Vertex Ramsey problems in the hypercube

John Goldwasser* John Talbot[†]

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Abstract

If we 2-color the vertices of a large hypercube what monochromatic substructures are we guaranteed to find? Call a set S of vertices from \mathcal{Q}_d , the d-dimensional hypercube, Ramsey if any 2-coloring of the vertices of \mathcal{Q}_n , for n sufficiently large, contains a monochromatic copy of S. Ramsey's theorem tells us that for any $r \geq 1$ every 2-coloring of a sufficiently large r-uniform hypergraph will contain a large monochromatic clique (a complete subhypergraph): hence any set of vertices from \mathcal{Q}_d that all have the same weight is Ramsey. A natural question to ask is: which sets S corresponding to unions of cliques of different weights from \mathcal{Q}_d are Ramsey?

The answer to this question depends on the number of cliques involved. In particular we determine which unions of 2 or 3 cliques are Ramsey and then show, using a probabilistic argument, that any non-trivial union of 39 or more cliques of different weights cannot be Ramsey.

A key tool is a lemma which reduces questions concerning monochromatic configurations in the hypercube to questions about monochromatic translates of sets of integers.

^{*}Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310. Email: jgoldwas@math.wvu.edu

[†]Department of Mathematics, University College London, WC1E 6BT, UK. Email: j.talbot@ucl.ac.uk. This author is a Royal Society University Research Fellow.

1 Introduction

Ramsey's theorem is a seminal result of extremal combinatorics. It implies that any 2-coloring of a sufficiently large r-uniform hypergraph will contain a monochromatic copy of a complete subgraph of a given size [11].

The question we wish to address is: what types of monochromatic sets are unavoidable in any 2-coloring of the vertices of a large hypercube? Such sets are said to be Ramsey. Since the set of vertices of weight r in a hypercube correspond to a complete r-uniform hypergraph it is natural to ask whether sets of vertices corresponding to unions of complete hypergraphs of different weights can be Ramsey. Our main results show that this can happen for some unions of two or three complete hypergraphs (Theorems 11 and 13), but that it cannot occur for arbitrarily large unions (Theorem 18).

In the next section we give the required definitions and show that when considering which subsets of vertices of the hypercube are Ramsey we may restrict our attention to particularly simple "layered" colorings (Theorem 2).

As far as we are aware this paper is the first to consider Ramsey problems for the vertices of the hypercube. There is an extensive literature, however, on the corresponding problems for edge-colorings of the hypercube.

Chung [4] showed that for all $k \geq 2$ and all $r \geq 1$, there exists N such that if $n \geq N$, every edge-coloring of \mathcal{Q}_n with r colors contains a monochromatic copy of C_{4k} . Moreover she gave a 4-coloring of \mathcal{Q}_n with no monochromatic copy of C_6 , while Conder [5] found a 3-coloring with this property.

Alon, Radoičić, Sudakov, and Vondrák [2] extended this to show that for all $k \geq 2$ and all $r \geq 1$, there exists N such that if $n \geq N$, every edge-coloring of Q_n with r colors contains a monochromatic copy of C_{4k+2} .

Axenovich and Martin [3] gave a 4-coloring of the edges of Q_n containing no induced monochromatic copy of C_{10} .

So-called d-polychromatic colorings have also been considered previously: these are edge colorings of the hypercube with p colors so that every d-dimensional subcube contains every color. Alon, Krech and Szabó [1] give upper and lower bounds for the maximum number of colors for which a d-polychromatic colorings exists. Their lower bound was later proved to be

exact by Offner [10]. They also considered d-polychromatic colorings for vertices of the hypercube. Recently Stanton and Özkahya [12] have also considered some of the questions raised by Alon, Krech and Szabó.

Related Turán-type problems for both edges and vertices of the hypercube, were also previously considered. Chung [4] gave bounds on the density of edges required to guarantee a copy of Q_2 and this was improved recently by Thomason and Wagner [13]. Chung also showed that any positive density of edges in a large hypercube guarantees a copy of C_{4k} , for $k \geq 2$. More recently this was extended to C_{4k+2} ($k \geq 3$) by Füredi and Özkahya. For a unified proof of the theorems of Chung, Füredi and Özkahya, see Conlon [6].

The first Turán-type result for vertices of the hypercube is due to E.A. Kostochka [8] who showed that any subset of the vertices of the hypercube of density greater than 2/3 will contain a copy of Q_2 . For related results see Johnson and Talbot [7].

2 Definitions and equivalences

For $a, b \in \mathbb{N}$, a < b we define $[a] = \{1, 2, \dots, a\}$ and $[a, b] = \{a, a + 1, \dots, b\}$.

For $n \geq 1$ let $V_n = \{0,1\}^n$. The *n*-dimensional hypercube, \mathcal{Q}_n , is the graph with vertex set V_n and edges between vertices that differ in exactly one coordinate.

If $1 \leq d \leq n$ then an *embedding* of \mathcal{Q}_d into \mathcal{Q}_n is an injective map $\psi : V_d \to V_n$ that preserves the edges of \mathcal{Q}_d . Note that the image of V_d under any such embedding consists of 2^d elements of V_n given by fixing n-d coordinates and allowing the other d coordinates to vary. We refer to the image of such an embedding as a (d-dimensional) subcube of \mathcal{Q}_n .

Given $F \subseteq V_d$ and $S \subseteq V_n$, with $1 \le d \le n$, we say that S contains a copy of F if there exists an embedding $\psi : V_d \to V_n$ satisfying $\psi(F) \subseteq S$.

For $t \geq 2$, a t-coloring of \mathcal{Q}_n is a map $c: V_n \to [t]$. A t-coloring of \mathcal{Q}_n contains a monochromatic copy of F if there is a color $i \in [t]$ such that $c^{-1}(i)$ contains a copy of F.

We say that a set $F \subseteq V_d$ is t-Ramsey if there exists $n_0(F,t)$ such that for

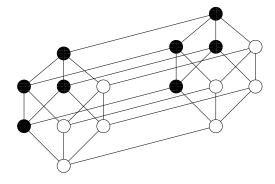


Figure 1: A 3-dimensional subcube of Q_4

all $n \geq n_0$, every t-coloring of \mathcal{Q}_n contains a monochromatic copy of F.

For the remainder of this paper we will work with a different model of the hypercube: the *Boolean lattice*, in which vertices of the hypercube are identified with subsets of [n]. To be precise, if $2^{[n]} = \{A : A \subseteq [n]\}$ is the powerset of [n], then the poset $(2^{[n]}, \subseteq)$ has \mathcal{Q}_n as its Hasse diagram. We identify V_n with $2^{[n]}$ via the natural isomorphism $s : V_n \to 2^{[n]}$, $s(x) = \{i : x_i = 1\}$.

We are interested in characterising those subsets of V_d which are t-Ramsey. The simplest example is given by Ramsey's theorem. For $a, t \geq 0$ a clique of order t and weight a is a family consisting of all a-sets from a set of size t. Given a set K with |K| = t we denote this by $K^{(a)}$.

Theorem 1 (Ramsey [11]). For $t \geq 2$, all cliques are t-Ramsey.

A trivial corollary is that any family of sets which are all the same size is t-Ramsey for $t \geq 2$. It is also obvious that any family of sets which contains members of even and odd weight is not 2-Ramsey since coloring all sets of even weight red and all sets of odd weight blue avoids monochromatic copies of such a family.

For $A \in V_d$ the weight of A is |A|. The collection of all sets of a fixed weight in V_d gives a special type of clique, called a *layer*. The layer containing all sets of weight i from \mathcal{Q}_n is called the i^{th} layer (of \mathcal{Q}_n) and we denote it by L_i .

