# Bifurcation from semi-trivial standing waves and ground states for a system of nonlinear Schrödinger equations 

Mathieu Colin* and Masahito Ohta ${ }^{\dagger}$


#### Abstract

We consider a system of nonlinear Schrödinger equations related to the Raman amplification in a plasma. We study the orbital stability and instability of standing waves bifurcating from the semi-trivial standing wave of the system. The stability and instability of the semitrivial standing wave at the bifurcation point are also studied. Moreover, we determine the set of the ground states completely.


## 1 Introduction

### 1.1 Motivation

In this paper, we consider the following system of nonlinear Schrödinger equations

$$
\left\{\begin{array}{l}
i \partial_{t} u_{1}=-\Delta u_{1}-\kappa\left|u_{1}\right| u_{1}-\gamma \overline{u_{1}} u_{2}  \tag{1.1}\\
i \partial_{t} u_{2}=-2 \Delta u_{2}-2\left|u_{2}\right| u_{2}-\gamma u_{1}^{2}
\end{array}\right.
$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}^{N}$, where $u_{1}$ and $u_{2}$ are complex-valued functions of $(t, x)$, $\kappa \in \mathbb{R}$ and $\gamma>0$ are constants and $N \leq 3$. System (1.1) is a reduced system studied in [7, 8] and related to the Raman amplification in a plasma.

[^0]Roughly speaking, the Raman amplification is an instability phenomenon taking place when an incident laser field propagates into a plasma. We refer to [5, 6] for a precise description of the phenomenon. A similar system to (1.1) also appears as an optics model with quadratic nonlinearity (see [21]).

In [7, 8], the authors studied the following three-component system

$$
\left\{\begin{array}{l}
i \partial_{t} v_{1}=-\Delta v_{1}-\left|v_{1}\right|^{p-1} v_{1}-\gamma v_{3} \overline{v_{2}}  \tag{1.2}\\
i \partial_{t} v_{2}=-\Delta v_{2}-\left|v_{2}\right|^{p-1} v_{2}-\gamma v_{3} \overline{v_{1}} \\
i \partial_{t} v_{3}=-\Delta v_{3}-\left|v_{3}\right|^{p-1} v_{3}-\gamma v_{1} v_{2},
\end{array}\right.
$$

where $1<p<1+4 / N$ and $N \leq 3$. Let $\omega>0$ and let $\varphi_{\omega} \in H^{1}\left(\mathbb{R}^{N}\right)$ be a unique positive radial solution of

$$
\begin{equation*}
-\Delta \varphi+\omega \varphi-|\varphi|^{p-1} \varphi=0, \quad x \in \mathbb{R}^{N} . \tag{1.3}
\end{equation*}
$$

Then, $\left(0,0, e^{i \omega t} \varphi_{\omega}\right)$ solves (1.2). We note that $e^{i \omega t} \varphi_{\omega}$ is a standing wave solution of the single nonlinear Schrödinger equation

$$
\begin{equation*}
i \partial_{t} u=-\Delta u-|u|^{p-1} u, \quad(t, x) \in \mathbb{R} \times \mathbb{R}^{N}, \tag{1.4}
\end{equation*}
$$

and that $e^{i \omega t} \varphi_{\omega}$ is orbitally stable for (1.4) if $1<p<1+4 / N$, and it is unstable if $1+4 / N \leq p<1+4 /(N-2)$ (see [1, 4] and also [3, Chapter 8]). In [7, 8], the authors proved the following result on the semi-trivial standing wave solution $\left(0,0, e^{i \omega t} \varphi_{\omega}\right)$ of (1.2).

Theorem 0. ([7, 8]) Let $N \leq 3,1<p<1+4 / N, \omega>0$, and let $\varphi_{\omega}$ be the positive radial solution of (1.3). Then, there exists a positive constant $\gamma^{*}$ such that the semi-trivial standing wave solution $\left(0,0, e^{i \omega t} \varphi_{\omega}\right)$ of (1.2) is stable if $0<\gamma<\gamma^{*}$, and it is unstable if $\gamma>\gamma^{*}$.

By the local bifurcation theorem by Crandall and Rabinowitz [10], it is easy to see that $\gamma=\gamma^{*}$ is a bifurcation point. We are interested in the structure of the bifurcation from the semi-trivial standing wave of 1.2 and its stability property. However, this problem is difficult to study in the general case $1<p<1+4 / N$, so we consider the special case $p=2$. Moreover, since $v_{1}$ and $v_{2}$ play the same role in the proof of Theorem 0 , we consider a reduced system (1.1) assuming $v_{1}=v_{2}$ in (1.2). We also introduce a parameter $\kappa$ in the first equation of (1.1), which makes the structure of standing wave solutions richer as we will see below. We remark that the positive constant $\gamma^{*}$ in Theorem 0 is given by

$$
\begin{equation*}
\gamma^{*}=\inf \left\{\frac{\|\nabla v\|_{L^{2}}^{2}+\omega\|v\|_{L^{2}}^{2}}{\int_{\mathbb{R}^{N}} \varphi_{\omega}(x)|v(x)|^{2} d x}: v \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}\right\} . \tag{1.5}
\end{equation*}
$$

For the case $p=2$, since $\varphi_{\omega}$ is the positive radial solution of

$$
\begin{equation*}
-\Delta \varphi+\omega \varphi-|\varphi| \varphi=0, \quad x \in \mathbb{R}^{N} \tag{1.6}
\end{equation*}
$$

we see that the infimum in (1.5) is attained at $v=\varphi_{\omega}$ and $\gamma^{*}=1$. In the same way as the proof of Theorem 0 , we can prove the following.

Theorem 1. Let $N \leq 3, \kappa \in \mathbb{R}, \gamma>0, \omega>0$, and let $\varphi_{\omega}$ be the positive radial solution of (1.6). Then, the semi-trivial standing wave solution $\left(0, e^{2 i \omega t} \varphi_{\omega}\right)$ of (1.1) is stable if $0<\gamma<1$, and it is unstable if $\gamma>1$.

We remark that the stability property of the semi-trivial standing wave of (1.1) is independent of $\kappa$ for the case $\gamma \neq 1$. On the other hand, we will see that the sign of $\kappa$ plays an important role for the case $\gamma=1$ (see Theorems 4 and 5 below).

### 1.2 Notation and Definitions

Before we state our main results, we prepare some notation and definitions. For a complex number $z$, we denote by $\Re z$ and $\Im z$ its real and imaginary parts. Thoughout this paper, we assume that $N \leq 3$. We regard $L^{2}\left(\mathbb{R}^{N}, \mathbb{C}\right)$ as a real Hilbert space with the inner product

$$
(u, v)_{L^{2}}=\Re \int_{\mathbb{R}^{N}} u(x) \overline{v(x)} d x
$$

and we define the inner products of real Hilbert spaces $H=L^{2}\left(\mathbb{R}^{N}, \mathbb{C}\right)^{2}$ and $X=H^{1}\left(\mathbb{R}^{N}, \mathbb{C}\right)^{2}$ by

$$
(\vec{u}, \vec{v})_{H}=\left(u_{1}, v_{1}\right)_{L^{2}}+\left(u_{2}, v_{2}\right)_{L^{2}}, \quad(\vec{u}, \vec{v})_{X}=(\vec{u}, \vec{v})_{H}+(\nabla \vec{u}, \nabla \vec{v})_{H} .
$$

Here and hereafter, we use the vectorial notation $\vec{u}=\left(u_{1}, u_{2}\right)$, and it is considered to be a column vector.

The energy $E$ and the charge $Q$ are defined by

$$
\begin{aligned}
& E(\vec{u})=\frac{1}{2}\|\nabla \vec{u}\|_{H}^{2}-\frac{\kappa}{3}\left\|u_{1}\right\|_{L^{3}}^{3}-\frac{1}{3}\left\|u_{2}\right\|_{L^{3}}^{3}-\frac{\gamma}{2} \Re \int_{\mathbb{R}^{N}} u_{1}^{2} \overline{u_{2}} d x, \\
& Q(\vec{u})=\frac{1}{2}\|\vec{u}\|_{H}^{2}, \quad \vec{u} \in X .
\end{aligned}
$$

For $\theta \in \mathbb{R}$, we define $G(\theta)$ and $J$ by

$$
G(\theta) \vec{u}=\left(e^{i \theta} u_{1}, e^{2 i \theta} u_{2}\right), \quad J \vec{u}=\left(i u_{1}, 2 i u_{2}\right), \quad \vec{u} \in X,
$$

and

$$
\langle G(\theta) \vec{f}, \vec{u}\rangle=\langle\vec{f}, G(-\theta) \vec{u}\rangle, \quad\langle J \vec{f}, \vec{u}\rangle=-\langle\vec{f}, J \vec{u}\rangle
$$

for $\vec{f} \in X^{*}$ and $\vec{u} \in X$, where $X^{*}$ is the dual space of $X$. For $y \in \mathbb{R}^{N}$, we define

$$
\tau_{y} \vec{u}(x)=\vec{u}(x-y), \quad \vec{u} \in X, x \in \mathbb{R}^{N} .
$$

Note that (1.1) is written as

$$
\partial_{t} \vec{u}(t)=-J E^{\prime}(\vec{u}(t)),
$$

and that $E\left(G(\theta) \tau_{y} \vec{u}\right)=E(\vec{u})$ for all $\theta \in \mathbb{R}, y \in \mathbb{R}^{N}$ and $\vec{u} \in X$.
By the standard theory (see, e.g., [3, Chapter 4]), we see that the Cauchy problem for (1.1) is globally well-posed in $X$, and the energy and the charge are conserved. For $\omega>0$, we define the action $S_{\omega}$ by

$$
S_{\omega}(\vec{v})=E(\vec{v})+\omega Q(\vec{v}), \quad \vec{v} \in X .
$$

Note that the Euler-Lagrange equation $S_{\omega}^{\prime}(\vec{\phi})=0$ is written as

$$
\left\{\begin{align*}
-\Delta \phi_{1}+\omega \phi_{1} & =\kappa\left|\phi_{1}\right| \phi_{1}+\gamma \overline{\phi_{1}} \phi_{2}  \tag{1.7}\\
-\Delta \phi_{2}+\omega \phi_{2} & =\left|\phi_{2}\right| \phi_{2}+(\gamma / 2) \phi_{1}^{2}
\end{align*}\right.
$$

and that if $\vec{\phi} \in X$ satisfies $S_{\omega}^{\prime}(\vec{\phi})=0$, then $G(\omega t) \vec{\phi}$ is a solution of (1.1).
Definition 1. We say that a standing wave solution $G(\omega t) \vec{\phi}$ of (1.1) is stable if for all $\varepsilon>0$ there exists $\delta>0$ with the following property. If $u_{0} \in X$ satisfies $\left\|\vec{u}_{0}-\vec{\phi}\right\|_{X}<\delta$, then the solution $\vec{u}(t)$ of (1.1) with $\vec{u}(0)=\vec{u}_{0}$ exists for all $t \geq 0$, and satisfies

$$
\inf _{\theta \in \mathbb{R}, y \in \mathbb{R}^{N}}\left\|\vec{u}(t)-G(\theta) \tau_{y} \vec{\phi}\right\|_{X}<\varepsilon
$$

for all $t \geq 0$. Otherwise, $G(\omega t) \vec{\phi}$ is called unstable.
In this article, we are also interested in the classification of ground states of (1.7). A ground state of (1.7) is a nontrivial solution which minimizes the action $S_{\omega}$ among all the nontrivial solutions of (1.7). The set $\mathcal{G}_{\omega}$ of the ground states for (1.7) is then defined as follows:

$$
\begin{aligned}
& \mathcal{A}_{\omega}=\left\{\vec{v} \in X: S_{\omega}^{\prime}(\vec{v})=0, \vec{v} \neq 0\right\}, \\
& d(\omega)=\inf \left\{S_{\omega}(\vec{v}): \vec{v} \in \mathcal{A}_{\omega}\right\}, \\
& \mathcal{G}_{\omega}=\left\{\vec{u} \in \mathcal{A}_{\omega}: S_{\omega}(\vec{u})=d(\omega)\right\} .
\end{aligned}
$$

### 1.3 Main Results

We first look for solutions of (1.7) of the form $\vec{\phi}=\left(\alpha \varphi_{\omega}, \beta \varphi_{\omega}\right)$ with $(\alpha, \beta) \in$ $] 0, \infty\left[{ }^{2}\right.$, where $\varphi_{\omega}$ is the positive radial solution of (1.6). It is clear that if $(\alpha, \beta) \in] 0, \infty\left[^{2}\right.$ satisfies

$$
\begin{equation*}
\kappa \alpha+\gamma \beta=1, \quad \gamma \alpha^{2}+2 \beta^{2}=2 \beta, \tag{1.8}
\end{equation*}
$$

then $\left(\alpha \varphi_{\omega}, \beta \varphi_{\omega}\right)$ is a solution of (1.7). For $\kappa \in \mathbb{R}$ and $\gamma>0$, we define

$$
\mathcal{S}_{\kappa, \gamma}=\{(x, y) \in] 0, \infty\left[^{2}: \kappa x+\gamma y=1, \gamma x^{2}+2 y^{2}=2 y\right\} .
$$

Note that $\gamma x^{2}+2 y^{2}=2 y$ is an ellipse with vertices $(x, y)=(0,0),(0,1)$, $( \pm 1 / \sqrt{2 \gamma}, 1 / 2)$, and that $\mathcal{S}_{\kappa, \gamma} \subset\{(x, y): 0<y<1\}$.

