ERGODICITY OF STOCHASTIC CURVE SHORTENING FLOW IN THE PLANE

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Abstract. We study a model of the motion by mean curvature of an (1+1) dimensional interface in a 2D Brownian velocity field. For the well-posedness of the model we prove existence and uniqueness for certain degenerate nonlinear stochastic evolution equations in the variational framework of Krylov-Rozovskiĭ, replacing the standard coercivity assumption by a Lyapunov type condition. Ergodicity is established for the case of additive noise, using the lower bound technique for Markov semigroups by Komorowski, Peszat and Szarek [6].

1. Introduction

Motion by mean curvature is a well studied and rich object in geometric PDE theory for which a variety of methods have been developed (see e.g. [17] for a survey). In physics it arises as sharp interface limit of the Allen-Cahn equation for the phase field of a binary alloy, describing the motion of the interface between the two phases. Stochastic mean curvature flow was derived heuristically in e.g., [5] as a refined model incorporating the influence of thermal noise. In the (d+1)-dimensional graph case the corresponding SPDE is of the form

$$du = \sqrt{1+|\nabla u|^2}\,\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)\,dt + B(u,\nabla u)\delta W, \tag{1.1}$$

where δ stands for Stratonovich or Itô differential, depending on the model. The degeneracy of the drift operator makes a rigorous treatment of this family of models very difficult. Motivated by the deterministic theory Lions and Souganidis introduced a notion of stochastic viscosity solutions [9, 10], but some technical details of this approach are still awaiting full justification [1, 2]. Existence of weak subsequential limits along tight approximations of stochastic mean curvature flow has been obtained by Yip [16] and more recently by Röger and Weber [15].

In this paper we consider the special case of a (1+1)-dimensional graph interface in an e.g. 2D Brownian velocity field, corresponding to the equation

$$du = \frac{\partial_x^2 u}{1+(\partial_x u)^2}\,dt + \sum_{i=1}^{\infty}\phi_i(.,u(.))\,db_t^i. \tag{1.2}$$

In the deterministic case this equation is also known as curve shortening flow. Note that the mild solution approach by da Prato-Zabzcyk [3] is not applicable because equation (1.2) is not semilinear, i.e. does not contain a dominating linear component. For the analysis of (1.2) we first establish an abstract existence and uniqueness result in the classical variational SPDE framework of Krylov-Rozovskiĭ [7] for a certain class of nonlinear stochastic evolution equations,

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which are not coercive but satisfy an alternative Lyapunov condition. This is then applied to equation (1.2) which is treated in the Gelfand triple

$$H_0^1([0,1]) \subset L^2([0,1]) \subset H^{-1}([0,1]),$$

although the operator $A : H_0^1([0,1]) \to H^{-1}([0,1])$

$$Au = \frac{\partial_x^2 u}{1 + (\partial_x u)^2}$$

fails to be coercive. By our method we prove well-posedness of (1.2), assuming $u_0 \in H_0^1$, $\phi_i \in \mathrm{Lip}([0,1] \times \mathbb{R})$, $\phi_i(0,.) = \phi_i(1,.) = 0$ and, for some finite Λ,

$$\sum_{i=1}^{\infty} (\mathrm{Lip}(\phi_i))^2 \leq \Lambda^2. \tag{1.3}$$

The latter condition should be compared to the weaker assumption that for all $z_1, z_2 \in [0,1] \times \mathbb{R}$

$$\sum_{i=1}^{\infty} (\phi_i(z_1) - \phi_i(z_2))^2 \leq \Lambda^2 |z_1 - z_2|^2, \tag{1.4}$$

which is well-known e.g. in the theory of isotropic flows, where it guarantees the existence of a forward stochastic flow $d\Phi = F(\Phi, dt)$ of homeomorphisms of $[0,1] \times \mathbb{R}_+$ driven by the martingale field $F(z,t) = \sum_{i=1}^{\infty} \phi_i(z) b_t^i$, cf. [8, Theorem 4.5.1].

In fact, we show below that the SPDE (1.2) with noise field satisfying only (1.4) and initial condition $u_0 \in L^2([0,1])$ still admits a unique generalized solution which is defined by approximation. More precisely, we obtain a unique Markov process $(\hat{u}_t^x; x \in L^2([0,1]); t \geq 0)$ on $L^2([0,1])$, inducing a Feller semigroup on the space of bounded continuos functions on $L^2([0,1])$ as the unique generalized solution of (1.2). However, in view of the poor regularity of the operator A, a more explicit characterization of the $L^2([0,1])$-valued process $(\hat{u}_t^x)_{t \geq 0}$ by some SPDE or even just an associated Kolmogorov operator on smooth finitely based test functions does not seem to be available. This is very similar to the generalized solutions for abstract SPDE with only m-accretive drift operators obtained in [14] by means of nonlinear semigroup theory. The advantage in the present case is, however, that the variational approach is embedded such that we know the solution $(\hat{u}_t^x)_{t \geq 0}$ is strong if (1.3) holds and the initial condition $x = u_0$ belongs to $H_0^1([0,1])$.

Finally we show the ergodicity of the generalized solution $(\hat{u}_t^x)$ of (1.2) in the case of additive noise by verifying the conditions of a recent abstract result by Komorowski, Peszat and Szarek for Markov semigroups with the so-called e-property [6, Theorem 1]. We point out that [6, Theorem 3] does not apply in our sitation because the deterministic flow does not converge to equilibrium locally uniformly with respect to the initial condition. However, for the verification of the lower bound in our case we exploit the fact that the stochastic flow admits a Lyapunov function with compact sublevel sets.

2. Well-Posedness of certain non-coercive variational SPDE

2.1. Strong solutions for a class of non-coercive SPDE with regular initial condition.

Although we are mainly interested in the example (1.2) we shall formulate here a general existence and uniqueness result in the abstract variational framework of [7] for stochastic evolution equations, following with only a few changes the excellent presentation in [13]. Let

$$V \subset H$$

be a continuous and dense embedding of two separable Hilbert spaces with corresponding inner products $\langle .,. \rangle_V$ and $\langle .,. \rangle_H$. Via the Riesz isomorphism on H, this induces the Gelfand triple

$$V \subset H \subset V^*$$

such that in particular

$$_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_H \quad \forall\, u \in H, v \in V.$$

In addition we shall also assume that the inner product $\langle .,. \rangle_V$ induces a closed quadratic form on H. This implies the existence of a densely defined selfadjoint operator $L : H \supset D(L) \to H$ on H such that $V = D(\sqrt{L})$, $\langle u, v \rangle_V = \langle u, Lv \rangle_H$ for $u \in V, v \in D(L)$ and such that the closure of $L : V \supset D(L) \to V^*$, still denoted by L, defines an isometry. Moreover we assume that L has discrete spectrum with corresponding eigenbasis $(e_i)_{i \geq n}$, which will be the case if e.g. the embedding $V \subset H$ is compact.

