Bifurcation from semi-trivial standing waves and ground states for a system of nonlinear Schrödinger equations

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Abstract

We consider a system of nonlinear Schrödinger equations related to the Raman amplification in a plasma. We study the orbital stability and instability of standing waves bifurcating from the semi-trivial standing wave of the system. The stability and instability of the semitrivial standing wave at the bifurcation point are also studied. Moreover, we determine the set of the ground states completely.

1 Introduction

1.1 Motivation

In this paper, we consider the following system of nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u_1 = -\Delta u_1 - \kappa |u_1| u_1 - \gamma \overline{u_1} u_2 \\ i\partial_t u_2 = -2\Delta u_2 - 2|u_2| u_2 - \gamma u_1^2 \end{cases}$$
(1.1)

for $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, where u_1 and u_2 are complex-valued functions of (t, x), $\kappa \in \mathbb{R}$ and $\gamma > 0$ are constants and $N \leq 3$. System (1.1) is a reduced system studied in [7, 8] and related to the Raman amplification in a plasma.

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Roughly speaking, the Raman amplification is an instability phenomenon taking place when an incident laser field propagates into a plasma. We refer to [5, 6] for a precise description of the phenomenon. A similar system to (1.1) also appears as an optics model with quadratic nonlinearity (see [21]).

In [7, 8], the authors studied the following three-component system

$$\begin{cases} i\partial_t v_1 = -\Delta v_1 - |v_1|^{p-1} v_1 - \gamma v_3 \overline{v_2} \\ i\partial_t v_2 = -\Delta v_2 - |v_2|^{p-1} v_2 - \gamma v_3 \overline{v_1} \\ i\partial_t v_3 = -\Delta v_3 - |v_3|^{p-1} v_3 - \gamma v_1 v_2, \end{cases}$$
(1.2)

where $1 and <math>N \leq 3$. Let $\omega > 0$ and let $\varphi_{\omega} \in H^1(\mathbb{R}^N)$ be a unique positive radial solution of

$$-\Delta \varphi + \omega \varphi - |\varphi|^{p-1} \varphi = 0, \quad x \in \mathbb{R}^N.$$
(1.3)

Then, $(0, 0, e^{i\omega t}\varphi_{\omega})$ solves (1.2). We note that $e^{i\omega t}\varphi_{\omega}$ is a standing wave solution of the single nonlinear Schrödinger equation

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$
 (1.4)

and that $e^{i\omega t}\varphi_{\omega}$ is orbitally stable for (1.4) if 1 , and it is $unstable if <math>1 + 4/N \le p < 1 + 4/(N-2)$ (see [1, 4] and also [3, Chapter 8]). In [7, 8], the authors proved the following result on the semi-trivial standing wave solution $(0, 0, e^{i\omega t}\varphi_{\omega})$ of (1.2).

Theorem 0. ([7, 8]) Let $N \leq 3$, $1 , <math>\omega > 0$, and let φ_{ω} be the positive radial solution of (1.3). Then, there exists a positive constant γ^* such that the semi-trivial standing wave solution $(0, 0, e^{i\omega t}\varphi_{\omega})$ of (1.2) is stable if $0 < \gamma < \gamma^*$, and it is unstable if $\gamma > \gamma^*$.

By the local bifurcation theorem by Crandall and Rabinowitz [10], it is easy to see that $\gamma = \gamma^*$ is a bifurcation point. We are interested in the structure of the bifurcation from the semi-trivial standing wave of (1.2) and its stability property. However, this problem is difficult to study in the general case 1 , so we consider the special case <math>p = 2. Moreover, since v_1 and v_2 play the same role in the proof of Theorem 0, we consider a reduced system (1.1) assuming $v_1 = v_2$ in (1.2). We also introduce a parameter κ in the first equation of (1.1), which makes the structure of standing wave solutions richer as we will see below. We remark that the positive constant γ^* in Theorem 0 is given by

$$\gamma^* = \inf\left\{\frac{\|\nabla v\|_{L^2}^2 + \omega \|v\|_{L^2}^2}{\int_{\mathbb{R}^N} \varphi_{\omega}(x) |v(x)|^2 \, dx} : v \in H^1(\mathbb{R}^N) \setminus \{0\}\right\}.$$
 (1.5)

For the case p = 2, since φ_{ω} is the positive radial solution of

$$-\Delta \varphi + \omega \varphi - |\varphi|\varphi = 0, \quad x \in \mathbb{R}^N,$$
(1.6)

we see that the infimum in (1.5) is attained at $v = \varphi_{\omega}$ and $\gamma^* = 1$. In the same way as the proof of Theorem 0, we can prove the following.

Theorem 1. Let $N \leq 3$, $\kappa \in \mathbb{R}$, $\gamma > 0$, $\omega > 0$, and let φ_{ω} be the positive radial solution of (1.6). Then, the semi-trivial standing wave solution $(0, e^{2i\omega t}\varphi_{\omega})$ of (1.1) is stable if $0 < \gamma < 1$, and it is unstable if $\gamma > 1$.

We remark that the stability property of the semi-trivial standing wave of (1.1) is independent of κ for the case $\gamma \neq 1$. On the other hand, we will see that the sign of κ plays an important role for the case $\gamma = 1$ (see Theorems 4 and 5 below).

1.2 Notation and Definitions

Before we state our main results, we prepare some notation and definitions. For a complex number z, we denote by $\Re z$ and $\Im z$ its real and imaginary parts. Thoughout this paper, we assume that $N \leq 3$. We regard $L^2(\mathbb{R}^N, \mathbb{C})$ as a real Hilbert space with the inner product

$$(u,v)_{L^2} = \Re \int_{\mathbb{R}^N} u(x)\overline{v(x)} \, dx,$$

and we define the inner products of real Hilbert spaces $H = L^2(\mathbb{R}^N, \mathbb{C})^2$ and $X = H^1(\mathbb{R}^N, \mathbb{C})^2$ by

$$(\vec{u}, \vec{v})_H = (u_1, v_1)_{L^2} + (u_2, v_2)_{L^2}, \quad (\vec{u}, \vec{v})_X = (\vec{u}, \vec{v})_H + (\nabla \vec{u}, \nabla \vec{v})_H.$$

Here and hereafter, we use the vectorial notation $\vec{u} = (u_1, u_2)$, and it is considered to be a column vector.

The energy E and the charge Q are defined by

$$E(\vec{u}) = \frac{1}{2} \|\nabla \vec{u}\|_{H}^{2} - \frac{\kappa}{3} \|u_{1}\|_{L^{3}}^{3} - \frac{1}{3} \|u_{2}\|_{L^{3}}^{3} - \frac{\gamma}{2} \Re \int_{\mathbb{R}^{N}} u_{1}^{2} \overline{u_{2}} \, dx,$$
$$Q(\vec{u}) = \frac{1}{2} \|\vec{u}\|_{H}^{2}, \quad \vec{u} \in X.$$

For $\theta \in \mathbb{R}$, we define $G(\theta)$ and J by

$$G(\theta)\vec{u} = (e^{i\theta}u_1, e^{2i\theta}u_2), \quad J\vec{u} = (iu_1, 2iu_2), \quad \vec{u} \in X,$$

and

$$\langle G(\theta)\vec{f},\vec{u}\rangle = \langle \vec{f},G(-\theta)\vec{u}\rangle, \quad \langle J\vec{f},\vec{u}\rangle = -\langle \vec{f},J\vec{u}\rangle$$

for $\vec{f} \in X^*$ and $\vec{u} \in X$, where X^* is the dual space of X. For $y \in \mathbb{R}^N$, we define

$$\tau_y \vec{u}(x) = \vec{u}(x-y), \quad \vec{u} \in X, \ x \in \mathbb{R}^N$$

Note that (1.1) is written as

$$\partial_t \vec{u}(t) = -JE'(\vec{u}(t)),$$

and that $E(G(\theta)\tau_y \vec{u}) = E(\vec{u})$ for all $\theta \in \mathbb{R}, y \in \mathbb{R}^N$ and $\vec{u} \in X$.

By the standard theory (see, e.g., [3, Chapter 4]), we see that the Cauchy problem for (1.1) is globally well-posed in X, and the energy and the charge are conserved. For $\omega > 0$, we define the action S_{ω} by

$$S_{\omega}(\vec{v}) = E(\vec{v}) + \omega Q(\vec{v}), \quad \vec{v} \in X.$$

Note that the Euler-Lagrange equation $S'_{\omega}(\vec{\phi}) = 0$ is written as

$$\begin{cases} -\Delta\phi_1 + \omega\phi_1 = \kappa |\phi_1|\phi_1 + \gamma \overline{\phi_1}\phi_2 \\ -\Delta\phi_2 + \omega\phi_2 = |\phi_2|\phi_2 + (\gamma/2)\phi_1^2 \end{cases}$$
(1.7)

and that if $\vec{\phi} \in X$ satisfies $S'_{\omega}(\vec{\phi}) = 0$, then $G(\omega t)\vec{\phi}$ is a solution of (1.1).

Definition 1. We say that a standing wave solution $G(\omega t)\vec{\phi}$ of (1.1) is *stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $u_0 \in X$ satisfies $\|\vec{u}_0 - \vec{\phi}\|_X < \delta$, then the solution $\vec{u}(t)$ of (1.1) with $\vec{u}(0) = \vec{u}_0$ exists for all $t \ge 0$, and satisfies

$$\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \| \vec{u}(t) - G(\theta) \tau_y \vec{\phi} \|_X < \varepsilon$$

for all $t \ge 0$. Otherwise, $G(\omega t)\vec{\phi}$ is called *unstable*.