To determine the structure of the set $\mathcal{S}_{\kappa, \gamma}$, which is one of the crucial points of our analysis, for $\kappa^{2} \geq 2 \gamma(1-\gamma)$ we define

$$
\begin{aligned}
& \alpha_{ \pm}=\frac{(2-\gamma) \kappa \pm \gamma \sqrt{\kappa^{2}+2 \gamma(\gamma-1)}}{2 \kappa^{2}+\gamma^{3}}, \\
& \beta_{ \pm}=\frac{\kappa^{2}+\gamma^{2} \pm \kappa \sqrt{\kappa^{2}+2 \gamma(\gamma-1)}}{2 \kappa^{2}+\gamma^{3}}, \\
& \alpha_{0}=\frac{(2-\gamma) \kappa}{2 \kappa^{2}+\gamma^{3}}, \quad \beta_{0}=\frac{\kappa^{2}+\gamma^{2}}{2 \kappa^{2}+\gamma^{3}} .
\end{aligned}
$$

We also divide the parameter domain $\mathcal{D}=\{(\kappa, \gamma): \kappa \in \mathbb{R}, \gamma>0\}$ into the following sets (see Figure 1).

$$
\begin{aligned}
& \mathcal{J}_{1}=\{(\kappa, \gamma): \kappa \leq 0, \gamma>1\} \cup\{(\kappa, \gamma): \kappa>0, \gamma \geq 1\}, \\
& \mathcal{J}_{2}=\{(\kappa, \gamma): 0<\gamma<1, \kappa>\sqrt{2 \gamma(1-\gamma)}\}, \\
& \mathcal{J}_{3}=\{(\kappa, \gamma): 0<\gamma<1, \kappa=\sqrt{2 \gamma(1-\gamma)}\}, \\
& \mathcal{J}_{0}=\{(\kappa, \gamma): \kappa \in \mathbb{R}, \gamma>0\} \backslash\left(\mathcal{J}_{1} \cup \mathcal{J}_{2} \cup \mathcal{J}_{3}\right) .
\end{aligned}
$$

Notice that the sets $\mathcal{J}_{0}, \mathcal{J}_{1}, \mathcal{J}_{2}$ and $\mathcal{J}_{3}$ are mutually disjoint, and $\mathcal{D}=$ $\mathcal{J}_{0} \cup \mathcal{J}_{1} \cup \mathcal{J}_{2} \cup \mathcal{J}_{3}$. Note also that for $0<\kappa \leq 1 / \sqrt{2}$, the equation $2 \gamma(1-\gamma)=\kappa^{2}$ has solutions $\gamma=\gamma_{ \pm}:=\left(1 \pm \sqrt{1-2 \kappa^{2}}\right) / 2$. It is then possible to determine the set $\mathcal{S}_{\kappa, \gamma}$ in terms of $\alpha_{ \pm}, \beta_{ \pm}, \alpha_{0}$ and $\beta_{0}$. Indeed, by elementary computations, we obtain the following.

Proposition 1. (0) If $(\kappa, \gamma) \in \mathcal{J}_{0}$, then $\mathcal{S}_{\kappa, \gamma}$ is empty.
(1) If $(\kappa, \gamma) \in \mathcal{J}_{1}$, then $\mathcal{S}_{\kappa, \gamma}=\left\{\left(\alpha_{+}, \beta_{-}\right)\right\}$.
(2) If $(\kappa, \gamma) \in \mathcal{J}_{2}$, then $\mathcal{S}_{\kappa, \gamma}=\left\{\left(\alpha_{+}, \beta_{-}\right),\left(\alpha_{-}, \beta_{+}\right)\right\}$.
(3) If $(\kappa, \gamma) \in \mathcal{J}_{3}$, then $\mathcal{S}_{\kappa, \gamma}=\left\{\left(\alpha_{0}, \beta_{0}\right)\right\}$.


Figure 1: The sets $\mathcal{J}_{0}, \mathcal{J}_{1}, \mathcal{J}_{2}$ and $\mathcal{J}_{3}$

Remark 1. (1) When $\kappa \leq 0,\left(\alpha_{+}, \beta_{-}\right) \rightarrow(0,1)$ as $\gamma \rightarrow 1+0$. That is, the branch $\left\{\left(\alpha_{+} \varphi_{\omega}, \beta_{-} \varphi_{\omega}\right): \gamma>1\right\}$ of positive solutions of (1.7) bifurcates from the semi-trivial solution $\left(0, \varphi_{\omega}\right)$ at $\gamma=1$.
(2) When $\kappa>0,\left(\alpha_{-}, \beta_{+}\right) \rightarrow(0,1)$ as $\gamma \rightarrow 1-0$. That is, the branch $\left\{\left(\alpha_{-} \varphi_{\omega}, \beta_{+} \varphi_{\omega}\right): \gamma_{m}<\gamma<1\right\}$ of positive solutions of (1.7) bifurcates from the semi-trivial solution $\left(0, \varphi_{\omega}\right)$ at $\gamma=1$, where $\gamma_{m}=\inf \left\{\gamma:(\kappa, \gamma) \in \mathcal{S}_{\kappa, \gamma}\right\}$, and it is given by $\gamma_{m}=0$ if $\kappa>1 / \sqrt{2}$, and $\gamma_{m}=\gamma_{+}$if $0<\kappa \leq 1 / \sqrt{2}$.

We obtain the following stability and instability results of standing waves of (1.1) associated with Proposition 1. Recall that $\varphi_{\omega}$ is the positive radial solution of (1.6).

Theorem 2. Let $N \leq 3$ and $(\kappa, \gamma) \in \mathcal{J}_{1} \cup \mathcal{J}_{2}$. For any $\omega>0$, the standing wave solution $G(\omega t)\left(\alpha_{+} \varphi_{\omega}, \beta_{-} \varphi_{\omega}\right)$ of (1.1) is stable.

Theorem 3. Let $N \leq 3$ and $(\kappa, \gamma) \in \mathcal{J}_{2}$. For any $\omega>0$, the standing wave solution $G(\omega t)\left(\alpha_{-} \varphi_{\omega}, \beta_{+} \varphi_{\omega}\right)$ of (1.1) is unstable.

Remark 2. In this paper, we do not study the stability/instability problem of $G(\omega t)\left(\alpha_{0} \varphi_{\omega}, \beta_{0} \varphi_{\omega}\right)$ for the case $(\kappa, \gamma) \in \mathcal{J}_{3}$.

Remark 3. The result for the case $\kappa=1$ in Theorem 3 is announced in [16] together with an outline of the proof.

We also obtain the stability and instability results of semi-tirivial standing wave at the bifurcation point $\gamma=1$. The results depend on the sign of $\kappa$.

Theorem 4. Let $N \leq 3, \kappa>0$ and $\gamma=1$. For any $\omega>0$, the standing wave solution $\left(0, e^{2 i \omega t} \varphi_{\omega}\right)$ of (1.1) is unstable.

Theorem 5. Let $N \leq 3, \kappa \leq 0$ and $\gamma=1$. For any $\omega>0$, the standing wave solution $\left(0, e^{2 i \omega t} \varphi_{\omega}\right)$ of (1.1) is stable.

Remark 4. The linearized operator $S_{\omega}^{\prime \prime}\left(0, \varphi_{\omega}\right)$ around the semi-trivial standing wave is independent of $\kappa$ (see (2.2) and (2.3) below). Therefore, Theorems 4 and 5 are never obtained from the linearized analysis only. The proof of Theorem 5 relies on the variational method of Shatah [18] and on the characterization of the ground states in Theorem 6 below.

Remark 5. For the case $\gamma=1$, using the notation in Section 2, we have $\mathcal{L}_{R} \vec{v}=\left(L_{1} v_{1}, L_{2} v_{2}\right)$ and $\mathcal{L}_{I} \vec{v}=\left(L_{-1} v_{1}, L_{1} v_{2}\right)$, and the kernel of $S_{\omega}^{\prime \prime}\left(0, \varphi_{\omega}\right)$ contains a nontrivial element $\left(\varphi_{\omega}, 0\right)$ other than the elements $\nabla\left(0, \varphi_{\omega}\right)$ and $J\left(0, \varphi_{\omega}\right)$ naturally coming from the symmetries of $S_{\omega}$ (see (2.4) below).

Next, we consider the ground state problem for (1.7). We define

$$
\begin{equation*}
\kappa_{c}(\gamma)=\frac{1}{2}(\gamma+2) \sqrt{1-\gamma}, \quad 0<\gamma<1 . \tag{1.9}
\end{equation*}
$$

Then, $\kappa_{c}$ is strictly decreasing on the open interval $] 0,1\left[, \kappa_{c}(0)=1\right.$ and $\kappa_{c}(1)=0$. We define a function $\gamma_{c}$ on $] 0,1\left[\right.$ by the inverse function of $\kappa_{c}$. For the ground state problem, it is convenient to divide the parameter domain $\mathcal{D}=\{(\kappa, \gamma): \kappa \in \mathbb{R}, \gamma>0\}$ into the following sets (see Figure 22).

$$
\begin{aligned}
\mathcal{K}_{1}= & \{(\kappa, \gamma): \kappa \leq 0, \gamma>1\} \cup\{(\kappa, \gamma): \kappa \geq 1, \gamma>0\} \\
& \cup\left\{(\kappa, \gamma): 0<\kappa<1, \gamma>\gamma_{c}(\kappa)\right\}, \\
\mathcal{K}_{2}= & \{(\kappa, \gamma): \kappa \leq 0,0<\gamma \leq 1\} \cup\left\{(\kappa, \gamma): 0<\kappa<1,0<\gamma<\gamma_{c}(\kappa)\right\}, \\
\mathcal{K}_{3}= & \left\{(\kappa, \gamma): 0<\kappa<1, \gamma=\gamma_{c}(\kappa)\right\} .
\end{aligned}
$$

Note that $\mathcal{K}_{1}, \mathcal{K}_{2}$ and $\mathcal{K}_{3}$ are mutually disjoint, and $\mathcal{D}=\mathcal{K}_{1} \cup \mathcal{K}_{2} \cup \mathcal{K}_{3}$. Remark also that since $\sqrt{2 \gamma(1-\gamma)}<\kappa_{c}(\gamma)$ for $0<\gamma<1$, we have $\mathcal{J}_{0} \subset \mathcal{K}_{2}$.

Moreover, we define

$$
\begin{aligned}
& \mathcal{G}_{\omega}^{0}=\left\{G(\theta) \tau_{y}\left(0, \varphi_{\omega}\right): \theta \in \mathbb{R}, y \in \mathbb{R}^{N}\right\}, \\
& \mathcal{G}_{\omega}^{1}=\left\{G(\theta) \tau_{y}\left(\alpha_{+} \varphi_{\omega}, \beta_{-} \varphi_{\omega}\right): \theta \in \mathbb{R}, y \in \mathbb{R}^{N}\right\} .
\end{aligned}
$$

Then, the set $\mathcal{G}_{\omega}$ of the ground states for $(1.7)$ is determined as follows.
Theorem 6. Let $N \leq 3$ and $\omega>0$.
(1) If $(\kappa, \gamma) \in \mathcal{K}_{1}$, then $\mathcal{G}_{\omega}=\mathcal{G}_{\omega}^{1}$.
(2) If $(\kappa, \gamma) \in \mathcal{K}_{2}$, then $\mathcal{G}_{\omega}=\mathcal{G}_{\omega}^{0}$.
(3) If $(\kappa, \gamma) \in \mathcal{K}_{3}$, then $\mathcal{G}_{\omega}=\mathcal{G}_{\omega}^{0} \cup \mathcal{G}_{\omega}^{1}$.


