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Boundary controllability and observability of a viscoelastic string*

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Abstract

In this paper we consider an integrodifferential system, which governs the vibration of a viscoelastic one-dimensional object. We assume that we can act on the system at the boundary and we prove that it is possible to control both the position and the velocity at every point of the body and at a certain time T, large enough. We shall prove this result using moment theory and we shall prove that the solution of this problem leads to identify a Riesz sequence which solves controllability

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and observability. So, the result as presented here are constructive and can lead to simple numerical algorithms.

Key words observability/controllability, integrodifferential system, moment problem, viscoelasticity

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1 Introduction

In this paper we study observability/controllability properties for a viscoelastic string, described by the equation

$$w_{tt}(t,x) = w_{xx}(t,x) + \int_0^t M(t-s) w_{xx}(s,x)\,\mathrm{d}s\,, \tag{1}$$

with initial and boundary conditions

$$\begin{cases} w(0,x) = \xi_0(x), & w_t(0,x) = \xi_1(x) \\ w(t,0) = g(t) \in L^2_{\mathrm{loc}}(0,+\infty)\,, & w(t,\pi) = 0\,. \end{cases} \tag{2}$$

The kernel $M(t)$ is of class $H^2(0,T)$ for every $T > 0$.

Our goal is to prove the existence of a control g which steers the system from the initial configuration to a given final configuration at a suitable time T and, more important, to give a formula for the steering control wich is amenable to numerical computations. A constructive approach to controllability for a system with memory is given in [27, 28], where the control problem is reduced to the solution of a moment problem respect to a Riesz sequence expecially taylored to the system we are studying. These papers study controllability/observability solely of the position $w(t,x)$ and ignore velocity. In fact, the equation in these papers is of first order in time (a case encountered in thermodynamics and nonfickian diffusion). In contrast with this, [22, 23] studies controllability of both position and velocity using moment methods, when the kernel has a special form. In this paper we extend the results in [22, 23], using ideas from [27, 28].
When studying the observability problem we shall consider the case

$$g(t) \equiv 0 \tag{3}$$

and the following boundary observation

$$y(t) = w_x(t, 0) + \int_0^t M(t - s) w_x(s, 0) \, \mathrm{d}s \, . \tag{4}$$

Observation of similar form has been considered in [24] for a particular kernel and in [23]. From a physical point of view, $y(t)$ represents the traction on the boundary, see [6, p. 286]. Due to the fact that $M(t)$ is smooth, it is equivalent to assume that the observation is $Y(t) = w_x(t, 0)$. This fictitious observation will be used in our computations but it does not have a physical meaning. So, it has an interest to see that the "physical" observation (4) is also natural from the point of view of duality, i.e. HUM method, see Section 1.3.

The content of the paper is as follows: we shall prove Theorems 1 and 2 on observability and the corresponding result for controllability, i.e. Theorem 3 is then immediate either using (HUM) or using a direct analysis of the formula for the solutions. Both the ways have their interest, the first one since it shows that the output (4) is the natural observation associated to the control problem; the direct analysis of the formula for the solution has its interest since it can be used in numerical methods. So, the organization of the paper is as follows: first, in Section 1.2, we state our observability and controllability results. In Section 1.3 we show how HUM method can be adapted to system (1). Then (Sections 2 and 3) the observability problem is reduced to a moment problem and the observability inequalities are proved, thanks to the fact that we can identify new Riesz sequences associated to Eq. (1) and this is one of the point of interest of our results: Riesz sequences are important objects in themselves, which can be used for the actual computation of controls, see Section 4.

The key result in this paper is Theorem 4 whose technical proof is in the appendix.

1.1 References

Controllability properties of distributed systems like (1) have been studied in several papers, both under the action of distributed or boundary controls, usually with nonconstructive methods. The following papers study the controllability of solely the position $w(t, \cdot)$ (systems are written as first order in time): [5] study the controllability to a smooth target, when the kernel is in particular of class C^∞; paper [8] studies even the multidimensional non-isotropic problem (kernel of class C^3) using Carleman estimates, when the control is distributed. Paper [25, 26] study the multidimensional isotropic problem (kernel of class C^3) and boundary control, using operator methods. The controllability time is not identified in these last papers. As we said already, papers [27, 28, 29] introduce the moment method approach we are going to extend here.

Papers which study second order systems (in time) are [16, 17] and, most interesting to us, [22, 23] whose results we are going to extend. Moreover, we cite [12, 13, 14, 15]. Paper [13] is particularly interesting since it proves the same controllability and observability problems as here, even when $\dim x > 1$, using nonconstructive methods (i.e. compactness and a contradiction argument).

Finally, we cite [3, 4], where controllability of the pair of $w(t)$ and of the traction (actually, temperature and flux in the interpretation of those papers) is studied and the recent preprint [1], which studies a problem similar to the one treated here.

Background on moment methods and Riesz basis can be found in the books [2, 10, 21, 31].

1.2 Main Results

Our main result is the following one:

Theorem 1 *Let $T > 2\pi$ and let $w(t,x)$ solve problem (1)-(3). There exist positive constants C_1, C_2 such that*

$$C_1 \left\{ \|\xi_0\|_{H^1_0(0,\pi)} + \|\xi_1\|_{L^2(0,\pi)} \right\} \le \left(\int_0^T \|y(s)\|^2 \, dt \right)^{1/2}$$
$$\le C_2 \left\{ \|\xi_0\|_{H^1_0(0,\pi)} + \|\xi_1\|_{L^2(0,\pi)} \right\} . \tag{5}$$

Equivalently, there exist positive constants C_1 and C_2 such that

$$C_1 \left\{ \|\xi_0\|_{H^1_0(0,\pi)} + \|\xi_1\|_{L^2(0,\pi)} \right\} \le \left(\int_0^T \|w_x(s,0)\|^2 \, \mathrm{d}s \right)^{1/2}$$
$$\le C_2 \left\{ \|\xi_0\|_{H^1_0(0,\pi)} + \|\xi_1\|_{L^2(0,\pi)} \right\} . \tag{6}$$

We observe that the inequalities from above hold for every T, also $T \le 2\pi$. So, we have the following result, in particular observability for $T > 2\pi$:

Theorem 2 *Let us consider system (1)-(2) and let $g(t) \equiv 0$. The transformations*

$$H_0^1(0,\pi) \times L^2(0,\pi) \ni (\xi_0, \xi_1) \mapsto y(\cdot) \in L^2(0,T)$$

is (linear) continuous for every $T > 0$ and it is injective with continuous inverse if $T > 2\pi$.

A consequence of Theorem 2 and known properties of Volterra integral equations applied to (4) is that, when $g(t) \equiv 0$,

$$t \mapsto Y(t) = w_x(t,0)$$

belongs to $L^2(0,T)$ and the relation between $y(\cdot)$ and $w_x(t,0)$ is continuous and continuously invertible on $L^2(0,T)$. This explains the reason why it is equivalent, and computationally easier, to assume that the observation is $Y(t) = w_x(t,0)$.

The controllability result is:

Theorem 3 *Let us consider system (1)-(2) and let $T > 2\pi$. For every target $(w_0, w_1) \in L^2(0,\pi) \times H^{-1}(0,\pi)$ it is possible to construct a function $g(t) \in L^2(0,T)$ which drives the solution $(w(t,\cdot), w_t(t,\cdot))$ from the null initial condition to the prescribed target.*

1.3 HUM method and systems with memory

In this section we apply HUM method in order to relate observability and controllability problems. From one side this shows that the "natural" observation associated to the control problem is (4) and we see that the proof of Theorem 3 can be derived from inequalities (6). From the other side HUM method combined with the series expansion of the solution which we give in Section 2 is also usefull for the numerical solution of the control problem, see Section 4. In the proof we shall also see that the solution $(w(t,\cdot), w_t(t,\cdot))$ to Eq. (1)-(2) takes values in $L^2(0,\pi) \times H^{-1}(0,\pi)$ when the boundary control is square integrable (see also [13]).

As usual for linear equations, it is enough that we prove reachability from null initial conditions: $\xi_0 = 0$, $\xi_1 = 0$.

We associate the following "adjoint system" to Eq. (1):

$$\eta_{tt}(t,x)=\eta_{xx}(t,x)+\int_t^T M(s-t)\eta_{xx}(s,x)\,\mathrm{d}s\,, \tag{7}$$

to be solved backward, with the following final and boundary conditions

$$\eta(T,x)=\eta_0(x)\in H_0^1(0,\pi)\,,\ \ \eta_t(T,x)=\eta_1(x)\in L^2(0,\pi)\,,$$
$$\eta(t,0)=\eta(t,\pi)=0\,.$$

It is clear that Theorem 2 can be adapted to this backward system. In particular, $\eta_x(\cdot,0)\in L^2(0,T)$ depends continuously on the "final data".