Let $(W(t))_{t \geq 0}$ be a cylindrical white noise on some separable Hilbert space $(U, \langle .,. \rangle_U)$ defined on some probability space $(\Omega, \mathbb{P}, \mathcal{F})$ and let $\mathcal{F}_t = \sigma(W_s, s \leq t)$ be the associated filtration. For $X = H$ resp. $X = V$ we denote by $L_2(U, X)$ the class of Hilbert-Schmidt mappings from U to X, equipped with the Hilbert-Schmidt norm $\|M\|^2_{L_2(U,X)} = \sum_{i \geq 1} \langle Mu_i, Mu_i \rangle_X$, where $(u_i)_{i \geq 1}$ is some orthonormal basis of U. Let

$$A : V \to V^*, \quad \sigma : V \to L_2(U, V)$$

be measurable maps, then the existence and uniqueness result below applies to H-valued Itô-type stochastic differential equations of the form

$$\begin{cases} du(t) = Au(t)dt + \sigma(u(t))dW_t \\ u(0) = u_0 \in H. \end{cases} \tag{2.1}$$

Below we shall work under the following set of assumptions on the coefficients A and B.

(H1) (Hemicontinuity) For all u, v, $x \in V$ the map

$$\mathbb{R} \ni \lambda \to_{V^*} \langle A(u + \lambda v), x \rangle_V$$

is continuous.

(H2) (Weak monotonicity) There exists $c_1 \in \mathbb{R}$ such that for all u, $v \in V$

$$2\ _{V^*}\langle Au - Av, u - v \rangle_V + \|\sigma(u) - \sigma(v)\|^2_{L_2(U,H)} \leq c_1 \|u - v\|^2_H$$

(H3) (Lyapunov condition) For $n \in \mathbb{N}$, the operator A maps $H^n := \operatorname{span}\{e_1, \ldots, e_n\} \subset V$ into V and there exists a constant $c_2 \in \mathbb{R}$ such that

$$2\ \langle Au, u \rangle_V + \|\sigma(u)\|^2_{L_2(U,V)} \leq c_2(1 + \|u\|^2_V) \quad \forall u \in H^n, n \in \mathbb{N}.$$

(H4) (Boundedness) There exists a constant $c_3 \in \mathbb{R}$ such that

$$\|A(u)\|_{V^*} \leq c_3(1 + \|u\|_V).$$

Remark 2.1. Note that (H3) replaces the standard coercivity assumption in [7]

$$2\ _{V^*}\langle Au, u \rangle_V + \|\sigma(u)\|^2_{L_2(U,H)} \leq c_2 \|u\|^2_H - c_4 \|u\|^\alpha_V, \quad \forall v \in V \tag{A}$$

for some positive constant c_4 and $\alpha > 1$. Both conditions (H3) and (A) yield the compactness of the Galerkin approximation in V. Condition (A) is used indirectly by applying the finite dimensional Itô formula to the square of the H-norm. In our case we use condition (H3) directly by application of the finite dimensional Itô formula to the squared V-norm functional.

Basically, a solution to (2.1) is a V-valued process such that the equation holds in V^* in integral form, c.f. [7]. The following precise definition is taken from [13].

Definition 2.2. *A continuous H-valued $(\mathcal{F}_t)$-adapted process $(u(t))_{t\in[0,T]}$ is called a solution of* (2.1)*, if for its $dt\otimes\mathbb{P}$-equivalence class $[u]$ we have $[u]\in L^2([0,T]\times\Omega, dt\otimes\mathbb{P}, V)$ and $\mathbb{P}$-a.s.*

$$u(t) = u(0) + \int_0^t A(\bar{u}(s))\,ds + \int_0^t \sigma(\bar{u}(s))\,dW_s, \quad t\in[0,T],$$

where $\bar{u}$ is any V-valued progressively measurable $dt\otimes\mathbb{P}$-version of $[u]$.

Now we can state the main result of this section as follows.

Theorem 2.3. *Assume that conditions (H1)-(H4) hold, then for any initial data $u_0\in V$, there exists a unique solution u to* (2.1) *in the sense of Definition 2.2. Moreover,*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|u(t)\|_H^2\right) < \infty.$$

Proof. The proof follows the standard path of spectral Galerkin approximation, the only difference towards [7, 13] is the compactness argument, c.f., lemma 2.4 below. To this aim let $(e_n)_{n\geq 1}$ be an orthonormal basis in H of eigenfunctions for the operator $L:\ H\supset D(L)\to H$. Clearly $(e_n)_{n\geq 1}\subset V$ and the set $\mathrm{span}\{e_n,\ n\geq 1\}$ is dense in V. Let $H_n := \mathrm{span}\{e_1,\cdots,e_n\}$ and define $P_n:\ V^*\to H_n$ by

$$P_n y := \sum_{i=1}^n {}_{V^*}\langle y, e_i\rangle_V e_i, \quad y\in V^*.$$

Then we have $P_n|_H$ is just the orthogonal projection onto H_n in H. We shall define the family of n-dimensional Brownian motions by setting

$$W_t^n := \sum_{i=1}^n \langle W_t, f_i\rangle_U f_i = \sum_{i=1}^n B^i(t) f_i,$$

where $(f_i)_{i\geq 1}$ is an orthonormal basis of the Hilbert space U. We now consider the n-dimensional SDE

$$\left\{\begin{array}{l} du^n(t) = P_n A u^n(t)dt + P_n\sigma(u^n(t))dW_t^n \\ u^n(0,x) = P_n u_0(x), \end{array}\right\}, \tag{2.2}$$

which is identified with a corresponding SDE $dx(t) = b^n(x(t))dt + \sigma^n(x(t))dB_t^n$ in $\mathbb{R}^n$ via the isometric map $\mathbb{R}^n\to H^n, x\to\sum_{i=1}^n x_i e_i$. By [13, remark 4.1.2] conditions (H1) and (H2) imply the continuity of the fields $x\to b^n(x)\in\mathbb{R}^n$ and $x\to\sigma^n(x)\in\mathbb{R}^{n\times n}$. Moreover, assumption (H2) implies

$$2\langle b^n(x)-b^n(y), x-y\rangle_{\mathbb{R}^n} + |\sigma^n(x)-\sigma^n(y)|^2_{L_2(\mathbb{R}^n,\mathbb{R}^n)} \leq c_1|x-y|^2, \quad \forall x,y\in\mathbb{R}^n$$

and, by the equivalence of norms on $\mathbb{R}^n$, (H3) gives the bound

$$2\langle b^n(x), x\rangle + |\sigma^n(x)|_{L_2(\mathbb{R}^n,\mathbb{R}^n)} \leq c_5(1+|x|^2),$$

for some $c_5\in\mathbb{R}$. Hence, equation (2.2) is a weakly monotone and coercive equation in $\mathbb{R}^n$ which has a unique globally defined solution, cf. [13, chapter 3].