Figure 2: The sets $\mathcal{K}_{1}, \mathcal{K}_{2}$ and $\mathcal{K}_{3}$

The rest of the paper is organized as follows. In Section 2, we study some spectral properties of the linearized operators around standing waves, which are needed in Sections 3 and 4. In Section 3, we prove Theorems 2 and 3, while Section 4 is devoted to the proof of Theorem 4. In Section 5, we study the ground state problem for (1.7), and prove Theorem 6. Finally, Theorem 5 is proved as a corollary of Theorem 6.

## 2 Linearized Operators

In this section, we study spectral properties of the linearized operator $S_{\omega}^{\prime \prime}(\Phi)$. Here and hereafter, for $\alpha \geq 0$ and $\beta>0$, we put

$$
\Phi=\left(\alpha \varphi_{\omega}, \beta \varphi_{\omega}\right), \quad \Phi_{1}=\left(-\beta \varphi_{\omega}, \alpha \varphi_{\omega}\right), \quad \Phi_{2}=\left(\alpha \varphi_{\omega}, 2 \beta \varphi_{\omega}\right) .
$$

First, by direct computations, we have

$$
\begin{equation*}
\left\langle S_{\omega}^{\prime \prime}(\Phi) \vec{u}, \vec{u}\right\rangle=\left\langle\mathcal{L}_{R} \Re \vec{u}, \Re \vec{u}\right\rangle+\left\langle\mathcal{L}_{I} \Im \vec{u}, \Im \vec{u}\right\rangle \tag{2.1}
\end{equation*}
$$

for $\vec{u}=\left(u_{1}, u_{2}\right) \in X$, where $\Re \vec{u}=\left(\Re u_{1}, \Re u_{2}\right), \Im \vec{u}=\left(\Im u_{1}, \Im u_{2}\right)$, and

$$
\begin{align*}
& \mathcal{L}_{R}=\left[\begin{array}{cc}
-\Delta+\omega & 0 \\
0 & -\Delta+\omega
\end{array}\right]-\left[\begin{array}{cc}
(2 \alpha+\gamma \beta) \varphi_{\omega} & \gamma \alpha \varphi_{\omega} \\
\gamma \alpha \varphi_{\omega} & 2 \beta \varphi_{\omega}
\end{array}\right],  \tag{2.2}\\
& \mathcal{L}_{I}=\left[\begin{array}{cc}
-\Delta+\omega & 0 \\
0 & -\Delta+\omega
\end{array}\right]-\left[\begin{array}{cc}
(\alpha-\gamma \beta) \varphi_{\omega} & \gamma \alpha \varphi_{\omega} \\
\gamma \alpha \varphi_{\omega} & \beta \varphi_{\omega}
\end{array}\right] . \tag{2.3}
\end{align*}
$$

Since $S_{\omega}^{\prime}\left(G(\theta) \tau_{y} \Phi\right)=0$ for $y \in \mathbb{R}^{N}$ and $\theta \in \mathbb{R}$, we see that

$$
\begin{equation*}
\nabla \Phi \in \operatorname{ker} \mathcal{L}_{R}, \quad \Phi_{2} \in \operatorname{ker} \mathcal{L}_{I} . \tag{2.4}
\end{equation*}
$$

For $a \in \mathbb{R}$, we define $L_{a}$ by

$$
L_{a} v=-\Delta v+\omega v-a \varphi_{\omega} v, \quad v \in H^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)
$$

We recall some known results on $L_{a}$.
Lemma 1. Let $N \leq 3$ and let $\varphi_{\omega}$ be the positive radial solution of (1.6). (1) $L_{2}$ has one negative eigenvalue, ker $L_{2}$ is spanned by $\left\{\nabla \varphi_{\omega}\right\}$, and there exists a constant $c_{1}>0$ such that $\left\langle L_{2} v, v\right\rangle \geq c_{1}\|v\|_{H^{1}}^{2}$ for all $v \in H^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)$ satisfying $\left(v, \varphi_{\omega}\right)_{L^{2}}=0$ and $\left(v, \nabla \varphi_{\omega}\right)_{L^{2}}=0$.
(2) $L_{1}$ is non-negative, ker $L_{1}$ is spanned by $\left\{\varphi_{\omega}\right\}$, and there exists $c_{2}>0$ such that $\left\langle L_{1} v, v\right\rangle \geq c_{2}\|v\|_{H^{1}}^{2}$ for all $v \in H^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)$ satisfying $\left(v, \varphi_{\omega}\right)_{L^{2}}=0$.
(3) If $a<1$, then $L_{a}$ is positive on $H^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)$.
(4) If $1<a<2$, then $\left\langle L_{a} \varphi_{\omega}, \varphi_{\omega}\right\rangle<0$, and there exists $c_{4}>0$ such that $\left\langle L_{a} v, v\right\rangle \geq c_{4}\|v\|_{H^{1}}^{2}$ for all $v \in H^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)$ satisfying $\left(v, \varphi_{\omega}\right)_{L^{2}}=0$.

Proof. Parts (1) and (2) are well-known (see [22]). Note that the quadratic nonlinearity in (1.6) is $L^{2}$-subcritical if and only if $N \leq 3$, and that the assumption $N \leq 3$ is essential for (1). Parts (3) and (4) follow from (1) and (2) immediately.

In the next lemma, we give the diagonalization of $\mathcal{L}_{R}$ and $\mathcal{L}_{I}$.
Lemma 2. By orthogonal matrices

$$
A=\frac{1}{\sqrt{\alpha^{2}+\beta^{2}}}\left[\begin{array}{cc}
\alpha & \beta \\
-\beta & \alpha
\end{array}\right], \quad B=\frac{1}{\sqrt{\alpha^{2}+4 \beta^{2}}}\left[\begin{array}{cc}
\alpha & 2 \beta \\
-2 \beta & \alpha
\end{array}\right]
$$

$\mathcal{L}_{R}$ and $\mathcal{L}_{I}$ are diagonalized as follows:

$$
\mathcal{L}_{R}=A^{*}\left[\begin{array}{cc}
L_{2} & 0 \\
0 & L_{(2-\gamma) \beta}
\end{array}\right] A, \quad \mathcal{L}_{I}=B^{*}\left[\begin{array}{cc}
L_{1} & 0 \\
0 & L_{(1-2 \gamma) \beta}
\end{array}\right] B .
$$

Proof. The computation is straightforward, and we omit the details.
The next three lemmas establish the coercivity properties of the operators $\mathcal{L}_{R}$ and $\mathcal{L}_{I}$. They represent the main results of this section, and are the key points in the proofs of Theorems 2 and 3 .

Lemma 3. If $(2-\gamma) \beta<1$, then there exists a constant $\delta_{1}>0$ such that $\left\langle\mathcal{L}_{R} \vec{v}, \vec{v}\right\rangle \geq \delta_{1}\|\vec{v}\|_{X}^{2}$ for all $\vec{v} \in H^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)^{2}$ satisfying $(\vec{v}, \Phi)_{H}=0$ and $(\vec{v}, \nabla \Phi)_{H}=0$.

Proof. By Lemma 2, we have $\left\langle\mathcal{L}_{R} \vec{v}, \vec{v}\right\rangle=\left\langle L_{2} w_{1}, w_{1}\right\rangle+\left\langle L_{(2-\gamma) \beta} w_{2}, w_{2}\right\rangle$, where $\vec{w}=A \vec{v}$. Since we have

$$
\left(w_{1}, \varphi_{\omega}\right)_{L^{2}}=\frac{(\vec{v}, \Phi)_{H}}{\sqrt{\alpha^{2}+\beta^{2}}}=0, \quad\left(w_{1}, \nabla \varphi_{\omega}\right)_{L^{2}}=\frac{(\vec{v}, \nabla \Phi)_{H}}{\sqrt{\alpha^{2}+\beta^{2}}}=0
$$

it follows from Lemma 1 (1) that $\left\langle L_{2} w_{1}, w_{1}\right\rangle \geq c_{1}\left\|w_{1}\right\|_{H^{1}}^{2}$. Moreover, by the assumption $(2-\gamma) \beta<1$ and by Lemma 1 (3), we have $\left\langle L_{(2-\gamma) \beta} w_{2}, w_{2}\right\rangle \geq$ $c_{3}\left\|w_{2}\right\|_{H^{1}}^{2}$. This completes the proof.

Lemma 4. If $1 \leq(2-\gamma) \beta<2$, then there exists a constant $\delta_{2}>0$ such that $\left\langle\mathcal{L}_{R} \vec{v}, \vec{v}\right\rangle \geq \delta_{2}\|\vec{v}\|_{X}^{2}$ for all $\vec{v} \in H_{\mathrm{rad}}^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)^{2}$ satisfying $(\vec{v}, \Phi)_{H}=0$ and $\left(\vec{v}, \Phi_{1}\right)_{H}=0$, where $\Phi_{1}=\left(-\beta \varphi_{\omega}, \alpha \varphi_{\omega}\right)$.

Proof. By Lemma 2, we have $\left\langle\mathcal{L}_{R} \vec{v}, \vec{v}\right\rangle=\left\langle L_{2} w_{1}, w_{1}\right\rangle+\left\langle L_{(2-\gamma) \beta} w_{2}, w_{2}\right\rangle$, where $\vec{w}=A \vec{v}$. Then we have $\left(w_{1}, \varphi_{\omega}\right)_{L^{2}}=(\vec{v}, \Phi)_{H} / \sqrt{\alpha^{2}+\beta^{2}}=0$. Moreover, since $\varphi_{\omega}$ and $w_{1}$ are radially symmetric, we have $\left(w_{1}, \nabla \varphi_{\omega}\right)_{L^{2}}=0$. Thus, it follows from Lemma 1 (1) that $\left\langle L_{2} w_{1}, w_{1}\right\rangle \geq c_{1}\left\|w_{1}\right\|_{H^{1}}^{2}$. Moreover, since $\left(w_{2}, \varphi_{\omega}\right)_{L^{2}}=\left(\vec{v}, \Phi_{1}\right)_{H} / \sqrt{\alpha^{2}+\beta^{2}}=0$, it follows from the assumption $1 \leq$ $(2-\gamma) \beta<2$ and Lemma 1 (2), (4) that $\left\langle L_{(2-\gamma) \beta} w_{2}, w_{2}\right\rangle \geq c_{2}\left\|w_{2}\right\|_{H^{1}}^{2}$.

Lemma 5. There exists a constant $\delta_{3}>0$ such that $\left\langle\mathcal{L}_{I} \vec{v}, \vec{v}\right\rangle \geq \delta_{3}\|\vec{v}\|_{X}^{2}$ for all $\vec{v} \in H^{1}\left(\mathbb{R}^{N}, \mathbb{R}\right)^{2}$ satisfying $\left(\vec{v}, \Phi_{2}\right)_{H}=0$, where $\Phi_{2}=\left(\alpha \varphi_{\omega}, 2 \beta \varphi_{\omega}\right)$.

Proof. By Lemma 2, we have $\left\langle\mathcal{L}_{I} \vec{v}, \vec{v}\right\rangle=\left\langle L_{1} w_{1}, w_{1}\right\rangle+\left\langle L_{(1-2 \gamma) \beta} w_{2}, w_{2}\right\rangle$, where $\vec{w}=B \vec{v}$. Since $\left(w_{1}, \varphi_{\omega}\right)_{L^{2}}=\left(\vec{v}, \Phi_{2}\right)_{H} / \sqrt{\alpha^{2}+4 \beta^{2}}=0$, Lemma 1 (2) implies $\left\langle L_{1} w_{1}, w_{1}\right\rangle \geq c_{2}\left\|w_{1}\right\|_{H^{1}}^{2}$. Moreover, since $(1-2 \gamma) \beta<1$, it follows from Lemma 1 (3) that $\left\langle L_{(1-2 \gamma) \beta} w_{2}, w_{2}\right\rangle \geq c_{3}\left\|w_{2}\right\|_{H^{1}}^{2}$.

The last two lemmas of this section make connections between parameters $(\kappa, \gamma)$ and the criteria used in Lemma 3, 4 and 5 on $\beta$.

