Simple and Asymptotically Optimal Online Bipartite Edge Coloring

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Abstract

We provide a simple online $\Delta(1 + o(1))$ -edge-coloring algorithm for bipartite graphs of maximum degree $\Delta = \omega(\log n)$ under adversarial vertex arrivals on one side of the graph. Our algorithm slightly improves the result of (Cohen, Peng and Wajc, FOCS19), which was the first, and currently only, to obtain an asymptotically optimal $\Delta(1 + o(1))$ guarantee for an adversarial arrival model. More importantly, our algorithm provides a new, simpler approach for tackling online edge coloring.

1 Introduction

Edge coloring is a classic problem in graph theory and algorithm design: Given a graph, assign colors to the edges, with no two adjacent edges sharing a color. Pioneering work by König [Kön16] and later Vizing [Viz64] showed that Δ and $\Delta + 1$ colors suffice for bipartite and general graphs of maximum degree Δ , respectively. (At least Δ colors are clearly needed.) Algorithms attaining or approximating these bounds were designed in numerous models of computation, including distributed [PS97, Chr23], parallel [KS87], dynamic [DHZ19, Chr23], and streaming algorithms [CL21, ASZZ22, CMZ23, GS23, BS23]. The latter includes several simple (asymptotically optimal) $\Delta(1 + o(1))$ -edge-coloring streaming algorithms for random-order streams [CL21, ASZZ22].

In contrast, online edge-coloring algorithms (especially for *adversarial order*) and their analyses are somewhat more involved [AMSZ03, BMM12, CPW19, BGW21, SW21, KLS⁺22, NSW23]. The only truly simple online edge coloring algorithm known is the trivial 2-approximate greedy algorithm, which is optimal only for the low-degree regime $\Delta = O(\log n)$ [BNMN92]. More involved algorithms were developed for the high-degree setting. For example, all known algorithms for $(\alpha + o(1))$ -approximate adversarial-order online edge coloring with $\alpha < 2$ for $\Delta = \omega(\log n)$ [CPW19, SW21, KLS⁺22, NSW23] rely on interleaved invocations of online matching subroutines that compute a matching that matches each edge *e* with probability at least $1/(\alpha\Delta)$.¹ The outer loop using such online matching algorithms, introduced by [CPW19], is not particularly complicated, and can be described and analyzed in about one page (see e.g., [SW21, Section 6]). However, the matching algorithms used within this framework and their analyses are quite non-trivial [CW18, CPW19, SW21, KLS⁺22, NSW23].

We break from the above template, avoiding this outer loop and subsequent complicated online matching subroutines. Instead, we obtain our results by a sequence of *offline* bipartite matching computations (more precisely, random sampling of matchings). This yields a simple asymptotically-optimal online edge coloring algorithm for the first (and so far only) adversarial arrival model for which positive results are known: one-sided vertex arrivals in bipartite graphs [CPW19]. Specifically, we prove the following.

Theorem 1.1 (See Theorem 2.4). There exists an online edge-coloring algorithm for the one-sided vertex arrival model with the following guarantee. On any *n*-node, maximum degree Δ bipartite graph, it computes a $(\Delta + q)$ -edge-coloring with high probability,² where $q = O(\Delta^{2/3} \log^{1/3} n)$.

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¹Such matchings can be obtained by sampling a color in an $\alpha\Delta$ coloring, so these problems are basically equivalent.

²By with high probability, we mean probability of at least $1 - n^{-c}$ for some constant c > 0.

The above theorem only gives non-trivial guarantees if $\Delta \ge q$ (i.e., when $\Delta = \Omega(\log n)$ is sufficiently large). Indeed, when $\Delta < q$, greedy already provides an edge coloring with $2\Delta - 1 \le \Delta + q$ colors.

Our simple online $\Delta(1 + o(1))$ -edge-coloring algorithm improves on the o(1) term of the algorithm of [CPW19], which uses $\Delta + O(\Delta^{3/4} \log^{1/4} n)$ colors if $\Delta = \omega(\log n)$. Moreover, our simpler algorithm nearly matches a lower bound of $\Delta + \Omega(\sqrt{\Delta})$ colors established in that prior paper. We leave the question of whether an algorithm (simple or otherwise) matching this lower bound's o(1) terms exists as an open problem.