We multiply both sides of Eq. (1) with $\eta(t,x)$ and we integrate on $(0,T)\times(0,\pi)$. We get three terms, one on the left and two on the right hand sides, which can be integrated by parts (as usual, working with smooth data and extending the equalities by continuity). Taking into account initial, final and boundary conditions we get

$$\int_0^\pi\int_0^T w_{tt}(t,x)\eta(t,x)\,\mathrm{d}t\ \mathrm{d}x$$
$$=\int_0^\pi [w_t(T,x)\eta_0(x)-w(T,x)\eta_1(x)]\ \mathrm{d}x+\int_0^\pi\int_0^T w(t,x)\eta_{tt}(t,x)\,\mathrm{d}t\ \mathrm{d}x\,,$$
$$\int_0^T\int_0^\pi w_{xx}(t,x)\eta(t,x)\,\mathrm{d}x\ \mathrm{d}t$$
$$=\int_0^T g(t)\eta_x(t,0)\,\mathrm{d}t+\int_0^T\int_0^\pi w(t,x)\eta_{xx}(t,x)\,\mathrm{d}x\ \mathrm{d}t\,,$$
$$\int_0^T\int_0^\pi\left[\int_0^t M(t-s)w_{xx}(s,x)\,\mathrm{d}s\right]\eta(t,x)\,\mathrm{d}x\ \mathrm{d}t=\int_0^T g(t)\int_t^T M(r-t)\eta_x(r,0)\,\mathrm{d}r\ \mathrm{d}t$$
$$+\int_0^T\int_0^\pi w(t,x)\left[\int_t^T M(r-t)\eta_{xx}(r,x)\,\mathrm{d}r\right]\ \mathrm{d}x\ \mathrm{d}t\,.$$

We equate the terms on the right and left sides, and we take into account that η solves Eq. (7). We remain with the following equality:

$$\int_0^\pi [w_t(T,x)\eta_0(x)-w(T,x)\eta_1(x)]\ \mathrm{d}x$$
$$=\int_0^T g(t)\left[\eta_x(t,0)+\int_t^T M(s-t)\eta_x(s,0)\,\mathrm{d}s\right]\ \mathrm{d}t\,. \tag{8}$$

In conclusion, we obtain:

- for $(\eta_0, \eta_1) \in H^1_0(0,\pi) \times L^2(0,\pi)$ fixed, the transformation

$$L^2(0,T) \ni g(\cdot) \mapsto (w(T,\cdot), w_t(T,\cdot))$$

is continuous, with values in $L^2(0,\pi) \times H^{-1}(0,\pi)$. Furthermore,

$$\sup_{t\in[0,T]} \| (w(t,\cdot), w_t(t,\cdot)) \|_{L^2(0,\pi)\times H^{-1}(0,\pi)} \le M \|g(\cdot)\|_{L^2(0,T)} .$$

As usual, this implies that the function $t \mapsto (w(t,\cdot), w_t(t,\cdot))$ is continuous, with values in $L^2(0,\pi) \times H^{-1}(0,\pi)$.

- If we choose

$$g(t) = \eta_x(t,0) + \int_t^T M(s-t)\eta_x(s,0)\, \mathrm{d}s$$

then the transformation

$$H^1_0(0,\pi) \times L^2(0,\pi) \ni (\eta_0, \eta_1) \mapsto g \in L^2(0,T)$$

is continuous. So, the linear operator L:

$$L(\eta_0,\eta_1) = (w_t(T,\cdot), -w(T,\cdot)) : \quad H^1_0(0,\pi)\times L^2(0,\pi) \mapsto H^{-1}(0,\pi)\times L^2(0,\pi) \tag{9}$$

is continuous and equality (8) gives

$$\langle\!\langle L(\eta_0,\eta_1), (\eta_0,\eta_1)\rangle\!\rangle \ge (\text{const}) \left\{ \|(\eta_0,\eta_1)\|^2_{H^1_0(0,\pi)\times L^2(0,\pi)} \right\}$$

($\langle\!\langle \cdot,\cdot \rangle\!\rangle$ denotes the duality pairing of $H^1_0(0,\pi) \times L^2(0,\pi)$ and its dual $H^{-1}(0,\pi) \times L^2(0,\pi)$). Consequently, the operator L is surjective, i.e. the controllability property stated in Theorem 3 is a consequence of Theorem 2.

2 Representation of the solutions and the observability/controllability theorems

The boundary value problem (1)-(2) has already been studied in the paper cited in Section 1.1. Nevertheless we shall give a suitable representation of the solutions in terms of a fundamental system of solutions, which is independent of initial conditions.

First, we consider the operator A on $L^2(0,\pi)$ defined as

$$Aw := w_{xx}, \qquad w \in \mathrm{dom}\, A := H^2(0,\pi) \cap H_0^1(0,\pi)\,.$$

The normalized eigenfunctions ϕ_n of A are given by

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)\,, \qquad n \geq 1\,,$$

and the corresponding eigenvalues are $\lambda_n = -n^2$. If we take the initial values $\xi_0 \in H_0^1(0,\pi)$ and $\xi_1 \in L^2(0,\pi)$, then we have

$$\xi_0(x) = \sum_{n=1}^{\infty} \phi_n(x)\xi_{0n}\,, \quad \xi_1(x) = \sum_{n=1}^{\infty} \phi_n(x)\xi_{1n} \quad \xi_{0n} = \langle \xi_0, \phi_n\rangle\,, \quad \xi_{1n} = \langle \xi_1, \phi_n\rangle\,,$$

($\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(0,\pi)$) are such that the sequences $\{n\xi_{0n}\}$ and $\{\xi_{1n}\}$ belong to l^2. So, the solution of Eq. (1) with initial conditions (2) can be written in the form

$$w(t,x) = \sum_{n=1}^{\infty} \phi_n(x) w_n(t)\,, \tag{10}$$

where the functions $w_n(t)$ solve the equation

$$z'' = -n^2 z - n^2 \int_0^t M(t-s)z(s)\,\mathrm{d}s + nf(t)\,, \qquad f(t) = \sqrt{\frac{2}{\pi}}\left[g(t) + \int_0^t M(t-s)g(s)\,\mathrm{d}s\right] \tag{11}$$

with initial conditions $w_n(0) = \xi_{0n}$, $w_n'(0) = \xi_{1n}$ (here $n \geq 1$). Then we have

$$w_n(t) = \xi_{0n} z_{1n}(t) + \frac{\xi_{1n}}{n} z_{2n}(t) + \int_0^t z_{2n}(t-s)f(s)\,\mathrm{d}s\,, \qquad n \geq 1\,, \tag{12}$$

where $z_{1n}(t)$ and $z_{2n}(t)$ solve Eq. (11) with $f = 0$ and initial conditions given by, respectively,

$$z_{1n}(0) = 1\,, \quad z_{1n}'(0) = 0\,, \qquad z_{2n}(0) = 0\,, \quad z_{2n}'(0) = n\,.$$

Note that

$$z_{2n}'(t) = nz_{1n}(t)\,.$$

Let

$$z_n(t) := z_{1n}(t) + iz_{2n}(t) \quad n \geq 1 \qquad \text{so that} \quad \left\{ \begin{array}{rcl} z_n(0) & = & 1\,, \\ z_n'(0) & = & in\,. \end{array} \right. \tag{13}$$

Then $w(t,x)$ and $w_t(t,x)$ are given by and

$$\left\{ \begin{array}{rcl} w(t,x) & = & \sum_{n=1}^{\infty} \phi_n(x) \Big\{ \Big[\frac{1}{2} \Big[\xi_{0n} - i\frac{\xi_{1n}}{n} \Big] z_n(t) + \frac{1}{2} \Big[\xi_{0n} + i\frac{\xi_{1n}}{n} \Big] \overline{z_n}(t) \Big] \\ & & + \int_0^t z_{2n}(t-s) f(s) \, \mathrm{d}s \Big\}\,, \\ w_t(t,x) & = & \sum_{n=1}^{\infty} \phi_n(x) \Big\{ \Big[\frac{1}{2} \Big[\xi_{0n} - i\frac{\xi_{1n}}{n} \Big] z_n'(t) + \frac{1}{2} \Big[\xi_{0n} + i\frac{\xi_{1n}}{n} \Big] \overline{z_n'}(t) \Big] \\ & & + n \int_0^t z_{1n}(t-s) f(s) \, \mathrm{d}s \Big\}\,. \end{array} \right. \tag{14}$$

We observe the system when the boundary control is zero, i.e. $g(t) = 0$ and so also $f(t) = 0$. Then we have

$$Y(t) = w_x(t,0) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \left[\frac{1}{2} \Big[n\xi_{0n} - i\xi_{1n} \Big] z_n(t) + \frac{1}{2} \Big[n\xi_{0n} + i\xi_{1n} \Big] \overline{z_n}(t) \right]\,.$$

Note that $\{n\xi_{0n}\} \in l^2$ since $\xi_0 \in H_0^1(0,\pi)$. Our goal is to study the sequence $\{z_n(t)\}$ in order to derive properties of the solutions of Eq. (1) and to deduce controllability and observability results.

2.1 Properties of the sequence $\{z_n(t)\}$

A key result of this paper is that we identify a a Riesz sequence associated to Eq. (1) . Riesz sequences can be defined in several equivalent ways, see [2, 10, 21, 31]. The most usefull to us is as follows: a sequence $\{z_n\}$ in a Hilbert space H is Riesz when there exist $c > 0$ and $C > 0$ such that for every finite sequence $\{\alpha_n\}$ of complex numbers we have

$$c \sum |\alpha_n|^2 \leq \left\| \sum \alpha_n z_n \right\|_H^2 \leq C \sum |\alpha_n|^2\,. \tag{15}$$

It is clear that inequalities (15) hold also if $\{\alpha_n\} \in l^2$ and that the right inequality extends Bessel inequality while the left inequality extends Parseval.

