# A note on bipartite graph tiling 

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#### Abstract

Bipartite graph tiling was studied by Zhao [7] who gave the best possible minimum degree conditions for a balanced bipartite graph on $2 m s$ vertices to contain $m$ vertex disjoint copies of $K_{s, s}$. Let $s<t$ be fixed positive integers. Hladký and Schacht [3] gave minimum degree conditions for a balanced bipartite graph on $2 m(s+t)$ vertices to contain $m$ vertex disjoint copies of $K_{s, t}$. Their results were best possible, except in the case when $m$ is odd and $t>2 s+1$. We give the best possible minimum degree condition in this case.


## 1 Introduction

If $G$ is a graph on $n=s m$ vertices, $H$ is a graph on $s$ vertices and $G$ contains $m$ vertex disjoint copies of $H$, then we say $G$ can be tiled with $H$. In this language, we state the seminal result of Hajnal and Szemerédi.

Theorem 1.1 (Hajnal-Szemerédi [2]). Let $G$ be a graph on $n=s m$ vertices. If $\delta(G) \geq(s-1) m$, then $G$ can be tiled with $K_{s}$.

For tiling with general $H$, results of Alon and Yuster [1] and Komlós, Sárközy, and Szemerédi [4] gave sufficient conditions on the minimum degree of a graph $G$ such that $G$ can be tiled with $H$. Specifically, in [4], it is shown that if $G$ is a graph on $n$ vertices with minimum degree at least $(1-1 / \chi(H)) n+K$ for a constant $K$ that only depends on $H$, then $G$ can be tiled with $H$. A more delicate minimum degree condition that involves the so-called critical chromatic number of $H$ was conjectured by Komlós and solved by Shokoufandeh and Zhao [6. Finally, Kühn and Osthus [5] determined exactly when the critical chromatic number or chromatic number is the appropriate parameter and thus settled the problem (for large graphs).

In this paper we study the tiling problem in bipartite graphs. Denote a bipartite graph $G$ with partition sets $U$ and $V$ by $G[U, V]$. We say $G[U, V]$ is balanced if $|U|=|V|$. Zhao proved the following Hajnal-Szemerédi type result for bipartite graphs.

[^0]Theorem 1.2 (Zhao [7]). For each $s \geq 2$, there exists $m_{0}$ such that the following holds for all $m \geq m_{0}$. If $G$ is a balanced bipartite graph on $2 n=2 m s$ vertices with

$$
\delta(G) \geq \begin{cases}\frac{n}{2}+s-1 & \text { if } m \text { is even } \\ \frac{n+3 s}{2}-2 & \text { if } m \text { is odd }\end{cases}
$$

then $G$ can be tiled with $K_{s, s}$.
Zhao proved that this minimum degree condition was tight.
Proposition 1.3 (Zhao [7]). Let $s \geq 2$, and $n=m s \geq 64 s^{2}$. There exists a balanced bipartite graph, $G$, on $2 n$ vertices with

$$
\delta(G)= \begin{cases}\frac{n}{2}+s-2 & \text { if } m \text { is even } \\ \frac{n+3 s}{2}-3 & \text { if } m \text { is odd }\end{cases}
$$

such that $G$ cannot be tiled with $K_{s, s}$.
Hladký and Schacht extended Zhao's result as follows.
Theorem 1.4 (Hladký-Schacht [3]). Let $1 \leq s<t$ be fixed integers. There exists $m_{0}$ such that the following holds for all $m \geq m_{0}$. If $G$ is a balanced bipartite graph on $2 n=2 m(s+t)$ vertices with

$$
\delta(G) \geq \begin{cases}\frac{n}{2}+s-1 & \text { if } m \text { is even } \\ \frac{n+t+s}{2}-1 & \text { if } m \text { is odd }\end{cases}
$$

then $G$ can be tiled with $K_{s, t}$.
They proved that this minimum degree condition was tight in all cases except when $m$ is odd and $t>2 s+1$. Note that since we are dealing with balanced bipartite graphs, in any tiling of $G[U, V]$ with $K_{s, t}$ there must be an equal number of copies of $K_{s, t}$ with $s$ vertices in $U$ as copies of $K_{s, t}$ with $t$ vertices in $U$. This explains why the authors [3] suppose $2 n=2 m(s+t)$ instead of $2 n=m(s+t)$.

Proposition 1.5 (Hladký-Schacht [3]). Let $1 \leq s<t$ be fixed integers. There exists $m_{0}$ such that the following holds for all $m \geq m_{0}$. There exists a balanced bipartite graph, $G$, on $2 n=2 m(s+t)$ vertices with

$$
\delta(G)= \begin{cases}\frac{n}{2}+s-2 & \text { if } m \text { is even } \\ \frac{n+t+s}{2}-2 & \text { if } m \text { is odd and } t \leq 2 s+1\end{cases}
$$

such that $G$ cannot be tiled with $K_{s, t}$.
Our objective is to give the tight minimum degree condition in the final remaining case, when $m$ is odd and $t>2 s+1$. We will do this in two parts. First in Section 2.3 we prove that when $m$ is odd and $t \geq 2 s+1$, the following minimum degree condition is sufficient.

Theorem 1.6. Let $1 \leq s<t$ be fixed integers with $2 s+1 \leq t$. There exists $m_{0}$ such that the following holds for all odd $m$ with $m \geq m_{0}$. If $G$ is a balanced bipartite graph on $2 n=2 m(s+t)$ vertices with

$$
\delta(G) \geq \frac{n+3 s}{2}-1
$$

then $G$ can be tiled with $K_{s, t}$.

Then in Section 3 we prove that the minimum degree condition in Theorem 1.6 is tight.
Proposition 1.7. Let $1 \leq s<t$ be fixed integers with $2 s+1 \leq t$. There exists $m_{0}$ such that the following holds for all odd $m$ with $m \geq m_{0}$. There exists a balanced bipartite graph, $G$, on $2 n=2 m(s+t)$ vertices with

$$
\delta(G)= \begin{cases}\frac{n+3 s}{2}-\frac{3}{2} & \text { if } t \text { is odd } \\ \frac{n+3 s}{2}-2 & \text { if } t \text { is even }\end{cases}
$$

such that $G$ cannot be tiled with $K_{s, t}$.
Let $m=2 k+1$ for some $k \in \mathbb{N}$ and let $n=m(s+t)$. We note that when $t=2 s+1$, $\frac{n+3 s}{2}-1=(k+1)(s+t)-\frac{3}{2}$ and $\frac{n+t+s}{2}-1=(k+1)(s+t)-1$. So the value for the lower bound in Theorem [1.6 is smaller than the value for the lower bound in Theorem [1.4 when $t=2 s+1$, but since $\delta(G)$ only takes integer values the minimum degree condition in Theorem 1.6 is not an improvement until $t>2 s+1$.

## 2 Proof of Theorem 1.6

For disjoint sets $A, B \subseteq V(G)$, we define $e(A, B)$ to be the number of edges with one end in $A$ and the other end in $B$ and for $v \in V(G) \backslash A$ we write $\operatorname{deg}(v, A)$ instead of $e(\{v\}, A)$. Also, $d(A, B)=\frac{e(A, B)}{|A| B \mid}, \delta(A, B)=\min \{\operatorname{deg}(v, B): v \in A\}$ and $\Delta(A, B)=\max \{\operatorname{deg}(v, B): v \in A\}$. An $h$-star from $A$ to $B$, is a copy of $K_{1, h}$ with the vertex of degree $h$, the center, in $A$ and the vertices of degree 1 , the leaves, in $B$.

The following theorem appears in 7 .
Theorem 2.1 (Zhao [7). For every $\alpha>0$ and every positive integer $r$, there exist $\beta>0$ and positive integer $m_{1}$ such that the following holds for all $n=m r$ with $m \geq m_{1}$. Given a bipartite graph $G[U, V]$ with $|U|=|V|=n$, if $\delta(G) \geq\left(\frac{1}{2}-\beta\right) n$, then either $G$ can be tiled with $K_{r, r}$, or there exist

$$
\begin{equation*}
U_{1}^{\prime} \subseteq U, V_{2}^{\prime} \subseteq V, \quad \text { such that }\left|U_{1}^{\prime}\right|=\left|V_{2}^{\prime}\right|=\lfloor n / 2\rfloor, d\left(U_{1}^{\prime}, V_{2}^{\prime}\right) \leq \alpha \tag{1}
\end{equation*}
$$

If a balanced bipartite graph $G[U, V]$ on $2 n$ vertices with $n$ divisible by $r$ satisfies (1), we say $G$ is extremal with parameter $\alpha$. In this case we set $U_{2}^{\prime}:=U \backslash U_{1}^{\prime}$ and $V_{1}^{\prime}:=V \backslash V_{2}^{\prime}$.

If we replace $r$ with $s+t$ in Theorem [2.1, we see that either $G$ can be tiled with $K_{s+t, s+t}$ or else we are in the extremal case. If it is the case that $G$ can be tiled with $K_{s+t, s+t}$, we split each copy of $K_{s+t, s+t}$ into two copies of $K_{s, t}$ to give the desired tiling. So we must only deal with the extremal case.

