

MAXIMUM PRINCIPLES FOR P_1 -CONFORMING FINITE ELEMENT APPROXIMATIONS OF QUASI-LINEAR SECOND ORDER ELLIPTIC EQUATIONS

JUNPING WANG* AND RAN ZHANG †

Abstract. This paper derives some discrete maximum principles for P_1 -conforming finite element approximations for quasi-linear second order elliptic equations. The results are extensions of the classical maximum principles in the theory of partial differential equations to finite element methods. The mathematical tools are based on the variational approach that was commonly used in the classical PDE theory. The discrete maximum principles are established by assuming a property on the discrete variational form that is of global nature. In particular, the assumption on the variational form is verified when the finite element partition satisfies some angle conditions. For the general quasi-linear elliptic equation, these angle conditions indicate that each triangle or tetrahedron needs to be $\mathcal{O}(h^\alpha)$ -acute in the sense that each angle α_{ij} (for triangle) or interior dihedral angle α_{ij} (for tetrahedron) must satisfy $\alpha_{ij} \leq \pi/2 - \gamma h^\alpha$ for some $\alpha \geq 0$ and $\gamma > 0$. For the Poisson problem where the differential operator is given by Laplacian, the angle requirement is the same as the existing ones: either all the triangles are non-obtuse or each interior edge is non-negative. It should be pointed out that the analytical tools used in this paper are based on the powerful De Giorgi's iterative method that has played important roles in the theory of partial differential equations. The mathematical analysis itself is of independent interest in the finite element analysis.

Key words. finite element methods, maximum principles, discrete maximum principles, quasi-linear elliptic equations

AMS subject classifications. Primary 65N30; Secondary 65N50

1. Introduction. In this paper we are concerned with maximum principles for P_1 conforming finite element solutions for quasi-linear second order elliptic equations. The continuous problem seeks an unknown function with appropriate regularity such that

$$(1.1) \quad -\nabla \cdot (a(x, u, \nabla u) \nabla u) + \mathbf{b}(x, u, \nabla u) \cdot \nabla u + c(x, u)u = f(x), \quad \text{in } \Omega,$$

where Ω is a polygonal or polyhedral domain in \mathbb{R}^d ($d = 2, 3$), $a = a(x, u, \nabla u)$ is a scalar function, $\mathbf{b} = (b_i(x, u, \nabla u))_{d \times 1}$ is a vector-valued function, $c = c(x, u)$ is a scalar function on Ω , and ∇u denotes the gradient of the function $u = u(x)$. We shall assume that the differential operator is strictly elliptic in Ω ; that is, there exists a positive number $\lambda > 0$ such that

$$(1.2) \quad a(x, \eta, p) \geq \lambda, \quad \forall x \in \Omega, \eta \in \mathbb{R}, p \in \mathbb{R}^d.$$

We also assume that the differential operator has bounded coefficients; that is for some constants Λ and $\nu \geq 0$ we have

$$(1.3) \quad |a(x, \eta, p)| \leq \Lambda, \quad \lambda^{-2} \sum |b_i(x, \eta, p)|^2 + \lambda^{-2} |c(x, \eta)|^2 \leq \nu^2,$$

for all $x \in \Omega, \eta \in \mathbb{R}$, and $p \in \mathbb{R}^d$.

*Division of Mathematical Sciences, National Science Foundation, Arlington, VA 22230 (jwang@nsf.gov). The research of Wang was supported by the NSF IR/D program, while working at the Foundation. However, any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

†Department of Mathematics, Jilin University, Changchun, China (zhangran@mail.jlu.edu.cn). The research of Zhang was supported in part by China Natural National Science Foundation.

Introduce the following form

$$(1.4) \quad \mathfrak{Q}(w; u, v) := \int_{\Omega} \{a \nabla u \cdot \nabla v + \mathbf{b} \cdot (\nabla u)v + cuv\} dx,$$

where $a = a(x, w, \nabla w)$, $\mathbf{b} = \mathbf{b}(x, w, \nabla w)$, and $c = c(x, w)$. Let the function f in (1.1) be locally integrable in Ω . Then a weakly differentiable function u is called a weak solution of (1.1) in Ω if