A particularly simple t-coloring of Q_n is one that is constant on each layer.

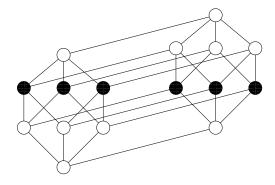


Figure 2: The 2^{nd} layer of \mathcal{Q}_4

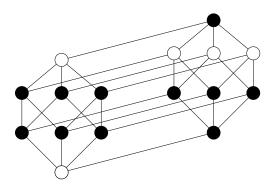


Figure 3: A layered 2-coloring of Q_4 with no monochromatic copy of Q_2 .

We call such a coloring layered. A set $F \subseteq V_d$ is t-layer-Ramsey if there exists $n_L(F,t)$ such that for all $n \geq n_L$, every layered t-coloring of \mathcal{Q}_n contains a monochromatic copy of F.

Our first result says that there is no difference between t-Ramsey and t-layer-Ramsey sets.

Theorem 2. A set $S \subseteq V_d$ is t-Ramsey iff it is t-layer-Ramsey.

For the non-trivial implication in Theorem 2 we require the following lemma.

Lemma 3. If $s, t \geq 1$ then there exists $c_L(s,t)$ such that any t-coloring of \mathcal{Q}_n , where $n \geq c_L$, contains a copy of \mathcal{Q}_s such that the restriction of the coloring to \mathcal{Q}_s is layered.

Proof. By Ramsey's theorem, for any $s \ge l \ge 0$ and $t \ge 2$ there exists an integer R(s, l, t) such that whenever the collection of all l-sets from [R(s, l, t)] are t-colored there is a monochromatic clique of order s. We define a sequence $f_0, f_1, ..., f_{s-1}$ by: $f_0 = s$, $f_i = R(f_{i-1}, i, t)$ for i > 0.

We claim that $c_L(s,t) = f_{s-1}$ will suffice. Suppose that χ is a t-coloring of $\mathcal{Q}_{f_{s-1}}$. By the definition of the $\{f_i\}$ there exists a nested sequence of sets $F_0 \subseteq F_1 \subseteq F_2 \cdots \subseteq F_{s-1} = [c_L(s,t)]$ such that $|F_j| = f_j$ for $j = 0, 1, \ldots, s-1$ and the restriction of χ to $F_{j-1}^{(j)}$ is monochromatic for $j = 1, 2, \ldots, s-1$. (To see this start with F_{s-1} and work down.) Hence the restriction of χ to $F_0^{(j)}$ is monochromatic for $j = 1, 2, \ldots, s-1$. Adding the empty set and F_0 then gives the desired copy of \mathcal{Q}_s on which the restriction of χ is layered.

We remark that our proof actually implies that in any t-coloring of Q_n , where $n \geq c_L(s,t)$, and for any $B \in V_n$ there is copy of Q_s with "B at the bottom" for which the restriction of the coloring is layered. The integer $c_L(s,t)$ produced by this "tower of Ramsey numbers" is obviously rather large if s is large. It would be interesting to find a good upper bound for the smallest possible value of $c_L(s,t)$.

Proof of Theorem 2. Since a layered t-coloring of the cube is still a t-coloring one implication is trivial.

For the converse suppose that $S \subseteq V_d$ is t-layer-Ramsey. Let χ be a t-coloring of \mathcal{Q}_n with $n \geq c_L(n_L(S,t),t)$. By Lemma 3 there is subcube $\mathcal{Q}_{n_L(S,t)}$ of \mathcal{Q}_n such that the restriction of χ to this subcube is layered. Since S is t-layer-Ramsey this subcube contains a monochromatic copy of S.

For $S \subseteq V_d$ we define $W_d(S) = \{|A| : A \in S\}$. For example, if

$$S = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{5\}\} \subseteq V_5$$

then $W_5(S) = \{1, 3\}.$

A layered t-coloring c of \mathcal{Q}_n is equivalent to a t-coloring \hat{c} of the integers $\{0, 1, \ldots, n\}$ (given by $\hat{c}(i) = c(L_i)$). Thus, using Theorem 2, we can translate our original question "which subsets of the hypercube are t-Ramsey", into a question concerning t-colorings of the integers that avoid certain distance sets. This is the key observation which underlies most of our results.

For $D \subseteq \mathbb{Z}$ and $b \in \mathbb{Z}$ we define $D+b=\{d+b \mid d \in D\}$ to be the translation of D by b. We say that a family of sets of integers $\mathcal{D}=\{D_1,D_2,\ldots,D_k\}$ is t-translate-Ramsey if for every t-coloring of the integers, $c:\mathbb{Z} \to [t]$ there exists $D \in \mathcal{D}$ and $j \in \mathbb{Z}$ such that D+j is monochromatic, i.e. every t-coloring of the integers contains a monochromatic translate of a set from the family.

For example the family $\mathcal{D} = \{\{0,1\}, \{0,2\}, \dots, \{0,t\}\}\}$ is t-translate-Ramsey but not (t+1)-translate-Ramsey (to get a (t+1)-coloring with no monochromatic translation of any set in \mathcal{D} just repeat a list of the t+1 distinct colors).

Lemma 4. If $\mathcal{D} = \{D_1, D_2, \dots, D_k\}$ is t-translate-Ramsey, then there exists $n_T(\mathcal{D}, t)$ such that every t-coloring of $[n_T(\mathcal{D}, t)]$ contains a monochromatic translate of a set from \mathcal{D} .

Proof. (This follows easily by compactness but for completeness we give an elementary self-contained proof.) Suppose that \mathcal{D} is t-translate-Ramsey. Let $d = \max\{\max D - \min D : D \in \mathcal{D}\}$. We will show that $n_T(\mathcal{D}, t) = d(t^d + 1)$ will suffice. Suppose, for a contradiction, that there is a t-coloring $c : [d(t^d + 1)] \to [t]$ with no monochromatic translate of any $D \in \mathcal{D}$. Let $B_i = \{1 + (i-1)d, 2 + (i-1)d, \ldots, id\}$ then $[n_T] = B_1 \cup B_2 \cup \cdots B_{t^d+1}$. Since there are $t^d + 1$ blocks B_i and each block contains d integers, there exist

blocks B_i , B_j with i < j such that B_i and B_j are colored identically. Now color the integers periodically using the colors of B_i , B_{i+1} , ..., B_{j-1}

If this coloring of \mathbb{Z} contains a monochromatic translate of $D \in \mathcal{D}$ then by definition of d this translate meets at most two consecutive blocks of the coloring. Moreover since the coloring of $[n_T]$ contained no such monochromatic translate it must meet two consecutive blocks which did not occur in the original coloring of $[n_T]$. But no such pair of blocks occur (since B_j and B_i are colored identically).

It would be interesting to find an order of magnitude estimate for the smallest possible $n_T(\mathcal{D}, t)$. We note that the proof of Lemma 4 also shows that if \mathcal{D} is not t-translate-Ramsey then there exists a periodic coloring of the integers with period of length at most dt^d which contains no monochromatic translates of sets from \mathcal{D} .

The link between t-Ramsey subsets of vertices of the hypercube and t-translate-Ramsey families is given by considering which collections of layers a given subset $S \subseteq V_d$ can meet in the hypercube under all possible embeddings.