Lemma 2.4. *Let u^n be the solution to equation* (2.2)*, then for any $T>0$ we have*

$$\sup_{0\leq t\leq T}\mathbb{E}\|u^n(t)\|_V^2 \leq (c_2 T + \mathbb{E}(\|u_0\|_V^2))e^{c_2 T}. \tag{2.3}$$

Proof. Due to the definition of P_n we may write

$$\langle u^n(t), e_i\rangle = \langle u^n(0), e_i\rangle + \int_0^t \Big\langle \sum_{k=1}^n {}_{V^*}\langle A(u^n(s)), e_k\rangle_V e_k\, ds, e_i\Big\rangle + \Big\langle \int_0^t P_n\sigma(u^n(s))\, dW_s^n, e_i\Big\rangle.$$

Hence, the Itô formula in $\mathbb{R}^n$ yields

$$\begin{aligned}\|u^n(t)\|_V^2 &= \|u_0^n\|_V^2 + 2\int_0^t \langle P_n A(u^n(s)), u^n(s)\rangle_V\, ds + \int_0^t \|P_n\sigma(u^n(s))\|^2_{L_2(U_n,V)}\, ds\\ &\quad + M^n(t), \quad t\in[0,T],\end{aligned}$$

$\mathbb{P}$-a.s, where $U_n := \operatorname{span}\{f_1, f_2, \cdots, f_n\}\subset U$ and

$$M^n(t) := 2\int_0^t \langle u^n(s), P_n\sigma(u^n(s))\, dW_s^n\rangle_V, \quad t\in[0,T],$$

is a local martingale. We consider a sequence of $\mathcal{F}_t$- stopping times τ_j with $\tau_j\uparrow+\infty$ as $j\to+\infty$ and such that $\|u^n(t\wedge\tau_j)(\omega)\|_V$ is bounded uniformly in $(t,\omega)\in[0,T]\times\Omega$, $M^n(t\wedge\tau_j)$, $t\in[0,T]$ is a martingale for each $j\in\mathbb{N}$. Then we have

$$\begin{aligned}\mathbb{E}\|u^n(t\wedge\tau_j)\|_V^2 &= \mathbb{E}\|u_0^n\|_V^2 + 2\int_0^t \mathbb{E}\mathbf{1}_{[0,\tau_j]}\langle P_n A(u^n(s)), u^n(s)\rangle_V\, ds\\ &\quad + \int_0^t \mathbb{E}\mathbf{1}_{[0,\tau_j]}\|P_n\sigma(u^n(s))\|^2_{L_2(U_n,V)}\, ds.\end{aligned} \tag{2.4}$$

Now using the definition of the operators A and P_n we can write

$$\begin{aligned}\langle P_n A(u^n(s)), u^n(s)\rangle_V &= \Big\langle \sum_{i=1}^n {}_{V^*}\langle A(u^n(s)), e_i\rangle_V e_i, u^n(s)\Big\rangle_V\\ &= \sum_{i=1}^n {}_{V^*}\langle A(u^n(s)), e_i\rangle_V \langle e_i, u^n(s)\rangle_V.\end{aligned}$$

Since $u^n(t)\in H_n$ for $t\in[0,T]$ and $(e_n)_{n\geq1}\subset V$ by assumption (H3) we can write

$${}_{V^*}\langle A(u^n(s)), e_i\rangle_V = \langle A(u^n(s)), e_i\rangle_H,$$

this yields

$$\begin{aligned}\langle P_n A(u^n(s)), u^n(s)\rangle_V &= \sum_{i=1}^n \langle A(u^n(s)), e_i\rangle_H \langle e_i, u^n(s)\rangle_V\\ &= \sum_{i=1}^n \langle A(u^n(s)), e_i\rangle_H \lambda_i \langle e_i, u^n(s)\rangle_H\end{aligned}$$

where $\{\lambda_i\geq 0\}$ are the eigenvalues of the operator L.
Therefore we have

$$\langle P_n A(u^n(s)), u^n(s)\rangle_V = \langle A(u^n(s)), u^n(s)\rangle_V.$$

Hence, the operator P_n may be dropped in the fist integral on the right hand side term of (2.4) such that by the second part of assumption (H3)

$$\mathbb{E}\|u^n(t\wedge\tau_j)\|_V^2 \leq \mathbb{E}\|u_0^n\|_V^2 + c_2\int_0^t (1+\mathbb{E}\|u^n\|_V^2) ds.$$

Hence letting $j\to+\infty$ and using Fatou's lemma we obtain

$$\mathbb{E}\|u^n(t)\|_V^2 \leq \mathbb{E}\|u_0^n\|_V^2 + c_2\int_0^t (1+\mathbb{E}\|(u^n(s))\|_V^2)\, ds.$$

Now Gronwall's lemma yields

$$\mathbb{E}\|u^n(t)\|_V^2 \leq (c_2 T + \mathbb{E}\|u_0^n\|_V^2) e^{c_2 T}. \tag{2.5}$$

For the estimate of $\mathbb{E}\|u_0^n\|_V^2$, we use the definition of P_n and write

$$\begin{aligned}
\|u_0^n\|_V^2 = \|P_n u_0\|_V^2 = \langle P_n u_0, P_n u_0\rangle_V &= \sum_{i=1}^n \sum_{j=1}^n {}_{V^*}\langle u_0, e_i\rangle_V \langle e_i, e_j\rangle_V \, {}_{V^*}\langle u_0, e_i\rangle_V \\
&= \sum_{i=1}^n \lambda_i \langle u_0, e_i\rangle_H^2 \leq \sum_{i=1}^\infty \lambda_i \langle u_0, e_i\rangle_H^2 = \|u_0\|_V^2.
\end{aligned}$$

□

From here all remaining arguments from [13, chapter 4] carry over without change in order to complete the proof of the theorem. To make the paper self-contained we briefly recall the main steps. Let

$$K := L^2([0,T]\times\Omega, dt\otimes\mathbb{P}, V) \quad \text{and} \quad J := L^2([0,T]\times\Omega, dt\otimes\mathbb{P}, L_2(U,H)).$$

Due to the bound (H4) and the reflexivity of K we find a subsequence $n_k \to +\infty$ such that $u^{n_k} \to \bar{u}$ weakly in K and weakly in $L^2([0,T]\times\Omega,\, dt\otimes\mathbb{P}, H)$, $v^{n_k} := A(u^{n_k}) \to v$ weakly in K^* and $\theta^{n_k} := P_{n_k}\sigma(u^{n_k}) \to \theta$ weakly in J. Passing to the limit in (2.2) one obtains in V^*

$$u(t) := u_0 + \int_0^t v(s)\,\mathrm{d}s + \int_0^t \theta(s)\,\mathrm{d}W(s), \quad t\in[0,T], \tag{2.6}$$

and in particular $u = \bar{u}$ $dt\otimes\mathbb{P}$-a.e. Now the following Itô formula for $\|u_t\|_H$ is crucial (c.f. [7]).

Theorem 2.5. *Let $u_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}, H)$ and $v \in L^2([0,T]\times\Omega, dt\otimes\mathbb{P}, V^*)$, $\theta \in L^2([0,T]\times\Omega, dt\otimes\mathbb{P}, L_2(U,H))$, both progressively measurable. Define the continuous V^*-valued process*

$$u(t) := u_0 + \int_0^t v(s)\,ds + \int_0^t \theta(s)\,dW_s, \quad t\in[0,T].$$

If for its $dt\otimes\mathbb{P}$-equivalence class $[u]$ we have $[u] \in L^2([0,T]\times\Omega, dt\otimes\mathbb{P}, V)$, then u is an H-valued continuous $\mathcal{F}_t$-adapted process,

$$\mathbb{E}\big(\sup_{t\in[0,T]} \|u(t)\|_H^2\big) < \infty$$

and the following Itô-formula holds for the square of its H-norm $\mathbb{P}$-a.s.

$$\|u(t)\|_H^2 = \|u_0\|_H^2 + 2\int_0^t \Big({}_{V^*}\langle v(s), \bar{u}(s)\rangle_V + \|\theta(s)\|_{L_2(U,H)}^2 \Big)\,ds + 2\int_0^t \langle u(s), \theta(s)\,dW_s\rangle \tag{2.7}$$

for all $t\in[0,T]$, where $\bar{u}$ is any V-valued progressively measurable $dt\otimes\mathbb{P}$-version of $[u]$.