$$(1.5) \quad \mathfrak{Q}(u; u, v) = F(v), \quad \forall v \in C_0^1(\Omega),$$

where $F(v) \equiv \int_{\Omega} f v dx$. For simplicity, we shall consider solutions of (1.1) with a non-homogeneous Dirichlet boundary condition

$$(1.6) \quad u = g, \quad \text{on } \partial\Omega,$$

where $g \in H^{\frac{1}{2}}(\partial\Omega)$ is a function defined on the boundary of Ω . Here $H^1(\Omega)$ is the Sobolev space consisting of functions which, together with its gradient, is square square integrable over Ω . $H^{\frac{1}{2}}(\partial\Omega)$ is the trace of $H^1(\Omega)$ on the boundary of Ω . The corresponding weak form seeks $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

$$(1.7) \quad \mathfrak{Q}(u; u, v) = F(v), \quad \forall v \in H_0^1(\Omega).$$

The usual maximum principle for the solution of (1.7) (e.g., see [9]) asserts that if $c(x, \eta) \geq 0$ and $f(x) \leq 0$ for all $x \in \Omega$ and $\eta \in \mathbb{R}$, then

$$(1.8) \quad \sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} g_+(x),$$

where $g_+(x) = \max(g(x), 0)$ is the non-negative part of the boundary data. Moreover, if $c = 0$, then one has

$$(1.9) \quad \sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} g(x).$$

For general non-homogeneous equation (1.1), by using the powerful De Giorgi's iterative technique [6] one can derive the following maximum principle (see [20] for details).

THEOREM 1.1. *Let $u \in H^1(\Omega)$ be a weak solution of (1.1) and (1.6) arising from the formula (1.7). Let $p > 2$ be any real number such that $p < +\infty$ for $d = 2$ and $p < \frac{2d}{d-2}$ for $d > 2$. Assume that $f \in L^{\frac{pr}{(p-1)(r-1)}}(\Omega)$ with a real number r , $1 \leq r < p-1$. Assume that the coefficient functions and the solution satisfy $c - \frac{1}{2} \nabla \cdot \mathbf{b} \geq 0$ for any $x \in \Omega$. Then, there exists a constant $C = C(\Omega)$ such that*

$$(1.10) \quad \sup_{x \in \Omega} u(x) \leq k_0 + C \|f\|_{L^{\frac{pr}{(p-1)(r-1)}}},$$

where

$$k_0 = \begin{cases} \sup_{x \in \partial\Omega} g_+(x), & \text{if } c \geq 0, \\ \sup_{x \in \partial\Omega} g(x), & \text{if } c \equiv 0. \end{cases}$$

Moreover, the dependence of $C = C(\Omega)$ is given by

$$C(\Omega) = C 2^{\frac{p-1}{p-1-r}} |\Omega|^{\frac{p-1-r}{pr}}.$$

Here and in what follows of this paper, C denotes a generic dimensionless constant.

The goal of this paper is to establish an analogy of the maximum principles (1.8), (1.9), and (1.10) for $P1$ -conforming finite element approximations of (1.7). We will establish similar maximum principles for such finite element approximations with an assumption on the form $\mathfrak{Q}(w; u, v)$ (see (3.6) for details) that can be verified through some geometric conditions imposed on the corresponding finite element partition. As an example, we shall explore some geometric conditions that apply to the angles of each element, as was commonly done in existing results on discrete maximum principles (DMP) (see for example, [5] and [18]). For the general quasi-linear elliptic equation (1.1), the triangles or tetrahedron need to be $\mathcal{O}(h^\alpha)$ -acute in the sense that each angle (for triangular case) or interior dihedral angle (for tetrahedral case) must satisfy $\alpha_{ij} \leq \pi/2 - \gamma h^\alpha$ for some $\alpha \geq 0$ and $\gamma > 0$. For the Poisson problem where the differential operator is given by Laplacian, the angle requirement is the same as the existing ones: either all the triangles are non-obtuse or each interior edge is non-negative as defined in [8].

