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Saddlepoint Approximations for Expectations and an Application to CDO Pricing*

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Abstract. We derive two types of saddlepoint approximations for expectations in the form of $\mathbb{E}[(X-K)^+]$, where X is the sum of n independent random variables and K is a known constant. We establish error convergence rates for both types of approximations in the independently and identically distributed case. The approximations are further extended to cover the case of lattice variables. An application of the saddlepoint approximations to CDO pricing is presented.

Key words. saddlepoint approximation, expectation, lattice variables, price of collateralized debt obligations

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1. Introduction.

We consider the saddlepoint approximations of $\mathbb{E}[(X-K)^+]$, where X is the sum of n independent random variables X_i, $i=1,\dots,n$, and K is a known constant. This expectation is frequently encountered in finance and insurance. It plays an integral role in the pricing of collateralized debt obligations (CDOs) [16], [1]. In option pricing, $\mathbb{E}[(X-K)^+]$ is the payoff of a call option [14]. In insurance, $\mathbb{E}[(X-K)^+]$ is known as the stop-loss premium. The expectation is also closely connected to $\mathbb{E}[X|X \geq K]$, which corresponds to the expected shortfall, also known as the tail conditional expectation, of a credit or insurance portfolio. It plays an increasingly important role in risk management in financial and insurance institutions.

In this article we derive two types of saddlepoint expansions for the quantity $\mathbb{E}[(X-K)^+]$. The first type of approximation is based on Esscher tilting and the Edgeworth expansion. The resulting approximations confirm the results in [1], which are obtained by a different approach. Our contributions are as follows. (1) We have provided the rates of convergence for the approximation formulas in the i.i.d. (independently and identically distributed) case. (2) We present explicit saddlepoint approximations for the log-return model considered in [14] and [15]. With our formulas, only one saddlepoint needs to be computed, whereas the measure change approach employed in [14] and [15] requires the calculation of two saddlepoints. (3) We have also provided the corresponding saddlepoint approximations for lattice variables. The lattice case is largely ignored in the literature so far, even in applications where lattice variables are highly relevant, for example, the pricing of CDOs.

Our main contribution is the second type of saddlepoint approximations. They are derived following the approach in [11] and [5] where the Lugannani–Rice formula for tail probabilities was derived. The higher order version of the approximations distinguishes itself from all

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existing saddlepoint approximations by its remarkable simplicity, high accuracy, and fast convergence. The application of the approximations for lattice variables to the valuation of CDOs leads to almost exact results.

The two expectations we have discussed are related as follows:

$$\mathbb{E}[X|X \geq K] = \frac{\mathbb{E}[(X-K)^+]}{\mathbb{P}(X \geq K)} + K. \tag{1.1}$$

Also closely related functions are $\mathbb{E}[(K-X)^+]$ and $\mathbb{E}[X|X < K]$. The connections are well known, and we include them here only for completeness:

$$\begin{aligned} \mathbb{E}[(K-X)^+] &= \mathbb{E}[(X-K)^+] - \mathbb{E}[X] + K, \\ \mathbb{E}[X|X < K] &= \left(\mathbb{E}[X] - \mathbb{E}[X\mathbf{1}_{\{X \geq K\}}]\right) / \mathbb{P}(X < K). \end{aligned}$$

For simplicity of notation, we define

$$C := \mathbb{E}[(X-K)^+]. \tag{1.2}$$

The article is organized as follows. In section 2 we recall the saddlepoint approximations for densities and tail probabilities. Section 3 reviews the existing literature for calculating C and related quantities by the formulas in section 2. In sections 4 and 5 we derive two types of formulas for the saddlepoint approximations to C. Section 6 gives the corresponding formulas for the lattice variables. Numerical results are presented in section 7, including in particular an application to CDO pricing.

2. Densities and tail probabilities.

Dating back to [6], the saddlepoint approximation has been recognized as a valuable tool in asymptotic analysis and statistical computing. It has found a wide range of applications in finance and insurance, reliability theory, physics, and biology. The saddlepoint approximation literature so far mainly focuses on the approximation of densities [4] and tail probabilities [11], [5]. For a comprehensive exposition of saddlepoint approximations, see [9].

We start with some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X_i, $i = 1, \ldots, n$, be n i.i.d. continuous random variables all defined on the given probability space and $X = \sum_{i=1}^n X_i$. Suppose that the moment generating function (MGF) of X_1 is analytic and given by $M_1(t)$ for t in some open neighborhood of zero. The MGF of the sum X is then simply the product of the MGF of X_i, i.e.,

$$M(t) = (M_1(t))^n.$$

Let $\kappa(t) = \log M(t)$ be the cumulant generating function (CGF) of X. The density and tail probability of X can be represented by the following inversion formulas:

$$f_X(K) = \frac{1}{2\pi \mathtt{i}} \int_{\tau - \mathtt{i}\infty}^{\tau + \mathtt{i}\infty} \exp(\kappa(t) - tK)\mathrm{d}t, \tag{2.1}$$

$$\mathbb{P}(X \geq K) = \frac{1}{2\pi \mathtt{i}} \int_{\tau - \mathtt{i}\infty}^{\tau + \mathtt{i}\infty} \frac{\exp(\kappa(t) - tK)}{t}\mathrm{d}t \qquad (\tau > 0). \tag{2.2}$$

Throughout this paper we adopt the following notation:

- $f(n) = g(n)+O(h(n))$ means that $(f(n)-g(n))/h(n)$ is bounded as n approaches some limiting value. When appropriate, we delete the $O(h(n))$ term and write $f(n) \approx g(n)$, denoting $g(n)$ as an approximation to $f(n)$.
- $\phi(\cdot)$ and $\Phi(\cdot)$ denote, respectively, the probability density function (pdf) and cumulative distribution function (cdf) of a standard normal random variable.
- $\kappa_1(t) = \log M_1(t)$ is the CGF of X_1.
- $\mu := \mathbb{E}[X]$ and $\mu_1 = \mathbb{E}[X_1]$ are the expectation of X and X_1 under $\mathbb{P}$.
- T represents the saddlepoint that gives $\kappa_1'(T) = K/n$ or $\kappa'(T) = K$.
- $\lambda_r := \kappa^{(r)}(T)/\kappa''(T)^{r/2}$ is the standardized cumulant of order r evaluated at T, and $\lambda_{1,r} := \kappa_1^{(r)}(T)/\kappa_1''(T)^{r/2}$.
- $Z := T\sqrt{\kappa''(T)}$ and $Z_1 := T\sqrt{\kappa_1''(T)}$.
- $W := sgn(T)\sqrt{2[KT-\kappa(T)]}$ and $W_1 := sgn(T)\sqrt{2[KT/n-\kappa_1(T)]}$, with $sgn(T)$ being the sign of T.

It is obvious that $\mu = n\mu_1$, $Z = \sqrt{n}Z_1$, $W = \sqrt{n}W_1$, $\lambda_3 = \lambda_{1,3}/\sqrt{n}$, and $\lambda_4 = \lambda_{1,4}/n$.

In what follows we will write formulas in terms of X_1 (i.e., formulas with subscript 1 such as Z_1, W_1, etc.) when deriving the approximations and *studying the order* of the approximation errors. In fact the i.i.d. assumption is necessary only for the study of the error convergence rates. The approximations are, however, readily applicable when the random variables X_i are not identically distributed. For this reason, we should delete the error terms once the order of the approximation errors has been established, and write the formulas in terms of X (i.e., Z, W, etc.) for both generality and notational simplicity.

The saddlepoint approximation for densities is given by the Daniels [4] formula:

$$\begin{aligned} f_X(K) &= \phi(\sqrt{n}W_1)\frac{T}{\sqrt{n}Z_1}\left[1+\frac{1}{n}\left(\frac{\lambda_{1,4}}{8}-\frac{5\lambda_{1,3}^2}{24}\right)+O\left(n^{-2}\right)\right] \\ &\approx \phi(W)\frac{T}{Z}\left(1+\frac{\lambda_4}{8}-\frac{5\lambda_3^2}{24}\right) =: f_D. \end{aligned} \tag{2.3}$$

For tail probabilities, two types of distinct saddlepoint expansions exist. The first type of expansion is given by

$$\begin{aligned} \mathbb{P}(X \geq K) &= e^{\frac{n}{2}(Z_1^2-W_1^2)}[1-\Phi(\sqrt{n}Z_1)]\left[1+O\left(n^{-\frac{1}{2}}\right)\right] \\ &\approx e^{-\frac{W^2}{2}+\frac{Z^2}{2}}[1-\Phi(Z)] =: P_1, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \mathbb{P}(X \geq K) &= \left[P_1\left(1-\frac{n\lambda_{1,3}}{6}Z_1^3\right)+\phi(\sqrt{n}W_1)\frac{\lambda_{1,3}}{6\sqrt{n}}\left(nZ_1^2-1\right)\right]\left[1+O\left(n^{-1}\right)\right] \\ &\approx P_1\left(1-\frac{\lambda_3}{6}Z^3\right)+\phi(W)\frac{\lambda_3}{6}\left(Z^2-1\right) =: P_2 \end{aligned} \tag{2.5}$$

in the case $T \geq 0$. For $T < 0$ similar formulas are available; see [5]. The second type of expansion is obtained by [11], with

$$
\begin{aligned}
&(2.6)\\
&\mathbb{P}(X \geq K) = [1-\Phi(\sqrt{n}W_1)] + \phi(\sqrt{n}W_1)\left[\frac{1}{\sqrt{n}}\left(\frac{1}{Z_1}-\frac{1}{W_1}\right)+O\left(n^{-\frac{3}{2}}\right)\right]\\
&\qquad\qquad \approx 1-\Phi(W)+\phi(W)\left[\frac{1}{Z}-\frac{1}{W}\right] =: P_3,\\
&\mathbb{P}(X \geq K) = P_3 + \phi(\sqrt{n}W_1)\left\{n^{-\frac{3}{2}}\left[\frac{1}{Z_1}\left(\frac{\lambda_{1,4}}{8}-\frac{5\lambda_{1,3}^2}{24}\right)-\frac{\lambda_{1,3}}{2Z_1^2}-\frac{1}{Z_1^3}+\frac{1}{W_1^3}\right]+O\left(n^{-\frac{5}{2}}\right)\right\}\\
&(2.7)\qquad \approx P_3+\phi(W)\left[\frac{1}{Z}\left(\frac{\lambda_4}{8}-\frac{5\lambda_3^2}{24}\right)-\frac{\lambda_3}{2Z^2}-\frac{1}{Z^3}+\frac{1}{W^3}\right] =: P_4.
\end{aligned}
$$

Widely known as the Lugannani–Rice formula, P_3 is most popular among the four tail probability approximations for both simplicity and accuracy. A good review of saddlepoint approximations for the tail probability is given in [5].