### 2.1 Pre-processing

Claim 2.2. Let $0<\alpha \ll 1, r \in \mathbb{N}$ and let $m_{1} \in \mathbb{N}$ be given by Theorem 2.1. Let $m \geq m_{1}$ and suppose that $G[U, V]$ is a balanced bipartite graph on $2 n=2 m r$ vertices such that $\delta(G)=\frac{n}{2}+C$, where $0 \leq C \leq 3 r / 2$. Suppose further that the deletion of any edge of $G$ will cause the resulting graph to have minimum degree less than $\frac{n}{2}+C$. If $G$ is extremal with parameter $\alpha$, then $d\left(U_{2}^{\prime}, V_{1}^{\prime}\right) \leq 5 \sqrt{\alpha}$.

Proof. Let $\gamma:=5 \sqrt{\alpha}$ and suppose $d\left(U_{2}^{\prime}, V_{1}^{\prime}\right)>\gamma$. Let $X^{\prime}=\left\{u \in U_{2}^{\prime}: \operatorname{deg}\left(u, V_{2}^{\prime}\right)<(1-\sqrt{\alpha}) \frac{n}{2}\right\}$, $Y^{\prime}=\left\{v \in V_{1}^{\prime}: \operatorname{deg}\left(v, U_{1}^{\prime}\right)<(1-\sqrt{\alpha}) \frac{n}{2}\right\}$. Since $e\left(U_{1}^{\prime}, V_{2}^{\prime}\right) \leq \alpha \frac{n^{2}}{4}$ and $e\left(U_{1}^{\prime}, V\right) \geq\left|U_{1}^{\prime}\right| \frac{n}{2}$, we have $e\left(U_{1}^{\prime}, V_{1}^{\prime}\right) \geq\left|U_{1}^{\prime}\right| \frac{n}{2}-\alpha \frac{n^{2}}{4}$. Thus we can bound the non-edges between $U_{1}^{\prime}$ and $V_{1}^{\prime}$,

$$
\sqrt{\alpha} \frac{n}{2}\left|Y^{\prime}\right| \leq \bar{e}\left(U_{1}^{\prime}, V_{1}^{\prime}\right) \leq \alpha \frac{n^{2}}{4}
$$

which gives $\left|Y^{\prime}\right| \leq \sqrt{\alpha} \frac{n}{2}$. Similarly we have $\left|X^{\prime}\right| \leq \sqrt{\alpha} \frac{n}{2}$. Let $U_{2}^{\prime \prime}=U_{2}^{\prime} \backslash X^{\prime}$ and $V_{1}^{\prime \prime}=V_{1}^{\prime} \backslash Y^{\prime}$. Since $d\left(U_{2}^{\prime}, V_{1}^{\prime}\right)>\gamma$, we have

$$
\begin{equation*}
e\left(U_{2}^{\prime \prime}, V_{1}^{\prime \prime}\right) \geq \gamma \frac{n^{2}}{4}-2 \sqrt{\alpha} \frac{n^{2}}{4}=3 \sqrt{\alpha} \frac{n^{2}}{4} . \tag{2}
\end{equation*}
$$

Let $X^{\prime \prime}=\left\{u \in U_{2}^{\prime \prime}: \operatorname{deg}\left(u, V_{1}^{\prime \prime}\right) \geq \sqrt{\alpha} \frac{n}{2}+C+1\right\}$ and $Y^{\prime \prime}=\left\{v \in V_{1}^{\prime \prime}: \operatorname{deg}\left(v, U_{2}^{\prime \prime}\right) \geq \sqrt{\alpha} \frac{n}{2}+C+1\right\}$. If there is an edge $u v \in E\left(X^{\prime \prime}, Y^{\prime \prime}\right)$, then $\operatorname{deg}(u), \operatorname{deg}(y) \geq \frac{n}{2}+C+1$ which contradicts the edge minimality of $G$, so suppose $e\left(X^{\prime \prime}, Y^{\prime \prime}\right)=0$. Finally, by (21) we have

$$
3 \sqrt{\alpha} \frac{n^{2}}{4} \leq e\left(U_{2}^{\prime \prime}, V_{1}^{\prime \prime}\right) \leq e\left(X^{\prime \prime}, Y^{\prime \prime}\right)+e\left(U_{2}^{\prime \prime} \backslash X^{\prime \prime}, V_{1}^{\prime \prime}\right)+e\left(V_{1}^{\prime \prime} \backslash Y^{\prime \prime}, U_{2}^{\prime \prime}\right) \leq 0+2\left(\sqrt{\alpha} \frac{n}{2}+C\right) \frac{n}{2}
$$

which is a contradiction, since $n$ is sufficiently large.

Let $1 \leq s<t$ be integers so that $2 s+1 \leq t$, and let $0<\alpha \ll 1$ (setting $\alpha:=\left(\frac{1}{32 t(s+t)}\right)^{3}$ is small enough). Let $G[U, V]$ be a balanced bipartite graph on $2 n=2 m(s+t)$ vertices, where $m=2 k+1$ and $k$ is a sufficiently large integer with respect to $\left(\frac{\alpha}{5}\right)^{2}$. Suppose that $G$ is extremal with parameter $\left(\frac{\alpha}{5}\right)^{2}$ and edge-minimal with respect to the condition $\delta(G) \geq \frac{n+3 s}{2}-1$. By Claim 2.2 we have $d\left(U_{i}^{\prime}, V_{3-i}^{\prime}\right) \leq \alpha$ for $i=1,2$. Then for $i=1,2$, we define

$$
\begin{aligned}
& U_{i}=\left\{u \in U: \operatorname{deg}\left(u, V_{3-i}^{\prime}\right)<\alpha^{\frac{1}{3}} \frac{n}{2}\right\}, V_{i}=\left\{v \in V: \operatorname{deg}\left(v, U_{3-i}^{\prime}\right)<\alpha^{\frac{1}{3}} \frac{n}{2}\right\}, \\
& U_{0}=U-U_{1}-U_{2}, \text { and } V_{0}=V-V_{1}-V_{2}
\end{aligned}
$$

As a consequence of these definitions, we have the following.
Claim 2.3. For $i=1,2$

$$
\begin{aligned}
& \text { (i) }\left(1-\alpha^{2 / 3}\right) \frac{n}{2} \leq\left|U_{i}\right|,\left|V_{i}\right| \leq\left(1+\alpha^{2 / 3}\right) \frac{n}{2}, \quad \text { (ii) }\left|U_{0}\right|,\left|V_{0}\right| \leq \alpha^{2 / 3} n, \\
& \text { (iii) }\left(1-2 \alpha^{1 / 3}\right) \frac{n}{2}<\delta\left(U_{i}, V_{i}\right), \delta\left(V_{i}, U_{i}\right), \quad \text { (iv) }\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) \frac{n}{2} \leq \delta\left(U_{0}, V_{i}\right), \delta\left(V_{0}, U_{i}\right), \\
& \text { (v) } \Delta\left(U_{i}, V_{3-i}\right), \Delta\left(V_{3-i}, U_{i}\right) \leq \alpha^{1 / 3} n
\end{aligned}
$$

Proof. A proof of (i)-(iv) can be found in [7] and was also used in [3]. So we prove (v) here.
Let $i \in\{1,2\}$ and note that

$$
\begin{equation*}
\left|U_{i}^{\prime} \backslash U_{i}\right| \alpha^{1 / 3} \frac{n}{2} \leq e\left(U_{i}^{\prime} \backslash U_{i}, V_{3-i}^{\prime}\right) \leq e\left(U_{i}^{\prime}, V_{3-i}^{\prime}\right) \leq \alpha \frac{n^{2}}{4} \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|V_{i}^{\prime} \backslash V_{i}\right| \alpha^{1 / 3} \frac{n}{2} \leq e\left(V_{i}^{\prime} \backslash V_{i}, U_{3-i}^{\prime}\right) \leq e\left(V_{i}^{\prime}, U_{3-i}^{\prime}\right) \leq \alpha \frac{n^{2}}{4} . \tag{4}
\end{equation*}
$$

Then (3) and (4) imply

$$
\begin{equation*}
\left|U_{i}^{\prime} \backslash U_{i}\right|,\left|V_{i}^{\prime} \backslash V_{i}\right| \leq \alpha^{2 / 3} \frac{n}{2}, \tag{5}
\end{equation*}
$$

which gives $\Delta\left(U_{i}, V_{3-i}\right) \leq \Delta\left(U_{i}, V_{3-i}^{\prime}\right)+\left|V_{3-i} \backslash V_{3-i}^{\prime}\right| \leq \Delta\left(U_{i}, V_{3-i}^{\prime}\right)+\left|V_{i}^{\prime} \backslash V_{i}\right| \leq \alpha^{1 / 3} n$ and $\Delta\left(V_{i}, U_{3-i}\right) \leq \Delta\left(V_{i}, U_{3-i}^{\prime}\right)+\left|U_{3-i} \backslash U_{3-i}^{\prime}\right| \leq \Delta\left(V_{i}, U_{3-i}^{\prime}\right)+\left|U_{i}^{\prime} \backslash U_{i}\right| \leq \alpha^{1 / 3} n$.

