# Fixed Points of Averages of Resolvents: Geometry and Algorithms 

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#### Abstract

To provide generalized solutions if a given problem admits no actual solution is an important task in mathematics and the natural sciences. It has a rich history dating back to the early 19th century when Carl Friedrich Gauss developed the method of least squares of a system of linear equations - its solutions can be viewed as fixed points of averaged projections onto hyperplanes. A powerful generalization of this problem is to find fixed points of averaged resolvents (i.e., firmly nonexpansive mappings).

This paper concerns the relationship between the set of fixed points of averaged resolvents and certain fixed point sets of compositions of resolvents. It partially extends recent work for two mappings on a question of C. Byrne. The analysis suggests a reformulation in a product space.

Furthermore, two new algorithms are presented. A complete convergence proof that is based on averaged mappings is provided for the first algorithm. The second algorithm, which currently has no convergence proof, iterates a mapping that is not even nonexpansive. Numerical experiments indicate the potential of these algorithms when compared to iterating the average of the resolvents.


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## 1 Introduction

## Throughout this paper,

and induced norm $\|\cdot\|$. We impose that $X \neq\{0\}$. To motivate the results of this paper, let us assume that $C_{1}, \ldots, C_{m}$ are finitely many nonempty closed convex subsets of $X$, with projections (nearest point mappings) $P_{1}, \ldots, P_{m}$. Many problems in mathematics and the physical sciences can be recast as the convex feasibility problem of finding a point in the intersection $C_{1} \cap \cdots \cap C_{m}$. However, in applications it may well be that this intersection is empty. In this case, a powerful and very useful generalization of the intersection is the set of fixed points of the operator

$$
\begin{equation*}
X \rightarrow X: x \mapsto \frac{P_{1} x+\cdots+P_{m} x}{m} \tag{2}
\end{equation*}
$$

(See, e.g., [14] for applications.) Indeed, these fixed points are precisely the minimizers of the convex function

$$
\begin{equation*}
X \rightarrow \mathbb{R}: x \mapsto \sum_{i=1}^{m}\left\|x-P_{i} x\right\|^{2} \tag{3}
\end{equation*}
$$

and-when each $C_{i}$ is a suitably described hyperplane-there is a well known connection to the set of least squares solutions in the sense of linear algebra (see Appendix A).

A problem open for a long time is to find precise relationships between the fixed points of the operator defined in (2) and the fixed points of the composition $P_{m} \circ \cdots \circ P_{2} \circ P_{1}$ when the intersection $C_{1} \cap \cdots \cap C_{m}$ is empty. (It is well known that both fixed points sets coincide with $C_{1} \cap \cdots \cap C_{m}$ provided this intersection is nonempty.) This problem was recently explicitly stated and nicely discussed in [11, Chapter 50] and [12, Open Question 2 on page 101 in Subsection 8.3.2]. For other related work1], see [2], [4], [13], [16], and the

[^1]references therein. When $m=2$, the recent work [30] contains some precise relationships. For instance, the results in [30, Section 3] show that
\[

$$
\begin{equation*}
\operatorname{Fix}\left(P_{2} \circ P_{1}\right) \rightarrow \operatorname{Fix}\left(\frac{1}{2} P_{1}+\frac{1}{2} P_{2}\right): x \mapsto \frac{1}{2} x+\frac{1}{2} P_{1} x \tag{4}
\end{equation*}
$$

\]

is a well defined bijection.
Our goal in this paper is two-fold. First, we wish to find a suitable extension to describe these fixed point sets when $m \geq 3$. Second, we build on these insights to obtain algorithms for finding these fixed points.

The results provided are somewhat surprising. While we completely generalize some of the two-set work from [30], the generalized intersection is not formulated as the fixed point set of a simple composition, but rather as the fixed point set of a more complicated operator described in a product space. Nonetheless, the geometric insight obtained will turn out to be quite useful in the design of new algorithms that show better convergence properties when compared to straight iteration of the averaged projection operator. Furthermore, the results actually hold for very general firmly nonexpansive operatorsequivalently, resolvents of maximally monotone operators-although the optimizationbased interpretation as a set of minimizers analogous to (3) is then unavailable.

The paper is organized as follows. In the remainder of this introductory section, we describe some central notions fundamental to our analysis. The main result of Section 2 is Theorem 2.1 where we provide a precise correspondence between the fixed point set of an averged resolvent $J_{A}$ and a certain set $\mathbf{S}$ in a product space. In Section 3, it is shown that $\mathbf{S}$ is in fact the fixed point set of an averaged mapping (see Corollary 3.8). This insight is brought to good use in Section 4, where we design a new algorithm for finding a point in $\mathbf{S}$ (and hence in Fix $J_{A}$ ) and where we provide a rigorous convergence proof. Akin to the Gauss-Seidel variant of the Jacobi iteration in numerical linear algebra, we propose another new algorithm. Numerical experiments illustrate that this heuristic algorithm performs very well; however, it still lacks a rigorous proof of convergence. An appendix concludes the paper. The first part of the appendix connects fixed points of averages of projections onto hyperplanes to classical least squares solutions, while the second part contains some more technical observations regarding the heuristic method. The notation we utilize is standard and as in [3], [6], [25], [26], [28], [29], or [31] to which we also refer for background.

Recall that a mapping

$$
\begin{equation*}
T: X \rightarrow X \tag{5}
\end{equation*}
$$

is firmly nonexpansive (see [32] for the first systematic study) if

$$
\begin{equation*}
(\forall x \in X)(\forall y \in X) \quad\|T x-T y\|^{2}+\|(\operatorname{Id}-T) x-(\operatorname{Id}-T) y\|^{2} \leq\|x-y\|^{2} \tag{6}
\end{equation*}
$$

where Id: $X \rightarrow X: x \mapsto x$ denotes the identity operator. The prime example of firmly nonexpansive mappings are projection operators (also known as nearest point mappings) with respect to nonemtpy closed convex subsets of $X$. It is clear that if $T$ is firmly nonexpansive, then it is nonexpansive, i.e., Lipschitz continuous with constant 1 ,

$$
\begin{equation*}
(\forall x \in X)(\forall y \in X) \quad\|T x-T y\| \leq\|x-y\| ; \tag{7}
\end{equation*}
$$

the converse, however, is false (consider - Id). The set of fixed points of $T$ is

$$
\begin{equation*}
\operatorname{Fix} T=\{x \in X \mid x=T x\} \tag{8}
\end{equation*}
$$

The following characterization of firm nonexpansiveness is well known and will be used repeatedly.

Fact 1.1 (See, e.g., [3, 18, 19].) Let $T: X \rightarrow X$. Then the following are equivalent:
(i) $T$ is firmly nonexpansive.
(ii) Id -T is firmly nonexpansive.
(iii) $2 T$ - Id is nonexpansive.
(iv) $(\forall x \in X)(\forall y \in X)\|T x-T y\|^{2} \leq\langle x-y, T x-T y\rangle$.
(v) $(\forall x \in X)(\forall y \in X) 0 \leq\langle T x-T y,(\operatorname{Id}-T) x-(\operatorname{Id}-T) y\rangle$.

Firmly nonexpansive mappings are also intimately tied with maximally monotone operators. Recall that a set-valued operator $A: X \rightrightarrows X$ (i.e., $(\forall x \in X) A x \subseteq X)$ with graph gr $A$ is monotone if

$$
\begin{equation*}
(\forall(x, u) \in \operatorname{gr} A)(\forall(y, v) \in \operatorname{gr} A) \quad\langle x-y, u-v\rangle \geq 0, \tag{9}
\end{equation*}
$$

and that $A$ is maximally monotone if it is monotone and every proper extension of $A$ fails to be monotone. We write $\operatorname{dom} A=\{x \in X \mid A x \neq \varnothing\}$ and $\operatorname{ran} A=A(X)=\bigcup_{x \in X} A x$ for the domain and range of $A$, respectively. The inverse of $A$ is defined via $\mathrm{gr} A^{-1}=$ $\{(u, x) \in X \times X \mid u \in A x\}$. Monotone operators are ubiquitous in modern analysis and optimization; see, e.g., the books [3], [6], [7], [10], [28], [29], [31], [33], [34], and [35]. Two key examples of maximally monotone operators are continuous linear monotone operators and subdifferential operators (in the sense of convex analysis) of functions that are convex, lower semicontinuous, and proper.

Now let $A: X \rightrightarrows X$ be maximally monotone and denote the associated resolvent by

$$
\begin{equation*}
J_{A}=(\operatorname{Id}+A)^{-1} \tag{10}
\end{equation*}
$$

In [23], Minty made the seminal observation that $J_{A}$ is in fact a firmly nonexpansive operator from $X$ to $X$ and that, conversely, every firmly nonexpansive operator arises this way:

Fact 1.2 (Minty) (See, e.g., [23] or [17].) Let $T: X \rightarrow X$ be firmly nonexpansive, and let $A: X \rightrightarrows X$ be maximally monotone. Then the following hold.
(i) $B=T^{-1}$ - Id is maximally monotone (and $J_{B}=T$ ).
(ii) $J_{A}$ is firmly nonexpansive (and $A=J_{A}^{-1}-\mathrm{Id}$ ).