3. Measure change approaches.

Before we derive the formulas for $\mathbb{E}[(X-K)^+]$, we would like to briefly review an existing approach to approximating the quantity. Usually the saddlepoint expansions for densities or tail probabilities are employed after a suitable change of measure.

An inversion formula similar to those for densities and tail probabilities also exists for $\mathbb{E}[(X-K)^+]$, which is given by

$$
\mathbb{E}\left[(X-K)^+\right] = \frac{1}{2\pi \mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty}\frac{\exp(\kappa(t)-tK)}{t^2}\mathrm{d}t \qquad (\tau>0). \tag{3.1}
$$

The authors of [16] rewrite the inversion formula to be

$$
\begin{aligned}
\mathbb{E}\left[(X-K)^+\right] &= \frac{1}{2\pi \mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty}\exp(\kappa(t)-\log t^2-tK)\mathrm{d}t\\
&= \frac{1}{2\pi \mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty}\exp(\tilde{\kappa}(t)-tK)\mathrm{d}t,
\end{aligned} \tag{3.2}
$$

where $\tilde{\kappa}(t)=\kappa(t)-\log t^2$. The right-hand side of (3.2) is then in the form of (2.1), and the Daniels formula (2.3) can be used for approximation. It should be pointed out, however, that in this case two saddlepoints always exist.

This approach is selected as a competitor to our approximation formulas later in our numerical experiments.

Bounded random variables. [15] considers the approximation of the expected shortfall, in two models of the associated random variable. The first model deals with bounded random variables. Without loss of generality, we consider only the case in which X has a nonnegative lower bound. Define the probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ by $\mathbb{Q}(A)=\int_A X/\mu \mathrm{d}\mathbb{P}$ for $A\in\mathcal{F}$;

then

$$\begin{aligned}\mathbb{E}[X|X \geq K] &= \frac{1}{\mathbb{P}(X \geq K)} \int_{\{X \geq K\}} X \mathrm{d}\mathbb{P} = \frac{\mu}{\mathbb{P}(X \geq K)} \int_{\{X \geq K\}} \frac{X}{\mu} \mathrm{d}\mathbb{P} \\ &= \frac{\mu}{\mathbb{P}(X \geq K)} \mathbb{Q}(X \geq K). \end{aligned} \tag{3.3}$$

Hence the expected shortfall is transformed to be a multiple of the ratio of two tail probabilities. The MGF of X under probability $\mathbb{Q}$ is given by

$$M_{\mathbb{Q}}(t) = \int e^{tX} \frac{X}{\mu} \mathrm{d}\mathbb{P} = \frac{M'(t)}{\mu} = \frac{M(t)\kappa'(t)}{\mu}$$

as $\kappa'(t) = [\log M(t)]' = M'(t)/M(t)$. It follows that

$$\kappa_{\mathbb{Q}}(t) = \log M_{\mathbb{Q}}(t) = \kappa(t) + \log\big(\kappa'(t)\big) - \log(\mu). \tag{3.4}$$

For more general cases, see [15, section 2.6.2].

The saddlepoint approximation for tail probabilities can be applied for both probabilities $\mathbb{P}$ and $\mathbb{Q}$ in (3.3). A disadvantage of this approach is that two saddlepoints need to be determined, as the saddlepoints under the two probability measures are generally different.

Log-return model. The second case in [15] deals with $\mathbb{E}[e^X|X \geq K]$ rather than with $\mathbb{E}[X|X \geq K]$. The expected shortfall $\mathbb{E}[e^X|X \geq K]$ can also be written as a multiple of the ratio of two tail probabilities. Define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by $\mathbb{Q}(A) = \int_A e^X/M(1)\mathrm{d}\mathbb{P}$ for $A \in \mathcal{F}$; then

$$\begin{aligned}\mathbb{E}[e^X|X \geq K] &= \frac{1}{\mathbb{P}(X \geq K)} \int_{\{X \geq K\}} e^X \mathrm{d}\mathbb{P} = \frac{M(1)}{\mathbb{P}(X \geq K)} \int_{\{X \geq K\}} \frac{e^X}{M(1)} \mathrm{d}\mathbb{P} \\ &= \frac{M(1)}{\mathbb{P}(X \geq K)} \mathbb{Q}(X \geq K). \end{aligned} \tag{3.5}$$

The MGF and CGF of X under probability $\mathbb{Q}$ are given by

$$\begin{aligned} M_{\mathbb{Q}}(t) &= \int e^{tX} \frac{e^X}{M(1)} \mathrm{d}\mathbb{P} = \frac{M(t+1)}{M(1)}, \\ \kappa_{\mathbb{Q}}(t) &= \kappa(t+1) - \kappa(1). \end{aligned}$$

This also forms the basis for the approach used in [14] for option pricing where the log-price process follows a Lévy process. Just like the case of bounded random variables, two saddlepoints need to be determined for the expectation.

4. Classical saddlepoint approximations. In the sections to follow we give, in the spirit of [5], two types of explicit saddlepoint approximations for $\mathbb{E}[(X-K)^+]$. For each type of approximation, we give a lower order and a higher order version. The approximations to $\mathbb{E}[X|X \geq K]$ then simply follow from (1.1). In contrast to [15] and [14], no measure change is required, and only one saddlepoint needs to be computed.

Following [9], we call this first type of approximation the *classical saddlepoint approximation*. Approximation formulas for $\mathbb{E}[(X-K)^+]$ of this type already appeared in [1], however without any discussion of the error terms. The formulas are obtained by means of application of the saddlepoint approximation to (3.1), i.e., on the basis of the Taylor expansion of $\kappa(t)-tK$ around $t=T$. Here we provide a statistically oriented derivation that employs Esscher tilting and the Edgeworth expansion. Rates of convergence for the approximations are readily available with our approach in the i.i.d. case. Another advantage of our approach is that it leads to explicit saddlepoint approximations in the log-return model from [15], which is not possible with the approach in [1].

For now we assume that the saddlepoint $t=T$ which solves $\kappa'(t)=K$ is positive. The expectation $\mathbb{E}[(X-K)^+]$ is reformulated under an exponentially tilted probability measure,

$$\mathbb{E}\left[(X-K)^+\right]=\int_K^\infty (x-K)f(x)\mathrm{d}x = e^{-\frac{nW_1^2}{2}}\int_K^\infty (x-K)e^{-T(x-K)}\tilde{f}(x)\mathrm{d}x, \tag{4.1}$$

where $\kappa'(T)=K$ and $\tilde{f}(x)=f(x)\exp(Tx-\kappa(T))$. The same exponential tilting is also applied in [13], [5] for the approximation of tail probabilities.

The MGF associated with $\tilde{f}(x)$ is given by $\tilde{M}(t)=M(T+t)/M(T)$. It immediately follows that the mean and variance of a random variable $\tilde{X}$ with density $\tilde{f}(\cdot)$ are given by $\mathbb{E}\tilde{X}=K$ and $Var(\tilde{X})=\kappa''(T)=n\kappa_1''(T)$. By writing $\xi=(x-K)/\sqrt{n\kappa_1''(T)}$ and $\tilde{f}(x)dx=g(\xi)d\xi$, we find that (4.1) reads

$$\mathbb{E}\left[(X-K)^+\right]=e^{-\frac{nW_1^2}{2}}\sqrt{n\kappa_1''(T)}\int_0^\infty \xi e^{-\sqrt{n}Z_1\xi}g(\xi)\mathrm{d}\xi. \tag{4.2}$$

For ξ with a density function, $g(\xi)$ can be approximated uniformly by a normal distribution such that $g(\xi)=\phi(\xi)[1+O(n^{-\frac{1}{2}})]$. The integral in (4.2) then becomes

$$\begin{aligned}\int_0^\infty \xi e^{-\sqrt{n}Z_1\xi}g(\xi)\mathrm{d}\xi &= \int_0^\infty \xi e^{-\sqrt{n}Z_1\xi}\phi(\xi)\left[1+O\left(n^{-\frac{1}{2}}\right)\right]\mathrm{d}\xi\\ &=\frac{\exp(\frac{nZ_1^2}{2})}{\sqrt{2\pi}}\int_0^\infty \xi e^{-\frac{(\xi+\sqrt{n}Z_1)^2}{2}}\mathrm{d}\xi\left[1+O\left(n^{-\frac{1}{2}}\right)\right]\\ &=\left\{\frac{1}{\sqrt{2\pi}}-\sqrt{n}Z_1e^{\frac{nZ_1^2}{2}}\left[1-\Phi(\sqrt{n}Z_1)\right]\right\}\left[1+O\left(n^{-\frac{1}{2}}\right)\right]. \end{aligned}\tag{4.3}$$

Inserting (4.3) into (4.2) leads to the following approximation:

$$\begin{aligned}&\mathbb{E}\left[(X-K)^+\right]\\ &=e^{-\frac{nW_1^2}{2}}\left\{\sqrt{\frac{n\kappa_1''(T)}{2\pi}}-Tn\kappa_1''(T)e^{\frac{nZ_1^2}{2}}\left[1-\Phi(\sqrt{n}Z_1)\right]\right\}\left[1+O\left(n^{-\frac{1}{2}}\right)\right].\end{aligned}\tag{4.4}$$

By deleting the error term in (4.4) and representing the remaining terms in quantities related to X, we obtain the following approximation:

$$\mathbb{E}\left[(X-K)^+\right]\approx e^{-\frac{W^2}{2}}\left\{\sqrt{\frac{\kappa''(T)}{2\pi}}-T\kappa''(T)e^{\frac{Z^2}{2}}\left[1-\Phi(Z)\right]\right\}=:C_1. \tag{4.5}$$

Higher order terms enter if $g(\xi)$ is approximated by its Edgeworth expansion, e.g., $g(\xi) = \phi(\xi)[1 + \frac{\lambda_{1,3}}{6\sqrt{n}}(\xi^3 - 3\xi) + O(n^{-1})]$. Then

$$
\begin{aligned}
&\mathbb{E}\left[(X-K)^+\right] \\
&= C_1\left[1+O(n^{-1})\right] + e^{-\frac{nW_1^2}{2}}\sqrt{\kappa_1''(T)}\frac{\lambda_{1,3}}{6}\int_0^\infty \xi e^{-Z\xi}\phi(\xi)(\xi^3-3\xi)\mathrm{d}\xi \\
&= C_1\left[1+O(n^{-1})\right] + e^{-\frac{nW_1^2}{2}}\sqrt{\kappa_1''(T)}\frac{\lambda_{1,3}}{6}\frac{e^{\frac{Z^2}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{(\xi+Z)^2}{2}}\left(-\xi^4+3\xi^2\right)\mathrm{d}\xi \\
&= C_1\left[1+O(n^{-1})\right] + e^{\frac{n}{2}(Z_1^2-W_1^2)}\sqrt{\kappa_1''(T)}\frac{\lambda_{1,3}}{6} \\
&\quad \times\left\{[1-\Phi(\sqrt{n}Z_1)](n^2Z_1^4+3nZ_1^2) - \phi(\sqrt{n}Z_1)(n^{\frac{3}{2}}Z_1^3+2\sqrt{n}Z_1)\right\}.
\end{aligned} \tag{4.6}
$$

Deleting the error term in (4.6), we get the higher order version of the approximation as

$$
C_2 := C_1 + e^{\frac{Z^2}{2}-\frac{W^2}{2}}\sqrt{\kappa''(T)}\frac{\lambda_3}{6}\left\{[1-\Phi(Z)](Z^4+3Z^2) - \phi(Z)(Z^3+2Z)\right\}. \tag{4.7}
$$

The approximations C_1 and C_2 are in agreement with the formulas given by [1].