For $S \subseteq V_d$ we define

$$W_d^*(S) = \{W_d(\psi(S)) : \psi : V_d \to V_d, \text{ is an automorphism}\}.$$

Any automorphism of \mathcal{Q}_d can be expressed (in the Boolean lattice model) as a set complement followed by a permutation of [d]. Since a permutation of the d labels does not alter the weight of $v \in V(\mathcal{Q}_d)$ we can restrict our attention to the *simple automorphisms* of \mathcal{Q}_d of the form $\psi_B(A) = A\Delta B$, $A, B \in 2^{[d]}$ when determining $W_d^*(S)$:

$$W_d^*(S) = \{W_d(\psi(S)) : \psi : V_d \to V_d, \text{ is a simple automorphism}\}.$$

For example, let $S = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{5\}\} \subseteq V_6$. Now $W_6(S) = \{1, 3\}$, while

$$W_6^*(S) = \{\{1,3\}, \{2,4\}, \{3,5\}, \{0,4\}, \{1,5\}, \{2,6\}, \{0,2,4\}, \{1,3,5\}, \{2,4,6\}\}.$$

Note that if $d_1 \leq d_2$ and S can be embedded in \mathcal{Q}_{d_1} then S can also be embedded in \mathcal{Q}_{d_2} , so $W_d^*(S)$ depends on the value of d. (In our example

above $W_5^*(S) = \{\{1,3\}, \{2,4\}, \{0,4\}, \{1,5\}, \{0,2,4\}, \{1,3,5\}\}.$) In general if $d_1 \leq d_2$ then $W_{d_1}^*(S) \subseteq W_{d_2}^*(S)$, while $W_{d_2}^*(S) \backslash W_{d_1}^*(S)$ consists of translates (in $[d_2]$) of sets from $W_{d_1}^*(S)$. For this reason $W_{d_1}^*(S)$ is t-translate-Ramsey iff $W_{d_2}^*(S)$ is t-translate-Ramsey and so we define $W^*(S)$ to be $W_d^*(S)$, with d minimal such that $S \subseteq V_d$.

When considering whether or not $W^*(S)$ is t-translate-Ramsey it is natural to define W'(S) to be the family of all translates of sets from $W^*(S)$ which have smallest element zero and which are minimal with respect to inclusion. Thus in our example above we have $W'(S) = \{\{0,2\},\{0,4\}\}.$

Lemma 5. If $S \subseteq V_d$ then the following are equivalent.

- (i) S is t-Ramsey;
- (ii) $W^*(S)$ is t-translate-Ramsey;
- (iii) W'(S) is t-translate-Ramsey.

Proof. Clearly $W^*(S)$ is t-translate-Ramsey iff W'(S) is t-translate-Ramsey (taking translations and removing supersets can have no effect on whether or not a family is t-translate-Ramsey). We will show that $S \subseteq V_d$ is t-Ramsey iff $W^*(S)$ is t-translate-Ramsey.

Suppose that $S \subseteq V_d$ is t-Ramsey and $n_0(S)$ is sufficiently large that any t-coloring of \mathcal{Q}_{n_0} contains a monochromatic copy of S. Now take a t-coloring c of \mathbb{Z} . This induces a layered coloring of \mathcal{Q}_{n_0} given by $\hat{c}(L_i) = c(i)$. By definition of n_0 there is a subcube of \mathcal{Q}_{n_0} containing a monochromatic copy of S. The set of layers of \mathcal{Q}_{n_0} in which this copy of S lies is a translate of some $D \in W^*(S)$ and hence there is a monochromatic translate of D in the original coloring of the integers. Hence $W^*(S)$ is t-translate-Ramsey.

Conversely, suppose that $W^*(S)$ is t-translate-Ramsey. Lemma 4 implies that there exists n_0 such that any t-coloring of $[n_0]$ contains a monochromatic translate of some $D \in W^*(S)$. Let c be a layered t-coloring of Q_{n_0} . Define a t-coloring \hat{c} of $[n_0]$ by $\hat{c}(i) = c(L_i)$. By definition of n_0 , this coloring contains a monochromatic translate of some $D \in W^*(S)$. Hence there is a subcube of Q_{n_0} containing a monochromatic copy of S. So S is t-layer-Ramsey and hence by Theorem 2 is t-Ramsey.

Given Ramsey's theorem (Theorem 1), telling us that all cliques are t-Ramsey for all $t \geq 2$, a natural question is to ask whether unions of cliques can also be t-Ramsey. The answer, rather surprisingly, depends on how many cliques we have.

3 Unions of cliques

3.1 Preliminaries

In order to decide which unions of cliques are Ramsey we need to consider the different sets of layers in which such unions may be embedded.

Recall that a clique of weight a and order s consists of all a-sets from a vertex set of size s. (Note that here we use the term vertex to mean a vertex of a hypergraph, rather than a vertex of the hypercube.) We say that a union of cliques is vertex disjoint if the vertex sets of distinct cliques are pairwise disjoint. For example if $S_1 = [3]$, $S_2 = [4, 10]$ and $S_3 = [13, 20]$ then $S_1 = [3] \cup S_2^{(1)} \cup S_3^{(2)} \cup S_3^{(3)}$ is a vertex disjoint union of cliques. We will focus mainly on vertex disjoint unions due to the following simple result.

Lemma 6. If $t \geq 2$ and S is a vertex disjoint union of cliques that is not t-Ramsey then any union of cliques with the same weights and orders as S (but not necessarily vertex disjoint) is also not t-Ramsey.

Let $S = K_1 \cup K_2 \cdots \cup K_s$ be a union of cliques from \mathcal{Q}_d . Suppose that K_i is of weight a_i and order $a_i + t_i$, i.e. $K_i \simeq K_{a_i + t_i}^{(a_i)}$. We wish to determine $W^*(S)$. Consider a simple automorphism given by $\psi_B(A) = A\Delta B$ for some $B \subseteq [d]$. Let us suppose that $|B \cap V(K_i)| = b_i$ and |B| = b. If $b_i \in \{0, a_i + t_i\}$ then $\psi_B(K_i)$ will be contained in a single layer (either $b + a_i$ or $b - a_i$). However if $0 < b_i < a_i + t_i$ then $\psi_B(K_i)$ will meet multiple layers. In order to succinctly describe which layers $\psi_B(K_i)$ will meet we need the following notation.

For integers x < y of the same parity we define

$$[x,y]_2 = \{x, x+2, \dots, y-2, y\}.$$

Lemma 7. If $S, \psi_B, b_1, \ldots, b_s$ and b are as above and D_i denotes the set of layers that $\psi_B(K_i)$ meets then

$$D_i = [\max\{a_i - 2b_i, -a_i\}, \min\{a_i, a_i + 2(t_i - b_i)\}]_2 + b.$$

Moreover precisely one of the following holds for each $1 \le i \le s$:

(i)
$$-a_i + b$$
 or $a_i + b$ belongs to D_i ,

(ii)
$$t_i < b_i < a_i$$
 and $D_i = [a_i - 2b_i, a_i - 2(b_i - t_i)]_2 + b$.

Proof. The first part follows by checking how large $|A\Delta B|$ can be as A varies over the sets from $K_i \simeq K_{a_i+t_i}^{(a_i)}$, where |B| = b and $|A \cap V(K_i)| = b_i$.

For the second part suppose that (i) fails to hold. Now $-a_i + b \notin D_i$ implies that $a_i - 2b_i > -a_i$, and hence $b_i < a_i$. Similarly since $a_i + b \notin D_i$ we must have $t_i < b_i$. Hence $D_i = [a_i - 2b_i, a_i - 2(b_i - t_i)]_2 + b$.