In view of (2.6) this implies that u is continuous in H, $\mathbb{E}(\sup_{0\leq t\leq T}\|u(t)\|_H^2) < +\infty$ and

$$\mathbb{E}\left(e^{-c_1 t}\|u(t)\|_H^2\right) - \mathbb{E}\left(\|u_0\|_H^2\right) = \mathbb{E}\Big(\int_0^t e^{-c_1 s}(2\,{}_{V^*}\langle v(s), \bar{u}(s)\rangle_V + \|\theta(s)\|_{L_2(U,H)}^2 - c_1\|u(s)\|_H^2)\,\mathrm{d}s\Big). \tag{2.8}$$

An analogous formula holds true for $(u^{n_k}(t))_{t\geq 0}$. Hence, for $\Phi\in K$, using (H2),

$$\begin{aligned}
&\mathbb{E}\left(e^{-c_1 t}\|u^{n_k}(t)\|_H^2\right) - \mathbb{E}\left(\|u_0^{n_k}\|_H^2\right) \\
&\leq \mathbb{E}\bigg(\int_0^t e^{-c_2 s}\Big(2\,{}_{V^*}\langle A(\Phi(s)), u^{n_k}(s)\rangle_V + 2\,{}_{V^*}\langle A(u^{n_k}(s)) - A(\Phi(s)), \Phi(s)\rangle_V \\
&- \|\sigma(\Phi(s))\|_{L_2(U,H)}^2 + 2\langle \sigma(u^{n_k}(s)), \sigma(\Phi(s))\rangle_{L_2(U,H)} - 2c_2\langle u^{n_k}(s), \Phi(s)\rangle_H + c_2\|\Phi(s)\|_H^2\Big)\,\mathrm{d}s\bigg).
\end{aligned}$$

Letting $k \to +\infty$ one concludes that for every nonnegative $\psi \in L^\infty([0,T],\mathbb{R})$

$$\begin{aligned}
&\liminf_{k\to+\infty} \mathbb{E}\left(\int_0^T \psi(t)(e^{-c_1 t}\|u^{n_k}(t)\|_H^2 - \|u_0^{n_k}\|_H^2)\,\mathrm{d}t\right) \\
&\leq \mathbb{E}\left(\int_0^T \psi(t)\left(\int_0^t e^{-c_1 s}\Big(2\,{}_{V^*}\langle A(\Phi(s)),\bar{u}(s)\rangle_V + 2\,{}_{V^*}\langle v(s)-A(\Phi(s)),\Phi(s)\rangle_V\right.\right. \qquad (2.9)\\
&- \|\sigma(\Phi(s))\|^2_{L_2(U,H)} + 2\langle\theta(s),\sigma(\Phi(s))\rangle_{L_2(U,H)} - 2c_1\langle u(s),\Phi(s)\rangle_H + c_1\|\Phi(s)\|_H^2\Big)\,\mathrm{d}s\Big).
\end{aligned}$$

On the other hand, due to the weak lower semicontinuity of the norm in K

$$\mathbb{E}\left(\int_0^T \psi(t)\|u(t)\|_H^2\,\mathrm{d}t\right) \leq \liminf_{k\to+\infty}\left(\mathbb{E}\int_0^T \psi(t)\|u^{n_k}(t)\|_H^2\,\mathrm{d}t\right). \tag{2.10}$$

Combining this with (2.8) and (2.9) one obtains that

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^T \psi(t)\int_0^t e^{-c_1 s}(2\,{}_{V^*}\langle v(s)-A(\Phi(s)),\bar{u}(s)-\Phi(s)\rangle_V \\
&\qquad + \|\sigma(\Phi(s))-\theta(s)\|^2_{L_2(U,H)} - c_1\|u(s)-\Phi(s)\|_H^2)\,\mathrm{d}s\,\mathrm{d}t\Big) \leq 0.
\end{aligned} \tag{2.11}$$

Taking $\Phi = \bar{u}$ in (2.11) we obtain $\theta = \sigma(\bar{u})$. By applying (2.11) to $\Phi = \bar{u} - \varepsilon\tilde{\Phi}h$ for $\varepsilon > 0$ and $\tilde{\Phi} \in L^\infty([0,T]\times\Omega,\mathbb{R})$, $h \in V$ and dividing both sides by ε and letting $\varepsilon \to 0$, by (H2) and Lebesgue's theorem we get

$$\mathbb{E}\left(\int_0^T \psi(t)\left(\int_0^t e^{-c_1 s}\tilde{\Phi}(s)\Big(2\,{}_{V^*}\langle v(s)-A(\bar{u}(s)),h\rangle_V\,\mathrm{d}s\right)\mathrm{d}t\right) \leq 0.$$

By the arbitrariness of ψ and $\tilde{\Phi}$ we conclude that $v = A(\bar{u})$.

As for the uniqueness consider two solutions $u^{(1)}$ and $u^{(2)}$ of (2.1) with initial condition $u_0^{(1)} \in V$ and $u_0^{(2)} \in V$ respectively. Applying theorem 2.5 to $u = u^{(1)} - u^{(2)}$ together with condition (H2) and Gronwall's lemma

$$\mathbb{E}\|u^{(1)}(t) - u^{(2)}(t)\|_H^2 \leq \|u_0^{(1)} - u_0^{(2)}\|_H^2 e^{2c_1 t}. \tag{2.12}$$

This implies uniqueness of the solution for given initial state. Theorem 2.3 is proved. ☐

2.2. Generalized solutions for initial condition in H. By means of (2.12) it is possible to construct a unique generalized solution to (2.1) for initial condition in $u_0 \in H$. In particular this yields a unique Feller process on H which extends the regular strong solutions of (2.1).

Proposition 2.6. *Assume (H1) - (H4) , then there exists a unique time homogeneous H-valued Markov process $(\hat{u}_t^x, t \geq 0, x \in H)$ such that $t \to \hat{u}_t^x$ solves the SPDE* (2.1) *in the sense of definition* (2.2) *whenever $x = u_0 \in V$. Moerover, $(\hat{u}_t^x)$ induces a Feller semigroup on H, i.e. the space $C_b(H)$ of bounded continuous function on H is invariant under the the operation $\varphi \to P_t\varphi$, where $P_t\varphi(x) = \mathbb{E}(\varphi(\hat{u}_t^x)), x \in H$ for any $t \geq 0$.*