The research on discrete maximum principles for finite element solutions can be dated back to the seventies of the last century. In [5], a linear second order elliptic equation was considered, and a discrete maximum principle was established for continuous piecewise linear finite element approximations if all angles in the finite element triangulation are not greater than $\pi/2$ (the so-called non-obtuse condition). In [18], it was noted (see page 78) that the discrete maximum principle holds true for continuous piecewise linear finite element approximations for the Poisson problem under the following weaker condition: for every pair $(\alpha_1; \alpha_2)$ of angles opposite a common edge of some given pair of adjacent triangles of the triangulation one has $\alpha_1 + \alpha_2 \leq \pi$. In [16], it was shown that the discrete maximum principle may hold true in some cases if both angles in such a pair are greater than $\pi/2$. In [3], the case of rectangular meshes and bilinear finite element approximations was considered for second order linear elliptic equations with Dirichlet boundary conditions. The notion of non-narrow rectangular element was introduced as a sufficient geometric condition for a discrete maximum principle to hold. In [14], a 3D nonlinear elliptic problem with Dirichlet boundary condition was considered and the effect of quadrature rules was taken into account. A corresponding discrete maximum principle was derived under the condition of non-obtuseness for the underlying tetrahedral meshes. It was further shown that the DMP may also hold true for continuous piecewise linear finite element approximations for elliptic problems under various weaker conditions on the simplicial meshes used. The acuteness assumption has been weakened in [13] and [16]. In particular, in certain situations, obtuse interior angles in the simplices of the meshes are acceptable. In [11], quasi-linear elliptic equation of second order in divergent form was considered, and corresponding DMPs were derived for mixed (Robin-type) boundary conditions. In [17], a weaker discrete maximum principle is shown to hold under quite general conditions on the mesh (quasi-uniformity) and arbitrary degree polynomials, namely

$$\|u_h\|_{\infty, \Omega} \leq C \|u_h\|_{\infty, \partial\Omega},$$

where $C > 0$ is independent of the meshsize h . In [8], positivity for discrete Green's function was investigated for Poisson equations. The authors addressed the question of whether the discrete Green's function is positive for triangular meshes allowing sufficiently good approximation of H^1 functions. They gave examples which show

that in general the answer is negative. The authors also extended the number of cases where it is known to be positive.

The contributions of this paper are as follows: (1) the DMP result with general non-homogeneous quasi-linear elliptic PDE (1.1) is new (see Theorem 3.2); (2) the DMP result, as summarized in Theorem 3.3, is new with the inclusion of the first order term $\mathbf{b}(x, u, \nabla u) \cdot \nabla u$ in the PDE; and (3) the mathematical tools for deriving DMPs are new in the finite element analysis. Our analytical tools are based on a variational approach which are extensions of similar tools that were used to derive maximum principles in pure theory of partial differential equations. We envision that the new analytical tool shall have applications to a much wider class of problems than the existing approach based on the inversion of M -matrices in the DMP analysis. In particular, we shall report some DMPs for $P1$ -nonconforming finite elements and mixed finite element approximations for (1.1) and (1.6) in a forthcoming paper.

The paper is organized as follows. In Section 2, we shall review the finite element method for (1.1) and (1.6) based on the form (2.3). In Section 3, we shall derive two discrete maximum principles (DMP) for $P1$ -conforming finite element approximations under an assumption to be verified in forthcoming sections. In Section 4, we discuss the relation of shape functions with angles and interior dihedral angles for each element (triangular or tetrahedral). Finally in Section 5, we shall verify the assumption under which the DMPs were derived in earlier sections by requiring some angle conditions for the underlying finite element partition.

2. Galerkin Finite Element Methods. In the standard Galerkin method (e.g., see [4, 1]), the trial space $H^1(\Omega)$ and the test space $H_0^1(\Omega)$ in (1.7) are each replaced by properly defined subspaces of finite dimensions. The resulting solution in the subspace/subset is called a Galerkin approximation. Galerkin finite element methods are particular examples of the Galerkin method in which the approximating functions (both trial and test) are given as continuous piecewise polynomials over a prescribed finite element partition for the domain, denoted by \mathcal{T}_h .