Given $S, \psi_B, b_1, \ldots, b_s$, and b as above define

$$E(b; b_1, b_2, \dots, b_s) = \bigcup_{i=1}^{s} [\max\{a_i - 2b_i, -a_i\}, \min\{a_i, a_i + 2(t_i - b_i)\}]_2 + b.$$

Since $\psi_B(S) = \psi_B(K_1) \cup \cdots \cup \psi_B(K_s)$, $\psi_B(S)$ will meet precisely those layers contained in $E(b; b_1, \ldots, b_s)$.

Thus the family of all possible sets of layers occupied by embeddings of S depends on which values of b and b_1, \ldots, b_s can occur:

$$W^*(S) = \{ E(b; b_1, \dots, b_s) \mid \exists B \subseteq [d], |B| = b, |B \cap K_i| = b_i, 1 \le i \le s \}.$$

Clearly each b_i must satisfy $0 \le b_i \le a_i + t_i$ and if the cliques in S are vertex disjoint then all such values are possible. If, however, two cliques overlap, say $|V(K_i) \cap V(K_j)| = c \ge 1$, then $b_i \ge a_i + t_i - d \implies b_j \ge c - d$, so for example $b_i = a_i + t_i$ and $b_j = 0$ is impossible.

We can now prove Lemma 6.

Proof of Lemma 6. Let S be a vertex disjoint union of cliques that is not t-Ramsey. If \hat{S} is any union of cliques with the same weights and orders

as those in S then by the above discussion we have $W'(\hat{S}) \subseteq W'(S)$ (any choice of $b_1, \ldots b_s$ that can occur for an embedding of \hat{S} can also occur for an embedding of S). Now if \hat{S} is t-Ramsey then Lemma 5 implies that $W'(\hat{S})$ is t-translate-Ramsey. But then $W'(\hat{S}) \subset W'(S)$ so W'(S) is t-translate-Ramsey and so S is t-Ramsey, a contradiction.

For the remainder of this section we will restrict attention to the case that S is a vertex disjoint unions of cliques. Note that in this case for any embedding we have $b = \sum_{i=1}^{s} b_i$, so we write $E(b_1, \ldots, b_s)$ for $E(b; b_1, \ldots, b_s)$.

Embeddings of S in which $b_i \in \{0, a_i + t_i\}$ for each $1 \le i \le s$ will play a special role and we call these *principal embeddings* of S. We define

$$P^*(S) = \{ E(b_1, \dots, b_s) \mid b_i \in \{0, a_i + t_i\}, 1 \le i \le s \},\$$

to denote those sets in $W^*(S)$ achieved by principal embeddings. Note that all $E \in P^*(S)$ are translates of sets of the form $\{x_1a_1, x_2a_2, \ldots, x_sa_s\}$, for some choice of signs $x_1, \ldots, x_s \in \{-1, +1\}$.

For example consider $S_1 = [6]^{(4)} \cup [7, 15]^{(8)}$. In this case the principal embeddings yield

$$P^*(S_1) = \{\{4,8\}, \{2,14\}, \{1,13\}, \{7,11\}\}.$$

We will let P'(S) denote those sets from W'(S) which are translates of sets from $P^*(S)$. So in this example we have

$$P'(S_1) = \{\{0, 4\}, \{0, 12\}\}.$$

Note that a coloring c of \mathbb{Z} which alternates colors on the integers in each congruence class modulo 4 contains no monochromatic translate of either set in $P'(S_1)$. However, while this coloring also contains no monochromatic translate of the set $\{4, 12, 14\}$ produced by the non-principal embedding obtained by taking $b_1 = 6$ and $b_2 = 2$, it does contain a monochromatic translate of the set $\{6, 8\}$ produced by the non-principal embedding obtained by taking $b_1 = 0$ and $b_2 = 2$. Since $\{0, 2\}$ and $\{0, 4\}$ are both in $W'(S_1)$, S_1 is 2-Ramsey. On the other hand, if $S_2 = [6]^{(4)} \cup [7, 16]^{(8)}$ then $P'(S_2) = P'(S_1)$ yet, as Theorem 11 will show, S_2 is not 2-Ramsey ($\{0, 2\}$ is not in $W'(S_2)$).

Our next result tells us that if the sizes of the cliques are not too small compared to their weights we need only consider principal embeddings.

Proposition 8. If S is as in Lemma 7 with $t_i \ge a_i - 1$ for each i then W'(S) = P'(S).

Proof. No b_i can satisfy the inequality in Lemma 7 (ii). Hence each set in W'(S) contains a set in P'(S), so in fact must equal a set in P'(S).

3.2 Two cliques

For integers a, b, c we denote "a is congruent to b modulo c" by $a \equiv_c b$. We extend this in the obvious way to sets: e.g. $\{8, 14\} \equiv_4 \{0, 2\}$.

Lemma 9. If $S = K_1 \cup K_2$ is the vertex disjoint union of two cliques of weights a_1 and a_2 and orders $a_1 + t_1$ and $a_2 + t_2$ respectively, with $a_1 = p_1 2^{r_1}$ and $a_2 = p_2 2^{r_2}$ where $t_1, t_2, r_1, r_2 \ge 0$, $r_1 \le r_2$, and p_1, p_2 are odd then

(a) the reduced family of sets of layers of principal embeddings is

$$P'(S) = \{\{0, |a_1 - a_2|\}, \{0, a_1 + a_2\}\}.$$

- (b) If $r_1 = r_2$ then S is 2-Ramsey.
- (c) If $r_1 < r_2$ and c is a 2-coloring of \mathbb{Z} , then there is no monochromatic translate of either set in P'(S) iff $c(x) \neq c(y)$ for all x, y such that $|x y| = d2^{r_1}$, where $d = \gcd(p_1, p_2)$.

Proof. By definition

$$P^*(S) = \{\{a_1, a_2\}, \{t_1, a_1 + a_2 + t_1\}, \{a_1 + a_2 + t_2, t_2\}, \{a_2 + t_1 + t_2, a_1 + t_1 + t_2\}\}$$

so (a) follows immediately.

(b) If $r_1 = r_2$ then, since $\{p_1 + p_2, p_1 - p_2\} \equiv_4 \{0, 2\}$, the integer

$$\frac{(a_1 - a_2)(a_1 + a_2)}{2^{r_1 + 1}} = \frac{p_1 - p_2}{2}(a_1 + a_2) = \frac{(p_1 + p_2)}{2}(a_1 - a_2)$$

is an odd multiple of one of $a_1 + a_2$ and $|a_1 - a_2|$, and an even multiple of the other. Hence any 2-coloring of \mathbb{Z} must contain a monochromatic translate of one of the sets in P'(S).

(c) If $r_1 < r_2$ then both $|a_1 - a_2|$ and $a_1 + a_2$ are odd multiples of $d2^{r_1}$. Now if $c(x) \neq c(y)$ for all x, y such that $|x - y| = d2^{r_1}$ then $c(x) \neq c(y)$ for all x, y such that |x - y| is an odd multiple of $d2^{r_1}$. Hence there is no monochromatic translate of either set in P'(S). Conversely, suppose c(x) = c(y) for some x, y with $x - y = d2^{r_1}$. If $r_1 < r_2$ then $\gcd(|a_1 - a_2|, a_1 + a_2) = d2^{r_1}$, so there exist integers k and m, one even and one odd, such that $(a_1 + a_2)k - (a_1 - a_2)m = d2^{r_1}$. By symmetry we may suppose k is even and m is odd. Now if $z = x + (a_1 - a_2)m = y + (a_1 + a_2)k$ then |z - x| is an odd multiple of $|a_1 - a_2|$, and |z - y| is an even multiple of $a_1 + a_2$. Since c(x) = c(y) there must be a monochromatic translate of a set from P'(S).