We need to define some new sets which were not specified in [7].
Definition 2.4. For $i=1,2$, let

$$
\begin{aligned}
& \tilde{U}_{i}=\left\{u \in U_{i}: \operatorname{deg}\left(u, V_{3-i}\right) \geq s\right\}, \quad \tilde{V}_{i}=\left\{v \in V_{i}: \operatorname{deg}\left(v, U_{3-i}\right) \geq s\right\}, \\
& \hat{U}_{i}=U_{i} \backslash \tilde{U}_{i}, \text { and } \hat{V}_{i}=V_{i} \backslash \tilde{V}_{i} .
\end{aligned}
$$

Note that the following inequalities are satisfied:

$$
\begin{align*}
& \delta\left(\hat{U}_{1}, V_{0}\right)+\delta\left(\hat{U}_{2}, V_{0}\right) \geq n+3 s-2-\left(\left|V_{1}\right|+s-1\right)-\left(\left|V_{2}\right|+s-1\right)=\left|V_{0}\right|+s \text { and }  \tag{6}\\
& \delta\left(\hat{V}_{1}, U_{0}\right)+\delta\left(\hat{V}_{2}, U_{0}\right) \geq n+3 s-2-\left(\left|U_{1}\right|+s-1\right)-\left(\left|U_{2}\right|+s-1\right)=\left|U_{0}\right|+s . \tag{7}
\end{align*}
$$

### 2.2 Preliminary Claims

The following useful lemma appears in [7.
Lemma 2.5 (Zhao [7], Fact 5.3). Let $F[A, B]$ be a bipartite graph with $\delta:=\delta(A, B)$ and $\Delta:=$ $\Delta(B, A)$ Then $F$ contains $f_{h}$ vertex disjoint $h$-stars from $A$ to $B$, and $g_{h}$ vertex disjoint $h$-stars from $B$ to $A$ (the stars from $A$ to $B$ and those from $B$ to $A$ need not be disjoint), where

$$
f_{h} \geq \frac{(\delta-h+1)|A|}{h \Delta+\delta-h+1}, \quad g_{h} \geq \frac{\delta|A|-(h-1)|B|}{\Delta+h \delta-h+1} .
$$

We now prove three claims that we will need in the main proof.
Claim 2.6. Let $i \in\{1,2\}$ and $\{A, B\}=\left\{U_{i}, V_{3-i}\right\}$. Let $0 \leq c \leq \alpha^{1 / 3} n, B_{0} \subseteq B$ and $A_{0}=\{v \in$ $\left.A: \operatorname{deg}\left(v, B_{0}\right) \geq s+c\right\}$. If $\left|A_{0}\right| \geq \frac{n}{4}$ then there is a set $\mathcal{S}_{A}$ of at least $\frac{c+1}{8 s \alpha^{1 / 3}}$ vertex disjoint $s$-stars from $A_{0}$ to $B_{0}$.

Proof. Let $\mathcal{S}_{A}$ be a maximum set of vertex disjoint $s$-stars from $A_{0}$ to $B_{0}$ and let $f_{s}=\left|\mathcal{S}_{A}\right|$. We apply Lemma 2.5] to the graph $G\left[A_{0}, B_{0}\right]$. Recall, by Claim [2.3, that $\Delta(B, A) \leq \alpha^{1 / 3} n$. Then

$$
f_{s} \geq \frac{(c+1)\left|A_{0}\right|}{s \alpha^{1 / 3} n+c+1} \geq \frac{(c+1) \frac{n}{4}}{2 s \alpha^{1 / 3} n}=\frac{c+1}{8 s \alpha^{1 / 3}} .
$$

Note that since $n=(2 k+1)(s+t)$, we can write $\delta(G) \geq \frac{n+3 s}{2}-1=k(s+t)+2 s+\frac{t}{2}-1$.
Claim 2.7. Let $i \in\{1,2\}$ and $\{A, B\}=\left\{U_{i}, V_{3-i}\right\}$. Let $|A|=k(s+t)+z$ and $|B|=k(s+t)+y$. Suppose $y \geq z$ and $y \geq \frac{t+1}{2}$. Then there is a set $\mathcal{S}_{B}$ of $y$ vertex disjoint s-stars with centers $C_{B} \subseteq B$ and leaves $L_{A} \subseteq A$. Furthermore if $z \geq 1$, then there is a set $\mathcal{S}_{A}$ of $z$ vertex disjoint $s$-stars from $A \backslash L_{A}$ to $B \backslash C_{B}$.

Proof. Let $\beta:=32 s \alpha^{1 / 3}$ and recall that by the choice of $\alpha$ we have $\frac{1}{t} \gg \beta \gg 2 \alpha^{1 / 3}$. We show that the desired set $\mathcal{S}_{B}$ exists by applying Lemma [2.5 to the graph $G[A, B]$. We have $\delta(A, B) \geq$ $k(s+t)+2 s+\frac{t}{2}-1-(n-|B|)=y+s-\frac{t}{2}-1$ and $\Delta(B, A) \leq \alpha^{1 / 3} n$ by Claim 2.3, Let $g_{s}=\left|\mathcal{S}_{B}\right|$, then

$$
\begin{aligned}
g_{s} & \geq \frac{\left(y-\frac{t}{2}+s-1\right)(k(s+t)+z)-(s-1)(k(s+t)+z+y-z)}{\alpha^{1 / 3} n+s\left(y-\frac{t}{2}+s-1\right)-s+1} \\
& =\frac{\left(y-\frac{t}{2}\right)(k(s+t)+z)-(s-1)(y-z)}{\alpha^{1 / 3} n+s\left(y-\frac{t}{2}\right)+s^{2}-2 s+1} \\
& \geq \frac{\left(y-\frac{t}{2}\right) \frac{n}{3}}{2 \alpha^{1 / 3} n \quad \quad\left(\text { since } y \leq \alpha^{2 / 3} \frac{n}{2} \text { and }-\alpha^{2 / 3} \frac{n}{2} \leq z,\right. \text { by Claim (2.3) }} \\
& \geq y \quad\left(\text { since } y \geq \frac{t+1}{2} \text { and } \alpha \ll 1\right) .
\end{aligned}
$$

Thus the desired set $\mathcal{S}_{B}$ exists.
Suppose $z \geq 1$. Let $c:=\frac{1}{2} y$ if $y \geq 1 / \beta$, and let $c:=0$ if $y<1 / \beta$. Let $B_{0}=B \backslash C_{B}$ and $A_{0}=\left\{v \in A \backslash L_{A} \mid \operatorname{deg}\left(v, B_{0}\right) \geq s+c\right\}$ and $\bar{A}=\left(A \backslash L_{A}\right) \backslash A_{0}$. Suppose that $|\bar{A}| \geq \frac{n}{16}$. Then there exists $u \in C_{B}$ such that if $y<1 / \beta$,

$$
\operatorname{deg}(u, A) \geq \frac{e\left(\bar{A}, C_{B}\right)}{\left|C_{B}\right|} \geq \frac{\left(y-\frac{t}{2}+s-1-(s-1)\right) \frac{n}{16}}{y}=\frac{\left(y-\frac{t}{2}\right) \frac{n}{16}}{y}>\frac{\beta n}{32} \geq \alpha^{1 / 3} n
$$

and if $y \geq 1 / \beta$,

$$
\operatorname{deg}(u, A) \geq \frac{e\left(\bar{A}, C_{B}\right)}{\left|C_{B}\right|}>\frac{\left(y-\frac{t}{2}+s-1-\left(s+\frac{1}{2} y\right)\right) \frac{n}{16}}{y}=\frac{\left(\frac{y}{2}-\frac{t}{2}-1\right) \frac{n}{16}}{y}>\frac{n}{64} \geq \alpha^{1 / 3} n,
$$

each contradicting Claim 2.3. So $|\bar{A}|<\frac{n}{16}$ and thus $\left|A_{0}\right| \geq|A|-\left|L_{A}\right|-\frac{n}{16} \geq k(s+t)-s \alpha^{2 / 3} \frac{n}{2}-\frac{n}{16} \geq \frac{n}{4}$. Now let $\mathcal{S}_{A}$ be a maximum set of disjoint $s$-stars from $A_{0}$ to $B_{0}$ and let $f_{s}=\left|\mathcal{S}_{A}\right|$. By Lemma 2.6 we have $f_{s} \geq \frac{c+1}{8 s \alpha^{1 / 3}}$. Recall that $1 \leq z \leq y$. If $y \geq 1 / \beta$, then $f_{s} \geq \frac{y}{16 s \alpha^{1 / 3}} \geq z$ and if $y<1 / \beta$, then $f_{s} \geq \frac{1}{8 s \alpha^{1 / 3}} \geq \frac{1}{\beta} \geq z$. So the desired set $\mathcal{S}_{A}$ exists.

