Time consistent portfolio management

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Abstract. This paper considers the portfolio management problem for an investor with finite time horizon who is allowed to consume and take out life insurance. Natural assumptions, such as different discount rates for consumption and life insurance lead to time inconsistency. This situation can also arise when the investor is in fact a group, the members of which have different utilities and/or different discount rates. As a consequence, the optimal strategies are not implementable. We focus on hyperbolic discounting, which has received much attention lately, especially in the area of behavioural finance. Following [10], we consider the resulting problem as a leader-follower game between successive selves, each of whom can commit for an infinitesimally small amount of time. We then define policies as subgame perfect equilibrium strategies. Policies are characterized by an integral equation which is shown to have a solution in the case of CRRA utilities. Our results can be extended for more general preferences as long as the equations admit solutions. Numerical simulations reveal that for the Merton problem with hyperbolic discounting, the consumption increases up to a certain time, after which it decreases; this pattern does not occur in the case of exponential discounting, and is therefore known in the litterature as the "consumption puzzle". Other numerical experiments explore the effect of time varying aggregation rate on the insurance premium.

[1]Work supported under NSERC grant 298427-04. We thank the referees for valuable advice, suggestions and a thorough reading of the first version.

AMS classification: 60G35, 60H20, 91B16, 91B70
Key words: Portfolio optimization, Pensioner's problem, Policies, Hyperbolic discounting.

1 Introduction

The investment/consumption problem in a stochastic context was considered by Merton [23] and [24]. His model consists in a risk-free asset with constant rate of return and one or more stocks, the prices of which are driven by geometric Brownian motion. The horizon T is prescribed, the portfolio is self-financing, and the investor seeks to maximize the expected utility of intertemporal consumption plus the final wealth. Merton provided a closed form solution when the utilities are of constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA) type. It turns out that for (CRRA) utilities the fraction of wealth invested in the risky asset is constant through time. Moreover for the case of (CARA) utilities, they are linear in wealth.

Richard [30] added life insurance to the investor's portfolio, assuming an arbitrary but known distribution of death time. In the same vein Pliska [28] studied optimal life insurance and consumption for an income earner whose lifetime is random and unbounded. More recently Kwak et al.[18] looked at the problem of finding optimal investment and consumption for a family whose parents receive deterministic labor income until some deterministic time horizon.

The aim of this paper is to revisit these problems in the case when the psychological discount rate is not constant. By now there is substantial evidence that people discount the future at a non-constant rate. More precisely, there is experimental evidence (see Frederick et. al. [11] for a review) that people are more sensitive to a given time delay if it occurs earlier: for instance, a person might prefer to get two oranges in 21 days than one orange in 20 days, but also prefer to get one orange right now than two oranges tomorrow. This is known as **the common difference effect**, and would not occur if future utilities are discounted at a constant rate. Individual behaviour is best described by **hyperbolic discounting**, where the discount factor is $h(t) = (1 + at)^{-\frac{b}{a}}$, with $a, b > 0$. The corresponding discount rate is $r(t) = \frac{b}{1+at}$, which starts from $r(0) = \frac{b}{a}$ and decreases to zero. Because of its empirical support, hyperbolic discounting has received a lot of attention in the areas of: microeconomics, macroeconomics and behavioural finance. We just mention here among others the works of Loewenstein and Prelec [21], Laibson [20] and Barro [2].

It is well-known that, for non-constant discount rates, optimal strategies are time inconsistent: for $t_1 < t_2$, the planner at time t_1 will find a strategy f_1 to be optimal on $[t_1, \infty)$, while the planner at time t_2 will find a different strategy f_2 to be optimal on that interval. As a result, the planner at time t_2 will not implement the strategy devised by the planner at time t_1, unless there exists some commitment mechanism. If there is none, then the strategy f_1, which is optimal from the perspective of the planner at time t_1, is not implementable, and the planner at time t_1 must look for a second-best strategy. This situation was first analyzed by Strotz [31], and this line of research has been pursued by many others (see Pollak [29], Phelps [27], Peleg and Yaari [26], Goldmann [12], Laibson [20], Barro [2], Krusell and Smith [19]), mostly in the framework of planning a discrete-time economy with production (Ramsey's problem). It is by now well established that time-consistent strategies are Stackelberg equilibria of a leader-follower game among successive selves (today's self has divergent interests from tomorrow's). More recently, the problem has been taken up again by Karp [13], [14], [15], [16]. Luttmer and Mariotti, [22], Ekeland and Lazrak [7], [8], [9], always within the framework of planning economic growth. In a series of papers Björk and Murgoci [3], Björk, Murgoci and Zhou [4] look at the mean variance problem which is also time inconsistent.

Ekeland and Pirvu [10] seem to have been the first to have considered the Merton problem with non-constant psychological discount rates. They studied the case of an investor who has a CRRA utility $u(c) = -\frac{1}{p}c^p,\ p < 1$, and a quasi-exponential discount factor $h(t)$, that is, $h(t)$ must belong to one of the families:

$$\begin{aligned} h(t) &= \lambda e^{-r_1 t} + (1-\lambda)\, e^{-r_2 t}, \\ h(t) &= (1+at)\, e^{-rt}. \end{aligned}$$

Extending the basic idea of Ekeland and Lazrak [7] to the stochastic framework, they find time-consistent strategies in the limiting case when the investor can commit only during an infinitesimal time interval. They show that time-consistent strategies exist if a certain BSDE has a solution, and they show that, because of the special form of the discount factor, this BSDE reduces to a system of two ODEs which has a solution.

The aim of this paper is to extend these results to more general discount rates, and more general problems. Quasi-exponential discount rates, although mathematically convenient, are not realistic. As we saw earlier, empirically observed discount rates among individuals tend to be hyperbolic. But there is another, perhaps more compelling, reason why general discount rates are of interest. Standard portfolio theory assumes that the investor is an individual. However, in most situations investment decisions are made by a group, such as the management team in the case when the investor entrusts his portfolio to professionals. Even when the investor manages the portfolio directly, the word "investor" which is suggestive of a single decision-maker very often hides a different reality, namely the family: one would expect the husband and the wife to take part in investment decisions concerning the couple. By now, the relevant economic literature has made abundantly clear (see Chiappori and Ekeland, [5]) that the group cannot be represented by a single utility function. Instead, there should be one utility function, and one discount factor per member of the group. The actual decision taken is the result of negociations within the group, a kind of black box which cannot be opened by outsiders. However, if the group is efficient, that is, if the outcome is Pareto optimal, then it can be modelled by maximising a suitable convex combination of the members' utilities, the weight conferred to each individual representing his/her power within the group. In the (very) particular case when all members of the group have the same utility, but different discount rates, the group behaves as a single individual with non-constant discount rate.

The difficulty in dealing with non-constant discount rates is to define time-consistent strategies and to prove that they exist. We follow the approach pioneered by Ekeland and Lazrak [7] in the deterministic framework, by considering the limiting case when the decision-maker can commit only during an infinitesimal amount of time. This approach was already followed in our earlier work [10], in the case of quasi-exponential discount factors. The proofs in that paper do not readily extend to the case of general discount factors, so in the present work we present a different method. Whereas [10] characterizes the time-consistent stategies in terms of a certain BSDE, we now characterize them in terms of a certain "value function", which is shown to satisfy a certain integral equation which has a natural interpretation. Assuming utilities to be (CRRA), we decouple time and space, and reduce it to a one-dimensional integral equation, which we solve by a fixed-point argument. Moreover this one dimensional equation is amenable to numerical treatments so one can compare the equilibrium policies arising from different choices of discounting. The numerical scheme we employ consists of the discretization of the one dimensional equation in three steps. This is based on a Riemann sum approximation of the integral. We obtain closed form solutions in certain cases.

We show that hyperbolic discounting may result in consumption patterns which are observationally different from the optimal strategies in the Merton model. The latter predicts that consumption grows smoothly over time if the interest rate exceeds the discount rate (or decays smoothly otherwise). However, household data indicates that consumption is hump-shaped. This is referred to in the literature as the consumption puzzle, and we show that it can arise as a time-consistent strategy in certain cases of hyperbolic discounting.

By running numerical simulations we study the effect of the weight given by the insurer to the beneficiaries on the life insurance process.

Organization of the paper: The remainder of this paper is organized as follows. In section 2 we describe the model and formulate the objective. Section 3 introduces the value function. Section 4 presents the main result. Section 5 deals with CRRA utilities. Numerical results are discussed in Section 6. An extension to multiple managers is discussed in Section 7. The paper ends with an appendix containing some proofs.

2 The Model

2.1 The decisions

Consider a financial market consisting of a savings account and one stock (the risky asset). The inclusion of more risky assets can be achieved by notational changes. The savings account accrues interest at the riskless rate $r > 0$. The stock price per share follows an exponential Brownian motion

$$dS(t) = S(t)\left[\alpha\, dt + \sigma\, dW(t)\right], \quad 0 \leq t \leq \infty,$$

where $\{W(t)\}_{t\geq 0}$ is a $1-$dimensional Brownian motion on a filtered probability space, $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}$ is the completed filtration generated by $\{W(t)\}$. Let us denote by $\mu \triangleq \alpha - r > 0$ *the excess return.*

A decision-maker in this market is continuously investing in the stock and the bond, consuming and buying life insurance, while receiving income at the continuous deterministic rate

$i(t)$. This assumption is key in deriving our results. Relaxing it to accommodate for problems relevant to small enterprises is not obvious and would be an interesting research project.

Life insurance is offered as a sucession of term contracts with infinitesimally small horizon. At every time t, a contract is offered, costing 1 unit of account. If the holder dies immediately after, the insurance company pays $l\,(t)$ to his/her beneficiaries. The deterministic function $l\,(t)$ is prescribed.

At every time t, the investor chooses $\zeta(t)$, the investment in the risky asset, $c(t)$ the consumption, and $p(t)$, the amount of life insurance. Given an adapted process $\{\zeta(t), c(t), p(t)\}_{t\geq 0}$, the equation describing the dynamics of wealth $X^{\zeta,c,p}(t)$ is given by

$$\begin{aligned} dX^{\zeta,c,p}(t) &= rX^{\zeta,c,p}(t)dt - c(t)dt - p(t)dt + i(t)dt + \zeta(t)(\alpha\, dt + \sigma dW(t)) \\ X^{\zeta,c,p}(0) &= X\,(0)\,, \end{aligned} \tag{2.1}$$

the initial wealth $X(0)$ being exogenously specified.

We assume a benchmark deterministic time horizon T. The investor is alive at time $t = 0$ and has a lifetime denoted by τ, which is a non-negative random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of the Brownian motion W. Denote by $g\,(t)$ its density and by $G\,(t)$ its distribution:

$$G(t) \triangleq \mathbb{P}(\tau < t) = \int_0^t g(u)\, du.$$

It will be useful for later computations to introduce the hazard function $\lambda\,(t)$, that is, the instantaneous death rate, defined by

$$\lambda(t) \triangleq \lim_{\delta t \to 0} \frac{\mathbb{P}(t \leq \tau < t + \varepsilon \mid \tau \ \geq t)}{\varepsilon} = \frac{g(t)}{1 - G(t)},$$

so that $g(t) = \lambda(t) \exp\{-\int_0^t \lambda(u)\, du\}$. We have, from the definition:

$$\mathbb{P}(\tau < s \mid \tau > t) = 1 - \exp\{-\int_t^s \lambda(u)\, du\}. \tag{2.2}$$

and

$$\mathbb{P}(\tau > T | \tau > t) = \exp\{-\int_t^T \lambda(u)\, du\}. \tag{2.3}$$

Next we turn to risk preferences.

2.2 Utility functions

Definition 2.1 *A utility function U is a strictly increasing, strictly concave differentiable real-valued function defined on $[0, \infty)$ which satisfies the Inada conditions*

$$U'(0) \triangleq \lim_{x \downarrow 0} U'(x) = \infty, \qquad U'(\infty) \triangleq \lim_{x \to \infty} U'(x) = 0. \tag{2.4}$$

The strictly decreasing C^1 function U' maps $(0,\infty)$ onto $(0,\infty)$ and hence has a strictly decreasing, C^1 inverse $I:(0,\infty)\to(0,\infty)$.

The legacy process of the decision-maker, $\{Z^{\zeta,c,p}(t)\}_{t\geq 0}$, is defined by

$$Z^{\zeta,c,p}(t) \triangleq \eta(t)X^{\zeta,c,p}(t) + l(t)p(t), \tag{2.5}$$

where $\eta(t)$ and $l(t)$ are prescribed deterministic and continuous functions. The legacy is the sum of two terms: the first one, $\eta(t)X^{\zeta,c,p}(t)$, is the part of his wealth which will benefit his heirs (after taxes, and various costs), and the second one $l(t)p(t)$ is the life insurance. Although the insurance premium $p(t)$ is allowed to be negative we require that the legacy $Z^{\zeta,c,p}(t)$ stays positive. A negative $p(t)$ means that the decision maker can sell life insurance.