Negative saddlepoint. We have assumed that the saddlepoint is positive when deriving C_1 and C_2 in (4.5) and (4.7), or, in other words, $\mu < K$. If the saddlepoint T equals 0, or equivalently $\mu = K$, it is straightforward to see that C_1 and C_2 both reduce to the following formula:

$$
\mathbb{E}[(X-\mu)^+] = \sqrt{\frac{\kappa''(0)}{2\pi}} =: C_0. \tag{4.8}
$$

In the case that $\mu > K$, we should work with $Y = -X$ and $\mathbb{E}[Y\mathbf{1}_{\{Y\ge -K\}}]$ instead since

$$
\mathbb{E}[X\mathbf{1}_{\{X\ge K\}}] = \mu + \mathbb{E}[-X\mathbf{1}_{\{-X\ge -K\}}] = \mu + \mathbb{E}[Y\mathbf{1}_{\{Y\ge -K\}}].
$$

The CGF of Y is given by $\kappa_Y(t) = \kappa_X(-t)$. The saddlepoint that solves $\kappa_Y(t) = -K$ is $-T > 0$, so that C_1 and C_2 can be applied to Y. Note that

$$
\kappa_Y^{(r)}(t) = (-1)^r\kappa_X^{(r)}(-t),
$$

where the superscript (r) denotes the rth derivative. Transforming back to X, we find the following saddlepoint approximation to $\mathbb{E}[(X-K)^+]$ in the case of a negative saddlepoint:

$$
C_1^- = \mu - K + e^{-\frac{W^2}{2}}\left\{\sqrt{\kappa''(T)/(2\pi)} + T\kappa''(T)e^{\frac{Z^2}{2}}\Phi(Z)\right\}, \tag{4.9}
$$

$$
C_2^- = C_1^- - e^{\frac{Z^2}{2}-\frac{W^2}{2}}\sqrt{\kappa''(T)}\frac{\lambda_3}{6}\left\{\Phi(Z)(Z^4+3Z^2) + \phi(Z)(Z^3+2Z)\right\}. \tag{4.10}
$$

Log-return model revisited. We now show how to deal with the log-return model in [15] without dealing with two probability measures simultaneously. We work with $\mathbb{E}\left[e^X \mathbf{1}_{\{X\geq K\}}\right]$ which equals $\mathbb{E}\left[e^X|X\geq K\right]\mathbb{P}(X\geq K)$. Replace x in (4.1) by e^x and make the same change of variables,

$$\mathbb{E}\left[e^X \mathbf{1}_{\{X\geq K\}}\right] = e^{-\frac{W^2}{2}}\int_0^\infty e^{K+\xi\sqrt{n\kappa''(T)}}e^{-Z\xi}g(\xi)\mathrm{d}\xi.$$

After approximating $g(\xi)$ by the standard normal density, we obtain

$$\begin{aligned}\mathbb{E}\left[e^X \mathbf{1}_{\{X\geq K\}}\right] &\approx e^{-\frac{W^2}{2}+K+\frac{\dot{Z}^2}{2}}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{(\xi+\dot{Z})^2}{2}}\mathrm{d}\xi \\ &= e^{-\frac{W^2}{2}+K+\frac{\dot{Z}^2}{2}}[1-\Phi(\dot{Z})], \qquad (4.11)\end{aligned}$$

where $\dot{Z} = (T-1)\sqrt{\kappa''(T)}$. Equation (4.11) is basically $e^K P_1$, where P_1 is given by (2.4), with Z replaced by $\dot{Z}$. It is easy to verify that this approximation is exact when X is normally distributed. A higher order approximation would be

$$\mathbb{E}\left[e^X \mathbf{1}_{\{X\geq K\}}\right] \approx e^{-\frac{W^2}{2}+K+\frac{\dot{Z}^2}{2}}\left\{[1-\Phi(\dot{Z})]\left(1-\frac{\lambda_3}{6\sqrt{n}}\dot{Z}^3\right)+\frac{\lambda_3}{6\sqrt{n}}\phi(\dot{Z})(\dot{Z}^2-1)\right\}.$$

5. The Lugannani–Rice-type formulas.

The second type of saddlepoint approximations to $\mathbb{E}[(X-K)^+]$ can be obtained with the same change of variable as was employed in section 4 of [5], where the Lugannani–Rice formula for tail probability was derived. As a result we shall call the obtained formulas *Lugannani–Rice-type formulas*. In this section we derive the approximation formulas by means of the Laurent expansion, without the analysis of the rates of error convergence in the i.i.d. case. An alternative (lengthy) derivation, *including the analysis of the convergence*, is presented in an appendix.

We look at $K = nx$ for fixed x and let $\kappa_1'(T) = x$, so that $\kappa'(T) = n\kappa_1'(T) = nx = K$. We follow the Bleistein approach employed in [5] to approximate $\kappa_1(t) - tx$ over an interval containing both $t = 0$ and $t = T$ by a quadratic function. Here, T need not be positive any more. Since $nx = K$ we have $-\frac{1}{2}W_1^2 = \kappa_1(T) - Tx$, with W_1 taking the same sign as T. Let w be defined between 0 and W_1 such that

$$\frac{1}{2}(w-W_1)^2 = \kappa_1(t) - tx - \kappa_1(T) + Tx. \qquad (5.1)$$

Then we have

$$\frac{1}{2}w^2 - W_1 w = \kappa_1(t) - t\kappa_1'(T), \qquad (5.2)$$

and $t = 0 \Leftrightarrow w = 0$, $t = T \Leftrightarrow w = W_1$. Differentiate both sides of (5.2) once and twice to obtain

$$w\frac{\mathrm{d}w}{\mathrm{d}t} - W_1\frac{\mathrm{d}w}{\mathrm{d}t} = \kappa_1'(t) - \kappa_1'(T), \qquad \left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 + (w-W_1)\frac{d^2w}{\mathrm{d}t^2} = \kappa_1''(t).$$

In the neighborhood of $t = T$ (or, equivalently, $w = W_1$), we have $\frac{\mathrm{d}w}{\mathrm{d}t} = \sqrt{\kappa_1''(T)}$. Note that $\mu_1 = \mathbb{E}[X_1] = \kappa_1'(0)$. In the neighborhood of $t = 0$ (or, equivalently, $w = 0$), we have

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \sqrt{\kappa_1''(0)} \quad \text{if } T = 0, \qquad (5.3)$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\kappa_1'(T) - \kappa_1'(0)}{W_1} = \frac{x-\mu_1}{W_1} \quad \text{if } T \neq 0.$$

Hence, in the neighborhood of $t = 0$ we have $w \propto t$. Moreover,

$$\frac{1}{t}\frac{\mathrm{d}t}{\mathrm{d}w} \sim \frac{1}{w}, \qquad \frac{\kappa_1'(t)}{t}\frac{\mathrm{d}t}{\mathrm{d}w} \sim \frac{\mu_1}{w}. \tag{5.4}$$

The inversion formula for $\mathbb{E}[(X-K)^+]$ can then be formulated as

$$\mathbb{E}\left[(X-K)^+\right] = \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} e^{n(\frac{1}{2}w^2 - W_1 w)}\frac{1}{t^2}\frac{\mathrm{d}t}{\mathrm{d}w}\mathrm{d}w \quad (\tau > 0). \tag{5.5}$$

Taking the first three terms of the Laurent expansion of $\frac{1}{t^2}\frac{dt}{dw}$ at $w = 0$ gives

$$\frac{1}{t^2}\frac{dt}{dw} \approx A_1 w^{-2} + A_2 w^{-1} + A_3, \tag{5.6}$$

where

$$A_1 = \frac{1}{2\pi\mathtt{i}}\oint_\gamma \frac{1}{t^2}\frac{\mathrm{d}t}{\mathrm{d}w}\frac{\mathrm{d}w}{w^{-1}} = \frac{1}{2\pi\mathtt{i}}\oint_\gamma \frac{w}{t^2}\mathrm{d}t, \tag{5.7}$$

$$A_2 = \frac{1}{2\pi\mathtt{i}}\oint_\gamma \frac{1}{t^2}\frac{\mathrm{d}t}{\mathrm{d}w}\mathrm{d}w = \frac{1}{2\pi\mathtt{i}}\oint_\gamma \frac{1}{t^2}\mathrm{d}t. \tag{5.8}$$

The path of integration, γ, traces out a circle around 0 in a counterclockwise manner. Since $\frac{w}{t^2}$ and $\frac{1}{t^2}$ have poles of order 1 and 2 at $t = 0$, respectively, we obtain

$$A_1 = \lim_{t\to 0} t\frac{w}{t^2} = w'(0) = \frac{x-\mu_1}{W_1}, \tag{5.9}$$

$$A_2 = \lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}t^2\frac{1}{t^2} = 0. \tag{5.10}$$

A_3 can now be chosen such that the approximation (5.6) is exact at T, where we have $\frac{dw}{dt} = \sqrt{\kappa_1''(T)}$. This leads to

$$A_3 = \frac{1}{T^2\sqrt{\kappa_1''(T)}} - \frac{(x-\mu_1)}{W_1}W_1^{-2} = \frac{1}{TZ_1} - \frac{(x-\mu_1)}{W_1^3}. \tag{5.11}$$