Lemma 10. For each positive integer m divisible by 4, there exists a 2-coloring c of \mathbb{Z} such that $c(x) \neq c(y)$ for all x, y with |x - y| = m, and c(z) = c(z + 2) = c(z + 4) does not occur for any z.

Proof. The period 2m coloring obtained by taking RRBBRRBB...RRBB on [0, m-1], then taking the complement of these colors on [m, 2m-1], and so on, satisfies the required properties.

Theorem 11. Let $S = K_1 \cup K_2$ be the vertex disjoint union of two cliques of weights a_1 and a_2 and orders $a_1 + t_1$ and $a_2 + t_2$ respectively, with $a_1 = p_1 2^{r_1}$ and $a_2 = p_2 2^{r_2}$ where $t_1, t_2, r_1, r_2 \ge 0$, p_1 , p_2 are odd integers, and $r_1 \le r_2$. Then S is 2-Ramsey iff at least one of the following is satisfied

- (1) $r_1 = r_2$;
- (2) at least one of t_1 or t_2 is equal to 0, and a_1 and a_2 are both even;
- (3) t_1 or t_2 is equal to 1, and $2 \le r_1 < r_2$.

Proof. If (1) is satisfied then S is 2-Ramsey by Lemma 9 (b). Assume $r_1 < r_2$ and that (2) is satisfied, say with $t_1 = 0$ and $a_1 < a_2$. The sets $\{0, a_2 - a_1 + 2\}$ (by taking $b = b_1 = 1$) and $\{0, a_2 - a_1\}$ are both in W'(S). If x is any integer such that c(x-2) is not equal to c(x), then $x + a_2 - a_1$ has the same color as x or x-2, so there is a monochromatic translate of a set in W'(S). The argument is virtually the same if $t_2 = 0$ or $a_2 < a_1$.

Now suppose (3) is satisfied. If c is a 2-coloring of \mathbb{Z} with no monochromatic translate of either set in P'(S), then it must have the form prescribed in

Lemma 9 (c), so there exist integers x and y with opposite colors such that y-x is a positive multiple of 4. That means there exists an integer $z \in [x, y]_2$ such that c(z) = c(z+2). Now there are four cases.

If $a_2 < a_1$ and $t_1 = 1$ then the set $\{(a_1 + a_2)/2, (a_1 + a_2)/2 + 2\}$ is in $W^*(S)$ (take $b = b_1 = |a_1 - a_2|/2$), and there is a monochromatic translate of this set. An identical argument works if $a_1 < a_2$ and $t_2 = 1$.

If $a_1 < a_2$ and $t_1 = 1$ then for each $i \in [a_1]$, a translate of the set $A_i = \{0, 2, a_2 - a_1 + 2i\}$ is in $W^*(S)$ (take $b = b_1 = i$). This means that if c(0) = c(2) = R, to avoid a red translate of some set $A_i \in W^*(S)$, all of the integers in $[a_2 - a_1 + 2, a_2 + a_1]_2$ must be blue. For each consecutive pair of blue integers in this set of size a_1 , to avoid a blue translate of some set A_i , there must be a set of a_1 consecutive red even integers. Taking their union forces every integer in $[2(a_2 - a_1 + 2), 2(a_2 + a_1 - 1)]_2$ to be red. Thus at this second stage we have $2a_1 - 2$ consecutive red integers of the same parity. Continuing this process, at the kth stage there must be $k(a_1 - 2) + 2$ consecutive integers of the same parity with the same color. Since $a_1 \ge 4$, this cannot be true for large k. An identical argument works if $a_2 < a_1$ and $a_2 < a_2 > a_2$

Conversely, assume that S does not satisfy (1), (2), or (3). If a_1 and a_2 have different parities then every member of $W^*(S)$ contains numbers of different parity, and thus S is not 2-Ramsey. So we can assume $1 \le r_1 < r_2$. If $r_1 = 1$ then we take a coloring which alternates colors on the even integers and on the odd integers. Since both $a_1 - a_2$ and $a_1 + a_2$ are odd multiples of 2, there is no monochromatic translate of either set in P'(S). Since both t_1 and t_2 are positive, each non-principal embedding contains two integers whose difference is 2, so these cannot be monochromatic. Hence S is not 2-Ramsey.

Now assume that $2 \le r_1 < r_2$ and so $t_1, t_2 \ge 2$. By Lemma 10 there exists a coloring c of the type prescribed in Lemma 9 (c) such that c(z) = c(z+2) = c(z+4) does not occur for any z. Since $t_1, t_2 \ge 2$, any set in $W'(S) \setminus P'(S)$ contains a translate of $\{0, 2, 4\}$, so there is no monochromatic translate of such a set. Furthermore, since $a_1 - a_2$ and $a_1 + a_2$ are both odd multiples of $d2^{r_1}$, there is no monochromatic translate of either set in P'(S). Hence S is not 2-Ramsey.

If $S = K_{a_1+t_1}^{(a_1)} \cup K_{a_2+t_2}^{(a_2)}$ is the union of two cliques which are not vertex

disjoint then Lemma 6 tells us that for S to be 2-Ramsey it must have the same parameters as a vertex disjoint union of cliques that is 2-Ramsey. In fact we can say more and state the following theorem without proof.

Theorem 12. If $S = K_{a_1+t_1}^{(a_1)} \cup K_{a_2+t_2}^{(a_2)}$ is the union of two cliques whose vertex sets overlap in $c \ge 1$ points, $a_1 > a_2$, $t_1, t_2 \ge 2$ then

- (i) If $c \geq 3$ then S is not 2-Ramsey.
- (ii) If c = 2 then S is 2-Ramsey iff there exists a positive integer m such that $a_1 a_2 = 4m$ and $a_1 + a_2 \equiv 2 \mod 8m$.
- (iii) If c=1 then S is 2-Ramsey iff there exists a positive integer m such that $a_1-a_2=4m$ and $a_1+a_2\equiv 0, 2, 4m-2$ or 4m+4 mod 8m, or there is an even integer m such that $a_1-a_2=4m$ and $a_1+a_2\equiv 6$ or 8m-4 mod 8m.

3.3 Three cliques

If the disjoint union of s cliques of different weights is t-Ramsey, then clearly the disjoint union of any s' of them, for any s' < s, is t-Ramsey as well. The converse obviously does not hold in general, so the following result is rather surprising.

Theorem 13. A vertex disjoint union of three cliques of pairwise distinct weights is 2-Ramsey iff the union of each pair of the cliques is 2-Ramsey.

Due to the various possibilities for the structure of each pair of two of the three cliques (Theorem 11 (1),(2),(3)), a complete proof of Theorem 13 would be long. The main idea of our proof is to assume that the union of each pair of two of the three cliques in S is 2-Ramsey, and then show that the only possible coloring of the integers with no monochromatic translate of any set in W'(S) is periodic, with a short period. It is then easy to show that no such coloring exists.

Lemma 14. Let a_1, a_2, a_3 be integers with $a_1 > a_2 > a_3$ such that a_2 and a_3 have the same number of factors of 2 in their prime factorizations. Let t_1, t_2, t_3 be nonnegative integers, and let $S = K_{a_1+t_1}^{(a_1)} \cup K_{a_2+t_2}^{(a_2)} \cup K_{a_3+t_3}^{(a_3)}$ be

a vertex disjoint union of cliques. Any 2-coloring of the integers with no monochromatic translate of any set in W'(S) is periodic with period $2a_1$.