Let U_1, U_2, U_3 be utility functions as in Definition 2.1; U_1 is the utility from intertemporal consumption, U_2 is the utility of the final wealth and U_3 is the utility of the legacy.

Next, we define the admissible strategies. Sometimes, to ease notations, we write $X^{t,x}(s)$ and $Z^{t,x}(s)$ for $\mathbb{E}[X^{\zeta,c,p}(s)|X^{\zeta,c,p}(t)=x]$ and $\mathbb{E}[Z^{\zeta,c,p}(s)|X^{\zeta,c,p}(t)=x]$.

Definition 2.2 *An $\mathbb{R}^3$-valued stochastic process $\{\zeta(t), c(t), p(t)\}_{t\geq 0}$ is called an admissible strategy process if*

- *it is progressively measurable with respect to the sigma algebra $\sigma(\{W(t)\}_{t\geq 0})$,*
- *$c(t)\geq 0, Z^{\zeta,c,p}(t)\geq 0$ for all, a.s.; $X^{\zeta,c,p}(T)\geq 0$, a.s.*
- *moreover we require that for all $t, x\geq 0$*

$$\mathbb{E}\sup_{\{t\leq s\leq T\}}|U_1(c(s))|<\infty,\ \mathbb{E}|U_2(X^{t,x}(T))|<\infty,\ \mathbb{E}\sup_{\{t\leq s\leq T\}}|U_3(Z^{t,x}(s))|<\infty. \tag{2.6}$$

The last set of inequalities are purely technical and are satisfied for e.g. bounded strategies. They are essential in proving our main result and are related to the fact that the expected utility criterion is continuously updated.

2.3 The intertemporal utility

In order to evaluate the performance of an investment-consumption-insurance strategy the decision maker uses an expected utility criterion. For an admissible strategy process $\{\zeta(s), c(s), p(s)\}_{s\geq 0}$ and its corresponding wealth process $\{X^{\zeta,c,p}(s)\}_{s\geq 0}$, we denote the expected intertemporal utility by

$$\begin{aligned} J(t,x,\zeta,c,p) &\triangleq \mathbb{E}\Big[\int_t^{T\wedge\tau} h(s-t)U_1(c(s))\,ds + nh(T-t)U_2(X^{\zeta,c,p}(T))1_{\{\tau>T|\tau>t\}} \\ &+ m(\tau-t)\hat{h}(\tau-t)U_3(Z^{\zeta,c,p}(\tau))1_{\{\tau\leq T|\tau>t\}}\Big|X^{\zeta,c,p}(t)=x\Big], \end{aligned} \tag{2.7}$$

where:

- $n > 0$ is a constant
- $m(t) > 0$ is a continuous function
- h and $\hat{h}$ are continuously differentiable, positive and decreasing functions, such that $h(0) = \hat{h}(0) = 1$.

The interpretation is as follows. The decision-maker will collect $X^{\zeta,c,p}(T)$ at time T, if he is still alive at time T, and the coefficient n is the weight he attributes to getting that lump sum, as compared to the utility of continuous consumption up to time T. The function $h(t)$ is his discount function, and it is no longer restricted to the exponential and quasi-exponential type.

He may, however, die before time T, in which case his wealth will accrue to others, and the decision-maker is taking the utility of his beneficiaries into account when managing his portfolio.

Since the death time τ is independent of the uncertainty driving the stock, we have the following simplified expression for the functional J, which is proved in Appendix A.

Lemma 2.3 *The functional J of* (2.7) *equals*

$$\begin{aligned} J(t,x,\zeta,c,p) &= \mathbb{E}\Big[\int_t^T Q(s,t)U_1(c(s))\,ds && (2.8)\\ &+ \int_t^T q(s,t)U_3(Z^{t,x}(s))\,ds + nQ(T,t)U_2(X^{t,x}(T))\Big], && (2.9)\end{aligned}$$

where

$$\begin{aligned} q(s,t) &\triangleq \bar{h}(s-t)\lambda(s)\exp\{-\int_t^s \lambda(z)\,dz\}, \quad \bar{h}(t) \triangleq m(t)\hat{h}(t) && (2.10)\\ Q(s,t) &\triangleq h(s-t)\exp\{-\int_t^s \lambda(z)\,dz\} && (2.11)\end{aligned}$$

A natural objective for the decision maker is to maximize the above expected utility criterion. However, because neither q nor Q are exponentials, time inconsistency will bite, that is, a strategy that will be considered to be optimal at time 0 will not be considered so at later times, so it will not be implemented unless the decision-maker at time 0 can constrain the decision-maker at all times $t > 0$.

2.4 Time-consistent strategies

We now introduce a special class of time-consistent strategies, which will henceforth be called *policies*. That is, we consider that the decision-maker at time t can commit his successors up to time ε, with $\varepsilon \to 0$, and we seek strategies which it is optimal to implement right now conditioned on them being implemented in the future.

More precisely, suppose that a strategy f is time-consistent. This means that, if it has been applied up to time t, the decision-maker at time t will apply it as well. Since there is no commitment mechanism to force him to do so, he will only apply strategy f if it is in his own best interests. Denote his current wealth by $X(t)$. He has two possibilities: either to stick to the strategy f, or to apply another one. To simplify matters, we will assume that the decision-maker considers only a very short time interval, $[t, t+\epsilon]$, so short in fact that all strategies can be assumed to be constant on that interval. The decision-maker then just compares the effect of investing $\bar{\zeta}$, consuming $\bar{c}$ and buying $\bar{p}$ worth of insurance, as required by the strategy f at time t, with the effect of investing ζ, consuming c and buying p worth of insurance, for different (constant) values. There will be, as usual, an immediate effect, corresponding to the change in consumption between t and $t+\epsilon$, and a long-term effect, corresponding to the change in wealth at time $t+\epsilon$.

Let us formalize this idea:

Definition 2.4 *An admissible trading strategy $\{\bar{\zeta}(s), \bar{c}(s), \bar{p}(s)\}_{t\leq s\leq T}$ is a* policy *if there exists a map $F = (F_1, F_2, F_3) : [0, T] \times \mathbb{R} \to \mathbb{R} \times [0, \infty) \times \mathbb{R}$ such that for any $t, x > 0$*

$$\liminf_{\epsilon \downarrow 0} \frac{J(t, x, \bar{\zeta}, \bar{c}, \bar{p}) - J(t, x, \zeta_\epsilon, c_\epsilon, p_\epsilon)}{\epsilon} \geq 0, \tag{2.12}$$

where:

$$\bar{\zeta}(s) = F_1(s, \bar{X}(s)), \quad \bar{c}(s) = F_2(s, \bar{X}(s)), \quad \bar{p}(s) = F_3(s, \bar{X}(s)) \tag{2.13}$$

and the wealth process $\{\bar{X}(s)\}_{s\in[t,T]}$ is a solution of the stochastic differential equation (SDE):

$$d\bar{X}(s) = [r\bar{X}(s)+\mu F_1(s, \bar{X}(s))-F_2(s, \bar{X}(s))-F_3(s, \bar{X}(s))+i(s)]ds+\sigma F_1(s, \bar{X}(s))dW(s). \tag{2.14}$$

Here, the process $\{\zeta_\epsilon(s), c_\epsilon(s), p_\epsilon(s)\}_{s\in[t,T]}$ is another investment-consumption strategy defined by

$$\zeta_\epsilon(s) = \begin{cases} \bar{\zeta}(s), & s \in [t, T]\backslash E_{\epsilon,t} \\ \zeta(s), & s \in E_{\epsilon,t}, \end{cases} \tag{2.15}$$

$$c_\epsilon(s) = \begin{cases} \bar{c}(s), & s \in [t, T]\backslash E_{\epsilon,t} \\ c(s), & s \in E_{\epsilon,t}, \end{cases} \tag{2.16}$$

$$p_\epsilon(s) = \begin{cases} \bar{p}(s), & s \in [t, T]\backslash E_{\epsilon,t} \\ p(s), & s \in E_{\epsilon,t}, \end{cases} \tag{2.17}$$

with $E_{\epsilon,t} = [t, t+\epsilon]$, and $\{\zeta(s), c(s), p(s)\}_{s\in E_{\epsilon,t}}$ is any strategy for which $\{\zeta_\epsilon(s), c_\epsilon(s), p_\epsilon(s)\}_{s\in[t,T]}$ is an admissible policy.

In other words, policies are Markov strategies such that unilateral deviations during an infinitesimally small time interval are penalized. Note that:

- this does not mean that unilateral deviations during a finite interval of time are penalized as well: it is possible that deviating from the policy between t_1 and t_2 will be to the advantage of all the decision-makers operating between t_1 and t_2.

- however, if a Markov strategy is not a policy, then it certainly will not be implemented, for at some point it will be to the advantage of some lone decision-maker to deviate, during a very small time interval, which is enough to compromise all the plans laid by his predecessors.

So time-consistency in the sense of Definition 2.4 is a minimal requirement for rationality: policies are the only Markov strategies that the decision-maker should consider.

3 The Value Function

We now extend to this situation the notion of a value function, which is classical in optimal control. Let $m \triangleq m(0)$, and I_1, I_3 be the inverse functions of U_1', U_3'.

Definition 3.1 *Let $v : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a $C^{1,2}$ function, concave in the second variable. We shall say that v is a value function if we have:*

$$v(t,x) = J(t,x,\bar{\zeta},\bar{c},\bar{p}). \tag{3.1}$$

Here the admissible process $\{\bar{\zeta}(s),\bar{c}(s),\bar{p}(s)\}_{s\in[t,T]}$ is given by:

$$\bar{\zeta}(s) \triangleq -\frac{\mu \frac{\partial v}{\partial x}(s,\bar{X}(s))}{\sigma^2 \frac{\partial^2 v}{\partial x^2}(s,\bar{X}(s))},\ \bar{c}(s) \triangleq I_1\left(\frac{\partial v}{\partial x}(s,\bar{X}(s))\right),\ \bar{p}(s) \triangleq \frac{1}{l(s)}\left[I_3\left(\frac{1}{m}\frac{\partial v}{\partial x}(s,\bar{X}(s))\right) - \eta(s)\bar{X}(s)\right], \tag{3.2}$$

where $\bar{X}(s)$ is the corresponding wealth process defined by the SDE

$$\begin{aligned} d\bar{X}(s) &= \left[r\bar{X}(s) - \frac{\mu^2 \frac{\partial v}{\partial x}(s,\bar{X}(s))}{\sigma^2 \frac{\partial^2 v}{\partial x^2}(s,\bar{X}(s))} - I_1\left(\frac{\partial v}{\partial x}(s,\bar{X}(s))\right)\right. && (3.3)\\ &- \left.\frac{1}{l(s)}\left[I_3\left(\frac{1}{m}\frac{\partial v}{\partial x}(s,\bar{X}(s))\right) - \eta(s)\bar{X}(s)\right] + i(s)\right]ds - \frac{\mu \frac{\partial v}{\partial x}(s,\bar{X}(s))}{\sigma \frac{\partial^2 v}{\partial x^2}(s,\bar{X}(s))}dW(s) \\ \bar{X}(t) &= x. && (3.4) \end{aligned}$$

The economic interpretation is very natural: if one applies the Markov strategy associated with v by the relations (3.2), and computes the corresponding value of the investor's criterion starting from x at time t, one gets precisely $v(t,x)$. In other words this is fundamentally a fixed-point characterization.