We substitute (5.6) into (5.5) to get

$$\begin{aligned}&\mathbb{E}\left[(X-K)^+\right] \\ &\approx \frac{A_1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} e^{n(\frac{1}{2}w^2 - W_1 w)}\frac{\mathrm{d}w}{w^2} + \frac{A_3}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} e^{n(\frac{1}{2}w^2 - W_1 w)}\mathrm{d}w.\end{aligned} \tag{5.12}$$

After yet another change of variables, $y = \sqrt{n}w$, the first term becomes

$$\frac{A_1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} e^{n(\frac{1}{2}w^2 - W_1 w)}\frac{\mathrm{d}w}{w^2} = A_1\sqrt{n}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty}\frac{1}{2\pi\mathtt{i}}e^{\frac{1}{2}y^2 - \sqrt{n}W_1 y}\frac{\mathrm{d}y}{y^2}. \tag{5.13}$$

The integral in (5.13) is precisely the inversion formula of $\mathbb{E}(Y-W)^+$, where Y is a standard Gaussian distributed variable. By basic calculus we find

$$\mathbb{E}(Y-W)^+ = \phi(W) - W[1-\Phi(W)]. \tag{5.14}$$

The second term in (5.12) is given by

$$\begin{aligned}\frac{A_3}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} e^{n(\frac{1}{2}w^2-W_1 w)}\mathrm{d}w &= \frac{A_3}{\sqrt{n}2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} e^{\frac{1}{2}(y-\sqrt{n}W_1)^2}\mathrm{d}y e^{-\frac{1}{2}nW_1^2}\\ &= \frac{A_3}{\sqrt{2\pi n}}e^{-\frac{1}{2}nW_1^2} = \frac{A_3}{\sqrt{n}}\phi(W).\end{aligned} \tag{5.15}$$

Adding up (5.13) and (5.15), we obtain the higher order version of the Lugannani–Rice-type saddlepoint approximation to the expectation $\mathbb{E}\left[(X-K)^+\right]$,

$$C_4 := (\mu-K)\left[1-\Phi(W)-\frac{\phi(W)}{W}\right]+\phi(W)\left[\frac{1}{TZ}+(\mu-K)\frac{1}{W^3}\right]. \tag{5.16}$$

This is a very compact approximation formula that involves only $\kappa''(T)$ and no cumulants of higher order. In this sense the complexity of the calculation of C_4 is comparable to that of C_1.

In the appendix we will show, however, that the order of error convergence of C_4 is $O\left(n^{-\frac{5}{2}}\right)$. A lower order version of the approximation, which we will denote by C_3, is given by

$$C_3 := (\mu-K)\left[1-\Phi(W)-\frac{\phi(W)}{W}\right]. \tag{5.17}$$

C_3 is an extremely neat formula requiring only the knowledge of W. More precisely, we don't need to compute $\kappa''(T)$. The order of error convergence of C_3 is shown to be $O\left(n^{-\frac{3}{2}}\right)$.

Remark 1. Interestingly, [12] gives an approximation formula for $\mathbb{E}[(X-K)^+]$, decomposing the expectation to one term involving the tail probability and another term involving the probability density,

$$\mathbb{E}\left[(X-K)^+\right] \approx (\mu-K)\mathbb{P}(X\geq K)+\frac{K-\mu}{T}f_X(K).$$

Martin [12] suggests approximating $\mathbb{P}(X\geq K)$ by the Lugannani–Rice formula P_3 in (2.6), and $f_X(K)$ by the Daniels formula f_D in (2.3). In the i.i.d. case, this leads to an approximation $C_M := n(\mu_1-x)P_3+n(x-\mu_1)f_D/T$ with a rate of convergence $n^{-1/2}$, as the first term has an error of order $n^{-1/2}$ and the second term has an error of order $n^{-3/2}$. We propose replacing P_3 by its higher order version, P_4, in (2.7). This gives the following formula:

$$\mathbb{E}\left[(X-K)^+\right] \approx C_3+(\mu-K)\phi(W)\left(\frac{1}{W^3}-\frac{\lambda_3}{2Z^2}-\frac{1}{Z^3}\right). \tag{5.18}$$

Equation (5.18) is simpler than C_M as λ_4 is not included. It has a rate of convergence of order $n^{-3/2}$. However, compared to C_4, (5.18) contains a term of λ_3 and is certainly more complicated to evaluate. Note further that if we neglect in C_M the terms of the higher order standard cumulants λ_3 and λ_4 in f_D, we get precisely C_3 as given in (5.17). For these reasons, C_4 is to be preferred.

Zero saddlepoint. Daniels [5] noted that if the saddlepoint equals $T = 0$, or, in other words, $\mu = K$, the approximations to tail probability P_1 to P_4 all reduce to

$$\mathbb{P}(X \geq K) = \frac{1}{2} - \frac{\lambda_3(0)}{6\sqrt{2\pi}}.$$

We would like to show that, under the same circumstances, C_3 and C_4 also reduce to the formula C_0 in (4.8). To show that $C_3 \equiv C_0$ when $T = 0$, we point out that

$$\lim_{T\to 0} C_3 = \lim_{T\to 0} \frac{\kappa'(0) - \kappa'(T)}{T}\left[T(1-\Phi(W)) - \phi(W)\frac{T}{W}\right].$$

Note that when $T \to 0$, $\frac{\kappa'(0)-\kappa'(T)}{T} \to -\kappa''(0)$, $T(1-\Phi(W)) \to 0$, and $\frac{T}{W} \to [\kappa''(0)]^{-\frac{1}{2}}$ (see (5.3)). This implies that $\lim_{T\to 0} C_3 = C_0$. Similarly we also have $\lim_{T\to 0} C_4 = C_0$.

6. Lattice variables.

So far we have considered approximations to continuous variables. Let us now turn to the lattice case. This case is largely ignored in the literature, even in applications in which lattice variables are highly relevant. For example, in the pricing of CDOs, the random variable concerned is essentially the number of defaults in the pool of companies and is thus discrete.

Suppose that $\hat{X}$ takes only integer values k with nonzero probabilities $p(k)$. The inversion formula of $\mathbb{E}[(\hat{X} - K)^+]$ can then be formulated as

$$\begin{aligned}\mathbb{E}[(\hat{X} - K)^+] &= \sum_{k=K+1}^{\infty}(k-K)p(k) = \sum_{k=K+1}^{\infty}(k-K)\frac{1}{2\pi \mathtt{i}}\int_{\tau-\mathtt{i}\pi}^{\tau+\mathtt{i}\pi}\exp(\kappa(t)-tk)\mathrm{d}t\\ &= \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\pi}^{\tau+\mathtt{i}\pi}\exp(\kappa(t)-tK)\sum_{m=1}^{\infty} m e^{-tm}\mathrm{d}t\\ &= \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\pi}^{\tau+\mathtt{i}\pi}\frac{\exp(\kappa(t)-tK)}{t^2}\frac{t^2e^{-t}}{(1-e^{-t})^2}\mathrm{d}t \qquad (\tau > 0).\end{aligned}$$

For $K > \mu$ we proceed by expanding the two terms in the integrand separately. According to a truncated version of Watson's lemma [10, see Lemmas 4.5.1 and 4.5.2], for an integrand in the form of $\exp(\frac{n\alpha}{2}(t-T)^2)\sum_{j=0}^{\infty}(t-T)^j$ the change in the contour of integration for t from $\tau \pm \mathtt{i}\infty$ to $\tau \pm \mathtt{i}\pi$ leads to a negligible difference which is exponentially small in n. The authors of [2] declare further that the integral over the range $\tau + \mathtt{i}y$, where $|y| > \log n/\sqrt{n}$, is negligible. This means that we are able to incorporate the formulas for continuous variables C_1 and C_2 into the approximations for the lattice variables. We find, for lattice variables, the following approximations corresponding to C_1 and C_2 in (4.5) and (4.7), respectively:

$$\hat{C}_1 = C_1\frac{T^2e^{-T}}{(1-e^{-T})^2}, \tag{6.1}$$

$$\begin{aligned}\hat{C}_2 &= C_2\frac{T^2e^{-T}}{(1-e^{-T})^2}\\ &\quad + e^{-\frac{W^2}{2}+\frac{Z^2}{2}}\{\phi(Z) - Z[(1-\Phi(Z))]\}\frac{Te^{-T}\left(2-T-2e^{-T}-Te^{-T}\right)}{\sqrt{\kappa''(T)}(1-e^{-T})^3}.\end{aligned} \tag{6.2}$$

For the approximations to $\mathbb{E}[\hat{X}|\hat{X} \geq K]$, we also need the lattice version for the tail probability

$$\mathbb{P}(\hat{X} \geq K) \approx e^{-\frac{W^2}{2}+\frac{Z^2}{2}}[1-\Phi(Z)]\frac{T}{1-e^{-T}} =: \hat{P}_1 \tag{6.3}$$

or its higher order version

$$\begin{aligned}\mathbb{P}(\hat{X} \geq K) \approx e^{-\frac{W^2}{2}+\frac{Z^2}{2}}\frac{T}{1-e^{-T}} \times \Bigg\{&[1-\Phi(Z)]\left(2-\frac{\lambda_3}{6}Z^3-\frac{T}{e^T-1}\right)\\ &+\phi(Z)\left[\frac{\lambda_3}{6}(Z^2-1)+\frac{1}{Z}-\frac{T}{Z(e^T-1)}\right]\Bigg\} =: \hat{P}_2.\end{aligned} \tag{6.4}$$

Recall that the Lugannani–Rice formula for lattice variables reads

$$\mathbb{P}(\hat{X} \geq K) \approx 1-\Phi(W)+\phi(W)\left[\frac{1}{\hat{Z}}-\frac{1}{W}\right] =: \hat{P}_3, \tag{6.5}$$

where $\hat{Z} = (1-e^{-T})\sqrt{\kappa''(T)}$. Similar lattice formulas can also be obtained for C_3 and C_4, which will be denoted by $\hat{C}_3$ and $\hat{C}_4$, respectively.