One of the motivations to study the correspondence between firmly nonexpansive mappings and maximally monotone operators is the very useful correspondence

$$
\begin{equation*}
A^{-1}(0)=\operatorname{Fix} J_{A}, \tag{11}
\end{equation*}
$$

where $A: X \rightrightarrows X$ is maximally monotone.
From now on we assume that
(12) $\quad A_{1}, \ldots, A_{m}$ are maximally monotone operators on $X$, where $m \in\{2,3, \ldots\}$,
that

$$
\begin{equation*}
\left.\lambda_{1}, \ldots, \lambda_{m} \text { belong to }\right] 0,1\left[\text { such that } \sum_{i \in I} \lambda_{i}=1, \quad \text { where } I=\{1,2, \ldots, m\}\right. \tag{13}
\end{equation*}
$$

and we set

$$
\begin{equation*}
A=\left(\sum_{i \in I} \lambda_{i} J_{A_{i}}\right)^{-1}-\mathrm{Id} \tag{14}
\end{equation*}
$$

Then the definition of the resolvent yields

$$
\begin{equation*}
J_{A}=\sum_{i \in I} \lambda_{i} J_{A_{i}} \tag{15}
\end{equation*}
$$

thus, since it is easy to see that $J_{A}$ is firmly nonexpansive, it follows from Fact 1.2 that $A$ is maximally monotone. We refer to the operator $A$ as the resolvent average of the
maximally monotone operators $A_{1}, \ldots, A_{m}$ and we note that $J_{A}$ is the weighted average of the resolvents $J_{A_{i}}$. The operator $J_{A}$ is the announced generalization of the averaged projection operator considered in (2), and Fix $J_{A}$ is the generalization of the minimizers of the function in (3).

This introductory section is now complete. In the next section, we shall derive an alternative description of Fix $J_{A}$.

## 2 The Fixed Point Set Viewed in a Product Space

It will be quite convenient to define numbers complementary to the convex coefficients fixed in (13); thus, we let

$$
\begin{equation*}
\mu_{i}=1-\lambda_{i}, \quad \text { for every } i \in I . \tag{16}
\end{equation*}
$$

Several of the results will be formulated in the Hilbert product space

$$
\begin{equation*}
\mathbf{X}=X^{m}, \quad \text { with inner product }\langle\mathbf{x}, \mathbf{y}\rangle=\sum_{i \in I}\left\langle x_{i}, y_{i}\right\rangle \tag{17}
\end{equation*}
$$

where $\mathbf{x}=\left(x_{i}\right)_{i \in I}$ and $\mathbf{y}=\left(y_{i}\right)_{i \in I}$ are generic vectors in $\mathbf{X}$. The set $\mathbf{S}$, defined by

$$
\begin{equation*}
\mathbf{S}=\left\{\mathbf{x}=\left(x_{i}\right)_{i \in I} \in \mathbf{X} \left\lvert\,(\forall i \in I) \quad x_{i}=J_{\mu_{i}^{-1} A_{i}}\left(\sum_{j \in I \backslash\{i\}} \frac{\lambda_{j}}{\mu_{i}} x_{j}\right)\right.\right\} \tag{18}
\end{equation*}
$$

turns out to be fundamental in describing Fix $J_{A}$.
Theorem 2.1 (correspondence between $\mathbf{S}$ and Fix $J_{A}$ ) The operator

$$
\begin{equation*}
L: \mathbf{S} \rightarrow \operatorname{Fix} J_{A}: \mathbf{x}=\left(x_{i}\right)_{i \in I} \mapsto \sum_{i \in I} \lambda_{i} x_{i} \tag{19}
\end{equation*}
$$

is well defined, bijective, and Lipschitz continuous with constant 1. Furthermore, the inverse operator of L satisfies

$$
\begin{equation*}
L^{-1}: \operatorname{Fix} J_{A} \rightarrow \mathbf{S}: x \mapsto\left(J_{A_{i}} x\right)_{i \in I} \tag{20}
\end{equation*}
$$

and $L^{-1}$ is Lipschitz continuous with constant $\sqrt{m}$.

Proof. We proceed along several steps.
Claim 1: $(\forall \mathbf{x} \in \mathbf{S}) L \mathbf{x} \in \operatorname{Fix} J_{A}$ and $\mathbf{x}=\left(J_{A_{i}} L \mathbf{x}\right)_{i \in I^{\prime}}$ consequently, $L$ is well defined.

Let $\mathbf{x}=\left(x_{i}\right)_{i \in I} \in \mathbf{S}$ and set $\bar{x}=\sum_{i \in I} \lambda_{i} x_{i}=L \mathbf{x}$. Using the definition of the resolvent, we have, for every $i \in I$,

$$
\begin{align*}
\sum_{j \in I \backslash\{i\}} \frac{\lambda_{j}}{\mu_{i}} x_{j} \in\left(\operatorname{Id}+\mu_{i}^{-1} A_{i}\right) x_{i} & \Leftrightarrow \sum_{j \in I \backslash\{i\}} \lambda_{j} x_{j} \in \mu_{i} x_{i}+A_{i} x_{i}  \tag{21a}\\
& \Leftrightarrow \sum_{j \in I \backslash\{i\}} \lambda_{j} x_{j} \in\left(1-\lambda_{i}\right) x_{i}+A_{i} x_{i}  \tag{21b}\\
& \Leftrightarrow \bar{x}=\sum_{j \in I} \lambda_{j} x_{j} \in\left(\operatorname{Id}+A_{i}\right) x_{i}  \tag{21c}\\
& \Leftrightarrow x_{i}=J_{A_{i}} \bar{x}=J_{A_{i}} L \mathbf{x} . \tag{21d}
\end{align*}
$$

Hence $\mathbf{x}=\left(J_{A_{i}} L \mathbf{x}\right)_{i \in I}$, as claimed. Moreover, $(\forall i \in I) \lambda_{i} x_{i}=\lambda_{i} J_{A_{i}} \bar{x}$, which, after summing over $i \in I$ and recalling (15), yields $\bar{x}=\sum_{i \in I} \lambda_{i} x_{i}=\sum_{i \in I} \lambda_{i} J_{A_{i}} \bar{x}=J_{A} \bar{x}$. Thus $L \mathbf{x}=\bar{x} \in \operatorname{Fix} J_{A}$ and Claim 1 is verified.

Claim 2: $\left(\forall x \in \operatorname{Fix} J_{A}\right)\left(J_{A_{i}} x\right)_{i \in I} \in \mathbf{S}$.
Assume that $x \in \operatorname{Fix} J_{A}$ and set $(\forall i \in I) y_{i}=J_{A_{i}} x$. Then, using (15), we see that

$$
\begin{equation*}
\sum_{i \in I} \lambda_{i} y_{i}=\sum_{i \in I} \lambda_{i} J_{A_{i}} x=J_{A} x=x \tag{22}
\end{equation*}
$$

Furthermore, for every $i \in I$, and using (22) in the derivation of (23c)

$$
\begin{align*}
y_{i}=J_{A_{i}} x & \Leftrightarrow x \in y_{i}+A_{i} y_{i} \Leftrightarrow x-\lambda_{i} y_{i} \in \mu_{i} y_{i}+A_{i} y_{i}  \tag{23a}\\
& \Leftrightarrow \mu_{i}^{-1}\left(x-\lambda_{i} y_{i}\right) \in\left(\operatorname{Id}+\mu_{i}^{-1} A_{i}\right) y_{i}  \tag{23b}\\
& \Leftrightarrow \mu_{i}^{-1} \sum_{j \in I \backslash\{i\}} \lambda_{j} y_{j} \in\left(\operatorname{Id}+\mu_{i}^{-1} A_{i}\right) y_{i}  \tag{23c}\\
& \Leftrightarrow y_{i}=J_{\mu_{i}^{-1} A_{i}}\left(\sum_{j \in I \backslash\{i\}} \frac{\lambda_{j}}{\mu_{i}} y_{j}\right) . \tag{23d}
\end{align*}
$$

Thus, $\left(y_{i}\right)_{i \in I} \in \mathbf{S}$ and Claim 2 is verified.
Having verified the two claims above, we now turn to proving the statements announced.

First, let $x \in$ Fix $J_{A}$. By Claim 2, $\left(J_{A_{i}} x\right)_{i \in I} \in \mathbf{S}$. Hence $L\left(J_{A_{i}} x\right)_{i \in I}=\sum_{i \in I} \lambda_{i} J_{A_{i}} x=J_{A} x=$ $x$ by (15). Thus, $L$ is surjective.

Second, assume that $\mathbf{x}=\left(x_{i}\right)_{i \in I}$ and $\mathbf{y}=\left(y_{i}\right)_{i \in I}$ belong to $\mathbf{S}$ and that $L \mathbf{x}=L \mathbf{y}$. Then, using Claim 1, we see that $\mathbf{x}=\left(J_{A_{i}} L \mathbf{x}\right)_{i \in I}=\left(J_{A_{i}} L \mathbf{y}\right)_{i \in I}=\mathbf{y}$ and thus $L$ is injective. Altogether, this shows that $L$ is bijective and we also obtain the formula for $L^{-1}$.