We consider only Galerkin finite element approximations arising from continuous piecewise linear finite element functions – known as $P1$ conforming finite element methods. To this end, let \mathcal{T}_h be a finite element partition of the domain Ω consisting of triangles ($d = 2$) or tetrahedra ($d = 3$). Assume that the partition \mathcal{T}_h is shape regular so that the routine inverse inequality in the finite element analysis holds true (see [4]). Denote by $h = \max_{T \in \mathcal{T}_h} h_T$ the meshsize of \mathcal{T}_h with h_T being the diameter of T . For each $T \in \mathcal{T}_h$, denote by $P_j(T)$ the set of polynomials on T with degree no more than j . The $P1$ conforming finite element space is given by

$$(2.1) \quad S_h := \{v : v \in H^1(\Omega), v|_T \in P_1(T), \forall T \in \mathcal{T}_h\}.$$

Denote by S_h^0 the subspace of S_h with vanishing boundary values on $\partial\Omega$; i.e.,

$$(2.2) \quad S_h^0 := \{v \in S_h, v|_{\partial\Omega} = 0\}.$$

The corresponding Galerkin method seeks $u_h \in S_h$ such that $u_h = I_h g$ on $\partial\Omega$ and

$$(2.3) \quad \mathfrak{Q}(u_h; u_h, v) = F(v), \quad \forall v \in S_h^0,$$

where $I_h g$ is an appropriately defined interpolation of the Dirichlet boundary condition (1.6) into continuous piecewise linear functions on $\partial\Omega$. For example, the standard

nodal point interpolation would be acceptable if the boundary data $u = g$ is sufficiently regular.

Let $v \in S_h$ be any finite element function and k be any real number. We shall decompose $v - k$ into two components

$$(2.4) \quad v - k = (v - k)_+ + (v - k)_-,$$

where $(v - k)_+$ is a finite element function in S_h taken as the non-negative part of $v - k$ at the nodal points of the finite element partition \mathcal{T}_h ; i.e., $(v - k)_+$ is defined as a function in S_h such that at each nodal point A ,

$$(v - k)_+(A) = \begin{cases} v(A) - k, & \text{if } v(A) \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

Likewise, the function $(v - k)_- := (v - k) - (v - k)_+$ is the non-positive part of $v - k$ at the nodal points of \mathcal{T}_h .

LEMMA 2.1. *Let $v \in S_h$ be any finite element function. Let k be any real number such that $k \geq 0$ if $c = c(x, \tau) \geq 0$ and k arbitrary if $c \equiv 0$. Then, we have*

$$(2.5) \quad \mathfrak{Q}(v; v, (v - k)_+) \geq \mathfrak{Q}(v; (v - k)_+, (v - k)_+) + \mathfrak{Q}(v; (v - k)_-, (v - k)_+).$$

Proof. Observe that $\mathfrak{Q}(w; u, v)$ is bilinear in terms of u and v . Thus,

$$\begin{aligned} \mathfrak{Q}(v; v, (v - k)_+) &= \mathfrak{Q}(v; v - k, (v - k)_+) + \mathfrak{Q}(v; k, (v - k)_+) \\ &= \mathfrak{Q}(v; v - k, (v - k)_+) + k(c, (v - k)_+). \end{aligned}$$

Here we have used the fact that $\mathfrak{Q}(v; k, (v - k)_+) = k(c, (v - k)_+)$. If $c \geq 0$ and $k \geq 0$, then we obtain

$$(2.6) \quad \mathfrak{Q}(v; v, (v - k)_+) \geq \mathfrak{Q}(v; v - k, (v - k)_+).$$

In the case of $c \equiv 0$, (2.6) clearly holds true for any real number k and the inequality can be replaced by equality. It follows from (2.6) and the decomposition (2.4) that (2.5) holds true. This completes the proof of the lemma. \square

For convenience of analysis, we shall need a discrete equivalence for the usual L^p norm $\|v\|_{L^p}$ in the finite element space S_h . To this end, let v be any finite element function in S_h . Denote by $\{v\}$ the vector

$$\{v\} = (v(A_1), \dots, v(A_j), \dots, v(A_N)),$$

where $\{A_j\}_{j=1, \dots, N}$ is the set of nodal points of the finite element partition \mathcal{T}_h . Denote by Ω_j the macro element associated with the nodal point A_j (i.e., Ω_j is the union of elements T_{ij} that share A_j as a vertex point). It is not hard to show that there exist constants C_0 and C_1 such that

$$(2.7) \quad C_0 \sum_{j=1}^N |v(A_j)|^p |\Omega_j| \leq \|v\|_{L^p}^p \leq C_1 \sum_{j=1}^N |v(A_j)|^p |\Omega_j|.$$

For completeness, let us outline a proof for the left inequality. For any $x \in \Omega_j$, we have

$$v(A_j) = v(x) + (A_j - x) \cdot \nabla v.$$

Thus,

$$|v(A_j)|^p \leq 2^p (|v(x)|^p + \|A_j - x\|^p \|\nabla v\|^p).$$

Integrating over Ω_j and then using the standard inverse inequality for the finite element function v yields

$$|v(A_j)|^p |\Omega_j| \leq C \int_{\Omega_j} |v(x)|^p dx.$$

By summing the above over all the nodal points A_j we obtain

$$\sum_{j=1}^N |v(A_j)|^p |\Omega_j| \leq C \int_{\Omega} |v|^p dx,$$

where we have used the fact that Ω_j overlaps with only a fixed number of other macro-elements.