We first write down the inversion formula of the tail probability of a lattice variable,

$$\mathbb{Q}(\hat{X} \geq K) = \sum_{k=K}^{\infty}\mathbb{Q}(\hat{X}=k) = \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\pi}^{\tau+\mathtt{i}\pi}\frac{\exp(\kappa_{\mathbb{Q}}(t)-tK)}{1-e^{-t}}\mathrm{d}t. \tag{6.6}$$

Combining (6.6) with Lemma 1 (from the appendix), we obtain

$$\mathbb{E}\left[\hat{X}\mathbf{1}_{\{\hat{X}\geq K\}}\right] = \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\pi}^{\tau+\mathtt{i}\pi}\kappa'(t)\frac{\exp(\kappa(t)-tK)}{1-e^{-t}}\mathrm{d}t.$$

By the same change of variables as in section 5, we have

$$\begin{aligned}\mathbb{E}\left[\hat{X}\mathbf{1}_{\{\hat{X}\geq K\}}\right] &= \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\pi}^{\tau+\mathtt{i}\pi}\kappa'(t)e^{\frac{1}{2}w^2-Ww}\frac{1}{1-e^{-t}}\frac{\mathrm{d}t}{\mathrm{d}w}\mathrm{d}w\\ &= \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\pi}^{\tau+\mathtt{i}\pi}e^{\frac{1}{2}w^2-Ww}\left[\frac{\mu}{w}+\frac{\kappa'(t)}{1-e^{-t}}\frac{\mathrm{d}t}{\mathrm{d}w}-\frac{\mu}{w}\right]\mathrm{d}w.\end{aligned}$$

As in the appendix, since $\lim_{t\to 0} 1-e^{-t} = t$, this leads to

$$\hat{C}_3 = (\mu-K)\left[1-\Phi(W)-\frac{\phi(W)}{W}\right] \equiv C_3. \tag{6.7}$$

Including higher order terms, we obtain

$$\hat{C}_4 = \hat{C}_3+\phi(W)\left[\frac{e^{-T}}{\hat{Z}(1-e^{-T})}+(\mu-K)\frac{1}{W^3}\right]. \tag{6.8}$$

A higher order version of $\hat{P}_3$ can be derived similarly,

$$\mathbb{P}(\hat{X} \geq K) \approx 1 - \Phi(W) + \phi(W) \left[\frac{1}{\hat{Z}} \left(1 + \frac{\lambda_4}{8} - \frac{5\lambda_3^2}{24} \right) - \frac{e^{-T}\lambda_3}{2\hat{Z}^2} - \frac{e^{-T}(1+e^{-T})}{2\hat{Z}^3} - \frac{1}{W} + \frac{1}{W^3} \right] =: \hat{P}_4. \tag{6.9}$$

This can be used to estimate $\mathbb{E}[\hat{X}|\hat{X} \geq K]$.

The rates of convergence of $\hat{C}_1$ to $\hat{C}_4$ in the i.i.d. case are identical to their nonlattice counterparts and shall not be elaborated further.

7. Numerical results.

7.1. Exponential and Bernoulli variables. Using two numerical experiments, we evaluate the quality of the various approximations derived in the earlier sections. The approach proposed by [16] is used as a competitor to our approximation formulas. Since their approach employs the saddlepoint approximation to densities, the approximations for continuous variables need not be modified for lattice variables. Their first order approximation to C will be denoted by C_{Y1}, and the second order approximation will be denoted by C_{Y2}. The calculation of C_{Y1} (resp., C_{Y2}) requires the second (resp., third and fourth) derivatives of the function $\kappa(t) - \log t^2$. As a result, the complexity of the calculation of C_{Y1} and C_{Y2} is comparable to that of C_1 and C_2, respectively.

In our first example $X = \sum_{i=1}^n X_i$, where X_i are *exponentially* i.i.d. with density $p(x) = e^{-x}$. The CGF of X reads $\kappa(t) = -n\log(1-t)$. The saddlepoint to $\kappa'(t) = K$ is given by $T = 1 - n/K$. Moreover, we have

$$\kappa''(T) = \frac{K^2}{n}, \quad \lambda_3 = \frac{2}{\sqrt{n}}, \quad \lambda_4 = \frac{6}{n}.$$

The exact distribution is available as $X \sim \textit{Gamma}(n,1)$. The tail probability is then given by

$$\mathbb{P}(X \geq K) = 1 - \frac{\gamma(n,K)}{\Gamma(n)}$$

and

$$\mathbb{E}[X\mathbf{1}_{\{X \geq K\}}] = n\left[1 - \frac{\gamma(n+1,K)}{\Gamma(n+1)}\right],$$

where Γ and γ are the gamma function and the incomplete gamma function, respectively.

We first fix $n = 100$. For different levels K, from 107 to 145, we calculate $\mathbb{E}[(X-K)^+]$. The expectation decreases from 4.50 to 9.53×10^{-5} as K increases. The tail probability $\mathbb{E}(X \geq 145)$ is 3.26×10^{-5}, indicating that we have entered the tail of the distribution. The relative errors of the various approximations are illustrated in Figure 1.

Then we fix the ratio $K/n = 1.15$ and set $n = 10 \times 2^i$ for $i = 1, \ldots, 8$. The expectation decreases from 0.70 to 1.05×10^{-6} as n increases. The tail probability $\mathbb{E}(X \geq 1472)$ is 1.46×10^{-7}. The relative errors of the various approximations are shown in Figure 2.

In the second example we consider the sum of Bernoulli random variables. This is particularly relevant for CDO pricing because the number of defaults in an underlying portfolio

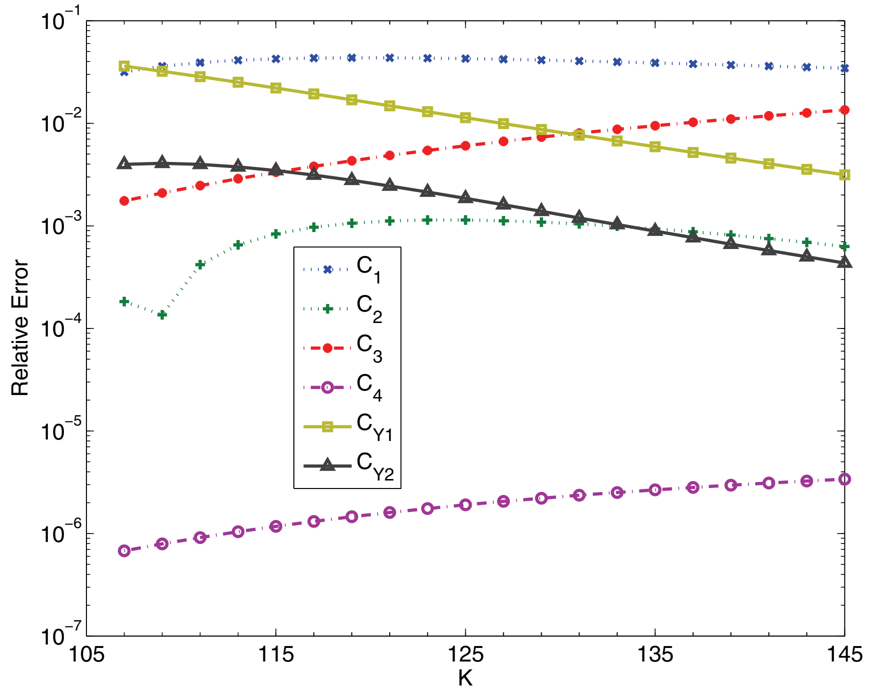

Figure 1. *Relative errors of various saddlepoint approximations for* $\mathbb{E}[(\sum_{i=1}^{n} X_i - K)^+]$ *for fixed* n *and different* K*.* X_i *is exponentially distributed with density* $f(x) = e^{-x}(x \geq 0)$*.* $n = 100$*,* K *ranges from* 107 *to* 145.

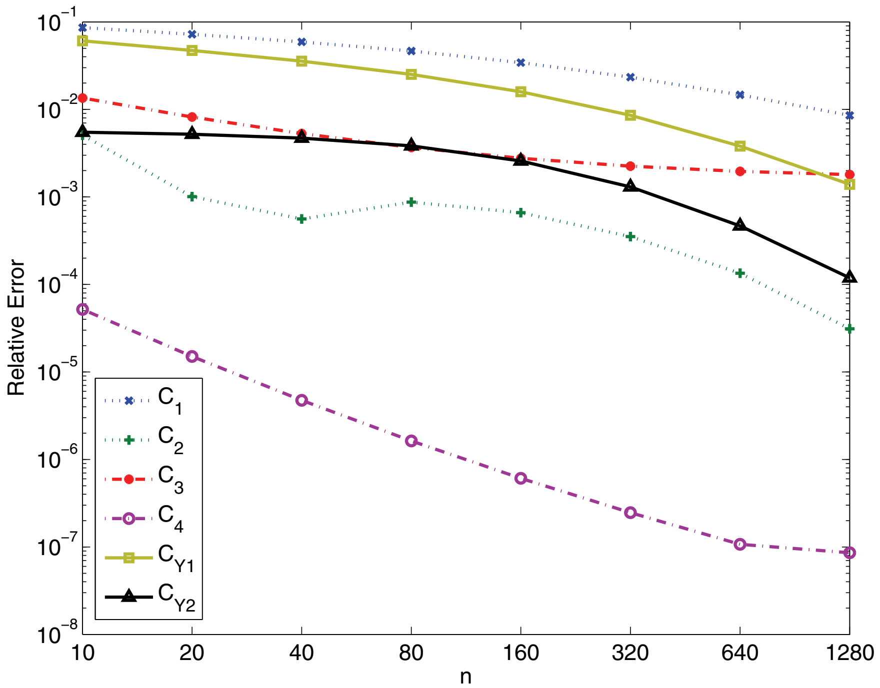

Figure 2. *Relative errors of various saddlepoint approximations for* $\mathbb{E}[(\sum_{i=1}^{n} X_i - K)^+]$ *for different* n*.* X_i *is exponentially distributed with density* $f(x) = e^{-x}(x \geq 0)$*.* $n = 10 \times 2^i$ *for* $i = 1, \ldots, 8$*,* $K = 1.15n$*.*

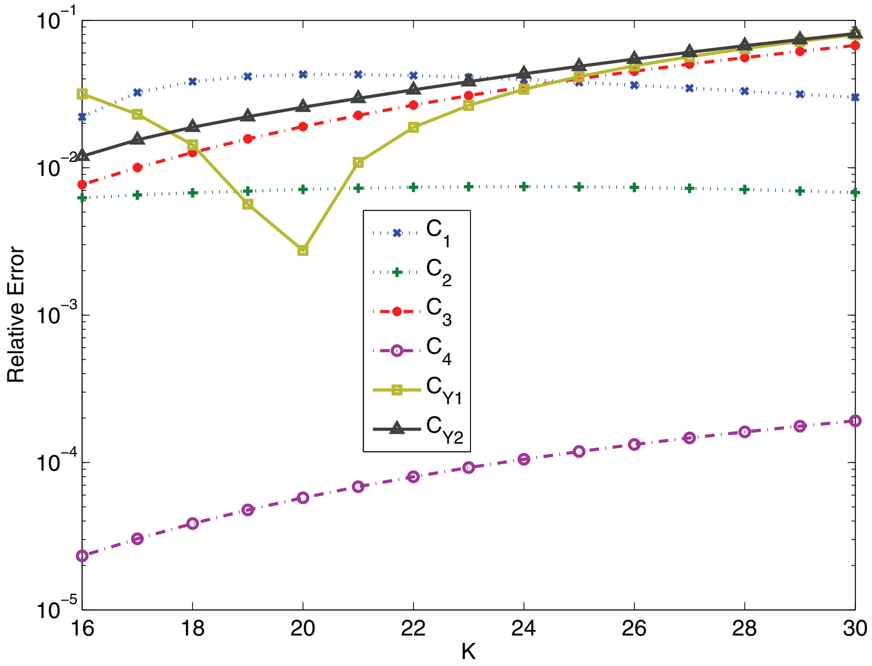

Figure 3. *Relative errors of various saddlepoint approximations for* $\mathbb{E}[(\sum_{i=1}^n X_i - K)^+]$ *for fixed* n *and different* K*.* X_i *is Bernoulli distributed with* $p(X_i = 1) = 0.15$*.* $n = 100$*,* K *ranges from* 16 *to* 30*.*

can be modeled by a sum of Bernoulli random variables. Consequently, by the results in this example we are able to estimate, at least partially, the performance of various approximations for CDO pricing.