Third, again let $\mathbf{x}=\left(x_{i}\right)_{i \in I}$ and $\mathbf{y}=\left(y_{i}\right)_{i \in I}$ be in $S$. Using the convexity of $\|\cdot\|^{2}$, we obtain

$$
\begin{align*}
\|L \mathbf{x}-L \mathbf{y}\|^{2} & =\left\|\sum_{i \in I} \lambda_{i}\left(x_{i}-y_{i}\right)\right\|^{2} \leq \sum_{i \in I} \lambda_{i}\left\|x_{i}-y_{i}\right\|^{2}  \tag{24a}\\
& \leq \sum_{i \in I}\left\|x_{i}-y_{i}\right\|^{2}=\|\mathbf{x}-\mathbf{y}\|^{2} \tag{24b}
\end{align*}
$$

Thus, $L$ is Lipschitz continuous with constant 1.
Finally, let $x$ and $y$ be in Fix $J_{A}$. Since $J_{A_{i}}$ is (firmly) nonexpansive for all $i \in I$, we estimate

$$
\begin{align*}
\left\|L^{-1} x-L^{-1} y\right\|^{2} & =\left\|\left(J_{A_{i}} x\right)_{i \in I}-\left(J_{A_{i}} x\right)_{i \in I}\right\|^{2}=\sum_{i \in I}\left\|J_{A_{i}} x-J_{A_{i}} y\right\|^{2}  \tag{25a}\\
& \leq \sum_{i \in I}\|x-y\|^{2}=m\|x-y\|^{2} \tag{25b}
\end{align*}
$$

Therefore, $L^{-1}$ is Lipschitz continuous with constant $\sqrt{m}$.
Remark 2.2 Some comments regarding Theorem 2.1are in order.
(i) Because of the simplicity of the bijection $L$ provided in Theorem 2.1, the task of finding Fix $J_{A}$ is essentially the same as finding $\mathbf{S}$.
(ii) Note that when each $A_{i}$ is a normal cone operator $N_{C_{i}}$, then the resolvents $J_{A_{i}}$ and $J_{\mu_{i}^{-1} A_{i}}$ simplify to the projections $P_{C_{i}}$, for every $i \in I$.
(iii) When $m=2$, the set $\mathbf{S}$ turns into

$$
\begin{equation*}
\mathbf{S}=\left\{\left(x_{1}, x_{2}\right) \in \mathbf{X} \mid x_{1}=J_{\lambda_{2}^{-1} A_{1}} x_{2} \text { and } x_{2}=J_{\lambda_{1}^{-1} A_{2}} x_{1}\right\} \tag{26}
\end{equation*}
$$

and Theorem 2.1] coincides with [30, Theorem 3.6]. Note that $\left(x_{1}, x_{2}\right) \in \mathbf{S}$ if and only if $x_{2} \in \operatorname{Fix}\left(J_{\lambda_{1}^{-1} A_{2}} J_{\lambda_{2}^{-1} A_{1}}\right)$ and $x_{1}=J_{\lambda_{2}^{-1} A_{1}} x_{2}$, which makes the connection between the fixed point set of the composition of the two resolvents and $\mathbf{S}$. It appears that this is a particularity of the case $m=2$; it seems that there is no simple connection between fixed points of $J_{\mu_{m}^{-1} A_{m}} J_{\mu_{m-1}^{-1} A_{m-1}} \cdots J_{\mu_{1}^{-1} A_{1}}$ and Fix $J_{A}$ when $m \geq 3$.

## 3 Fixed Points of a Composition

From now on, we let

$$
\begin{equation*}
\mathbf{R}: \mathbf{X} \rightarrow \mathbf{X}: \mathbf{x}=\left(x_{i}\right)_{i \in I} \mapsto\left(\sum_{j \in I \backslash\{i\}} \frac{\lambda_{j}}{\mu_{i}} x_{j}\right)_{i \in I} \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{J}: \mathbf{X} \rightarrow \mathbf{X}: \mathbf{x}=\left(x_{i}\right)_{i \in I} \mapsto\left(J_{\mu_{i}^{-1} A_{i}} x_{i}\right)_{i \in I} \tag{28}
\end{equation*}
$$

It is immediate from the definition of the set $\mathbf{S}$ (see (18)) that

$$
\begin{equation*}
\mathbf{S}=\operatorname{Fix}(\mathbf{J} \circ \mathbf{R}) . \tag{29}
\end{equation*}
$$

We are thus ultimately interested in developing algorithms for finding a fixed point of $\mathbf{J} \circ \mathbf{R}$. We start by collecting relevant information about the operator $\mathbf{R}$.

Proposition 3.1 The adjoint of $\mathbf{R}$ is given by

$$
\begin{equation*}
\mathbf{R}^{*}: \mathbf{X} \rightarrow \mathbf{X}: \mathbf{x}=\left(x_{i}\right)_{i \in I} \mapsto\left(\sum_{j \in I \backslash\{i\}} \frac{\lambda_{i}}{\mu_{j}} x_{j}\right)_{i \in I} \tag{30}
\end{equation*}
$$

and the set of fixed points of $\mathbf{R}$ is the "diagonal" in $\mathbf{X}$, i.e.,

$$
\begin{equation*}
\operatorname{Fix} \mathbf{R}=\left\{(x)_{i \in I} \in \mathbf{X} \mid x \in X\right\} \tag{31}
\end{equation*}
$$

Proof. Denote the operator defined in (30) by $\mathbf{L}$, and take $\mathbf{x}=\left(x_{i}\right)_{i \in I}$ and $\mathbf{y}=\left(y_{i}\right)_{i \in I}$ in $\mathbf{X}$. Then

$$
\begin{align*}
\langle\mathbf{x}, \mathbf{L y}\rangle & =\sum_{i \in I}\left\langle x_{i},(\mathbf{L y})_{i}\right\rangle=\sum_{i \in I} \sum_{j \in I \backslash\{i\}} \frac{\lambda_{i}}{\mu_{j}}\left\langle x_{i}, y_{j}\right\rangle  \tag{32a}\\
& =\sum_{\{(i, j) \in I \times I} \frac{\lambda_{i}}{\mu_{j}}\left\langle x_{i}, y_{j}\right\rangle  \tag{32b}\\
& =\sum_{j \in I \in j\}} \sum_{i \in I \backslash\{j\}} \frac{\lambda_{i}}{\mu_{j}}\left\langle x_{i}, y_{j}\right\rangle=\sum_{j \in I}\left\langle(\mathbf{R} \mathbf{x})_{j}, y_{j}\right\rangle \tag{32c}
\end{align*}
$$

$$
\begin{equation*}
=\langle\mathbf{R} \mathbf{x}, \mathbf{y}\rangle \tag{32d}
\end{equation*}
$$

which shows that $\mathbf{R}^{*}=\mathbf{L}$ as claimed.
Next, let $x \in X$ and denote the right side of (31) by $\Delta$. Since

$$
\begin{equation*}
(\forall i \in I) \quad \sum_{j \in I \backslash\{i\}} \mu_{i}^{-1} \lambda_{j}=1, \tag{33}
\end{equation*}
$$

it is clear that

$$
\begin{equation*}
\Delta \subseteq \operatorname{Fix} \mathbf{R} \tag{34}
\end{equation*}
$$

Now let $\mathbf{x}=\left(x_{i}\right)_{i \in I} \in \operatorname{Fix} \mathbf{R}$ and set $\bar{x}=\sum_{i \in I} \lambda_{i} x_{i}$. Then $\mathbf{x}=\mathbf{R} \mathbf{x}$, i.e., for every $i \in I$, we have

$$
\begin{align*}
x_{i}=(\mathbf{x})_{i}=(\mathbf{R} \mathbf{x})_{i} & \Leftrightarrow x_{i}=\sum_{j \in I \backslash\{i\}} \frac{\lambda_{j}}{\mu_{i}} x_{j} \Leftrightarrow \mu_{i} x_{i}=\sum_{j \in I \backslash\{i\}} \lambda_{j} x_{j}  \tag{35a}\\
& \Leftrightarrow\left(1-\lambda_{i}\right) x_{i}=\sum_{j \in I \backslash\{i\}} \lambda_{j} x_{j} \Leftrightarrow x_{i}=\sum_{j \in I} \lambda_{j} x_{j}  \tag{35b}\\
& \Leftrightarrow x_{i}=\bar{x} \tag{35c}
\end{align*}
$$

by (16). Hence $\mathbf{x}=(\bar{x})_{i \in I} \in \Delta$ and thus
$\operatorname{Fix} \mathbf{R} \subseteq \Delta$.
Combining (34) and (36), we obtain (31).
Remark 3.2 If $m=2$, then $\mathbf{R}^{*}=\mathbf{R}$. However, when $m \geq 3$, one has the equivalence $\mathbf{R}^{*}=\mathbf{R} \Leftrightarrow\left(\lambda_{i}\right)_{i \in I}=\left(\frac{1}{m}\right)_{i \in I}$.

The following observation will be useful when discussing nonexpansiveness of $\mathbf{R}$.
Lemma 3.3 We have $1 \leq m \sum_{i \in I} \lambda_{i}^{2}$; furthermore, equality holds if and only if $(\forall i \in I) \lambda_{i}=\frac{1}{m}$.
Proof. Indeed,

$$
\begin{equation*}
1=\sum_{i \in I} \lambda_{i} \cdot 1 \leq\left(\sum_{i \in I} \lambda_{i}^{2}\right)^{1 / 2}\left(\sum_{i \in I} 1^{2}\right)^{1 / 2} \Leftrightarrow 1=1^{2} \leq\left(\sum_{i \in I} \lambda_{i}^{2}\right) m \tag{37}
\end{equation*}
$$

and the result follows from the Cauchy-Schwarz inequality and its characterization of equality.