3. Maximum Principles for $P1$ Conforming Approximations. The goal of this section is to establish a maximum principle for $P1$ conforming finite element approximations u_h arising from the formula (2.3). This shall be accomplished by using a technique known as the De Giorgi's iterative method ([6]) originally developed for second order elliptic equations associated with maximum principles. In its essence, the De Giorgi's iterative technique is to estimate the set

$$G(k) := \{x : x \in \Omega, u(x) \geq k\}$$

by showing that the measure of the set $G(k)$ is zero for some values of k . The center piece of the De Giorgi's iterative method is the following technical lemma which can be proved through an iterative argument, and hence the name of the method.

LEMMA 3.1. *([6]) Let $\phi(t)$ be a non-negative monotone function on $[k_0, +\infty)$. Assume that ϕ is non-increasing and satisfies*

$$(3.1) \quad \phi(s) \leq \left(\frac{M}{s-k} \right)^\alpha [\phi(k)]^\beta, \quad \forall s > k \geq k_0,$$

where $\alpha > 0, \beta > 1$ are two fixed parameters and $M > 0$ is a constant. Then, there exists a number ρ such that

$$\phi(k_0 + \rho) = 0.$$

Moreover, one has the following estimate

$$\rho \geq M[\phi(k_0)]^{(\beta-1)/\alpha} 2^{\beta/(\beta-1)}.$$

A proof of Lemma 3.1 can be found in [20]. Readers can also find more applications of this lemma in the study of partial differential equations. For completeness,

we outline a proof of Lemma 3.1 as follows. Let ρ be a real number to be determined later, and set

$$k_\tau = k_0 + \rho - \frac{\rho}{2^\tau}, \quad \tau = 0, 1, 2, \dots$$

It then follows from (3.1) that the following recursive formula holds true

$$(3.2) \quad \phi(k_{\tau+1}) \leq \frac{M^\alpha 2^{(\tau+1)\alpha}}{\rho^\alpha} [\phi(k_\tau)]^\beta, \quad \tau = 0, 1, 2, \dots$$

We claim that (3.2) implies the following

$$(3.3) \quad \phi(k_\tau) \leq \frac{\phi(k_0)}{r^\tau}, \quad \tau = 0, 1, 2, \dots$$

with some real number $r > 1$ to be chosen. In fact, (3.3) can be proved by a mathematical induction. The formula (3.3) is clearly true with any real number $r > 1$ when $\tau = 0$. Assume that (3.3) is valid for τ . Now using (3.2) one obtains

$$\begin{aligned} \phi(k_{\tau+1}) &\leq \frac{M^\alpha 2^{(\tau+1)\alpha}}{\rho^\alpha} [\phi(k_\tau)]^\beta \\ &\leq \frac{\phi(k_0)}{r^{\tau+1}} \cdot \frac{M^\alpha 2^{(\tau+1)\alpha}}{\rho^\alpha r^{\tau(\beta-1)-1}} [\phi(k_0)]^{\beta-1}. \end{aligned}$$

Now if we choose $r = 2^{\alpha/(\beta-1)}$, then

$$\phi(k_{\tau+1}) \leq \frac{\phi(k_0)}{r^{\tau+1}} \cdot \frac{M^\alpha 2^{\alpha\beta/(\beta-1)}}{\rho^\alpha} [\phi(k_0)]^{\beta-1}.$$

From this, we see that (3.3) is also valid for $\tau + 1$ if $\rho = M[\phi(k_0)]^{(\beta-1)/\alpha} 2^{\beta/(\beta-1)}$. Now by taking $\tau \rightarrow +\infty$ in (3.3), we see that the left limit of ϕ at $k_0 + \rho$ must be zero. This, together with the given monotonicity of ϕ , completes a proof for the De Giorgi Lemma.