Let us define the functions F_1, F_2, F_3 by:

$$F_1(t,x) \triangleq -\frac{\mu \frac{\partial v}{\partial x}(t,x)}{\sigma^2 \frac{\partial^2 v}{\partial x^2}(t,x)},\ F_2(t,x) \triangleq I_1\left(\frac{\partial v}{\partial x}(t,x)\right),\ F_3(t,x) \triangleq \frac{1}{l(t)}\left[I_3\left(\frac{1}{m}\frac{\partial v}{\partial x}(t,x)\right) - \eta(t)x\right]. \tag{3.5}$$

Next we impose a technical assumption; for a $C^{1,2}$ function $f:[0,T]\times\mathbb{R}\to\mathbb{R}$, let us define the operator Lf by

$$Lf(t,x)\triangleq\frac{\partial f}{\partial t}(t,x)+(rx+\mu F_1(t,x)-F_2(t,x)-F_3(t,x)+i(t))\frac{\partial f}{\partial x}(t,x)+\frac{\sigma^2F_1^2(t,x)}{2}\frac{\partial^2 f}{\partial x^2}(t,x).$$

Assumption 3.2 *Assume that the PDEs*

$$Lf(t,x)=0,\quad f(s,x)=g(x), \tag{3.6}$$

have a $C^{1,2}$ solution on $[0,s]\times\mathbb{R}\to\mathbb{R}$ with exponential growth. Here $t<s\leq T$, and $g(x)$ is one of the functions

$$U_1(F_2(s,x)):\ t<s\leq T,\quad U_3(\eta(s)x+l(s)F_2(s,x)):\ t<s\leq T,\quad U_2(x):\ s=T.$$

4 Main Result

The following Theorem states the central result of our paper. It involves the notions of policies and value function for which we gave economic intuition.

Theorem 4.1 *Let v be a value function which satisfies Assumption 3.2. Then, $\{\bar\zeta(s),\bar c(s),\bar p(s)\}_{s\in[t,T]}$ given by* (3.2) *is a policy.*

We proceed in two steps. First we show that the value function v satisfies a partial differential equation with a non-local term and this is done in the following Lemma, which is proved in Appendix B.

Lemma 4.2 *The function v solves the following equation*

$$\frac{\partial v}{\partial t}(t,x)+\Big(rx+\mu F_1(t,x)-F_2(t,x)-F_3(t,x)+i(t)\Big)\frac{\partial v}{\partial x}(t,x)+ \tag{4.1}$$

$$\frac{\sigma^2F_1^2(t,x)}{2}\frac{\partial^2 v}{\partial x^2}(t,x)+U_1(F_2(t,x))+mU_3(x+l(t)F_3(t,x))=$$

$$\mathbb{E}\Big[\int_t^T\frac{\partial Q}{\partial t}(s,t)U_1(F_2(s,\bar X^{t,x}(s)))\,ds+\int_t^T\frac{\partial q}{\partial t}(s,t)U_3(\bar Z^{t,x}(s))\,ds+n\frac{\partial Q}{\partial t}(T,t)U_2(\bar X^{t,x}(T))\Big],$$

with boundary condition $v(T,x)=nU_2(x)$, and the processes $\bar X$ of (2.14)*, and $\bar Z^{t,x}(s)\triangleq\eta(s)\bar X^{t,x}(s)+l(s)F_3(s,\bar X^{t,x}(s))$.*

We now proceed to the second step. In view of function's v concavity in variable x, and the definition of (F_1, F_2, F_3) (see (3.5)), the equation (4.1) can be re-written as

$$\frac{\partial v}{\partial t}(t,x) + \sup_{\zeta,c,p}\Bigg[\Bigg(r + \mu\zeta - c - p + i(t)\Bigg)\frac{\partial v}{\partial x}(t,x) +$$

$$\frac{1}{2}\sigma^2\zeta^2\frac{\partial^2 v}{\partial x^2}(t,x) + U_1(c) + mU_3(\eta(t)x + l(t)p)\Bigg] = \tag{4.2}$$

$$\mathbb{E}\Bigg[\int_t^T \frac{\partial Q}{\partial t}(s,t)U_1(F_2(s,\bar{X}^{t,x}(s)))\,ds + \int_t^T \frac{\partial q}{\partial t}(s,t)U_3(\bar{Z}^{t,x}(s))\,ds + n\frac{\partial Q}{\partial t}(T,t)U_2(\bar{X}^{t,x}(T))\Bigg].$$

We notice that

$$J(t,x,\zeta_\epsilon,c_\epsilon,p_\epsilon) - J(t,x,\bar{\zeta},\bar{c},\bar{p}) =$$
$$\mathbb{E}\Bigg[\int_t^{t+\epsilon} Q(s,t)[U_1(c(s)) - U_1(F_2(s,X^{t,x}(s)))]\,ds\Bigg] +$$
$$\mathbb{E}\Bigg[\int_t^{t+\epsilon} q(s,t)[U_3(Z^{t,x}(s)) - U_3(\bar{Z}^{t,x}(s))]\,ds\Bigg] +$$
$$\mathbb{E}[v(t+\epsilon, X^{t,x}(t+\epsilon)) - v(t+\epsilon, \bar{X}^{t,x}(t+\epsilon))] +$$
$$\mathbb{E}\left[\int_{t+\epsilon}^T [Q(s,t) - Q(s,t-\epsilon)][U_1(F_2(s,\bar{X}^{t,x}(s))) - U_1(F_2(s,X^{t,x}(s)))]\,ds\right] +$$
$$\mathbb{E}\left[\int_{t+\epsilon}^T [q(s,t) - q(s,t-\epsilon)][U_3(Z^{t,x}(s)) - U_3(\bar{Z}^{t,x}(s))]\,ds\right] +$$
$$\mathbb{E}\left[n[Q(T,t) - Q(T,t-\epsilon)][U_2(X^{t,x}(T)) - U_2(\bar{X}^{t,x}(T))]\right].$$

The RHS of this equation has six terms and we will treat each of these terms separately:
1. In the light of inequality (2.6) and the Lebesgue Dominated Convergence Theorem

$$\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}\Big[\int_t^{t+\epsilon} Q(s,t)[U_1(c(s)) - U_1(F_2(s,X^{t,x}(s)))]\,ds\Big]}{\epsilon}$$
$$= [U_1(c(t)) - U_1(F_2(t,x))].$$

2. In the light of inequality (2.6) and the Lebesgue Dominated Convergence Theorem

$$\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}\Big[\int_t^{t+\epsilon} q(s,t)[U_3(Z^{t,x}(s)) - U_3(\bar{Z}^{t,x}(s))]\,ds\Big]}{\epsilon} = m[U_3(x + p(t)l(t)) - U_3(x + F_3(t,x)l(t))].$$

3. One has

$$\mathbb{E}[v(t+\epsilon,\bar{X}^{t,x}(t+\epsilon)) - v(t+\epsilon,X^{t,x}(t+\epsilon))] =$$
$$\mathbb{E}[v(t+\epsilon,\bar{X}^{t,x}(t+\epsilon)) - v(t,x)] - \mathbb{E}[v(t+\epsilon,X^{t,x}(t+\epsilon) - v(t,x)].$$

Moreover

$$\mathbb{E}[v(t+\epsilon,\bar{X}^{t,x}(t+\epsilon)) - v(t,x)] = \mathbb{E}\int_t^{t+\epsilon} d[v(u,\bar{X}^{t,x}(u))].$$

Itô's formula yields

$$\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}\int_t^{t+\epsilon} d[v(u,\bar{X}^{t,x}(u))]}{\epsilon} =$$
$$\left[\frac{\partial v}{\partial t}(t,x) + \left(rx + \mu F_1(t,x) - F_2(t,x) - F_3(t,x) + i(t)\right)\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2 F_1^2(t,x)}{2}\frac{\partial^2 v}{\partial x^2}(t,x)\right].$$

Similarly

$$\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}[v(t+\epsilon,X^{t,x}(t+\epsilon)) - v(t,x)]}{\epsilon} = \lim_{\epsilon\downarrow 0}\frac{\mathbb{E}\int_t^{t+\epsilon} d[v(u,X^{t,x}(u))]}{\epsilon} =$$
$$\left[\frac{\partial v}{\partial t}(t,x) + \left(r + \mu\zeta(t) - c(t) - p(t) + i(t)\right)\frac{\partial v}{\partial x}(t,x) + \frac{1}{2}\sigma^2\zeta^2(t)\frac{\partial^2 v}{\partial x^2}(t,x)\right].$$

4. In the light of inequality (2.6) and the Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}\left[\int_{t+\epsilon}^T [Q(s,t) - Q(s,t-\epsilon)][U_1(F_2(s,\bar{X}^{t,x}(s))) - U_1(F_2(s,X^{t,x}(s)))]\,ds\right]}{\epsilon} = 0.$$

5. Similarly

$$\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}\left[\int_{t+\epsilon}^T [q(s,t) - q(s,t-\epsilon)][U_3(Z^{t,x}(s)) - U_3(\bar{Z}^{t,x}(s))]\,ds\right]}{\epsilon} = 0.$$

6. Finally, by the same token

$$\lim_{\epsilon\downarrow 0}\frac{\mathbb{E}\left[n[Q(T,t) - Q(T,t-\epsilon)][U_2(X^{t,x}(T)) - U_2(\bar{X}^{t,x}(T))]\right]}{\epsilon} = 0.$$

Therefore

$$\lim_{\epsilon\downarrow 0}\frac{J(t,x,\bar{\zeta},\bar{c},\bar{p}) - J(t,x,\zeta_\epsilon,c_\epsilon,p_\epsilon)}{\epsilon} =$$
$$\left[\frac{\partial v}{\partial t}(t,x) + \left(rx + \mu F_1(t,x) - F_2(t,x) - F_3(t,x) + i(t)\right)\frac{\partial v}{\partial x}(t,x) +\right.$$

$$\frac{\sigma^2 F_1^2(t,x)}{2}\frac{\partial^2 v}{\partial x^2}(t,x) + U_1(F_2(t,x))) + mU_3(x + l(t)F_3(t,x))\Bigg] -$$

$$\left[\frac{\partial v}{\partial t}(t,x) + \Big(r + \mu\zeta(t) - c(t) - p(t) + i(t)\Big)\frac{\partial v}{\partial x}(t,x) + \right.$$

$$\left.\frac{1}{2}\sigma^2\zeta^2(t)\frac{\partial^2 v}{\partial x^2}(t,x) + U_1(c(t)) + mU_3(x + l(t)p(t))\right] \geq 0,$$

where the last inequality comes from (4.1) and (4).

□

5 CRRA Preferences

Finding a value function is a complicated problem. We are able to deal with the case of power type utilities, that is (with some abuse of notations) $U_1(x) = U_2(x) = U_3(x) = U_\gamma(x) \triangleq \frac{x^\gamma}{\gamma}$, with $\gamma < 1$. In this case we search for the value function v of the form

$$v(t,x) = a(t)U_\gamma(x + b(t)), \tag{5.1}$$

where the functions $a(t)$, $b(t)$ are to be found. We consider here the case $\gamma \neq 0$ (the case of logarithmic utility will be treated separately). In the light of equations (3.5) one gets

$$F_1(t,x) = \frac{\mu(x + b(t))}{\sigma^2(1-\gamma)},\ F_2(t,x) = [a(t)]^{\frac{1}{\gamma-1}}(x + b(t)), \tag{5.2}$$

$$F_3(t,x) = \frac{1}{l(t)}\left[([\frac{a(t)}{m}]^{\frac{1}{\gamma-1}} - \eta(t))x + [\frac{a(t)}{m}]^{\frac{1}{\gamma-1}}b(t))\right]. \tag{5.3}$$

By (3.4) the associated wealth process has the following dynamics:

$$\begin{aligned} d\bar{X}(s) &= \Bigg[\left(r + \frac{\eta(s)}{l(s)}\right)\bar{X}(s) + \frac{\mu^2}{\sigma^2(1-\gamma)}(\bar{X}(s) + b(s)) \\ &- (a(s))^{\frac{1}{\gamma-1}}\left(1 + \frac{1}{m^{\frac{1}{\gamma-1}}l(s)}\right)(\bar{X}(s) + b(s))\Bigg]ds \\ &+ i(s)ds + \frac{\mu(\bar{X}(s) + b(s))}{\sigma(1-\gamma)}dW(s). \end{aligned}$$

Let us define the process $Y(s) \triangleq \bar{X}(s) + b(s)$ which has the dynamics

$$\begin{aligned} dY(s) = &\Bigg[(r + \frac{\eta(s)}{l(s)})Y(s) + \frac{\mu^2}{\sigma^2(1-\gamma)}Y(s) - [a(s)]^{\frac{1}{\gamma-1}}\left(1 + \frac{1}{m^{\frac{1}{\gamma-1}}l(s)}\right)Y(s) \\ &+ i(s) + b'(s) - (r + \frac{\eta(s)}{l(s)})b(s)\Bigg]ds + \frac{\mu Y(s)}{\sigma(1-\gamma)}dW(s). \end{aligned} \tag{5.4}$$

For considerations that will become clear later on we choose $b(s)$ such that

$$i(s)+b'(s)-(r+\frac{\eta(s)}{l(s)})b(s)=0,\ \text{ and } b(T)=0.$$

By solving this ODE we get

$$b(s)=\int_s^T i(u)e^{-\int_u^s\left(r+\frac{\eta(x)}{l(x)}\right)dx}du. \tag{5.5}$$