In this article, we are also interested in the classification of ground states of (1.7). A ground state of (1.7) is a nontrivial solution which minimizes the action S_{ω} among all the nontrivial solutions of (1.7). The set \mathcal{G}_{ω} of the ground states for (1.7) is then defined as follows:

$$\mathcal{A}_{\omega} = \{ \vec{v} \in X : S'_{\omega}(\vec{v}) = 0, \ \vec{v} \neq 0 \},\$$

$$d(\omega) = \inf\{ S_{\omega}(\vec{v}) : \vec{v} \in \mathcal{A}_{\omega} \},\$$

$$\mathcal{G}_{\omega} = \{ \vec{u} \in \mathcal{A}_{\omega} : S_{\omega}(\vec{u}) = d(\omega) \}.$$

1.3 Main Results

We first look for solutions of (1.7) of the form $\vec{\phi} = (\alpha \varphi_{\omega}, \beta \varphi_{\omega})$ with $(\alpha, \beta) \in]0, \infty[^2$, where φ_{ω} is the positive radial solution of (1.6). It is clear that if $(\alpha, \beta) \in]0, \infty[^2$ satisfies

$$\kappa \alpha + \gamma \beta = 1, \quad \gamma \alpha^2 + 2\beta^2 = 2\beta, \tag{1.8}$$

then $(\alpha \varphi_{\omega}, \beta \varphi_{\omega})$ is a solution of (1.7). For $\kappa \in \mathbb{R}$ and $\gamma > 0$, we define

$$S_{\kappa,\gamma} = \{(x,y) \in]0, \infty[^2: \kappa x + \gamma y = 1, \ \gamma x^2 + 2y^2 = 2y\}.$$

Note that $\gamma x^2 + 2y^2 = 2y$ is an ellipse with vertices $(x, y) = (0, 0), (0, 1), (\pm 1/\sqrt{2\gamma}, 1/2),$ and that $S_{\kappa, \gamma} \subset \{(x, y) : 0 < y < 1\}.$

To determine the structure of the set $S_{\kappa,\gamma}$, which is one of the crucial points of our analysis, for $\kappa^2 \geq 2\gamma(1-\gamma)$ we define

$$\alpha_{\pm} = \frac{(2-\gamma)\kappa \pm \gamma\sqrt{\kappa^2 + 2\gamma(\gamma - 1)}}{2\kappa^2 + \gamma^3},$$

$$\beta_{\pm} = \frac{\kappa^2 + \gamma^2 \pm \kappa\sqrt{\kappa^2 + 2\gamma(\gamma - 1)}}{2\kappa^2 + \gamma^3},$$

$$\alpha_0 = \frac{(2-\gamma)\kappa}{2\kappa^2 + \gamma^3}, \quad \beta_0 = \frac{\kappa^2 + \gamma^2}{2\kappa^2 + \gamma^3}.$$

We also divide the parameter domain $\mathcal{D} = \{(\kappa, \gamma) : \kappa \in \mathbb{R}, \gamma > 0\}$ into the following sets (see Figure 1).

$$\begin{aligned} \mathcal{J}_1 &= \{(\kappa, \gamma) : \kappa \leq 0, \ \gamma > 1\} \cup \{(\kappa, \gamma) : \kappa > 0, \ \gamma \geq 1\}, \\ \mathcal{J}_2 &= \{(\kappa, \gamma) : 0 < \gamma < 1, \ \kappa > \sqrt{2\gamma(1 - \gamma)}\}, \\ \mathcal{J}_3 &= \{(\kappa, \gamma) : 0 < \gamma < 1, \ \kappa = \sqrt{2\gamma(1 - \gamma)}\}, \\ \mathcal{J}_0 &= \{(\kappa, \gamma) : \kappa \in \mathbb{R}, \ \gamma > 0\} \setminus (\mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3). \end{aligned}$$

Notice that the sets \mathcal{J}_0 , \mathcal{J}_1 , \mathcal{J}_2 and \mathcal{J}_3 are mutually disjoint, and $\mathcal{D} = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$. Note also that for $0 < \kappa \leq 1/\sqrt{2}$, the equation $2\gamma(1-\gamma) = \kappa^2$ has solutions $\gamma = \gamma_{\pm} := (1 \pm \sqrt{1 - 2\kappa^2})/2$. It is then possible to determine the set $\mathcal{S}_{\kappa,\gamma}$ in terms of α_{\pm} , β_{\pm} , α_0 and β_0 . Indeed, by elementary computations, we obtain the following.

Proposition 1. (0) If $(\kappa, \gamma) \in \mathcal{J}_0$, then $\mathcal{S}_{\kappa,\gamma}$ is empty.

- (1) If $(\kappa, \gamma) \in \mathcal{J}_1$, then $\mathcal{S}_{\kappa, \gamma} = \{(\alpha_+, \beta_-)\}.$
- (2) If $(\kappa, \gamma) \in \mathcal{J}_2$, then $\mathcal{S}_{\kappa, \gamma} = \{(\alpha_+, \beta_-), (\alpha_-, \beta_+)\}.$
- (3) If $(\kappa, \gamma) \in \mathcal{J}_3$, then $\mathcal{S}_{\kappa, \gamma} = \{(\alpha_0, \beta_0)\}.$

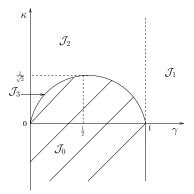


Figure 1: The sets \mathcal{J}_0 , \mathcal{J}_1 , \mathcal{J}_2 and \mathcal{J}_3

Remark 1. (1) When $\kappa \leq 0$, $(\alpha_+, \beta_-) \to (0, 1)$ as $\gamma \to 1 + 0$. That is, the branch $\{(\alpha_+\varphi_\omega, \beta_-\varphi_\omega) : \gamma > 1\}$ of positive solutions of (1.7) bifurcates from the semi-trivial solution $(0, \varphi_\omega)$ at $\gamma = 1$.

(2) When $\kappa > 0$, $(\alpha_{-}, \beta_{+}) \rightarrow (0, 1)$ as $\gamma \rightarrow 1 - 0$. That is, the branch $\{(\alpha_{-}\varphi_{\omega}, \beta_{+}\varphi_{\omega}) : \gamma_{m} < \gamma < 1\}$ of positive solutions of (1.7) bifurcates from the semi-trivial solution $(0, \varphi_{\omega})$ at $\gamma = 1$, where $\gamma_{m} = \inf\{\gamma : (\kappa, \gamma) \in \mathcal{S}_{\kappa, \gamma}\}$, and it is given by $\gamma_{m} = 0$ if $\kappa > 1/\sqrt{2}$, and $\gamma_{m} = \gamma_{+}$ if $0 < \kappa \leq 1/\sqrt{2}$.

We obtain the following stability and instability results of standing waves of (1.1) associated with Proposition 1. Recall that φ_{ω} is the positive radial solution of (1.6).

Theorem 2. Let $N \leq 3$ and $(\kappa, \gamma) \in \mathcal{J}_1 \cup \mathcal{J}_2$. For any $\omega > 0$, the standing wave solution $G(\omega t)(\alpha_+\varphi_\omega, \beta_-\varphi_\omega)$ of (1.1) is stable.

Theorem 3. Let $N \leq 3$ and $(\kappa, \gamma) \in \mathcal{J}_2$. For any $\omega > 0$, the standing wave solution $G(\omega t)(\alpha_-\varphi_{\omega}, \beta_+\varphi_{\omega})$ of (1.1) is unstable.

Remark 2. In this paper, we do not study the stability/instability problem of $G(\omega t)(\alpha_0 \varphi_{\omega}, \beta_0 \varphi_{\omega})$ for the case $(\kappa, \gamma) \in \mathcal{J}_3$.

Remark 3. The result for the case $\kappa = 1$ in Theorem 3 is announced in [16] together with an outline of the proof.

We also obtain the stability and instability results of semi-tirivial standing wave at the bifurcation point $\gamma = 1$. The results depend on the sign of κ .

Theorem 4. Let $N \leq 3$, $\kappa > 0$ and $\gamma = 1$. For any $\omega > 0$, the standing wave solution $(0, e^{2i\omega t}\varphi_{\omega})$ of (1.1) is unstable.

Theorem 5. Let $N \leq 3$, $\kappa \leq 0$ and $\gamma = 1$. For any $\omega > 0$, the standing wave solution $(0, e^{2i\omega t}\varphi_{\omega})$ of (1.1) is stable.

Remark 4. The linearized operator $S''_{\omega}(0, \varphi_{\omega})$ around the semi-trivial standing wave is independent of κ (see (2.2) and (2.3) below). Therefore, Theorems 4 and 5 are never obtained from the linearized analysis only. The proof of Theorem 5 relies on the variational method of Shatah [18] and on the characterization of the ground states in Theorem 6 below.

Remark 5. For the case $\gamma = 1$, using the notation in Section 2, we have $\mathcal{L}_R \vec{v} = (L_1 v_1, L_2 v_2)$ and $\mathcal{L}_I \vec{v} = (L_{-1} v_1, L_1 v_2)$, and the kernel of $S''_{\omega}(0, \varphi_{\omega})$ contains a nontrivial element $(\varphi_{\omega}, 0)$ other than the elements $\nabla(0, \varphi_{\omega})$ and $J(0, \varphi_{\omega})$ naturally coming from the symmetries of S_{ω} (see (2.4) below).

Next, we consider the ground state problem for (1.7). We define

$$\kappa_c(\gamma) = \frac{1}{2}(\gamma+2)\sqrt{1-\gamma}, \quad 0 < \gamma < 1.$$
(1.9)

Then, κ_c is strictly decreasing on the open interval $]0,1[, \kappa_c(0) = 1$ and $\kappa_c(1) = 0$. We define a function γ_c on]0,1[by the inverse function of κ_c . For the ground state problem, it is convenient to divide the parameter domain $\mathcal{D} = \{(\kappa, \gamma) : \kappa \in \mathbb{R}, \gamma > 0\}$ into the following sets (see Figure 2).

$$\begin{split} \mathcal{K}_1 &= \{(\kappa, \gamma) : \kappa \leq 0, \ \gamma > 1\} \cup \{(\kappa, \gamma) : \kappa \geq 1, \ \gamma > 0\} \\ &\cup \{(\kappa, \gamma) : 0 < \kappa < 1, \ \gamma > \gamma_c(\kappa)\}, \\ \mathcal{K}_2 &= \{(\kappa, \gamma) : \kappa \leq 0, \ 0 < \gamma \leq 1\} \cup \{(\kappa, \gamma) : 0 < \kappa < 1, \ 0 < \gamma < \gamma_c(\kappa)\}, \\ \mathcal{K}_3 &= \{(\kappa, \gamma) : 0 < \kappa < 1, \ \gamma = \gamma_c(\kappa)\}. \end{split}$$

Note that \mathcal{K}_1 , \mathcal{K}_2 and \mathcal{K}_3 are mutually disjoint, and $\mathcal{D} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$. Remark also that since $\sqrt{2\gamma(1-\gamma)} < \kappa_c(\gamma)$ for $0 < \gamma < 1$, we have $\mathcal{J}_0 \subset \mathcal{K}_2$. Moreover, we define

$$\mathcal{G}^{0}_{\omega} = \{ G(\theta)\tau_{y}(0,\varphi_{\omega}) : \theta \in \mathbb{R}, \ y \in \mathbb{R}^{N} \}, \\ \mathcal{G}^{1}_{\omega} = \{ G(\theta)\tau_{y}(\alpha_{+}\varphi_{\omega},\beta_{-}\varphi_{\omega}) : \theta \in \mathbb{R}, \ y \in \mathbb{R}^{N} \}.$$

Then, the set \mathcal{G}_{ω} of the ground states for (1.7) is determined as follows.