Let $p > 2$ be any real number such that

$$(3.4) \quad p < \begin{cases} +\infty, & d = 2, \\ \frac{2d}{d-2}, & d > 2. \end{cases}$$

Next, we introduce a number k_* defined as follows

$$(3.5) \quad k_* = \begin{cases} \sup_{x \in \partial\Omega} \max\{I_h g(x), 0\}, & \text{if } c \geq 0, \\ \sup_{x \in \partial\Omega} I_h g(x), & \text{if } c \equiv 0. \end{cases}$$

ASSUMPTION 1. *Let the form $\mathfrak{Q}(w; u, v)$ be given by (1.4), and u_h be the finite element approximation of u arising from (2.3). For any real number $k \geq k_*$, assume the following holds true:*

$$(3.6) \quad \mathfrak{Q}(u_h; (u_h - k)_-, (u_h - k)_+) \geq 0.$$

We are now in a position to derive a maximum principle for $P1$ conforming finite element approximations.

THEOREM 3.2. *Let $u_h \in S_h$ be the $P1$ -conforming finite element approximation of (1.1) and (1.6) arising from the formula (2.3). Denote by $I_h g$ the interpolation of the Dirichlet boundary data (1.6) that was used in the finite element formula (2.3). Let p and r be real numbers satisfying (3.4) and $1 \leq r < p - 1$. Assume that $f \in L^{\frac{pr}{(p-1)(r-1)}}(\Omega)$ and the Assumption 1 holds true. Also assume that*

$$(3.7) \quad c(x, u_h) - \frac{1}{2} \nabla \cdot \mathbf{b}(x, u_h, \nabla u_h) \geq 0, \quad \forall x \in \Omega.$$

Then, there exists a constant $C = C(\Omega)$ such that

$$(3.8) \quad \sup_{x \in \Omega} u_h(x) \leq k_* + C \|f\|_{L^{\frac{pr}{(p-1)(r-1)}}},$$

where k_* is given by (3.5). Moreover, the dependence of $C = C(\Omega)$ is given by

$$C(\Omega) = C 2^{\frac{p-1}{p-1-r}} |\Omega|^{\frac{p-1-r}{pr}}.$$

Proof. Let $k \geq k_*$ be any real number. Denote by $\varphi = (u_h - k)_+$ the positive part of $u_h - k$ at nodal points. Since $k \geq k_*$ and k_* is no smaller than the maximum value of the finite element solution u_h on $\partial\Omega$, then φ must vanish on the boundary of Ω ; i.e.,

$$(3.9) \quad \varphi(x) \in S_h^0.$$

Thus, φ is eligible as a test function in the finite element formulation (2.3). By taking $v = \varphi$ in (2.3), we obtain from (2.5) and the Assumption 1 that

$$\begin{aligned} F(\varphi) &= \mathfrak{Q}(u_h; u_h, \varphi) \\ &= \mathfrak{Q}(u_h; u_h, (u_h - k)_+) \\ &\geq \mathfrak{Q}(u_h; (u_h - k)_+, (u_h - k)_+) + \mathfrak{Q}(u_h; (u_h - k)_-, (u_h - k)_+) \\ (3.10) \quad &\geq \mathfrak{Q}(u_h; (u_h - k)_+, (u_h - k)_+). \end{aligned}$$

Using the notation $\varphi = (u_h - k)_+$ in (3.10) we obtain

$$(3.11) \quad (a \nabla \varphi, \nabla \varphi) + (\mathbf{b} \cdot \nabla \varphi, \varphi) + (c \varphi, \varphi) \equiv \mathfrak{Q}(u_h; \varphi, \varphi) \leq F(\varphi).$$

Since the usual integration by parts implies

$$(\mathbf{b} \cdot \nabla \varphi, \varphi) = -(\varphi, \mathbf{b} \cdot \nabla \varphi) - (\varphi, (\nabla \cdot \mathbf{b}) \varphi),$$

then we have

$$(\mathbf{b} \cdot \nabla \varphi, \varphi) = -\frac{1}{2}((\nabla \cdot \mathbf{b}) \varphi, \varphi).$$