2 Simple yet optimal online bipartite edge coloring

Problem statement. A bipartite graph of maximum degree $\Delta = \omega(\log n)$ is revealed.³ Initially, only n, Δ and the nodes on the *offline* side are known. At time t, the node w_t on the *online* side is revealed, together with its edges, which must be assigned colors immediately and irrevocably. The objective is to compute a valid edge coloring using as few colors as possible.

Our algorithm. We attempt to provide a valid $(\Delta + q)$ -edge-coloring, for $q = o(\Delta)$ to be chosen later. In particular, we will color edges of each offline node u with distinct colors, chosen uniformly at random from $C := [\Delta + q]$. To also color edges of each online node w_t with distinct colors, we correlate the random choices at different offline nodes as follows.

At each time t we consider a bipartite graph H_t with one side given by the set of neighbors $N_G(w_t)$ of the arriving online node w_t in G, and the other side being the set of colors C. The neighbor $v \in N_G(w_t)$ and color $c \in C$ are connected by an edge $cv \in H_t$ if and only if v has no edge colored c.⁴ To color the edges incident to the arriving node w_t in a valid manner, these edges must be given distinct colors and the color chosen for the edge $\{u, w_t\}$ must not already be used at the offline node u. These requirements correspond exactly to matchings in H_t . We thus attempt to sample a matching M_t in H_t where each edge $\{u, w_t\}$ is assigned a uniformly random available color of neighbor u. This can be achieved by a number of randomized rounding algorithms for the bipartite matching polytope, provided the desired marginal matching probabilities lie in this polytope. Fittingly, the crux of our analysis is show that the latter holds w.h.p. for $q = o(\Delta)$ sufficiently large. For simplicity of analysis, we allow for a low-probability "failure mode" if this condition fails, in which case we still insist on coloring offline nodes with colors uniformly at random, but without necessarily providing a valid edge coloring. Our pseudocode is given in Algorithm 1.

Algorithm 1.

At the arrival of online node w_t :

- Let H_t be a bipartite graph with node sets $N_G(w_t)$ and C, with $cv \in H_t$ iff v has no edge colored c (yet).
- For each $c \in C$ and $v \in V$, let $x_{cv}^t \leftarrow \frac{\mathbb{1}[cv \in H_t]}{\Delta d_t(v) + q}$, for $d_t(v)$ the degree of v by time t.
- If $\sum_{v} x_{cv}^t \leq 1$ for each color $c \in C$: sample matching M_t in H_t with marginals $\Pr[cv \in M_t] = x_{cv}^t$, and color each edge $\{v, w_t\}$ using the color c that is matched to v in M_t .
- Else (FAILURE MODE): color each edge $\{v, w_t\}$ with u.a.r. color $c \in N_{H_t}(v)$.

Observation 2.1. By definition, we always have $\sum_{c} x_{cv}^{t} = 1$ for a vertex $v \in N_{G}(w_{t})$. And so, if $\sum_{v} x_{cv}^{t} \leq 1$ for all colors $c \in C$, then the vector \vec{x}^{t} is in the bipartite matching polytope (of H_{t}), and a matching M_{t} as above can be sampled efficiently (and simply, [GKPS06]). In this case, all edges $\{v, w_{t}\}$ incident to w_{t} get colored at time t (since $\sum_{c} x_{cv}^{t} = 1$) and they all receive distinct colors from their endpoints' prior and other current edges (due to the definition of H_{t} and $\sum_{v} x_{cv}^{t} \leq 1$).

 $^{^{\}scriptscriptstyle 3}\!\operatorname{As}$ noted above, if Δ is smaller, the problem is solved optimally by the greedy algorithm.

⁴We use the notation $cv \in H_t$ instead of the more standard but notationally cumbersome $\{c, v\} \in E(H_t)$.