Claim 2.8. Suppose $\left|U_{0}\right|,\left|V_{0}\right| \geq s$. If $\left|\hat{U}_{1}\right| \geq \frac{n}{8}$ and $\left|\hat{U}_{2}\right| \geq \frac{n}{8}$ (see Definition 2.4), then there is a $K_{s, t}=: K^{1}$ with $s$ vertices in $V_{0},\lceil t / 2\rceil$ vertices in $U_{1}$ and $\lfloor t / 2\rfloor$ vertices in $U_{2}$. Likewise, if $\left|\hat{V}_{1}\right| \geq \frac{n}{8}$ and $\left|\hat{V}_{2}\right| \geq \frac{n}{8}$ then there is a $K_{s, t}=: K^{2}$ with $s$ vertices in $U_{0}$, $\lceil t / 2\rceil$ vertices in $V_{1}$ and $\lfloor t / 2\rfloor$ vertices in $V_{2}$.

Proof. Without loss of generality we will only prove the first statement. Let

$$
\ell:=s\binom{\left|U_{2}\right|}{\lfloor t / 2\rfloor} /\binom{\left\lceil\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) n / 2\right\rceil}{\lfloor t / 2\rfloor}
$$

and recall that $\left|U_{1}\right|,\left|U_{2}\right| \leq\left(1+\alpha^{2 / 3}\right) \frac{n}{2}$ by Claim 2.3. Thus we have

$$
\begin{equation*}
\ell \leq s\left(\frac{\left|U_{2}\right|}{\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) \frac{n}{2}-\lfloor t / 2\rfloor}\right)^{\lfloor t / 2\rfloor} \leq s\left(\frac{\left(1+\alpha^{2 / 3}\right) \frac{n}{2}}{\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) \frac{n}{3}}\right)^{\lfloor t / 2\rfloor} \leq s\left(\frac{3\left(1+\alpha^{2 / 3}\right)}{2\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right)}\right)^{\lfloor t / 2\rfloor} \tag{8}
\end{equation*}
$$

Case 1. $\left|V_{0}\right| \geq \ell\left(\begin{array}{c}\left.\left\lvert\, \begin{array}{c}\left|U_{1}\right| \\ {[t / 2\rceil}\end{array}\right.\right)\end{array}\right)\binom{\left\lceil\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) n / 2\right\rceil}{\lceil t / 2\rceil}$. Recall that $\delta\left(V_{0}, U_{i}\right) \geq\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) n / 2$ for $i=1,2$ by Claim 2.3 and suppose that there is no $K_{\lceil t / 2\rceil, \ell}$ with $\lceil t / 2\rceil$ vertices in $U_{1}$ and $\ell$ vertices in $V_{0}$. We count the $\lceil t / 2\rceil$-stars from $V_{0}$ to $U_{1}$ in two ways which gives

$$
\left|V_{0}\right|\binom{\left\lceil\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) n / 2\right\rceil}{\lceil t / 2\rceil}<\ell\binom{\left|U_{1}\right|}{\lceil t / 2\rceil}
$$

contradicting the lower bound for $\left|V_{0}\right|$. Consequently there is a complete bipartite graph $K^{\prime}=$ $K_{\lceil t / 2\rceil, \ell}$ with $\lceil t / 2\rceil$ vertices in $U_{1}$ and $\ell$ vertices in $V_{0}$. If there is no $K_{\lfloor t / 2\rfloor, s}$ with $s$ vertices in $V\left(K^{\prime}\right) \cap V_{0}$ and $\lfloor t / 2\rfloor$ vertices in $U_{2}$, then a similar counting argument gives

$$
\ell\binom{\left\lceil\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) n / 2\right\rceil}{\lfloor t / 2\rfloor}<s\binom{\left|U_{2}\right|}{\lfloor t / 2\rfloor}
$$

contradicting the definition of $\ell$.
Case 2. $\left|V_{0}\right|<\ell\binom{\left|U_{1}\right|}{[t / 2\rceil} /\binom{\left\lceil\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) n / 2\right\rceil}{\lceil t / 2\rceil}$. By (8) , we have

$$
\left|V_{0}\right|<\ell\left(\frac{3\left(1+\alpha^{2 / 3}\right)}{2\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right)}\right)^{\lceil t / 2\rceil} \leq s\left(\frac{3\left(1+\alpha^{2 / 3}\right)}{2\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right)}\right)^{t} .
$$

Let $p:=\delta\left(\hat{U}_{1}, V_{0}\right)$, and note that $p \geq s$ by (6). We claim that there is a complete bipartite graph $K^{\prime}:=K_{\lceil t / 2\rceil, p}$ with $\lceil t / 2\rceil$ vertices in $\hat{U}_{1}$ and $p$ vertices in $V_{0}$. Let $c$ be the number of $p$-stars with centers in $\hat{U}_{1}$ and leaves in $V_{0}$. We have $c \geq\left|\hat{U}_{1}\right| \geq \frac{n}{8}$ and if no $p$-subset of $V_{0}$ is in $\lceil t / 2\rceil$ of such stars, i.e. $K^{\prime}$ does not exist, we have $c \leq(\lceil t / 2\rceil-1)\binom{\left|V_{0}\right|}{p}$ which contradicts the fact that $\left|V_{0}\right|$ is $O(1)$ and $n$ is sufficiently large (with respect to $\alpha$, $t$, and consequently $\left.\left|V_{0}\right|\right)$. From (6) we have $\delta\left(\hat{U}_{2}, V_{0}\right) \geq\left|V_{0}\right|-p+s$, so every vertex $u \in \hat{U}_{2}$ has at least $s$ neighbors in $V\left(K^{\prime}\right) \cap V_{0}$. Repeating the argument above by counting $s$-stars with centers in $\hat{U}_{2}$ and leaves in $V\left(K^{\prime}\right) \cap V_{0}$ gives $K^{\prime \prime}:=K_{s,\lfloor t / 2\rfloor}$. Now choose $K^{1} \subseteq K^{\prime} \cup K^{\prime \prime}$ having the property that $\left|V_{0} \cap V\left(K^{1}\right)\right|=s,\left|U_{1} \cap V\left(K^{1}\right)\right|=\lceil t / 2\rceil$, and $\left|U_{2} \cap V\left(K^{1}\right)\right|=\lfloor t / 2\rfloor$ as desired.

### 2.3 Extremal Case

Recall that $t \geq 2 s+1, n=(2 k+1)(s+t)$ for some sufficiently large $k \in \mathbb{N}$, and $\delta(G) \geq \frac{n+3 s}{2}-1=$ $k(s+t)+2 s+\frac{t}{2}-1$. We start with the partition given in Section 2.1 and we call $U_{0}$ and $V_{0}$ the exceptional sets. Let $i \in\{1,2\}$. We will attempt to update the partition by moving a constant number (depending only on $t$ ) of special vertices between $U_{1}$ and $U_{2}$, denote them by $X$, and special vertices between $V_{1}$ and $V_{2}$, denote them by $Y$, as well as partitioning the exceptional sets as $U_{0}=U_{0}^{1} \cup U_{0}^{2}$ and $V_{0}=V_{0}^{1} \cup V_{0}^{2}$. Let $U_{1}^{*}, U_{2}^{*}, V_{1}^{*}$ and $V_{2}^{*}$ be the resulting sets after moving the special vertices. Our goal is to obtain two graphs, $G_{1}:=G\left[U_{1}^{*} \cup U_{0}^{1}, V_{1}^{*} \cup V_{0}^{1}\right]$ and $G_{2}:=\left[U_{2}^{*} \cup U_{0}^{2}, V_{2}^{*} \cup V_{0}^{2}\right]$ so that $G_{1}$ satisfies

$$
\left|U_{1}^{*} \cup U_{0}^{1}\right|=\ell_{1}(s+t)+a s+b t,\left|V_{1}^{*} \cup V_{0}^{1}\right|=\ell_{1}(s+t)+b s+a t
$$

and $G_{2}$ satisfies

$$
\left|U_{2}^{*} \cup U_{0}^{2}\right|=\ell_{2}(s+t)+b s+a t,\left|V_{2}^{*} \cup V_{0}^{2}\right|=\ell_{2}(s+t)+a s+b t,
$$

for some nonnegative integers $a, b, \ell_{1}, \ell_{2}$. We tile $G_{1}$ as follows. We find $a$ copies of $K_{s, t}$, each with $t$ vertices in $U_{1}^{*}$, so that each special vertex in $X \cap U_{1}^{*}$ is in a unique copy (some copies may not contain any special vertex). Also, we find $b$ copies of $K_{s, t}$, each with $t$ vertices in $V_{1}^{*}$ so that each special vertex in $Y \cap V_{1}^{*}$ is in a unique copy (some copies may not contain any special vertex). Note that we only move vertices which will make this step possible. Deleting these $a+b$ copies of $K_{s, t}$ from $G_{1}$ gives us a balanced bipartite graph on $2 \ell_{1}(s+t)$ vertices. As noted in [7] and [3], this graph can easily be tiled: By Claim 2.3 there are at most $\alpha^{2 / 3} \frac{n}{2}$ exceptional vertices in $U_{0}^{1}$ (resp. $V_{0}^{1}$ ), each with degree at least $\left(\alpha^{1 / 3}-\alpha^{2 / 3}\right) \frac{n}{2}$ to $V_{1}$ (resp. $U_{1}$ ), so they may greedily be incorporated into unique copies of $K_{s+t, s+t}$. The remaining graph is still balanced, divisible by $s+t$, and almost complete, thus can be tiled.