Substituting the above into (3.11) yields,

$$(a \nabla \varphi, \nabla \varphi) + \left(\left(c - \frac{1}{2} \nabla \cdot \mathbf{b} \right) \varphi, \varphi \right) \leq F(\varphi),$$

which, along with the condition (3.7), leads to

$$(a\nabla\varphi, \nabla\varphi) \leq F(\varphi).$$

Now let $G(k)$ be the subset of Ω where $\varphi > 0$; i.e.,

$$G(k) = \{T : T \in \mathcal{T}_h, \varphi > 0 \text{ for some } x \in T\}.$$

Denote by $|G(k)|$ the Lebesgue measure of the set $G(k)$. We are going to show that $|G(k)| = 0$ for sufficiently large values of k . To this end, we apply the ellipticity (1.2) and the usual Hölder inequality to (3.11) to obtain

$$(3.12) \quad \lambda \int_{\Omega} |\nabla\varphi|^2 dx \leq \|\varphi\|_{L^p(\Omega)} \|f\|_{L^q(G(k))},$$

where $p > 2$ satisfies (3.4) and q is the conjugate of p ; i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Here the L^q norm of f was taken on the support of φ for the obvious reason. Combining the usual Sobolev embedding with the estimate (3.12) yields

$$(3.13) \quad \|\varphi\|_{L^p}^2 \leq C \|\nabla\varphi\|_{L^2}^2 \leq C \|f\|_{L^q(G(k))} \|\varphi\|_{L^p}.$$

It follows that

$$\|\varphi\|_{L^p} \leq C \|f\|_{L^q(G(k))} \leq C \|f\|_{L^{qs}} |G(k)|^{\frac{1}{qr}},$$

where $r \geq 1$ and $\frac{1}{r} + \frac{1}{s} = 1$ are arbitrary real numbers. The above inequality can be rewritten as

$$\|\varphi\|_{L^p}^p \leq C \|f\|_{L^{qs}}^p |G(k)|^{\frac{p}{qr}}.$$

Now using the norm equivalence (2.7) we obtain

$$(3.14) \quad C_0 \sum_{j=1}^N [(u_h - k)_+(A_j)]^p |\Omega_j| \leq C \|f\|_{L^{qs}}^p |G(k)|^{\frac{p}{qr}}.$$

It is not hard to see that $G(k)$ is the union of all the macro-elements Ω_j so that $u_h(A_j) > k$. For any $\rho > k$, one would have a corresponding set $G(\rho)$. Moreover, if $\Omega_{j_0} \subset G(\rho)$, then we must have $u_h(A_{j_0}) > \rho > k$. This implies that $\Omega_{j_0} \subset G(k)$. Therefore, we have

$$\begin{aligned} C_0 \sum_{j=1}^N [(u_h - k)_+(A_j)]^p |\Omega_j| &\geq C_0 \sum_{j=1, \dots, N; u_h(A_j) > \rho} [(u_h - k)_+(A_j)]^p |\Omega_j| \\ &\geq C_0 (\rho - k)^p \sum_{j=1, \dots, N; u_h(A_j) > \rho} |\Omega_j| \\ &\geq \tilde{C}_0 (\rho - k)^p |G(\rho)|. \end{aligned}$$

Substituting the above inequality into (3.14) gives

$$(3.15) \quad (\rho - k)^p |G(\rho)| \leq C \|f\|_{L^{qs}}^p |G(k)|^{\frac{p}{qr}}.$$

Thus, for any $\rho > k$, we have

$$|G(\rho)| \leq \left(\frac{C\|f\|_{L^{qs}}}{\rho - k} \right)^p |G(k)|^{\frac{p}{qr}}.$$

Note that $q = \frac{p}{p-1}$ and $s = \frac{r}{r-1}$. Thus,

$$|G(\rho)| \leq \left(\frac{C\|f\|_{L^{\frac{pr}{(p-1)(r-1)}}}}{\rho - k} \right)^p |G(k)|^{\frac{p-1}{r}}.$$

Since, by assumption, $p > 2$ and $1 \leq r < p-1$, then we have $\frac{p-1}{r} > 1$. Thus, with $\phi(s) = |G(s)|$, it follows from the De Giorgi's Lemma 3.1 that

$$(3.16) \quad |G(d + k_*)| = 0,$$

where

$$d = C2^{\frac{p-1}{p-1-r}} |\Omega|^{\frac{p-1-r}{pr}} \|f\|_{L^{\frac{pr}{(p-1)(r-1)}}}.$$

The equation (3.16) implies that $u_h \leq d + k_*$ on Ω , which can be rewritten as

$$\sup_{\Omega} u_h \leq k_* + C2^{\frac{p-1}{p-1-r}} |\Omega|^{\frac{p-1-r}{pr}} \|f\|_{L^{\frac{pr}{(p-1)(r-1)}}}.$$

This completes the proof. \square

The rest of this section will establish another discrete maximum principle for the underlying quasi-linear second order equation when $f \leq 0$. The result can be stated as follows.