Solving for the process $Y(s)$ we get

$$\begin{aligned} Y(s) &= Y(t)\exp\left(\int_t^s\left(r+\frac{\mu^2}{2\sigma^2(1-\gamma)^2}+\frac{\eta(u)}{l(u)}-(a(u))^{\frac{1}{\gamma-1}}\left(1+\frac{1}{m^{\frac{1}{\gamma-1}}l(u)}\right)\right)du \right.\\ &+ \left.\frac{\mu(W(s)-W(t))}{\sigma(1-\gamma)}\right) \end{aligned}$$

Therefore

$$\begin{aligned} \bar{X}^{t,x}(T) &= x\exp\left(\int_t^T\left(r+\frac{\mu^2}{2\sigma^2(1-\gamma)^2}+\frac{\eta(u)}{l(u)}-(a(u))^{\frac{1}{\gamma-1}}\left(1+\frac{1}{m^{\frac{1}{\gamma-1}}l(u)}\right)\right)du \right.\\ &+ \left.\frac{\mu(W(T)-W(t))}{\sigma(1-\gamma)}\right). \end{aligned}$$

By plugging v of (5.1) (with (F_1,F_2,F_3) of (5.2), (5.3)) into (3.1) and (3.4), we obtain the following integral equation (IE) for $a(t)$

$$\begin{aligned} a(t) &= \int_t^T[Q(s,t)+q(s,t)](a(s))^{\frac{\gamma}{\gamma-1}}e^{K(s-t)+\left(\int_t^s\frac{\gamma\eta(z)}{l(z)}-\gamma(a(z))^{\frac{1}{\gamma-1}}\left(1+\frac{1}{m^{\frac{1}{\gamma-1}}l(z)}\right)dz\right)}ds \\ &+ nQ(T,t)e^{K(T-t)+\left(\int_t^T\frac{\gamma\eta(z)}{l(z)}-\gamma(a(z))^{\frac{1}{\gamma-1}}\left(1+\frac{1}{m^{\frac{1}{\gamma-1}}l(z)}\right)dz\right)}, \qquad a(T)=n. \end{aligned} \tag{5.6}$$

with

$$K\triangleq\gamma\left(r+\frac{\mu^2}{2(1-\gamma)\sigma^2}\right). \tag{5.7}$$

Let us summarize this finding:

Lemma 5.1 *Let $a(t)$ be a solution of the fixed-point problem (5.6). Define $b(t)$ by (5.5). Then $v(t,x)=a(t)U_\gamma(x+b(t))$ is a value function.*

We turn our attention to the integral equation (5.6). Set

$$M(z) \triangleq 1 + \frac{1}{m^{\frac{1}{\gamma-1}} l(z)} \tag{5.8}$$

Assumption 5.2 *We require that:*

$$\min_{t\in[0,T]} (1 - \gamma M(t) + \lambda(t)) \geq 0. \tag{5.9}$$

The Assumption 5.2 is met if $\gamma \leq 0$. If $m = 1$ and $l(t) = \frac{1}{\lambda(t)}$, then Assumption 5.2 is also satisfied (this is the situation considered by [28]). In the case when $\min_{t\in[0,T]}(1-\gamma M(t)+\lambda(t)) < 0$ it can happen that $a(t)$ reaches 0 which leads to unbounded consumption.

The following Proposition is proved in Appendix C.

Proposition 5.3 *If Assumption 5.2 is satisfied, then there exists a unique global C^1 solution of the integral equation* (5.6)*.*

In other words, for the problem under consideration, there always exists a value function of the special type $v(t,x) = a(t)U_\gamma(x + b(t))$ (note that there may be others as well). We now proceed to deduce the existence of policies.

Theorem 5.4 *Let v be the value function of Lemma 5.1. Then, $\{\bar{\zeta}(s), \bar{c}(s), \bar{p}(s)\}_{s\in[t,T]}$ given by* (3.2) *is a policy.*

The proof follows from Theorem 4.1, Lemma 5.1 and Proposition 5.3 as long as we prove that $\{\bar{\zeta}(s), \bar{c}(s), \bar{p}(s)\}_{s\in[t,T]}$ is an admissible strategy and Assumption 3.2 is met. The first claim follows if one establishes (2.6). Taking into account the special form of v, U_γ, and $\bar{X}(s) + b(s)$, (see (5.4)) then Burkholder Davis Gundy inequality yields (2.6). Next, to show that Assumption 3.2 holds true boils down to construct a $C^{1,2}$ solution to some PDEs. Again by exploiting the special structure, one can construct solutions (for the PDEs of Assumption 3.2) of the form $l(t)U_\gamma(x + b(t))$, with $l(t)$ being the solution of some ODE.

5.1 The Case of Logarithmic Utility

In this special case we can solve the integral equation (5.6) in closed form. Indeed with $\gamma = 0$ (the case of logarithmic utility) we follow the ansatz

$$v(t,x) = a(t)U_\gamma(x + b(t)) + d(t). \tag{5.10}$$

Then (5.6) becomes

$$a(t) = \int_t^T [Q(s,t) + q(s,t)]\, ds + nQ(T,t), \tag{5.11}$$

with $b(t)$ given in (5.5) and an appropriate choice of function $d(t)$. The equilibrium policy is then given through (3.5) which becomes

$$F_1(t,x) = \frac{\mu(x+b(t))}{\sigma^2},\ F_2(t,x) = [a(t)]^{-1}(x+b(t)), \tag{5.12}$$

$$F_3(t,x) = \frac{1}{l(t)}\left[([\frac{a(t)}{m}]^{-1} - \eta(t))x + [\frac{a(t)}{m}]^{-1}b(t))\right]. \tag{5.13}$$

Remark 5.5 *Let us notice that the amount invested in the stock is the same as in the case of the standard Merton problem with exponential discounting. This somehow surprising result is explained by constant return and volatility for the stock. We conjecture that in a stochastic volatility model these amounts will be different. The consumption and insurance policies differ from the optimal ones except for the case of exponential discounting. In fact, this is the topic of the next subsection.*

5.2 The Classical Merton Problem

The case of exponential discounting, $h(t) = \hat{h}(t) = e^{-\rho t}$ and constant Pareto weight $m(t) = m$, deserves special consideration. In that case, the equation (4.1) becomes the classical HJB equation given by dynamic programming

$$-(\lambda(t)+\rho)v(t,x) + \frac{\partial v}{\partial t}(t,x) + \Big(rx + \mu F_1(t,x) - F_2(t,x) - F_3(t,x) + i(t)\Big)\frac{\partial v}{\partial x}(t,x) +$$
$$\frac{\sigma^2 F_1^2(t,x)}{2}\frac{\partial^2 v}{\partial x^2}(t,x) + U_\gamma(F_2(t,x)) + mU_\gamma(\eta(t)x + l(t)F_3(t,x)) = 0, \tag{5.14}$$

with the boundary condition $v(T,x) = nU_\gamma(x)$, and (F_1, F_2, F_3) given through (3.5). Therefore for the case of exponential discounting the optimal strategy given by dynamic programming coincides with the policy (given through (3.1), (3.4) (3.5)). This non-linear equation can be linearized by Fenchel-Legendre transform and therefore it can be shown that it has a unique solution. Moreover, it can be computed by the ansatz (5.1). The function $a(t)$ solves an ODE which can be solved explicitly to yield

$$a(t) = \left[n^{\frac{1}{1-\gamma}} e^{\int_t^T \frac{K + \frac{\gamma\eta(s)}{l(s)} - \rho - \lambda(s)}{1-\gamma} ds} + \int_t^T \left(\frac{1+\lambda(u) - \gamma M(u)}{1-\gamma}\right) e^{\int_t^u \frac{K + \frac{\gamma\eta(s)}{l(s)} - \rho - \lambda(s)}{1-\gamma} ds} du\right]^{1-\gamma},$$

with K given by (5.7) and $M(z)$ given by (5.8).

5.3 The Merton Problem with Hyperbolic Discounting

In this section we assume that the decision-maker gets no income ($i(t) = 0$), he/she does not buy life insurance and there is no possibility of him/her dying before T. Moreover, we assume that discounting is hyperbolic, i.e., $h(t) = (1 + k_1 t)^{-\frac{k_2}{k_1}}$, with k_1, k_2 positive. In [21] it is shown that CRRA type utilities and hyperbolic or exponential discounting exhibit *the common difference effect*. Due to this effect, people are more sensitive to a given time delay if it occurs earlier than later. More precisely, if a person is indifferent between receiving $x > 0$ immediately, and $y > x$ at some later time s, then he or she will strictly prefer the better outcome if both outcomes are postponed by some time t :

$$U(x) = h(s)U(y), \qquad \text{implies that} \qquad U(x)h(t) < h(t+s)U(y).$$

Furthermore, they assume that the delay needed to compensate for the larger outcome is a linear function of time, that is

$$U(x) = h(s)U(y), \qquad \text{implies that} \qquad U(x)h(t) = h(kt+s)U(y),$$

for some constant k. They show that the only solution of this functional equation is $U(x) = \frac{x^\gamma}{\gamma}$. We pay special attention to this case because it explains the consumption puzzle; there is a satiation in the consumption rate before maturity and exponential discounting can not capture it (in fact with exponential discounting the optimal consumption rate is either increasing or decreasing at all times depending on the relationship between the discount rate and the interest rate). Moreover, it says that optimal strategies and policies are not observationally equivalent. In the following we illustrate this point by a numerical experiment.

We consider one stock following a geometric Brownian motion with drift $\alpha = 0.12$, volatility $\sigma = 0.2$, interest rate $r = 0.05$, and the horizon $T = 4$. This set of parameters is chosen for illustration. Inspired by [20] let the discount function $h(x) = (1 + k_1 x)^{-\frac{k_2}{k_1}}$ be one of the three choices of hyperbolic discount: case 1. $k_1 = 5$; case 2. $k_2 = 10$; case 3. $k_3 = 15$; and b is chosen such that $h(1) = 0.3$. We set $\gamma = -1$ (this choice reflects risk aversion). Let us consider three cases: $n = 1, 10, 30$. We apply the numerical scheme developed in the Numerical Results Section.

As we see from these graphs, the consumption rate policy is increasing up to a satiation point after which it is decreasing. This phenomena is observed from the data (people are consuming more and more up to some age (around 60 years) after which the consumption is decreasing). This may be explained by a drop in income. As the parameter n gets higher (when the agents get more utility from terminal wealth) the satiation point comes earlier.

Lemma 5.6 shows that consumption rate policy is not always monotone.

Lemma 5.6 *One can find a hyperbolic discount function such that the consumption rate policy is neither increasing nor decreasing in time.*

Appendix D proves this Lemma.

5.4 The Stationary Case

Let us now consider the stationary problem. The coefficients in the model are assumed constant taking their stationary value, i.e., $n = 0$, $m(t) = m$, $l(t) = l$, $\lambda(t) = \lambda$, $i(t) = i, \eta(t) = \eta$, and $T = \infty$. For simplicity we assume that

$$q(s,t) = m\lambda \exp\{-(\lambda + r_1)(s-t)\}, \qquad Q(s,t) = \exp\{-(\lambda + r_2)(s-t)\},$$

for some r_1 and r_2 positive. Before engaging into the formal definition, let us point the following key fact. For an admissible time homogeneous (stationary) policy process $\{\zeta(t), c(t), p(t)\}_{t\in[0,\infty)}$ and its corresponding wealth process $\{X(t)\}_{t\in[0,\infty)}$ (see (2.1)) the expected utility functional $J(t,x,\zeta,c)$ is time homogeneous, i.e.,

$$\begin{aligned} J(t,x,\zeta,c,p) &= J(0,x,\zeta,c,p) \\ &\triangleq \mathbb{E}\left[\int_0^\infty \exp\{-(\lambda + r_2)s\} U(c(s))\, ds + \int_0^\infty m\lambda \exp\{-(\lambda + r_1)s\} U(Z^{0,x}(s))\, ds\right]. \end{aligned}$$

The intuition is that the clock can be reset so that the expected utility criterion takes zero as the starting point. We have a similar definition for policies as in the case of finite horizon.