Theorem 6. Let $N \leq 3$ and $\omega > 0$.

- (1) If $(\kappa, \gamma) \in \mathcal{K}_1$, then $\mathcal{G}_{\omega} = \mathcal{G}_{\omega}^1$. (2) If $(\kappa, \gamma) \in \mathcal{K}_2$, then $\mathcal{G}_{\omega} = \mathcal{G}_{\omega}^0$.
- (3) If $(\kappa, \gamma) \in \mathcal{K}_3$, then $\mathcal{G}_{\omega} = \mathcal{G}_{\omega}^{0} \cup \mathcal{G}_{\omega}^1$.

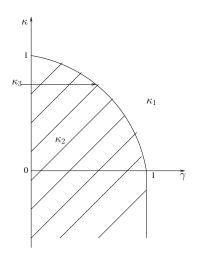


Figure 2: The sets \mathcal{K}_1 , \mathcal{K}_2 and \mathcal{K}_3

The rest of the paper is organized as follows. In Section 2, we study some spectral properties of the linearized operators around standing waves, which are needed in Sections 3 and 4. In Section 3, we prove Theorems 2 and 3, while Section 4 is devoted to the proof of Theorem 4. In Section 5, we study the ground state problem for (1.7), and prove Theorem 6. Finally, Theorem 5 is proved as a corollary of Theorem 6.

2 Linearized Operators

In this section, we study spectral properties of the linearized operator $S''_{\omega}(\Phi)$. Here and hereafter, for $\alpha \geq 0$ and $\beta > 0$, we put

$$\Phi = (\alpha \varphi_{\omega}, \beta \varphi_{\omega}), \quad \Phi_1 = (-\beta \varphi_{\omega}, \alpha \varphi_{\omega}), \quad \Phi_2 = (\alpha \varphi_{\omega}, 2\beta \varphi_{\omega}).$$

First, by direct computations, we have

$$\langle S''_{\omega}(\Phi)\vec{u},\vec{u}\rangle = \langle \mathcal{L}_R \Re \vec{u}, \Re \vec{u}\rangle + \langle \mathcal{L}_I \Im \vec{u}, \Im \vec{u}\rangle$$
(2.1)

for $\vec{u} = (u_1, u_2) \in X$, where $\Re \vec{u} = (\Re u_1, \Re u_2), \ \Im \vec{u} = (\Im u_1, \Im u_2)$, and

$$\mathcal{L}_R = \begin{bmatrix} -\Delta + \omega & 0\\ 0 & -\Delta + \omega \end{bmatrix} - \begin{bmatrix} (2\alpha + \gamma\beta)\varphi_\omega & \gamma\alpha\varphi_\omega\\ \gamma\alpha\varphi_\omega & 2\beta\varphi_\omega \end{bmatrix}, \quad (2.2)$$

$$\mathcal{L}_{I} = \begin{bmatrix} -\Delta + \omega & 0\\ 0 & -\Delta + \omega \end{bmatrix} - \begin{bmatrix} (\alpha - \gamma\beta)\varphi_{\omega} & \gamma\alpha\varphi_{\omega}\\ \gamma\alpha\varphi_{\omega} & \beta\varphi_{\omega} \end{bmatrix}.$$
 (2.3)

Since $S'_{\omega}(G(\theta)\tau_y\Phi) = 0$ for $y \in \mathbb{R}^N$ and $\theta \in \mathbb{R}$, we see that

$$\nabla \Phi \in \ker \mathcal{L}_R, \quad \Phi_2 \in \ker \mathcal{L}_I.$$
 (2.4)

For $a \in \mathbb{R}$, we define L_a by

$$L_a v = -\Delta v + \omega v - a\varphi_\omega v, \quad v \in H^1(\mathbb{R}^N, \mathbb{R}).$$

We recall some known results on L_a .

Lemma 1. Let $N \leq 3$ and let φ_{ω} be the positive radial solution of (1.6). (1) L_2 has one negative eigenvalue, ker L_2 is spanned by $\{\nabla \varphi_{\omega}\}$, and there exists a constant $c_1 > 0$ such that $\langle L_2 v, v \rangle \geq c_1 ||v||_{H^1}^2$ for all $v \in H^1(\mathbb{R}^N, \mathbb{R})$ satisfying $(v, \varphi_{\omega})_{L^2} = 0$ and $(v, \nabla \varphi_{\omega})_{L^2} = 0$.

(2) L_1 is non-negative, ker L_1 is spanned by $\{\varphi_{\omega}\}$, and there exists $c_2 > 0$ such that $\langle L_1 v, v \rangle \geq c_2 \|v\|_{H^1}^2$ for all $v \in H^1(\mathbb{R}^N, \mathbb{R})$ satisfying $(v, \varphi_{\omega})_{L^2} = 0$. (3) If a < 1, then L_a is positive on $H^1(\mathbb{R}^N, \mathbb{R})$.

(4) If 1 < a < 2, then $\langle L_a \varphi_\omega, \varphi_\omega \rangle < 0$, and there exists $c_4 > 0$ such that $\langle L_a v, v \rangle \ge c_4 ||v||_{H^1}^2$ for all $v \in H^1(\mathbb{R}^N, \mathbb{R})$ satisfying $(v, \varphi_\omega)_{L^2} = 0$.

Proof. Parts (1) and (2) are well-known (see [22]). Note that the quadratic nonlinearity in (1.6) is L^2 -subcritical if and only if $N \leq 3$, and that the assumption $N \leq 3$ is essential for (1). Parts (3) and (4) follow from (1) and (2) immediately.

In the next lemma, we give the diagonalization of \mathcal{L}_R and \mathcal{L}_I .

Lemma 2. By orthogonal matrices

$$A = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad B = \frac{1}{\sqrt{\alpha^2 + 4\beta^2}} \begin{bmatrix} \alpha & 2\beta \\ -2\beta & \alpha \end{bmatrix},$$

 \mathcal{L}_R and \mathcal{L}_I are diagonalized as follows:

$$\mathcal{L}_R = A^* \begin{bmatrix} L_2 & 0 \\ 0 & L_{(2-\gamma)\beta} \end{bmatrix} A, \quad \mathcal{L}_I = B^* \begin{bmatrix} L_1 & 0 \\ 0 & L_{(1-2\gamma)\beta} \end{bmatrix} B.$$

Proof. The computation is straightforward, and we omit the details.

The next three lemmas establish the coercivity properties of the operators \mathcal{L}_R and \mathcal{L}_I . They represent the main results of this section, and are the key points in the proofs of Theorems 2 and 3.

Lemma 3. If $(2 - \gamma)\beta < 1$, then there exists a constant $\delta_1 > 0$ such that $\langle \mathcal{L}_R \vec{v}, \vec{v} \rangle \geq \delta_1 \|\vec{v}\|_X^2$ for all $\vec{v} \in H^1(\mathbb{R}^N, \mathbb{R})^2$ satisfying $(\vec{v}, \Phi)_H = 0$ and $(\vec{v}, \nabla \Phi)_H = 0$.

Proof. By Lemma 2, we have $\langle \mathcal{L}_R \vec{v}, \vec{v} \rangle = \langle L_2 w_1, w_1 \rangle + \langle L_{(2-\gamma)\beta} w_2, w_2 \rangle$, where $\vec{w} = A\vec{v}$. Since we have

$$(w_1, \varphi_{\omega})_{L^2} = \frac{(\vec{v}, \Phi)_H}{\sqrt{\alpha^2 + \beta^2}} = 0, \quad (w_1, \nabla \varphi_{\omega})_{L^2} = \frac{(\vec{v}, \nabla \Phi)_H}{\sqrt{\alpha^2 + \beta^2}} = 0,$$

it follows from Lemma 1 (1) that $\langle L_2 w_1, w_1 \rangle \geq c_1 ||w_1||_{H^1}^2$. Moreover, by the assumption $(2 - \gamma)\beta < 1$ and by Lemma 1 (3), we have $\langle L_{(2-\gamma)\beta}w_2, w_2 \rangle \geq c_3 ||w_2||_{H^1}^2$. This completes the proof. \Box

Lemma 4. If $1 \leq (2 - \gamma)\beta < 2$, then there exists a constant $\delta_2 > 0$ such that $\langle \mathcal{L}_R \vec{v}, \vec{v} \rangle \geq \delta_2 \|\vec{v}\|_X^2$ for all $\vec{v} \in H^1_{rad}(\mathbb{R}^N, \mathbb{R})^2$ satisfying $(\vec{v}, \Phi)_H = 0$ and $(\vec{v}, \Phi_1)_H = 0$, where $\Phi_1 = (-\beta \varphi_\omega, \alpha \varphi_\omega)$.

Proof. By Lemma 2, we have $\langle \mathcal{L}_R \vec{v}, \vec{v} \rangle = \langle L_2 w_1, w_1 \rangle + \langle L_{(2-\gamma)\beta} w_2, w_2 \rangle$, where $\vec{w} = A\vec{v}$. Then we have $(w_1, \varphi_{\omega})_{L^2} = (\vec{v}, \Phi)_H / \sqrt{\alpha^2 + \beta^2} = 0$. Moreover, since φ_{ω} and w_1 are radially symmetric, we have $(w_1, \nabla \varphi_{\omega})_{L^2} = 0$. Thus, it follows from Lemma 1 (1) that $\langle L_2 w_1, w_1 \rangle \geq c_1 ||w_1||_{H^1}^2$. Moreover, since $(w_2, \varphi_{\omega})_{L^2} = (\vec{v}, \Phi_1)_H / \sqrt{\alpha^2 + \beta^2} = 0$, it follows from the assumption $1 \leq (2-\gamma)\beta < 2$ and Lemma 1 (2), (4) that $\langle L_{(2-\gamma)\beta} w_2, w_2 \rangle \geq c_2 ||w_2||_{H^1}^2$.

Lemma 5. There exists a constant $\delta_3 > 0$ such that $\langle \mathcal{L}_I \vec{v}, \vec{v} \rangle \geq \delta_3 \|\vec{v}\|_X^2$ for all $\vec{v} \in H^1(\mathbb{R}^N, \mathbb{R})^2$ satisfying $(\vec{v}, \Phi_2)_H = 0$, where $\Phi_2 = (\alpha \varphi_{\omega}, 2\beta \varphi_{\omega})$.