The next result is surprising as it shows that the actual values of the convex parameters $\lambda_{i}$ matter when $m \geq 3$.

Proposition 3.4 (nonexpansiveness of $\mathbf{R}$ ) The following hold.
(i) If $m=2$, then $\mathbf{R}:\left(x_{1}, x_{2}\right) \mapsto\left(x_{2}, x_{1}\right)$; thus, $\mathbf{R}$ is an isometry and nonexpansive.
(ii) If $m \geq 3$, then: $\mathbf{R}$ is nonexpansive if and only if $(\forall i \in I) \lambda_{i}=\frac{1}{m}$, in which case $\|\mathbf{R}\|=1$.

Proof. (i); When $m=2$, we have $\lambda_{1}=\mu_{2}$ and $\lambda_{2}=\mu_{1}$; thus, the definition of $\mathbf{R}$ (see (27)) yields the announced formula and it is clear that then $\mathbf{R}$ is an isometry and hence nonexpansive.
(ii): Suppose that $m \geq 3$. Assume first that $(\forall i \in I) \lambda_{i}=\frac{1}{m}$; hence, $\mu_{i}=1-\frac{1}{m}=$ $(m-1) / m$. Then

$$
\begin{equation*}
(\forall j \in I) \quad \lambda_{j} \sum_{i \in I \backslash\{j\}} \frac{1}{\mu_{i}}=\frac{1}{m} \sum_{i \in I \backslash\{j\}} \frac{1}{(m-1) / m}=1 . \tag{38}
\end{equation*}
$$

Now let $\mathbf{x}=\left(x_{i}\right)_{i \in I} \in \mathbf{X}$. Using the definition of $\mathbf{R}$ (see (27)), the convexity of $\|\cdot\|^{2}$ in (39b), and (38) in (39e), we obtain

$$
\begin{align*}
\|\mathbf{R} \mathbf{x}\|^{2} & =\sum_{i \in I}\left\|(\mathbf{R} \mathbf{x})_{i}\right\|^{2}=\sum_{i \in I}\left\|\sum_{j \in I \backslash\{i\}} \frac{\lambda_{j}}{\mu_{i}} x_{j}\right\|^{2}  \tag{39a}\\
& \leq \sum_{i \in I} \sum_{j \in I \backslash\{i\}} \frac{\lambda_{j}}{\mu_{i}}\left\|x_{j}\right\|^{2}  \tag{39b}\\
& =\sum_{\{(i, j) \in I \times I \mid} \frac{\lambda_{j}}{\mu_{i}}\left\|x_{j}\right\|^{2}  \tag{39c}\\
& =\sum_{j \in j\}} \lambda_{j}\left\|x_{j}\right\|^{2} \sum_{i \in I \backslash\{j\}} \frac{1}{\mu_{i}}  \tag{39~d}\\
& =\sum_{j \in I}\left\|x_{j}\right\|^{2}  \tag{39e}\\
& =\|\mathbf{x}\|^{2} . \tag{39f}
\end{align*}
$$

Since $\mathbf{R}$ is linear, it follows that $\mathbf{R}$ is nonexpansive; furthermore, since Fix $\mathbf{R} \neq\{0\}$ by (31), we then have $\|\mathbf{R}\|=1$.

To prove the remaining implication, we demonstrate the contrapositive and thus assume that

$$
\begin{equation*}
\left(\lambda_{i}\right)_{i \in I} \neq\left(\frac{1}{m}\right)_{i \in I} \tag{40}
\end{equation*}
$$

Take $u \in X$ such that $\|u\|=1$ and set $(\forall i \in I) x_{i}=\mu_{i} u$ and $\mathbf{x}=\left(x_{i}\right)_{i \in I}$. We compute

$$
\begin{align*}
\|\mathbf{x}\|^{2} & =\sum_{i \in I}\left\|x_{i}\right\|^{2}=\sum_{i \in I}\left\|\mu_{i} u\right\|^{2}=\sum_{i \in I} \mu_{i}^{2}  \tag{41a}\\
& =\sum_{i \in I}\left(1-\lambda_{i}\right)^{2}=\sum_{i \in I}\left(1-2 \lambda_{i}+\lambda_{i}^{2}\right)  \tag{41b}\\
& =m-2+\sum_{i \in I} \lambda_{i}^{2} \tag{41c}
\end{align*}
$$

Using (30), the fact that $\|u\|=1$, we obtain

$$
\begin{align*}
\left\|\mathbf{R}^{*} \mathbf{x}\right\|^{2} & =\sum_{i \in I} \lambda_{i}^{2}\left\|\sum_{j \in I \backslash\{i\}} \mu_{j}^{-1} x_{j}\right\|^{2}=\sum_{i \in I} \lambda_{i}^{2}\left\|\sum_{j \in I \backslash\{i\}} \mu_{j}^{-1} \mu_{j} u\right\|^{2}  \tag{42a}\\
& =\sum_{i \in I} \lambda_{i}^{2}\|(m-1) u\|^{2}=(m-1)^{2} \sum_{i \in I} \lambda_{i}^{2} \tag{42b}
\end{align*}
$$

Altogether,

$$
\begin{align*}
\left\|\mathbf{R}^{*} \mathbf{x}\right\|^{2}-\|\mathbf{x}\|^{2} & =2-m+\left((m-1)^{2}-1\right) \sum_{i \in I} \lambda_{i}^{2}  \tag{43a}\\
& =(m-2)\left(-1+m \sum_{i \in I} \lambda_{i}^{2}\right) \tag{43b}
\end{align*}
$$

Now $m \geq 3$ implies that $m-2>0$; furthermore, by (40) and Lemma 3.3, $-1+$ $m \sum_{i \in I} \lambda_{i}^{2}>0$. Therefore,

$$
\begin{equation*}
\left\|\mathbf{R}^{*} \mathbf{x}\right\|>\|\mathbf{x}\| \tag{44}
\end{equation*}
$$

This implies $\left\|\mathbf{R}^{*}\right\|>1$ and hence $\|\mathbf{R}\|>1$ by [21, Theorem 3.9-2]. Since $\mathbf{R}$ is linear, it cannot be nonexpansive.

For algorithmic purposes, nonexpansiveness is a desirable property but it does not guarantee the convergence of the iterates to a fixed point (consider, e.g., - Id). The very useful notion of an averaged mapping, which is intermediate between nonexpansiveness and firm nonexpansiveness, was introduced by Baillon, Bruck, and Reich in [1].

Definition 3.5 (averaged mapping) Let $T: X \rightarrow X$. Then $T$ is averaged if there exist a nonexpansive mapping $N: X \rightarrow X$ and $\alpha \in[0,1[$ such that

$$
\begin{equation*}
T=(1-\alpha) \operatorname{Id}+\alpha N \tag{45}
\end{equation*}
$$

if we wish to emphasis the constant $\alpha$, we say that $T$ is $\alpha$-averaged.

It is clear from the definition that every averaged mapping is nonexpansive; the converse, however, is false: indeed, - Id is nonexpansive, but not averaged. It follows from Fact 1.1 that every firmly nonexpansive mapping is $\frac{1}{2}$-averaged.

The class of averaged mappings is closed under compositions; this is not true for firmly nonexpansive mappings: e.g., consider two projections onto two lines that meet at 0 at a $\pi / 4$ angle. Let us record the following well known key properties.

Fact 3.6 Let $T, T_{1}$, and $T_{2}$ be mappings from $X$ to $X$, let $\alpha_{1}$ and $\alpha_{2}$ be in $\left[0,1\left[\right.\right.$, and let $x_{0} \in X$. Then the following hold.
(i) $T$ is firmly nonexpansive if and only if $T$ is $\frac{1}{2}$-averaged.
(ii) If $T_{1}$ is $\alpha_{1}$-averaged and $T_{2}$ is $\alpha_{2}$-averaged, then $T_{1} \circ T_{2}$ is $\alpha$-averaged, where

$$
\alpha= \begin{cases}0, & \text { if } \alpha_{1}=\alpha_{2}=0  \tag{46}\\ \frac{2}{1+1 / \max \left\{\alpha_{1}, \alpha_{2}\right\}}, & \text { otherwise }\end{cases}
$$

is the harmonic mean of 1 and $\max \left\{\alpha_{1}, \alpha_{2}\right\}$.
(iii) If $T_{1}$ and $T_{2}$ are averaged, and $\operatorname{Fix}\left(T_{1} \circ T_{2}\right) \neq \varnothing$, then $\operatorname{Fix}\left(T_{1} \circ T_{2}\right)=\operatorname{Fix}\left(T_{1}\right) \cap \operatorname{Fix}\left(T_{2}\right)$.
(iv) If $T$ is averaged and Fix $T \neq \varnothing$, then the sequence of iterates $\left(T^{n} x_{0}\right)_{n \in \mathbb{N}}$ converges weakly 2 to a point in Fix $T$; otherwise, $\left\|T^{n} x_{0}\right\| \rightarrow+\infty$.

Proof. (i). This is well known and immediate from Fact 1.1 ,
(ii): The fact that the composition of averaged mappings is again averaged is well known and implicit in the proof of [1, Corollary 2.4]. For the exact constants, see [15, Lemma 2.2] or [3, Proposition 4.32].
(iii), This follows from [9, Proposition 1.1, Proposition 2.1, and Lemma 2.1]. See also [24, Theorem 3] for the case when Fix $T \neq \varnothing$.
(iv) This follows from [9, Corollary 1.3 and Corollary 1.4].