Definition 5.7 *An admissible trading strategy $\{\bar{\zeta}(s), \bar{c}(s), \bar{p}(s)\}_{s\in[0,\infty]}$ is a policy if there exists a map $F = (F_1, F_2, F_3) : \mathbb{R} \to \mathbb{R} \times [0,\infty) \times \mathbb{R}$ such that for any $x > 0$*

$$\liminf_{\epsilon \downarrow 0} \frac{J(0,x,\bar{\zeta},\bar{c},\bar{p}) - J(0,x,\zeta_\epsilon,c_\epsilon,p_\epsilon)}{\epsilon} \geq 0, \tag{5.15}$$

where

$$\bar{\zeta}(s) = F_1(\bar{X}(s)), \quad \bar{c}(s) = F_2(\bar{X}(s)), \quad \bar{p}(s) = F_3(\bar{X}(s)) \tag{5.16}$$

and the wealth process $\{\bar{X}(s)\}_{s\in[0,\infty]}$ is a solution of the stochastic differential equation (SDE)

$$d\bar{X}(s) = [r\bar{X}(s) + \mu F_1(\bar{X}(s)) - F_2(\bar{X}(s)) - F_3(\bar{X}(s)) + i(s)]ds + \sigma F_1(\bar{X}(s))dW(s). \tag{5.17}$$

Moreover, the process $\{\zeta_\epsilon(s), c_\epsilon(s), p_\epsilon(s)\}_{s\in[0,\infty]}$ is another investment-consumption strategy defined by

$$\zeta_\epsilon(s) = \begin{cases} \bar{\zeta}(s), & s \in [0,\infty]\backslash E_\epsilon \\ \zeta(s), & s \in E_\epsilon, \end{cases} \tag{5.18}$$

$$c_\epsilon(s) = \begin{cases} \bar{c}(s), & s \in [0,\infty]\backslash E_\epsilon \\ c(s), & s \in E_\epsilon, \end{cases} \tag{5.19}$$

$$p_\epsilon(s) = \begin{cases} \bar{p}(s), & s \in [0,\infty]\backslash E_\epsilon \\ p(s), & s \in E_\epsilon, \end{cases} \tag{5.20}$$

with $E_\epsilon = [0,\epsilon]$, and $\{\zeta(s), c(s), p(s)\}_{s\in E_\epsilon}$ is any strategy for which $\{\zeta_\epsilon(s), c_\epsilon(s), p_\epsilon(s)\}_{s\in[0,\infty]}$ is an admissible policy.

Similarly we define the value function.

Definition 5.8 *A function $v : \mathbb{R} \to \mathbb{R}$ is a value function if it solves the following system of equations*

$$v(x) = J(0, x, \bar{\zeta}, \bar{c}, \bar{p}) \tag{5.21}$$

$$\bar{\zeta}(s) = F_1(\bar{X}(s)), \quad \bar{c}(s) = F_2(\bar{X}(s)), \quad \bar{p}(s) = F_3(\bar{X}(s)),$$

$$d\bar{X}(s) = [r\bar{X}(s) + \mu F_1(\bar{X}(s)) - F_2(\bar{X}(s)) - F_3(\bar{X}(s)) + i(s)]ds + \sigma F_1(\bar{X}(s))dW(s), \tag{5.22}$$

$$F_1(x) = -\frac{\mu v'(x)}{\sigma^2 v''(x)}, \; F_2(x) = \left(v'(x)\right)^{\frac{1}{\gamma-1}}, \; F_3(x) = \frac{1}{l}\left[\left(\frac{1}{m}v'(x)\right)^{\frac{1}{\gamma-1}} - \eta x\right]. \tag{5.23}$$

Let us look for the value function of the form

$$v(x) = aU_\gamma(x+b), \tag{5.24}$$

for some constants a and b. By solving (3.4), we get $b = \frac{i}{r+\frac{\eta}{l}}$. Let $\beta = 1 + \frac{m^{\frac{1}{1-\gamma}}}{l}$ and K be given by (5.7). The constant a should solve the following equation

$$a = \frac{a^{\frac{\gamma}{\gamma-1}}}{\lambda + r_1 - K - \frac{\gamma\eta}{l} + \gamma\beta a^{\frac{1}{\gamma-1}}} + m\lambda\frac{(\frac{a}{m})^{\frac{\gamma}{\gamma-1}}}{\lambda + r_2 - K - \frac{\gamma\eta}{l} + \gamma\beta a^{\frac{1}{\gamma-1}}} \tag{5.25}$$

with the transversality conditions

$$\lambda + r_j - K - \frac{\gamma\eta}{l} + \gamma\beta a^{\frac{1}{\gamma-1}} > 0 \qquad j = 1, 2. \tag{5.26}$$

Lemma 5.9 *There is a unique solution of* (5.25) *and* (5.26).

Appendix E proves this Lemma.

We are ready to state the main result of this section.

Theorem 5.10 *Let v be defined by* (5.24) *with a the solution of* (5.25). *The function* (F_1, F_2, F_3) *of* (5.23) *defines a policy through* (5.17) *and* (5.16).

Proof: The proof for the most part parallels Theorem 4.1. The only part which requires more analysis is showing that

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left[\int_{\epsilon}^{\infty}\exp\{-(\lambda+r_i)s\}[U_\gamma(F_2(\bar{X}^{0,x}(s)))-U_\gamma(F_2(X^{0,x}(s)))]\,ds\right]=0,\quad i=1,2.$$

which is equivalent to

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left[\int_{\epsilon}^{\infty}\exp\{-(\lambda+r_i)s\}[(\bar{X}^{0,x}(s)+b)^\gamma-(X^{0,x}(s)+b)^\gamma]\,ds\right]=0,,\quad i=1,2.$$

The result follows from Lebesque Dominated Convergence Theorem if we prove that

$$\mathbb{E}\left[\int_{0}^{\infty}\exp\{-(\lambda+r_i)s\}[(\bar{X}^{0,x}(s)+b)^\gamma+(X^{0,x}(s)+b)^\gamma]\,ds\right]<\infty,\quad i=1,2. \tag{5.27}$$

Notice that from the transversality conditions (5.26) one gets

$$\mathbb{E}\left[\int_{0}^{\infty}\exp\{-(\lambda+r_i)s\}(\bar{X}^{0,x}(s)+b)^\gamma\right]<\infty,\quad i=1,2.$$

Moreover, if $s\in[\epsilon,\infty]$ then

$$\left(\frac{X^{0,x}(s)+b}{\bar{X}^{0,x}(s)+b}\right)^\gamma=\left(\frac{X^{0,x}(\epsilon)+b}{\bar{X}^{0,x}(\epsilon)+b}\right)^\gamma\triangleq R(\epsilon),$$

and $R(\epsilon)$ and $\bar{X}^{0,x}(s)$ are independent. Thus, Holder inequality and a standard argument yields

$$\mathbb{E}\left[\int_{0}^{\infty}\exp\{-(\lambda+r_i)s\}(X^{0,x}(s)+b)^\gamma\right]<\infty,\quad i=1,2,$$

so (5.27) holds true.

□

6 Numerical Results

We provide a numerical scheme to approximate the integral equation (5.6). For simplicity we assume that $\eta(t)=1$. In a first step let us discretize the interval $[0,T]$ by introducing the points $t_n=T+nh$, where $\epsilon=-\frac{T}{N}$; recall that with K given by (5.7) and $M(\cdot)$ of (5.8), the equation (5.6) becomes in a differential form

$$
\begin{aligned}
a'(t) &= (\gamma M(t) - \lambda(t) - 1)(a(t))^{\frac{\gamma}{\gamma-1}} + \left(\lambda(t) - \frac{h'(T-t)}{h(T-t)} - K - \frac{\gamma}{l(t)}\right) a(t) \qquad (6.1)\\
&+ \int_t^T L(s,t)(a(s))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(s)}{A(t)}\right) ds,
\end{aligned}
$$

where

$$A(s) \triangleq \exp(\int_s^T \gamma(a(z))^{\frac{1}{\gamma-1}} M(z)dz)$$

and

$$L(s,t) \triangleq \left[\left(\frac{h'(T-t)}{h(T-t)} - \frac{h'(s-t}{h(s-t)}\right) Q(s,t) + \left(\frac{h'(T-t)}{h(T-t)} - \frac{\bar{h}'(s-t)}{\bar{h}(s-t)}\right) q(s,t)\right] e^{\int_t^s K + \frac{\gamma}{l(u)} du}.$$

From the definition of $A(s)$, it follows that

$$A'(s) = -\gamma a(s))^{\frac{1}{\gamma-1}} M(s)A(s).$$

Our approximation scheme is done in three steps. In a first step, we construct the sequence a_n^1 and A_n^1 recursively by

$$a_{n+1}^1 \triangleq a_n^1 + \epsilon a'(t_n), \qquad A_{n+1}^1 \triangleq A_n^1 + \epsilon A'(t_n).$$

Lemma 6.1 *If a_n^1 and A_n^1, $n = 0 \cdots N$ are defined by $a_0^1 = 1$, $A_0^1 = 1$ and*

$$
\begin{cases}
a_{n+1}^1 &= a_n^1 + \epsilon\big((\gamma M(t_n) - \lambda(t_n) - 1)(a_n^1)^{\frac{\gamma}{\gamma-1}} + \left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K - \frac{\gamma}{l(t_n)}\right) a_n^1 \\
&+ \int_{t_n}^T L(s,t_n)(a(s))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(s)}{A(t_n)}\right) ds \\
A_{n+1}^1 &= A_n^1 - \gamma\epsilon(a(t_n))^{\frac{1}{\gamma-1}} M(t_n)A_n^1
\end{cases}
$$

Then there exists a constant C such that

$$|a_n^1 - a(t_n)| \le C|\epsilon| \text{ and } |A_n^1 - A(t_n)| \le C|\epsilon|, \quad \forall n \in 0,1,\cdots,N.$$

Appendix F proves this Lemma.

In a second step we discretize the integral $\int_{t_n}^T L(s,t_n)(a(s))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(s)}{A(t_n)}\right) ds$. This will lead to the following Lemma.

Lemma 6.2 *If a_n^2 and A_n^2, $n = 0 \cdots N$ are defined by $a_0^2 = 1, A_0^2 = 1$ and*

$$
\begin{cases}
a^2_{n+1} & = & a^2_n + \epsilon(\gamma M(t_n) - \lambda(t_n) - 1)(a^2_n)^{\frac{\gamma}{\gamma-1}} + \epsilon\left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K - \frac{\gamma}{l(t_n)}\right) a^2_n \\
 & - & \epsilon^2 \sum_{j=0}^{n-1} L(t_j, t_n)(a(t_j))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(t_j)}{A(t_n)}\right) \\
A^2_{n+1} & = & A^2_n - \gamma\epsilon(a^2_n)^{\frac{1}{\gamma-1}} M(t_n) A^2_n
\end{cases}
$$

Then there exists a constant C such that

$$|a^2_n - a^1_n)| \leq C|\epsilon| \text{ and } |A^2_n - A^1_n| \leq C|\epsilon|, \quad \forall n \in 0, 1, \cdots, N.$$

Appendix G proves this Lemma.

In a third step we introduce an explicit scheme.

Lemma 6.3 *If a^3_n and A^3_n, $n = 0 \cdots N$ are defined by $a^3_0 = 1, A^3_0 = 1$ and*

$$
\begin{cases}
a^3_{n+1} & = & a^3_n + \epsilon(\gamma M(t_n) - \lambda(t_n) - 1)(a^3_n)^{\frac{\gamma}{\gamma-1}} + \epsilon\left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K - \frac{\gamma}{l(t_n)}\right) a^3_n \\
 & - & \epsilon^2 \sum_{j=0}^{n-1} L(t_j, t_n)(a^3_j)^{\frac{\gamma}{\gamma-1}} \left(\frac{A^3_j}{A^3_n}\right) \\
A^3_{n+1} & = & A^3_n - \gamma\epsilon(a^3_n)^{\frac{1}{\gamma-1}} M(t_n) A^3_n
\end{cases}
$$

Then there exists a constant C such that

$$|a^3_n - a^2_n)| \leq C|\epsilon| \text{ and } |A^3_n - A^2_n| \leq C|\epsilon|, \quad \forall n \in 0, 1, \cdots, N.$$

Appendix H proves this Lemma.

By using the preceding lemmas and Lipschitz continuity of function $a(t)$ we summarize the results of this section by the following Theorem.

Theorem 6.4 *Let $a_N(t)$ be the function obtained by the linear interpolation of the points $(t_n = T - \frac{nT}{N}, a^3_n)$. Then*

$$|a_N(t) - a(t)| \leq C|\epsilon|, \qquad \forall t \in [0, T],$$

for some positive constant C independent of N.