Proof. By Lemma 2, we have $\langle \mathcal{L}_I \vec{v}, \vec{v} \rangle = \langle L_1 w_1, w_1 \rangle + \langle L_{(1-2\gamma)\beta} w_2, w_2 \rangle$, where $\vec{w} = B\vec{v}$. Since $(w_1, \varphi_{\omega})_{L^2} = (\vec{v}, \Phi_2)_H / \sqrt{\alpha^2 + 4\beta^2} = 0$, Lemma 1 (2) implies $\langle L_1 w_1, w_1 \rangle \geq c_2 \|w_1\|_{H^1}^2$. Moreover, since $(1 - 2\gamma)\beta < 1$, it follows from Lemma 1 (3) that $\langle L_{(1-2\gamma)\beta} w_2, w_2 \rangle \geq c_3 \|w_2\|_{H^1}^2$.

The last two lemmas of this section make connections between parameters (κ, γ) and the criteria used in Lemma 3, 4 and 5 on β .

Lemma 6. Let $(\kappa, \gamma) \in \mathcal{J}_1 \cup \mathcal{J}_2$. Then, $(2 - \gamma)\beta_- < 1$ and $(1 - 2\gamma)\beta_- < 1$.

Proof. We put $D = \kappa^2 + 2\gamma(\gamma - 1)$. By the second equation of (1.8), we have $0 < \beta_- < 1$. Thus, we have $(1 - 2\gamma)\beta_- < \beta_- < 1$. If $\gamma > 1$, then $(2 - \gamma)\beta_- < \beta_- < 1$. While, if $0 < \gamma \leq 1$, then $\kappa > 0$, D > 0 and $(2 - \gamma)\beta_- < (2 - \gamma)(\kappa^2 + \gamma^2)/(2\kappa^2 + \gamma^3) < 1$. Note that the last inequality is equivalent to D > 0.

Lemma 7. Let $(\kappa, \gamma) \in \mathcal{J}_2$. Then, $1 < (2 - \gamma)\beta_+ < 2$ and $(1 - 2\gamma)\beta_+ < 1$.

Proof. We put $D = \kappa^2 + 2\gamma(\gamma - 1)$. Since $0 < \beta_+ < 1$, we have $(2 - \gamma)\beta_+ < 2\beta_+ < 2$ and $(1 - 2\gamma)\beta_+ < \beta_+ < 1$. Next, we see that $(2 - \gamma)\beta_+ > 1$ is equivalent to $(2 - \gamma)\kappa > \gamma\sqrt{D}$. Since $0 < \gamma < 1$ and $\kappa > 0$, we have $\gamma\sqrt{D} < \gamma\kappa < (2 - \gamma)\kappa$.

Remark 6. When $(\kappa, \gamma) \in \mathcal{J}_3$, we have $D = \kappa^2 + 2\gamma(\gamma - 1) = 0$, $(2 - \gamma)\beta_0 = 1$ and $(1 - 2\gamma)\beta_0 < 1$.

3 Proofs of Theorems 2 and 3

In this section we prove Theorems 2 and 3 using the results of Section 2 and the following propositions. Proposition 2 follows from Theorem 3.4 of Grillakis, Shatah and Strauss [11] (see also [23] and [7, Section 3]). While, Proposition 3 follows from Theorem 1 of [16] (see also [11, 15, 19]).

Proposition 2. Let $\vec{\phi} \in \mathcal{A}_{\omega}$. Assume that there exists a constant $\delta > 0$ such that $\langle S''_{\omega}(\vec{\phi})\vec{w},\vec{w}\rangle \geq \delta \|\vec{w}\|_X^2$ for all $\vec{w} \in X$ satisfying $(\vec{\phi},\vec{w})_H = (J\vec{\phi},\vec{w})_H = 0$ and $(\nabla\vec{\phi},\vec{w})_H = 0$. Then the standing wave solution $G(\omega t)\vec{\phi}$ of (1.1) is stable.

Proposition 3. Let $\vec{\phi} \in \mathcal{A}_{\omega}$ be radially symmetric. Assume that there exist $\vec{\psi} \in X_{\text{rad}}$ and a constant $\delta > 0$ such that $\|\vec{\psi}\|_H = 1$, $(\vec{\psi}, \vec{\phi})_H = (\vec{\psi}, J\vec{\phi})_H = 0$, $\langle S''_{\omega}(\vec{\phi})\vec{\psi}, \vec{\psi} \rangle < 0$, and $\langle S''_{\omega}(\vec{\phi})\vec{w}, \vec{w} \rangle \geq \delta \|\vec{w}\|_X^2$ for all $\vec{w} \in X_{\text{rad}}$ satisfying $(\vec{\phi}, \vec{w})_H = (J\vec{\phi}, \vec{w})_H = (\vec{\psi}, \vec{w})_H = 0$. Then the standing wave solution $G(\omega t)\vec{\phi}$ of (1.1) is unstable.

Proof of Theorem 2. For $(\kappa, \gamma) \in \mathcal{J}_1 \cup \mathcal{J}_2$, let $(\alpha, \beta) = (\alpha_+, \beta_-)$. Let $\vec{w} \in X$ satisfy $(\Phi, \vec{w})_H = (J\Phi, \vec{w})_H = 0$ and $(\nabla \Phi, \vec{w})_H = 0$. By (2.1), we have

$$\langle S''_{\omega}(\Phi)\vec{w},\vec{w}\rangle = \langle \mathcal{L}_R\Re\vec{w},\Re\vec{w}\rangle + \langle \mathcal{L}_I\Im\vec{w},\Im\vec{w}\rangle$$

Since $(\Phi, \Re \vec{w})_H = (\Phi, \vec{w})_H = 0$ and $(\nabla \Phi, \Re \vec{w})_H = (\nabla \Phi, \vec{w})_H = 0$, it follows from Lemmas 6 and 3 that $\langle \mathcal{L}_R \Re \vec{w}, \Re \vec{w} \rangle \geq \delta_1 ||\Re \vec{w}||_X^2$. While, since $(\Im \vec{w}, \Phi_2)_H = (J\Phi, \vec{w})_H = 0$, Lemma 5 implies $\langle \mathcal{L}_I \Im \vec{w}, \Im \vec{w} \rangle \geq \delta_3 ||\Im \vec{w}||_X^2$. Therefore, Theorem 2 follows from Proposition 2. Proof of Theorem 3. For $(\kappa, \gamma) \in \mathcal{J}_2$, let $(\alpha, \beta) = (\alpha_-, \beta_+)$. We take $\vec{\psi} = \Phi_1/\|\Phi_1\|_H$. Then we have $\|\vec{\psi}\|_H = 1$, $(\vec{\psi}, \Phi)_H = 0$ and $(\vec{\psi}, J\Phi)_H = 0$. Moreover, by Lemma 7 and Lemma 1 (4), we have

$$\langle S_{\omega}''(\Phi)\vec{\psi},\vec{\psi}\rangle = \langle \mathcal{L}_R\vec{\psi},\vec{\psi}\rangle = \langle L_{(2-\gamma)\beta}\varphi_{\omega},\varphi_{\omega}\rangle/\|\varphi_{\omega}\|_{L^2}^2 < 0.$$

Finally, let $\vec{w} \in X_{\text{rad}}$ satisfy $(\Phi, \vec{w})_H = (J\Phi, \vec{w})_H = (\vec{\psi}, \vec{w})_H = 0$. Since $(\Phi, \Re \vec{w})_H = (\Phi, \vec{w})_H = 0$ and $(\Phi_1, \Re \vec{w})_H = (\Phi_1, \vec{w})_H = 0$, by Lemmas 7 and 4, we have $\langle \mathcal{L}_R \Re \vec{w}, \Re \vec{w} \rangle \geq \delta_2 ||\Re \vec{w}||_X^2$. While, since $(\Im \vec{w}, \Phi_2)_H = (J\Phi, \vec{w})_H = 0$, by Lemma 5, we have $\langle \mathcal{L}_I \Im \vec{w}, \Im \vec{w} \rangle \geq \delta_3 ||\Im \vec{w}||_X^2$. Thus, by (2.1), we have $\langle S''_{\omega}(\vec{\phi})\vec{w}, \vec{w} \rangle \geq \delta ||\vec{w}||_X^2$, and Theorem 3 follows from Proposition 3.

4 Proof of Theorem 4

We introduce the following Proposition 4 to prove Theorem 4. It is a modification of Theorem 2 of [16]. In what follows, $sgn(\mu)$ denotes the sign of any real μ .

Proposition 4. Let $\vec{\phi} \in \mathcal{A}_{\omega}$ be radially symmetric. Assume that there exist $\vec{\psi} \in X_{\text{rad}}$ such that

- (i) $\|\vec{\psi}\|_H = 1, \ (\vec{\psi}, \vec{\phi})_H = 0, \ (\vec{\psi}, J\vec{\phi})_H = (\vec{\psi}, J\vec{\phi})_X = 0, \ S''_{\omega}(\vec{\phi})\vec{\psi} = \vec{0},$
- (ii) there exists a positive constant k_0 such that $\langle S''_{\omega}(\vec{\phi})\vec{w},\vec{w}\rangle \geq k_0 \|\vec{w}\|_X^2$ for all $\vec{w} \in X_{\text{rad}}$ satisfying $(\vec{\phi},\vec{w})_H = (J\vec{\phi},\vec{w})_H = (\vec{\psi},\vec{w})_H = 0$,
- (iii) there exist positive constants k_1 , k_2 and ε such that

$$\operatorname{sgn}(\lambda) \cdot \langle S'_{\omega}(\vec{\phi} + \lambda \vec{\psi} + \vec{z}), \vec{\psi} \rangle \leq -k_1 \lambda^2 + k_2 \|\vec{z}\|_X^2 + o(\lambda^2 + \|\vec{z}\|_X^2)$$

for all $\lambda \in \mathbb{R}$ and $\vec{z} \in X_{rad}$ satisfying $|\lambda| + ||\vec{z}||_X < \varepsilon$. Then the standing wave solution $G(\omega t)\vec{\phi}$ of (1.1) is unstable.

We first prove Theorem 4 using Proposition 4.