Lemma 6. Let $(\kappa, \gamma) \in \mathcal{J}_{1} \cup \mathcal{J}_{2}$. Then, $(2-\gamma) \beta_{-}<1$ and $(1-2 \gamma) \beta_{-}<1$.
Proof. We put $D=\kappa^{2}+2 \gamma(\gamma-1)$. By the second equation of (1.8), we have $0<\beta_{-}<1$. Thus, we have $(1-2 \gamma) \beta_{-}<\beta_{-}<1$. If $\gamma>1$, then $(2-\gamma) \beta_{-}<\beta_{-}<1$. While, if $0<\gamma \leq 1$, then $\kappa>0, D>0$ and $(2-\gamma) \beta_{-}<(2-\gamma)\left(\kappa^{2}+\gamma^{2}\right) /\left(2 \kappa^{2}+\gamma^{3}\right)<1$. Note that the last inequality is equivalent to $D>0$.

Lemma 7. Let $(\kappa, \gamma) \in \mathcal{J}_{2}$. Then, $1<(2-\gamma) \beta_{+}<2$ and $(1-2 \gamma) \beta_{+}<1$.
Proof. We put $D=\kappa^{2}+2 \gamma(\gamma-1)$. Since $0<\beta_{+}<1$, we have $(2-\gamma) \beta_{+}<$ $2 \beta_{+}<2$ and $(1-2 \gamma) \beta_{+}<\beta_{+}<1$. Next, we see that $(2-\gamma) \beta_{+}>1$ is equivalent to $(2-\gamma) \kappa>\gamma \sqrt{D}$. Since $0<\gamma<1$ and $\kappa>0$, we have $\gamma \sqrt{D}<\gamma \kappa<(2-\gamma) \kappa$.

Remark 6. When $(\kappa, \gamma) \in \mathcal{J}_{3}$, we have $D=\kappa^{2}+2 \gamma(\gamma-1)=0,(2-\gamma) \beta_{0}=1$ and $(1-2 \gamma) \beta_{0}<1$.

## 3 Proofs of Theorems 2 and 3

In this section we prove Theorems 2 and 3 using the results of Section 2 and the following propositions. Proposition 2 follows from Theorem 3.4 of Grillakis, Shatah and Strauss [11] (see also [23] and [7, Section 3]). While, Proposition 3 follows from Theorem 1 of [16] (see also [11, 15, 19]).

Proposition 2. Let $\vec{\phi} \in \mathcal{A}_{\omega}$. Assume that there exists a constant $\delta>0$ such that $\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{w}, \vec{w}\right\rangle \geq \delta\|\vec{w}\|_{X}^{2}$ for all $\vec{w} \in X$ satisfying $(\vec{\phi}, \vec{w})_{H}=(J \vec{\phi}, \vec{w})_{H}=0$ and $(\nabla \vec{\phi}, \vec{w})_{H}=0$. Then the standing wave solution $G(\omega t) \vec{\phi}$ of (1.1) is stable.

Proposition 3. Let $\vec{\phi} \in \mathcal{A}_{\omega}$ be radially symmetric. Assume that there exist $\vec{\psi} \in X_{\mathrm{rad}}$ and a constant $\delta>0$ such that $\|\vec{\psi}\|_{H}=1,(\vec{\psi}, \vec{\phi})_{H}=(\vec{\psi}, J \vec{\phi})_{H}=0$, $\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{\psi}, \vec{\psi}\right\rangle \leq 0$, and $\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{w}, \vec{w}\right\rangle \geq \delta\|\vec{w}\|_{X}^{2}$ for all $\vec{w} \in X_{\mathrm{rad}}$ satisfying $(\vec{\phi}, \vec{w})_{H}=(J \vec{\phi}, \vec{w})_{H}=(\vec{\psi}, \vec{w})_{H}=0$. Then the standing wave solution $G(\omega t) \vec{\phi}$ of (1.1) is unstable.

Proof of Theorem 2. For $(\kappa, \gamma) \in \mathcal{J}_{1} \cup \mathcal{J}_{2}$, let $(\alpha, \beta)=\left(\alpha_{+}, \beta_{-}\right)$. Let $\vec{w} \in X$ satisfy $(\Phi, \vec{w})_{H}=(J \Phi, \vec{w})_{H}=0$ and $(\nabla \Phi, \vec{w})_{H}=0$. By (2.1), we have

$$
\left\langle S_{\omega}^{\prime \prime}(\Phi) \vec{w}, \vec{w}\right\rangle=\left\langle\mathcal{L}_{R} \Re \vec{w}, \Re \vec{w}\right\rangle+\left\langle\mathcal{L}_{I} \Im \vec{w}, \Im \vec{w}\right\rangle .
$$

Since $(\Phi, \Re \vec{w})_{H}=(\Phi, \vec{w})_{H}=0$ and $(\nabla \Phi, \Re \vec{w})_{H}=(\nabla \Phi, \vec{w})_{H}=0$, it follows from Lemmas 6 and 3 that $\left\langle\mathcal{L}_{R} \Re \vec{w}, \Re \vec{w}\right\rangle \geq \delta_{1}\|\Re \vec{w}\|_{X}^{2}$. While, since $\left(\Im \vec{w}, \Phi_{2}\right)_{H}=(J \Phi, \vec{w})_{H}=0$, Lemma 5 implies $\left\langle\mathcal{L}_{I} \Im \vec{w}, \Im \vec{w}\right\rangle \geq \delta_{3}\|\Im \vec{w}\|_{X}^{2}$. Therefore, Theorem 2 follows from Proposition 2 .

Proof of Theorem 3. For $(\kappa, \gamma) \in \mathcal{J}_{2}$, let $(\alpha, \beta)=\left(\alpha_{-}, \beta_{+}\right)$. We take $\vec{\psi}=$ $\Phi_{1} /\left\|\Phi_{1}\right\|_{H}$. Then we have $\|\vec{\psi}\|_{H}=1,(\vec{\psi}, \Phi)_{H}=0$ and $(\vec{\psi}, J \Phi)_{H}=0$. Moreover, by Lemma 7 and Lemma 1 (4), we have

$$
\left\langle S_{\omega}^{\prime \prime}(\Phi) \vec{\psi}, \vec{\psi}\right\rangle=\left\langle\mathcal{L}_{R} \vec{\psi}, \vec{\psi}\right\rangle=\left\langle L_{(2-\gamma) \beta} \varphi_{\omega}, \varphi_{\omega}\right\rangle /\left\|\varphi_{\omega}\right\|_{L^{2}}^{2}<0 .
$$

Finally, let $\vec{w} \in X_{\mathrm{rad}}$ satisfy $(\Phi, \vec{w})_{H}=(J \Phi, \vec{w})_{H}=(\vec{\psi}, \vec{w})_{H}=0$. Since $(\Phi, \Re \vec{w})_{H}=(\Phi, \vec{w})_{H}=0$ and $\left(\Phi_{1}, \Re \vec{w}\right)_{H}=\left(\Phi_{1}, \vec{w}\right)_{H}=0$, by Lemmas 7 and 4. we have $\left\langle\mathcal{L}_{R} \Re \vec{w}, \Re \vec{w}\right\rangle \geq \delta_{2}\|\Re \vec{w}\|_{X}^{2}$. While, since $\left(\Im \vec{w}, \Phi_{2}\right)_{H}=(J \Phi, \vec{w})_{H}=$ 0 , by Lemma 5, we have $\left\langle\mathcal{L}_{I} \Im \vec{w}, \Im \vec{w}\right\rangle \geq \delta_{3}\|\Im \vec{w}\|_{X}^{2}$. Thus, by (2.1), we have $\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{w}, \vec{w}\right\rangle \geq \delta\|\vec{w}\|_{X}^{2}$, and Theorem 3 follows from Proposition 3.

## 4 Proof of Theorem 4

We introduce the following Proposition 4 to prove Theorem 4. It is a modification of Theorem 2 of [16]. In what follows, $\operatorname{sgn}(\mu)$ denotes the sign of any real $\mu$.

Proposition 4. Let $\vec{\phi} \in \mathcal{A}_{\omega}$ be radially symmetric. Assume that there exist $\vec{\psi} \in X_{\mathrm{rad}}$ such that
(i) $\|\vec{\psi}\|_{H}=1,(\vec{\psi}, \vec{\phi})_{H}=0,(\vec{\psi}, J \vec{\phi})_{H}=(\vec{\psi}, J \vec{\phi})_{X}=0, S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{\psi}=\overrightarrow{0}$,
(ii) there exists a positive constant $k_{0}$ such that $\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{w}, \vec{w}\right\rangle \geq k_{0}\|\vec{w}\|_{X}^{2}$
for all $\vec{w} \in X_{\mathrm{rad}}$ satisfying $(\vec{\phi}, \vec{w})_{H}=(J \vec{\phi}, \vec{w})_{H}=(\vec{\psi}, \vec{w})_{H}=0$,
(iii) there exist positive constants $k_{1}, k_{2}$ and $\varepsilon$ such that

$$
\operatorname{sgn}(\lambda) \cdot\left\langle S_{\omega}^{\prime}(\vec{\phi}+\lambda \vec{\psi}+\vec{z}), \vec{\psi}\right\rangle \leq-k_{1} \lambda^{2}+k_{2}\|\vec{z}\|_{X}^{2}+o\left(\lambda^{2}+\|\vec{z}\|_{X}^{2}\right)
$$

for all $\lambda \in \mathbb{R}$ and $\vec{z} \in X_{\text {rad }}$ satisfying $|\lambda|+\|\vec{z}\|_{X}<\varepsilon$.
Then the standing wave solution $G(\omega t) \vec{\phi}$ of (1.1) is unstable.
We first prove Theorem 4 using Proposition 4 .
Proof of Theorem 4. The proof consists of verifying the assumptions (i), (ii), (iii) of Proposition 4. Let $(\alpha, \beta)=(0,1)$ and $\Phi=\left(0, \varphi_{\omega}\right)$. We take

$$
\vec{\psi}=\left(\psi_{1}, \psi_{2}\right)=\left(\varphi_{\omega}, 0\right) /\left\|\varphi_{\omega}\right\|_{L^{2}} .
$$

Then, $\|\vec{\psi}\|_{H}=1,(\vec{\psi}, \Phi)_{H}=0,(\vec{\psi}, J \Phi)_{H}=(\vec{\psi}, J \Phi)_{X}=0$, and

$$
S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{\psi}=\left(L_{1} \varphi_{\omega}, 0\right) /\left\|\varphi_{\omega}\right\|_{L^{2}}=(0,0) .
$$

Thus, (i) is satisfied. The assumption (ii) is proved in the same way as the proof of Theorem 3. Finally, we prove (iii). Let $\lambda \in \mathbb{R}$ and $\vec{z}=\left(z_{1}, z_{2}\right) \in X_{\mathrm{rad}}$, and put $\vec{v}=\left(v_{1}, v_{2}\right)=\lambda \vec{\psi}+\vec{z}$. Then, we have

$$
v_{1}=\lambda \psi_{1}+z_{1}, \quad v_{2}=z_{2}, \quad \psi_{1}=\varphi_{\omega} /\left\|\varphi_{\omega}\right\|_{L^{2}},
$$

and

$$
\begin{aligned}
& \left\|\varphi_{\omega}\right\|_{L^{2}}\left\langle S_{\omega}^{\prime}(\Phi+\vec{v}), \vec{\psi}\right\rangle \\
& =\Re \int_{\mathbb{R}^{N}}\left\{\nabla v_{1} \cdot \nabla \varphi_{\omega}+\omega v_{1} \varphi_{\omega}-\kappa\left|v_{1}\right| v_{1} \varphi_{\omega}-v_{1}\left(\varphi_{\omega}+\overline{v_{2}}\right) \varphi_{\omega}\right\} d x \\
& =\Re \int_{\mathbb{R}^{N}}\left\{v_{1}\left(-\Delta \varphi_{\omega}+\omega \varphi_{\omega}-\varphi_{\omega}^{2}\right)-\kappa\left|v_{1}\right| v_{1} \varphi_{\omega}-v_{1} \overline{v_{2}} \varphi_{\omega}\right\} d x \\
& =-\kappa \Re \int_{\mathbb{R}^{N}}\left|v_{1}\right| v_{1} \varphi_{\omega} d x-\Re \int_{\mathbb{R}^{N}} v_{1} \overline{v_{2}} \varphi_{\omega} d x .
\end{aligned}
$$

Thus, we have

$$
\begin{equation*}
\left\langle S_{\omega}^{\prime}(\Phi+\vec{v}), \vec{\psi}\right\rangle=-\kappa \Re \int_{\mathbb{R}^{N}}\left|v_{1}\right| v_{1} \psi_{1} d x-\Re \int_{\mathbb{R}^{N}} v_{1} \overline{v_{2}} \psi_{1} d x . \tag{4.1}
\end{equation*}
$$