Theorem 3.7 (averagedness of $\mathbf{R}$ ) The following hold.
(i) If $m=2$, then $\mathbf{R}$ is not averaged.
(ii) If $m \geq 3$ and $(\forall i \in I) \lambda_{i}=\frac{1}{m}$, then $\mathbf{R}=(1-\alpha) \operatorname{Id}+\alpha \mathbf{N}$, where $\alpha=\frac{m}{2 m-2}$ and $\mathbf{N}$ is an isometry; in particular, $\mathbf{R}$ is $\alpha$-averaged.

[^2]Proof. (i): Assume that $m=2$. By Proposition 3.4[(i), R: $\left(x_{1}, x_{2}\right) \mapsto\left(x_{2}, x_{1}\right)$. We argue by contradiction and thus assume that $\mathbf{R}$ is averaged, i.e., there exist a nonexpansive mapping $\mathbf{N}: \mathbf{X} \rightarrow \mathbf{X}$ and $\alpha \in[0,1[$ such that $\mathbf{R}=(1-\alpha)$ Id $+\alpha \mathbf{N}$. Since $\mathbf{R} \neq \mathrm{Id}$, it is clear that $\alpha>0$. Thus,

$$
\begin{equation*}
\mathbf{N}: \mathbf{X} \rightarrow \mathbf{X}:\left(x_{1}, x_{2}\right) \mapsto \alpha^{-1}\left(x_{2}-x_{1}+\alpha x_{1}, x_{1}-x_{2}+\alpha x_{2}\right) . \tag{47}
\end{equation*}
$$

Now take $u \in X$ such that $\|u\|=1$ and set $\mathbf{x}=\left(x_{1}, x_{2}\right)=(0, \alpha u)$. Then

$$
\begin{equation*}
\|\mathbf{x}\|^{2}=\|0\|^{2}+\|\alpha u\|^{2}=\alpha^{2} \tag{48}
\end{equation*}
$$

and $\mathbf{N} \mathbf{x}=(u,(\alpha-1) u)$. Thus,

$$
\begin{equation*}
\|\mathbf{N} \mathbf{x}\|^{2}=\|u\|^{2}+\|(\alpha-1) u\|^{2}=1+(1-\alpha)^{2}=\alpha^{2}+2(1-\alpha)>\alpha^{2}=\|\mathbf{x}\|^{2} . \tag{49}
\end{equation*}
$$

Hence $\|\mathbf{N}\|>1$ and, since $\mathbf{N}$ is linear, $\mathbf{N}$ cannot be nonexpansive. This contradiction completes the proof of (i).
(ii): Assume that $m \geq 3$ and that $(\forall i \in I) \lambda_{i}=\frac{1}{m}$. For future reference, we observe that

$$
\begin{equation*}
(\forall i \in I)(\forall j \in I) \quad \frac{\lambda_{j}}{\mu_{i}}=\frac{\frac{1}{m}}{1-\frac{1}{m}}=\frac{1}{m-1} \tag{50}
\end{equation*}
$$

We start by defining

$$
\begin{equation*}
L: \mathbf{X} \rightarrow X:\left(x_{i}\right)_{i \in I} \mapsto \sum_{i \in I} x_{i} . \tag{51}
\end{equation*}
$$

Then it is easily verified that

$$
\begin{equation*}
L^{*}: X \rightarrow \mathbf{X}: x \mapsto(x)_{i \in I} \tag{52}
\end{equation*}
$$

and hence that

$$
\begin{equation*}
L^{*} L L^{*} L=m L^{*} L \tag{53}
\end{equation*}
$$

Now set

$$
\begin{equation*}
\alpha=\frac{m}{2 m-2} \quad \text { and } \quad \mathbf{N}=\alpha^{-1}(\mathbf{R}-(1-\alpha) \text { Id }) \tag{54}
\end{equation*}
$$

Then $\alpha \in] 0,1[$ and $\mathbf{R}=\alpha \mathbf{N}+(1-\alpha)$ Id; thus, it suffices to show that $\mathbf{N}$ is an isometry. Note that

$$
\begin{equation*}
\alpha-1=-\alpha+\frac{1}{m-1} . \tag{55}
\end{equation*}
$$

Take $\mathbf{x}=\left(x_{i}\right)_{i \in I} \in \mathbf{X}$. Using (27), (50), and (55), we obtain for every $i \in I$,

$$
\begin{align*}
(\mathbf{N} \mathbf{x})_{i} & =\alpha^{-1}\left(-(1-\alpha) x_{i}+(\mathbf{R} \mathbf{x})_{i}\right)  \tag{56a}\\
& =\alpha^{-1}\left((\alpha-1) x_{i}+\sum_{j \in I \backslash\{i\}} \frac{\lambda_{j}}{\mu_{i}} x_{j}\right)  \tag{56b}\\
& =\alpha^{-1}\left(-\alpha x_{i}+\frac{1}{m-1} x_{i}+\sum_{j \in I \backslash\{i\}} \frac{1}{m-1} x_{j}\right)  \tag{56c}\\
& =\alpha^{-1}\left(-\alpha x_{i}+\sum_{j \in I} \frac{1}{m-1} x_{j}\right)  \tag{56d}\\
& =-x_{i}+\frac{\alpha^{-1}}{m-1} L \mathbf{x}  \tag{56e}\\
& =-x_{i}+\frac{2}{m} L \mathbf{x} ; \tag{56f}
\end{align*}
$$

hence, $\mathbf{N} \mathbf{x}=-\mathbf{x}+\frac{2}{m} L^{*} L \mathbf{x}$. It follows that $\mathbf{N}=-\operatorname{Id}+\frac{2}{m} L^{*} L$ and thus $\mathbf{N}^{*}=\mathbf{N}$. Using (53), we now obtain

$$
\begin{align*}
\mathbf{N}^{*} \mathbf{N} & =\mathbf{N N}=\left(-\operatorname{Id}+\frac{2}{m} L^{*} L\right)\left(-\operatorname{Id}+\frac{2}{m} L^{*} L\right)  \tag{57a}\\
& =\operatorname{Id}-\frac{2}{m} L^{*} L-\frac{2}{m} L^{*} L+\frac{4}{m^{2}}\left(L^{*} L L^{*} L\right)  \tag{57b}\\
& =\operatorname{Id}-\frac{4}{m} L^{*} L+\frac{4}{m^{2}}\left(m L^{*} L\right)  \tag{57c}\\
& =\mathrm{Id} \tag{57d}
\end{align*}
$$

Therefore, $\|\mathbf{N} \mathbf{x}\|^{2}=\langle\mathbf{N} \mathbf{x}, \mathbf{N} \mathbf{x}\rangle=\left\langle\mathbf{x}, \mathbf{N}^{*} \mathbf{N} \mathbf{x}\right\rangle=\langle\mathbf{x}, \mathbf{x}\rangle=\|\mathbf{x}\|^{2}$ and hence $\mathbf{N}$ is an isometry; in particular, $\mathbf{N}$ is nonexpansive and $\mathbf{R}$ is $\alpha$-averaged.

We are now in a position to describe the set $\mathbf{S}$ as the fixed point set of an averaged mapping.

Corollary 3.8 Suppose that $m \geq 3$ and that $(\forall i \in I) \lambda_{i}=\frac{1}{m}$. Then $\mathbf{J} \circ \mathbf{R}$ is $\frac{2 m}{3 m-2}$-averaged and $\operatorname{Fix}(\mathbf{J} \circ \mathbf{R})=\mathbf{S}$.

Proof. On the one hand, since $\mathbf{J}$ is clearly firmly nonexpansive, $\mathbf{J}$ is $\frac{1}{2}$-averaged. On the other hand, by Theorem 3.7(ii), $\mathbf{R}$ is $\frac{m}{2 m-2}$-averaged. Since $0<\frac{1}{2}<\frac{m}{2 m-2}$, it follows from Fact 3.6(ii) that $\mathbf{J} \circ \mathbf{R}$ is $\alpha$-averaged, where

$$
\begin{equation*}
\alpha=\frac{2}{1+1 /(m /(2 m-2))}=\frac{2 m}{3 m-2} \tag{58}
\end{equation*}
$$

as claimed. To complete the proof, recall (29).

## 4 Two New Algorithms

In Section2, we saw that Fix $J_{A}=L(\mathbf{S})$ (see Theorem 2.1), and in Section 3 we discovered that $\mathbf{S}=\operatorname{Fix}(\mathbf{J} \circ \mathbf{R})$ is the fixed point set of an averaged operator. This analysis leads to new algorithms for finding a point in Fix $J_{A}$.

Theorem 4.1 Suppose that $m \geq 3$ and that $(\forall i \in I) \lambda_{i}=\frac{1}{m}$. Let $\mathbf{x}_{0}=\left(x_{0, i}\right)_{i \in I} \in \mathbf{X}$ and generate the sequence $\left(\mathbf{x}_{n}\right)_{n \in \mathbb{N}}$ by

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n+1}=(\mathbf{J} \circ \mathbf{R}) \mathbf{x}_{n} . \tag{59}
\end{equation*}
$$

Then exactly one of the following holds.
(i) Fix $J_{A} \neq \varnothing,\left(\mathbf{x}_{n}\right)_{n \in \mathbb{N}}$ converges weakly to a point $\mathbf{x}=\left(x_{i}\right)_{i \in I}$ in $\mathbf{S}$ and $\left(\sum_{i \in I} \lambda_{i} \mathbf{x}_{n, i}\right)_{n \in \mathbb{N}}$ converges weakly to $\sum_{i \in I} \lambda_{i} x_{i} \in \operatorname{Fix} J_{A}$.
(ii) $\operatorname{Fix} J_{A}=\varnothing$ and $\left\|\mathbf{x}_{n}\right\| \rightarrow+\infty$.