Proof. For $x \in V \subset H$ define $t \to \hat{u}_t^x \in H$ as the unique solution to (2.1) with initial condition $u_0 = x$. For arbitrary $x \in H$, choose a sequence $(x_k)_k$ in V such that $\|x_k - x\|_H \to 0$, then by (2.12) the sequence of processes $(t \to \hat{u}_t^{x_k})_{k\in N}$ is Cauchy in $C([0,\infty); L^2(\Omega, H))$ with respect to the topology of locally uniform convergence and define $(t \to \hat{u}_t^x)$ as the unique limit. For $\varphi \in C_b(H)$ define $P_t\varphi(x)$ as above, then (2.12) obviously yields

$$\mathbb{E}\|\hat{u}_t^x - \hat{u}_t^y\|_H^2 \leq e^{2c_1 t}\|x-y\|_H^2, \quad t \geq 0, \tag{2.13}$$

which implies that $P_t\varphi \in C_b(H)$ for $\varphi \in C_b(H)$. To prove that $(\hat{u}^x_t)^{x\in H}_{t\geq 0}$ is Markov, by the monotone class argument it suffices to show for all $x \in H$

$$\mathbb{E}\big(\psi(\hat{u}^x_t)\cdot\varphi_1(\hat{u}^x_{s_1})\cdots\varphi_n(\hat{u}^x_{s_n})\big) = \mathbb{E}\big(P_{t-s_n}\psi(\hat{u}^x_{s_n})\cdot\varphi_1(\hat{u}^x_{s_1})\cdots\varphi_n(\hat{u}^x_{s_n})\big), \tag{2.14}$$

for any $0 \leq s_1 \leq s_2 \cdots \leq s_n < t$ and $\varphi_1,\ldots,\varphi_n,\psi \in C_b(H)\cap \mathrm{Lip}(H)$. By (2.13) we have

$$|P_{t-s}\varphi(x) - P_{t-s}\varphi(y)| \leq e^{c_2(t-s)}\,\mathrm{Lip}(\varphi)\|x-y\|_H \quad \forall\,\varphi\in \mathrm{Lip}(H),$$

hence will be enough to show (2.14) for $x \in V$, where it follows by standard arguments from the uniqueness of solutions of (2.1), their adaptedness to the filtration $\mathcal{F}_s$, $s \geq 0$, which for $s \leq t$ is independent of the sigma algebra of increments $\mathcal{G}_{s,t} = \sigma(W_\sigma - W_s; s \leq \sigma \leq t)$, c.f., [13, proposition 4.3.5]. This proves the existence of $(\hat{u}^x_t; t \geq 0; x \in H)$ as in the claim of the theorem. Trivially, uniqueness of $(\hat{u}^x_t)$ follows from (2.13) which holds for any H-valued closure of solutions to equation (2.1). □

3. Application: Stochastic Curve Shortening Flow in (1+1) Dimension

3.1. Strong solutions for $u_0 \in H^{1,2}([0,1])$ and smooth noise. Let us now show how we can treat the model rigorously in the case $d = 1$, which is also known as curve shortening flow, using the results of the previous section. The simple but essential observation is that for $d = 1$ the drift operator in the SPDE (1.1) above may be written

$$Au = \frac{\partial_x^2 u}{1+(\partial_x u)^2} = \partial_x(\arctan(\partial_x u)), \tag{3.1}$$

which fits into our slightly modified Krylov-Rozovskiĭ framework. To this aim let

$$H^1_0([0,1]) \subset L^2([0,1]) \subset H^{-1}([0,1]),$$

be the Gelfand triple, which is induced from the Dirichlet Laplacian $L = \Delta$ on $L^2([0,1])$.

For $u \in H^1_0([0,1])$, let $Au \in H^{-1}([0,1])$ be defined by

$${}_{H_0^{-1}}\langle Au, v\rangle_{H^1_0} = -\int_{[0,1]} \arctan(\partial_x u)\partial_x v dx, \quad \forall v \in H^1,$$

which is clearly hemicontinuous in the sense of condition (H1), due to the continuity and uniform boundedness of $\zeta \to \arctan\zeta$. Trivially A is also bounded in the sense of (H4) because

$$\|Au\|_{H^{-1}([0,1])} = \sup_{v\in H^1_0([0,1]),\|v\|_{H^1_0}\leq 1} \int_{[0,1]} \arctan(\partial_x u)\partial_x v dx \leq (\frac{\pi}{2})^{1/2}. \tag{3.2}$$

Moreover, by the monotonicity of arctan

$${}_{H^{-1}}\langle Au - Av, u - v\rangle_{H^1} = -\int_{[0,1]} (\arctan(\partial_x u) - \arctan(\partial_x v))(\partial_x u - \partial_x v)dx \leq 0. \tag{3.3}$$

The eigenvectors of $L = \Delta_0$ are $e_i = (x \to \sin(i2\pi x)), i \in \mathbb{N}$, hence $Au = \partial_x^2 u/(1+(\partial_x u)^2) \in H^1_0([0,1])$ for any $u \in H^n = \mathrm{span}\{e_1,\ldots,e_n\} \subset H^1_0([0,1])$. Moreover,

$$\langle Au, u\rangle_{H^1_0} = -\int_{[0,1]} \frac{\partial_x^2 u}{1+(\partial_x u)^2}\partial_x^2 u(x)dx \leq 0 \quad \forall u \in H^n. \tag{3.4}$$

Let $(\phi_i)_{i\in\mathbb{N}}$ denote a sequence of linear independent Lipschitz functions on $[0,1]\times\mathbb{R}$ such that $\phi(0,y) = \phi(1,y) = 0$ for all $y \in \mathbb{R}$ and such that the stronger regularity assumption (1.3) holds for the noise field, and let furthermore U denote the Hilbert space obtained from the closure of the span of $\{\phi_i, i \in \mathbb{N}\}$ with respect to the inner product $\langle \sum_{i=1}^{n} \lambda_i \phi_i, \sum_{j=1}^{m} \eta_j \phi_j \rangle_U := \sum_{i=1}^{n\wedge m} \lambda_i \eta_i$.

Define the diffusion operator $B : H_0^1([0,1]) \to L(U, L^2([0,1])$ by

$$B(u)[\phi](x) = \phi(x, u(x)) \in L^2([0,1])$$

Note that $B(u)$ is in fact in $L_2(U, L^2([0,1]))$ since

$$\begin{aligned}\|B(u)(\phi_i)\|^2_{L^2([0,1])} &= \int_{[0,1]} \phi_i(x,u(x))^2 dx = \int_{[0,1]} |\phi_i(x,u(x)) - \phi_i(0,u(0))|^2 dx \\ &\le (\mathrm{Lip}(\phi_i))^2 \int_{[0,1]} (x^2 + u^2(x)) dx = (\mathrm{Lip}(\phi_i))^2 (\frac{1}{3} + \|u\|^2_{L^2([0,1])}),\end{aligned}$$

such that

$$\|B(u)\|^2_{L_2(U,L^2([0,1]))} = \sum_i \|B(u)(\phi_i)\|^2_{L^2([0,1])} \le (\frac{1}{3} + \|u\|^2_{L^2([0,1])}) \cdot \Lambda^2 \tag{3.5}$$

Moreover,

$$\begin{aligned}\|B(u) - B(v)\|^2_{L_2(U,L^2([0,1]))} &= \sum_i \|B(u)[\phi_i] - B(v)[\phi_i]\|^2_{L^2([0,1])} \\ &= \sum_i \int_{[0,1]} (\phi_i(x,u(x)) - \phi_i(x,v(x)))^2 dx \\ &\le \Lambda^2 \|u - v\|^2_{L^2([0,1])}. \end{aligned} \tag{3.6}$$