Proof of Theorem 4. The proof consists of verifying the assumptions (i), (ii), (iii) of Proposition 4. Let $(\alpha, \beta) = (0, 1)$ and $\Phi = (0, \varphi_{\omega})$. We take

$$\psi = (\psi_1, \psi_2) = (\varphi_\omega, 0) / \|\varphi_\omega\|_{L^2}.$$

Then, $\|\vec{\psi}\|_H = 1$, $(\vec{\psi}, \Phi)_H = 0$, $(\vec{\psi}, J\Phi)_H = (\vec{\psi}, J\Phi)_X = 0$, and

$$S''_{\omega}(\vec{\phi})\vec{\psi} = (L_1\varphi_{\omega}, 0)/\|\varphi_{\omega}\|_{L^2} = (0, 0).$$

Thus, (i) is satisfied. The assumption (ii) is proved in the same way as the proof of Theorem 3. Finally, we prove (iii). Let $\lambda \in \mathbb{R}$ and $\vec{z} = (z_1, z_2) \in X_{\text{rad}}$, and put $\vec{v} = (v_1, v_2) = \lambda \vec{\psi} + \vec{z}$. Then, we have

$$v_1 = \lambda \psi_1 + z_1, \quad v_2 = z_2, \quad \psi_1 = \varphi_\omega / \|\varphi_\omega\|_{L^2},$$

and

$$\begin{split} \|\varphi_{\omega}\|_{L^{2}} \langle S_{\omega}'(\Phi+\vec{v}), \vec{\psi} \rangle \\ &= \Re \int_{\mathbb{R}^{N}} \{\nabla v_{1} \cdot \nabla \varphi_{\omega} + \omega v_{1} \varphi_{\omega} - \kappa | v_{1} | v_{1} \varphi_{\omega} - v_{1} (\varphi_{\omega} + \overline{v_{2}}) \varphi_{\omega} \} \, dx \\ &= \Re \int_{\mathbb{R}^{N}} \{ v_{1} (-\Delta \varphi_{\omega} + \omega \varphi_{\omega} - \varphi_{\omega}^{2}) - \kappa | v_{1} | v_{1} \varphi_{\omega} - v_{1} \overline{v_{2}} \varphi_{\omega} \} \, dx \\ &= -\kappa \Re \int_{\mathbb{R}^{N}} | v_{1} | v_{1} \varphi_{\omega} \, dx - \Re \int_{\mathbb{R}^{N}} v_{1} \overline{v_{2}} \varphi_{\omega} \, dx. \end{split}$$

Thus, we have

$$\langle S'_{\omega}(\Phi+\vec{v}),\vec{\psi}\rangle = -\kappa\Re \int_{\mathbb{R}^N} |v_1|v_1\psi_1\,dx - \Re \int_{\mathbb{R}^N} v_1\overline{v_2}\psi_1\,dx.$$
(4.1)

Here, we have

$$\operatorname{sgn}(\lambda) \cdot \kappa \Re \int_{\mathbb{R}^N} |\lambda \psi_1| \lambda \psi_1 \psi_1 \, dx = C_0 \lambda^2, \quad \text{where } C_0 := \kappa \|\varphi_\omega\|_{L^3}^3 / \|\varphi_\omega\|_{L^2}^3,$$

and the first term of the right hand side of (4.1) is estimated as follows.

$$\begin{aligned} \left| \operatorname{sgn}(\lambda) \cdot \kappa \Re \int_{\mathbb{R}^N} |v_1| v_1 \psi_1 \, dx - C_0 \lambda^2 \right| &\leq \kappa \int_{\mathbb{R}^N} \left| |v_1| v_1 - |\lambda \psi_1| \lambda \psi_1 \right| \psi_1 \, dx \\ &\leq C \int_{\mathbb{R}^N} (|v_1| + |\lambda \psi_1|) |v_1 - \lambda \psi_1| \psi_1 \, dx \leq C \int_{\mathbb{R}^N} (|\lambda \psi_1| + |z_1|) |z_1| \psi_1 \, dx \\ &\leq C |\lambda| \|z_1\|_{L^3} \|\psi_1\|_{L^3}^2 + C \|z_1\|_{L^3}^2 \|\psi_1\|_{L^3} \leq C_0 \lambda^2 / 4 + C_1 \|z_1\|_{H^1}^2 \end{aligned}$$

for some constant C_1 depending on φ_{ω} . Here, in the last inequality, we used the inequality of the type $2ab \leq \varepsilon^2 a^2 + b^2/\varepsilon^2$. While, the second term of the right hand side of (4.1) is estimated as follows.

$$\left| \Re \int_{\mathbb{R}^N} v_1 \overline{v_2} \psi_1 \, dx \right| \le |\lambda| ||z_2||_{L^3} ||\psi_1||_{L^3}^2 + ||z_1||_{L^3} ||z_2||_{L^3} ||\psi_1||_{L^3} \\ \le C_0 \lambda^2 / 4 + C_2 ||z_2||_{H^1}^2 + C_3 ||z_1||_{H^1} ||z_2||_{H^1}$$

for some positive constants C_2 and C_3 . Thus, we have

$$\operatorname{sgn} \lambda \cdot \langle S'_{\omega}(\Phi + \lambda \vec{\psi} + \vec{z}), \vec{\psi} \rangle \leq -C_0 \lambda^2 / 2 + C_4 \|\vec{z}\|_X^2$$

for some constant $C_4 > 0$. This completes the proof.

In the rest of this section, we give the proof of Proposition 4 by modifying the proof of Theorem 2 of [16]. We define

$$\mathcal{N}_{\varepsilon}(\vec{\phi}) = \{ \vec{u} \in X_{\mathrm{rad}} : \inf_{\theta \in \mathbb{R}} \| G(\theta)\vec{u} - \vec{\phi} \|_X < \varepsilon \},\$$

and the identification operator $I:X\to X^*$ by

$$\langle I\vec{u}, \vec{v} \rangle = (\vec{u}, \vec{v})_H, \quad \vec{u}, \vec{v} \in X$$

Lemma 8. There exist $\varepsilon > 0$ and a C^2 map $\Theta : \mathcal{N}_{\varepsilon}(\vec{\phi}) \to \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\begin{split} \|G(\Theta(\vec{u}))\vec{u} - \vec{\phi}\|_X &\leq \|G(\theta)\vec{u} - \vec{\phi}\|_X, \\ (G(\Theta(\vec{u}))\vec{u}, J\vec{\phi})_X &= 0, \quad \Theta(G(\theta)\vec{u}) = \Theta(\vec{u}) - \theta, \\ I^{-1}\Theta'(\vec{u}) &= \frac{JG(-\Theta(\vec{u}))(1 - \Delta)\vec{\phi}}{(G(\Theta(\vec{u}))\vec{u}, J^2\vec{\phi})_X} \end{split}$$
(4.2)

for all $\vec{u} \in \mathcal{N}_{\varepsilon}(\vec{\phi})$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Proof. See Lemma 3.2 of [11]. Note that $\vec{\phi} \in H^3(\mathbb{R}^N)^2$ by the elliptic regularity for (1.7).

We put $M(\vec{u}) = G(\Theta(\vec{u}))\vec{u}$. Then we have $M(\vec{\phi}) = \vec{\phi}$ and $M(G(\theta)\vec{u}) = M(\vec{u})$ for $\vec{u} \in \mathcal{N}_{\varepsilon}(\vec{\phi})$ and $\theta \in \mathbb{R}$. We define \mathcal{A} and Λ by

$$\mathcal{A}(\vec{u}) = (M(\vec{u}), J^{-1}\vec{\psi})_H, \quad \Lambda(\vec{u}) = (M(\vec{u}), \vec{\psi})_H \tag{4.3}$$

for $\vec{u} \in \mathcal{N}_{\varepsilon}(\vec{\phi})$. Then we have

$$JI^{-1}\mathcal{A}'(\vec{u}) = G(-\Theta(\vec{u}))\vec{\psi} - \Lambda(\vec{u})JI^{-1}\Theta'(\vec{u}), \qquad (4.4)$$

$$0 = \frac{d}{d\theta} \mathcal{A}(G(\theta)\vec{u})|_{\theta=0} = \langle \mathcal{A}'(\vec{u}), J\vec{u} \rangle = -\langle I\vec{u}, JI^{-1}\mathcal{A}'(\vec{u}) \rangle.$$
(4.5)

We define \mathcal{P} by

$$\mathcal{P}(\vec{u}) = \langle E'(\vec{u}), JI^{-1}\mathcal{A}'(\vec{u}) \rangle$$

for $\vec{u} \in \mathcal{N}_{\varepsilon}(\vec{\phi})$. Note that by (4.2), (4.4) and (4.5), we have

$$\mathcal{P}(\vec{u}) = \langle S'_{\omega}(\vec{u}), JI^{-1}\mathcal{A}'(\vec{u}) \rangle$$

= $\langle S'_{\omega}(M(\vec{u})), \vec{\psi} \rangle - \frac{\Lambda(\vec{u})}{(M(\vec{u}), J^2\vec{\phi})_X} \langle S'_{\omega}(M(\vec{u})), J^2(1-\Delta)\vec{\phi} \rangle.$ (4.6)

Lemma 9. Let \mathcal{I} be an interval of \mathbb{R} . Let $\vec{u} \in C(\mathcal{I}, X) \cap C^1(\mathcal{I}, X^*)$ be a solution of (1.1), and assume that $\vec{u}(t) \in \mathcal{N}_{\varepsilon}(\vec{\phi})$ for all $t \in \mathcal{I}$. Then

$$\frac{d}{dt}\mathcal{A}(\vec{u}(t)) = \mathcal{P}(\vec{u}(t)), \quad t \in \mathcal{I}.$$

Proof. See Lemma 4.6 of [11] and Lemma 2 of [16].

Lemma 10. There exist positive constants k^* and ε_0 such that

$$E(\vec{u}) \ge E(\vec{\phi}) + k^* \operatorname{sgn} \Lambda(\vec{u}) \cdot \mathcal{P}(\vec{u})$$

for all $\vec{u} \in \mathcal{N}_{\varepsilon_0}(\vec{\phi})$ satisfying $Q(\vec{u}) = Q(\vec{\phi})$.