Analysis overview. We wish to show that the condition $\sum_{v} x_{cv}^{t} \leq 1$ for all times t and colors c, necessary to avoid the failure mode and output a valid edge coloring, occurs with high probability. For this, we prove two invariants in Lemma 2.2: we prove (1) a closed form for $\mathbb{E}[x_{cv}^t]$, implying $\mathbb{E}[\sum_{v} x_{cv}^t] \leq 1 - \Omega(q/\Delta)$. If for all t and c these x_{cv}^t were independent, standard Chernoff bounds would suffice to show that w.h.p., $\sum_{v} x_{cv}^t$ does not deviate much from its expectation, and in particular is at most one. As these variables may be dependent, we also prove (2) negative correlation of the random variables x_{cv}^t , allowing us to apply Chernoff-like bounds to these dependent variables and prove that the desired condition holds w.h.p., in Lemma 2.3.

Lemma 2.2. Let Z_{cv}^t be the indicator variable for color c not being used by edges of v when w_t arrives. At any time t, the following invariants hold:

• (Marginals) For any color $c \in C$ and offline node v, we have:

$$\Pr[Z_{cv}^t = 1] = \frac{\Delta - d_t(v) + q}{\Delta + q}.$$
(1)

• (Negative dependence) For any color $c \in C$ and offline nodes v_1, \ldots, v_k , we have:

$$\Pr\left[\bigwedge_{i\in[k]} (Z_{cv_i}^t = 1)\right] \leqslant \prod_{i\in[k]} \Pr[Z_{cv_i}^t = 1].$$
(2)

Proof. We prove both invariants by induction on $t \ge 1$. The base case t = 1 trivially holds for both. To prove both inductive steps, we first note that $\mathbb{1}[cv \in H_t] = Z_{cv}^t \cdot \mathbb{1}[v \in N_G(w^t)]$. So, the value of the random variable x_{cv}^t conditioned on any history up to time t implying $Z_{cv}^t = 1$ is precisely $\overline{x}_{cv}^t := \frac{\mathbb{1}[v \in N_G(w^t)]}{\Delta - d_t(v) + q}$. In particular, conditioning on any such history, the color c is used for edge $\{v, w_t\}$ with probability precisely \overline{x}_{cv}^t (also in the failure mode, and also if $v \notin N_G(w_t)$).

The first invariant's inductive step then follows from the above observation and the inductive hypothesis,by a routine calculation, as follows:

$$\Pr[Z_{cv}^{t+1} = 1] = (1 - \overline{x}_{cv}^t) \cdot \Pr[Z_{cv}^t = 1]$$

$$= \left(1 - \frac{\mathbb{1}[v \in N_G(w^t)]}{\Delta - d_t(v) + q}\right) \cdot \frac{\Delta - d_t(v) + q}{\Delta + q}$$

$$= \frac{\Delta - d_{t+1}(v) + q}{\Delta + q}.$$
(3)

For the second invariant's inductive step, we claim that for any history \mathcal{H} up to time t that implies $\bigwedge_{i \in [k]} (Z_{cv_i}^t = 1)$, we have that $\Pr\left[\bigwedge_{i \in [k]} (Z_{cv_i}^{t+1} = 1) \mid \mathcal{H}\right] \leq \prod_{i \in [k]} (1 - \overline{x}_{cv}^t)$. This inequality is clearly an equality for the failure mode, where colors are assigned independently; otherwise, the LHS equals $1 - \sum_{i \in [k]} \overline{x}_{cv_i}^t$, which is upper bounded by the RHS, where this standard inequality follows from the union bound. Therefore, by total probability over histories \mathcal{H} as above and the inductive hypothesis and Equation (3), we obtain the claimed statement:

$$\begin{split} \Pr\left[\bigwedge_{i\in[k]}(Z_{cv_i}^{t+1}=1)\right] &= \Pr\left[\bigwedge_{i\in[k]}(Z_{cv_i}^{t+1}=1) \ \middle| \bigwedge_{i\in[k]}(Z_{cv_i}^t=1)\right] \cdot \Pr\left[\bigwedge_{i\in[k]}(Z_{cv_i}^t=1)\right] \\ &\leqslant \prod_{i\in[k]}\left(1-\overline{x}_{cv}^t\right) \cdot \prod_{i\in[k]}\Pr[Z_{cv_i}^t=1] \\ &= \prod_{i\in[k]}\Pr[Z_{cv_i}^{t+1}=1]. \end{split}$$