Here, we have

$$
\operatorname{sgn}(\lambda) \cdot \kappa \Re \int_{\mathbb{R}^{N}}\left|\lambda \psi_{1}\right| \lambda \psi_{1} \psi_{1} d x=C_{0} \lambda^{2}, \quad \text { where } C_{0}:=\kappa\left\|\varphi_{\omega}\right\|_{L^{3}}^{3} /\left\|\varphi_{\omega}\right\|_{L^{2}}^{3},
$$

and the first term of the right hand side of (4.1) is estimated as follows.

$$
\begin{aligned}
& \left|\operatorname{sgn}(\lambda) \cdot \kappa \Re \int_{\mathbb{R}^{N}}\right| v_{1}\left|v_{1} \psi_{1} d x-C_{0} \lambda^{2}\right| \leq \kappa \int_{\mathbb{R}^{N}}\left|v_{1}\right| v_{1}-\left|\lambda \psi_{1}\right| \lambda \psi_{1} \mid \psi_{1} d x \\
& \leq C \int_{\mathbb{R}^{N}}\left(\left|v_{1}\right|+\left|\lambda \psi_{1}\right|\right)\left|v_{1}-\lambda \psi_{1}\right| \psi_{1} d x \leq C \int_{\mathbb{R}^{N}}\left(\left|\lambda \psi_{1}\right|+\left|z_{1}\right|\right)\left|z_{1}\right| \psi_{1} d x \\
& \leq C|\lambda|\left\|z_{1}\right\|_{L^{3}}\left\|\psi_{1}\right\|_{L^{3}}^{2}+C\left\|z_{1}\right\|_{L^{3}}^{2}\left\|\psi_{1}\right\|_{L^{3}} \leq C_{0} \lambda^{2} / 4+C_{1}\left\|z_{1}\right\|_{H^{1}}^{2}
\end{aligned}
$$

for some constant $C_{1}$ depending on $\varphi_{\omega}$. Here, in the last inequality, we used the inequality of the type $2 a b \leq \varepsilon^{2} a^{2}+b^{2} / \varepsilon^{2}$. While, the second term of the right hand side of 4.1) is estimated as follows.

$$
\begin{aligned}
& \left|\Re \int_{\mathbb{R}^{N}} v_{1} \overline{v_{2}} \psi_{1} d x\right| \leq|\lambda|\left\|z_{2}\right\|_{L^{3}}\left\|\psi_{1}\right\|_{L^{3}}^{2}+\left\|z_{1}\right\|_{L^{3}}\left\|z_{2}\right\|_{L^{3}}\left\|\psi_{1}\right\|_{L^{3}} \\
& \leq C_{0} \lambda^{2} / 4+C_{2}\left\|z_{2}\right\|_{H^{1}}^{2}+C_{3}\left\|z_{1}\right\|_{H^{1}}\left\|z_{2}\right\|_{H^{1}}
\end{aligned}
$$

for some positive constants $C_{2}$ and $C_{3}$. Thus, we have

$$
\operatorname{sgn} \lambda \cdot\left\langle S_{\omega}^{\prime}(\Phi+\lambda \vec{\psi}+\vec{z}), \vec{\psi}\right\rangle \leq-C_{0} \lambda^{2} / 2+C_{4}\|\vec{z}\|_{X}^{2}
$$

for some constant $C_{4}>0$. This completes the proof.

In the rest of this section, we give the proof of Proposition 4 by modifying the proof of Theorem 2 of [16]. We define

$$
\mathcal{N}_{\varepsilon}(\vec{\phi})=\left\{\vec{u} \in X_{\mathrm{rad}}: \inf _{\theta \in \mathbb{R}}\|G(\theta) \vec{u}-\vec{\phi}\|_{X}<\varepsilon\right\},
$$

and the identification operator $I: X \rightarrow X^{*}$ by

$$
\langle I \vec{u}, \vec{v}\rangle=(\vec{u}, \vec{v})_{H}, \quad \vec{u}, \vec{v} \in X .
$$

Lemma 8. There exist $\varepsilon>0$ and a $C^{2} \operatorname{map} \Theta: \mathcal{N}_{\varepsilon}(\vec{\phi}) \rightarrow \mathbb{R} / 2 \pi \mathbb{Z}$ such that

$$
\begin{align*}
& \|G(\Theta(\vec{u})) \vec{u}-\vec{\phi}\|_{X} \leq\|G(\theta) \vec{u}-\vec{\phi}\|_{X}, \\
& (G(\Theta(\vec{u})) \vec{u}, J \vec{\phi})_{X}=0, \quad \Theta(G(\theta) \vec{u})=\Theta(\vec{u})-\theta, \\
& I^{-1} \Theta^{\prime}(\vec{u})=\frac{J G(-\Theta(\vec{u}))(1-\Delta) \vec{\phi}}{\left(G(\Theta(\vec{u})) \vec{u}, J^{2} \vec{\phi}\right)_{X}} \tag{4.2}
\end{align*}
$$

for all $\vec{u} \in \mathcal{N}_{\varepsilon}(\vec{\phi})$ and $\theta \in \mathbb{R} / 2 \pi \mathbb{Z}$.
Proof. See Lemma 3.2 of [11]. Note that $\vec{\phi} \in H^{3}\left(\mathbb{R}^{N}\right)^{2}$ by the elliptic regularity for (1.7).

We put $M(\vec{u})=G(\Theta(\vec{u})) \vec{u}$. Then we have $M(\vec{\phi})=\vec{\phi}$ and $M(G(\theta) \vec{u})=$ $M(\vec{u})$ for $\vec{u} \in \mathcal{N}_{\varepsilon}(\vec{\phi})$ and $\theta \in \mathbb{R}$. We define $\mathcal{A}$ and $\Lambda$ by

$$
\begin{equation*}
\mathcal{A}(\vec{u})=\left(M(\vec{u}), J^{-1} \vec{\psi}\right)_{H}, \quad \Lambda(\vec{u})=(M(\vec{u}), \vec{\psi})_{H} \tag{4.3}
\end{equation*}
$$

for $\vec{u} \in \mathcal{N}_{\varepsilon}(\vec{\phi})$. Then we have

$$
\begin{align*}
& J I^{-1} \mathcal{A}^{\prime}(\vec{u})=G(-\Theta(\vec{u})) \vec{\psi}-\Lambda(\vec{u}) J I^{-1} \Theta^{\prime}(\vec{u}),  \tag{4.4}\\
& 0=\left.\frac{d}{d \theta} \mathcal{A}(G(\theta) \vec{u})\right|_{\theta=0}=\left\langle\mathcal{A}^{\prime}(\vec{u}), J \vec{u}\right\rangle=-\left\langle I \vec{u}, J I^{-1} \mathcal{A}^{\prime}(\vec{u})\right\rangle . \tag{4.5}
\end{align*}
$$

We define $\mathcal{P}$ by

$$
\mathcal{P}(\vec{u})=\left\langle E^{\prime}(\vec{u}), J I^{-1} \mathcal{A}^{\prime}(\vec{u})\right\rangle
$$

for $\vec{u} \in \mathcal{N}_{\varepsilon}(\vec{\phi})$. Note that by (4.2), (4.4) and (4.5), we have

$$
\begin{align*}
\mathcal{P}(\vec{u}) & =\left\langle S_{\omega}^{\prime}(\vec{u}), J I^{-1} \mathcal{A}^{\prime}(\vec{u})\right\rangle \\
& =\left\langle S_{\omega}^{\prime}(M(\vec{u})), \vec{\psi}\right\rangle-\frac{\Lambda(\vec{u})}{\left(M(\vec{u}), J^{2} \vec{\phi}\right)_{X}}\left\langle S_{\omega}^{\prime}(M(\vec{u})), J^{2}(1-\Delta) \vec{\phi}\right\rangle . \tag{4.6}
\end{align*}
$$

Lemma 9. Let $\mathcal{I}$ be an interval of $\mathbb{R}$. Let $\vec{u} \in C(\mathcal{I}, X) \cap C^{1}\left(\mathcal{I}, X^{*}\right)$ be a solution of (1.1), and assume that $\vec{u}(t) \in \mathcal{N}_{\varepsilon}(\vec{\phi})$ for all $t \in \mathcal{I}$. Then

$$
\frac{d}{d t} \mathcal{A}(\vec{u}(t))=\mathcal{P}(\vec{u}(t)), \quad t \in \mathcal{I} .
$$

Proof. See Lemma 4.6 of [11] and Lemma 2 of [16].
Lemma 10. There exist positive constants $k^{*}$ and $\varepsilon_{0}$ such that

$$
E(\vec{u}) \geq E(\vec{\phi})+k^{*} \operatorname{sgn} \Lambda(\vec{u}) \cdot \mathcal{P}(\vec{u})
$$

for all $\vec{u} \in \mathcal{N}_{\varepsilon_{0}}(\vec{\phi})$ satisfying $Q(\vec{u})=Q(\vec{\phi})$.
Proof. We put $\vec{v}=M(\vec{u})-\vec{\phi}$, and decompose $\vec{v}$ as

$$
\vec{v}=a \vec{\phi}+b J \vec{\phi}+c \vec{\psi}+\vec{w},
$$

where $a, b, c \in \mathbb{R}$, and $\vec{w} \in X_{\text {rad }}$ satisfies $(\vec{\phi}, \vec{w})_{H}=(J \vec{\phi}, \vec{w})_{H}=(\vec{\psi}, \vec{w})_{H}=0$. Since $Q(\vec{\phi})=Q(\vec{u})=Q(M(\vec{u}))=Q(\vec{\phi})+(\vec{\phi}, \vec{v})_{H}+Q(\vec{v})$ and $(\vec{\phi}, \vec{v})_{H}=$ $a\|\vec{\phi}\|_{H}^{2}$, we have $a=O\left(\|\vec{v}\|_{X}^{2}\right)$. Moreover, by Lemma 8 and by the assumption (i) of Proposition 4, we have $(M(\vec{u}), J \vec{\phi})_{X}=0$ and $(J \vec{\phi}, \vec{\psi})_{X}=0$. Thus, we have $0=(\vec{v}, J \vec{\phi})_{X}=b\|J \vec{\phi}\|_{X}^{2}+(\vec{w}, J \vec{\phi})_{X},|b|\|J \vec{\phi}\|_{X} \leq\|\vec{w}\|_{X}$ and

$$
\begin{equation*}
\|\vec{v}\|_{X} \leq|c|\|\vec{\psi}\|_{X}+2\|\vec{w}\|_{X}+O\left(\|\vec{v}\|_{X}^{2}\right) \tag{4.7}
\end{equation*}
$$

Since $S_{\omega}^{\prime}(\vec{\phi})=0$ and $Q(\vec{u})=Q(\vec{\phi})$, by the Taylor expansion, we have

$$
\begin{equation*}
E(\vec{u})-E(\vec{\phi})=S_{\omega}(M(\vec{u}))-S_{\omega}(\vec{\phi})=\frac{1}{2}\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{v}, \vec{v}\right\rangle+o\left(\|\vec{v}\|_{X}^{2}\right) . \tag{4.8}
\end{equation*}
$$

Here, since $a=O\left(\|\vec{v}\|_{X}^{2}\right)$ and $S_{\omega}^{\prime \prime}(\vec{\phi})(J \vec{\phi})=S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{\psi}=\overrightarrow{0}$, by the assumption (ii) of Proposition 4, we have

$$
\begin{align*}
& E(\vec{u})-E(\vec{\phi})=\frac{1}{2}\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{v}, \vec{v}\right\rangle+o\left(\|\vec{v}\|_{X}^{2}\right) \\
& =\frac{1}{2}\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{w}, \vec{w}\right\rangle+o\left(\|\vec{v}\|_{X}^{2}\right) \geq \frac{k_{0}}{2}\|\vec{w}\|_{X}^{2}-o\left(\|\vec{v}\|_{X}^{2}\right) . \tag{4.9}
\end{align*}
$$