So if we are able to split $G$ into graphs $G_{1}$ and $G_{2}$ as detailed above, we will conclude that $G$ can be tiled. However, if it is not possible to carry out this goal, then we will use an alternate method which is explained in Case 2.

Proof of Theorem 1.6. There are two main cases.
Case 1. $\max \left\{\left|U_{1}\right|,\left|U_{2}\right|,\left|V_{1}\right|,\left|V_{2}\right|\right\} \geq k(s+t)+\frac{t+1}{2}$. Without loss of generality, suppose $\left|U_{1}\right|=$ $\max \left\{\left|U_{1}\right|,\left|U_{2}\right|,\left|V_{1}\right|,\left|V_{2}\right|\right\}$.

Case 1.1. $\left|V_{2} \cup V_{0}\right| \geq k(s+t)+s$. We apply Claim 2.7 to $G\left[U_{1}, V_{2}\right]$ with $A=V_{2}$ and $B=U_{1}$ to obtain $\left|U_{1}\right|-(k(s+t)+s)$ vertex disjoint $s$-stars with centers $C_{U} \subseteq U_{1}$ and leaves in $V_{2}$ and a set of $\max \left\{0,\left|V_{2}\right|-(k(s+t)+s)\right\}$ vertex disjoint $s$-stars with centers $C_{V} \subseteq V_{2}$ and leaves in $U_{1}$. We move the vertices in $C_{U}$ to $U_{2}$ and the vertices in $C_{V}$ to $V_{1}$. If $\left|V_{2}\right|<k(s+t)+s$, we choose $V_{0}^{\prime} \subseteq V_{0}$ so that $\left.\mid\left(V_{2} \cup V_{0}\right) \backslash V_{0}^{\prime}\right) \mid=k(s+t)+s$ otherwise we set $V_{0}^{\prime}=\emptyset$. Then $G_{1}:=G\left[U_{1} \backslash C_{U}, V_{1} \cup C_{V} \cup V_{0}^{\prime}\right]$ satisfies

$$
\left|U_{1}\right|-\left|C_{U}\right|=k(s+t)+s,\left|V_{1}\right|+\left|V_{0}^{\prime}\right|+\left|C_{V}\right|=k(s+t)+t,
$$

and $G_{2}:=G-G_{1}$ satisfies

$$
\left|U_{2} \cup U_{0}\right|+\left|C_{U}\right|=k(s+t)+t,\left|V_{2}\right|+\left|V_{0} \backslash V_{0}^{\prime}\right|-\left|C_{V}\right|=k(s+t)+s .
$$

Thus $G_{1}$ and $G_{2}$ can be tiled, which completes the tiling of $G$.

## Case 1.2. $\left|V_{2} \cup V_{0}\right|<k(s+t)+s$.

This implies $\left|V_{1}\right|>k(s+t)+t$. So we apply Claim 2.7 to $G\left[V_{1}, U_{2}\right]$ with $A=U_{2}$ and $B=V_{1}$ to obtain a set of $\left|V_{1}\right|-k(s+t)$ vertex disjoint $s$-stars with centers $C_{V} \subseteq V_{1}$ and leaves in $U_{2}$. Likewise we apply Claim [2.7 to $G\left[U_{1}, V_{2}\right]$ with $A=V_{2}$ and $B=U_{1}$ to obtain a set of $\left|U_{1}\right|-k(s+t)$ vertex $s$-stars with centers $C_{U} \subseteq U_{1}$ and leaves in $V_{2}$. We move the vertices in $C_{U}$ to $U_{2}$ and the vertices in $C_{V}$ to $V_{2}$. Then $G_{1}:=G\left[U_{1} \backslash C_{U}, V_{1} \backslash C_{V}\right]$ satisfies

$$
\left|U_{1}\right|-\left|C_{U}\right|=k(s+t),\left|V_{1}\right|-\left|C_{V}\right|=k(s+t)
$$

and $G_{2}:=G-G_{1}$ satisfies

$$
\left|U_{2} \cup U_{0}\right|+\left|C_{U}\right|=(k+1)(s+t),\left|V_{2} \cup V_{0}\right|+\left|C_{V}\right|=(k+1)(s+t) .
$$

Thus $G_{1}$ and $G_{2}$ can be tiled, which completes the tiling of $G$.
Case 2. $\max \left\{\left|U_{1}\right|,\left|U_{2}\right|,\left|V_{1}\right|,\left|V_{2}\right|\right\} \leq k(s+t)+\frac{t}{2}$. Note that this implies $\left|U_{0}\right|,\left|V_{0}\right| \geq s$.
Case 2.1. $\max \left\{\left|\tilde{U}_{1}\right|,\left|\tilde{U}_{2}\right|,\left|\tilde{V}_{1}\right|,\left|\tilde{V}_{2}\right|\right\} \geq \frac{n}{4}$ (see Definition (2.4). Without loss of generality we can assume $\left|\tilde{U}_{1}\right|=\max \left\{\left|\tilde{U}_{1}\right|,\left|\tilde{U}_{2}\right|,\left|\tilde{V}_{1}\right|,\left|\tilde{V}_{2}\right|\right\}$. Set $h:=\lceil t /(2 s)\rceil$. Since $\left|\tilde{U}_{1}\right|>\frac{n}{4}$ and $\frac{1}{8 s \alpha^{1 / 3}} \geq$ $(h-1)(s+t)$, we can apply Claim 2.6 to $G\left[\tilde{U}_{1}, V_{2}\right]$ with $c=0$ to obtain a set of $(h-1)(s+t)$ vertex
disjoint $s$-stars with centers $C_{U} \subseteq \tilde{U}_{1}$ and leaves in $V_{2}$. We first move the vertices in $C_{U}$ from $\tilde{U}_{1}$ to $U_{2}$. Then since

$$
\frac{t}{2}=s \frac{t}{2 s} \leq s h \leq s \frac{t+2 s-1}{2 s}=\frac{t}{2}+s-\frac{1}{2},
$$

we can choose sets $U_{0}^{\prime} \subseteq U_{0}$ with $\left|U_{0}^{\prime}\right|=k(s+t)+\lfloor t / 2\rfloor-\left|U_{1}\right|+s h-\lfloor t / 2\rfloor$ and $V_{0}^{\prime} \subseteq V_{0}$ with $\left|V_{0}^{\prime}\right|=k(s+t)+\lfloor t / 2\rfloor-\left|V_{1}\right|+s+\lceil t / 2\rceil-s h$ so that $G_{1}:=G\left[\left(U_{1} \cup U_{0}^{\prime}\right) \backslash C_{U}, V_{1} \cup V_{0}^{\prime}\right]$ satisfies

$$
\left|U_{1}\right|+\left|U_{0}^{\prime}\right|-\left|C_{U}\right|=(k-h+1)(s+t)+h s,\left|V_{1}\right|+\left|V_{0}^{\prime}\right|=(k-h+1)(s+t)+h t,
$$

and $G_{2}:=G-G_{1}$ satisfies

$$
\left|U_{2}\right|+\left|U_{0} \backslash U_{0}^{\prime}\right|+\left|C_{U}\right|=k(s+t)+h t,\left|V_{2}\right|+\left|V_{0} \backslash V_{0}^{\prime}\right|=k(s+t)+h s .
$$

Thus $G_{1}$ and $G_{2}$ can be tiled, which completes the tiling of $G$.
Case 2.2. $\max \left\{\left|\tilde{U}_{1}\right|,\left|\tilde{U}_{2}\right|,\left|\tilde{V}_{1}\right|,\left|\tilde{V}_{2}\right|\right\}<\frac{n}{4}$. Thus for $i=1,2$, we have

$$
\left|\hat{U}_{i}\right|,\left|\hat{V}_{i}\right| \geq\left(1-\alpha^{2 / 3}\right) \frac{n}{2}-\frac{n}{4} \geq \frac{n}{8}
$$

So we may apply Claim 2.8 to obtain the two special copies of $K_{s, t}, K^{1}$ and $K^{2}$. Note that $\left|U_{i} \backslash V\left(K^{1}\right)\right|,\left|V_{i} \backslash V\left(K^{2}\right)\right| \leq k(s+t)$ for $i=1,2$. Let $U_{0}^{\prime}=U_{0} \backslash V\left(K^{2}\right)$ and $V_{0}^{\prime}=V_{0} \backslash V\left(K^{1}\right)$. We remove the graphs $K^{1}$ and $K^{2}$, then we partition the vertices $U_{0}^{\prime}=U_{0}^{1} \cup U_{0}^{2}$ and $V_{0}^{\prime}=V_{0}^{1} \cup V_{0}^{2}$ so that $G_{1}:=G\left[\left(U_{1} \cup U_{0}^{1}\right) \backslash V\left(K^{1}\right),\left(V_{1} \cup V_{0}^{1}\right) \backslash V\left(K^{2}\right)\right]$ satisfies

$$
\left|U_{1}\right|-\lceil t / 2\rceil+\left|U_{0}^{1}\right|=k(s+t),\left|V_{1}\right|-\lceil t / 2\rceil+\left|V_{0}^{1}\right|=k(s+t)
$$

and $G_{2}=G-G_{1}-K^{1}-K^{2}$ satisfies

$$
\left|U_{2}\right|-\lfloor t / 2\rfloor+\left|U_{0}^{2}\right|=k(s+t),\left|V_{2}\right|-\lfloor t / 2\rfloor+\left|V_{0}^{2}\right|=k(s+t) .
$$

Thus $G_{1}$ and $G_{2}$ can be tiled, so along with $K^{1}$ and $K^{2}$, this completes the tiling of $G$.