Using these invariants, we now show that Algorithm 1 is unlikely to enter the failure mode. Lemma 2.3. If $q = 3\Delta^{2/3} \log^{1/3} n \leq \Delta$, then with high probability, for each time t and color $c \in C$

$$\sum_{v} x_{cv}^t \leqslant 1$$

Proof. Fix a time t and color c. Notice that $\mathbb{1}[cv \in H_t] = Z_{cv}^t \cdot \mathbb{1}[v \in N_G(w^t)]$, and hence $x_{cv}^t = \frac{Z_{cv}^t}{\Delta - d_t(v) + q}$ for all $v \in N_G(w_t)$ (and $x_{cv}^t = 0$ for all $v \notin N_G(w_t)$). For all $v \in N_G(w_t)$, define the random variables $Y_v := q \cdot x_{cv}^t = \frac{q}{\Delta - d_t(v) + q} \cdot Z_{cv}^t$. It suffices to prove that $\sum_v Y_v \leq q$ with high probability. This follows from a variant of Chernoff bounds, as follows.

First, by Invariant (2), because $Y_v \neq 0$ if and only if $Z_{cv}^t = 1$, we have that:

$$\Pr\left[\bigwedge_{v} (Y_v \neq 0)\right] \leqslant \prod_{v} \Pr\left[Y_v \neq 0\right].$$

For such weighted binary variables $Y_v \in \{0, \frac{q}{\Delta - d_t(v) + q}\}$, the above is equivalent to the definition of 1correlation in the sense of [PS97, Definition 3.1], namely $\mathbb{E}[\prod_{v \in U} Y_v] \leq \prod_{v \in U} \mathbb{E}[Y_v]$ for all $U \subseteq N_G(w^t)$. As shown in [PS97], this suffices to upper bound the moment-generating function of $\sum_v Y_v$ and derive strong tail bounds. In particular, by [PS97, Corollary 3.3], since we also have that $Y_v \in [0, 1]$ for all v, the following Chernoff bound holds for any $\varepsilon > 0$:

$$\Pr\left[\sum_{v} Y_{v} \ge (1+\varepsilon) \cdot \mathbb{E}\left[\sum_{v} Y_{v}\right]\right] \le \exp\left(-\frac{\varepsilon^{2} \cdot \mathbb{E}\left[\sum_{v} Y_{v}\right]}{2+\varepsilon}\right).$$
(4)

Next, by Invariant (1), $\mathbb{E}[Y_v] = \frac{q}{\Delta + q}$ for each node $v \in N_G(w_t)$. Hence, $\mathbb{E}[\sum_v Y_v] = \frac{kq}{\Delta + q}$, where $k := |N_G(w_t)| \leq \Delta$. By setting $\varepsilon := \frac{\Delta + q - k}{k}$ in the Chernoff bound (4) we obtain:

$$\Pr\left[\sum_{v \in N_G(w^t)} Y_v \ge q\right] = \Pr\left[\sum_{v \in N_G(w^t)} Y_v \ge \left(1 + \frac{\Delta + q - k}{k}\right) \cdot \frac{kq}{\Delta + q}\right]$$
$$\leqslant \exp\left(-\frac{(\Delta + q - k)^2}{k^2} \cdot \frac{kq}{\Delta + q} \cdot \frac{k}{\Delta + q + k}\right)$$
$$\leqslant \exp\left(-\frac{q^3}{2\Delta^2 + 3\Delta q + q^2}\right)$$
$$\leqslant \exp\left(-\frac{q^3}{6\Delta^2}\right).$$

Above, the second-to-last inequality follows because $\left(-\frac{(\Delta+q-k)^2 \cdot q}{(\Delta+q)(\Delta+q+k)}\right)$ is decreasing in $k \leq \Delta$, and the last inequality relies on $q \leq \Delta$ by the lemma's hypothesis. Thus, for our choice of q,

$$\Pr\left[\sum_{v} x_{cv}^{t} \ge 1\right] = \Pr\left[\sum_{v} Y_{v} \ge q\right] \le \frac{1}{n^{4.5}} \le \frac{1}{2n^{3}}.$$

The lemma then follows by union bounding over all n online nodes and at most 2n colors.