On the other hand, we have $c=(\vec{v}, \vec{\psi})_{H}=\Lambda(\vec{u})=O\left(\|\vec{v}\|_{X}\right)$,

$$
S_{\omega}^{\prime}(\vec{\phi}+\vec{v})=S_{\omega}^{\prime}(\vec{\phi})+S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{v}+o\left(\|\vec{v}\|_{X}\right)=S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{w}+o\left(\|\vec{v}\|_{X}\right)
$$

and $\left(M(\vec{u}), J^{2} \vec{\phi}\right)_{X}=\left(\vec{\phi}, J^{2} \vec{\phi}\right)_{X}+O\left(\|\vec{v}\|_{X}\right)$. Thus, by (4.6) we have

$$
\mathcal{P}(\vec{u})=\left\langle S_{\omega}^{\prime}(\vec{\phi}+\vec{v}), \vec{\psi}\right\rangle+\frac{c}{\|J \vec{\phi}\|_{X}^{2}}\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{w}, J^{2}(1-\Delta) \vec{\phi}\right\rangle+o\left(\|\vec{v}\|_{X}^{2}\right) .
$$

Here, by the assumption (iii) of Proposition 4, we have

$$
\begin{aligned}
& \operatorname{sgn}(c) \cdot\left\langle S_{\omega}^{\prime}(\vec{\phi}+\vec{v}), \vec{\psi}\right\rangle \\
& \leq-k_{1} c^{2}+k_{2}\|a \vec{\phi}+b J \vec{\phi}+\vec{w}\|_{X}^{2}+o\left(c^{2}+\|a \vec{\phi}+b J \vec{\phi}+\vec{w}\|_{X}^{2}\right) \\
& \leq-k_{1} c^{2}+k_{3}\|\vec{w}\|_{X}^{2}+o\left(\|\vec{v}\|_{X}^{2}\right) .
\end{aligned}
$$

Moreover, we have

$$
\left|\frac{c}{\|J \vec{\phi}\|_{X}^{2}}\left\langle S_{\omega}^{\prime \prime}(\vec{\phi}) \vec{w}, J^{2}(1-\Delta) \vec{\phi}\right\rangle\right| \leq k|c|\|\vec{w}\|_{X} \leq \frac{k_{1}}{2} c^{2}+k_{4}\|\vec{w}\|_{X}^{2} .
$$

Thus, we have

$$
\begin{equation*}
-\operatorname{sgn} \Lambda(\vec{u}) \cdot \mathcal{P}(\vec{u}) \geq \frac{k_{1}}{2} c^{2}-k_{5}\|\vec{w}\|_{X}^{2}-o\left(\|\vec{v}\|_{X}^{2}\right) \tag{4.10}
\end{equation*}
$$

with some constant $k_{5}>0$. By (4.9) and (4.10), we have

$$
\begin{equation*}
E(\vec{u})-E(\vec{\phi})-k^{*} \operatorname{sgn} \Lambda(\vec{u}) \cdot \mathcal{P}(\vec{u}) \geq k_{6} c^{2}+k_{7}\|\vec{w}\|_{X}^{2}-o\left(\|\vec{v}\|_{X}^{2}\right), \tag{4.11}
\end{equation*}
$$

where we put $k^{*}=k_{0} /\left(4 k_{5}\right), k_{6}=k^{*} k_{1} / 2$ and $k_{7}=k_{0} / 4$. Finally, since $\|\vec{v}\|_{X}=\left\|M(\vec{u})-\varphi_{\omega}\right\|_{X}<\varepsilon_{0}$, it follows from (4.7) that the right hand side of (4.11) is non-negative, if $\varepsilon_{0}$ is sufficiently small. This completes the proof.

Lemma 11. There exist $\lambda_{1}>0$ and a continuous curve $\left(-\lambda_{1}, \lambda_{1}\right) \ni \lambda \mapsto$ $\vec{\phi}_{\lambda} \in X_{\mathrm{rad}}$ such that $\vec{\phi}_{0}=\vec{\phi}$ and

$$
E\left(\vec{\phi}_{\lambda}\right)<E(\vec{\phi}), \quad Q\left(\vec{\phi}_{\lambda}\right)=Q(\vec{\phi}), \quad \lambda \mathcal{P}\left(\vec{\phi}_{\lambda}\right)<0
$$

for $0<|\lambda|<\lambda_{1}$.
Proof. For $\lambda$ close to 0 , we define

$$
\vec{\phi}_{\lambda}=\vec{\phi}+\lambda \vec{\psi}+\sigma(\lambda) \vec{\phi}, \quad \sigma(\lambda)=\left(1-\frac{Q(\vec{\psi})}{Q(\vec{\phi})} \lambda^{2}\right)^{1 / 2}-1
$$

Then, we have $Q\left(\vec{\phi}_{\lambda}\right)=Q(\vec{\phi}), \sigma(\lambda)=O\left(\lambda^{2}\right), \sigma^{\prime}(\lambda)=O(\lambda)$ and

$$
S_{\omega}\left(\vec{\phi}_{\lambda}\right)-S_{\omega}(\vec{\phi})=\int_{0}^{\lambda} \frac{d}{d s} S_{\omega}\left(\vec{\phi}_{s}\right) d s=\int_{0}^{\lambda}\left\langle S_{\omega}^{\prime}\left(\vec{\phi}_{s}\right), \vec{\psi}+\sigma^{\prime}(s) \vec{\phi}\right\rangle d s .
$$

Here, by the assumption (iii) of Proposition 4, we have

$$
\operatorname{sgn}(s) \cdot\left\langle S_{\omega}^{\prime}\left(\vec{\phi}_{s}\right), \vec{\psi}\right\rangle \leq-k_{1} s^{2}+o\left(s^{2}\right) .
$$

Moreover, since $S_{\omega}^{\prime}\left(\vec{\phi}_{s}\right)=S_{\omega}^{\prime}(\vec{\phi})+S_{\omega}^{\prime \prime}(\vec{\phi})(s \vec{\psi}+\sigma(s) \vec{\phi})+o(s)=o(s)$, we have $\left\langle S_{\omega}^{\prime}\left(\vec{\phi}_{s}\right), \sigma^{\prime}(s) \vec{\phi}\right\rangle=o\left(s^{2}\right)$. Thus, we have $S_{\omega}\left(\vec{\phi}_{\lambda}\right)-S_{\omega}(\vec{\phi}) \leq-k_{1}|\lambda|^{3} / 3+o\left(\lambda^{3}\right)$. Finally, by (4.10), we have $\lambda \mathcal{P}\left(\vec{\phi}_{\lambda}\right) \leq-k_{1}|\lambda|^{3} / 2+o\left(\lambda^{3}\right)$.

Proof of Proposition 4. Suppose that $G(\omega t) \vec{\phi}$ is stable. For $\lambda$ close to 0 , let $\vec{\phi}_{\lambda} \in X_{\text {rad }}$ be the function given in Lemma 11, and let $\vec{u}_{\lambda}(t)$ be the solution of (1.1) with $\vec{u}_{\lambda}(0)=\vec{\phi}_{\lambda}$. Then, there exists $\lambda_{0}>0$ such that if $|\lambda|<\lambda_{0}$, then $\vec{u}_{\lambda}(t) \in \mathcal{N}_{\varepsilon_{0}}(\vec{\phi})$ for all $t \geq 0$, where $\varepsilon_{0}$ is the positive constant given in Lemma 10. Moreover, by the definition (4.3), there exist positive constants $C_{1}$ and $C_{2}$ such that $|\mathcal{A}(\vec{u})| \leq C_{1}$ and $|\Lambda(\vec{u})| \leq C_{2}$ for all $\vec{u} \in \mathcal{N}_{\varepsilon_{0}}(\vec{\phi})$. Let $-\lambda_{0}<\lambda<0$ and put $\delta_{\lambda}=E(\vec{\phi})-E\left(\vec{\phi}_{\lambda}\right)$. Since $\mathcal{P}\left(\vec{\phi}_{\lambda}\right)>0$ and $t \mapsto \mathcal{P}\left(\vec{u}_{\lambda}(t)\right)$ is continuous, by Lemma 10 and by the conservation laws of $E$ and $Q$, we see that $\mathcal{P}\left(\vec{u}_{\lambda}(t)\right)>0$ for all $t \geq 0$ and

$$
\delta_{\lambda}=E(\vec{\phi})-E\left(\vec{u}_{\lambda}(t)\right) \leq-k^{*} \operatorname{sgn} \Lambda\left(\vec{u}_{\lambda}(t)\right) \cdot \mathcal{P}\left(\vec{u}_{\lambda}(t)\right) \leq k^{*} C_{2} \mathcal{P}\left(\vec{u}_{\lambda}(t)\right)
$$

for all $t \geq 0$. Moreover, by Lemma 9, we have

$$
\frac{d}{d t} \mathcal{A}\left(\vec{u}_{\lambda}(t)\right)=\mathcal{P}\left(\vec{u}_{\lambda}(t)\right) \geq \frac{\delta_{\lambda}}{k^{*} C_{2}}
$$

for all $t \geq 0$, which implies that $\mathcal{A}\left(\vec{u}_{\lambda}(t)\right) \rightarrow \infty$ as $t \rightarrow \infty$. This contradicts the fact that $\left|\mathcal{A}\left(\vec{u}_{\lambda}(t)\right)\right| \leq C_{1}$ for all $t \geq 0$. Hence, $G(\omega t) \vec{\phi}$ is unstable.

## 5 Ground States

### 5.1 Existence and Stability of Ground States

In this subsection, we briefly recall the existence and stability of ground states for (1.7). We define

$$
\begin{aligned}
& \|\vec{u}\|_{X_{\omega}}^{2}=\|\nabla \vec{u}\|_{H}^{2}+\omega\|\vec{u}\|_{H}^{2}, \\
& V(\vec{u})=\kappa\left\|u_{1}\right\|_{L^{3}}^{3}+\left\|u_{2}\right\|_{L^{3}}^{3}+\frac{3}{2} \gamma \Re \int_{\mathbb{R}^{N}} u_{1}^{2} \overline{u_{2}} d x, \\
& K_{\omega}(\vec{u})=\|\vec{u}\|_{X_{\omega}}^{2}-V(\vec{u})
\end{aligned}
$$

for $\vec{u} \in X$. Then the action $S_{\omega}$ associated with (1.7) is written as

$$
S_{\omega}(\vec{u})=\frac{1}{2}\|\vec{u}\|_{X_{\omega}}^{2}-\frac{1}{3} V(\vec{u}) .
$$

Remark that for $\vec{u} \in X$ satisfying $K_{\omega}(\vec{u})=0$, one has

$$
\begin{equation*}
S_{\omega}(\vec{u})=\frac{1}{6}\|\vec{u}\|_{X_{\omega}}^{2} . \tag{5.1}
\end{equation*}
$$

Moreover, we define

$$
\begin{aligned}
& \mu(\omega)=\inf \left\{S_{\omega}(\vec{u}): \vec{u} \in X, K_{\omega}(\vec{u})=0, \vec{u} \neq(0,0)\right\} \\
& \mathcal{M}_{\omega}=\left\{\vec{\phi} \in X: S_{\omega}(\vec{\phi})=\mu(\omega), K_{\omega}(\vec{\phi})=0\right\}
\end{aligned}
$$

The following Lemma 12 establishes the existence of a ground state for (1.7). Since it can be proved by the standard variational method (see [2, 13, [14, 24] and also [9, 17]), we omit the proof.

Lemma 12. Let $\kappa \in \mathbb{R}, \gamma>0$ and $\omega>0$. If $\left\{\vec{u}_{n}\right\} \subset X$ satisfies $S_{\omega}\left(\vec{u}_{n}\right) \rightarrow$ $\mu(\omega)$ and $K_{\omega}\left(\vec{u}_{n}\right) \rightarrow 0$, then there exist a sequence $\left\{y_{n}\right\} \subset \mathbb{R}^{N}$ and $\vec{\phi} \in$ $\mathcal{M}_{\omega}$ such that $\left\{\tau_{y_{n}} \vec{u}_{n}\right\}$ has a subsequence that converges to $\vec{\phi}$ strongly in $X$. Moreover, $\mathcal{M}_{\omega}=\mathcal{G}_{\omega}$ and $\mu(\omega)=d(\omega)$. As a consequence, the set $\mathcal{G}_{\omega}$ is not empty.

Next, we consider the stability of ground states. By the scale invariance of (1.7), we see that $d(\omega)=\omega^{3-N / 2} d(1)$ for all $\omega>0$. Since $N \leq 3$ and $d(1)>0$, we have $d^{\prime \prime}(\omega)>0$ for all $\omega>0$. Using this fact and Lemma 12, the following Proposition 5 can be proved by the method of Shatah [18] (see also (9). Since it is standard, we omit the proof.