If a sequence $\{z_n\}$ is Riesz and complete in H then it is called a Riesz basis (so, every Riesz sequence is a Riesz basis in its closed span) and it is possible to prove that a Riesz basis is the image of an orthonormal basis under a linear, bounded and boundedly invertible transformation, and conversely.

An important fact is that Riesz bases come in pairs: every Riesz basis $\{z_n\}$ has a unique biorthogonal sequence ζ_n, i.e. a sequence such that

$$\langle z_n, \zeta_k \rangle_H = \delta_{n,k}$$

($\delta_{n,k}$ is Kronecker delta) and it turns out that $\{\zeta_n\}$ is a Riesz basis too. If $\{z_n\}$ is a Riesz sequence then biorthogonal sequences $\{\zeta_n\}$ which are also Riesz sequences exist, but these are not unique.

For us, one of the crucial properties of Riesz sequences is as follows: let $\{z_n\}$ be a Riesz sequence and let us consider the *moment problem*

$$\langle z_n, u \rangle_H = d_n \,.$$

For every $\{d_n\} \in l^2$ a solution $u \in H$ exists, given by

$$u = \sum d_n \zeta_n$$

where $\{\zeta_n\}$ is any biorthogonal Riesz sequence of $\{z_n\}$.

After these preliminaries, we shall state:

Theorem 4 *The sequence $\{z_n(t), \overline{z_n}(t)\}_{n\geq 1}$ in* (13) *is a Riesz sequence in $L^2(0,T)$ for every $T > 2\pi$.*

The proof of this result is long and technical, and can be found in Appendix A.

We finish this section with an observation: let $H = L^2(0,T_0)$ and let $\{z_n(t)\}$ be a Riesz sequence in $L^2(0,T_0)$. Then, using the second inequality in (15), we see that the following holds for $T < T_0$:

$$\int_0^T \left|\sum \alpha_n z_n(t)\right|^2 \, \mathrm{d}t \leq \int_0^{T_0} \left|\sum \alpha_n z_n(t)\right|^2 \, \mathrm{d}t \leq C \sum |\alpha_n|^2 \,;$$

i.e., *if the second inequality in (15) holds in $L^2(0,T_0)$ then it holds also in $L^2(0,T)$ for every $T < T_0$.*

3 Properties of the solution map and the proofs of Theorems 1–3

We first prove the following property of the solution map. This property will imply Theorems 1, and 2. In particular they justify the computation in HUM method.

Lemma 5 *Let $g(t) = 0$. If $\xi_0 \in H_0^1(0,\pi)$ and $\xi_1 \in L^2(0,\pi)$ then the solution $w(x,t)$ of (1), i.e. the function in (10), belongs to $C([0,T];H_0^1(0,\pi)) \cap C^1([0,T];L^2(0,\pi))$ for any $T > 0$. Moreover, the transformation*

$$H_0^1(0,\pi) \times L^2(0,\pi) \ni (\xi_0,\xi_1) \mapsto w \in C([0,T];H_0^1(0,\pi)) \cap C^1([0,T];L^2(0,\pi))$$

is continuous.

Proof. In view of Eq. (12) we can represent $w(t,x)$ as

$$w(t,x) = \sum_{n=1}^{\infty} \frac{\phi_n(x)}{n} \left[n\xi_{0n} z_{1n}(t) + \xi_{1n} z_{2n}(t) \right] . \tag{16}$$

Since $\{\phi_n(x)/n\}$ is a Riesz sequence in $H_0^1(0,\pi)$ and $\{z_n(t)\}$ is bounded on $[0,T]$, see (27), we get for any $n'' > n'$ and $t \in [0,T]$

$$\left\| \sum_{n=n'}^{n''} \frac{\phi_n(x)}{n} \left[n\xi_{0n} z_{1n}(t) + \xi_{1n} z_{2n}(t) \right] \right\|^2_{H_0^1(0,\pi)} \leq C \sum_{n=n'}^{n''} \left[n^2 |\xi_{0n}|^2 + |\xi_{1n}|^2 \right] .$$

By taking into account that the sequences $\{n\xi_{0n}\}$ and $\{\xi_{1n}\}$ belong to l^2, we have that $w \in C([0,T];H_0^1(0,\pi))$. As regards w_t, by (16) we have

$$\begin{aligned} w_t(t,x) &= \sum_{n=1}^{\infty} \frac{\phi_n(x)}{n} \left[n\xi_{0n} z'_{1n}(t) + \xi_{1n} z'_{2n}(t) \right] \\ &= \sum_{n=1}^{\infty} \phi_n(x) \left[\xi_{0n} z'_{1n}(t) + \frac{\xi_{1n}}{n} z'_{2n}(t) \right] , \end{aligned}$$

so by using also $|z'_n(t)| \leq Cn$, see (28), similar argumentations show that $w_t \in C([0,T];L^2(0,\pi))$.

Continuous dependence is clear. ∎

We are now in position to prove Theorem 2. Let $g(t) \equiv 0$. We must prove that the transformation

$$H_0^1(0,\pi) \times L^2(0,\pi) \ni (\xi_0, \xi_1) \mapsto w_x(\cdot, 0) \in L^2(0,T)$$

satisfies the estimates (5). The result is known when $\xi_1 = 0$ (see [25, Theorem 24]). Now we consider the general case $\xi_1 \neq 0$.

We consider the first series in the expression of $w_x(t,0)$, i.e.

$$F(t) = \sum_{n=1}^{\infty} \left[n\xi_{0n} - i\xi_{1n} \right] z_n(t) \,.$$

We observe

$$\sum_{n=1}^{\infty} |n\xi_{0n} - i\xi_{1n}|^2 = \sum_{n=1}^{\infty} \left[|n\xi_{0n}|^2 + |\xi_{1n}|^2 - 2\mathrm{Im}\,(n\xi_{0n}\xi_{1n}) \right]$$
$$\asymp \sum_{n=1}^{\infty} \left[|n\xi_{0n}|^2 + |\xi_{1n}|^2 \right] .$$

The above property and the fact that $\{z_n(t)\}$ is a Riesz sequence in $L^2(0,T)$, i.e. Teorem 4, imply the following inequalities:

$$c\sum_{n=1}^{\infty} \left[|n\xi_{0n}|^2 + |\xi_{1n}|^2 \right] \leq \int_0^T |F(t)|^2 \, \mathrm{d}t \leq C \sum_{n=1}^{\infty} \left[|n\xi_{0n}|^2 + |\xi_{1n}|^2 \right] \qquad (17)$$

where $m > 0$. Of course, the estimate from above holds for every $T > 0$.

The inequality from above implies continuity of the transformation

$$H_0^1(0,\pi) \times L^2(0,\pi) \ni (\xi_0, \xi_1) \mapsto w_x(\cdot, 0) \in L^2(0,T)$$

for any $T > 0$. The inequality from below (which holds for $T > 2\pi$) implies injectivity and boundedness of the inverse transform.

The second series is treated analogously.

These inequalities imply at the same time continuous dependence of the output y on the initial conditions $(\xi_0, \xi_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ and boundedly invertibility of the transformation $(\xi_0, \xi_1) \mapsto y(\cdot)$.

This ends the proof of Theorem 1, hence also of Theorem 2.

As we noted, Theorem 3 follows, using HUM method. However, we would like to stress the fact that moment method leads to a conceptually simple constructive method for the steering control. In order to see this, let the initial conditions be zero and let us impose the condition

$$w(T) = \xi \in L^2(0,\pi)\,, \qquad w_t(T) = \eta \in H^{-1}(0,\pi)$$

to be attained using a suitable control $g(t)$. Let

$$\xi_n = \int_0^\pi \xi(x)\phi_n(x)\,\mathrm{d}x\,, \qquad \eta_n = \int_0^\pi \eta(x)\phi_n(x)\,\mathrm{d}x\,.$$

Using (13) and (14), we see that the target (ξ,η) can be reached in time T if we can solve the moment problem

$$\int_0^T z_n(s)f(T-s)\,\mathrm{d}s = a_n\,, \quad a_n = i\xi_n + \frac{1}{n}\eta_n\,, \qquad n \geq 1\,. \tag{18}$$

Note that $\{a_n\}$ is an arbitrary sequence in $l^2(\mathbb{C})$ if we require that $\xi \in L^2(0,\pi)$ and $\eta \in H^{-1}(0,\pi)$ are arbitrary. Moreover, once the function $f(t)$ has been identified from here, the steering control $g(t)$ is obtained by solving a Volterra integral equation of the second kind, see (11).

It is convenient to consider the moment problem (18) for every $n \in \mathbb{Z}' = \mathbb{Z}\setminus\{0\}$. For this, we define $c_n = \frac{1}{\sqrt{2\pi}}[n\xi_{0n} + i\xi_{1n}]$ for $n > 0$ and

$$z_{-n}(t) = \overline{z_n(t)}\,,\ c_n = \overline{c_{-n}} \qquad n \in \mathbb{Z}'\,.$$

Theorem 4 states that $\{z_n(t)\}_{n\in\mathbb{Z}'}$ is a Riesz sequence in $L^2(0,T)$ when $T > 2\pi$. Then, a solution of problem (18) is

$$f(T-t) = \sum a_n \zeta_n(t) \tag{19}$$

where $\{\zeta_n(t)\}_{n\in\mathbb{Z}'}$ is a biorthogonal sequence to $\{z_n(t)\}_{n\in\mathbb{Z}'}$ (the solution is not unique, since $\{z_n(t)\}_{n\in\mathbb{Z}'}$ is a Riesz sequence which is not a basis.)