We set $X = \sum_{i=1}^n X_i$, where X_i are i.i.d. *Bernoulli* variables with $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p = 0.15$. The CGF of X is given by $\kappa(t) = n \log\left(1 - p + pe^t\right)$. Here the saddlepoint to $\kappa'(t) = K$ equals $T = \log\left[\frac{K(1-p)}{(n-K)p}\right]$ and

$$\kappa''(T) = \frac{K(n-K)}{n}, \quad \lambda_3 = \frac{n-2K}{\sqrt{nK(n-K)}}, \quad \lambda_4 = \frac{n^2 - 6nK + 6K^2}{nK(n-K)}.$$

In this specific case, X is binomially distributed with

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

which means that C as defined in (1.2) can also be calculated exactly.

Similar to the exponential case, we first fix $n = 100$. For different levels K from 16 to 30 we calculate $\mathbb{E}[(X-K)^+]$. The expectation decreases from 0.24 to 1.92×10^{-6} as K increases. The tail probability $\mathbb{E}(X \geq 30)$ is 1.05×10^{-4}.

Then we fix the ratio $K/n = 0.2$ and set $n = 10 \times 2^i$ for $i = 1, \ldots, 8$. The expectation decreases from 0.98 to 6.42×10^{-5} as n increases. The tail probability $\mathbb{E}(X \geq 256)$ is 8.68×10^{-7}. The relative errors of the various approximations are presented in Figures 3 and 4. Note

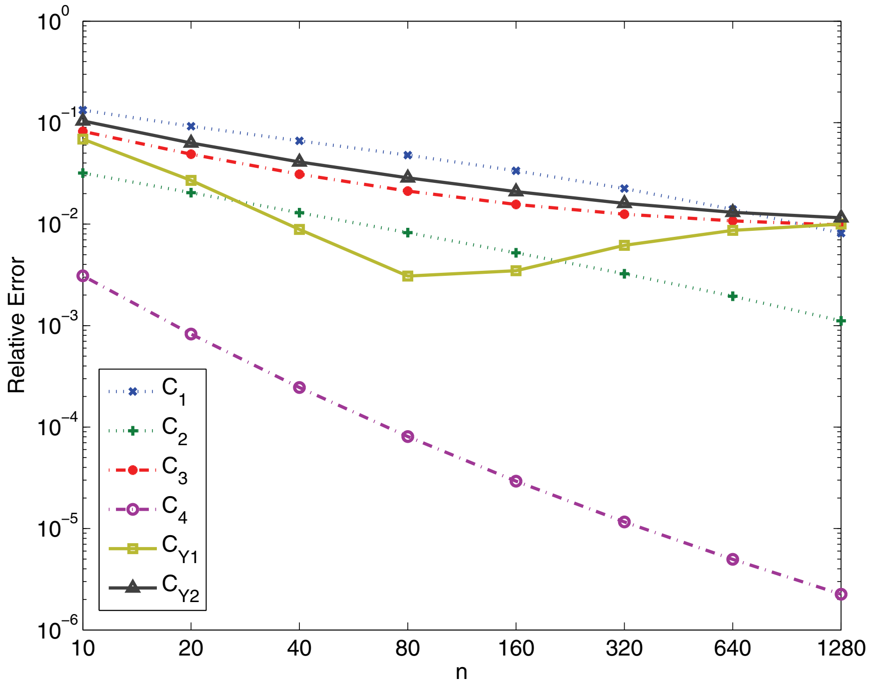

Figure 4. *Relative errors of various saddlepoint approximations for* $\mathbb{E}[(\sum_{i=1}^{n} X_i - K)^+]$ *for different* n. X_i *is Bernoulli distributed with* $p(X_i = 1) = 0.15$. $n = 10 \times 2^i$ *for* $i = 1, \ldots, 8$, $K = n/5$.

that the saddlepoint approximations in the Bernoulli case are based on the formulas $\hat{C}_1$–$\hat{C}_4$ for lattice variables, derived in section 6.

In summary, all approximations work quite well in our experiments in the sense that they all produce small relative errors, also in the case that the expectation is very small. The error convergence rates of the approximations C_1–C_4 shown in Figures 2 and 4 confirm the derived theoretical convergence rates. The higher order Lugannani–Rice-type formulas, C_4 and its lattice sister, are clearly the winners. They produce almost exact approximations and have the highest error convergence rate. Moreover, the calculation of C_4 requires the same information as for C_1 and $\hat{C}_1$. The performance of C_{Y1} and C_{Y2} is in general comparable to that of C_1 and C_3 but inferior to C_2.

7.2. CDO tranche pricing.

In this section we show how the saddlepoint approximations can be used for CDO tranche pricing.

The value and payments of a CDO are derived from a portfolio of fixed-income underlying assets, for example bonds. CDO securities are split into different risk classes, or tranches, and the pricing of the CDOs involves determining the fair spread of the tranches. Details of the CDOs can be found in [3], [8].

Here we focus on the calculation of the fair spread of a CDO tranche. Let us denote by $t_m = m\Delta t$, $m = 1, 2, \ldots$, the payment dates, and let $L_i(t_m)$ be the loss due to obligor i up to t_m and $L(t_m) = \sum L_i(t_m)$ the portfolio loss. Then the fair spread of a CDO tranche with a lower attachment point K_1 and an upper attachment point K_2 is given by

$$s = \frac{\sum_m d(0,t_m)\left[\mathbb{E}L_{[K_1,K_2]}(t_m) - \mathbb{E}L_{[K_1,K_2]}(t_{m-1})\right]}{\Delta t \sum_m d(0,t_m)\left[K_2 - K_1 - \mathbb{E}L_{[K_1,K_2]}(t_m)\right]},$$

where $d(0,t_m)$ denotes the discount factor from time t_m to 0 and

$$\mathbb{E}L_{[K_1,K_2]}(t_m) := \mathbb{E}[\min(L_{t_m}, K_2)] - \mathbb{E}[\min(L_{t_m}, K_1)]$$

represents the tranche loss at t_m. As $\mathbb{E}[\min(X,K)] := \mathbb{E}X - \mathbb{E}(X-K)^+$, we obtain

$$s = \frac{\sum_m d(0,t_m)\left[\mathbb{E}(L_{t_m}-K_1)^+ - \mathbb{E}(L_{t_m}-K_2)^+ - \mathbb{E}(L_{t_{m-1}}-K_1)^+ + \mathbb{E}(L_{t_{m-1}}-K_2)^+\right]}{\Delta t \sum_m d(0,t_m)\left[K_2 - K_1 - \mathbb{E}(L_{t_m}-K_1)^+ + \mathbb{E}(L_{t_m}-K_2)^+\right]}.$$

So we see that the pricing of a CDO tranche can be reduced to the calculation of $\mathbb{E}(L_t - K)^+$ for a number of payment dates and two attachment points, which is exactly what we have been working on in the previous sections.

For simplicity of notation from now on we omit the subscript time index t. Let D_i be the default indicator of obligor i. Assuming a constant recovery rate, $1-\lambda$, the loss due to obligor i is given by $L_i = \lambda D_i$. With $D = \sum D_i$ the number of defaults in the portfolio, then we have

$$\mathbb{E}(L-K)^+ = \mathbb{E}\left(\sum L_i - K\right)^+ = \lambda\mathbb{E}\left(\sum D_i - K/\lambda\right)^+ = \lambda\mathbb{E}(D - K/\lambda)^+. \tag{7.1}$$

The quantity K/λ is in general not an integer. Consequently we need to make an adjustment before we can apply the saddlepoint approximations for lattice variables. We have, denoting by $\lceil x \rceil$ the nearest integer that is greater than or equal to x,

$$\begin{aligned}
\mathbb{E}(D-K/\lambda)^+ &= \sum_{k \ge \lceil K/\lambda \rceil} (k - K/\lambda)\mathbb{P}(D=k) \\
&= \sum_{k \ge \lceil K/\lambda \rceil} (k - \lceil K/\lambda \rceil)\mathbb{P}(D=k) + (\lceil K/\lambda \rceil - K/\lambda) \sum_{k \ge \lceil K/\lambda \rceil} \mathbb{P}(D=k) \\
&= \mathbb{E}(X - \lceil K/\lambda \rceil)^+ + (\lceil K/\lambda \rceil - K/\lambda)\mathbb{P}(D \ge \lceil K/\lambda \rceil). \qquad (7.2)
\end{aligned}$$

For example, for the attachment point 3% of the iTraxx index (with a notional 125) and a recovery $\lambda = 0.6$, we have

$$\mathbb{E}(L - 3\% \times 125)^+ = 0.6\mathbb{E}(D - 3.75/0.6)^+ = 0.6\left[\mathbb{E}(D-7)^+ + 0.75\mathbb{P}(D \ge 7)\right].$$

Both the expectation and the tail probability in (7.2) can be approximated by the saddlepoint approximations based on the same saddlepoint. Finally we substitute (7.2) into (7.1).

Now we consider the approximation of (7.1) in the industrial standard Gaussian copula model. In this model, A_i, the standardized asset return of counterparty i is normally distributed and can be decomposed as $A_i = \sqrt{\rho}Y + \sqrt{1-\rho}\epsilon_i$, where Y is a systematic factor which affects all counterparties and ϵ_i is a specific risk which affects only obligor i; ρ is called the asset correlation. The counterparty defaults at time t if $A_i < c$ with $p = \mathbb{P}(A_i < c)$ being the default probability. Note that both c and p are time-dependent.