Similarly, $B(u)[\phi] \in H_0^1([0,1])$ for $u \in H_0^1([0,1])$, and by the chain rule for weakly differentiable functions,

$$\begin{aligned}\|B(u)(\phi_i)\|^2_{H_0^1([0,1])} &= \int_{[0,1]} (\partial_x \phi_i(x,u(x)))^2 dx \\ &\le (\mathrm{Lip}(\phi_i))^2 \int_{[0,1]} (1 + |\partial_x u(x)|^2) dx = (\mathrm{Lip}(\phi_i))^2 (1 + \|u\|^2_{H^1([0,1])}),\end{aligned}$$

which yields

$$\|B(u)\|^2_{L_2(U,H^1([0,1]))} = \sum_i \|B(u)(\phi_i)\|^2_{H^1([0,1])} \le (1 + \|u\|^2_{H^1([0,1])}) \cdot \Lambda^2 \tag{3.7}$$

In view of (3.2) – (3.7) we conclude that the conditions (H1) – (H4) are satisfied in the given case with constants $c_1 = c_2 = \Lambda^2$ and $c_3 = \sqrt{\pi/2}$. Hence, by theorem 2.3 we arrive at the following result.

Theorem 3.1. *Assume the regularity condition* (1.3) *holds for the noise field, then for any* $T > 0$ *there is a (up to* $dt \otimes \mathbb{P}$*-equivalence in* $[0,T] \times \Omega$*) unique* $H_0^1([0,1])$*-valued process* $(u_t)_{t\in[0,T]}$ *solving the SPDE* (1.2) *in the sense of definition 2.2.*

3.2. Generalized Markovian solution in $L^2([0,1])$ for non-smooth noise. Proposition 2.6 readily yields generalized solutions for initial condition in $L^2([0,1])$ as follows.

Proposition 3.2. *Under condition* (1.3) *there is a unique* $L^2([0,1])$*-valued Markov process* $(\hat{u}_t^x, t \ge 0, x \in L^2([0,1]))$ *such that* $t \to \hat{u}_t^x$ *is a strong solution to the equation* (1.2) *when* $x = u_0 \in H_0^{1,2}([0,1])$*. Moreover,* $(\hat{u}_t^x)_{t\ge 0}$ *induces a Feller semigroup on* $C_b(L^2([0,1]))$*.*

However, noticing that estimates (3.5) and (3.6) remain true under the weaker regularity condition (1.4), by similar arguments as in the proof of proposition 2.6 we arrive at the following well-posedness result for the SPDE (1.2) under the Kunita-type regularity condition (1.4).

Proposition 3.3. *Under condition* (1.4) *there is a unique* $L^2([0,1])$*-valued Markov process* $(\hat{u}_t^x; t \geq 0, x \in L^2([0,1]))$ *such that for* $x \in H_0^1([0,1])$, $(u_t^x)_{t\geq 0}$ *is the limit, in the sense of locally uniform convergence on* $C\big([0,\infty); L^2(\Omega,\mathbb{P}; L^2([0,1]))\big)$, *of the strong solutions to the SPDE*

$$du^{(k)} = \frac{\partial_x^2 u^{(k)}}{1+(\partial_x u^{(k)})^2}\, dt + \sum_i^k \phi_i(.,u^{(k)}(.))\, db_t^i, \quad u_0^k = x.$$

Moreover,

$$E\|\hat{u}_t^x - \hat{u}_t^y\|^2_{L^2([0,1])} \leq e^{\Lambda^2 t}\|x-y\|^2_{L^2([0,1])} \quad \forall x,y \in L^2([0,1]), t \geq 0. \tag{3.8}$$

In particular, the induced semigroup, $P_t\varphi(x) = \mathbb{E}(\varphi(\hat{u}_t^x))$ *for measurable* $\varphi : H \to \mathbb{R}$, *is Feller.*

4. Ergodicity for Stochastic Curve Shorting Flow with Additive Noise

In this final section we show existence and uniqueness of an invariant measure for the generalized $L^2([0,1])$-valued solution $(\hat{u}_t^x; t \geq 0, x \in L^2([0,1]))$ obtained in proposition 3.2 for the SPDE (1.2) in the additive noise case, i.e. when

$$du = \frac{\partial_x^2 u}{1+(\partial_x u)^2} dt + Q dW_t, \quad u(0) = u_0 \in H_0^{1,2}([0,1]), \tag{4.1}$$

where W is cylindrical white noise on some abstract Hilbert space U and $Q \in L_2(U, H_0^{1,2}([0,1]))$. As an example consider the case of $U = L^2([0,1])$ and $Q = (-\Delta)^{-\beta}$ for $\beta > 3/4$, with Δ being the Dirichlet Laplacian on $[0,1]$.

Note also that for additive noise the condition (H2) is satisfied with $c_1 = 0$. As a consequence of (2.12), the Feller semigroup on L^2 induced from the generalized solutios $\hat{u}$ of (1.2) by $P_t\varphi(x) = \mathbb{E}(\varphi(\hat{u}_t^x))$ has the so-called e-property [6], i.e. for all bounded Lipschitz continuous functions $\varphi : L^2 \mapsto \mathbb{R}$

$$|P_t\varphi(x) - P_t\varphi(y)| \leq \mathrm{Lip}(\varphi)\|x-y\| \quad \forall x,y \in L^2. \tag{4.2}$$

Theorem 4.1. *Let* $(P_t)_{t\geq 0}$ *denote the Feller semigroup on* $L^2([0,1])$ *corresponding to the generalized solution to* (4.1), *then* (P_t) *is ergodic, i. e. there is a unique* (P_t)*-invariant probablity measure* μ *on* $L^2([0,1])$. *In particular,* $\lim_{t\to\infty} \frac{1}{t}\int_0^t \langle P_t\varphi, \nu\rangle = \langle \varphi, \mu\rangle$ *for any Borel probability measure* $\nu \in \mathcal{M}_1(L^2([0,1]))$ *and any bounded continuous* $\varphi : L^2([0,1]) \mapsto \mathbb{R}$.

Let $Q^T(x,\cdot) := \frac{1}{T}\int_0^T \mu_{\hat{u}_t}\, dt$, where $\mu_{\hat{u}_t}$ denotes the distribution at time t of the generalized solution $\hat{u}_t^x$ to (4.1) with initial conditon $u_0 = x \in L^2$.

Proposition 4.2. *For any* $x \in L^2$ *the family of measures* $\big\{Q^T(x,\cdot),\ T \geq 1\big\}$ *is tight on* $L^2([0,1])$.

Proof. Assume first that $\in H_0^{1,2}([0,1])$. In view of

$$|\xi| - \alpha \leq \operatorname{arctg}\xi \cdot \xi \leq \beta + |\xi|, \quad \xi \in \mathbb{R}, \quad \alpha,\ \beta > 0$$

it holds that

$$\begin{aligned} {}_{H^{-1}}\langle Av, v\rangle_{H^1} &= -\int_0^1 \operatorname{arctg}(\partial_x v)\cdot \partial_x v\, dx \leq -\int_0^1 |\partial_x v|\, dx + \alpha \\ &\leq -c\|v\|_{W^{1,1}(0,1)} + \alpha \end{aligned} \tag{4.3}$$

for some $c > 0$, by Poincaré inequality.