Combining Observation 2.1 and Lemma 2.3, we obtain our result.

Theorem 2.4. Algorithm 1 with $q = 3\Delta^{2/3} \log^{1/3} n \leq \Delta$ (i.e., if $\Delta \ge 81 \log n$) computes a $(\Delta + q)$ -edgecoloring of any *n*-node, maximum degree Δ bipartite graph with high probability.

Proof. By Lemma 2.3, the condition $\sum_{v} x_{cv}^t \leq 1$ holds for all time t and colors c with high probability, which by Observation 2.1 results in a valid edge coloring using $\Delta + q$ colors.

Remark 2.5. In Appendix A, using standard anti-concentration bounds, we also show that our analysis is tight, *i.e.*, that Algorithm 1 indeed requires $\Delta + \Omega(\Delta^{2/3} \log^{1/3} n)$ colors to work.

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References

- [AMSZ03] Gagan Aggarwal, Rajeev Motwani, Devavrat Shah, and An Zhu. Switch scheduling via randomized edge coloring. In Proceedings of the 44th Symposium on Foundations of Computer Science (FOCS), pages 502-512, 2003. [↑]1
- [ASZZ22] Mohammad Ansari, Mohammad Saneian, and Hamid Zarrabi-Zadeh. Simple streaming algorithms for edge coloring. In Proceedings of the 30th Annual European Symposium on Algorithms (ESA), 2022. ↑1
- [BGW21] Sayan Bhattacharya, Fabrizio Grandoni, and David Wajc. Online edge coloring algorithms via the nibble method. In Proceedings of the 32nd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 2830–2842, 2021. ↑1
- [BMM12] Bahman Bahmani, Aranyak Mehta, and Rajeev Motwani. Online graph edge-coloring in the random-order arrival model. *Theory of Computing*, 8(1):567–595, 2012. [↑]1
- [BNMN92] Amotz Bar-Noy, Rajeev Motwani, and Joseph Naor. The greedy algorithm is optimal for on-line edge coloring. *Information Processing Letters (IPL)*, 44(5):251–253, 1992. [↑]1
- [BS23] Soheil Behnezhad and Mohammad Saneian. Streaming edge coloring with asymptotically optimal colors. *arXiv preprint arXiv:2305.01714*, 2023. ↑1
- [Chr23] Aleksander Bjørn Grodt Christiansen. The power of multi-step vizing chains. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing (STOC)*, pages 1013–1026, 2023. [↑]1
- [CL21] Moses Charikar and Paul Liu. Improved algorithms for edge colouring in the w-streaming model. In *Proceedings of the 4th Symposium on Simplicity in Algorithms (SOSA)*, pages 181–183, 2021. ¹
- [CMZ23] Shiri Chechik, Doron Mukhtar, and Tianyi Zhang. Streaming edge coloring with subquadratic palette size. *arXiv preprint arXiv:2305.07090*, 2023. [↑]1
- [CPW19] Ilan Reuven Cohen, Binghui Peng, and David Wajc. Tight bounds for online edge coloring. In Proceedings of the 60th Symposium on Foundations of Computer Science (FOCS), pages 1–25, 2019. ↑1, ↑2
- [CW18] Ilan Reuven Cohen and David Wajc. Randomized online matching in regular graphs. In Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 960–979, 2018. ↑1
- [DHZ19] Ran Duan, Haoqing He, and Tianyi Zhang. Dynamic edge coloring with improved approximation. In Proceedings of the 30th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1937–1945, 2019. ↑1
- [GKPS06] Rajiv Gandhi, Samir Khuller, Srinivasan Parthasarathy, and Aravind Srinivasan. Dependent rounding and its applications to approximation algorithms. *Journal of the ACM (JACM)*, 53(3):324–360, 2006. ↑2
- [GS23] Prantar Ghosh and Manuel Stoeckl. Low-memory algorithms for online and w-streaming edge coloring. *arXiv preprint arXiv:2304.12285*, 2023. ↑1
- [KLS⁺22] Janardhan Kulkarni, Yang P Liu, Ashwin Sah, Mehtaab Sawhney, and Jakub Tarnawski. Online edge coloring via tree recurrences and correlation decay. In Proceedings of the 54th Annual ACM Symposium on Theory of Computing (STOC), pages 2958–2977, 2022. [↑]1
- [Kön16] Dénes König. Über graphen und ihre anwendung auf determinantentheorie und mengenlehre. Mathematische Annalen, 77(4):453-465, 1916. ↑1

