a such that for all $b \neq a$, $G(a, b)$ has a perfect matching.

Theorem 10 ([MW19]). *Every candidate in the matching uncovered set has distortion at most 3.*

Given this theorem, all that remains is to show that the matching uncovered set is always nonempty. [GHS20] proved this by considering a more manageable definition, below.

Definition 4. Given an election instance and a candidate a , the *domination graph* $G(a)$ is a bipartite graph which has a vertex on each side for each voter, and the edge (v, v') exists if $a \succeq_v \text{top}(v')$ where $\text{top}(v')$ is the favorite candidate of v' .

It is not hard to see that $G(a)$ is a subgraph of $G(a, b)$ for all $b \neq a$, [Theorem 10](#) implies that if $G(a)$ has a perfect matching, then a has distortion at most 3. Since there is only one graph per candidate, this version can be easier to work with. [GHS20] proved the upper bound of 3 by showing that there exists a such that $G(a)$ always has a perfect matching, which also implies that the matching uncovered set is always nonempty.

Here, we will give an alternate proof of [Theorem 10](#) via the biased metric framework.

Proof of Theorem 10. Consider the constraints given by [Eq. \(4\)](#) with $\lambda = 1$ and setting $p_c = 1$ and $p_i = 0$ for $i \neq c$. This tells us that a candidate a achieves distortion 3 if for all subsets of voters I with $a \notin I$, and all $i^* \in I$, we have $s_{I \succ a} + s_{i^* \succ I^c} \leq 1$.

Suppose that a is in the matching uncovered set. We will use the fact that $G(a, i^*)$ has a perfect matching to prove that $s_{I \succ a} + s_{i^* \succ I^c} \leq 1$.

We claim that in $G(a, i^*)$, there is no edge (v, v') such that $v \in S_{I \succ a}$ and $v' \in S_{i^* \succ I^c}$. If there were, then we have that there exists a candidate c such that $a \succeq_v c$ and $c \succeq_{v'} i^*$. But then we have that $I \succ_v a \succeq_v c$ which means that $c \notin I$, but also $c \succeq_{v'} i^* \succ_{v'} I^c$ which means $c \in I$. This is a contradiction, so the claim is true.

It follows that in $G(a, i^*)$, the neighbors of the set $S_{I \succ a}$ (on the left) are disjoint from the set $S_{i^* \succ I^c}$ (on the right), which means that $|N(S_{I \succ a})| + |S_{i^* \succ I^c}| \leq n$. Since $G(a, i^*)$ has a perfect matching, by Hall's theorem, $|N(S_{I \succ a})| \geq |S_{I \succ a}|$. It follows that $s_{I \succ a} + s_{i^* \succ I^c} \leq 1$ as desired. \square

C.3 Plurality Veto

[KK22] later introduced a novel voting rule, Plurality Veto, and showed that it has distortion at most 3 via [Theorem 10](#) (the domination graph version). Their rule, and the proof of its distortion are very clean and simple. Here, we take their proof and translate it into one that goes through the biased metrics instead of [Theorem 10](#). This shows that if one takes the biased metrics as a primitive, one can prove that there exists a deterministic rule with distortion 3 within a couple of paragraphs.

Plurality Veto ([KK22])

- Initially the score of candidate i is $n \cdot \text{plu}(i)$.
- One by one, each voter decrements the score of their least favorite candidate with positive score.
- The last candidate with positive score wins.

Theorem 11 ([KK22]). *Plurality Veto guarantees distortion 3.*

Proof. Suppose we have an election instance, and let c be the candidate the Plurality Veto chooses. Like before it suffices to show that for all sets I such that $c \notin I$, and all $i^* \in I$, we have $s_{I \succ c} \leq 1 - s_{i^* \succ Ic}$.

The key observation is that the voters in $S_{I \succ c}$ do not decrement the score of any candidate in I . This is because c always has positive score, and so for any voter in this set, no candidate in I can be their least favorite candidate with positive score. On the other hand, since none of the candidates in I are eventually chosen, at least $n \sum_{i \in I} \text{plu}(i)$ voters must decrement the score of some candidate in I . It follows that

$$s_{I \succ c} \leq 1 - \sum_{i \in I} \text{plu}(i) \leq 1 - s_{i^* \succ Ic}$$

where the last inequality follows because the top candidate of a voter in $S_{i^* \succ Ic}$ must be in I . \square

[KK22] also considered a class of randomized rules that are variants of Plurality Veto, called k -Round Plurality Veto. Instead of every voter having the opportunity to decrement some candidate's score, only k voters do so. Then, the rule randomly chooses a candidate with probability proportional to their score. Notice that if $k = 0$, this rule is exactly Random Dictatorship, so the choice of k can be thought of as a measure of interpolation between Random Dictatorship and Plurality Veto.