## 3 Tightness

In this section we will prove Proposition 1.7. We will need to use the graphs $P(m, p)$, where $m, p \in \mathbb{N}$, introduced by Zhao in [7].

Lemma 3.1. For all $p \in \mathbb{N}$ there exists $m_{0}$ such that for all $m \in \mathbb{N}, m>m_{0}$, there exists a balanced bipartite graph, $P(m, p)$, on $2 m$ vertices, so that the following hold:
(i) $P(m, p)$ is $p$-regular
(ii) $P(m, p)$ does not contain a copy of $K_{2,2}$.

Proof of Proposition 1.7. Let $G[U, V]$ be a balanced bipartite graph on $2 n$ vertices satisfying the following conditions. Let $n=(2 k+1)(s+t)$ for some sufficiently large $k$ (as determined by Lemma 3.1 with $p=s-1$ ). Partition $U$ into $U=U_{0} \cup U_{1} \cup U_{2}$ and partition $V$ into $V=V_{0} \cup V_{1} \cup V_{2}$ where, $\left|U_{1}\right|=\left|V_{2}\right|=k(s+t)+\left\lfloor\frac{t+1}{2}\right\rfloor,\left|V_{1}\right|=\left|U_{2}\right|=k(s+t)+\left\lceil\frac{t+1}{2}\right\rceil$ and $\left|U_{0}\right|=\left|V_{0}\right|=s-1$. Let $G\left[U_{i}, V_{i}\right]$ be complete for $i \in\{1,2\}, G\left[U_{1}, V_{2}\right] \cong P\left(k(s+t)+\left\lfloor\frac{t+1}{2}\right\rfloor, s-1\right)$ and $G\left[U_{2}, V_{1}\right] \cong$
$P\left(k(s+t)+\left\lceil\frac{t+1}{2}\right\rceil, s-1\right)$. Let $G\left[U_{0}, V_{1} \cup V_{2}\right]$ be complete, $G\left[V_{0}, U_{1} \cup U_{2}\right]$ be complete and $G\left[U_{0}, V_{0}\right]$ be empty. Note that

$$
\delta(G)= \begin{cases}\frac{n+3 s}{2}-\frac{3}{2} & \text { if } t \text { is odd } \\ \frac{n+3 s}{2}-2 & \text { if } t \text { is even. }\end{cases}
$$

Finally we reiterate the following properties of $G\left[U_{1}, V_{2}\right]$ and $G\left[U_{2}, V_{1}\right]$. For $i=1,2$,

$$
\begin{equation*}
\Delta\left(U_{i}, V_{3-i}\right)=\Delta\left(V_{i}, U_{3-i}\right)=s-1 \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
G\left[U_{i}, V_{3-i}\right] \text { is } K_{2,2} \text {-free. } \tag{10}
\end{equation*}
$$

For $i \in\{1,2\}$ and $A \in\left\{U_{i}, V_{i}\right\}$, let $A^{D}:=V_{3-i}$ if $A=U_{i}$ and let $A^{D}:=U_{3-i}$ if $A=V_{i}$. We call $A^{D}$ the diagonal set of $A$. Let $A^{N}:=V_{i}$ if $A=U_{i}$ and $A^{N}:=U_{i}$ if $A=V_{i}$. We call $A^{N}$ the non-diagonal set of $A$. Finally, we let $A^{M}:=V_{0}$ if $A=U_{i}$ and $A^{M}:=U_{0}$ if $A=V_{i}$. We call $A^{M}$ the opposite middle set of $A$.

Suppose $K \cong K_{s, t}$ is a subgraph of $G$. We say $K$ is a crossing $K_{s, t}$ if $V(K) \cap\left(U_{1} \cup V_{1}\right) \neq \emptyset$ and $V(K) \cap\left(U_{2} \cup V_{2}\right) \neq \emptyset$. Let $\mathcal{W}=\left\{U_{1}, U_{2}, V_{1}, V_{2}\right\}$.
Claim 3.2. If $K$ is a crossing $K_{s, t}$, then
(i) $V(K)$ must intersect some member of $\mathcal{W}$ in exactly one vertex, and
(ii) there is a unique $A_{0} \in\left\{U_{0}, V_{0}\right\}$ such that $V(K) \cap A_{0} \neq \emptyset$.

Furthermore, if $|V(K) \cap A|=1$ for some $A \in \mathcal{W}$, then
(iii) $V(K) \cap A^{D} \neq \emptyset$, and
(iv) either $\left|V(K) \cap A^{N}\right| \geq 2$ and $V(K) \cap\left(A^{N}\right)^{D}=\emptyset$, or $V(K) \cap A^{N}=\emptyset$ and $\left|V(K) \cap\left(A^{N}\right)^{D}\right| \geq 2$.

Proof. (i) Suppose not. Then without loss of generality, suppose that $\left|V(K) \cap V_{1}\right| \geq 2$. By (10) we have, $\left|V(K) \cap U_{2}\right| \leq 1$ and thus $V(K) \cap U_{2}=\emptyset$. Since $K$ is crossing, we have $V(K) \cap V_{2} \neq \emptyset$ and thus $\left|V(K) \cap V_{2}\right| \geq 2$. By (10) we have, $\left|V(K) \cap U_{1}\right| \leq 1$ and thus $V(K) \cap U_{1}=\emptyset$. This is a contradiction, since $K \cong K_{s, t}$ and $|V(K) \cap U| \leq\left|U_{0}\right|=s-1$.
(ii) Suppose first that $V(K) \cap U_{0}=\emptyset=V(K) \cap V_{0}$. By Claim 3.2(i), we can assume without loss of generality that $\left|V(K) \cap U_{1}\right|=1$. Then either $\left|V(K) \cap U_{2}\right|=t-1$ or $\left|V(K) \cap U_{2}\right|=s-1$. If $\left|V(K) \cap U_{2}\right|=t-1$, then by (9) we must have $V(K) \cap V_{1}=\emptyset$ which implies $\left|V(K) \cap V_{2}\right|=s$, contradicting (99). If $\left|V(K) \cap U_{2}\right|=s-1$, then since $t \geq 2 s+1$ we have $\left|V(K) \cap V_{1}\right| \geq s+1$ or $\left|V(K) \cap V_{2}\right| \geq s+1$, both of which contradict (9). Thus there exists $A_{0} \in\left\{U_{0}, V_{0}\right\}$ such that $V(K) \cap A_{0} \neq \emptyset$. Finally since $G\left[U_{0}, V_{0}\right]$ is empty, $A_{0}$ must be unique.
(iii) Suppose that $V(K) \cap A^{D}=\emptyset$. Since $\left|V_{0}\right|=s-1$, we have $V(K) \cap A^{N} \neq \emptyset$ and since $K$ is crossing, we have $V(K) \cap\left(A^{N}\right)^{D} \neq \emptyset$. Then by (9), we have $\left|V(K) \cap A^{N}\right|,\left|V(K) \cap\left(A^{N}\right)^{D}\right| \leq$ $s-1$. Thus $|V(K) \cap U| \leq 2 s-1$ and $|V(K) \cap V| \leq 2 s-2$, contradicting the fact that $K \cong K_{s, t}$ and $t \geq 2 s+1$.
(iv) We first show that it is not possible for either $\left|V(K) \cap A^{N}\right|=1$ or $\left|V(K) \cap\left(A^{N}\right)^{D}\right|=1$. If $\left|V(K) \cap A^{N}\right|=1$, then by (9) and $\left|U_{0}\right|=\left|V_{0}\right|=s-1$, we have $|V(K) \cap U|,|V(K) \cap V| \leq 2 s-1$, contradicting the fact that $K \cong K_{s, t}$ and $t \geq 2 s+1$. So suppose $\left|V(K) \cap\left(A^{N}\right)^{D}\right|=1$. If
$V(K) \cap U_{0}=\emptyset$, then $|V(K) \cap U|=2$ and since $t \geq 3$ we must have $s=2$. Then by (9) we have $|V(K) \cap V| \leq 3$ contradicting the fact that $K \cong K_{s, t}$ and $t \geq 2 s+1$. If $V(K) \cap U_{0} \neq \emptyset$, then $V(K) \cap V_{0}=\emptyset$. So $|V(K) \cap U| \leq s+1$ and by (9), $|V(K) \cap V| \leq 2 s-2$ contradicting the fact that $K \cong K_{s, t}$ and $t \geq 2 s+1$.
Now suppose $V(K) \cap A^{N} \neq \emptyset$ and $V(K) \cap\left(A^{N}\right)^{D} \neq \emptyset$. Thus, by the previous paragraph we have $\left|V(K) \cap A^{N}\right|,\left|V(K) \cap\left(A^{N}\right)^{D}\right| \geq 2$, contradicting (10).
So suppose that $V(K) \cap A^{N}=\emptyset=V(K) \cap\left(A^{N}\right)^{D}$. Then it must be the case that $\mid V(K) \cap$ $\left(A^{N}\right)^{M} \mid=s-1$ and consequently $\left|V(K) \cap A^{D}\right|=t$, contradicting (9).