- [KS87] Howard J. Karloff and David B. Shmoys. Efficient parallel algorithms for edge coloring problems. *J. Algorithms*, 8(1):39–52, 1987. [↑]1
- [KY15] Philip Klein and Neal E Young. On the number of iterations for dantzig–wolfe optimization and packing-covering approximation algorithms. SIAM Journal on Computing (SICOMP), 44(4):1154– 1172, 2015. ↑6
- [NSW23] Joseph (Seffi) Naor, Aravind Srinivasan, and David Wajc. Online dependent rounding schemes. arXiv preprint arXiv:2301.08680, 2023. [↑]1
- [PS97] Alessandro Panconesi and Aravind Srinivasan. Randomized distributed edge coloring via an extension of the chernoff–hoeffding bounds. SIAM Journal on Computing (SICOMP), 26(2):350–368, 1997. ↑1, ↑4
- [SW21] Amin Saberi and David Wajc. The greedy algorithm is not optimal for on-line edge coloring. In Proceedings of the 48th International Colloquium on Automata, Languages and Programming (ICALP), pages 109:1-109:18, 2021. ↑1
- [Viz64] Vadim G Vizing. On an estimate of the chromatic class of a p-graph. *Diskret analiz*, 3:25–30, 1964. ↑1

APPENDIX

A Tight example for our algorithm

In the following we show that the bound of $(\Delta + O(\Delta^{2/3} \log^{1/3} n))$ colors is tight for Algorithm 1, for a wide range of Δ superlogarithmic (and even polynomial) in n.

Lemma A.1. For any constant $r \ge 3$, there exists an infinite family of instances with n nodes and maximum degree $\Delta = \Theta(n^{1/r})$, on which Algorithm 1 run with $q = \frac{1}{6r^{1/3}} \cdot \Delta^{2/3} \log^{1/3} n$ fails to output a valid edge coloring with constant probability.

Proof. For all (sufficiently large) integer k, we let $\Delta := k - 1$ and construct an instance graph with $n := \max\{k, \lfloor k^{r-2} \rfloor\} \cdot (\Delta^2 + 1) \leq k^{r-2} \cdot (\Delta^2 + 1) \leq k^r$ many nodes. In the instance, $\Delta = k - 1 = \Theta(n^{1/r})$ is the maximum degree of any node in the instance. We turn to describing this instance.

A gadget consists of an online node w_t connected to Δ offline neighbors of degree $\Delta - 1$ (before w_t arrives), each of these belonging to disjoint subgraphs. Hence, for any color c these offline nodes are neighbors of c in H_t independently. Our instance consists of $\max\{k, \lfloor k^{r-2} \rfloor\} \ge k$ disjoint (hence independent) such gadgets, each having $\Delta^2 + 1$ nodes and therefore totaling n nodes.

We now fix the gadget corresponding to some w_t . Since the Δ neighbors v of w_t neighbor c independently in H_t , each with probability $\frac{\Delta - d_t(v) + q}{\Delta + q} = \frac{q+1}{\Delta + q}$ (by Invariant (1)), the number of neighbors of c in H_t is distributed as $X = \sum_{v \in N_G(w_t)} Z_{cv}^t \sim \text{Bin}(\Delta, \frac{q+1}{\Delta + q})$. Since all neighbors v of c in H_t have $d_t(v) = \Delta - 1$ and hence $x_{cv} = \frac{1}{q+1} \cdot Z_{cv}^t$, Algorithm 1 does not enter failure mode if and only if $|X| \leq q + 1$. We thus wish to lower bound

$$\Pr[X > q+1] = \Pr[X > (1+q/\Delta) \cdot \mathbb{E}[X]] \ge \Pr[X \ge (1+2q/\Delta) \cdot \mathbb{E}[X]].$$
(5)