Proof. By Theorem 2.1, S $\neq \varnothing$ if and only if Fix $J_{A} \neq \varnothing$. Furthermore, Corollary 3.8 shows that $\mathbf{S}=\operatorname{Fix}(\mathbf{J} \circ \mathbf{R})$, where $\mathbf{J} \circ \mathbf{R}$ is averaged. The result thus follows from Fact 3.6(iv), Theorem 2.1, and the weak continuity of the operator $L$ defined in (19).

Remark 4.2 The assumption that $m \geq 3$ in Theorem 4.1 is critical: indeed, suppose that $m=2$. Then, by Proposition 3.4](i), $\mathbf{R}:\left(x_{1}, x_{2}\right) \mapsto\left(x_{2}, x_{1}\right)$. Now assume further that $A_{1}=A_{2} \equiv 0$. Then $\mathbf{J}=\mathrm{Id}$ and hence $\mathbf{J} \circ \mathbf{R}=\mathbf{R}$. Thus, if $y$ and $z$ are two distinct points in $X$ and the sequence $\left(\mathbf{x}_{n}\right)_{n \in \mathbb{N}}$ is generated by iterating $\mathbf{J} \circ \mathbf{R}$ with a starting point $\mathbf{x}_{0}=(y, z)$, then

$$
(\forall n \in \mathbb{N}) \quad \mathbf{x}_{n}= \begin{cases}(y, z), & \text { if } n \text { is even; }  \tag{60}\\ (z, y), & \text { if } n \text { is odd }\end{cases}
$$

Consequently, $\left(\mathbf{x}_{n}\right)_{n \in \mathbb{N}}$ is a bounded sequence that is not weakly convergent. On the other hand, keeping the assumption $m=2$ but allowing again for general maximally monotone operators $A_{1}$ and $A_{2}$, and assuming that Fix $J_{A} \neq \varnothing$, we observe that

$$
\begin{equation*}
\mathbf{J} \circ \mathbf{R} \circ \mathbf{J} \circ \mathbf{R}: \mathbf{X} \rightarrow \mathbf{X}:\left(x_{1}, x_{2}\right) \mapsto\left(J_{\lambda_{2}^{-1} A_{1}} J_{\lambda_{1}^{-1} A_{2}} x_{1}, J_{\lambda_{1}^{-1} A_{2}} J_{\lambda_{2}^{-1} A_{1}} x_{2}\right) . \tag{61}
\end{equation*}
$$

Hence, by [30, Theorem 5.3], the even iterates of $\mathbf{J} \circ \mathbf{R}$ will converge weakly to point ( $\bar{x}_{1}, \bar{x}_{2}$ ) with $\bar{x}_{1}=J_{\lambda_{2}^{-1} A_{1}} J_{\lambda_{1}^{-1} A_{2}} \bar{x}_{1}$ and $\bar{x}_{2}=J_{\lambda_{1}^{-1} A_{2}} J_{\lambda_{2}^{-1} A_{1}} \bar{x}_{2}$. However, $\left(\bar{x}_{1}, \bar{x}_{2}\right) \notin \mathbf{S}$ in general.

Just as the Gauss-Seidel iteration can be viewed as a modification of the Jacobi iteration where new information is immediately utilized (see, e.g., [27, Section 4.1]), we shall propose a similar modification of the iteration of the operator $\mathbf{J} \circ \mathbf{R}$ analyzed above. To this end, we introduce, for every $k \in I$, the following operators from $\mathbf{X}$ to $\mathbf{X}$ :

$$
\left(\forall \mathbf{x}=\left(x_{i}\right)_{i \in I} \in \mathbf{X}\right)(\forall i \in I) \quad\left(\mathbf{R}_{k} \mathbf{x}\right)_{i}= \begin{cases}x_{i}, & \text { if } i \neq k ;  \tag{62}\\ \sum_{j \in I \backslash\{k\}} \frac{\lambda_{j}}{\mu_{k}} x_{j}, & \text { if } i=k\end{cases}
$$

and

$$
\left(\forall \mathbf{x}=\left(x_{i}\right)_{i \in I} \in \mathbf{X}\right)(\forall i \in I) \quad\left(\mathbf{J}_{k} \mathbf{x}\right)_{i}= \begin{cases}x_{i}, & \text { if } i \neq k ;  \tag{63}\\ J_{\mu_{k}^{-1} A_{k}} x_{k}, & \text { if } i=k\end{cases}
$$

It follows immediately from the definition of $\mathbf{S}$ (see (18)) that

$$
\begin{equation*}
\mathbf{S}=\bigcap_{k \in I} \operatorname{Fix}\left(\mathbf{J}_{k} \circ \mathbf{R}_{k}\right) \tag{64}
\end{equation*}
$$

This implies

$$
\begin{equation*}
\mathbf{S} \subseteq \operatorname{Fix}\left(\mathbf{J}_{m} \circ \mathbf{R}_{m} \circ \cdots \circ \mathbf{J}_{1} \circ \mathbf{R}_{1}\right) \tag{65}
\end{equation*}
$$

and it motivates-but does not justify-to iterate the composition

$$
\begin{equation*}
\mathbf{T}=\mathbf{J}_{m} \circ \mathbf{R}_{m} \circ \cdots \circ \mathbf{J}_{1} \circ \mathbf{R}_{1} \tag{66}
\end{equation*}
$$

in order to find points in $\mathbf{S}$.
Remark 4.3 In general, the composition $\mathbf{T}=\mathbf{J}_{m} \circ \mathbf{R}_{m} \circ \cdots \circ \mathbf{J}_{1} \circ \mathbf{R}_{1}$ is not nonexpansive: indeed, assume that $(\forall k \in I) A_{k} \equiv 0$ so that $\mathbf{J}_{k}=$ Id. Then $\mathbf{T}=\mathbf{R}_{m} \circ \cdots \circ \mathbf{R}_{1}$ and we show in Appendix B that this composition is not nonexpansive and neither is any $\mathbf{R}_{k}$.

Remark 4.4 One may verify that $\mathbf{J}_{k} \circ \mathbf{R}_{k}$ is Lipschitz continuous with constant $\sqrt{m /(m-1)}$ when $\left(\lambda_{i}\right)_{i \in I}=\left(\frac{1}{m}\right)_{i \in I}$ (see Appendix B). In turn, this implies that
$\mathbf{T}$ is Lipschitz continuous with constant $\left(\frac{m}{m-1}\right)^{m / 2}$.
As $m \rightarrow+\infty$, the Lipschitz constant of $\mathbf{T}$ decreases to $\sqrt{\exp (1)} \approx 1.6487$.

Remark 4.5 (numerical experiments) In our numerical experiments, we assumed that $X=\mathbb{R}^{50}$, that $m=55$, and that $\left(\lambda_{i}\right)_{i \in I}=\left(\frac{1}{m}\right)_{i \in I}$. We considered $m$ hyperplanes and the associated normal cone operators; this corresponds to a mildly overdetermined system of linear equations and to resolvents that are projection mappings $\left(P_{i}\right)_{i \in I}$. As the aim is to find fixed points of the the averaged resolvent $J_{A}$, which in this case is the (equally weighted) average of the projections $\left(P_{i}\right)_{i \in I}$ (see (2) and (15)), we measured performance at the $n$ iteration of $x_{n} \in X$ by the relative error function in decibel (dB), i.e., by

$$
\begin{equation*}
10 \log _{10}\left(\frac{\left\|J_{A} x_{n}-x_{n}\right\|^{2}}{\left\|J_{A} x_{0}-x_{0}\right\|^{2}}\right) \tag{68}
\end{equation*}
$$

For all experiments, the starting point $x_{0}$ is the zero vector. We compared three algorithms denoted $\operatorname{alg}\left(J_{A}\right), \operatorname{alg}(\mathbf{J} \circ \mathbf{R})$, and $\operatorname{alg}(\mathbf{T})$, which correspond to iterating $J_{A}, \mathbf{J} \circ \mathbf{R}$, and $\mathbf{T}$, respectively. The last two new algorithms operate in the product space $\mathbf{X}$; thus, we project the $n$th iterate down to $X$ via $\left(x_{n}\right)_{i \in I}=\left(x_{n, i}\right)_{i \in I} \mapsto \sum_{i \in I} \lambda_{i} x_{n, i}$ to compare to alg $\left(J_{A}\right)$. The random sets (i.e., the hyperplanes) were generated in 5 instances, and the values of (68) were averaged for each iteration number. These values are plotted in Figure 1.


Figure 1: Values of the relative error function for the three algorithms.

As seen in Figure 1, the new rigorous algorithm $\operatorname{alg}(\mathbf{J} \circ \mathbf{R})$ performs better than $\operatorname{alg}\left(J_{A}\right)$, although the performance gain is slight. (Convergence is guaranteed by Fact 3.6(iv) and Theorem 4.1)(i).) Furthermore, the new heuristic algorithm $\mathrm{alg}(\mathbf{T})$, which currently lacks a convergence analysis (see Remark 4.3) substantially outperfoms $\operatorname{alg}(\mathbf{J} \circ \mathbf{R})$.