Table 1

The saddlepoint approximations to $\mathbb{E}(L-K)^+$ for three payment dates and a variety of attachment points (AP) and their relative errors.

AP	$\mathrm{p}(t_1)=0.0005$	$\mathrm{p}(t_2)=0.005$	$\mathrm{p}(t_3)=0.05$
3%	6.1962e-04 (4.44e-05)	4.3983e-02 (2.06e-05)	1.7946e+00 (4.44e-06)
6%	8.5987e-05 (1.05e-05)	1.2159e-02 (4.68e-06)	9.6209e-01 (1.15e-06)
9%	1.6686e-05 (6.66e-06)	4.1627e-03 (2.72e-06)	5.3731e-01 (7.53e-07)
12%	3.1798e-06 (9.80e-06)	1.5707e-03 (3.54e-06)	3.0515e-01 (1.13e-06)
22%	2.5578e-10 (1.61e-05)	7.4415e-05 (8.74e-07)	4.5675e-02 (3.80e-07)

Table 2

The saddlepoint approximations (SA) to the spreads (in basis points) of various tranches.

Tranche	SA	Benchmark
$[3\%, 6\%]$	742.0349	742.0414
$[6\%, 9\%]$	363.9013	363.9019
$[9\%, 12\%]$	195.4237	195.4238
$[12\%, 22\%]$	64.6433	64.6434
$[22\%, 100\%]$	1.4492	1.4492

We consider a homogeneous portfolio of 125 counterparties, although the saddlepoint approximations can also handle inhomogeneous portfolios well. An application of saddlepoint approximations to inhomogeneous credit portfolios can be found in [16] for CDO pricing and in [7] for the calculation of the portfolio value at risk. We choose to work with a homogeneous portfolio only because we can obtain the exact solution by binomial expansion in this case.

For simplicity we consider only three payment dates and take the following default probabilities: $p(t_1) = 0.0005$, $p(t_2) = 0.005$, $p(t_3) = 0.05$. Further we assume an asset correlation $\rho = 0.3$ and a constant recovery rate $1 - \lambda = 0.4$. The homogeneity assumption allows us to calculate the exact tranche losses and spreads by the binomial distribution, which can be used as benchmarks to evaluate the performance of the saddlepoint approximations.

For all standard attachment points of the iTraxx index, i.e., 3%, 6%, 9%, 12%, and 22%, we calculate

$$\mathbb{E}(L-K)^+ = \int \mathbb{E}[L(Y)-K]^+ \mathrm{d}\mathbb{P}(Y)$$

by approximating the integral by the Gauss–Legendre quadrature with 250 nodes in the interval $Y \in [-5, 5]$. In Table 1 we present the estimates derived from the saddlepoint approximations $\hat{C}_4$ and $\hat{P}_3$. In parentheses are the relative errors of the approximations with respect to the exact results obtained with the binomial distribution.

Suppose that $d(0, t_1) = 1.05$, $d(0, t_2) = 1.1$, $d(0, t_3) = 1.2$, and $\Delta t = 1$. The saddlepoint approximations to the spreads of various tranches (in basis points) are shown in Table 2. The results confirm the high accuracy of the saddlepoint approximations.

8. Conclusions. We have derived two types of saddlepoint approximations to $\mathbb{E}[(X-K)^+]$, where X is the sum of n independent random variables and K is a known constant. For each type of approximation, we have given a lower order as well as a higher order version. We have also established the error convergence rates for the approximations in the i.i.d. case. The approximations have been further extended to cover the case of lattice variables. Numerical examples, including in particular an application of the saddlepoint approximations to CDO pricing, show that all these approximations work very well. The higher order Lugannani–Rice-type formulas for $\mathbb{E}[(X-K)^+]$ are particularly attractive because of their remarkable simplicity, extremely high accuracy, and fast convergence.

Appendix. Error convergence of the Lugannani–Rice-type formulas. In this section we present an alternative derivation of the Lugannani–Rice-type saddlepoint approximations to $\mathbb{E}[(X-K)^+]$. An analysis of the error convergence of the approximation formulas is also provided here.

In this alternative derivation to (5.16), instead of working directly with $\mathbb{E}[(X-K)^+]$ we first work on the saddlepoint approximations to $\mathbb{E}\left[X\mathbf{1}_{\{X\geq K\}}\right]$, which is related to $\mathbb{E}[(X-K)^+]$ in the following way:

$$\mathbb{E}[(X-K)^+] = \mathbb{E}\left[X\mathbf{1}_{\{X\geq K\}}\right] - K\mathbb{P}(X\geq K). \tag{A.1}$$

To start, we derive the following inversion formula for $\mathbb{E}\left[X\mathbf{1}_{\{X\geq K\}}\right]$.

Lemma 1. *Let $\kappa(t)=\log M(t)$ be the cumulant generating function of a continuous random variable X. Then*

$$\mathbb{E}\left[X\mathbf{1}_{\{X\geq K\}}\right] = \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} \kappa'(t)\frac{\exp(\kappa(t)-tK)}{t}\mathrm{d}t \qquad (\tau>0). \tag{A.2}$$

Proof. We start with the case that X has a nonnegative lower bound. Employing the same change of measure as in (3.3), we have $\mathbb{E}\left[X\mathbf{1}_{\{X\geq K\}}\right]=\mu\mathbb{Q}(X\geq K)$, where

$$\mathbb{Q}(X\geq K) = \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} \frac{\exp(\kappa_{\mathbb{Q}}(t)-tK)}{t}\mathrm{d}t \qquad (\tau>0).$$

Substituting $\kappa_{\mathbb{Q}}(t)$, which is given by (3.4), into (A.2), we find

$$\begin{aligned}\mathbb{E}\left[X\mathbf{1}_{\{X\geq K\}}\right] &= \mu\frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} \frac{\exp\left[\kappa(t)+\log\kappa'(t)-\log\mu-tK\right]}{t}\mathrm{d}t\\ &= \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} \kappa'(t)\frac{\exp(\kappa(t)-tK)}{t}\mathrm{d}t.\end{aligned}$$

In the case that X has a negative lower bound, $-a$, with $a>0$, we define $Y=X+a$ so that Y has a nonnegative lower bound. Then, the CGF of Y and its first derivative are given by $\kappa_Y(t)=\kappa(t)+ta$ and $\kappa_Y'(t)=\kappa'(t)+a$, respectively. Since

$$\mathbb{E}\left[X\mathbf{1}_{\{X\geq K\}}\right] = \mathbb{E}\left[(Y-a)\mathbf{1}_{\{Y-a\geq K\}}\right] = \mathbb{E}\left[Y\mathbf{1}_{\{Y-a\geq K\}}\right] - a\mathbb{P}(Y-a\geq K)$$

and

$$\mathbb{E}\left[Y\mathbf{1}_{\{Y-a\geq K\}}\right] = \frac{1}{2\pi\mathtt{i}}\int_{\tau-\mathtt{i}\infty}^{\tau+\mathtt{i}\infty} \kappa'(t)\frac{\exp(\kappa(t)-tK)}{t}\mathrm{d}t + a\mathbb{P}(Y-a\geq K),$$

we are again led to (A.2).

For unbounded X, we take $X_L = \max(X, L)$, where $L < -1/\tau$ is a constant. Since X_L is bounded from below, we have

$$\begin{aligned}\mathbb{E}\left[X_L \mathbf{1}_{\{X_L \geq K\}}\right] &= \frac{1}{2\pi \mathtt{i}} \int_{\tau - \mathtt{i}\infty}^{\tau + \mathtt{i}\infty} \kappa'_{X_L}(t) \frac{\exp(\kappa_{X_L}(t) - tK)}{t} \mathrm{d}t \\ &= \frac{1}{2\pi \mathtt{i}} \int_{\tau - \mathtt{i}\infty}^{\tau + \mathtt{i}\infty} M'_{X_L}(t) \frac{\exp(-tK)}{t} \mathrm{d}t, \end{aligned} \tag{A.3}$$

where $M'_{X_L}(\tau) = M'(\tau) + \int_{-\infty}^{L} (Le^{\tau L} - xe^{\tau x}) d\mathbb{P}(x)$. For $L < -1/\tau$, $M'_{X_L}(\tau)$ increases monotonically as L decreases, and approaches $M'(\tau)$ as $L \to -\infty$. Note also that $\mathbb{E}\left[X \mathbf{1}_{\{X \geq K\}}\right] = \mathbb{E}\left[X_L \mathbf{1}_{\{X_L \geq K\}}\right]$ for all $L < K$. Now take the limit of both sides of (A.3) as $L \to -\infty$. Due to the monotone convergence theorem, we again obtain

$$\begin{aligned}\mathbb{E}\left[X \mathbf{1}_{\{X \geq K\}}\right] &= \frac{1}{2\pi \mathtt{i}} \int_{\tau - \mathtt{i}\infty}^{\tau + \mathtt{i}\infty} M'(t) \frac{\exp(-tK)}{t} \mathrm{d}t \\ &= \frac{1}{2\pi \mathtt{i}} \int_{\tau - \mathtt{i}\infty}^{\tau + \mathtt{i}\infty} \kappa'(t) \frac{\exp(\kappa(t) - tK)}{t} \mathrm{d}t. \qquad \blacksquare\end{aligned}$$

We apply the same change of variables as in section 5.[1] Based on Lemma 1, the inversion formula for $\mathbb{E}\left[X \mathbf{1}_{\{X \geq nx\}}\right]$ can be formulated as

$$\begin{aligned}\mathbb{E}\left[X \mathbf{1}_{\{X \geq nx\}}\right] &= \frac{1}{2\pi \mathtt{i}} \int_{\tau - \mathtt{i}\infty}^{\tau + \mathtt{i}\infty} n\kappa'_1(t) e^{n(\frac{1}{2}w^2 - W_1 w)} \frac{1}{t} \frac{\mathrm{d}t}{\mathrm{d}w} \mathrm{d}w \\ &= \frac{n}{2\pi \mathtt{i}} \int_{\tau - \mathtt{i}\infty}^{\tau + \mathtt{i}\infty} e^{n(\frac{1}{2}w^2 - W_1 w)} \left[\frac{\mu_1}{w} + \frac{\kappa'_1(t)}{t} \frac{\mathrm{d}t}{\mathrm{d}w} - \frac{\mu_1}{w}\right] \mathrm{d}w \\ &= n\mu_1 \int_{\tau - \mathtt{i}\infty}^{\tau + \mathtt{i}\infty} \frac{1}{2\pi \mathtt{i}} e^{n(\frac{1}{2}w^2 - W_1 w)} \frac{\mathrm{d}w}{w} \\ &\quad + \frac{ne^{-\frac{nW_1^2}{2}}}{2\pi \mathtt{i}} \int_{W_1 - \mathtt{i}\infty}^{W_1 + \mathtt{i}\infty} e^{\frac{1}{2}n(w - W_1)^2} \left[\frac{\kappa'_1(t)}{t} \frac{\mathrm{d}t}{\mathrm{d}w} - \frac{\mu_1}{w}\right] \mathrm{d}w. \end{aligned} \tag{A.4}$$

The first integral takes the value $1 - \Phi(\sqrt{n} W_1) = 1 - \Phi(W)$. The second integral does not have a singularity, because of (5.4). Hence there is no problem in changing the integration contour from the imaginary axis along $\tau > 0$ to one along W_1, as done in (A.4), even if W_1 and T are both negative.