Proposition 5. Let $\kappa \in \mathbb{R}$ and $\gamma>0$. For any $\omega>0$, the set $\mathcal{G}_{\omega}$ is stable in the following sense. For any $\varepsilon>0$ there exists $\delta>0$ such that if $\vec{u}_{0} \in X$ satisfies dist $\left(\vec{u}_{0}, \mathcal{G}_{\omega}\right)<\delta$, then the solution $\vec{u}(t)$ of (1.1) with $\vec{u}(0)=\vec{u}_{0}$ exists for all $t \geq 0$, and satisfies $\operatorname{dist}\left(\vec{u}(t), \mathcal{G}_{\omega}\right)<\varepsilon$ for all $t \geq 0$, where we put

$$
\operatorname{dist}\left(\vec{u}, \mathcal{G}_{\omega}\right)=\inf \left\{\|\vec{u}-\vec{\phi}\|_{X}: \vec{\phi} \in \mathcal{G}_{\omega}\right\} .
$$

### 5.2 Preliminaries from Elementary Geometry

In this section, we explain some basic geometric properties concerning the line and the ellipse defined by $(1.8)$. In the proof of Theorem 6 , one has to compare, for a given $(\alpha, \beta) \in \mathcal{S}_{\kappa, \gamma}$, the quantities $\alpha^{2}+\beta^{2}$ and 1 . This is the purpose of Lemmas 14 and 15 .

Lemma 13. Let $\gamma>0$ and $0<r \leq 1$, and put

$$
B=\{(x, y) \in] 0, \infty\left[^{2}: \gamma x^{2}+2 y^{2}=2 y, x^{2}+y^{2}=r^{2}\right\}
$$

(1) If $0<r<1$, then $B$ consists of one point.
(2) If $r=1$ and $0<\gamma<1$, then $B=\{(2 \sqrt{1-\gamma} /(2-\gamma), \gamma /(2-\gamma))\}$.
(3) If $r=1$ and $\gamma \geq 1$, then $B$ is empty.

Proof. First we prove (1). Let $0<r<1$. Recall that $\gamma x^{2}+2 y^{2}=2 y$ is an ellipse with vertices $(x, y)=(0,0),(0,1)$ and $( \pm 1 / \sqrt{2 \gamma}, 1 / 2)$, and that $B \subset\{(x, y): 0<y<1\}$. By the equations in $B$, we have $g(y):=(2-\gamma) y^{2}-$ $2 y+\gamma r^{2}=0$. Since $g(0)=\gamma r^{2}>0$ and $g(1)=\gamma\left(r^{2}-1\right)<0$, the equation $g(y)=0$ has only one solution in $] 0,1[$. This proves (1). Parts (2) and (3) are obtained by direct computations.
Lemma 14. Let $(\kappa, \gamma) \in \mathcal{J}_{1} \cup \mathcal{J}_{2}$. Then, $\alpha_{+}^{2}+\beta_{-}^{2}=1$ if and only if $(\kappa, \gamma) \in$ $\mathcal{K}_{3}$.
Proof. Assume that $\left(\alpha_{+}, \beta_{-}\right)$satisfies $\alpha_{+}^{2}+\beta_{-}^{2}=1$. Since $\left(\alpha_{+}, \beta_{-}\right)$satisfies $\gamma \alpha_{+}^{2}+2 \beta_{-}^{2}=2 \beta_{-}$, it follows from (2) and (3) of Lemma 13 that $0<\gamma<1$ and $\left(\alpha_{+}, \beta_{-}\right)=(2 \sqrt{1-\gamma} /(2-\gamma), \gamma /(2-\gamma))$. Substituting this into $\kappa \alpha_{+}+\gamma \beta_{-}=$ 1 , we have $\kappa=\kappa_{c}(\gamma)$. Thus, $(\kappa, \gamma) \in \mathcal{K}_{3}$. Conversely, it is easy to see that $\alpha_{+}^{2}+\beta_{-}^{2}=1$ if $(\kappa, \gamma) \in \mathcal{K}_{3}$.
Lemma 15. If $(\kappa, \gamma) \in \mathcal{K}_{1}$, then $\alpha_{+}^{2}+\beta_{-}^{2}<1$.
Proof. First, we remark that the function $f(\kappa, \gamma):=\alpha_{+}^{2}+\beta_{-}^{2}-1$ is continuous in $\mathcal{J}_{1} \cup \mathcal{J}_{2}$, and that $\mathcal{K}_{1}$ is a connected subset of $\mathcal{J}_{1} \cup \mathcal{J}_{2}$. By Lemma 14, $f$ has no zeros in $\mathcal{K}_{1}$. Thus, $f$ has a constant sign in $\mathcal{K}_{1}$. Finally, since $f(0, \gamma) \rightarrow-1$ as $\gamma \rightarrow \infty$, we conclude that $f(\kappa, \gamma)<0$ for all $(\kappa, \gamma) \in \mathcal{K}_{1}$.

In the same way as Lemma 15 , we see that $\alpha_{+}^{2}+\beta_{-}^{2}>1$ for $(\kappa, \gamma) \in \mathcal{K}_{2} \cap \mathcal{J}_{2}$, but this fact is not used in what follows. The following Lemma 16 plays an important role in the proof of Lemma 17 .

Lemma 16. Let $(\kappa, \gamma) \in \mathcal{J}_{1} \cup \mathcal{J}_{2}$. Then, $\gamma \alpha_{+}>\kappa \beta_{-}$.
Proof. Since $\gamma, \alpha_{+}$and $\beta_{-}$are positive, the inequality is trivial for the case $\kappa \leq 0$. Let $\kappa>0$ and put $D=\kappa^{2}+2 \gamma(\gamma-1)$. Then, $D>0$ and

$$
\begin{aligned}
\gamma \alpha_{+}>\kappa \beta_{-} & \Longleftrightarrow(2-\gamma) \gamma \kappa+\gamma^{2} \sqrt{D}>\kappa\left(\kappa^{2}+\gamma^{2}\right)-\kappa^{2} \sqrt{D} \\
& \Longleftrightarrow\left(\gamma^{2}+\kappa^{2}\right) \sqrt{D}>\kappa D .
\end{aligned}
$$

Since $\left(\gamma^{2}+\kappa^{2}\right)^{2}-\kappa^{2} D=\gamma^{4}+2 \gamma \kappa^{2}>0$, the last inequality holds.

We define

$$
\ell= \begin{cases}\alpha_{+}^{2}+\beta_{-}^{2} & \text { if }(\kappa, \gamma) \in \mathcal{K}_{1}  \tag{5.2}\\ 1 & \text { if }(\kappa, \gamma) \in \mathcal{K}_{2} \cup \mathcal{K}_{3}\end{cases}
$$

and for a given $(\kappa, \gamma)$,

$$
\begin{aligned}
& E_{1}=\{(x, y) \in] 0, \infty\left[^{2}: \kappa x+\gamma y \geq 1\right\} \\
& E_{2}=\{(x, y) \in] 0, \infty\left[^{2}: \gamma x^{2}+2 y^{2} \geq 2 y, x^{2}+y^{2} \leq \ell\right\}
\end{aligned}
$$

In Lemmas 17 and 18, we establish the structure of the set $E_{1} \cap E_{2}$ with respect to $(\kappa, \gamma)$.

Lemma 17. If $(\kappa, \gamma) \in \mathcal{K}_{1} \cup \mathcal{K}_{3}$, then $E_{1} \cap E_{2}=\left\{\left(\alpha_{+}, \beta_{-}\right)\right\}$.
Proof. Since it is clear that $\left\{\left(\alpha_{+}, \beta_{-}\right)\right\} \subset E_{1} \cap E_{2}$, we prove the inverse inclusion. By Lemmas 13, 14 and 15, we see that the ellipse $\gamma x^{2}+2 y^{2}=2 y$ and the circle $x^{2}+y^{2}=\alpha_{+}^{2}+\beta_{-}^{2}$ intersect at only one point $(x, y)=\left(\alpha_{+}, \beta_{-}\right)$ in $\{(x, y): x>0\}$. The normal $y=f_{1}(x)$ and the tangent $y=f_{2}(x)$ of the circle $x^{2}+y^{2}=\alpha_{+}^{2}+\beta_{-}^{2}$ at the point $(x, y)=\left(\alpha_{+}, \beta_{-}\right)$are given by

$$
f_{1}(x)=\frac{\beta_{-}}{\alpha_{+}}\left(x-\alpha_{+}\right)+\beta_{-}, \quad f_{2}(x)=-\frac{\alpha_{+}}{\beta_{-}}\left(x-\alpha_{+}\right)+\beta_{-},
$$

and we see that $E_{2} \subset E_{3}:=\left\{(x, y): y \leq f_{1}(x), y \leq f_{2}(x)\right\}$. By Lemma 16 and by $\kappa \alpha_{+}+\gamma \beta_{-}=1$, we have $-\alpha_{+} / \beta_{-}<-\kappa / \gamma<\beta_{-} / \alpha_{+}$. That is, the slope of the line $\kappa x+\gamma y=1$ is less than that of the normal $y=f_{1}(x)$, and is greater than or equal to that of the tangent $y=f_{2}(x)$. Recalling that $\left(\alpha_{+}, \beta_{-}\right)$is on the line $\kappa x+\gamma y=1$, we see that $E_{1} \cap E_{2} \subset E_{1} \cap E_{3}=$ $\left\{\left(\alpha_{+}, \beta_{-}\right)\right\}$. This completes the proof.

Lemma 18. If $(\kappa, \gamma) \in \mathcal{K}_{2}$, then $E_{1} \cap E_{2}$ is empty.
Proof. First, we consider the case where $\kappa \leq 0$ and $0<\gamma \leq 1$. Then, $E_{1} \subset\{(x, y): y \geq 1 / \gamma\} \subset\{(x, y): y \geq 1\}$, and we see that $E_{1} \cap E_{2}$ is empty.

Next, we consider the case where $0<\gamma<1$ and $0<\kappa<\kappa_{c}(\gamma)$. We fix $\gamma \in$ ] $0,1\left[\right.$ and denote $E_{1}=E_{1}(\kappa)$ for $0<\kappa \leq \kappa_{c}$. Remark that $E_{2}$ is independent of $\left.\kappa \in] 0, \kappa_{c}\right]$. When $\kappa=\kappa_{c}$, by Lemma 17, we have $E_{1}\left(\kappa_{c}\right) \cap E_{2}=\left\{\left(\alpha_{+}, \beta_{-}\right)\right\}$. Moreover, when $0<\kappa<\kappa_{c}, E_{1}(\kappa)$ is strictly smaller than $E_{1}\left(\kappa_{c}\right)$. Thus, we see that $E_{1}(\kappa) \cap E_{2}$ is empty if $0<\kappa<\kappa_{c}$.

### 5.3 Determination of Ground States

We are now able to determine the structure of the set $\mathcal{G}_{\omega}$. We use an idea of Sirakov [20] (see also [12]). We denote

$$
\|u\|_{H_{\omega}^{1}}^{2}=\|\nabla u\|_{L^{2}}^{2}+\omega\|u\|_{L^{2}}^{2}, \quad u \in H^{1}\left(\mathbb{R}^{N}\right) .
$$

Lemma 19. Let $\vec{u}=\left(u_{1}, u_{2}\right) \in \mathcal{A}_{\omega}$. Then we have

$$
\begin{aligned}
& \left\|u_{1}\right\|_{H_{\omega}^{1}}^{2}=\kappa\left\|u_{1}\right\|_{L^{3}}^{3}+\gamma \int_{\mathbb{R}^{N}}{\overline{u_{1}}}^{2} u_{2} d x \\
& \left\|u_{2}\right\|_{H_{\omega}^{1}}^{2}=\left\|u_{2}\right\|_{L^{3}}^{3}+\frac{\gamma}{2} \int_{\mathbb{R}^{N}} u_{1}^{2} \overline{\bar{u}_{2}} d x
\end{aligned}
$$

Proof. The first identity is obtained by mutliplying the first equation of (1.7) by $\overline{u_{1}}$ and by integrating by parts. The second identity is obtained in the same way.

Lemma 20. For any $\omega>0,6 d(\omega) \leq \ell\left\|\varphi_{\omega}\right\|_{L^{3}}^{3}$, where $\ell$ is the number defined by (5.2).