So, *also Theorem Theorem 3 is proved using moment methods.*

4 The construction of the steering control

As we noted, most of the observability/controllability proofs are either non-constructive or they lead to quite involved algorithms. Let us consider this problem from the point of view of the moment method used here.

The computation of the steering control boils down to the computation of a biorthogonal sequence $\{\zeta_n(t)\}$. This problem being important in many practical applications, it has been widely studied, see for example [11] and references therein. But, clearly every practical method can only study a truncated sequence $\{z_n(t)\}_{|n|<N}$, and construct a biorthogonal to this sequence. In general, there is no way for approximating ζ_n with n large. Instead, in our case, we have more: it is proved in Lemma 12 that for every $T>0$ the following series converges (here $\gamma=M(0)/2$):

$$\sum_{n\in\mathcal{Z}'}\int_0^T\left|z_n(t)-e^{(\gamma+in)t}\right|^2\,\mathrm{d}t\,.$$

We express this fact by sayng that the sequences $\{z_n(t)\}$ and $\left\{e^{(\gamma+in)t}\right\}$ are *quadratically close* in $L^2(0,T)$. This observation suggests that for sufficiently large n it should be possible to replace $\zeta_n(t)$ with $e^{(\gamma+in)t}$. This is justified by the following result.

Theorem 6 *There exists a number c such that*

$$\left\{\begin{array}{l}\left|\int_0^{2\pi}z_n(t)e^{-(\gamma+in)t}\,\mathrm{d}t-2\pi\right|\le\frac{c}{|n|}\\ \left|\int_0^{2\pi}z_n(t)e^{-(\gamma+ik)t}\,\mathrm{d}t\right|\le\dfrac{c}{\left||k|-|n|\right|}\quad\text{if } |n|\ne|k|\,.\end{array}\right.\tag{20}$$

The proof uses estimates and lemmas from proof of Theorem 4 and so is postponed to Appendix B.

Using (18), (19) and this result, we can (approximately) compute $\{\zeta_n(t)\}_{|n|<N}$ using existing numerical methods, and then we can approximate $f(t)$ with the series

$$f(T-t)=\sum_{|n|<N}a_n\zeta_n(t)+\sum_{|k|\ge N}a_ke^{(\gamma+ik)t}\,.\tag{21}$$

When checking the validity of (18) and computing

$$\int_0^{2\pi}f(T-t)z_{n_0}(t)\,\mathrm{d}t\,,\qquad|n_0|<N$$

(which in principle should be equal to a_{n_0}) the error introduced by the second series (which in practice will be truncated) with the exponential replacing the functions of the biorthogonal sequence, is estimated, using (20), as

$$\sum_{|k|\ge N}|a_k|\frac{c}{|k|-|n_0|}\,.$$

Other methods for the computation of $f(t)$ have been proposed. The representation formula (14) for the solutions and the sequence $\{z_n(t)\}$ can be used also in conjunction with these methods.

A numerical approach based on HUM method can profit of the representation formula (14). In following the procedure of the HUM method, one has to construct the operator L defined in (9). If we consider the problem

$$\inf_{(\eta_0,\eta_1)\in H_0^1(0,\pi)\times L^2(0,\pi)} \left\{ \frac{1}{2} \langle\!\langle L(\eta_0,\eta_1),(\eta_0,\eta_1)\rangle\!\rangle - \langle w_1,\eta_0\rangle + \langle w_0,\eta_1\rangle \right\}, \tag{22}$$

($w_0 \in L^2(0,\pi)$ and $w_1 \in H^{-1}(0,\pi)$ are the final data of the original controllability problem) thanks to the expansion (14), the problem can be solved finding the (generalized) Fourier coefficients. By cutting the number of coefficients and using the conjugate gradient or other numerical methods combined with (21) and estimate (20), we could find the related approximated minimum problem, see [9] for an application of these ideas.

Another approximation method follows by observing that HUM method, to find the control which steers the system to rest in time T, can be deduced by the following penalization argument: we minimize the functional

$$J_\varepsilon(g,w) = \frac{1}{2}\int_0^T g^2(t)dt + \frac{1}{2\varepsilon}\int_0^T \int_0^\pi [w'' - w_{xx} - M * w_{xx}] \ \mathrm{d}t \ \mathrm{d}x\,, \tag{23}$$

where $w'' = \frac{\partial^2 w}{\partial t^2}$. One has to assume $g \in L^2((0,T)\times(0,\pi))$ and also that w is such that the following function h belongs to $L^2((0,T)\times(0,\pi))$:

$$h(x,t) = w'' - w_{xx} - M * w_{xx} \tag{24}$$

and furthermore w satisfies the initial and boundary conditions (2). So, $w(x,t)$ has the form in (14).

We search for a control such that

$$w(T,x) = \frac{\partial w}{\partial t}(T,x) = 0 \qquad \text{in} \quad (0,\pi)\,. \tag{25}$$

Let $(g_\varepsilon, w_\varepsilon)$ minimize $J_\varepsilon(g,w)$. Our problem being exactly controllable, then following the same arguments in [19] for the memoryless case, we obtain

$$\|g_\varepsilon\|_{L^2(\Sigma)} \le C\,, \qquad \|w''_\varepsilon - (w_\varepsilon)_{xx} - M*(w_\varepsilon)_{xx}\|_{L^2(Q)} \le C\sqrt{\varepsilon}\,.$$

By using the above estimates, the approximation problem leads to the solution of the original controllability problem as $\varepsilon \to 0^+$.

A The proof of Theorem 4

Before proving the Riesz property stated in Theorem 4, we need to recall Paley-Wiener and Bari theorems (see [10, 31] for the proofs). Bari theorem is specific of Hilbert spaces, while Paley-Wiener concerns Schauder bases of Banach spaces. We give a formulation of Paley-Wiener Theorem for Riesz sequences in Hilbert spaces.

Theorem 7 (Paley-Wiener) *Let $\{e_n\}$ be a Riesz sequence in a Hilbert space H. A second system $\{z_n\}$ is a Riesz sequence too when the following holds: there exists $q \in (0,1)$ such that for every sequence $\{\alpha_n\}$ with only finitely many non zero elements we have*

$$\left|\sum \alpha_n(e_n - z_n)\right|^2 < q\left|\sum \alpha_n e_n\right|^2$$

In fact, we shall use a corollary of this theorem, which we state as a lemma. We recall: a sequences $\{z_n\}$ is *quadratically close* to a sequence $\{e_n\}$ when

$$\sum \|z_n - e_n\|_H^2 < +\infty\,.$$

Corollary 8 *If $\{z_n\}_{n\geq 1}$ is quadratically close to a Riesz sequence $\{e_n\}_{n\geq 1}$, then there exists $N \geq 1$ such that $\{z_n\}_{n\geq N}$ is a Riesz sequence too.*

A further definition we need is: a sequence $\{z_n\}$ in a Hilbert space is ω-independent when

$$\{\alpha_n\} \in l^2 \quad \text{and} \qquad \sum_{n=1}^{\infty} \alpha_n z_n = 0$$

(the series being norm convergent) implies $\{\alpha_n\} = 0$. With this definition we can state

Theorem 9 (Bari) *If a sequence is quadratically close to a Riesz sequence and ω-independent, then it is a Riesz sequence.*

We prove now Theorem 4, which requires several lemmas. We recall the notations

$$z_{-n}(t) = \overline{z_n}(t)\,, \qquad \mathbb{Z}' = \mathbb{Z} - \{0\}$$

so that now $z_n(t)$ is defined for every $n \neq 0$. Then we have

Lemma 10 *The sequence of the functions $\{z_n(t)\}_{n\in\mathbb{Z}'}$ in (13) is linearly independent in $L^2(0,T)$, for every $T>0$.*

Proof. The proof is by contradiction. If the sequence $\{z_n(t)\}$ is linearly dependent on an interval $[0,T]$, $T>0$, then there is a *least* K and a *larger* R such that

$$\sum_{n=R}^{K} \alpha_n z_n(t) = 0 \tag{26}$$

where α_n are suitable complex numbers. So, we have also

$$\sum_{n=R}^{K} \alpha_n z_n'(t) = 0 \qquad \sum_{n=R}^{K} \alpha_n z_n''(t) = 0\,.$$

Taking into account (11), we get also

$$\left[\sum_{n=R}^{K} n^2\alpha_n z_n(t)\right] + \int_0^t M(t-s)\left[\sum_{n=R}^{K} n^2\alpha_n z_n(s)\right]\,\mathrm{d}s = 0$$

for every $t\in[0,T]$. So,

$$\sum_{n=R}^{K} n^2\alpha_n z_n(t) = 0 \qquad \forall t\in[0,T]\,.$$

This equality and (26) show that

$$\sum_{n=R}^{K-1}\left(n^2-K^2\right)\alpha_n z_n(t) = 0$$

and this contradicts the definition of K. ∎

Furthermore

Lemma 11 *For every $T>0$ there exists a constant $C=C_T$ such that*

$$|z_n(t)| \le C\,, \qquad \forall n\ge 1\,, \quad \forall t\in[0,T]\,, \tag{27}$$

$$|z_n'(t)| \le Cn\,, \qquad \forall n\ge 1\,, \quad \forall t\in[0,T]\,. \tag{28}$$

<u>Proof</u>. We note that $z_n(t)$, $n \neq 0$, is the solution of Eq. (11) with initial conditions $z_n(0) = 1$ and $z'_n(0) = in$, so it solves the Volterra integral equation

$$z_n(t) = e^{int} - n \int_0^t \sin n(t-s) \int_0^s M(s-r) z_n(r)\, \mathrm{d}r\ \mathrm{d}s\,. \tag{29}$$

If we integrate by parts, then we get

$$z_n(t) = e^{int} + M(0) \int_0^t \cos n(t-s) z_n(s)\, \mathrm{d}s - \int_0^t M(t-s) z_n(s)\, \mathrm{d}s \\ + \int_0^t \cos n(t-s) \int_0^s M'(s-r) z_n(r)\, \mathrm{d}r\ \mathrm{d}s\,. \tag{30}$$

Estimate (27) follows from Gronwall Lemma.