Proof. We put $\vec{v} = M(\vec{u}) - \vec{\phi}$, and decompose \vec{v} as

$$\vec{v} = a\vec{\phi} + bJ\vec{\phi} + c\vec{\psi} + \vec{w},$$

where $a, b, c \in \mathbb{R}$, and $\vec{w} \in X_{\text{rad}}$ satisfies $(\vec{\phi}, \vec{w})_H = (J\vec{\phi}, \vec{w})_H = (\vec{\psi}, \vec{w})_H = 0$. Since $Q(\vec{\phi}) = Q(\vec{u}) = Q(M(\vec{u})) = Q(\vec{\phi}) + (\vec{\phi}, \vec{v})_H + Q(\vec{v})$ and $(\vec{\phi}, \vec{v})_H = a \|\vec{\phi}\|_H^2$, we have $a = O(\|\vec{v}\|_X^2)$. Moreover, by Lemma 8 and by the assumption (i) of Proposition 4, we have $(M(\vec{u}), J\vec{\phi})_X = 0$ and $(J\vec{\phi}, \vec{\psi})_X = 0$. Thus, we have $0 = (\vec{v}, J\vec{\phi})_X = b \|J\vec{\phi}\|_X^2 + (\vec{w}, J\vec{\phi})_X$, $|b|\|J\vec{\phi}\|_X \leq \|\vec{w}\|_X$ and

$$\|\vec{v}\|_X \le |c| \|\vec{\psi}\|_X + 2\|\vec{w}\|_X + O(\|\vec{v}\|_X^2).$$
(4.7)

Since $S'_{\omega}(\vec{\phi}) = 0$ and $Q(\vec{u}) = Q(\vec{\phi})$, by the Taylor expansion, we have

$$E(\vec{u}) - E(\vec{\phi}) = S_{\omega}(M(\vec{u})) - S_{\omega}(\vec{\phi}) = \frac{1}{2} \langle S_{\omega}''(\vec{\phi})\vec{v}, \vec{v} \rangle + o(\|\vec{v}\|_X^2).$$
(4.8)

Here, since $a = O(\|\vec{v}\|_X^2)$ and $S''_{\omega}(\vec{\phi})(J\vec{\phi}) = S''_{\omega}(\vec{\phi})\vec{\psi} = \vec{0}$, by the assumption (ii) of Proposition 4, we have

$$E(\vec{u}) - E(\vec{\phi}) = \frac{1}{2} \langle S''_{\omega}(\vec{\phi})\vec{v}, \vec{v} \rangle + o(\|\vec{v}\|_X^2)$$

$$= \frac{1}{2} \langle S''_{\omega}(\vec{\phi})\vec{w}, \vec{w} \rangle + o(\|\vec{v}\|_X^2) \ge \frac{k_0}{2} \|\vec{w}\|_X^2 - o(\|\vec{v}\|_X^2).$$
(4.9)

On the other hand, we have $c = (\vec{v}, \vec{\psi})_H = \Lambda(\vec{u}) = O(\|\vec{v}\|_X),$

$$S'_{\omega}(\vec{\phi} + \vec{v}) = S'_{\omega}(\vec{\phi}) + S''_{\omega}(\vec{\phi})\vec{v} + o(\|\vec{v}\|_X) = S''_{\omega}(\vec{\phi})\vec{w} + o(\|\vec{v}\|_X)$$

and $(M(\vec{u}), J^2 \vec{\phi})_X = (\vec{\phi}, J^2 \vec{\phi})_X + O(\|\vec{v}\|_X)$. Thus, by (4.6) we have

$$\mathcal{P}(\vec{u}) = \langle S'_{\omega}(\vec{\phi} + \vec{v}), \vec{\psi} \rangle + \frac{c}{\|J\vec{\phi}\|_X^2} \langle S''_{\omega}(\vec{\phi})\vec{w}, J^2(1-\Delta)\vec{\phi} \rangle + o(\|\vec{v}\|_X^2).$$

Here, by the assumption (iii) of Proposition 4, we have

$$sgn(c) \cdot \langle S'_{\omega}(\vec{\phi} + \vec{v}), \vec{\psi} \rangle$$

$$\leq -k_1 c^2 + k_2 \|a\vec{\phi} + bJ\vec{\phi} + \vec{w}\|_X^2 + o(c^2 + \|a\vec{\phi} + bJ\vec{\phi} + \vec{w}\|_X^2)$$

$$\leq -k_1 c^2 + k_3 \|\vec{w}\|_X^2 + o(\|\vec{v}\|_X^2).$$

Moreover, we have

$$\frac{c}{\|J\vec{\phi}\|_X^2} \langle S_{\omega}''(\vec{\phi})\vec{w}, J^2(1-\Delta)\vec{\phi} \rangle \right| \le k|c| \|\vec{w}\|_X \le \frac{k_1}{2}c^2 + k_4 \|\vec{w}\|_X^2.$$

Thus, we have

$$-\operatorname{sgn} \Lambda(\vec{u}) \cdot \mathcal{P}(\vec{u}) \ge \frac{k_1}{2}c^2 - k_5 \|\vec{w}\|_X^2 - o(\|\vec{v}\|_X^2)$$
(4.10)

with some constant $k_5 > 0$. By (4.9) and (4.10), we have

$$E(\vec{u}) - E(\vec{\phi}) - k^* \operatorname{sgn} \Lambda(\vec{u}) \cdot \mathcal{P}(\vec{u}) \ge k_6 c^2 + k_7 \|\vec{w}\|_X^2 - o(\|\vec{v}\|_X^2), \quad (4.11)$$

where we put $k^* = k_0/(4k_5)$, $k_6 = k^*k_1/2$ and $k_7 = k_0/4$. Finally, since $\|\vec{v}\|_X = \|M(\vec{u}) - \varphi_{\omega}\|_X < \varepsilon_0$, it follows from (4.7) that the right hand side of (4.11) is non-negative, if ε_0 is sufficiently small. This completes the proof. \Box

Lemma 11. There exist $\lambda_1 > 0$ and a continuous curve $(-\lambda_1, \lambda_1) \ni \lambda \mapsto \vec{\phi}_{\lambda} \in X_{\text{rad}}$ such that $\vec{\phi}_0 = \vec{\phi}$ and

$$E(\vec{\phi}_{\lambda}) < E(\vec{\phi}), \quad Q(\vec{\phi}_{\lambda}) = Q(\vec{\phi}), \quad \lambda \mathcal{P}(\vec{\phi}_{\lambda}) < 0$$

for $0 < |\lambda| < \lambda_1$.

Proof. For λ close to 0, we define

$$\vec{\phi}_{\lambda} = \vec{\phi} + \lambda \vec{\psi} + \sigma(\lambda) \vec{\phi}, \quad \sigma(\lambda) = \left(1 - \frac{Q(\vec{\psi})}{Q(\vec{\phi})} \lambda^2\right)^{1/2} - 1.$$

Then, we have $Q(\vec{\phi}_{\lambda}) = Q(\vec{\phi}), \ \sigma(\lambda) = O(\lambda^2), \ \sigma'(\lambda) = O(\lambda)$ and

$$S_{\omega}(\vec{\phi}_{\lambda}) - S_{\omega}(\vec{\phi}) = \int_{0}^{\lambda} \frac{d}{ds} S_{\omega}(\vec{\phi}_{s}) \, ds = \int_{0}^{\lambda} \langle S_{\omega}'(\vec{\phi}_{s}), \vec{\psi} + \sigma'(s)\vec{\phi} \rangle \, ds.$$

Here, by the assumption (iii) of Proposition 4, we have

$$\operatorname{sgn}(s) \cdot \langle S'_{\omega}(\vec{\phi}_s), \vec{\psi} \rangle \leq -k_1 s^2 + o(s^2)$$

Moreover, since $S'_{\omega}(\vec{\phi}_s) = S'_{\omega}(\vec{\phi}) + S''_{\omega}(\vec{\phi})(s\vec{\psi} + \sigma(s)\vec{\phi}) + o(s) = o(s)$, we have $\langle S'_{\omega}(\vec{\phi}_s), \sigma'(s)\vec{\phi} \rangle = o(s^2)$. Thus, we have $S_{\omega}(\vec{\phi}_{\lambda}) - S_{\omega}(\vec{\phi}) \leq -k_1|\lambda|^3/3 + o(\lambda^3)$. Finally, by (4.10), we have $\lambda \mathcal{P}(\vec{\phi}_{\lambda}) \leq -k_1|\lambda|^3/2 + o(\lambda^3)$.

Proof of Proposition 4. Suppose that $G(\omega t)\vec{\phi}$ is stable. For λ close to 0, let $\vec{\phi}_{\lambda} \in X_{\text{rad}}$ be the function given in Lemma 11, and let $\vec{u}_{\lambda}(t)$ be the solution of (1.1) with $\vec{u}_{\lambda}(0) = \vec{\phi}_{\lambda}$. Then, there exists $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$, then $\vec{u}_{\lambda}(t) \in \mathcal{N}_{\varepsilon_0}(\vec{\phi})$ for all $t \geq 0$, where ε_0 is the positive constant given in Lemma 10. Moreover, by the definition (4.3), there exist positive constants C_1 and C_2 such that $|\mathcal{A}(\vec{u})| \leq C_1$ and $|\Lambda(\vec{u})| \leq C_2$ for all $\vec{u} \in \mathcal{N}_{\varepsilon_0}(\vec{\phi})$. Let $-\lambda_0 < \lambda < 0$ and put $\delta_{\lambda} = E(\vec{\phi}) - E(\vec{\phi}_{\lambda})$. Since $\mathcal{P}(\vec{\phi}_{\lambda}) > 0$ and $t \mapsto \mathcal{P}(\vec{u}_{\lambda}(t))$ is continuous, by Lemma 10 and by the conservation laws of E and Q, we see that $\mathcal{P}(\vec{u}_{\lambda}(t)) > 0$ for all $t \geq 0$ and

$$\delta_{\lambda} = E(\vec{\phi}) - E(\vec{u}_{\lambda}(t)) \le -k^* \operatorname{sgn} \Lambda(\vec{u}_{\lambda}(t)) \cdot \mathcal{P}(\vec{u}_{\lambda}(t)) \le k^* C_2 \mathcal{P}(\vec{u}_{\lambda}(t))$$

for all $t \ge 0$. Moreover, by Lemma 9, we have

$$\frac{d}{dt}\mathcal{A}(\vec{u}_{\lambda}(t)) = \mathcal{P}(\vec{u}_{\lambda}(t)) \geq \frac{\delta_{\lambda}}{k^*C_2}$$

for all $t \ge 0$, which implies that $\mathcal{A}(\vec{u}_{\lambda}(t)) \to \infty$ as $t \to \infty$. This contradicts the fact that $|\mathcal{A}(\vec{u}_{\lambda}(t))| \le C_1$ for all $t \ge 0$. Hence, $G(\omega t)\vec{\phi}$ is unstable. \Box

5 Ground States

5.1 Existence and Stability of Ground States

In this subsection, we briefly recall the existence and stability of ground states for (1.7). We define

$$\|\vec{u}\|_{X_{\omega}}^{2} = \|\nabla\vec{u}\|_{H}^{2} + \omega\|\vec{u}\|_{H}^{2},$$

$$V(\vec{u}) = \kappa \|u_{1}\|_{L^{3}}^{3} + \|u_{2}\|_{L^{3}}^{3} + \frac{3}{2}\gamma\Re\int_{\mathbb{R}^{N}} u_{1}^{2}\overline{u_{2}} dx,$$

$$K_{\omega}(\vec{u}) = \|\vec{u}\|_{X_{\omega}}^{2} - V(\vec{u})$$

for $\vec{u} \in X$. Then the action S_{ω} associated with (1.7) is written as

$$S_{\omega}(\vec{u}) = \frac{1}{2} \|\vec{u}\|_{X_{\omega}}^2 - \frac{1}{3}V(\vec{u})$$

Remark that for $\vec{u} \in X$ satisfying $K_{\omega}(\vec{u}) = 0$, one has

$$S_{\omega}(\vec{u}) = \frac{1}{6} \|\vec{u}\|_{X_{\omega}}^2.$$
(5.1)

Moreover, we define

$$\mu(\omega) = \inf\{S_{\omega}(\vec{u}) : \vec{u} \in X, \ K_{\omega}(\vec{u}) = 0, \ \vec{u} \neq (0,0)\}$$
$$\mathcal{M}_{\omega} = \{\vec{\phi} \in X : S_{\omega}(\vec{\phi}) = \mu(\omega), \ K_{\omega}(\vec{\phi}) = 0\}.$$

The following Lemma 12 establishes the existence of a ground state for (1.7). Since it can be proved by the standard variational method (see [2, 13, 14, 24] and also [9, 17]), we omit the proof.