In the following we perform a numerical experiment. Let $T = 4, r = 0.05, \mu = 0.07, \sigma = 0.2, p = -1, N = 1000, \rho = 0.8, \lambda(t) = \frac{1}{200} + \frac{9}{8000}t, l(t) = \frac{1}{\lambda(t)}, \eta(t) = 1$, The discount function is exponential $h(t) = \hat{h}(t) = \exp(-\rho t)$ with $\rho = 0.8$. The Pareto weight is $m(t) = \log(\frac{T+\epsilon-t}{\epsilon})$ with $\epsilon = 10^{-15}$. We choose this set of parameters for illustration. As people get older, perhaps they weigh more their heirs utility; so a time decreasing aggregation rate seems the natural choice. It is for this reason that we consider a decreasing function m. We plot the maps F_2 and F_3 which lead to the policies. Furthermore, we plot the difference in F_3 when m is variable as opposed to m is constant. The results show that higher utility weight m leads to higher amount spent on life insurance.

7 Conclusion and future research

We have studied the portfolio management problem in which an agent invests in a risky asset, consumes, and buys life insurance in order to maximize utility of his/her and heirs. Different discount rates for the agent and heirs lead to time inconsistency. Moreover a time changing aggregation weight will lead to time inconsistency as well. The way we deal with this predicament is by looking for subgame perfect Nash equilibrium strategies that we call policies. We find them in special cases. Our model is rich enough to capture different aspects in portfolio theory. We perform numerical experiments in order to explain the effect of discounting in one hand and the effect of aggregation on a different hand. Hyperbolic discounting is emphasized in a Merton type problem (for simplicity we shut off some parameters). The surprising result is that the policies and optimal strategies are not always observationally equivalent. For example, in certain cases, watching one's consumption rate we can infer (from its time monotonicity) wether or not is a optimal or a equilibrium one. Indeed, a non monotone consumption rate can not be optimal for the case of exponential discounting. The consumption rate policy exhibit a satiation point (it has a hump shaped behaviour) and this may explain the consumption puzzle. A time varying aggregation rate is benchmarked to a constant one. Our numerical experiment comes to support the intuition that the more the manager cares for his/her heirs, the more he/she will pay on life insurance.

We have introduced a system of equations: the integral equation (3.1) together with the SDE (3.4) and PDE (3.5). Their validity has been established in the case when the utility function and the bequest function are of (CRRA) type, but we think that it extends to any concave utilities, as in the deterministic case. This paper can be seen as a first step in the general direction of extending stochastic control away from the optimization paradigm towards time-consistent strategies. The mathematical difficulties are considerable: we have no general existence nor uniqueness theory for the equations (3.1) or (4.1), which replace the classical HJB equation of optimal control. In the present paper we sidestep the difficulty by using an Ansatz, but we hope that future work, by ourselves or others, will solve these problems.

We conclude by pointing out that most portfolios are not managed by an individual, but by a group, either a professional management team, or the investor himself and his family. As we mentioned in the introduction, we should then introduce one utility function, one psychological discount rate, and one Pareto weight for each member of the group. Consider for instance a group with two members (husband and wife). Member i (for $i = 1, 2$) has utility $u_i(c_1, c_2)$, where c_1 is the consumption of the husband and c_2 is the consumption of the wife, and discount factor $h_i(t)$, so the utility derived at time t by member i from the couple consuming (c_1, c_2) at time $s > t$ is $h_i(s-t)\, u_i(c_1, c_2)$. The utilities of the husband and the wife have to be aggregated by Pareto weights. If for instance we assume that, as is the case in most couples, one member specializes in long-term decisions and the other in short-term ones, we find that there should exist some decreasing function $\mu(t)$, with $0 \leq \mu(t) \leq 1$, such that the behaviour of the couple between t and T is adequately described by maximising the intertemporal criterion:

$$\int_t^T \left[\mu(s-t)\, h_1(s-t)\, u_1(c_1(s), c_2(s)) + (1-\mu(s-t))\, h_2(s-t)\, u_2(c_1(s), c_2(s))\right] ds$$

plus some terminal criterion (legacy) at time T. Even in the case when both members of the group have a constant psychological discount rate, so that $h_i(t) = \exp(-r_i t)$, and even if $r_1 = r_2$, the group will exhibit time-inconsistency.

Our model covers the particular case when $u_1 = u_2 = U$. This group is time-inconsistent: their expected intertemporal utility is:

$$\begin{aligned} J(t,x,\zeta,c_1,c_2) \;&=\; \mathbb{E}\bigg[\int_t^T h_1(s-t)U(c_1(s))\,ds + \int_t^T m(s-t)\; h_2(s-t)U(c_2(s))\,ds \\ &\quad +\;\; h_1(T-t)U(X^{\zeta,c_1,c_2}(T))\bigg], \end{aligned}$$

which falls within our model. Assuming that the function $m(\cdot)$ is decreasing with $m(0) \simeq \infty$, and $m(T) \simeq 0$ would capture the situation when one member plans for short time and the other plans for long time.

The more general case when $u_1 \neq u_2$ together with other macroeconomic problems with heterogeneous agents will be the subject of future research.

8 Appendix

Appendix A: Proof of Lemma 2.3: We first establish that

$$\mathbb{E}\bigg[\int_t^{T\wedge\tau} h(s-t)U_1(c(s))\,ds\bigg] = \mathbb{E}\bigg[\int_t^T Q(s,t)U_1(c(s))\,ds\bigg] \tag{8.1}$$

In the light of equations (2.2), (2.3) and the random variable τ being independent of Brownian motion W it follows that

$$\begin{aligned} \mathbb{E}\bigg[\int_t^{T\wedge\tau} h(s-t)U_1(c(s))\,ds\bigg] &= \mathbb{E}\bigg[\exp\{-\int_t^T \lambda(z)\,dz\}\int_t^T h(s-t)U_1(c(s))\,ds + \\ &\int_t^T \lambda(u)\exp\{-\int_t^u \lambda(z)\,dz\}\int_t^u h(s-t)U_1(c(s))\,dsdu\bigg]. \end{aligned}$$

Moreover

$$\begin{aligned} &\mathbb{E}\bigg[\int_t^T \lambda(u)\exp\{-\int_t^u \lambda(z)\,dz\}\int_t^u h(s-t)U_1(c(s))\,dsdu\bigg] = \\ &-\mathbb{E}\bigg[\frac{\partial}{\partial u}\left(\exp\{-\int_t^u \lambda(z)\,dz\}\right)\int_t^u h(s-t)U_1(c(s))\,dsdu\bigg], \end{aligned}$$

and integration by parts will lead to (8.1). It is easy to see that

$$\mathbb{E}\Big[\bar{h}(\tau-t)U_3(Z^{t,x}(\tau))1_{\{\tau\le T|\tau>t\}} = \mathbb{E}\bigg[\int_t^T q(s,t)U_3(Z^{t,x}(s))\,ds\bigg]. \tag{8.2}$$

Finally let us prove that

$$\mathbb{E}\Bigg[nh(\tau - t)U_2(X^{t,x}(T))1_{\{\tau>T|\tau>t\}}\Bigg] = \mathbb{E}\Bigg[nQ(T,t)U_2(X^{t,x}(T))\Bigg]. \tag{8.3}$$

This follows from (2.3). In the light of (8.1) (8.2) and (8.3), it follows that (2.8) holds true.

□

Appendix B: Proof of Lemma 4.2: Let the functions $(t,x) \to f^i(t,\cdot,x)$ satisfy the following PDEs:

$$\frac{\partial f^i}{\partial t} + (rx + \mu F_1(t,x) - F_2(t,x) - F_3(t,x) + i(t))\frac{\partial f^i}{\partial x} + \frac{\sigma^2 F_1^2(t,x)}{2}\frac{\partial^2 f^i}{\partial x^2} = 0, \quad i = 1,2,3. \tag{8.4}$$

with the boundary conditions

$$f^1(t,s,x) = U_1(F_2(s,x)), \quad f^2(t,s,x) = U_3(\eta(s)x + l(s)F_2(s,x)), \quad f^3(t,T,x) = U_2(x).$$

In the light of Assumption 3.2, these PDEs have $C^{1,2}$ solutions. According to the Feyman-Kac's formula

$$f^1(t,s,x) = \mathbb{E}[U_1(F_2(s,\bar{X}^{t,x}(s)))],\ f^2(t,s,x) = \mathbb{E}[U_3(\bar{Z}^{t,x}(s))],\ f^3(t,T,x) = \mathbb{E}[U_2(\bar{X}^{t,x}(T))].$$

Therefore

$$v(t,x) = \int_t^T (Q(s,t)f^1(t,s,x) + q(s,t)f^2(t,s,x))\,ds + nQ(T,t)f^3(t,T,x) \tag{8.5}$$

By differentiating under the integral sign in (8.5) we obtain

$$\begin{aligned}
\frac{\partial v}{\partial t}(t,x) &= \int_t^T (Q(s,t)\frac{\partial f^1}{\partial t}(t,s,x) + q(s,t)\frac{\partial f^2}{\partial t}(t,s,x))\,ds + nQ(T,t)\frac{\partial f^3}{\partial t}(t,T,x) \qquad (8.6)\\
&+ (Q(t,t)f^1(t,t,x) + q(t,t)f^2(t,t,x))\\
&+ \int_t^T (\frac{\partial Q}{\partial t}(s,t)f^1(t,s,x) + \frac{\partial q}{\partial t}(s,t)f^2(t,s,x))\,ds + n\frac{\partial Q}{\partial t}(T,t)f^3(t,T,x),
\end{aligned}$$

$$\frac{\partial v}{\partial x}(t,x) = \int_t^T (Q(s,t)\frac{\partial f^1}{\partial x}(t,s,x) + q(s,t)\frac{\partial f^2}{\partial x}(t,s,x))\,ds + nQ(T,t)\frac{\partial f^3}{\partial x}(t,T,x), \tag{8.7}$$

and

$$\frac{\partial^2 v}{\partial x^2}(t,x) = \int_t^T (Q(s,t)\frac{\partial^2 f^1}{\partial x^2}(t,s,x) + q(s,t)\frac{\partial^2 f^2}{\partial x^2}(t,s,x))\,ds + nQ(T,t)\frac{\partial^2 f^3}{\partial x^2}(t,T,x) \tag{8.8}$$

By combining (8.6), (8.7), (8.8) and (8.4), we obtain (4.1).

$\square$

Appendix C: Proof of Proposition 5.3. We proceed in a couple of steps. In a first step, we obtain lower and upper bounds for $a(t)$. The second step shows that for the case of discount functions being a linear combination of exponentials the equation becomes an ODE system for which we have local existence. The local solution together with the bounds obtained in the first step will lead to global existence. In a last step, we approximate the discount functions by linear combination of exponentials and the solution is constructed by the limit of the ODE systems solutions. In the following, we will go over step one only since the second step follow as in [10] and the last step follows from a density argument. For simplicity we assume that $\eta(t) = 1$. Now, let us define

$$\bar{a}(t) = a(t)e^{\left[K(t-T)+\int_t^T\left(-\frac{\gamma}{l(z)}+\gamma a(z)^{\frac{1}{\gamma-1}}M(z)\right)dz\right]}.$$

It follows that

$$\bar{a}(t) \quad = \quad \int_t^T \left[Q(s,t)+q(s,t)\right]\left[a(s)\right]^{\frac{\gamma}{\gamma-1}}e^{\left[K(s-T)-\int_s^T\left(\frac{\gamma}{l(z)}+\gamma(a(z))^{\frac{1}{\gamma-1}}M(z)dz\right)\right]}ds + nQ(T,t)$$

Consequently

$$\bar{a}'(t) = -(Q(t,t)+q(t,t))(a(t))^{\frac{\gamma}{\gamma-1}}e^{\left[K(t-T)-\int_t^T\left(\frac{\gamma}{l(z)}+\gamma(a(z))^{\frac{1}{\gamma-1}}M(z)\right)dz\right]}ds + n\frac{\partial}{\partial t}Q(T,t)$$
$$+\int_t^T \frac{\partial}{\partial t}\left[Q(s,t)+q(s,t)\right](a(s))^{\frac{\gamma}{\gamma-1}}e^{\left[K(s-T)+\int_s^T\left(\frac{\gamma}{l(z)}+\gamma(a(z))^{\frac{1}{\gamma-1}}M(z)\right)dz\right]}ds$$

From direct calculations

$$\begin{aligned}\frac{\partial}{\partial t}Q(T,t) &= (\lambda(t)h(T-t)-h'(T-t))exp\left[-\int_t^T\lambda(z)dz\right]\\ &= (\lambda(t)-\frac{h'}{h}(T-t))Q(T,t)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left[Q(s,t)+q(s,t)\right] &= (\lambda(t)h(s-t)-h'(s-t))exp\left[-\int_t^s\lambda(z)dz\right]\\ &+ \lambda(s)(\lambda(t)\bar{h}(s-t)-\bar{h}'(s-t))exp\left[-\int_t^s\lambda(z)dz\right].\end{aligned}$$