As regards $z'_n(t)$, first we observe that we can rewrite equality (30) as

$$z_n(t) = e^{int} + M(0) \int_0^t z_n(t-s) \cos(ns)\, \mathrm{d}s - \int_0^t z_n(t-s) M(s)\, \mathrm{d}s \\ + \int_0^t z_n(t-s) \int_0^s \cos n(s-r) M'(r)\, \mathrm{d}r\ \mathrm{d}s\,.$$

Computing the derivatives, we obtain

$$z'_n(t) = in\, e^{int} + M(0) \int_0^t z'_n(t-s) \cos(ns)\, \mathrm{d}s + M(0)\cos(nt) - \int_0^t z'_n(t-s) M(s)\, \mathrm{d}s \\ - M(t) + \int_0^t z'_n(t-s) \int_0^s \cos n(s-r) M'(r)\, \mathrm{d}r\ \mathrm{d}s + \int_0^t \cos n(t-r) M'(r)\, \mathrm{d}r\,.$$

The desired inequality (28) follows again from Gronwall Lemma. ∎

The following result opens the way to the use of Corollary 8:

Lemma 12 *Let $\{z_n\}_{n\geq 1}$ be the sequence defined in* (13) *and introduce $\gamma = M(0)/2$. Then, $\{z_n(t)\}$ and $\left\{e^{(\gamma+in)t}\right\}$ are quadratically close in $L^2(0,T)$ for any $T > 0$.*

<u>Proof</u>. We shall prove that for every $T > 0$ the sequence $\{z_n(t)\}$ is quadratically close to $\left\{e^{(\gamma+in)t}\right\}$ in $L^2(0,T)$, $\gamma = M(0)/2$. In fact, we shall prove the existence of a constant c such that

$$\left| z_n(t) - e^{(\gamma+in)t} \right| \leq \frac{c}{n} \tag{31}$$

which imply Lemma 12.

Let $e_n(t) = z_n(t) - e^{(\gamma+in)t}$. Using (30) we see that

$$e_n(t) = \left[e^{int} - e^{(\gamma+in)t} + 2\gamma \int_0^t \cos n(t-s) e^{(\gamma+in)s}\, \mathrm{d}s\right] \tag{32}$$
$$+2\gamma \int_0^t \cos n(t-s) e_n(s)\, \mathrm{d}s + \int_0^t K_n(t-s) e_n(s)\, \mathrm{d}s + b_n(t)\,,$$
$$K_n(t) = \int_0^t M'(t-s)\cos ns\, \mathrm{d}s - M(t) \tag{33}$$
$$b_n(t) = \int_0^t \cos n(t-s) \int_0^s M'(s-r) e^{(\gamma+in)r}\, \mathrm{d}r\, \mathrm{d}s - \int_0^t M(t-s) e^{(\gamma+in)s} \tag{34}$$

The integral at the line (32) is easily computed (using Euler formulas):

$$\left[e^{int} - e^{(\gamma+in)t} + 2\gamma \int_0^t \cos n(t-s) e^{(\gamma+in)s}\, \mathrm{d}s\right] = \frac{\gamma}{\gamma+2in}\left[e^{(\gamma+in)t} - e^{-int}\right]. \tag{35}$$

Using $M \in H^2$, we integrate by parts both the integrals in $b_n(t)$ and we see that

$$|b_n(t)| \le \frac{c}{n}\,.$$

Boundedness of the sequence $\{K_n(t)\}$ and Gronwall lemma imply (31). ∎

The sequence $\left\{e^{(\gamma+in)t}\right\} = \{e^{\gamma t}e^{int}\}$ is a Riesz sequence in $L^2(0, 2\pi)$ since multiplication by $e^{\gamma t}$ is a linear bounded and boundedly invertible transformation in $L^2(0, 2\pi)$. Consequently, the conditions in Corollary 8 hold for the sequence $\{z_n(t)\}_{|n|\ge N}$ and we can state:

Lemma 13 *There exists $N \ge 1$ such that the sequence $\{z_n(t)\}_{|n|\ge N}$ is Riesz in $L^2(0, T)$ for every $T \ge 2\pi$.*

We recapitulate: we already proved

- the sequence $\{z_n(t)\}$ is linearly independent;
- there exists $N \ge 1$ such that $\{z_n(t)\}_{|n|\ge N}$ is a Riesz sequence in $L^2(0, T)$, $T \ge 2\pi$.

If $N = 1$ then Theorem 4 is proved. Otherwise *we rely on Bari Theorem so that in order to prove Theorem 4 it is enough to see that the sequence $\{z_n(t)\}_{n\ne 0}$ is ω-independent in $L^2(0, T)$ when $T > 2\pi$.*

The idea of the proof, taken from [28], is as follows. First we note that the sequence $\{z_n(t)\}_{|n|\geq N}$ being Riesz in $L^2(0,T)$ implies that

$$\sum_{n\neq 0} \alpha_n z_n(t)$$

converges in $L^2(0,T)$ if and only if $\{\alpha_n\} \in l^2$ (here and below $l^2 = l^2(\mathbb{Z} - \{0\})$). We consider a sequence $\{\alpha_n\} \in l^2$ such that

$$\sum_{n\neq 0} \alpha_n z_n(t) = 0 \tag{36}$$

(convergence is in $L^2(0,T)$ since $\{z_n(t)\}_{|n|\geq N}$ is Riesz). We shall prove the existence of a sequence $\left\{\alpha_n^{(1)}\right\} \in l^2$ such that the following properties hold:

- $\alpha_1^{(1)} = 0$;
- for any $n \notin \{0,1\}$ $\alpha_n^{(1)} = 0$ if and only if $\alpha_n = 0$;
- $\displaystyle\sum_{n\notin\{0,1\}} \alpha_n^{(1)} z_n(t) = 0.$

The same procedure can be done for $n = -1$.

So, we peel off one of the elements of the series. Since the sequence $\left\{\alpha_n^{(1)}\right\}$ is in l^2, this procedure can be repeated and we can peel off as many elements as we whish. So, after a finite number of steps we end up with

$$\sum_{|n|>N} \alpha_n^{(N)} z_n(t) = 0$$

which implies $\alpha_n^{(N)} = 0$ when $|n| > N$, because $\{z_n(t)\}_{|n|\geq N}$ is a Riesz sequence. But, $\alpha_n^{(N)} = 0$ if and only if $\alpha_n = 0$ so that the coefficients in (36) are zero if $|n| > N$. And then, also the coefficients with $|n| \leq N$ are zero, since the sequence $\{z_n(t)\}$ is linearly independent.

Therefore, the crucial point of the proof is the construction of the sequence $\left\{\alpha_n^{(1)}\right\}$. This construction depends on the following lemma:

Lemma 14 *Let condition (36) hold. Then, there exists a sequence* $\{\sigma_n\} \in l^2$ *such that*

$$\alpha_n = \frac{\sigma_n}{n^2}\,.$$

Before we proceed, we show the use of this lemma to construct the sequence $\left\{\alpha_n^{(1)}\right\}$. The lemma and inequality (28) imply that we can compute the derivatives of the series termwise:

$$\sum_{n\neq 0} \alpha_n z_n'(t) = 0\,.$$

Now we use again Lemma 14 and formula (11) to see that we can compute a second derivative termwise, so that

$$\sum_{n\neq 0} \alpha_n z_n''(t) = 0\,.$$

So, from (11),

$$\left[\sum_{n\neq 0}(\alpha_n n^2) z_n(t)\right] + \int_0^t M(t-s)\left[\sum_{n\neq 0}(\alpha_n n^2) z_n(s)\right]\,\mathrm{d}s = 0$$

i.e.

$$\sum_{n\neq 0}(\alpha_n n^2) z_n(s) = 0\,.$$

We combine this last equality and (36) in order to peel off the first element of index 1, so we have

$$\sum_{n\notin\{0,1\}} \alpha_n(1-n^2) z_n(t) = 0\,,$$

and hence

$$\alpha_n^{(1)} := \alpha_n(1-n^2)\,.$$

This is the construction of the sequence $\left\{\alpha_n^{(1)}\right\}$.