Lemma 12. Let $\kappa \in \mathbb{R}$, $\gamma > 0$ and $\omega > 0$. If $\{\vec{u}_n\} \subset X$ satisfies $S_{\omega}(\vec{u}_n) \rightarrow \mu(\omega)$ and $K_{\omega}(\vec{u}_n) \rightarrow 0$, then there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and $\vec{\phi} \in \mathcal{M}_{\omega}$ such that $\{\tau_{y_n}\vec{u}_n\}$ has a subsequence that converges to $\vec{\phi}$ strongly in X. Moreover, $\mathcal{M}_{\omega} = \mathcal{G}_{\omega}$ and $\mu(\omega) = d(\omega)$. As a consequence, the set \mathcal{G}_{ω} is not empty.

Next, we consider the stability of ground states. By the scale invariance of (1.7), we see that $d(\omega) = \omega^{3-N/2}d(1)$ for all $\omega > 0$. Since $N \leq 3$ and d(1) > 0, we have $d''(\omega) > 0$ for all $\omega > 0$. Using this fact and Lemma 12, the following Proposition 5 can be proved by the method of Shatah [18] (see also [9]). Since it is standard, we omit the proof.

Proposition 5. Let $\kappa \in \mathbb{R}$ and $\gamma > 0$. For any $\omega > 0$, the set \mathcal{G}_{ω} is stable in the following sense. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\vec{u}_0 \in X$ satisfies dist $(\vec{u}_0, \mathcal{G}_{\omega}) < \delta$, then the solution $\vec{u}(t)$ of (1.1) with $\vec{u}(0) = \vec{u}_0$ exists for all $t \ge 0$, and satisfies dist $(\vec{u}(t), \mathcal{G}_{\omega}) < \varepsilon$ for all $t \ge 0$, where we put

dist
$$(\vec{u}, \mathcal{G}_{\omega}) = \inf\{\|\vec{u} - \vec{\phi}\|_X : \vec{\phi} \in \mathcal{G}_{\omega}\}.$$

5.2 Preliminaries from Elementary Geometry

In this section, we explain some basic geometric properties concerning the line and the ellipse defined by (1.8). In the proof of Theorem 6, one has to compare, for a given $(\alpha, \beta) \in S_{\kappa,\gamma}$, the quantities $\alpha^2 + \beta^2$ and 1. This is the purpose of Lemmas 14 and 15.

Lemma 13. Let $\gamma > 0$ and $0 < r \le 1$, and put

$$B = \{ (x, y) \in]0, \infty[^2: \gamma x^2 + 2y^2 = 2y, \ x^2 + y^2 = r^2 \}.$$

- (1) If 0 < r < 1, then B consists of one point.
- (2) If r = 1 and $0 < \gamma < 1$, then $B = \{(2\sqrt{1-\gamma}/(2-\gamma), \gamma/(2-\gamma))\}$.
- (3) If r = 1 and $\gamma \ge 1$, then B is empty.

Proof. First we prove (1). Let 0 < r < 1. Recall that $\gamma x^2 + 2y^2 = 2y$ is an ellipse with vertices (x, y) = (0, 0), (0, 1) and $(\pm 1/\sqrt{2\gamma}, 1/2)$, and that $B \subset \{(x, y) : 0 < y < 1\}$. By the equations in B, we have $g(y) := (2 - \gamma)y^2 - 2y + \gamma r^2 = 0$. Since $g(0) = \gamma r^2 > 0$ and $g(1) = \gamma (r^2 - 1) < 0$, the equation g(y) = 0 has only one solution in]0, 1[. This proves (1). Parts (2) and (3) are obtained by direct computations.

Lemma 14. Let $(\kappa, \gamma) \in \mathcal{J}_1 \cup \mathcal{J}_2$. Then, $\alpha_+^2 + \beta_-^2 = 1$ if and only if $(\kappa, \gamma) \in \mathcal{K}_3$.

Proof. Assume that (α_+, β_-) satisfies $\alpha_+^2 + \beta_-^2 = 1$. Since (α_+, β_-) satisfies $\gamma \alpha_+^2 + 2\beta_-^2 = 2\beta_-$, it follows from (2) and (3) of Lemma 13 that $0 < \gamma < 1$ and $(\alpha_+, \beta_-) = (2\sqrt{1-\gamma}/(2-\gamma), \gamma/(2-\gamma))$. Substituting this into $\kappa \alpha_+ + \gamma \beta_- = 1$, we have $\kappa = \kappa_c(\gamma)$. Thus, $(\kappa, \gamma) \in \mathcal{K}_3$. Conversely, it is easy to see that $\alpha_+^2 + \beta_-^2 = 1$ if $(\kappa, \gamma) \in \mathcal{K}_3$.

Lemma 15. If $(\kappa, \gamma) \in \mathcal{K}_1$, then $\alpha_+^2 + \beta_-^2 < 1$.

Proof. First, we remark that the function $f(\kappa, \gamma) := \alpha_+^2 + \beta_-^2 - 1$ is continuous in $\mathcal{J}_1 \cup \mathcal{J}_2$, and that \mathcal{K}_1 is a connected subset of $\mathcal{J}_1 \cup \mathcal{J}_2$. By Lemma 14, f has no zeros in \mathcal{K}_1 . Thus, f has a constant sign in \mathcal{K}_1 . Finally, since $f(0, \gamma) \to -1$ as $\gamma \to \infty$, we conclude that $f(\kappa, \gamma) < 0$ for all $(\kappa, \gamma) \in \mathcal{K}_1$. \Box

In the same way as Lemma 15, we see that $\alpha_+^2 + \beta_-^2 > 1$ for $(\kappa, \gamma) \in \mathcal{K}_2 \cap \mathcal{J}_2$, but this fact is not used in what follows. The following Lemma 16 plays an important role in the proof of Lemma 17.

Lemma 16. Let $(\kappa, \gamma) \in \mathcal{J}_1 \cup \mathcal{J}_2$. Then, $\gamma \alpha_+ > \kappa \beta_-$.

Proof. Since γ , α_+ and β_- are positive, the inequality is trivial for the case $\kappa \leq 0$. Let $\kappa > 0$ and put $D = \kappa^2 + 2\gamma(\gamma - 1)$. Then, D > 0 and

$$\gamma \alpha_{+} > \kappa \beta_{-} \iff (2 - \gamma)\gamma \kappa + \gamma^{2}\sqrt{D} > \kappa(\kappa^{2} + \gamma^{2}) - \kappa^{2}\sqrt{D}$$
$$\iff (\gamma^{2} + \kappa^{2})\sqrt{D} > \kappa D.$$

Since $(\gamma^2 + \kappa^2)^2 - \kappa^2 D = \gamma^4 + 2\gamma\kappa^2 > 0$, the last inequality holds.

We define

$$\ell = \begin{cases} \alpha_+^2 + \beta_-^2 & \text{if } (\kappa, \gamma) \in \mathcal{K}_1, \\ 1 & \text{if } (\kappa, \gamma) \in \mathcal{K}_2 \cup \mathcal{K}_3, \end{cases}$$
(5.2)

and for a given (κ, γ) ,

$$E_1 = \{ (x, y) \in]0, \infty[^2: \kappa x + \gamma y \ge 1 \},\$$

$$E_2 = \{ (x, y) \in]0, \infty[^2: \gamma x^2 + 2y^2 \ge 2y, \ x^2 + y^2 \le \ell \}.$$

In Lemmas 17 and 18, we establish the structure of the set $E_1 \cap E_2$ with respect to (κ, γ) .

Lemma 17. If $(\kappa, \gamma) \in \mathcal{K}_1 \cup \mathcal{K}_3$, then $E_1 \cap E_2 = \{(\alpha_+, \beta_-)\}$.

Proof. Since it is clear that $\{(\alpha_+, \beta_-)\} \subset E_1 \cap E_2$, we prove the inverse inclusion. By Lemmas 13, 14 and 15, we see that the ellipse $\gamma x^2 + 2y^2 = 2y$ and the circle $x^2 + y^2 = \alpha_+^2 + \beta_-^2$ intersect at only one point $(x, y) = (\alpha_+, \beta_-)$ in $\{(x, y) : x > 0\}$. The normal $y = f_1(x)$ and the tangent $y = f_2(x)$ of the circle $x^2 + y^2 = \alpha_+^2 + \beta_-^2$ at the point $(x, y) = (\alpha_+, \beta_-)$ are given by

$$f_1(x) = \frac{\beta_-}{\alpha_+}(x - \alpha_+) + \beta_-, \quad f_2(x) = -\frac{\alpha_+}{\beta_-}(x - \alpha_+) + \beta_-,$$

and we see that $E_2 \subset E_3 := \{(x, y) : y \leq f_1(x), y \leq f_2(x)\}$. By Lemma 16 and by $\kappa \alpha_+ + \gamma \beta_- = 1$, we have $-\alpha_+/\beta_- < -\kappa/\gamma < \beta_-/\alpha_+$. That is, the slope of the line $\kappa x + \gamma y = 1$ is less than that of the normal $y = f_1(x)$, and is greater than or equal to that of the tangent $y = f_2(x)$. Recalling that (α_+, β_-) is on the line $\kappa x + \gamma y = 1$, we see that $E_1 \cap E_2 \subset E_1 \cap E_3 =$ $\{(\alpha_+, \beta_-)\}$. This completes the proof. \Box

Lemma 18. If $(\kappa, \gamma) \in \mathcal{K}_2$, then $E_1 \cap E_2$ is empty.