Using a generalization of the flow technique used in [Kem20b], [KK22] showed that for any k , k -Round Plurality Veto has distortion at most 3. Below, we show that this can also be done via the biased metrics, which is arguably simpler.

Theorem 12 ([KK22]). *k -Round Plurality Veto has distortion 3 for all $0 \leq k \leq n$.*

Proof. Let p_j be the probability that an k -Round Plurality Veto chooses candidate j (for some fixed ordering of the voters). As we have seen before, it suffices to show that for all sets I and all $i^* \in I$, we have that

$$\sum_{j \notin I} s_{I \succ j} p_j \leq 1 - \sum_{i \in I} \text{plu}(i) = \sum_{j \notin I} \text{plu}(j).$$

Among the k voters that decremented a candidate's score, let V_j denote the set that decremented candidate j 's score, and let $V_I = \bigcup_{i \in I} V_i$ denote the set that decremented the score of some candidate in set I . We will use the lower case v to denote the proportion of voters in the corresponding set. For instance, note that $\frac{k}{n} = v_C = v_I + v_{I^c}$. With this notation, we can observe that

$$p_j = \frac{\text{plu}(j) - v_j}{1 - v_C} = \frac{\text{plu}(j) - v_j}{1 - v_I - v_{I^c}}.$$

On the other hand, if $p_j > 0$ then no voter in the set $S_{I \succ j}$ decrements the score of a candidate in I . i.e., $s_{I \succ j} \leq 1 - v_I$. This tells us that

$$\sum_{j \notin I} s_{I \succ j} p_j \leq \frac{1 - v_I}{1 - v_I - v_{I^c}} \sum_{j \notin I} (\text{plu}(j) - v_j) = \frac{1 - v_I}{1 - v_I - v_{I^c}} \left(-v_{I^c} + \sum_{j \notin I} \text{plu}(j) \right).$$

Therefore, it suffices to show that

$$(1 - v_I) \left(-v_{I^c} + \sum_{j \notin I} \text{plu}(j) \right) \leq (1 - v_I - v_{I^c}) \sum_{j \notin I} \text{plu}(j).$$

Rearranging, we see that this is equivalent to

$$\sum_{j \notin I} \text{plu}(j) \leq 1 - v_I \iff v_I \leq \sum_{i \in I} \text{plu}(i).$$

This is of course true because $n \sum_{i \in I} \text{plu}(i)$ is the total initial score of the candidates in I , and $nv_I = |V_I|$ is the amount that these scores are decremented in the rule. \square

C.4 Random Dictatorship and its Variants

[AP17, FFG16] first showed that Random Dictatorship, which chooses candidate i with probability $\text{plu}(i)$, gets distortion 3. This can be proved quite easily using Eq. (4):

$$\sum_{j \notin I} s_{I \succ j} \text{plu}(j) \leq \sum_{j \notin I} \text{plu}(j) \leq 1 - s_{i^* \succ I^c}.$$

The last inequality follows because the set of voters where i^* is preferred over I^c is disjoint from the set of candidates that ranks $j \notin I$ first, for all j .

[GHS20] showed that Smart Dictatorship, which chooses candidate i with probability proportional to $\frac{\text{plu}(i)}{1 - \text{plu}(i)}$, has distortion at most $3 - 2/m$ within the class of instances with m candidates. [CR22] showed that this can also be proved using Eq. (4). Since $1 - \text{plu}(j) \geq s_{I \succ j}$, we have very similarly that

$$\sum_{j \notin I} s_{I \succ j} \frac{\text{plu}(j)}{1 - \text{plu}(j)} \leq \sum_{j \notin I} \text{plu}(j) \leq 1 - s_{i^* \succ I^c}.$$

This means that in Eq. (4), we can take $\frac{1}{\lambda} = \sum_i \frac{\text{plu}(i)}{1 - \text{plu}(i)}$, which is at least $\frac{1}{1 - \|\mathbf{plu}\|_2^2}$ by Jensen's inequality with the function $f(x) = \frac{1}{1-x}$. This gives us distortion $3 - 2\|\mathbf{plu}\|_2^2 \leq 3 - 2/m$.

Finally, we note that this approach can also give us stronger distortion guarantees in special cases of election instances. For instance, we can prove the following.