Let us now list some open problems.
Remark 4.6 (open problems) Suppose that $m \geq 3$. We do not know the answers to the following questions.

Q1: Concerning (65), is it actually true that

$$
\begin{equation*}
\mathbf{S}=\operatorname{Fix} \mathbf{T}=\operatorname{Fix}\left(\mathbf{J}_{m} \circ \mathbf{R}_{m} \circ \cdots \circ \mathbf{J}_{1} \circ \mathbf{R}_{1}\right) ? \tag{69}
\end{equation*}
$$

Q2: Can one give simple sufficient or necessary conditions for the convergence of the heuristic algorithm, i.e., the iteration of $\mathbf{T}$, when Fix $\mathbf{T} \neq \varnothing$ ?

Q3: Under the most general assumption (13), we observed convergence in numerical experiments of the new rigorous algorithm even though there is no underlying theory-see Proposition 3.4(ii) and Theorem 4.1. Can one provide simple sufficient or necessary conditions for the convergence of the sequence defined by (59)?

Remark 4.7 Concerning Remark 4.6, we note that the first two questions posed have affirmative answers when $m=2$. Indeed, one then computes

$$
\begin{equation*}
\mathbf{T}: \mathbf{X} \rightarrow \mathbf{X}:\left(x_{1}, x_{2}\right) \mapsto\left(J_{\lambda_{2}^{-1} A_{1}} x_{2}, J_{\lambda_{1}^{-1} A_{2}} J_{\lambda_{2}^{-1} A_{1}} x_{2}\right) \tag{70}
\end{equation*}
$$

and hence $\left(x_{1}, x_{2}\right) \in$ Fix $\mathbf{T}$ if and only if $x_{1}=J_{\lambda_{2}^{-1} A_{1}} x_{2}$ and $x_{2}=J_{\lambda_{1}^{-1} A_{2}} x_{1}$, which is the same as requiring that $\left(x_{1}, x_{2}\right) \in \mathbf{S}$. When $\mathbf{S}=$ Fix $\mathbf{T} \neq \varnothing$, then the iterates of $\mathbf{T}$ converge weakly to a fixed point by [30, Theorem 5.3(i)].

## Appendix A

Most of this part of the appendix is part of the folklore; however, we include it here for completeness and because we have not quite found a reference that makes all points we wish to stress.

We assume that $m \in\{1,2, \ldots\}$, that $I=\{1,2, \ldots, m\}$ and that $\left(C_{i}\right)_{i \in I}$ is a family of closed hyperplanes given by

$$
\begin{equation*}
(\forall i \in I) \quad C_{i}=\left\{x \in X \mid\left\langle a_{i}, x\right\rangle=b_{i}\right\}, \quad \text { where } a_{i} \in X \backslash\{0\} \text { and } b_{i} \in \mathbb{R}, \tag{71}
\end{equation*}
$$

with corresponding projections $P_{i}$. Set $A: X \rightarrow \mathbb{R}^{m}: x \mapsto\left\langle a_{i}, x\right\rangle$ and $b=\left(b_{i}\right)_{i \in I} \in \mathbb{R}^{m}$. Then $A^{*}: \mathbb{R}^{m} \rightarrow X:\left(y_{i}\right)_{i \in I} \mapsto \sum_{i \in I} y_{i} a_{i}$. Denote, for every $i \in I$, the $i^{\text {th }}$ unit vector in $\mathbb{R}^{m}$ by $e_{i}$, and the projection $\mathbb{R}^{m} \rightarrow \mathbb{R}^{m}: y \mapsto\left\langle y, e_{i}\right\rangle e_{i}$ onto $\mathbb{R} e_{i}$ by $Q_{i}$. Note that

$$
\sum_{i \in I} Q_{i}=\mathrm{Id} \quad \text { and } \quad(\forall(i, j) \in I \times I) \quad Q_{i} Q_{j}= \begin{cases}Q_{i}, & \text { if } i=j  \tag{72}\\ 0, & \text { otherwise }\end{cases}
$$

We now assume that

$$
\begin{equation*}
(\forall i \in I) \quad\left\|a_{i}\right\|=1 \tag{73}
\end{equation*}
$$

which gives rise to the pleasant representation of the projectors as

$$
\begin{equation*}
(\forall i \in I) \quad P_{i}: x \mapsto x-A^{*} Q_{i}(A x-b) \tag{74}
\end{equation*}
$$

and to (see (3))

$$
\begin{equation*}
(\forall x \in X) \quad\|A x-b\|^{2}=\sum_{i \in I}\left|\left\langle a_{i}, x\right\rangle-b_{i}\right|^{2}=\sum_{i \in I}\left\|x-P_{i} x\right\|^{2} \tag{75}
\end{equation*}
$$

Now let $x \in X$. Using (74) and (72), we thus obtain the following characterization of fixed points of averaged projections:

$$
\begin{align*}
& \text { (76a) } \quad x \in \operatorname{Fix}\left(\sum_{i \in I} \lambda_{i} P_{i}\right)  \tag{76a}\\
& \text { (76b) } \quad \Leftrightarrow x=\sum_{i \in I} \lambda_{i} P_{i} x  \tag{76b}\\
& \text { (76c) } \quad \Leftrightarrow x=\sum_{i \in I} \lambda_{i}\left(x-A^{*} Q_{i}(A x-b)\right) \\
& \text { (76d) } \quad \Leftrightarrow x=\left(\sum_{i \in I} \lambda_{i} x\right)-A^{*} \sum_{i \in I} \lambda_{i} Q_{i}(A x-b) \\
& \text { (76e) } \quad \Leftrightarrow A^{*}\left(\sum_{i \in I} \lambda_{i} Q_{i}\right)(A x-b)=0 \\
& \text { (76f) } \quad \Leftrightarrow A^{*}\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right)\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right)(A x-b)=0 \\
& \text { (76g) } \quad \Leftrightarrow A^{*}\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right)\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right) A x=A^{*}\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right)\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right) b \\
& \text { (76h) } \quad \Leftrightarrow\left(\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right) A\right)^{*}\left(\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right) A\right) x=\left(\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right) A\right)^{*}\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right) b \\
& \text { (76i) } \quad \Leftrightarrow x \text { satisfies the normal equation of the system } \\
& \text { (76j) } \quad\left(\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right) A\right) x=\left(\sum_{i \in I} \sqrt{\lambda_{i}} Q_{i}\right) b \tag{76j}
\end{align*}
$$

$$
\begin{align*}
& \Leftrightarrow\left(\left(\sum_{i \in I} \sqrt{m \lambda_{i}} Q_{i}\right) A\right)^{*}\left(\left(\sum_{i \in I} \sqrt{m \lambda_{i}} Q_{i}\right) A\right) x  \tag{76k}\\
& \quad=\left(\left(\sum_{i \in I} \sqrt{m \lambda_{i}} Q_{i}\right) A\right)^{*}\left(\sum_{i \in I} \sqrt{m \lambda_{i}} Q_{i}\right) b \tag{761}
\end{align*}
$$

(76m) $\Leftrightarrow x$ satisfies the normal equation of the system

$$
\begin{equation*}
\left(\left(\sum_{i \in I} \sqrt{m \lambda_{i}} Q_{i}\right) A\right) x=\left(\sum_{i \in I} \sqrt{m \lambda_{i}} Q_{i}\right) b \tag{76n}
\end{equation*}
$$

Note that when $\left(\lambda_{i}\right)_{i \in I}=\left(\frac{1}{m}\right)_{i \in I}$, i.e., we have equal weights, then (76) and (72) yield
(77a) $x \in \operatorname{Fix}\left(\frac{1}{m} \sum_{i \in I} P_{i}\right) \Leftrightarrow\left(\left(\sum_{i \in I} Q_{i}\right) A\right)^{*}\left(\left(\sum_{i \in I} Q_{i}\right) A\right) x=\left(\left(\sum_{i \in I} Q_{i}\right) A\right)^{*}\left(\sum_{i \in I} Q_{i}\right) b$

$$
\begin{equation*}
\Leftrightarrow x \text { satisfies the normal equation of the system } A x=b \tag{77c}
\end{equation*}
$$

$$
\begin{equation*}
\Leftrightarrow A^{*} A x=A^{*} b \tag{77b}
\end{equation*}
$$

$$
\begin{equation*}
\Leftrightarrow x \text { is a least squares solution of the system } A x=b \tag{77d}
\end{equation*}
$$

In other words, the fixed points of the equally averaged projections onto hyperplanes are precisely the classical least squares solutions encountered in linear algebra, i.e., the solutions to the classical normal equation $A^{*} A x=A^{*} b$ of the system $A x=b$. The idea of least squares solutions goes back to the famous prediction of the asteroid Ceres due to Carl Friedrich Gauss in 1801 (see [5, Subsection 1.1.1] and also [22, Epilogue in Section 4.6]).