Therefore

$$\frac{\partial}{\partial t}\left[Q(s,t)+q(s,t)\right] = \left(\lambda(t)-\frac{h'(s-t)}{h(s-t)}(s-t)\right)Q(s,t)+\left(\lambda(t)-\frac{\bar{h}'(s-t)}{\bar{h}(s-t)}\right)q(s,t)$$

Thus

$$\begin{aligned}
\bar{a}'(t) &= -(\lambda(t)+1)(a(t))^{\frac{\gamma}{\gamma-1}} e^{\left[K(t-T)-\int_t^T\left(\frac{\gamma}{l(z)}+\gamma[a(z)]^{\frac{1}{\gamma-1}}M(z)\right)dz\right]} ds \\
&+ n\left(\lambda(t)-\frac{h'(T-t)}{h(T-t)}\right)Q(T,t) \\
&+ \int_t^T \left[\left(\lambda(t)-\frac{h'(s-t)}{h(s-t)}\right)Q(s,t)+\left(\lambda(t)-\frac{\bar{h}'(s-t)}{\bar{h}(s-t)}\right)q(s,t)\right] \\
&\times (a(s))^{\frac{\gamma}{\gamma-1}} e^{\left[K(s-T)+\int_s^T\left(\frac{\gamma}{l(z)}+\gamma(a(z))^{\frac{1}{\gamma-1}}M(z)\right)dz\right]} ds
\end{aligned}$$

Hence

$$\begin{aligned}
\big((a'(t) &+ (K+\frac{\gamma}{l(t)}-\gamma(a(t))^{\frac{1}{\gamma-1}}M(t))a(t)\big)e^{\left[-\int_t^T\left(K+\frac{\gamma}{l(z)}\right)dz+\int_t^T\gamma(a(z))^{\frac{1}{\gamma-1}}M(z)dz\right]} \\
&= -(\lambda(t)+1)(a(t))^{\frac{\gamma}{\gamma-1}} e^{\left[-\int_t^T\left(K+\frac{\gamma}{l(z)}\right)dz+\int_t^T\gamma(a(z))^{\frac{1}{\gamma-1}}M(z)dz\right]} \\
&+ n\left(\lambda(t)-\frac{h'(T-t)}{h(T-t)})\right)Q(T,t) \\
&+ \int_t^T \left[\left(\lambda(t)-\frac{h'(s-t)}{h(s-t)}\right)Q(s,t)+\left(\lambda(t)-\frac{\bar{h}'(s-t)}{\bar{h}(s-t)}\right)q(s,t)\right] \\
&\times (a(s))^{\frac{\gamma}{\gamma-1}} e^{\left[-\int_s^T\left(K+\frac{\gamma}{l(z)}\right)dz+\int_s^T\gamma(a(z))^{\frac{1}{\gamma-1}}M(z)dz\right]} ds
\end{aligned}$$

Consequently

$$\left[(a'(t)+(K+\frac{\gamma}{l(t)}-\gamma(a(t))^{\frac{1}{\gamma-1}}M(t))a(t)\right] =$$

$$-(\lambda(t)+1)(a(t))^{\frac{\gamma}{\gamma-1}}+n\left(\lambda(t)-\frac{h'(T-t)}{h(T-t)}\right)Q(T,t)e^{\left[\int_t^T K+\frac{\gamma}{l(z)}dz-\int_t^T\gamma(a(z))^{\frac{1}{\gamma-1}}M(z)dz\right]}ds+$$

$$\int_t^T\left[\left(\lambda(t)-\frac{h'(s-t)}{h(s-t)}\right)Q(s,t)+\left(\lambda(t)-\frac{\bar{h}'(s-t)}{\bar{h}(s-t)}\right)q(s,t)\right](a(s))^{\frac{\gamma}{\gamma-1}}e^{\left[\int_t^s\left(K+\frac{\gamma}{l(z)}\right)dz+\int_s^t\gamma(a(z))^{\frac{1}{\gamma-1}}M(z)dz\right]}ds$$

From the definition of $a(t)$ we get that

$$\left[(a'(t)+(K+\frac{\gamma}{l(t)}-\gamma(a(t))^{\frac{1}{\gamma-1}}M(t))a(t)\right] = -(\lambda(t)+1)(a(t))^{\frac{\gamma}{\gamma-1}}+\lambda(t)a(t)-\frac{h'(T-t)}{h(T-t)}a(t) \tag{8.9}$$

$$+\int_t^T \left(\frac{h'(T-t}{h(T-t})\big(Q(s,t)+q(s,t)\big)-\frac{h'(s-t)}{h(s-t)}(Q(s,t)-\right.$$

$$\left.\frac{\bar{h}'(s-t)}{\bar{h}(s-t)})q(s,t)\right)(a(s))^{\frac{\gamma}{\gamma-1}}e^{\left[K(s-t)+\int_s^t \gamma[a(z)]^{\frac{1}{\gamma-1}}M(z)dz\right]}ds$$

Since $-\rho \leq \frac{h'}{h}(z) \leq \rho$ and $-\rho \leq \frac{\bar{h}'}{h}(z) \leq \rho$ and $-\rho' \leq \frac{\gamma}{l(z)} \leq \rho'$ for $0 \leq z \leq T$ the equation (8.9) leads to

$$a'(t) \leq -(1+\lambda(t)-\gamma M(t))(a(t))^{\frac{\gamma}{\gamma-1}}+(\lambda(t)+3\rho-K+\rho')a(t) \tag{8.10}$$

and

$$a'(t) \geq -(1+\lambda(t))(a(t))^{\frac{\gamma}{\gamma-1}}-(K+\lambda(t)+3\rho-\rho')a(t) \tag{8.11}$$

Let us denote

$$C_1 \triangleq \min_{t\in[0,T]}(1-\gamma M(t)+\lambda(t)), \qquad C_0 \triangleq \max_{t\in[0,T]}(\lambda(t)+3\rho-K+\rho')$$

$$D_0 \triangleq \max_{t\in[0,T]}(K+\lambda(t)+3\rho), \qquad D_1 \triangleq \max_{t\in[0,T]}(1+\lambda(t))$$

so that equations (8.10) and (8.11) will become

$$a'(t) \leq -C_1(a(t))^{\frac{\gamma}{\gamma-1}}+C_0 a(t) \tag{8.12}$$

and

$$a'(t) \geq -D_1(a(t))^{\frac{\gamma}{\gamma-1}}-D_0 a(t) \tag{8.13}$$

Now, $C_0, D_1, D_0 > 0$ and by assumption 5.2, it follows that $C_1 > 0$. Consequently, we get lower and upper bounds on $a(t)$ by integrating (8.12) and (8.13) as in [10].

□

Appendix D: Proof of Lemma 5.6: We take $n = 2$ and the coefficients as in the numerical experiment. In the light of the differential equation

$$\begin{aligned} a'(t) &= -\left[\frac{h'(T-t)}{h(T-t)}+K\right]a(t)+(\gamma-1)(a(t))^{\frac{\gamma}{\gamma-1}} \\ &+ \int_t^T h(T-t)\frac{\partial}{\partial t}\left[\frac{h(s-t)}{h(T-t)}\right](a(s))^{\frac{\gamma}{\gamma-1}}e^{-\left(\int_t^s \gamma(a(u))^{\frac{1}{\gamma-1}}du\right)}ds. \end{aligned} \tag{8.14}$$

We notice that on $[T-\epsilon, T]$

$$a'(t) \approx -\left[\frac{h'(T-t)}{h(T-t)} + K\right] a(t) + (\gamma - 1)(a(t))^{\frac{\gamma}{\gamma-1}} + O(\epsilon).$$

Consequently, for the choice of our parameters in the numerical experiment we get that $\frac{h'(T-t)}{h(T-t)} + K < -1$ for small ϵ (keeping in mind that $a(1) = 2$) we see that $a(t)$ is increasing on $[T-\epsilon, T]$. For hyperbolic discounting it can be shown that $\frac{\partial}{\partial t}\left[\frac{h(s-t)}{h(T-t)}\right] < 0$. It is obvious that $a(t)$ is decreasing in a neighborhood of 0 due to the negative contribution of the term

$$\int_t^T h(T-t)\frac{\partial}{\partial t}\left[\frac{h(s-t)}{h(T-t)}\right](a(s))^{\frac{\gamma}{\gamma-1}} e^{-\left(\int_t^s \gamma(a(u))^{\frac{1}{\gamma-1}}\,du\right)}\,ds.$$

In conclusion, the consumption rate policy, $\frac{F_2(t,x)}{x} = [a(t)]^{\frac{1}{\gamma-1}}$, (see (5.2)) is neither increasing nor decreasing in time.

$\square$

Appendix E: Proof of Lemma 5.9: Let $x \triangleq a^{\frac{1}{1-\gamma}}$, $\alpha_j \triangleq \lambda + r_j - K - \frac{\gamma\eta}{l}$, $j = 1, 2$. Equation (5.25) becomes

$$\frac{1}{x} = \frac{1}{\alpha_1 + \gamma\beta x} + \frac{\lambda m^{\frac{1}{1-\gamma}}}{\alpha_2 + \gamma\beta x} \tag{8.15}$$

For the sake of simplicity, consider the case $m = 1$.

We want to find $x > 0$ which solves

$$\gamma\beta(1 - \gamma\beta + \lambda)x^2 + (\alpha_2(1-\gamma\beta) + \alpha_1(\lambda - \gamma\beta))x - \alpha_1\alpha_2 = 0 \tag{8.16}$$

and transversality conditions (5.26) i.e

$$\alpha_1 + \gamma\beta x > 0 \qquad \alpha_2 + \gamma\beta x > 0 \tag{8.17}$$

This can be done by splitting the analysis into 3 cases: $\gamma \in (0,1), \gamma = 0$, and $\gamma < 0$. We omit the details.

$\square$

Appendix F: Proof of Lemma 6.1: Let $a = inf_{t\in[0,T]}a(t)$. Lemma 5.3 guarantees that $a > 0$. We show that

$$a_n^1 \geq \frac{2A}{3} \quad \forall n \in 0, 1, \cdots, N, \tag{8.18}$$

and this makes the recursive scheme well defined. We prove (8.18) by mathematical induction. Assume that

$$a_k^1 \geq \frac{2A}{3} \quad \forall k \in 0, 1, \cdots, n, \tag{8.19}$$

and prove that $a_{n+1}^1 \geq \frac{2a}{3}$. Let us define

$$e_n \triangleq a_n^1 - a(t_n), \quad f_n \triangleq A_n^1 - A(t_n)$$

.

By considering a second order Taylor expansion of $a(t_{n+1})$ at $a(t_n)$, we get

$$a(t_{n+1}) = a(t_n) + \epsilon a'(t_n) + c_n \epsilon^2$$

with c_n a constant depending on a and bounded by c independently of n. Consequently

$$\begin{aligned}
e_{n+1} &= a_{n+1}^1 - a(t_{n+1}) \\
&= a_n^1 + \epsilon\big((\gamma M(t_n) - \lambda(t_n) - 1)(a_n^1)^{\frac{\gamma}{\gamma-1}} \\
&+ \left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K\right) a_n^1 + \int_{t_n}^T L(s,t_n)(a(s))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(s)}{A(t_n)}\right) ds \\
&- \Bigg(a(t_n) + \epsilon(\gamma M(t_n) - \lambda(t_n) - 1)(a(t_n))^{\frac{\gamma}{\gamma-1}} \\
&+ \left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K\right) a(t_n) \\
&+ \int_{t_n}^T L(s,t_n)(a(s))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(s)}{A(t_n)}\right) ds + c_n\epsilon^2\Bigg) \\
&= e_n + \epsilon(\gamma M(t_n) - \lambda(t_n) - 1)((a_n^1)^{\frac{\gamma}{\gamma-1}} - (a(t_n))^{\frac{\gamma}{\gamma-1}}) \\
&+ \epsilon\left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K\right) e_n - c_n\epsilon^2
\end{aligned}$$

By the mean value Theorem applied to the function $x \to x^{\frac{\gamma}{\gamma-1}}$, one gets that

$$|(a_k^1)^{\frac{\gamma}{\gamma-1}} - (a(t_k))^{\frac{\gamma}{\gamma-1}}| \leq \frac{\gamma}{\gamma-1}\left(\frac{2a}{3}\right)^{\frac{1}{\gamma-1}} |a_k^1 - a(t_k)|.$$

Therefore there exists $M > 0$, such that

$$|e_{k+1}| \leq |e_k|(1 + M|\epsilon|) + c\epsilon^2, \quad \forall k \in 0, 1, \cdots, n. \tag{8.20}$$

By iterating (8.20) for $k = 0 \cdots n$, one gets

$$|e_{n+1}| \leq c\epsilon^2 \frac{(1+M\epsilon)^{n+1} - 1}{M\epsilon} \leq |c|\epsilon^2 \frac{e^{MT} - 1}{\frac{MT}{N}} \leq C|\epsilon|,$$

for some constant C independent of n. Therefore

$$a^1_{n+1} \geq a(t_{n+1}) - |e_{n+1}| \geq a - C|\epsilon| \geq 2a/3$$

for $|\epsilon|$ small enough. This proves (8.18).