Now we prove Lemma 14. We need the following preliminary result:

Lemma 15 *The following properties hold:*

- *For every* $f \in L^2(0,T)$ *there exists a number* $c = c_T > 0$ *such that*

$$\sum_{n\in\mathbb{Z}} \left| \int_0^t f(r) e^{-inr}\,\mathrm{d}r \right|^2 < c\,, \qquad 0 \le t \le T\,. \tag{37}$$

- *Let $L \in L^2(0,T)$, $T > 0$, and $\gamma \in \mathbb{R}$. Let us define for any $n \in \mathbb{Z}$ and $t \in [0,T]$*

$$a_n(t) = \int_0^t \cos n(t-s) \left[\int_0^s L(s-r) e^{(\gamma+in)r} \, \mathrm{d}r \right] \, \mathrm{d}s \,.$$

Then, there exists a constant $c = c_{T,\gamma} > 0$ such that

$$\sum_{n\in\mathbb{Z}} |a_n(t)|^2 \leq c \,. \tag{38}$$

<u>Proof</u>. We prove that (37) holds. Let $K \in \mathbb{N}$ be such that $2K\pi \geq T$. If the inequality is strict, we extend f to $[0, 2K\pi]$ with $f(t) = 0$ for $t > T$ so that the extension (still denoted f) belongs to $L^2(0, 2K\pi)$. Let $\chi_t(r)$ be the characteristic function of $[0,t]$ for any $t \in [0,T]$. Since the sequence $\{\epsilon_n\}$, defined as

$$\epsilon_n(t) = \frac{1}{\sqrt{2K\pi}} e^{-int} \,,$$

is orthonormal in $L^2(0, 2K\pi)$, Bessel inequality gives

$$\sum |\langle f\chi_t, \epsilon_n \rangle|^2 \leq \|f\chi_t\|^2_{L^2(0,2K\pi)} \,,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(0, 2K\pi)$. Therefore,

$$\sum \left| \int_0^t f(r) e^{-inr} \, dr \right|^2 = 2K\pi \sum |\langle f\chi_t, \epsilon_n \rangle|^2 \leq 2K\pi \int_0^t |f(r)|^2 \, \mathrm{d}r \leq 2K\pi \int_0^T |f(r)|^2 \, dr \,,$$

as required.

Now we prove the second statement. First of all, we shall see that $a_n(t)$ can be represented as a linear combination of terms of type

$$\int_0^t f(r) e^{-inr} \, \mathrm{d}r \,,$$

with $f \in L^2(0,T)$, and of terms dominated by (const)$/n$. Indeed, if we exchange the order of integration and use Euler formula for the cosine, then we get

$$\begin{aligned}
a_n(t) &= \int_0^t L(r) \int_0^{t-r} e^{(\gamma+in)(t-r-s)} \cos(ns)\, ds\, dr \\
&= \frac{1}{2} e^{(\gamma+in)t} \int_0^t e^{-(\gamma+in)r} L(r) \int_0^{t-r} e^{-(\gamma+in)s} \left(e^{ins} + e^{-ins}\right) \mathrm{d}s \; \mathrm{d}r \\
&= \frac{1}{2\gamma} e^{(\gamma+in)t} \int_0^t \left[e^{-\gamma r} L(r)\right] e^{-inr} \,\mathrm{d}r - \frac{1}{2\gamma} e^{int} \int_0^t L(r) e^{-inr} \,\mathrm{d}r \\
&+ \frac{1}{2(\gamma+2in)} \left[e^{(\gamma+in)t} \int_0^t L(r) e^{-(\gamma+in)r} \,\mathrm{d}r - e^{-int} \int_0^t L(r) e^{inr} \,\mathrm{d}r \right] .
\end{aligned}$$

The last bracket is dominated by (const)/n (the constant will depend on T and γ), so the desired condition (38) is verified by the last bracket.

Inequality (38) now follows using (37). ∎

Now we prove Lemma 14, proceeding in four steps:

Step 1: a preliminary transformation and a new sequence of functions. Our goal here is to transform the sequence $\{z_n(t)\}$ into a new sequence $\{w_n(t)\}$, such that $w_n(t)$ solves a Volterra integrodifferential equation with a new kernel $R(t)$ having $R(0) = 0$. This is purely technical but very useful to simplify the computations.

We define

$$N(t) = 1 + \int_0^t M(s)\,\mathrm{d}s$$

(note that 1 is the coefficient of the laplacian outside the integral). So, integrating both the sides of (11) and using (13), we get

$$z_n'(t) = in - n^2 \int_0^t N(t-r) z_n(r)\,\mathrm{d}r\,. \tag{39}$$

We introduce

$$w_n(t) = e^{-\gamma t} z_n(t)\,, \qquad \gamma = \frac{1}{2} M(0) \tag{40}$$

(here and in the following, $w_n(t)$ is the function in (40). I.e., it is not the same function as in (10). This is not going to do any confusion). Using Lemma 12 for the asymptotic estimate and (39), we get:

Lemma 16 *The sequence $\{w_n(t)\}$ is quadratically close to the sequence $\{e^{int}\}$ in $L^2(0,T)$, for every $T > 0$.*

It is simple to see that $w_n(t)$ solves

$$w_n'' = -(n^2-\gamma^2)w_n - n^2\int_0^t R(t-s)w_n(s)\,\mathrm{d}s - i\left(2\gamma n e^{-\gamma t}\right) \tag{41}$$

where

$$R(t) = e^{-\gamma t}\left[M(t) - 2\gamma N(t)\right]$$

so that

$$R(0) = 0\,.$$

The initial conditions of $w_n(t)$ are

$$w_n(0) = 1\,, \qquad w_n'(0) = in - \gamma\,.$$

$$\begin{aligned}
&w_n'(t) = -\gamma w_n(t) + ine^{-\gamma t} - n^2\int_0^t e^{-\gamma(t-r)}N(t-r)w_n(r)\,\mathrm{d}r\,,\\
&w_n''(t) = -\gamma w_n'(t) - in\gamma e^{-\gamma t} - n^2 w_n(t)\\
&-n^2\int_0^t e^{-\gamma(t-r)}\left[-2\gamma N(t-r) + M(t-r)\right]w_n(r)\,\mathrm{d}r\,.
\end{aligned}$$

Inserting here the expression of $w_n'(t)$, we get (41).

We introduce

$$\beta_n = (\operatorname{sign} n)\sqrt{n^2-\gamma^2}$$

and we note that

$$\beta_n - n = \frac{-\gamma^2}{n+\sqrt{n^2-\gamma^2}} \asymp \frac{1}{n}$$

so that Lemma 16 gives

Lemma 17 *The sequence $\{w_n(t)\}$ is quadratically close to the sequence $\left\{e^{i\beta_n t}\right\}$.*

We have the following representation formulas for $w_n(t)$. If $\beta_n \neq 0$ i.e. $n \neq \pm\gamma$ we have

$$\begin{aligned}
w_n(t) = &\left\{\frac{n+\beta_n}{2\beta_n} + i\frac{\gamma}{2\beta_n} - \frac{\gamma n}{\beta_n(\gamma + i\beta_n)}\right\}e^{i\beta_n t}\\
&\qquad + \left\{\frac{\beta_n - n}{2\beta_n} - i\frac{\gamma}{2\beta_n} + i\frac{\gamma n}{\beta_n(\beta_n + i\gamma)}\right\}e^{-i\beta_n t}\\
&\quad - \frac{2\gamma i}{n}e^{-\gamma t} - \frac{n^2}{\beta_n}\int_0^t \sin\beta_n(t-s)\int_0^s R(s-r)w_n(r)\,\mathrm{d}r\,\mathrm{d}s\,.
\end{aligned} \tag{42}$$

The important fact to note is that the coefficients of $e^{-i\beta_n t}$ and of $e^{-\gamma t}$ are of the order of $1/n$ while the coefficient of $e^{i\beta_n t}$ is $1+\mathrm{O}(1/n)$. In fact,

$$\frac{n+\beta_n}{2\beta_n}=1+\frac{n-\beta_n}{2\beta_n}=1+\frac{\gamma^2}{2\beta_n\,(n+\beta_n)}\,.$$

If γ is not an integer or if it is zero, then this is all. Otherwise we have also the following two functions, which correspond to $n=\pm\gamma$.

Integrating twice formula (41) with $n=\gamma$ and $n=-\gamma$ respectively, we obtain

$$\begin{aligned}w_\gamma(t)=r-2i\gamma^2\int_0^t(t-r)e^{-\gamma r}\,\mathrm{d}r&\\ \left[1+2i-\gamma(1+i)t-2ie^{-\gamma t}\right]&\\ -\gamma^2\int_0^t(t-r)\int_0^r R(r-s)w_\gamma(s)\,\mathrm{d}s\;\mathrm{d}r&\end{aligned}\tag{43}$$

$$\begin{aligned}w_{-\gamma}(t)=r+2i\gamma^2\int_0^t(t-r)e^{-\gamma r}\,\mathrm{d}r&\\ \left[1-2i+\gamma(i-1)t+2ie^{-\gamma t}\right]&\\ -\gamma^2\int_0^t(t-r)\int_0^r R(r-s)w_{-\gamma}(s)\,\mathrm{d}s\;\mathrm{d}r\,.&\end{aligned}\tag{44}$$

Step 2: the comparison functions. The functions $w_n(t)$, $n\neq\pm\gamma$, are compared with the exponentials $e^{\pm i\beta_n t}$. If γ is a nonzero integer then we introduce also the two comparison functions

$$f_+(t)=1+2i-\gamma(1+i)t-2ie^{-\gamma t}\tag{45}$$

$$f_-(t)=1-2i+\gamma(i-1)t+2ie^{-\gamma t}\,.\tag{46}$$

It is easily seen that their wronskian is not identically zero, so that these functions are linearly independent.