Let $A \in \mathcal{W}$. We say $K$ is crossing from $A$ if either $|V(K) \cap A|=1$ and $\left|V(K) \cap A^{D}\right| \geq 2$, or $|V(K) \cap A|=1,\left|V(K) \cap A^{D}\right|=1$ and $V(K) \cap A^{M} \neq \emptyset$. We say that a crossing $K_{s, t}$ from $A$ is Type 1 if $\left|V(K) \cap\left(A^{N}\right)^{M}\right|=s-1,\left|V(K) \cap A^{N}\right|=t-p$ and $\left|V(K) \cap A^{D}\right|=p$ for some $2 \leq p \leq s-1$. We say that a crossing $K_{s, t}$ from $A$ is Type 2 if $\left|V(K) \cap\left(A^{N}\right)^{D}\right|=t-1,\left|V(K) \cap A^{M}\right|=s-p$, and $\left|V(K) \cap A^{D}\right|=p$ for some $1 \leq p \leq s-1$.


Figure 1

Claim 3.3. Every crossing $K_{s, t}$ is either Type 1 or Type 2.
Proof. (See Figure 1) Let $K$ be a crossing $K_{s, t}$ and without loss of generality suppose $K$ is crossing from $U_{1}$. Let $p:=\left|V(K) \cap V_{2}\right|$. By Claim 3.2 (iii) and (9) we have $1 \leq p \leq s-1$. Suppose $K$ is not Type 1. If $V(K) \cap U_{2}=\emptyset$, then $\left|V(K) \cap U_{0}\right|=s-1$ which implies $V(K) \cap V_{0}=\emptyset$ by Claim 3.2 (ii). Since $K$ is not Type 1, it must be the case that $\left|V(K) \cap V_{2}\right|=1$ and $\left|V(K) \cap V_{1}\right|=t-1$ in which case $K$ is not crossing from $U_{1}$, contradicting our assumption. So we suppose that $V(K) \cap U_{2} \neq \emptyset$. By Claim 3.2 (iv) we have $\left|V(K) \cap U_{2}\right| \geq 2$ and $V(K) \cap V_{1}=\emptyset$, which implies that $\left|V(K) \cap V_{0}\right|=s-p$. So by Claim 3.2 (ii), we have $V(K) \cap U_{0}=\emptyset$ and thus $\left|V(K) \cap U_{2}\right|=t-1$, so $K$ is Type 2 .

Suppose for a contradiction that $G$ can be tiled with $K_{s, t}$. Let $\mathcal{F}$ be a tiling of $G$ which minimizes the number of crossing $K_{s, t}$ 's.

Claim 3.4. For $i=1,2$, if there is a crossing $K_{s, t}$ of Type 2 from $U_{i}$ or $V_{i}$, then there is no crossing $K_{s, t}$ of Type 2 from $U_{3-i}$ or $V_{3-i}$.

Proof. Without loss of generality suppose $K^{1}$ is a crossing $K_{s, t}$ of Type 2 from $U_{1}$. Suppose that $K^{2}$ is a crossing $K_{s, t}$ of Type 2 from $U_{2}$ (See Figure 2). For $i \in\{1,2\}$, let

$$
K_{*}^{i}:=G\left[U_{i} \cap\left(V\left(K^{1}\right) \cup V\left(K^{2}\right)\right), V\left(K^{3-i}\right) \cap\left(V_{0} \cup V_{i}\right)\right] .
$$




Figure 2: Two cases in the proof of Claim 3.4.
We have $K_{*}^{1} \cong K_{s, t} \cong K_{*}^{2}$, neither of $K_{*}^{1}, K_{*}^{2}$ are crossing, and $V\left(K^{1}\right) \cup V\left(K^{2}\right)=V\left(K_{*}^{1}\right) \cup V\left(K_{*}^{2}\right)$. Thus we obtain a tiling with fewer crossing $K_{s, t}$ 's, contradicting the minimality of $\mathcal{F}$.

Now, suppose $K^{1}$ is a crossing $K_{s, t}$ of Type 2 from $U_{1}$ and $K^{2}$ is a crossing $K_{s, t}$ of Type 2 from $V_{2}$ (See Figure 2). Specify an element $L^{1} \in \mathcal{F}$, such that $V\left(L^{1}\right) \subseteq U_{1} \cup V_{1}$ and $\left|V\left(L^{1}\right) \cap V_{1}\right|=t$ and specify an element $L^{2} \in \mathcal{F}$, such that $V\left(L^{2}\right) \subseteq U_{2} \cup V_{2}$ and $\left|V\left(L^{2}\right) \cap U_{2}\right|=t$. Choose arbitrary vertices $v^{\prime} \in V\left(K^{1}\right) \cap V_{0}$ and $u^{\prime} \in V\left(K^{2}\right) \cap U_{0}$. We now define four subgraphs of $G$. Let

$$
\begin{aligned}
K_{*}^{1} & :=G\left[V\left(L^{1}\right) \cap V_{1},\left(V\left(K^{1}\right) \cup V\left(K^{2}\right)\right) \cap\left(\left(U_{1} \cup U_{0}\right) \backslash\left\{u^{\prime}\right\}\right)\right], \\
L_{*}^{1} & :=G\left[V\left(L^{1}\right) \cap U_{1},\left(V\left(K^{2}\right) \cap V_{1}\right) \cup\left\{v^{\prime}\right\}\right], \\
K_{*}^{2} & :=G\left[V\left(L^{2}\right) \cap U_{2},\left(V\left(K^{1}\right) \cup V\left(K^{2}\right)\right) \cap\left(\left(V_{2} \cup V_{0}\right) \backslash\left\{v^{\prime}\right\}\right)\right], \text { and } \\
L_{*}^{2} & :=G\left[V\left(L^{2}\right) \cap V_{1},\left(V\left(K^{1}\right) \cap U_{2}\right) \cup\left\{u^{\prime}\right\}\right] .
\end{aligned}
$$

All of $K_{*}^{1}, K_{*}^{2}, L_{*}^{1}, L_{*}^{2}$ are isomorphic to $K_{s, t}$, none of $K_{*}^{1}, K_{*}^{2}, L_{*}^{1}, L_{*}^{2}$ are crossing, and $V\left(K_{*}^{1}\right) \cup$ $V\left(K_{*}^{2}\right) \cup V\left(L_{*}^{1}\right) \cup V\left(L_{*}^{2}\right)=V\left(K^{1}\right) \cup V\left(K^{2}\right) \cup V\left(L^{1}\right) \cup V\left(L^{2}\right)$. Thus we obtain a tiling with fewer crossing $K_{s, t}$ 's, contradicting the minimality of $\mathcal{F}$.