The major contribution to the second integral comes from the saddlepoint. The terms in the brackets are expanded around T and integrated to give an expansion of the form

$$n\phi(\sqrt{n} W_1)(b_1 n^{-\frac{1}{2}} + b_3 n^{-\frac{3}{2}} + b_5 n^{-\frac{5}{2}} + \cdots). \tag{A.5}$$

By Watson's lemma this is an asymptotic expansion in a neighborhood of W_1. For more details, see Lemma 4.5.2 in [10]. Coefficient b_1 in (A.5) can be obtained by only taking into

[1] Let w be defined between 0 and W_1 such that $\frac{1}{2}(w - W_1)^2 = \kappa_1(t) - tx - \kappa_1(T) + Tx$. Then we have $\frac{1}{2}w^2 - W_1 w = \kappa_1(t) - t\kappa'_1(T)$.

account the leading terms of the Taylor expansion of

$$\frac{\kappa_1'(t)}{t}\frac{\mathrm{d}t}{\mathrm{d}w} - \frac{\mu_1}{w} = \frac{\kappa_1'(t)}{t}\frac{\mathrm{d}t}{\mathrm{d}w}\Big|_T - \frac{\mu_1}{w}\Big|_{W_1} + \cdots = \frac{x}{Z_1} - \frac{\mu_1}{W_1} + \cdots . \tag{A.6}$$

Therefore we are led to

$$\mathbb{E}\left[X\mathbf{1}_{\{X\geq nx\}}\right] = n\mu_1\left[1-\Phi(\sqrt{n}W_1)\right] + n\phi(\sqrt{n}W_1)\left[\frac{1}{\sqrt{n}}\left(\frac{x}{Z_1}-\frac{\mu_1}{W_1}\right) + O\left(n^{-\frac{3}{2}}\right)\right]. \tag{A.7}$$

Subtracting $K\mathbb{P}(X\geq K)$ from (A.7) with the tail probability approximated by the Lugannani–Rice formula P_3 from (2.6), we see immediately that

$$\mathbb{E}\left[(X-nx)^+\right] = n(\mu_1 - x)\left[1-\Phi(\sqrt{n}W_1) - \frac{\phi(\sqrt{n}W_1)}{\sqrt{n}W_1} + O\left(n^{-\frac{3}{2}}\right)\right]. \tag{A.8}$$

Rewriting (A.7) and (A.8) in quantities related to X and deleting the error terms, we obtain the following approximation:

$$\mathbb{E}\left[X\mathbf{1}_{\{X\geq K\}}\right] \approx \mu\left[1-\Phi(W)\right] + \phi(W)\left[\frac{K}{Z}-\frac{\mu}{W}\right], \tag{A.9}$$

$$\mathbb{E}\left[(X-K)^+\right] \approx (\mu - K)\left[1-\Phi(W) - \frac{\phi(W)}{W}\right] =: C_3. \tag{A.10}$$

Next, we consider the coefficient b_3 in (A.5). Write $U := \kappa_1''(T)T - \kappa_1'(T)$. The Taylor expansion of $\kappa_1'(t)/t$ around T gives

$$\frac{\kappa_1'(t)}{t} = \frac{\kappa_1'(T)}{T} + (t-T)\frac{U}{T^2} + \frac{(t-T)^2}{2}\left[\frac{\kappa_1'''(T)}{T} - \frac{2U}{T^3}\right] + \cdots . \tag{A.11}$$

Furthermore, we expand $\exp(n[\kappa_1(t) - tx])$ in the same way as in [4]:

$$\begin{aligned}
&\exp(n[\kappa_1(t) - tx]) \\
&= \exp\left(n[\kappa_1(T) - Tx] + \frac{1}{2}n\kappa_1''(T)(t-T)^2 + \frac{n}{6}\kappa_1'''(T)(t-T)^3 + \frac{n}{24}\kappa_1^{(4)}(t-T)^4 + \cdots\right) \\
&= \exp\left(n[\kappa_1(T) - Tx] + \frac{1}{2}n\kappa_1''(T)(t-T)^2\right) \\
&\quad \times\left[1 + \frac{n}{6}\kappa_1'''(T)(t-T)^3 + \frac{n}{24}\kappa_1^{(4)}(T)(t-T)^4 + \frac{n^2}{72}\kappa_1'''(T)^2(t-T)^6 + \cdots\right].
\end{aligned} \tag{A.12}$$

We put (A.11) and (A.12) together and have, at the line $t = T + \mathtt{i}y$,

$$\frac{n}{2\pi\mathtt{i}}\int_{T-\mathtt{i}\infty}^{T+\mathtt{i}\infty} e^{n[\kappa_1(t)-tx]}\frac{\kappa_1'(t)}{t}\mathrm{d}t = \frac{ne^{-\frac{nW_1^2}{2}}}{2\pi\mathtt{i}}\int_{T-\mathtt{i}\infty}^{T+\mathtt{i}\infty} e^{\frac{1}{2}n\kappa_1''(T)(t-T)^2}$$
$$\times\left[1+\frac{n}{6}\kappa_1'''(T)(t-T)^3+\frac{n}{24}\kappa_1^{(4)}(T)(t-T)^4+\frac{n^2}{72}\kappa_1'''(T)^2(t-T)^6+\cdots\right]$$
$$\times\left\{\frac{\kappa_1'(T)}{T}+(t-T)\frac{U}{T^2}+\frac{(t-T)^2}{2}\left[\frac{\kappa_1'''(T)}{T}-\frac{2U}{T^3}\right]+\cdots\right\}\mathrm{d}t$$
$$=\frac{ne^{-\frac{nW_1^2}{2}}}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}n\kappa_1''(T)y^2}\left[1-\frac{n}{6}\kappa_1'''(T)\mathtt{i}y^3+\frac{n}{24}\kappa_1^{(4)}(T)y^4\right.$$
$$\left.-\frac{n^2}{72}\kappa_1'''(T)^2y^6+\cdots\right]\left\{\frac{\kappa_1'(T)}{T}+\mathtt{i}y\frac{U}{T^2}-\frac{y^2}{2}\left[\frac{\kappa_1'''(T)}{T}-\frac{2U}{T^3}\right]+\cdots\right\}\mathrm{d}y$$
$$=n\phi(\sqrt{n}W_1)\left\{\frac{\kappa_1'(T)}{\sqrt{n}Z_1}+n^{-\frac{3}{2}}\left[\frac{\kappa_1'(T)}{Z_1}\left(\frac{\lambda_{1,4}}{8}-\frac{5}{24}\lambda_{1,3}^2\right)+\frac{U\lambda_{1,3}}{2Z_1^2}-\frac{\lambda_{1,3}}{2T}+\frac{U}{Z_1^3}\right]+O\left(n^{-\frac{5}{2}}\right)\right\}$$

(A.13)
$$=n\phi(W)\left[\frac{x}{\sqrt{n}Z_1}+n^{-\frac{3}{2}}\left(\frac{x\lambda_{1,4}}{8Z_1}-\frac{5x\lambda_{1,3}^2}{24Z_1}+\frac{1}{TZ_1}-\frac{x\lambda_{1,3}}{2Z_1^2}-\frac{x}{Z_1^3}\right)+O\left(n^{-\frac{5}{2}}\right)\right].$$

Notice that (A.13) is itself a saddlepoint approximation to $\mathbb{E}[X\mathbf{1}_{\{X\geq K\}}]$ for $K > \mu$. However, it becomes inaccurate when T approaches zero due to the presence of a pole at zero in the integrand. Meanwhile expanding $1/w$ in the second integral in (A.4) around W_1 gives

$$\frac{ne^{-\frac{nW_1^2}{2}}}{2\pi\mathtt{i}}\int_{W_1-\mathtt{i}\infty}^{W_1+\mathtt{i}\infty} e^{\frac{1}{2}n(w-W_1)^2}\frac{\mu_1}{w}\mathrm{d}w$$
$$=\frac{n\mu_1e^{-\frac{nW_1^2}{2}}}{2\pi\mathtt{i}}\int_{W_1-\mathtt{i}\infty}^{W_1+\mathtt{i}\infty} e^{\frac{1}{2}n(w-W_1)^2}\left[\frac{1}{W_1}-\frac{(w-W_1)}{W_1^2}+\frac{(w-W_1)^2}{W_1^3}+\cdots\right]\mathrm{d}w$$
$$=n\mu_1\phi(\sqrt{n}W_1)\left[\frac{1}{\sqrt{n}W_1}-\frac{1}{(\sqrt{n}W_1)^3}+O\left(n^{-\frac{5}{2}}\right)\right]. \tag{A.14}$$

Adding (A.13) and (A.14) to $1-\Phi(\sqrt{n}W_1)$ and then subtracting nx times (2.7), we obtain

$$\mathbb{E}[(X-nx)^+] = n(\mu_1-x)\left\{[1-\Phi(\sqrt{n}W_1)]+\frac{\phi(\sqrt{n}W_1)}{\sqrt{n}W_1}\right\}$$
$$+n\phi(\sqrt{n}W_1)\left\{n^{-\frac{3}{2}}\left[\frac{1}{TZ_1}+\frac{\mu_1-x}{W_1^3}\right]+O\left(n^{-\frac{5}{2}}\right)\right\}, \tag{A.15}$$

which can be rewritten as

$$\mathbb{E}[(X-K)^+]\approx C_3+\phi(W)\left[\frac{1}{TZ}+(\mu-K)\frac{1}{W^3}\right]=: C_4. \tag{A.16}$$

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