Proof. First, we consider the case where $\kappa \leq 0$ and $0 < \gamma \leq 1$. Then, $E_1 \subset \{(x, y) : y \geq 1/\gamma\} \subset \{(x, y) : y \geq 1\}$, and we see that $E_1 \cap E_2$ is empty.

Next, we consider the case where $0 < \gamma < 1$ and $0 < \kappa < \kappa_c(\gamma)$. We fix $\gamma \in$]0, 1[and denote $E_1 = E_1(\kappa)$ for $0 < \kappa \leq \kappa_c$. Remark that E_2 is independent of $\kappa \in$]0, κ_c]. When $\kappa = \kappa_c$, by Lemma 17, we have $E_1(\kappa_c) \cap E_2 = \{(\alpha_+, \beta_-)\}$. Moreover, when $0 < \kappa < \kappa_c$, $E_1(\kappa)$ is strictly smaller than $E_1(\kappa_c)$. Thus, we see that $E_1(\kappa) \cap E_2$ is empty if $0 < \kappa < \kappa_c$.

5.3 Determination of Ground States

We are now able to determine the structure of the set \mathcal{G}_{ω} . We use an idea of Sirakov [20] (see also [12]). We denote

$$\|u\|_{H^1_{\omega}}^2 = \|\nabla u\|_{L^2}^2 + \omega \|u\|_{L^2}^2, \quad u \in H^1(\mathbb{R}^N).$$

Lemma 19. Let $\vec{u} = (u_1, u_2) \in \mathcal{A}_{\omega}$. Then we have

$$\|u_1\|_{H^1_{\omega}}^2 = \kappa \|u_1\|_{L^3}^3 + \gamma \int_{\mathbb{R}^N} \overline{u_1}^2 u_2 \, dx,$$

$$\|u_2\|_{H^1_{\omega}}^2 = \|u_2\|_{L^3}^3 + \frac{\gamma}{2} \int_{\mathbb{R}^N} u_1^2 \overline{u_2} \, dx.$$

Proof. The first identity is obtained by multiplying the first equation of (1.7) by $\overline{u_1}$ and by integrating by parts. The second identity is obtained in the same way.

Lemma 20. For any $\omega > 0$, $6d(\omega) \le \ell \|\varphi_{\omega}\|_{L^3}^3$, where ℓ is the number defined by (5.2).

Proof. Let $(\alpha \varphi_{\omega}, \beta \varphi_{\omega}) \in \mathcal{A}_{\omega}$. Then, we have $K_{\omega}(\alpha \varphi_{\omega}, \beta \varphi_{\omega}) = 0$, and so by (5.1),

$$S_{\omega}(\alpha\varphi_{\omega},\beta\varphi_{\omega}) = \frac{1}{6} \|(\alpha\varphi_{\omega},\beta\varphi_{\omega})\|_{X_{\omega}}^{2} = \frac{\alpha^{2}+\beta^{2}}{6} \|\varphi_{\omega}\|_{H_{\omega}^{1}}^{2}.$$

Moreover, since φ_{ω} is a solution of (1.6), we have $\|\varphi_{\omega}\|_{H^{1}_{\omega}}^{2} = \|\varphi_{\omega}\|_{L^{3}}^{3}$, and $6S_{\omega}(\alpha\varphi_{\omega},\beta\varphi_{\omega}) = (\alpha^{2} + \beta^{2})\|\varphi_{\omega}\|_{L^{3}}^{3}$. Finally, by the definitions of $d(\omega)$ and ℓ , we obtain the desired estimate.

The following variational characterization of φ_{ω} is well-known (see [3, 12, 20, 24]), and we omit the proof.

Lemma 21. Let $\omega > 0$. Then

$$\|\varphi_{\omega}\|_{L^{3}} = \inf\left\{\|v\|_{H^{1}_{\omega}}^{2}/\|v\|_{L^{3}}^{2} : v \in H^{1}(\mathbb{R}^{N}) \setminus \{0\}\right\}.$$
(5.3)

Moreover,

$$\{v \in H^{1}(\mathbb{R}^{N}) : \|v\|_{H^{1}_{\omega}}^{2} = \|v\|_{L^{3}}^{3} = \|\varphi_{\omega}\|_{L^{3}}^{3} \}$$
$$= \{e^{i\theta}\varphi_{\omega}(\cdot + y) : \theta \in \mathbb{R}, \ y \in \mathbb{R}^{N} \}.$$
(5.4)

The next lemma is linked to the key Lemmas 17 and 18.

Lemma 22. Let $(u_1, u_2) \in \mathcal{G}_{\omega}$, and put

$$a = ||u_1||_{L^3} / ||\varphi_\omega||_{L^3}, \quad b = ||u_2||_{L^3} / ||\varphi_\omega||_{L^3}.$$

Then, $a \ge 0$, b > 0, and (a, b) satisfies

$$a^{2} \leq a^{2}(\kappa a + \gamma b), \quad 2b \leq 2b^{2} + \gamma a^{2}, \quad a^{2} + b^{2} \leq \ell.$$
 (5.5)

Moreover,

- (1) If $(\kappa, \gamma) \in \mathcal{K}_1$, then $(a, b) = (\alpha_+, \beta_-)$.
- (2) If $(\kappa, \gamma) \in \mathcal{K}_2$, then (a, b) = (0, 1).
- (3) If $(\kappa, \gamma) \in \mathcal{K}_3$, then $(a, b) \in \{(\alpha_+, \beta_-), (0, 1)\}$.

Proof. We first prove (5.5). If $u_2 = 0$, then the second equation of (1.7) implies $u_1 = 0$. This contradicts $(u_1, u_2) \in \mathcal{A}_{\omega}$. Thus, $u_2 \neq 0$ and b > 0. By (5.3), Lemma 19 and the Hölder inequality, we have

$$\begin{aligned} \|\varphi_{\omega}\|_{L^{3}} \|u_{1}\|_{L^{3}}^{2} &\leq \|u_{1}\|_{H^{1}_{\omega}}^{2} = \kappa \|u_{1}\|_{L^{3}}^{3} + \gamma \int_{\mathbb{R}^{N}} \overline{u_{1}}^{2} u_{2} \, dx \\ &\leq \kappa \|u_{1}\|_{L^{3}}^{3} + \gamma \|u_{1}\|_{L^{3}}^{2} \|u_{2}\|_{L^{3}}, \end{aligned}$$
(5.6)

which provides $a^2 \leq a^2(\kappa a + \gamma b)$. In the same way, we have

$$\begin{aligned} \|\varphi_{\omega}\|_{L^{3}} \|u_{2}\|_{L^{3}}^{2} &\leq \|u_{2}\|_{H^{1}_{\omega}}^{2} = \|u_{2}\|_{L^{3}}^{3} + \frac{\gamma}{2} \int_{\mathbb{R}^{N}} u_{1}^{2} \overline{u_{2}} \, dx \\ &\leq \|u_{2}\|_{L^{3}}^{3} + \frac{\gamma}{2} \|u_{1}\|_{L^{3}}^{2} \|u_{2}\|_{L^{3}}. \end{aligned}$$

$$(5.7)$$

Since b > 0, this gives $2b \le 2b^2 + \gamma a^2$. Finally, by Lemma 20 and (5.3), we obtain

$$\ell \|\varphi_{\omega}\|_{L^{3}}^{3} \ge 6d(\omega) = 6S_{\omega}(\vec{u}) = \|\vec{u}\|_{X_{\omega}}^{2} \ge \|\varphi_{\omega}\|_{L^{3}} \Big(\|u_{1}\|_{L^{3}}^{2} + \|u_{2}\|_{L^{3}}^{2}\Big), \quad (5.8)$$

which implies $a^2 + b^2 \leq \ell$. Hence, (5.5) is proved.

We now prove (1), (2) and (3). Let $(\kappa, \gamma) \in \mathcal{K}_1$. Then, since $\ell < 1$, by Lemma 22, we see that a > 0 and $(a, b) \in E_1 \cap E_2$. Thus, (1) follows from (5.5). Next, let $(\kappa, \gamma) \in \mathcal{K}_2$. Suppose that a > 0. Then, by (5.5), we have $(a, b) \in E_1 \cap E_2$. However, this contradicts Lemma 18. Thus, we have a = 0and b = 1, which proves (2). Part (3) can be proved similarly. \Box Proof of Theorem 6. We consider the case $(\kappa, \gamma) \in \mathcal{K}_1$. Let $\vec{u} \in \mathcal{G}_{\omega}$. By (5.6), (5.7) and Lemma 22, we see that

$$\|u_1/\alpha_+\|_{H^1_{\omega}}^2 = \|\varphi_{\omega}\|_{L^3}^3 = \|u_1/\alpha_+\|_{L^3}^3,$$
(5.9)

$$\|u_2/\beta_-\|_{H^1_{\omega}}^2 = \|\varphi_{\omega}\|_{L^3}^3 = \|u_2/\beta_-\|_{L^3}^3, \tag{5.10}$$

$$\int_{\mathbb{R}^N} u_1^2 \overline{u_2} \, dx = \|u_1\|_{L^3}^2 \|u_2\|_{L^3}.$$
(5.11)

By (5.4) and by (5.9) and (5.10), there exist (θ_1, y_1) , $(\theta_2, y_2) \in \mathbb{R} \times \mathbb{R}^N$ such that $u_1 = e^{i\theta_1} \alpha_+ \varphi_\omega(\cdot + y_1)$ and $u_2 = e^{i\theta_2} \beta_- \varphi_\omega(\cdot + y_2)$. Moreover, by (5.11), we see that $2\theta_1 - \theta_2 \in 2\pi\mathbb{Z}$ and $y_1 = y_2$. Thus, we have $\mathcal{G}_\omega \subset \mathcal{G}_\omega^1$. Since \mathcal{G}_ω is not empty, (1) is proved. (2) and (3) can be proved in the same way. \Box

Finally, Theorem 5 is obtained as a corollary of Theorem 6.

Proof of Theorem 5. Let $\kappa \leq 0$ and $\gamma = 1$. Then, $(\kappa, \gamma) \in \mathcal{K}_2$, and by Theorem 6, $\mathcal{G}_{\omega} = \mathcal{G}_{\omega}^0$. Thus, Theorem 5 follows from Proposition 5.

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