For $i \in\{1,2\}$, let $\mathcal{F}_{i}$ be the set of all copies of $K_{s, t}$ in $\mathcal{F}$ which touch $U_{i} \cup V_{i}$. And let $U_{i}^{*}$ (resp. $V_{i}^{*}$ ) be all the vertices in $U$ (resp. $V$ ) which touch elements of $\mathcal{F}_{i}$. Precisely, let $\mathcal{F}_{i}=\{K \in$ $\left.\mathcal{F}: V(K) \cap\left(U_{i} \cup V_{i}\right) \neq \emptyset\right\}$ for $i=1,2$, and let

$$
U_{i}^{*}=\left(\cup_{K \in \mathcal{F}_{i}} V(K)\right) \cap U \quad \text { and } \quad V_{i}^{*}=\left(\cup_{K \in \mathcal{F}_{i}} V(K)\right) \cap V .
$$

Note that $U_{i} \subseteq U_{i}^{*}$ and $V_{i} \subseteq V_{i}^{*}$. We will use the following claim to show that all of the remaining possible configurations of crossing $K_{s, t}$ 's lead to contradictions.
Claim 3.5. For all $i \in\{1,2\}$, either

$$
\max \left\{\left|U_{i}^{*}\right|,\left|V_{i}^{*}\right|\right\} \geq k(s+t)+2 t \text { or } \min \left\{\left|U_{i}^{*}\right|,\left|V_{i}^{*}\right|\right\} \geq(k+1)(s+t) .
$$

Proof. Suppose that $\max \left\{\left|U_{i}^{*}\right|,\left|V_{i}^{*}\right|\right\}<k(s+t)+2 t$. Then since $U_{i} \subseteq U_{i}^{*}$ and $V_{i} \subseteq V_{i}^{*}$, we have

$$
\begin{equation*}
k(s+t)+s<\left|U_{i}^{*}\right|,\left|V_{i}^{*}\right|<k(s+t)+2 t \tag{11}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\| U_{i}^{*}\left|-\left|V_{i}^{*}\right|\right|<2 t-s \tag{12}
\end{equation*}
$$

By definition $G\left[U_{i}^{*}, V_{i}^{*}\right]$ can be tiled, thus there exists nonnegative integers $\ell, a, b$ such that $\left|U_{i}^{*}\right|=$ $\ell(s+t)+a s+b t$ and $\left|V_{i}^{*}\right|=\ell(s+t)+a t+b s$. By choosing $\ell$ to be maximal, we have $a=0$ or $b=0$. If $\ell \leq k-1$, then in order to satisfy the lower bound in (11) we must have $a \geq 3$ or $b \geq 3$. Since $a=0$ or $b=0$, we have $\| U_{i}^{*}\left|-\left|V_{i}^{*}\right|\right| \geq 3 t-3 s \geq 2 t-s$, which contradicts (12). If $\ell=k$, then in order to satisfy the lower bound in (11), we must have $a \geq 2$ or $b \geq 2$, but then we violate the upper bound. So $\ell \geq k+1$ and we have $\min \left\{\left|U_{i}^{*}\right|,\left|V_{i}^{*}\right|\right\} \geq(k+1)(s+t)$.

We will also use the following facts. For $i=1,2$, we have

$$
\begin{equation*}
\left|V_{i} \cup V_{0}\right|+s,\left|U_{i} \cup U_{0}\right|+s \leq k(s+t)+\frac{t+2}{2}+2 s-1<(k+1)(s+t) . \tag{13}
\end{equation*}
$$

which in particular implies

(Case 1.2.ii)
Figure 3: Case 1
Let $i \in\{1,2\}$ and let $X_{i}=\left\{K \in \mathcal{F}: K\right.$ is crossing from $U_{i}$ and $K$ is Type 2$\}$ and $Y_{i}=\{K \in$ $\mathcal{F}: K$ is crossing from $V_{i}$ and $K$ is Type 2\}. Since $\left|U_{0}\right|=\left|V_{0}\right|=s-1$, Claim 3.2 (ii) implies,

$$
\begin{equation*}
0 \leq\left|X_{i}\right|,\left|Y_{i}\right| \leq s-1 . \tag{15}
\end{equation*}
$$

Case 0. There are no crossing $K_{s, t}$ 's. So $\left|U_{1}^{*}\right| \leq\left|U_{1} \cup U_{0}\right|$ and $\left|V_{1}^{*}\right| \leq\left|V_{1} \cup V_{0}\right|$. Then by (13) we have $\left|U_{1}^{*}\right|,\left|V_{1}^{*}\right|<(k+1)(s+t)$, contradicting Claim 3.5.
Case 1. There is a crossing $K_{s, t}$ of Type 1. Without loss of generality, suppose $K^{1}$ is a crossing $K_{s, t}$ of Type 1 from $U_{1}$ and let $p:=\left|V\left(K^{1}\right) \cap V_{2}\right|$. Since $U_{0} \backslash V\left(K^{1}\right)=\emptyset$, there can be no other crossing $K_{s, t}$ 's of Type 1 from $U_{1}$ or $U_{2}$ and no crossing $K_{s, t}$ 's of Type 2 from $V_{1}$ or $V_{2}$. By Claim 3.3, we must only consider five subcases:

Case 1.0. $K^{1}$ is the only crossing $K_{s, t}$. So $\left|U_{1}^{*}\right| \leq\left|U_{1} \cup U_{0}\right|$ and $\left|V_{1}^{*}\right| \leq\left|V_{1} \cup V_{0}\right|+p<\left|V_{1} \cup V_{0}\right|+s$. Then by (13) we have $\left|U_{1}^{*}\right|,\left|V_{1}^{*}\right|<(k+1)(s+t)$, contradicting Claim 3.5,

Case 1.1.i. There is a crossing $K_{s, t}$ of Type 1 from $V_{1}$. Let $K^{2}$ be a crossing $K_{s, t}$ from $V_{1}$ and let $q:=\left|V\left(K^{2}\right) \cap U_{2}\right|$. Since $V_{0} \backslash V\left(K^{2}\right)=\emptyset, K^{1}$ and $K^{2}$ are the only crossing $K_{s, t}$ 's. So
$\left|U_{1}^{*}\right| \leq\left|U_{1} \cup U_{0}\right|+q<\left|U_{1} \cup U_{0}\right|+s$ and $\left|V_{1}^{*}\right| \leq\left|V_{1} \cup V_{0}\right|+p<\left|V_{1} \cup V_{0}\right|+s$. Then by (13) we have, $\left|U_{1}^{*}\right|,\left|V_{1}^{*}\right|<(k+1)(s+t)$, contradicting Claim 3.5,

Case 1.1.ii. There is a crossing $K_{s, t}$ of Type 1 from $V_{2}$. Let $K^{2}$ be a crossing $K_{s, t}$ from $V_{2}$ and let $q:=\left|V\left(K^{2}\right) \cap U_{1}\right|$. Since $V_{0} \backslash V\left(K^{2}\right)=\emptyset, K^{1}$ and $K^{2}$ are the only crossing $K_{s, t}$ 's. So $\left|V_{1}^{*}\right| \leq\left|V_{1} \cup V_{0}\right|+p+1 \leq\left|V_{1} \cup V_{0}\right|+s$ and $\left|U_{1}^{*}\right| \leq\left|U_{1} \cup U_{0}\right|+t-q<\left|U_{1} \cup U_{0}\right|+t$. Then by (13) and (14) we have $\left|V_{1}^{*}\right|<(k+1)(s+t)$ and $\left|U_{1}^{*}\right|<k(s+t)+2 t$, contradicting Claim 3.5.

Case 1.2.i. $1 \leq\left|X_{1}\right|$. By Claim [3.4], since there exists a crossing $K_{s, t}$ of Type 2 from $U_{1}$, there can be no crossing $K_{s, t}$ 's of Type 2 from $U_{2}$. So $\left|U_{2}^{*}\right| \leq\left|U_{2} \cup U_{0}\right|+\left|X_{1}\right|+1 \leq\left|U_{2} \cup U_{0}\right|+s$ and $\left|V_{2}^{*}\right| \leq\left|V_{2} \cup V_{0}\right|+t-p<\left|V_{2} \cup V_{0}\right|+t$. Then by (13) and (14) we have $\left|U_{2}^{*}\right|<(k+1)(s+t)$ and $\left|V_{2}^{*}\right|<k(s+t)+2 t$, contradicting Claim 3.5.

Case 1.2.ii. $1 \leq\left|X_{2}\right|$. By Claim 3.4, since there exists a crossing $K_{s, t}$ of Type 2 from $U_{2}$, then there can be no crossing $K_{s, t}$ 's of Type 2 from $U_{1}$. So $\left|U_{1}^{*}\right| \leq\left|U_{1} \cup U_{0}\right|+\left|X_{2}\right|<\left|U_{1} \cup U_{0}\right|+s$ and $\left|V_{1}^{*}\right| \leq\left|V_{1} \cup V_{0}\right|+p<\left|V_{1} \cup V_{0}\right|+s$. Then by (13) we have $\left|U_{1}^{*}\right|,\left|V_{1}^{*}\right|<(k+1)(s+t)$, contradicting Claim 3.5,


Figure 4: Case 2
Case 2. There are no crossing $K_{s, t}$ 's of Type 1. By Claim 3.3, there can only be crossing $K_{s, t}$ 's of Type 2. Without loss of generality suppose that $1 \leq\left|X_{1}\right|$. Then there can be no crossing $K_{s, t}$ of Type 2 from $U_{2}$ or $V_{2}$. So $\left|U_{2}^{*}\right| \leq\left|U_{2} \cup U_{0}\right|+\left|X_{1}\right|<\left|U_{2} \cup U_{0}\right|+s$ and $\left|V_{2}^{*}\right| \leq\left|V_{2} \cup V_{0}\right|+\left|Y_{1}\right|<$ $\left|V_{2} \cup V_{0}\right|+s$. Then by (13) we have $\left|U_{2}^{*}\right|,\left|V_{2}^{*}\right|<(k+1)(s+t)$, contradicting Claim 3.5],

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