Proof. Let C be any 2-coloring of the integers with no monochromatic translate of any set in W'(S). Let $e = \gcd(a_1, a_2, a_3)$ and for each $0 \le i \le 2e - 1$ let Z_i be the set of all integers congruent to $i \mod 2e$. By the proof of Lemma 9(b) there must be two integers in Z_0 with the same color whose difference is $a_2 - a_3$ or $a_2 + a_3$. Assume it is the former, say $C(0) = C(a_2 - a_3) = R$. Then $C(a_1 - a_3) = C(a_1 + a_2) = B$ to avoid red translates of $\{a_3, a_2, a_1\}$ and $\{-a_2, -a_3, a_1\}$ respectively. (Note that $a_1 - a_3$ and $a_1 + a_2$ have the same color and their difference is $a_2 + a_3$. If we had instead assumed two integers with difference $a_2 + a_3$ are both R, then two integers with difference $a_2 - a_3$ would be R.) Then $R(2a_1) = R(2a_1 + a_2 - a_3) = R(2a_1 + a_2 - a_3) = R(2a_1 - a_3) = R$

Now consider any integer m colored R by the above argument. Then every integer congruent to m mod $2a_1$ is colored R, and if $C(m + a_2 - a_3) = R$ then, by the same argument as above every integer congruent to $m + a_2 - a_3$ mod $2a_1$ is also colored R. This in turn implies that if $C(m + a_2 - a_3) = B$ then all integers congruent to $m + a_2 - a_3 \mod 2a_1$ are colored B (since if any of them were red they would all be red). Thus all integers congruent to $m + a_2 - a_3 \mod 2a_1$ have the same color. Similarly all integers congruent to $m + a_2 + a_3 \mod 2a_1$ must have the same color. Continuing in this way we see that for any fixed integers x, y the set of integers congruent to $x(a_2 - a_3) + y(a_2 + a_3) \mod 2a_1$ all have the same color (of course for some values of x and y the color is B, for others it is B). In particular if $d = \gcd(a_2 - a_3, a_2 + a_3)$ and f is any fixed integer then all integers congruent to f mod f mod f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f have the same color (and all these integers are in f and f

Now $d = \gcd(a_2 + a_3, a_2 - a_3) = 2\gcd(a_2, a_3)$, so $2e = \gcd(2a_1, d)$. Hence for each fixed integer j, all integers congruent to $2je \mod 2a_1$ have the same color. So we have shown that the coloring C is periodic with period $2a_1$ on Z_0 . The same argument can be applied to Z_i for each $1 \le i \le 2e - 1$, showing that C has period $2a_1$ on the integers.

Lemma 15. Let a_1, a_2, a_3 be integers with $a_1 > a_2 > a_3$ such that a_1 and

 a_2 have the same number of factors of 2 in their prime factorizations. Let t_1, t_2, t_3 be nonnegative integers, and let $S = K_{a_1+t_1}^{(a_1)} \cup K_{a_2+t_2}^{(a_2)} \cup K_{a_3+t_3}^{(a_3)}$ be a vertex disjoint union of cliques. Any 2-coloring of the integers with no monochromatic translate of any set in W'(S) is periodic with period $2a_3$.

Lemma 16. Let a_1, a_2, a_3 be integers with $a_1 > a_2 > a_3$ such that a_1 and a_3 have the same number of factors of 2 in their prime factorizations. Let t_1, t_2, t_3 be nonnegative integers, and let $S = K_{a_1+t_1}^{(a_1)} \cup K_{a_2+t_2}^{(a_2)} \cup K_{a_3+t_3}^{(a_3)}$ be a vertex disjoint union of cliques. Any 2-coloring of the integers with no monochromatic translate of any set in W'(S) is periodic with period $2a_2$.

The proofs of Lemma 15 and 16 are similar to that of Lemma 14. For Lemma 15, just as in the proof of Lemma 14, there exist two integers with a difference of a_1-a_2 which must be the same color, say $C(0) = C(a_1-a_2) = R$. Then $C(a_3 - a_2) = C(a_1 + a_3) = B$ (to avoid red translates of $\{a_3, a_2, a_1\}$ and $\{-a_1, -a_2, a_3\}$ respectively). Then $C(2a_3) = C(a_1 - a_2 + 2a_3) = R$, and so on, eventually showing that the coloring on Z_i has period $2a_3$, for each $0 \le i \le 2e - 1$. For Lemma 16, just as in the proof of Lemma 14, there exist two integers with a difference of $a_1 - a_3$ which have the same color. If $C(0) = C(a_1 - a_3) = R$ then $C(a_2 - a_3) = C(a_1 + a_2) = B$, so then $C(2a_2) = C(a_1 + 2a_2 - a_3) = R$, and so on.

Proof of Theorem 13. Suppose $a_1 > a_2 > a_3$ and $a_1 = p_1 2^{r_1}$, $a_2 = p_2 2^{r_2}$, $a_3 = p_3 2^{r_3}$ with p_1, p_2, p_3 odd and $r_1, r_2, r_3 \ge 0$. Let $S = K_{a_1 + t_1}^{(a_1)} \cup K_{a_2 + t_2}^{(a_2)} \cup K_{a_3 + t_3}^{(a_3)}$ be a vertex disjoint union of cliques, with $t_1, t_2, t_3 \ge 0$.

Clearly if any pair of the cliques in S is not 2-Ramsey, then neither is S. So we just need to show that if each pair of cliques is 2-Ramsey then so is S.

Case 1.
$$r_1 = r_2 = r_3$$

By the above lemmas, any 2-coloring of the integers which does not have a monochromatic translate of any set in W'(S) has period $2a_1$, has period $2a_2$, and has period $2a_3$. Hence it has period d where $d = \gcd(2a_1, 2a_2, 2a_3)$. Thus there is a monochromatic translate of the set $\{0, a_2 - a_3, a_1 - a_3\}$ from W'(S), since $a_2 - a_3$ and $a_1 - a_3$ are multiples of d (in fact there are monochromatic translates of every set from P'(S)).

Case 2. $r_1 = r_2 \neq r_3$ and either (i) $t_3 = 1$ and $r_1, r_2, r_3 \geq 2$; (ii) $t_1 = t_2 = 1$

and $r_1, r_2, r_3 \ge 2$; (iii) $t_3 = 0$ or one of t_1 and t_2 is 0 and the other is at most 1 (with restrictions on the exponents according to Theorem 11)

By Lemma 15, if there is a 2-coloring C of the integers with no monochromatic translate of any set in W'(S) then C has period $2a_3$. For subcase (i), since $t_3 = 1$, $\{-a_2, j, j+2, a_1\}$ and $\{j, j+2, a_2, a_1\}$ are both translates of sets in W'(S) for all $j \in [-a_3, a_3 - 2]_2$. As in the proof of Lemma 14 there exist two integers with difference $a_1 - a_2$ with the same color, say R. Due to the a_3 forbidden sets containing a_2 listed above, no two consecutive even integers in the period $2a_3$ coloring C can be colored R. As in the the proof of Lemma 14, there also exist two integers with difference $a_1 + a_2$ with the other color, B, so due to the a_3 forbidden sets containing $-a_2$ listed above, no two consecutive even integers can be colored B. That means C must alternate colors on the even integers. However, then $\{a_3, a_2, a_1\}$ is monochromatic because a_1, a_2, a_3 are all multiples of 4.