Now we consider the set of functions

$$\left\{1\,,\;t\,,\;e^{-\gamma t}\,,\;e^{\pm i\beta_n t}\right\}_{n\neq 0}\,.$$

Haraux Theorem (see [21]) implies that this set of functions is Riesz in $L^2(0,T)$, for every $T>2\pi$. So, from now on we shall fix a value of $T>2\pi$.

In the computations below, we shall use the definitions

$$\begin{cases} e_n(t) &=& w_n(t) - e^{i\beta_n t} \\ e_\gamma(t) &=& w_\gamma(t) - f_+(t) \\ e_{-\gamma}(t) &=& w_{-\gamma}(t) - f_-(t)\,. \end{cases}$$

Note that this is different from the previous definition of $e_n(t)$, which was $z_n(t) - e^{int}$, but this is not going to make any confusion. Furthermore, $e_{\pm\gamma}(t)$ are introduced only if γ is a nonzero integer.
We subtract $e^{i\beta_n t}$ from both sides of (42). Integration by parts of the last integral in (42), and using $R(0) = 0$, we find (in the case $n \neq \pm\gamma$)

$$e_n(t) = \left[\frac{i\gamma}{2\beta_n} - \frac{\gamma(n+\beta_n) - i\beta_n(n-\beta_n)}{2\beta_n(\gamma + i\beta_n)}\right] e^{i\beta_n t}$$
$$+ \left[\frac{\beta_n - n - i\gamma}{2\beta_n} + i\frac{\gamma n}{\beta_n(\beta_n + i\gamma)}\right] e^{-i\beta_n t} - 2i\frac{\gamma}{n} e^{-\gamma t}$$
$$-\frac{n^2}{\beta_n^2}\left\{\int_0^t R(t-r) w_n(r)\,\mathrm{d}r - \int_0^t \cos\beta_n s \int_0^{t-s} R'(t-s-r) w_n(r)\,\mathrm{d}r\ \mathrm{d}s\right\}\,.$$

Step 3: ω-independence. The proof is by contradiction: we assume the existence of a sequence $\{\alpha_n\}$ such that

$$0 = \sum_{n\neq 0} \alpha_n z_n \qquad \text{in } L^2(0,T),\ T > 2\pi \tag{47}$$

and we prove that $\alpha_n = \sigma_n/n^2$, with $\{\sigma_n\} \in l^2$. Note that the index is integer and not zero. So, here $l^2 = l^2(\mathbb{Z} - \{0\})$.

We noted that $\{z_n\}_{|n|>N}$ is a Riesz sequence. So, convergence of the series implies that $\{\alpha_n\} \in l^2$. Furthermore, condition (47) is equivalent to

$$0 = \sum_{n\neq 0} \alpha_n w_n = \alpha_{-\gamma} w_{-\gamma} + \alpha_\gamma w_\gamma + \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n w_n \tag{48}$$

(of course, the indices $\pm\gamma$ have to be considered only if γ is a nonzero integer). Using an idea in [28], we introduce the functions $\Phi(t)$ and $\Psi(t)$ as follows:

$$\Phi(t) = \alpha_{-\gamma} f_-(t) + \alpha_\gamma f_+(t) + \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n e^{i\beta_n t}\,,$$

$$\Psi(t) = \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n e_n(t)\,. \tag{49}$$

so that $\Phi(t) = -\left[\alpha_{-\gamma} f_-(t) + \alpha_\gamma f_+(t) + \Psi(t)\right]$.

Now we proceed in two substeps:

sub-step 1: we prove $\alpha_n = (\delta_n / n)$, and $\{\delta_n\} \in l^2$.

We prove that $\Phi(t)$ is of class $W^{1,2}(0,T)$. To do this, we prove the same property of the function $\Psi(t)$. We note that

$$\begin{aligned}
&\Psi(t) \\
&= \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \left[i\frac{\gamma}{2\beta_n} - \frac{\gamma(n+\beta_n) - i\beta_n(n-\beta_n)}{2\beta_n(\gamma+i\beta_n)} \right] e^{i\beta_n t} \qquad (50)\\
&+ \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \left[\frac{\beta_n - n - i\gamma}{2\beta_n} + i\frac{\gamma n}{\beta_n(\beta_n + i\gamma)} \right] e^{-i\beta_n t} \qquad (51)\\
&-2\gamma i \left[\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\alpha_n}{n} \right] e^{-\gamma t} - \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} \int_0^t R(t-r) w_n(r)\, \mathrm{d}r \qquad (52)\\
&+ \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} \int_0^t \cos\beta_n s \int_0^{t-s} R'(t-s-r) w_n(r)\, \mathrm{d}r\ \ \mathrm{d}s\,. \qquad (53)
\end{aligned}$$

We prove that each one of the series (50)-(53) defines a $W^{1,2}$-functions. This is clear for the first series in (52) and in fact also for the second one, since we already know that $\{w_n(r)\}_{|n|>N}$ is a Riesz sequence. Hence we can exchange the series with the integral and we see that this series is equal to

$$\int_0^t \left[\sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} R(t-r) w_n(r) \right]\ \mathrm{d}r\,,$$

the integrand being square integrable. As to the remaining series, we first note that they converge uniformly. This is clear for the series (50) and (51), since $\{\alpha_n\} \in l^2$ and the brackets are of the order $1/n$. Convergence of the series in (53) follows from the first statement in Lemma 15, since this series is equal to

$$\sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} \int_0^t w_n(r) \int_0^{t-r} R'(t-r-s) \cos\beta_n s\ \mathrm{d}s\ \ \mathrm{d}r\,.$$

Now we compute termwise the derivative of each addendum and we see that the resulting series converges in $L^2(0,T)$. In fact, formally we have:

$$\Psi'(t) = i \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \left[i\frac{\gamma}{2} - \frac{\gamma(n+\beta_n) - i\beta_n(n-\beta_n)}{2(\gamma+i\beta_n)} \right] e^{i\beta_n t} \tag{54}$$

$$-i \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \left[\frac{\beta_n - n - i\gamma}{2} - i\frac{\gamma n}{\beta_n + i\gamma} \right] e^{-i\beta_n t} \tag{55}$$

$$+2i\gamma^2 e^{-\gamma t} \left[\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\alpha_n}{n} \right] \tag{56}$$

$$-\int_0^t R'(t-r) \left[\sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} w_n(r) \right] \,\mathrm{d}r \tag{57}$$

$$+R'(0) \left[\sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} \int_0^t w_n(r) \cos\beta_n(t-r) \,\mathrm{d}r \right] \tag{58}$$

$$+\sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} \int_0^t w_n(r) \int_0^{t-r} R''(t-r-s) \cos\beta_n s \,\mathrm{d}s \,\mathrm{d}r\,. \tag{59}$$

We show that each line defines a square integrable function. This is clear for the lines (54)-(56). The series under the integral at line (57) converges in $L^2(0,T)$, since $\{w_n(t)\}_{n\geq N}$ is a Riesz sequence. So, the integral is a $W^{1,2}$-function.

Convergence of the series (58) follows by using $w_n(t) = e_n(t) + e^{i\beta_n t}$. Indeed, we get

$$\begin{aligned}
\sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} \int_0^t e_n(r) \cos\beta_n(t-r) \,\mathrm{d}r \\
&+ \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} \int_0^t e^{i\beta_n r} \cos\beta_n(t-r) \,\mathrm{d}r \\
&= \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n \frac{n^2}{\beta_n^2} \int_0^t e_n(r) \cos\beta_n(t-r) \,\mathrm{d}r \\
&+ t\left[\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\alpha_n n^2}{2\beta_n^2} e^{i\beta_n t} \right] + \frac{1}{2} \sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\alpha_n n^2}{\beta_n^3} \sin\beta_n t\,.
\end{aligned} \tag{60}$$

The first series on the right hand side converges uniformly thanks to Lemma 17; the last series converges uniformly and the intermediate series converges in $L^2(0,T)$ since $\{e^{i\beta_n t}\}$ is a Riesz sequence.

Convergence of the series (59) follows from the first statement in Lemma 15. So,

$$\Phi(t) = -\left(\alpha_{-\gamma} f_-(t) + \alpha_{+\gamma} f_+(t) + \sum_{n\notin\{-\gamma,0,\gamma\}} \alpha_n e^{i\beta_n t}\right) \in W^{1,2}(0,T)\,.$$

Using the fact that $\left\{e^{i\beta_n t}\right\}$ is a Riesz sequence in $L^2(0,T)$ with $T > 2\pi$, the same method as in [28] shows that

$$\alpha_n = \frac{\delta_n}{n}\,, \qquad \{\delta_n\} \in l^2\,. \tag{61}$$

In fact, we introduce the two closed subspaces X_0 and X_1 of $L^2(0,T)$, generated respectively by the functions in the sets

$$\left\{1\,,\ \ e^{-\gamma t}\,,\ e^{\pm i\beta_n t}\right\}_{n\neq 0}\,, \qquad \left\{1\,,\ t\,,\ \ e^{-\gamma t}\,,\ e^{\pm i\beta_n t}\right\}_{n\neq 0}\,.$$

Then, $\Phi \in X_1$. Its derivative is the limit, in $L^2(0,T)$, of its incremental quotient. It is easy to see that the incremental quotient is a series of elements of X_0. So, $\Phi' \in X_0$,

$$\Phi'(t) = \tilde{A} + \tilde{B}e^{-\gamma t} + \sum_{n\notin\{-\gamma,0,\gamma\}} \delta_n e^{i\beta_n t}\,.$$

We note that $\{\delta_n\} \in l^2$.