Let now $t \to u_t$ be the solution to equation (4.1) with regular initial condition $x = u_0 \in H_0^{1,2}([0,1])$, then theorem 2.5 holds. Hence by the Itô formula for $\|u_t\|^2_{L^2([0,1]}$ and (4.3) we have

$$\begin{aligned}\mathbb{E}\|u(t)\|^2 &= \mathbb{E}\|u(0)\|^2 + 2\mathbb{E}\int_0^t {}_{V^*}\langle A(\bar{u}(s)), \bar{u}(s)\rangle_V ds + \mathbb{E}\int_0^t \|Q\|^2_{\mathcal{L}_{HS}(U,H)}\, ds \\ &\leq \mathbb{E}\|u(0)\|^2 - c\,\mathbb{E}\int_0^t \|\bar{u}(s)\|_{W^{1,1}(0,1)} + Dt\end{aligned} \tag{4.4}$$

where $D := \alpha + \|Q\|^2_{\mathcal{L}_{HS}(U,H)}$. In particular,

$$\mathbb{E}\left(\frac{1}{t}\int_0^t \|\bar{u}(s)\|_{W^{1,1}(0,1)}\, ds\right) \leq \frac{1}{c}\Big(\mathbb{E}\|x\|^2 + D\Big) \quad \forall t \geq 1. \tag{4.5}$$

Since the functional $L^2([0,1]) \ni u \to \|u\|_{W^{1,1}(0,1)} \in \mathbb{R} \cup \{\infty\}$ has compact sublevel sets in $L^2([0,1])$, the claim follows for regular initial condition $x = u_0 \in H_0^{1,2}([0,1])$.

For the tightness of $Q^T(x,.)$ with general $x \in L^2$ recall (e.g. [12, Remark on p. 49]) that it is sufficient (and necessary) to find for arbitrary $\epsilon > 0, \delta > 0$ a finite union of δ-balls $S_\delta = \bigcup_{i=1,\dots,k} B_\delta(x_i) \subset L^2$ such that

$$Q^T(x, S_\delta) > 1 - \epsilon \quad \forall T > 1.$$

To this aim choose $z \in B_{\delta\epsilon/4}(x) \cap H_0^{1,2}(0,1)$ and a finite union of $\delta/2$-balls $S_{\delta/2} = \bigcup\limits_{i=1,\dots,k} B_{\delta/2}(x_i)$ such that $Q^T(z, S_{\delta/2}) \geq 1 - \frac{\epsilon}{2}$. Let $S_\delta = \bigcup\limits_{i=1,\dots,k} B_\delta(x_i)$ and choose a bounded Lipschitz function φ on L^2 with $\chi_{S_{\delta/2}} \leq \varphi \leq \chi_{S_\delta}$ and $\mathrm{Lip}(\varphi) \leq \frac{2}{\delta}$. Hence, using (4.2), for all $T > 1$

$$\begin{aligned}Q^T(x, S_\delta) &\geq \frac{1}{T}\int_0^T P_s\varphi(x)ds \geq \frac{1}{T}\int_0^T P_s\varphi(z)ds - \frac{2}{\delta}\|x - z\| \\ &\geq Q^T(z, S_{\frac{\delta}{2}}) - \frac{2\|x-z\|}{\delta} > 1 - \epsilon.\end{aligned}$$

□

Lemma 4.3. *For $x \in L^2(0,1)$, let $(v^x(t))_{t\geq 0}$ the (generalized) solution of* (4.1) *corresponding to $Q = 0$. Then it holds*

$$\lim_{t\to+\infty} \|v^x(t)\| = 0.$$

Proof. First, we consider the case where the initial data $v_0 \in C_0^\infty(0,1)$ (space of C^∞-differentiable function compactly supported in $[0,1]$. We set $M := \|v_0'\|_\infty$ and define a function $h(t)$ with

(i) h is of class $C^\infty(\mathbb{R})$ and satisfies

$$h(t) = \arctan t \quad \text{for } |t| \leq M$$

$$|h(t)| \leq |t|, \quad t \in \mathbb{R}.$$

(ii) h' is a bounded function on $\mathbb{R}$ satisfies $\inf_{x\in\mathbb{R}} h'(x) \geq \mu > 0$ for a positive constant μ.

(iii) h'' is a bounded function on $\mathbb{R}$.

For $T > 0$ fixed, consider the equation

$$\begin{cases} dv(t) = (h(v_x(t)))_x\, dt, \\ v(0) = v_0. \end{cases} \tag{4.6}$$

Following a similar argument as in [11] and a maximum principle for uniformly parabolic equation we can prove that the classical solution v of (4.6) satisfies

$$\sup_{0\le t\le T}\|v_x\|_\infty \le M.$$

Hence from the construction of h we deduce that this solution is also the solution of (4.1) with $Q=0$ corresponding to the initial data $v_0\in C^\infty(0,1)$. Now we remark that for the function $z\mapsto \arctan z$ we can write

$$\arctan z = k(z)\cdot z \quad \text{for all } z\in\mathbb{R},$$

for some positive decreasing function k on $\mathbb{R}$. Therefore by using the energy estimate for the function $v(t)$ we can write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 &= -\langle \arctan v_x(t), v_x(t)\rangle_{L^2(0,1)}\\ &\le -\inf_{z\in B(0,M)} k(z)\,\|v_x(t)\|^2\\ &\le -\inf_{z\in B(0,M)} k(z)\,\|v(t)\|^2.\end{aligned}\tag{4.7}$$

Thus we obtain

$$\|v(t)\|^2 \le e^{-2t\inf\limits_{z\in B(0,M)} k(z)}\|v_0\|^2.$$

This implies the statement of the lemma for regular initial datum v_0. For general $v_0\in L^2(0,1)$ we proceed by approximation and let v_0^n a sequence of functions in $C_0^\infty(0,1)$ which converges to v_0 in $L^2(0,1)$ for $n\to+\infty$. For $n\ge 0$ we denote by $v_n(t)$ the solution corresponding to the initial condition v_0^n. By using the fact that $v_n(t)\to 0$ as $t\to 0$ and a triangle inequality argument we deduce the statement of the lemma for general initial datum $v_0\in L^2(0,1)$.