For subcase (ii), since $t_1 = t_2 = 1$, the set $\{0, 2\}$ is in W'(S): to see this take an automorphism given by flippling $(a_1 - a_3)/2$ coordinates in the first clique and $(a_2 - a_3)/2$ coordinates in the second clique. So the only way to avoid a monochromatic translate is to alternate colors on the even integers which, as in subcase (i), produces a monochromatic translate after all.

The proof of subcase (iii) is similar (but easier).

Case 3. $r_1 = r_3 \neq r_2$. Subcase (i) is exactly as in Case 2. For subcase (ii), since $t_1 = t_3 = 1$, a translate of each of the sets $\{j, j+2, a_2\}$ and $\{-a_2, j, j+2\}$ is in W'(S) for each $j \in [-a_3, a_3 - 2]_2$, and now an argument identical to the one in the proof of Theorem 11 (3) for the case $a_1 < a_2$ and $t_1 = 1$, produces a monochromatic translate after all.

Case 4. $r_2 = r_3 \neq r_1$. Almost identical to Case 3.

Case 5. r_1, r_2, r_3 all distinct and greater than or equal to 2, at least two of t_1, t_2, t_3 equal to 0 or 1. If t_1 and t_2 are equal to 0 or 1 then the set $\{0, 2\}$ is in W'(S) and we can finish as in Case 1(ii). If $t_1 = t_3 = 1$ then translates of $\{j, j+2, a_2\}$ and $\{-a_2, j, j+2\}$ are in W'(S) for each $j \in [-a_3, a_3-2]_2$. We know that any coloring candidate has two consecutive even integers with the same color, say 0 and 2 are colored R, so $[a_2 - a_3 + 2, a_2 + a_3]_2$ is all R, so $[-2a_3 + 2, 2a_3 - 2]_2$ is all R, and so on, producing arbitrarily long sequences of consecutive even integers with the same color, an impossibility. The other

3.4 Arbitrary unions of cliques

There is no analogue of Theorem 13 for the disjoint union of four cliques of different weights. For example, if S is the disjoint union of cliques of weights 1,5,7,9 then, no matter what the orders of the cliques may be, S is 2-Ramsey (this can be verified rather laboriously by hand by considering the 16 sets in P'(S)). However, if S is the disjoint union of cliques of weights 1, 5, 7, 11, and if the orders are large enough so that W'(S) = P'(S), (so by Proposition 8, orders at least 1, 9, 13, 21 respectively) then S is not 2-Ramsey. The period 38 coloring of the integers obtained by repeating the sequence RRRRBBRRBRBRBRBRBRBBB on the even integers, and on the odd integers, has no monochromatic translate of any of the 13 sets in P'(S) (in fact these colorings are the only colorings of the integers with no monochromatic translate of any set in W'(S)). By Theorem 13 the disjoint union of any three of these four cliques is 2-Ramsey, no matter what the orders of the cliques may be.

Which disjoint unions of s cliques are not 2-Ramsey, but the disjoint union of any s-1 of the cliques is 2-Ramsey? By Theorem 13, none with s=3. By our next result, none if s is sufficiently large.

If S is the vertex disjoint union of s cliques $K_{a_1+t_1}^{(a_1)}, \ldots, K_{a_s+t_s}^{(a_s)}$ where a_i is odd and $t_i \leq 1$ for each i, then S is 2-Ramsey. This is so because $\{0,2\} \in W'(S)$ (take $b_i = (a_i+1)/2$ for each i), but the only 2-coloring of the integers with no monochromatic translate of $\{0,2\}$, is one that alternates colors on the even integers and so contains a monochromatic translate of the set $\{x_1a_1,\ldots,x_sa_s\}$ obtained by letting $x_i = 1$ if $a_i \equiv 1 \mod 4$, and $x_i = -1$ if $a_i \equiv 3 \mod 4$, since any pair of elements from this set differ by a multiple of 4.

Our final result (Theorem 18) tells us that if we require $t_i \geq 2$ for each i then, for sufficiently large s, the vertex disjoint union of s cliques of different weights cannot be 2-Ramsey. First we show that to prove this we need only consider configurations S where a_i is odd for all i.

Proposition 17. Let $S = K_{a_1+t_1}^{(a_1)} \cup \cdots \cup K_{a_s+t_s}^{(a_s)}$, be a vertex disjoint union of s cliques. For a positive integer m, let $S^m = K_{e_1+u_1}^{(e_1)} \cup \cdots \cup K_{e_s+u_s}^{(e_s)}$, be a

vertex disjoint union of $s \geq 2$ cliques with $e_i = ma_i$ for each i.

- (a) If $u_i \ge 2$ for each i and S^m is 2-Ramsey, then so is S (for all values of the t_i 's).
- (b) If $t_i \ge a_i 1$ for each i and S is 2-Ramsey, then so is S^m (for all values of the u_i 's).

Proof. We note that the sets in $P'(S^m)$ are obtained by multiplying each element in each set in P'(S) by m.

For (a) suppose that S^m is 2-Ramsey. Since $u_i \geq 2$ for $1 \leq i \leq s$, Theorem 11 implies that $r_1 = r_2 = \ldots = r_s$, where 2^{r_i} is the largest power of two that divides a_i . In particular all the a_i are of the same parity. Now, for a contradiction, suppose that S is not 2-Ramsey and take a coloring c of the integers avoiding all monochromatic translates of sets from P'(S). Since the a_i are all of the same parity the sets in P'(S) only contain even integers. Hence the sets in $P'(S^m)$ only contain numbers congruent to 0 mod 2m and so their translates lie in a congruence class mod 2m.

Since each $u_i \geq 2$, Lemma 7 implies that for any embedding $\psi : V_d \to V_d$, $W(\psi(S^m))$ either contains a translate of $\{0, 2, 4\}$ or it contains a translate of $A^m \in P'(S^m)$. Thus, if we construct a coloring c' of the integers avoiding all monochromatic translates of sets in $P'(S^m)$ and $\{0, 2, 4\}$ then S^m is not 2-Ramsey, a contradiction.

We can define such a coloring as follows: for integers j, n, with $0 \le j \le 2m-1$, let c'(2mn+j) = c(2n), if $j \equiv 0, 1 \mod 4$ and $c'(2mn+j) \ne c(2n)$ if $j \equiv 2, 3 \mod 4$. For any m > 1, c' avoids monochromatic translates of $\{0, 2, 4\}$ (if c'(x) = c'(x-2) then $x \equiv 0, 1 \mod 2m$ and so $c'(x+2) \ne c'(x)$). Moreover c' restricted to any mod 2m congruence class gives a restriction of c or its complement to the even integers. Since any monochromatic translate under c' of a set $A^m \in P'(S^m)$ lies in a congruence class mod 2m it would correspond to a monochromatic translate under c of a set $A \in P'(S)$, but c contains no such monochromatic translates.

For (b) suppose S^m is not 2-Ramsey and let c' be a coloring of the integers with no monochromatic translate of any set in $P'(S^m)$. Define a coloring c on the integers by c(n) = c'(mn). Clearly c does not produce a monochromatic

translate of any set in P'(S), and since P'(S) = W'(S) (by Proposition 8), S is not 2-Ramsey.

Theorem 18. If S is the vertex disjoint union of $s \geq 39$ cliques $K_{a_1+t_1}^{(a_1)}, \ldots, K_{a_s+t_s}^{(a_s)}$ contained in \mathcal{Q}_d , and each $t_i \geq 2$ then S is not 2-Ramsey

Proof. We will use a probabilistic argument employing the Lovász Local Lemma [9] (see Lemma 19 below).