Theorem 13. *Within the class of instances where $s_{i \succ j} \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ for all i, j , Random Dictatorship guarantees distortion at most $2 + 2\varepsilon$.*

Proof. We have $s_{I \succ j} \leq \frac{1}{2} + \varepsilon$, and so

$$\sum_{j \notin I} s_{I \succ j} \text{plu}(j) \leq (\frac{1}{2} + \varepsilon) \sum_{j \notin I} \text{plu}(j) \leq (\frac{1}{2} + \varepsilon)(1 - s_{i^* \succ I^c}).$$

This means that in Eq. (4), we can take $\lambda = \frac{1}{2} + \varepsilon$, which corresponds to distortion $1 + 2(\frac{1}{2} + \varepsilon) = 2 + 2\varepsilon$. \square

D Proofs for the Biased Metrics

In this section, we will prove two key properties of the biased metrics: (1) that [Definition 1](#) actually defines a valid metric space, and (2) that these metrics are the hardest for the problem. These proofs are adapted from [\[CR22\]](#) with minor adjustments.

Proposition 12. *For any vector (x_1, \dots, x_m) of nonnegative real numbers where $x_{i^*} = 0$, and any election instance, the biased metric corresponding to (x_1, \dots, x_m) is indeed a valid distance metric.*

Proof. Clearly, the distances we have defined are nonnegative, since the expression for $d(i^*, v)$ allows for $i = j$, which means that it is the maximum over a set that includes 0, and for any $j \neq i^*$ we have $d(j, v) \geq d(i^*, v)$. Thus, it suffices to show that the metric satisfies the triangle inequality.

We can view the metric as a weighted graph, and in order to show that it satisfies the triangle inequality, we need to show that the weight of any edge is at most the sum of the weights of any path between the endpoints of the edge.

Suppose we have some path between a candidate j and voter v . We will show that $d(j, v)$ is at most the total weight of the path. Recall that $d(j, v) = d(i^*, v) + \min_{k: j \succeq_v k} x_k \leq d(i^*, v) + x_j$. Now, suppose that the first two edges on the path are (j, u) and (u, k) , and the last edge is (i, v) . Then the total length of the path is at least

$$d(j, u) + d(u, k) + d(i, v) \geq d(j, u) + d(i^*, u) + d(i^*, v).$$

Now, $d(j, u) = d(i^*, u) + x_\ell$ for some ℓ for which $j \succeq_u \ell$. By the definition of $d(i^*, u)$, we also have $2d(i^*, u) \geq x_j - x_\ell$. Putting all of these together, we have

$$d(j, u) + d(i^*, u) + d(i^*, v) \geq 2d(i^*, u) + x_\ell + d(i^*, v) \geq x_j + d(i^*, v) \geq d(j, v)$$

as desired. □

Proposition 13. *Suppose we have an election instance and a distance metric d that is consistent with the instance. Let $i^* = \arg \min_i \text{SC}(i, d)$. Then there is a biased metric \widehat{d} such that $\text{SC}(i^*, \widehat{d}) \leq \text{SC}(i^*, d)$ and $\text{SC}(j, \widehat{d}) - \text{SC}(i^*, \widehat{d}) \geq \text{SC}(j, d) - \text{SC}(i^*, d)$ for each $j \neq i^*$.*

In particular, this shows that for any election rule, the distortion of the rule is greater with \widehat{d} than with d . Thus, showing that a rule has low distortion on all biased metrics is sufficient to show that it has low distortion for all metrics.

Proof. Let $x_i = d(i, i^*)$, and let \widehat{d} be the biased metric for (x_1, x_2, \dots, x_m) . We will show that for any voter v , $\widehat{d}(i^*, v) \leq d(i^*, v)$ and $\widehat{d}(j, v) - \widehat{d}(i^*, v) \geq d(j, v) - d(i^*, v)$, which will immediately imply the proposition.

Fix v , and let i and j be such that $i \succeq_v j$. Then we have

$$d(i, i^*) \leq d(i, v) + d(i^*, v) \leq d(j, v) + d(i^*, v) \leq d(j, i^*) + 2d(i^*, v).$$

This implies that $d(i^*, v) \geq \frac{d(i, i^*) - d(j, i^*)}{2} = \frac{x_i - x_j}{2}$. Taking the maximum over all choices of i and j such that $i \succeq_v j$, we get

$$d(i^*, v) \geq \frac{1}{2} \max_{i, j: i \succeq_v j} (x_i - x_j) = \widehat{d}(i^*, v)$$

as claimed. Next, fix a candidate $j \neq i^*$ and a voter v . Let $k = \arg \min_{k: j \succeq_v k} x_k$, so that $\widehat{d}(j, v) - \widehat{d}(i^*, v) = x_k$. Then we have

$$d(j, v) \leq d(k, v) \leq d(k, i^*) + d(i^*, v) = x_k + d(i^*, v)$$

which means that

$$d(i, v) - d(i^*, v) \leq x_j = \widehat{d}(i, v) - \widehat{d}(i^*, v)$$

as claimed. This establishes the proposition. □