Proof. Let $\left(\alpha \varphi_{\omega}, \beta \varphi_{\omega}\right) \in \mathcal{A}_{\omega}$. Then, we have $K_{\omega}\left(\alpha \varphi_{\omega}, \beta \varphi_{\omega}\right)=0$, and so by (5.1),

$$
S_{\omega}\left(\alpha \varphi_{\omega}, \beta \varphi_{\omega}\right)=\frac{1}{6}\left\|\left(\alpha \varphi_{\omega}, \beta \varphi_{\omega}\right)\right\|_{X_{\omega}}^{2}=\frac{\alpha^{2}+\beta^{2}}{6}\left\|\varphi_{\omega}\right\|_{H_{\omega}^{1}}^{2} .
$$

Moreover, since $\varphi_{\omega}$ is a solution of (1.6), we have $\left\|\varphi_{\omega}\right\|_{H_{\omega}^{1}}^{2}=\left\|\varphi_{\omega}\right\|_{L^{3}}^{3}$, and $6 S_{\omega}\left(\alpha \varphi_{\omega}, \beta \varphi_{\omega}\right)=\left(\alpha^{2}+\beta^{2}\right)\left\|\varphi_{\omega}\right\|_{L^{3}}^{3}$. Finally, by the definitions of $d(\omega)$ and $\ell$, we obtain the desired estimate.

The following variational characterization of $\varphi_{\omega}$ is well-known (see [3, 12, [20, 24]), and we omit the proof.

Lemma 21. Let $\omega>0$. Then

$$
\begin{equation*}
\left\|\varphi_{\omega}\right\|_{L^{3}}=\inf \left\{\|v\|_{H_{\omega}^{1}}^{2} /\|v\|_{L^{3}}^{2}: v \in H^{1}\left(\mathbb{R}^{N}\right) \backslash\{0\}\right\} . \tag{5.3}
\end{equation*}
$$

Moreover,

$$
\begin{align*}
& \left\{v \in H^{1}\left(\mathbb{R}^{N}\right):\|v\|_{H_{\omega}^{1}}^{2}=\|v\|_{L^{3}}^{3}=\left\|\varphi_{\omega}\right\|_{L^{3}}^{3}\right\} \\
& =\left\{e^{i \theta} \varphi_{\omega}(\cdot+y): \theta \in \mathbb{R}, y \in \mathbb{R}^{N}\right\} . \tag{5.4}
\end{align*}
$$

The next lemma is linked to the key Lemmas 17 and 18 .
Lemma 22. Let $\left(u_{1}, u_{2}\right) \in \mathcal{G}_{\omega}$, and put

$$
a=\left\|u_{1}\right\|_{L^{3}} /\left\|\varphi_{\omega}\right\|_{L^{3}}, \quad b=\left\|u_{2}\right\|_{L^{3}} /\left\|\varphi_{\omega}\right\|_{L^{3}} .
$$

Then, $a \geq 0, b>0$, and $(a, b)$ satisfies

$$
\begin{equation*}
a^{2} \leq a^{2}(\kappa a+\gamma b), \quad 2 b \leq 2 b^{2}+\gamma a^{2}, \quad a^{2}+b^{2} \leq \ell . \tag{5.5}
\end{equation*}
$$

Moreover,
(1) If $(\kappa, \gamma) \in \mathcal{K}_{1}$, then $(a, b)=\left(\alpha_{+}, \beta_{-}\right)$.
(2) If $(\kappa, \gamma) \in \mathcal{K}_{2}$, then $(a, b)=(0,1)$.
(3) If $(\kappa, \gamma) \in \mathcal{K}_{3}$, then $(a, b) \in\left\{\left(\alpha_{+}, \beta_{-}\right),(0,1)\right\}$.

Proof. We first prove (5.5). If $u_{2}=0$, then the second equation of (1.7) implies $u_{1}=0$. This contradicts $\left(u_{1}, u_{2}\right) \in \mathcal{A}_{\omega}$. Thus, $u_{2} \neq 0$ and $b>0$. By (5.3), Lemma 19 and the Hölder inequality, we have

$$
\begin{align*}
\left\|\varphi_{\omega}\right\|_{L^{3}}\left\|u_{1}\right\|_{L^{3}}^{2} & \leq\left\|u_{1}\right\|_{H_{\omega}^{1}}^{2}=\kappa\left\|u_{1}\right\|_{L^{3}}^{3}+\gamma \int_{\mathbb{R}^{N}}{\bar{u}_{1}}^{2} u_{2} d x \\
& \leq \kappa\left\|u_{1}\right\|_{L^{3}}^{3}+\gamma\left\|u_{1}\right\|_{L^{3}}^{2}\left\|u_{2}\right\|_{L^{3}}, \tag{5.6}
\end{align*}
$$

which provides $a^{2} \leq a^{2}(\kappa a+\gamma b)$. In the same way, we have

$$
\begin{align*}
\left\|\varphi_{\omega}\right\|_{L^{3}}\left\|u_{2}\right\|_{L^{3}}^{2} & \leq\left\|u_{2}\right\|_{H_{\omega}^{1}}^{2}=\left\|u_{2}\right\|_{L^{3}}^{3}+\frac{\gamma}{2} \int_{\mathbb{R}^{N}} u_{1}^{2} \overline{u_{2}} d x \\
& \leq\left\|u_{2}\right\|_{L^{3}}^{3}+\frac{\gamma}{2}\left\|u_{1}\right\|_{L^{3}}^{2}\left\|u_{2}\right\|_{L^{3}} . \tag{5.7}
\end{align*}
$$

Since $b>0$, this gives $2 b \leq 2 b^{2}+\gamma a^{2}$. Finally, by Lemma 20 and (5.3), we obtain

$$
\begin{equation*}
\ell\left\|\varphi_{\omega}\right\|_{L^{3}}^{3} \geq 6 d(\omega)=6 S_{\omega}(\vec{u})=\|\vec{u}\|_{X_{\omega}}^{2} \geq\left\|\varphi_{\omega}\right\|_{L^{3}}\left(\left\|u_{1}\right\|_{L^{3}}^{2}+\left\|u_{2}\right\|_{L^{3}}^{2}\right), \tag{5.8}
\end{equation*}
$$

which implies $a^{2}+b^{2} \leq \ell$. Hence, (5.5) is proved.
We now prove (1), (2) and (3). Let $(\kappa, \gamma) \in \mathcal{K}_{1}$. Then, since $\ell<1$, by Lemma 22, we see that $a>0$ and $(a, b) \in E_{1} \cap E_{2}$. Thus, (1) follows from (5.5). Next, let $(\kappa, \gamma) \in \mathcal{K}_{2}$. Suppose that $a>0$. Then, by (5.5), we have $(a, b) \in E_{1} \cap E_{2}$. However, this contradicts Lemma 18. Thus, we have $a=0$ and $b=1$, which proves (2). Part (3) can be proved similarly.

Proof of Theorem 6. We consider the case $(\kappa, \gamma) \in \mathcal{K}_{1}$. Let $\vec{u} \in \mathcal{G}_{\omega}$. By (5.6), (5.7) and Lemma 22, we see that

$$
\begin{align*}
&\left\|u_{1} / \alpha_{+}\right\|_{H_{\omega}^{1}}^{2}=\left\|\varphi_{\omega}\right\|_{L^{3}}^{3}=\left\|u_{1} / \alpha_{+}\right\|_{L^{3}}^{3},  \tag{5.9}\\
&\left\|u_{2} / \beta_{-}\right\|_{H_{\omega}^{1}}^{2}=\left\|\varphi_{\omega}\right\|_{L^{3}}^{3}=\left\|u_{2} / \beta_{-}\right\|_{L^{3}}^{3},  \tag{5.10}\\
& \int_{\mathbb{R}^{N}} u_{1}^{2} \overline{\overline{u_{2}}} d x=\left\|u_{1}\right\|_{L^{3}}^{2}\left\|u_{2}\right\|_{L^{3}} . \tag{5.11}
\end{align*}
$$

By (5.4) and by (5.9) and (5.10), there exist $\left(\theta_{1}, y_{1}\right),\left(\theta_{2}, y_{2}\right) \in \mathbb{R} \times \mathbb{R}^{N}$ such that $u_{1}=e^{i \theta_{1}} \alpha_{+} \varphi_{\omega}\left(\cdot+y_{1}\right)$ and $u_{2}=e^{i \theta_{2}} \beta_{-} \varphi_{\omega}\left(\cdot+y_{2}\right)$. Moreover, by (5.11), we see that $2 \theta_{1}-\theta_{2} \in 2 \pi \mathbb{Z}$ and $y_{1}=y_{2}$. Thus, we have $\mathcal{G}_{\omega} \subset \mathcal{G}_{\omega}^{1}$. Since $\mathcal{G}_{\omega}$ is not empty, (1) is proved. (2) and (3) can be proved in the same way.

Finally, Theorem 5 is obtained as a corollary of Theorem 6 .
Proof of Theorem 5. Let $\kappa \leq 0$ and $\gamma=1$. Then, $(\kappa, \gamma) \in \mathcal{K}_{2}$, and by Theorem 6, $\mathcal{G}_{\omega}=\mathcal{G}_{\omega}^{0}$. Thus, Theorem 5 follows from Proposition 5 .

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## References

[1] H. Berestycki and T. Cazenave, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires, C. R. Acad. Sci. Paris sér. I Math. 293 (1981) 489-492.
[2] H. Brezis and E. H. Lieb, Minimum action solutions of some vector field equations, Comm. Math. Phys. 96 (1984) 97-113.
[3] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics 10, Amer. Math. Soc., 2003.
[4] T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. 85 (1982) 549-561.
[5] M. Colin and T. Colin, On a quasi-linear Zakharov system describing laser-plasma interactions, Differential Integral Equations 17 (2004) 297330.
[6] M. Colin and T. Colin, A numerical model for the Raman amplification for laser-plasma interaction, J. Comput. App. Math. 193 (2006) 535562.
[7] M. Colin, T. Colin and M. Ohta, Stability of solitary waves for a system of nonlinear Schrödinger equations with three wave interaction, Ann. Inst. H. Poincaré, Anal. Non Linéaire 26 (2009) 2211-2226.
[8] M. Colin, T. Colin and M. Ohta, Instability of standing waves for a system of nonlinear Schrödinger equations with three-wave interaction, Funkcial. Ekvac. 52 (2009) 371-380.
[9] M. Colin and M. Ohta, Stability of solitary waves for derivative nonlinear Schrödinger equation, Ann. Inst. H. Poincaré, Anal. Non Linéaire 23 (2006) 753-764.
[10] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal. 8 (1971) 321-340.
[11] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry, I, J. Funct. Anal. 74 (1987) 160-197.
[12] H. Kikuchi, Orbital stability of semitrivial standing waves for the Klein-Gordon-Schrödinger system, preprint.
[13] P. L. Lions, The concentration-compactness principle in the calculus of variations, the locally compact case, part I, Ann. Inst. H. Poincaré, Anal. Nonlinéaire 1 (1984) 109-145.
[14] P. L. Lions, The concentration-compactness principle in the calculus of variations, the locally compact case, part II, Ann. Inst. H. Poincaré, Anal. Nonlinéaire 1 (1984) 223-282.
[15] M. Maeda, Instability of bound states of nonlinear Schrödinger equations with Morse index equal to two, Nonlinear Anal. 72 (2010) 2100-2113.
[16] M. Ohta, Instability of bound states for abstract nonlinear Schrödinger equations, preprint, arXiv:1010.1511.
[17] A. Pomponio, Ground states for a system of nonlinear Schrödinger equations with three wave interaction, J. Math. Phys. 51 (2010) 093513, 20pp.
[18] J. Shatah, Stable standing waves of nonlinear Klein-Gordon equations, Comm. Math. Phys. 91 (1983) 313-327.
[19] J. Shatah and W. Strauss, Instability of nonlinear bound states, Comm. Math. Phys. 100 (1985) 173-190.
[20] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^{N}$, Comm. Math. Phys. 271 (2007) 199-221.
[21] A. C. Yew, Stability analysis of multipulses in nonlinearly-coupled Schrödinger equations, Indiana Univ. Math. J. 49 (2000) 1079-1124.
[22] M. I. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations, SIAM J. Math. Anal. 16 (1985) 472-491.
[23] M. I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Comm. Pure Appl. Math. 39 (1986) 51-68.
[24] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser, Boston, MA, 1996.


[^0]:    *Institut de Mathématiques de Bordeaux, Université de Bordeaux and INRIA Bordeaux-Sud Ouest, EPI MC2, 351 cours de la libération, 33405 Talence Cedex, France (mcolin@math.u-bordeaux1.fr).
    ${ }^{\dagger}$ Institut de Mathématiques de Bordeaux, Université Bordeaux 1. Permanent address: Department of Mathematics, Faculty of Science, Saitama University, Saitama 338-8570, Japan (mohta@mail.saitama-u.ac.jp).