Moreover it follows that

$$|a^1_n - a(t_n)| = |e_n| \leq C|\epsilon|, \ \forall n \in 0, 1, \cdots, N.$$

Similar arguments show that

$$|A^1_n - A(t_n)| \leq C|\epsilon|, \quad \forall n \in 0, 1, \cdots, N.$$

□

Appendix G: Proof of Lemma 6.2: We show that

$$a^2_n \geq \frac{a}{2} \quad \forall n \in 0, 1, \cdots, N, \tag{8.21}$$

and this makes the recursive scheme well defined. We prove (8.21) by mathematical induction. Assume that

$$a^2_k \geq \frac{a}{2} \quad \forall k \in 0, 1, \cdots, n, \tag{8.22}$$

and prove that $a^2_{n+1} \geq \frac{A}{2}$. Let $r_n \triangleq a^2_n - a^1_n$, so

$$\begin{aligned}
r_{n+1} &= a^2_{n+1} - a^1_{n+1} \\
&= a^2_n + \epsilon(\gamma M(t_n) - \lambda(t_n) - 1)(a^2_n)^{\frac{\gamma}{\gamma-1}} + \epsilon\left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K\right) a^2_n \\
&- \epsilon^2 \sum_{j=0}^{n-1} L(t_j, t_n)(a(t_j))^{\frac{\gamma}{\gamma-1}} A(t_j) - \epsilon \int_{t_n}^{T} L(s, t_n)(a(s))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(s)}{A(t_n)}\right) ds \\
&- a^1_n - \epsilon(\gamma M(t_n) - \lambda(t_n) - 1)(a^1_n)^{\frac{\gamma}{\gamma-1}} - \epsilon\left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K\right) a^1_n \\
&= r_n + \epsilon(\gamma M(t_n) - \lambda(t_n) - 1)\big((a^2_n)^{\frac{\gamma}{\gamma-1}} - (a^1_n)^{\frac{\gamma}{\gamma-1}}\big) \\
&+ \epsilon\left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K\right) r_n \\
&+ \epsilon \sum_{j=0}^{n-1}\left[-\epsilon L(t_j, t_n)(a(t_j))^{\frac{\gamma}{\gamma-1}}\left(\frac{A(t_j)}{A(t_n)}\right) - \int_{t_{j+1}}^{t_j} L(s, t_n)(a(s))^{\frac{\gamma}{\gamma-1}}\left(\frac{A(s)}{A(t_n)}\right) ds\right]
\end{aligned}$$

Moreover

$$
\begin{aligned}
& \left| -\epsilon L(t_j,t_n)(a(t_j))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(t_j)}{A(t_n)}\right) - \int_{t_{j+1}}^{t_j} L(s,t_n)(a(s))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(s)}{A(t_n)}\right) ds \right| \\
= \; & \left| \int_{t_{j+1}}^{t_j} \left(L(t_j,t_n)(a(t_j))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(t_j)}{A(t_n)}\right) - L(s,t_n)(a(s))^{\frac{\gamma}{\gamma-1}} \left(\frac{A(s)}{A(t_n)}\right)\right) ds \right| \\
\leq \; & \frac{1}{A(t_n)} \Big| \int_{t_{j+1}}^{t_j} \Big([L(t_j,t_n)(a(t_j))^{\frac{\gamma}{\gamma-1}} A(t_j) - L(s,t_n)(a(t_j))^{\frac{\gamma}{\gamma-1}} A(t_j)] + [L(s,t_n)(a(t_j))^{\frac{\gamma}{\gamma-1}} A(t_j) \\
- \; & L(s,t_n)(a(s))^{\frac{\gamma}{\gamma-1}} A(t_j)] + [L(s,t_n)(a(s))^{\frac{\gamma}{\gamma-1}} A(t_j) - L(s,t_n)(a(s))^{\frac{\gamma}{\gamma-1}} A(s)] \Big) ds \Big| \\
\leq \; & \frac{1}{A(t_n)} \big(K_0(t_j-t_{j+1})^2 + K_1(t_j-t_{j+1})^2 + K_2(t_j-t_{j+1})^2 \big) \leq K_4 \epsilon^2,
\end{aligned}
$$

for some positive constants K_0, K_1, K_2, K_4. The last inequalities follow from the boundedness of $a(t)$ (see Lemma 5.3) and from the boundedness of coefficients. Arguing as in Lemma 6.1, we can then find $M > 0$ such that

$$|r_{k+1}| \leq |r_k|(1 + M|\epsilon|) + c\epsilon^2, \quad \forall k \in 0, 1, \cdots, n. \tag{8.23}$$

By iterating (8.23) for $k = 0 \cdots n$, one gets

$$|r_{n+1}| \leq c\epsilon^2 \frac{(1+M\epsilon)^{n+1} - 1}{M\epsilon} \leq |c|\epsilon^2 \frac{e^{MT} - 1}{\frac{MT}{N}} \leq C|\epsilon|,$$

for some constant C independent of n. Therefore

$$a^2_{n+1} \geq a^1_{n+1} - |r_{n+1}| \geq 2a/3 - C|\epsilon| \geq a/2$$

for $|\epsilon|$ small enough. This proves (8.21). Moreover it follows that

$$|a^2_n - a^1_n| = |r_n| \leq C|\epsilon|, \; \forall n \in 0, 1, \cdots, N.$$

Similar arguments show that

$$|A^2_n - A^1_n| \leq C|\epsilon|, \quad \forall n \in 0, 1, \cdots, N.$$

□

Appendix H: Proof of Lemma 6.3: We show that

$$a^3_n \geq \frac{a}{4} \quad \forall n \in 0, 1, \cdots, N, \tag{8.24}$$

and this makes the recursive scheme well defined. We prove (8.24) by mathematical induction. Assume that

$$a^3_k \geq \frac{a}{4} \quad \forall k \in 0, 1, \cdots, n, \tag{8.25}$$

and prove that $a^3_{n+1} \geq \frac{a}{4}$. Let us introduce $u_n \triangleq a^3_n - a^2_n$ and $v_n \triangleq A^3_n - A^2_n$. It follows that

$$\begin{aligned}
u_{n+1} &= a^3_{n+1} - a^2_{n+1} \\
&= a^3_n + \epsilon(\gamma M(t_n) - \lambda(t_n) - 1)(a^3_n)^{\frac{\gamma}{\gamma-1}} + \epsilon\left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K\right) a^3_n \\
&- \epsilon^2 \sum_{j=0}^{n-1} L(t_j,t_n)(a^3_j)^{\frac{\gamma}{\gamma-1}} A^3_j - a^2_n - \epsilon(\gamma m(t_n) - \lambda(t_n) - 1)(a^2_n)^{\frac{\gamma}{\gamma-1}} \\
&- \epsilon\left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K\right) a^2_n + \epsilon^2 \sum_{j=0}^{n-1} L(t_j,t_n)(a(t_j))^{\frac{\gamma}{\gamma-1}} A(t_j) \\
&= u_n + \epsilon(\gamma m(t_n) - \lambda(t_n) - 1)((a^3_n)^{\frac{\gamma}{\gamma-1}} - (a^2_n)^{\frac{\gamma}{\gamma-1}}) \\
&+ \epsilon\left(\lambda(t_n) - \frac{h'(T-t_n)}{h(T-t_n)} - K\right) u^3_n - \epsilon^2 \sum_{j=0}^{n-1} L(t_j,t_n) r_{j,n},
\end{aligned}$$

where $r_{j,n} \triangleq (a^3_j)^{\frac{\gamma}{\gamma-1}}\left(\frac{A^3_j}{A^3_n}\right) - (a(t_j))^{\frac{\gamma}{\gamma-1}}\left(\frac{A(t_j)}{A(t_n)}\right)$. By triangle inequality it follows that

$$\begin{aligned}
|r_{j,n}| &\leq |(a^3_j)^{\frac{\gamma}{\gamma-1}}\left(\frac{A^3_j}{A^3_n}\right) - (a^2_j)^{\frac{\gamma}{\gamma-1}}\left(\frac{A^3_j}{A^3_n}\right)| + |(a^2_j)^{\frac{\gamma}{\gamma-1}}\left(\frac{A^3_j}{A^3_n}\right) - (a^2_j)^{\frac{\gamma}{\gamma-1}}\left(\frac{A^2_j}{A^3_n}\right)| \\
&+ |(a^2_j)^{\frac{\gamma}{\gamma-1}}\left(\frac{A^2_j}{A^3_n}\right) - (a^2_j)^{\frac{\gamma}{\gamma-1}}\left(\frac{A^2_j}{A^2_n}\right)| + |(a^2_j)^{\frac{\gamma}{\gamma-1}}\left(\frac{A^2_j}{A^2_n}\right) - (a(t_j))^{\frac{\gamma}{\gamma-1}}\left(\frac{A(t_j)}{A(t_n)}\right)| \\
&\leq M_1|u_j| + M_2|v_j| + M_3|v_n| + M_4|\epsilon|,
\end{aligned}$$

for some positive constants M_1, M_2, M_3, M_4, where the last inequality from the boundedness of $a(t)$ (see Lemma 5.3) and of other coefficients in our model. Arguing as in the previous Lemmas one can find a positive constant C such that

$$|u_{n+1}| \leq |u_n| + C|\epsilon u_n| + C|\epsilon|(max_{j\in 0,\cdots,n}|u_j| + max_{j\in 0,\cdots,n}|v_j|) + C\epsilon^2. \tag{8.26}$$

On the other hand

$$v_{n+1} = v_n - \gamma\epsilon M(t_n)\big((a^3_n)^{\frac{\gamma}{\gamma-1}} p^3_n - (a^2_n)^{\frac{\gamma}{\gamma-1}} p^2_n\big),$$

hence

$$|v_{n+1}| \leq |v_n| + |\gamma\epsilon M(t_n)|\big((a^3_n)^{\frac{\gamma}{\gamma-1}}|A^3_n - A^2_n| + A^2_n|(a^3_n)^{\frac{\gamma}{\gamma-1}} - (a^2_n)^{\frac{\gamma}{\gamma-1}}|\big).$$

However this implies that

$$|v_{n+1}| \leq |v_n| + M|\epsilon|(|u_n| + |v_n|), \tag{8.27}$$

for some positive constant M. Let us define $x_n = max_{j\in 0,\cdots,n}|u_j|$ and $y_n = max_{j\in 0,\cdots,n}|v_j|$ and $z_n = x_n + y_n$. Inequalities (8.26) and (8.27) hold also for $k \le n$, i.e.,

$$|u_{k+1}| \le |u_k| + C|\epsilon u_k| + C|\epsilon|(x_k + y_k) + C\epsilon^2, \; |v_{k+1}| \le |v_k| + M|\epsilon|(|u_k| + |v_k|).$$

By taking maximum over $k \in 0, \cdots, n$ in these inequalities one obtains

$$x_{n+1} \le x_n + 2C|\epsilon|x_n + C|\epsilon|y_n + C\epsilon^2$$

and

$$y_{n+1} \le y_n + M|\epsilon|(x_n + y_n)$$

By adding these inequalities, it follows that

$$z_{n+1} \le z_n + (2C + M)|\epsilon|z_n + M\epsilon^2.$$

This in turn yields that $z_n \le C|\epsilon|$, for some positive constant still denoted (with some abuse of notations) C. Therefore

$$a^3_{n+1} \ge a^2_{n+1} - |u_{n+1}| \ge a/2 - K|\epsilon| \ge a/4$$

for $|\epsilon|$ small enough. This proves (8.24). Moreover $z_n \le C|\epsilon|$,
implies that

$$|a^3_n - a^2_n| = |u_n| \le C|\epsilon|, \; \forall n \in 0, 1, \cdots, N.$$

and

$$|A^3_n - A^2_n| = |v_n| \le C|\epsilon|, \quad \forall n \in 0, 1, \cdots, N.$$

□

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