Example. Consider the following inconsistent linear system of equations

$$
\begin{align*}
& x=1  \tag{78a}\\
& x=2 \tag{78b}
\end{align*}
$$

which was also studied by Byrne [12, Subsection 8.3.2 on page 100]. Here $m=2$ and (73) holds, and the above discussion yields that Fix $\left(\frac{1}{2} P_{1}+\frac{1}{2} P_{2}\right)$ and the set of least squares solutions coincide, namely with the singleton $\left\{\frac{3}{2}\right\}$. Now change the representation to

$$
\begin{align*}
2 x & =2  \tag{79a}\\
x & =2 \tag{79b}
\end{align*}
$$

so that (73) is violated. The set of fixed points remains unaltered as the two hyperplanes $C_{1}$ and $C_{2}$ are unchanged and thus it equals $\left\{\frac{3}{2}\right\}$. However, the set of least squares solutions is now $\left\{\frac{6}{5}\right\}$. Similarly and returning to the first representation in (78), the set of fixed points will changes if we consider different weights, say $\lambda_{1}=\frac{1}{3}$ and $\lambda_{2}=\frac{2}{3}$ : indeed, we then obtain Fix $\left(\frac{1}{3} P_{1}+\frac{2}{3} P_{2}\right)=\left\{\frac{5}{3}\right\}$ while the set of least squares solutions is still $\left\{\frac{3}{2}\right\}$.

## Appendix B

The proof of the following result is simple and hence omitted.
Lemma B. 1 Let $\left(\alpha_{i, j}\right)_{(i, j) \in I \times I}$ and $\left(\beta_{i, j}\right)_{(i, j) \in I \times I}$ be in $\mathbb{R}^{m \times m}$, and define

$$
\begin{equation*}
\mathbf{A}: \mathbf{X} \rightarrow \mathbf{X}:\left(x_{i}\right)_{\in I} \mapsto\left(\sum_{j \in I} \alpha_{i, j} x_{j}\right)_{i \in I} \text { and } \mathbf{B}: \mathbf{X} \rightarrow \mathbf{X}:\left(x_{i}\right)_{\in I} \mapsto\left(\sum_{j \in I} \beta_{i, j} x_{j}\right)_{i \in I} \tag{80}
\end{equation*}
$$

Then

$$
\begin{equation*}
\mathbf{A} \circ \mathbf{B}: \mathbf{X} \rightarrow \mathbf{X}:\left(x_{i}\right)_{\in I} \mapsto\left(\sum_{j \in I} \gamma_{i, j} x_{j}\right)_{i \in I} \tag{81}
\end{equation*}
$$

where $(\forall(i, j) \in I \times I) \gamma_{i, j}=\sum_{k \in I} \alpha_{i, k} \beta_{k, j}$. Furthermore, the following hold:
(i) If for every $i \in I, \sum_{j \in I} \alpha_{i, j}=1=\sum_{j \in I} \beta_{i, j}$, then $\sum_{j \in I} \gamma_{i, j}=1$ as well.
(ii) If for every $(i, j) \in I \times I, \alpha_{i, j} \geq 0$ and $\beta_{i, j} \geq 0$, then $\gamma_{i, j} \geq 0$ as well.

Proof of Remark 4.3. No $\mathbf{R}_{k}$ is nonexpansive and neither is $\mathbf{R}_{m} \circ \cdots \circ \mathbf{R}_{2} \circ \mathbf{R}_{1}$.
Proof. Take $\mathbf{x}=\left(x_{i}\right)_{i \in I} \in \mathbf{X}$, and let $i \in I$. If $i \neq k$, then $\left(\mathbf{R}_{k} \mathbf{x}\right)_{i}=x_{i}$; otherwise, $i=k$ and $\left(\mathbf{R}_{k} \mathbf{x}\right)_{k}$ is a convex combination of the vectors $\left\{x_{j}\right\}_{j \in I \backslash\{k\}}$. In either case, $\left(\mathbf{R}_{k} \mathbf{x}\right)_{k}$ is a convex combination of the vectors $\left\{x_{j}\right\}_{j \in I \backslash\{k\}}$. Thus if $u \in X$ satisfies $\|u\|=1$ and

$$
(\forall i \in I) \quad x_{i}= \begin{cases}u, & \text { if } i \neq k  \tag{82}\\ 0, & \text { if } i=k\end{cases}
$$

then $\mathbf{R}_{k} \mathbf{x}=(u)_{i \in I}$ and hence $\left\|\mathbf{R}_{k} \mathbf{x}\right\|^{2}=\sum_{i \in I}\|u\|^{2}=m>m-1=\sum_{j \in I \backslash\{k\}}\|u\|^{2}=\|\mathbf{x}\|^{2}$. Therefore $\mathbf{R}_{k}$ is not nonexpansive.

Now assume that $k=1$ and that $\mathbf{x}$ is defined as in (82). The above reasoning shows that $\mathbf{R}_{1} \mathbf{x}=(u)_{i \in I}$. In view of Lemma B.1, $(u)_{i \in I} \in \operatorname{Fix}\left(\mathbf{R}_{m} \circ \cdots \circ \mathbf{R}_{2}\right)$. Hence $\left(\mathbf{R}_{m} \circ \cdots \circ \mathbf{R}_{2} \circ\right.$ $\left.\mathbf{R}_{1}\right) \mathbf{x}=(u)_{i \in I}$ and thus once again $\left\|\left(\mathbf{R}_{m} \circ \cdots \circ \mathbf{R}_{2} \circ \mathbf{R}_{1}\right) \mathbf{x}\right\|^{2}=m>m-1=\|\mathbf{x}\|^{2}$. This completes the proof.

Proof of Remark 4.4, Let $\mathbf{x}=\left(x_{i}\right)_{i \in I}$ and $\mathbf{y}=\left(y_{i}\right)_{i \in I}$ be in $\mathbf{X}$, and take $k \in I$. Using that $J_{\mu_{k}^{-1} A_{k}}$ is (firmly) nonexpansive in (83c), and that $\|\cdot\|^{2}$ is convex in (83e), we obtain

$$
\begin{equation*}
\left\|\left(\mathbf{J}_{k} \circ \mathbf{R}_{k}\right) \mathbf{x}-\left(\mathbf{J}_{k} \circ \mathbf{R}_{k}\right) \mathbf{y}\right\|^{2} \tag{83a}
\end{equation*}
$$

$$
\begin{align*}
& =\left\|J_{\mu_{k}^{-1} A_{k}}\left(\sum_{j \in I \backslash\{k\}} \frac{\lambda_{j}}{\mu_{k}} x_{j}\right)-J_{\mu_{k}^{-1} A_{k}}\left(\sum_{j \in I \backslash\{k\}} \frac{\lambda_{j}}{\mu_{k}} y_{j}\right)\right\|^{2}+\sum_{j \in I \backslash\{k\}}\left\|x_{j}-y_{j}\right\|^{2}  \tag{83b}\\
& \leq\left\|\left(\sum_{j \in I \backslash\{k\}} \frac{\lambda_{j}}{\mu_{k}} x_{j}\right)-\left(\sum_{j \in I \backslash\{k\}} \frac{\lambda_{j}}{\mu_{k}} y_{j}\right)\right\|^{2}+\sum_{j \in I \backslash\{k\}}\left\|x_{j}-y_{j}\right\|^{2}  \tag{83c}\\
& =\left\|\sum_{j \in I \backslash\{k\}} \frac{\lambda_{j}}{\mu_{k}}\left(x_{j}-y_{j}\right)\right\|^{2}+\sum_{j \in I \backslash\{k\}}\left\|x_{j}-y_{j}\right\|^{2}  \tag{83d}\\
& \leq \sum_{j \in I \backslash\{k\}} \frac{\lambda_{j}}{\mu_{k}}\left\|x_{j}-y_{j}\right\|^{2}+\sum_{j \in I \backslash\{k\}}\left\|x_{j}-y_{j}\right\|^{2} \\
& =\sum_{j \in I \backslash\{k\}} \frac{\lambda_{j}+\mu_{k}}{\mu_{k}}\left\|x_{j}-y_{j}\right\|^{2} .
\end{align*}
$$

Since $\left(\lambda_{i}\right)_{i \in I}=\left(\frac{1}{m}\right)_{i \in I^{\prime}}$, we further deduce that

$$
\begin{align*}
\left\|\left(\mathbf{J}_{k} \circ \mathbf{R}_{k}\right) \mathbf{x}-\left(\mathbf{J}_{k} \circ \mathbf{R}_{k}\right) \mathbf{y}\right\|^{2} & \leq \sum_{j \in I \backslash\{k\}} \frac{m}{m-1}\left\|x_{j}-y_{j}\right\|^{2}  \tag{84a}\\
& \leq \sum_{j \in I} \frac{m}{m-1}\left\|x_{j}-y_{j}\right\|^{2}  \tag{84b}\\
& =\frac{m}{m-1}\|\mathbf{x}-\mathbf{y}\|^{2} \tag{84c}
\end{align*}
$$

which implies that $\mathbf{J}_{k} \circ \mathbf{R}_{k}$ is Lipschitz continuous with constant $\sqrt{m /(m-1)}$. The rest of Remark 4.4 now follows from elementary calculus.

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[^1]:    ${ }^{1}$ In passing, we mention that when $C_{1}, C_{2}, C_{3}$ are line segments forming a triangle in the Euclidean plane, then the minimizer of (3) is known as the symmedian point (also known as the Grebe-Lemoine point) of the given triangle; see [20, Theorem 349 on page 216].

[^2]:    ${ }^{2}$ When $T$ is firmly nonexpansive, the weak convergence goes back at least to [8].