Let $\epsilon := 2q/\Delta$. By [KY15, Lemma 4], for $\varepsilon < 1/2$ such that $\varepsilon^2 \cdot \mathbb{E}[X] \ge 3$ (as we shortly verify is the case here), we have the following asymptotic converse of Chernoff's bound:

$$\Pr\left[X \ge (1+\varepsilon) \cdot \mathbb{E}[X]\right] \ge \exp\left(-9\varepsilon^2 \cdot \mathbb{E}[X]\right).$$
(6)

To see that the required conditions for applying this inequality hold, first notice that $\varepsilon = \frac{2q}{\Delta} = O(\sqrt[3]{\log n/\Delta})$, and so $\varepsilon < 1/2$ for sufficiently large k (and hence for sufficiently large $\Delta = \Theta(n^{1/r}) >> \log n$). On the other hand, we have that for large enough k (and hence n):

$$\varepsilon^2 \cdot \mathbb{E}[X] = \frac{(2q)^2}{\Delta^2} \cdot \Delta \cdot \frac{q+1}{\Delta+q} \ge \frac{4q^2}{\Delta^2} \cdot \Delta \cdot \frac{q}{2\Delta} = \frac{2q^3}{\Delta^2} = \frac{1}{108r} \cdot \log n \ge 3.$$

Similarly, using that $n \leq k^r$, we have:

$$\varepsilon^2 \cdot \mathbb{E}[X] = \frac{(2q)^2}{\Delta^2} \cdot \Delta \cdot \frac{q+1}{\Delta+q} \leqslant \frac{4q^2}{\Delta^2} \cdot \Delta \cdot \frac{2q}{\Delta} = \frac{8q^3}{\Delta^2} = \frac{\log n}{27r} \leqslant \frac{\log k}{9}.$$
(7)

Combining the above, we obtain:

$$\Pr[X > q+1] \stackrel{(5)}{\geqslant} \Pr[X \ge (1+\epsilon) \cdot \mathbb{E}[X]] \stackrel{(6)}{\geqslant} \exp\left(-9\varepsilon^2 \cdot \mathbb{E}[X]\right) \stackrel{(7)}{\geqslant} \exp\left(-9 \cdot \frac{\log k}{9}\right) = \frac{1}{k}$$

Hence, Algorithm 1 enters failure mode on any fixed gadget with probability at least $\frac{1}{k}$. As the instance consists of $\max\{k, \lfloor k^{r-2} \rfloor\} \ge k$ many independent gadgets, the probability that the algorithm does *not enter* failure mode on *any* of them is upper bounded by a constant, $(1 - \frac{1}{k})^k \le 1/e$, or put otherwise $\Pr[\text{enter failure mode}] \ge 1 - 1/e$.

Now, condition on Algorithm 1 entering failure mode, and fix some time t and color $c \in C$ for which $\sum_{v} x_{cv}^t > 1$ (i.e., this is a witness for the algorithm entering failure mode). Then, by the preceding discussion, at least q+2 neighbors v of w_t in G are neighbors of c in H_t , where they all have have degree $\Delta - d_t(v) + 1 = q + 1$. Therefore, by the independent coloring in the failure mode, the probability that the algorithm fails in outputting a valid edge coloring since it assigns c to two or more edges of w_t is at least

$$\begin{aligned} \Pr[\text{fail} \mid \text{enter failure mode}] &\geq \Pr\left[\text{Bin}\left(q+2, \frac{1}{q+1}\right) \geqslant 2\right] \\ &= 1 - \left(1 - \frac{1}{q+1}\right)^{q+2} - \frac{q+2}{q+1} \cdot \left(1 - \frac{1}{q+1}\right)^{q+1} \\ &= 1 - \left(\frac{q}{q+1} + \frac{q+2}{q+1}\right) \cdot \left(1 - \frac{1}{q+1}\right)^{q+1} \\ &\geqslant 1 - 2/e. \end{aligned}$$

Consequently, Algorithm 1 fails with constant probability, at least (1 - 1/e)(1 - 2/e), as claimed.