Term by term integration gives the following representation for Φ:

$$\Phi(t) = C + At + Be^{-\gamma t} + \sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\delta_n}{\beta_n} e^{i\beta_n t} \tag{62}$$

where A, B, C are suitable constants.

We note that $f_\pm(t)$ are linear combinations of 1, t and $e^{-\gamma t}$. Using this observation, we compare the definition (49) of Φ and (62). Equating the coefficients we find in particular the equality we were looking for, i.e. (61).

Sub-step 2 we prove $\alpha_n = (\sigma_n/n^2)$, and $\{\sigma_n\} \in l^2$.

We prove that $\Phi(t) \in H^2(0,T)$. We replace $\alpha_n = \delta_n/n$ in the expression of $\Psi'(t)$ and we replace the bracket at line (58) with (60). We get:

$$
\begin{aligned}
\Psi'(t) = i \sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\delta_n}{n}\left[i\frac{\gamma}{2} - \frac{\gamma(n+\beta_n) - i\beta_n(n-\beta_n)}{2(\gamma+i\beta_n)}\right] e^{i\beta_n t} \\
- i \sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\delta_n}{n}\left[\frac{\beta_n - n - i\gamma}{2} + i\frac{\gamma n}{\beta_n + i\gamma}\right] e^{-i\beta_n t} + 2\gamma^2 i e^{-\gamma t}\left[\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\delta_n}{n^2}\right] \\
- \int_0^t R'(t-s)\left[\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\delta_n n}{\beta_n^2} w_n(s)\right] \mathrm{d}s \\
+ R'(0)\left[\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\delta_n n}{\beta_n^2}\int_0^t e_n(r)\cos\beta_n(t-r)\,\mathrm{d}r\right] \\
+ \left(R'(0)\frac{t}{2}\right)\left[\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{n\delta_n}{\beta_n^2} e^{i\beta_n t}\right] + \left(\frac{R'(0)}{2}\right)\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{n\delta_n}{\beta_n^3}\sin\beta_n t \\
+ \sum_{n\notin\{-\gamma,0,\gamma\}} \frac{n\delta_n}{\beta_n^2}\int_0^t \cos\beta_n(t-s)\int_0^s R''(s-r)w_n(r)\,\mathrm{d}r\ \mathrm{d}s\,. \quad (63)
\end{aligned}
$$

Termwise differentiation of these series gives again series which converge in $L^2(0,T)$. This is clear for every series a part the last one, which is studied as follows. Termwise differentiation gives

$$
\begin{aligned}
\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{n\delta_n}{\beta_n^2}\int_0^t R''(t-r)w_n(r)\,\mathrm{d}r \\
- \sum_{n\notin\{-\gamma,0,\gamma\}} \frac{n\delta_n}{\beta_n}\int_0^t \sin\beta_n(t-s)\int_0^s R''(s-r)w_n(r)\,\mathrm{d}r\ \mathrm{d}s\,. \quad (64)
\end{aligned}
$$

The first series can be exchanged with the integral, since $\{w_n\}_{|n|>N}$ is a Riesz sequence. The second one converges in square norm thanks to an argument which similar to the one used in the proof of Lemma 15. In fact, we proceed as follows: let

$$
\Gamma_n(t) = \int_0^t R''(t-s)\sin\beta_n s\,\mathrm{d}s
$$

so that for every $T>0$ we have

$$\sum \int_0^T |\Gamma_n(s)|^2 \,\mathrm{d}s \le (\mathrm{const}) \int_0^T |R''(s)|^2 \,\mathrm{d}s \,.$$

Using this fact and uniform boundedness of the sequence $\{w_n(t)\}$ on $[0,T]$, we get

$$\begin{aligned}
&\left\| \sum_{n=M}^{N} \frac{n\delta_n}{\beta_n} \int_0^t w_n(r)\Gamma_n(t-r)\,\mathrm{d}r \right\|^2_{L^2(0,T)} \\
&\qquad \le M \left[\sum_{n=M}^{N} \left(\frac{n\delta_n}{\beta_n} \right)^2 \right] \left[\sum_{n=M}^{N} \int_0^T \left| \int_0^t |\Gamma_n(t-r)| \,\mathrm{d}r \right|^2 \mathrm{d}t \right] \\
&\qquad\qquad \le (\mathrm{const}) \left[\sum_{n=M}^{N} \left(\frac{n\delta_n}{\beta_n} \right)^2 \right] \left[\sum_{n=M}^{N} \int_0^T |\Gamma_n(r)|^2 \,\mathrm{d}r \right] .
\end{aligned}$$

As in the previous step, we see that

$$\delta_n = \frac{\sigma_n}{n} \qquad \{\sigma_n\} \in l^2 \,.$$

Hence we have

$$\alpha_n = \frac{\sigma_n}{n^2}$$

as wanted.

Step 4: the conclusion. We proved that

$$\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\sigma_n}{n^2} z_n(t) = 0 \,, \qquad \{\sigma_n\} \in l^2 \,. \tag{65}$$

Using (39), we see that the series of the derivatives is

$$\sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\sigma_n}{n^2} z_n'(t) = \sum_{n\notin\{-\gamma,0,\gamma\}} \frac{\sigma_n}{n^2} \left\{ in - n^2 \int_0^t N(t-s) z_n(s) \,\mathrm{d}s \right\} .$$

The above series is the sum of a convergent numerical series and of the series

$$\sum_{n\notin\{-\gamma,0,\gamma\}} \sigma_n \int_0^t N(t-s) z_n(s) \,\mathrm{d}s \,. \tag{66}$$

The fact that $\{z_n(t)\}_{|n|>N}$ is a Riesz sequence implies L^2-convergence of $\sum_{n\notin\{-\gamma,0,\gamma\}}\sigma_n z_n(t)$ so that also series (66) converges and we can exchange the series and the integral. in conclusion, the derivative of (65) gives

$$i\left(\sum_{n\notin\{-\gamma,0,\gamma\}}\frac{\sigma_n}{n}\right)-\left(\sum_{n\notin\{-\gamma,0,\gamma\}}n^2\int_0^t N(t-s)z_n(s)\,\mathrm{d}s\right)=0\,.$$

We compute the derivative again and we get

$$\left[\sum_{n\notin\{-\gamma,0,\gamma\}}\sigma_n z_n(t)\right]+\int_0^t M(t-s)\left[\sum_{n\notin\{-\gamma,0,\gamma\}}\sigma_n z_n(s)\right]\,\mathrm{d}s=0\,.$$

We get from here

$$\sum_{n\notin\{-\gamma,0,\gamma\}}\sigma_n z_n(t)=0\,,$$

an equality that we couple with (47) to peel off a term from this series, i.e. in order to perform the procedure described at the beginning of the proof. So, we conclude that $\alpha_n=0$ for every n, as wanted.

The proof of Theorem 4 is now complete.

B Proof of Theorem 6

The proof of the first inequality follows from (31). The case $n\neq k$ is obtained from equalities (32)-(34). We see from here that the left integral in (20) is sum of five terms (the last one, corresponding to $b(t)$ is again sum of two terms). Using (35) The first terms add to

$$\int_0^{2\pi}e^{(-\gamma+ik)t}\left[e^{(\gamma+in)t}+\frac{\gamma}{\gamma+2in}\left(e^{(\gamma+in)t}-e^{-int}\right)\right]\,\mathrm{d}t\,.$$

This integral is easily computed and it is seen that its modulus is dominated by $c/(|\,|k|-|n|\,|)$.

A further term is

$$\int_0^{2\pi}e_n(s)\int_s^{2\pi}e^{-(\gamma+ik)t}\cos n(t-s)\,\mathrm{d}t\ \mathrm{d}s\,.$$

The inner integral is explicitly computed and gives the required estimate.

Then we have to estimate

$$\int_0^{2\pi} e^{-(\gamma+ik)t} \int_0^t K_n(t-s)e_n(s)\,\mathrm{d}s\ \mathrm{d}t\,.$$

We integrate by parts once so to get a factor $1/(\gamma+ik)$ and we use $|e_n(s)| < M/n$. The required estimate follows since $|kn| \geq |\,|k|-|n|\,|$.

The terms corresponding to $b(t)$: the second one is

$$\int_0^{2\pi} e^{-(\gamma+ik)t} \left[\int_0^t M(s)e^{(\gamma+in)(t-s)}\,\mathrm{d}s\right]\ \mathrm{d}t\,.$$

The required estimate is seen upon interchanging the integrals.

The remaining integral is

$$\int_0^{2\pi} e^{-(\gamma+ik)t} \left[\int_0^t \cos ns \int_0^{t-s} M'(t-s-r)e^{(\gamma+in)r}\,\mathrm{d}r\ \mathrm{d}s\right]\ \mathrm{d}t$$
$$= \int_0^{2\pi} \Phi_n(r) \int_r^{2\pi} e^{i(n-k)t}\,\mathrm{d}t\,\mathrm{d}r\,, \qquad \Phi_n(r) = e^{-(\gamma+in)r}\int_0^r M'(r-s)\cos ns\,\mathrm{d}s\,.$$

The required inequality follows from the inner integral.

The estimates in (20) follows by adding the previous estimates.

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