□

Lemma 4.4. *For $x\in L^2(0,1)$, let $(\hat v^x(t))_{t\ge 0}$ the (generalized) solution of (4.1) corresponding to $Q=0$. Then for every $x\in L^2, T>0$ and $\epsilon>0$, it holds that*

$$\mathbb{P}(\|\hat u_T^x - \hat v_T^x\| < \epsilon) > 0.$$

Proof. First we suppose that $x\in V$ and denote by $(v_t^x)_{t\ge 0}$ the solution corresponding to (4.1) with $Q=0$. We write

$$z(t) = u(t) - v(t), \quad t\ge 0.$$

Then the process $z(t)_{t\ge 0}$ solves the equation

$$\begin{cases} dz(t) = (Au(t) - Av(t))dt + QdW_t\\ z(0) = 0.\end{cases}$$

We set

$$z(t) = y(t) + QW_t.$$

Then we have

$$dy(t) = (Au(t) - Av(t))\,dt.$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y(t)\|^2 &= {}_{V^*}\langle Au(t) - Av(t), y(t)\rangle_V \, dt \\
&= {}_{V^*}\langle Au(t) - Av(t), z(t)\rangle_V \, dt - {}_{V^*}\langle Au(t) - Av(t), QW_t\rangle_V \\
&\le 2\Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}\|QW_t\|_V \le 2\Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}\|QW_t\|_V.
\end{aligned}
$$

Where we used the monotonicity of A and (3.2) to obtain the estimate in the last line. Thus we deduce for $0 \le t \le T$

$$
\|y(t)\| \le c\,T \sup_{0\le t\le T}\|QW_t\|_V,
$$

for some positive constant c. We now use the splitting of $z(\cdot)$ and the Poincaré inequality to obtain for $0 \le t \le T$

$$
\|z(t)\| \le (c\,T + \frac{1}{2}) \sup_{0\le t\le T}\|QW_t\|_V. \tag{4.8}
$$

For the case where $x \in H$ we proceed by approximation and use the uniform bound (3.2) to obtain the same estimate as in (4.8) for the process $z(t) = \hat{u}^x(t) - \hat{v}^x(t)$, $x \in H$. Since Q is a Hilbert-Schmidt operator from U to V, $(QW_t)_{t\ge 0}$ is a continuous Gaussian random process with values in V. Hence, for all $\delta > 0$

$$
\mathbb{P}\Big(\sup_{0\le t\le T}\|QW_t\|_V < \delta\Big) > 0.
$$

Now let $\varepsilon > 0$ and take $\delta > 0$ such that $(cT + 1/2)\delta < \varepsilon$. Then

$$
\mathbb{P}\Big(\|z(t)\| < \varepsilon\Big) > \mathbb{P}\Big(\sup_{0\le t\le T}\|QW_t\|_V < \delta\Big) > 0. \qquad \square
$$

Proposition 4.5. *For every $\delta > 0$ and every $x \in L^2([0,1])$ it holds that*

$$
\liminf_{T\to\infty} Q^T(x, B_\delta(0)) > 0.
$$

Proof. We proceed in three steps. Let $\delta > 0$ and $x \in L^2([0,1])$ be given.
Step 1. For $R > 0$ let $C_R = \{u \in L^2 | u \in W_0^{1,1}(0,1), \|u\|_{1,1} \le R\}$, which is a compact subset of $L^2([0,1])$. From (4.5) and Chebychev's inequality we deduce

$$
Q^T(0, L^2([0,1]) \setminus C_R) \le \frac{c}{R} \quad \forall\, T > 1.
$$

Hence we may pick some $R > 0$ such that $Q^T(0, C_R) > \frac{3}{4}$ for all $T > 1$. From now we omit the subscript R, i.e. $C = C_R$.
Step 2. Claim: There is some $\epsilon_1 > 0$, a $\gamma_1 > 0$ and a finite sequence $T_1, \cdots, T_k$, $T_i > 0$ such that

$$
\frac{1}{k}\sum_{i=1,\dots,k} P_{T_i}(x, B_\delta(0)) > \gamma_1 \quad \forall\, x \in C_{\epsilon_1},
$$

where $C_{\epsilon_1} = \{u \in L^2([0,1]) \,|\, d_{L^2}(u, C) < \epsilon_1\}$ and $P_T(x,\cdot)$ the transition probability corresponding to $(\hat{u}^x(t))_{t\ge 0}$ at time T. In fact, by lemma 4.3 for each $x \in L^2([0,1])$ there exists a T_x and a $r_x > 0$ such that $\hat{v}^x_{T_x} \in B_{\delta/4}(0)$. For $T > 0$ and $\delta > 0$ let

$$
D(x, T, \delta) := \mathbb{P}\{\|\hat{v}^x_T - \hat{u}^x_T\|_{L^2([0,1])} \le \delta\},
$$

which is strictly positive by lemma 4.4. Hence it follows that $P_{T_x}(x, B_{\frac{\delta}{2}}(0)) \ge D(x, T_x, \delta/4) =: \gamma_x > 0$. Similarly as in the second part of proposition 4.2 we may use (4.2) to deduce that for each $x \in L^2([0,1])$ there exists $r_x > 0$ such that $P_{T_x}(y, B_\delta(0)) > \gamma_x/2$ for all $y \in B_{r_x}(x)$. Since C

is compact we may select a finite sequence (x_i, r_i), $i = 1, \dots, k$, such that $C \subset \bigcup_{i=1,\dots,k} B(x_i, r_i)$. Setting $T_i := T_{x_i}$ the claim follows with $\epsilon_1 := \min_{i=1,\dots,k} r_i$ and $\gamma_1 := \min_{i=1,\dots,k} \gamma_i/2k$.

Step 3: Choose $\rho > 0$ such that

$$Q^T(x, C_{\epsilon_1}) > \frac{1}{2} \quad \forall\, x \in B_\rho(0).$$

This is possible by a similar argument as in the second part proposition 4.2. Finally, by analogous reasons as in step 2, we may find some $T_0 > 0$ and $\gamma_2 > 0$ such that $P_{T_0}(x, B_\rho(0)) > \gamma_2$.

Hence,

$$\begin{aligned}
\liminf_T Q^T(x, B_\delta(0)) &= \liminf_T \frac{1}{T} \int_0^T P_s(x, B_\delta(0)) ds \\
&= \liminf_T \frac{1}{k} \sum_{i=1,\dots,k} \frac{1}{T} \int_0^T P_{s+T_i+T_0}(x, B_\delta(0)) ds \\
&= \liminf_T \frac{1}{k} \sum_{i=1,\dots,k} \frac{1}{T} \int_0^T \int_{L^2([0,1])} \int_{L_2([0,1])} P_{T_i}(z, B_\delta(0)) P_s(y, dz) P_{T_0}(x, dy) ds \\
&\geq \liminf_T \frac{1}{T} \int_0^T \int_{B_\rho(0)} \int_{C_{\epsilon_1}} \frac{1}{k} \sum_{i=1,\dots,k} P_{T_i}(z, B_\delta(0)) P_s(y, dz) P_{T_0}(x, dy) ds \\
&\geq \gamma_1 \liminf_T \frac{1}{T} \int_0^T \int_{B_\rho(0)} P_s(y, C_{\epsilon_1}) P_{T_0}(x, dy) ds
\end{aligned}$$

which, by Fatou's lemma is bounded from below by

$$\begin{aligned}
&\geq \gamma_1 \int_{B_\rho(0)} \liminf_T \frac{1}{T} \int_0^T P_s(y, C_{\epsilon_1}) P_{T_0}(x, dy) ds \\
&= \gamma_1 \int_{B_\rho(0)} \liminf_T Q^T(y, C_{\epsilon_1}) P_{T_0}(x, dy) ds \\
&> \frac{1}{2} \gamma_1 P_{T_0}(x, B_\rho(0)) > \frac{1}{2} \gamma_1 \gamma_2 > 0. \qquad \square
\end{aligned}$$

In view of (4.2) and proposition 4.5, Theorem 4.1 is now a consequence of [6, Theorem 1], where $\mathcal{T} = L^2([0,1])$ according to proposition 4.2.

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