Let S be a vertex disjoint union of $s \geq 39$ cliques $K_{a_1+t_1}^{(a_1)}, \ldots, K_{a_s+t_s}^{(a_s)}$, with each $t_i \geq 2$. Suppose, for a contradiction, that S is 2-Ramsey. By Proposition 17 (a) we may suppose that $\gcd(a_1,\ldots,a_s)=1$. If any pair of the a_i are of different parities then S is trivially not 2-Ramsey (simply color all even layers red and all odd layers blue). So we may suppose that $a_1 < a_2 < \cdots < a_s$ are all odd and in particular $a_{i+1} - a_i \geq 2$ for $1 \leq i \leq s - 1$. By Lemma 5, $W^*(S)$ is 2-translate-Ramsey. Hence, by Lemma 4, there exists n_T such that any 2-coloring of $[n_T]$ contains a monochromatic translate of $D \in W^*(S) = \{W(\psi(S)) : \psi : V_d \to V_d \text{ is an embedding}\}$.

Since each $t_i \geq 2$, Lemma 7 implies that for any embedding $\psi : V_d \to V_d$, $W(\psi(S))$ either contains a translate of $\{0, 2, 4\}$ or it contains a translate of $\{x_1a_1, x_2a_2, \ldots, x_sa_s\}$, for some choice of signs $x_1, \ldots, x_s \in \{-1, +1\}$. To show that S is not 2-Ramsey it is sufficient to prove that there exists a coloring of $[n_T]$ with no monochromatic translate of $\{0, 2, 4\}$ or $\{x_1a_1, \ldots, x_sa_s\}$, for any choice of signs. We will do this by defining a random 2-coloring of the integers and showing that with positive probability no translate of sets of the above types are found in the restriction of this coloring to $[n_T]$.

Define a random coloring of the integers $c: \mathbb{Z} \to \{R, B\}$ as follows. For each $i \in \mathbb{Z}$ such that $i \equiv 0$ or $1 \mod 4$, toss a fair coin (all coin tosses are independent). If the coin toss is heads set c(i) = R and c(i+2) = B otherwise set c(i) = B and c(i+2) = R. We refer to each pair (i, i+2) of integers colored in this way as a block.

Note that if $y_1, y_2, ..., y_k$ are distinct integers no pair of which differ by exactly two then they are all colored independently. Moreover for any choice of colors $c_1, ..., c_k \in \{R, B\}$ we have

$$\mathbf{Pr}[c(y_1) = c_1, c(y_2) = c_2, \dots, c(y_k) = c_k] = 2^{-k}.$$

The coloring has the property that for any $x \in \mathbb{Z}$ it is not true that c(x) = c(x+2) = c(x+4) (since either (x, x+2) or (x+2, x+4) is a block). Hence no translate of $\{0, 2, 4\}$ is monochromatic.

For each integer b let R_b be the event that there exists a choice of signs $x_1, \ldots, x_s \in \{-1, +1\}$ such that $\{x_1 a_1, \ldots, x_s a_s\} + b$ is red. Let E_b^i be the event that at least one of $b - a_i$ and $b + a_i$ is red. Then

$$\mathbf{Pr}[R_b] = \mathbf{Pr}[E_b^1 \wedge E_b^2 \wedge \dots \wedge E_b^s].$$

Clearly $\mathbf{Pr}[E_b^i] = 3/4$ unless i = 1 and $a_1 = 1$ (in which case it is equal to 1 if (b-1,b+1) is a block, and 3/4 otherwise). We note that $(b+a_i,b+a_{i+1})$ is a block iff $(b-a_{i+1},b-a_i)$ is a block, since $a_{i+1}-a_i = 2$ implies $(b+a_{i+1})-(b-a_i)$ is a multiple of 4. If i < j and $b+a_i$, $b+a_j$ are in different blocks, then E_b^i and E_b^j are independent, while if they are in the same block then j = i+1 and $\mathbf{Pr}[E_b^i \wedge E_b^j] = 1/2$.

Hence if $(b - a_1, b + a_1)$ is not a block, and there are precisely t blocks of the form $(b + a_i, b + a_{i+1})$, for some $1 \le i \le s - 1$, then

$$\mathbf{Pr}[R_b] = \frac{1}{2^t} \left(\frac{3}{4}\right)^{s-2t} \le \left(\frac{3}{4}\right)^s,$$

while if $(b - a_1, b + a_1)$ is a block, then $\mathbf{Pr}[R_b] \leq (3/4)^{s-1}$. Hence this last inequality holds no matter what.

For an integer b let M_b be the event that there exists a choice of signs $x_1, \ldots, x_s \in \{-1, +1\}$ such that $\{x_1 a_1, \ldots, x_s a_s\} + b$ is monochromatic. By symmetry we have $\mathbf{Pr}[M_b] \leq 2(3/4)^{s-1}$.

Our next aim is to show that the event M_b is independent of "most" other events $M_{b'}$, in the following sense.

Claim: M_b is independent of all but at most $6s^2$ events $M_{b'}$.

For any integer b let $D_b = \{\pm a_1, \pm a_2, \ldots, \pm a_s\} + b$. Let b be fixed. We first count the number of ways to choose $b' \neq b$ such that $D_b \cap D_{b'} \neq \emptyset$. If $D_b \cap D_{b'} \neq \emptyset$ and $b' \neq b$ then there exist $u, v \in \{\pm a_1, \ldots, \pm a_s\}$ such that b' = b + u - v. Now $b' \neq b$ implies that $u \neq v$. If we suppose also that $u \neq -v$ then there are 2s(2s-2) such ordered pairs (u, v), but they produce at most s(2s-2) distinct values of b' (since (u, v) and (-v, -u) produce the same

value of b'). There are at most 2s other values of b' produced when u = -v, so there are a total of at most $2s^2$ distinct values of b' such that $D_b \cap D_{b'} \neq \emptyset$. Since M_b is independent of all $M_{b'}$ except those for which there exist $x \in D_b$ and $y \in D_{b'}$ such that $x \in \{y - 2, y, y + 2\}$, there are at most $6s^2$ values of b' such that M_b and $M_{b'}$ are dependent.

It is straightforward to check that for $s \geq 39$ we have $2(6s^2+1)e\left(\frac{3}{4}\right)^{s-1} < 1$. Hence, by the Lovász Local Lemma (Lemma 19), with non-zero probability c gives a coloring of $[n_T]$ with no monochromatic translate of $\{x_1a_1,\ldots,x_sa_s\}$ for any choice of signs $x_1,\ldots,x_s\in\{-1,+1\}$. Hence S is not 2-Ramsey. \square

Lemma 19 (Erdős–Lovász [9]). Let A_1, \ldots, A_k be events in a probability space that each occur with probability at most p. If each event is independent of all but at most d other events and $ep(d+1) \leq 1$ then there is a non-zero probability that none of the events occur.

4 Questions

Given Theorem 18, a natural question to ask is: do there exist 2-Ramsey subsets of V_d that cannot be embedded into a small number of layers? To make this precise we define l(S) to be the smallest number layers into which $S \subseteq V_d$ can be embedded:

$$l(S) = \min_{B \in W'(S)} |B|.$$

Question 20. Do there exist subsets $S_d \subseteq V_d$ such that S_d is 2-Ramsey and $\lim_{d\to\infty} l(S_d) = \infty$?

Another natural question to ask is: how large can a 2-Ramsey subset of V_d be? By Ramsey's theorem examples of size $\binom{d}{\lfloor d/2 \rfloor}$ exist.

Question 21. If $S \subseteq V_d$ is 2-Ramsey how large can |S| be?

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