THEOREM 3.3. *Let $u_h \in S_h$ be the P1-conforming finite element approximation of (1.1) and (1.6) arising from the formula (2.3). Let $f \leq 0$ be any locally integrable function, and the ellipticity (1.2) and the boundedness (1.3) are satisfied. Assume that the Assumption 1 holds true. Then, we have*

$$(3.17) \quad \sup_{x \in \Omega} u_h(x) \leq \begin{cases} \sup_{x \in \partial\Omega} \max(I_h g(x), 0), & \text{if } c \geq 0, \\ \sup_{x \in \partial\Omega} I_h g(x), & \text{if } c \equiv 0, \end{cases}$$

provided that the meshsize h is sufficiently small such that

$$(3.18) \quad h\nu < 1.$$

Proof. Assume that the maximum principle (3.17) does not hold true. We show that such an assumption shall lead to a contradiction. To this end, using the notation as given in (3.5), we see that $k_* < k_M \equiv \sup_{x \in \Omega} u_h(x)$. Let $k_{\#}$ be the largest nodal value of u_h (including the nodal points on the boundary of Ω) that is smaller than k_M . Let k be any real number such that $k_{\#} \leq k < k_M$. Let $\varphi = (u_h - k)_+ \in S_h$ be the positive part of $u_h - k$ at nodal points. Since $k \geq k_{\#} \geq k_*$ and k_* is no smaller than the maximum value of the finite element solution u_h on $\partial\Omega$, then (3.9) holds true. By choosing $v = \varphi$ in (2.3), we obtain from (2.5) and the assumption of $f \leq 0$ that

$$(3.19) \quad \begin{aligned} 0 &\geq F(\varphi) = \mathfrak{Q}(u_h; u_h, \varphi) = \mathfrak{Q}(u_h; u_h, (u_h - k)_+) \\ &\geq \mathfrak{Q}(u_h; (u_h - k)_+, (u_h - k)_+) + \mathfrak{Q}(u_h; (u_h - k)_-, (u_h - k)_+). \end{aligned}$$

Now using the Assumption 1 and the notation of $\varphi = (u_h - k)_+$ we obtain

$$\mathfrak{Q}(u_h; \varphi, \varphi) \leq 0,$$

which leads to

$$(3.20) \quad (a \nabla \varphi, \nabla \varphi) + (\mathbf{b} \cdot \nabla \varphi, \varphi) + (c \varphi, \varphi) \leq 0.$$

Thus, we have from the ellipticity (1.2), the boundedness (1.3), and the condition of $c \geq 0$ that

$$(3.21) \quad \begin{aligned} \lambda \|\nabla \varphi\|_{L^2}^2 &\leq (a \nabla \varphi, \nabla \varphi) \\ &\leq |(\mathbf{b} \cdot \nabla \varphi, \varphi)| \\ &\leq \lambda \nu \|\nabla \varphi\|_{L^2} \|\varphi\|_{L^2(D_k)}, \end{aligned}$$

where D_k is the subset of Ω on which $\nabla \varphi \neq 0$. Note that D_k is a collection of triangular or tetrahedral elements. It follows from the last inequality that

$$(3.22) \quad \|\nabla \varphi\|_{L^2} \leq \nu \|\varphi\|_{L^2(D_k)}.$$

The inequality (3.22) can be rewritten by using element integrals as follows

$$(3.23) \quad \sum_{T \in D_k} \int_T |\nabla \varphi|^2 dT \leq \nu^2 \sum_{T \in D_k} \int_T |\varphi|^2 dT.$$

On each $T \subset D_k$, since $\nabla \varphi \neq 0$, then φ is not a constant on T . Therefore, the selection of k implies that $\varphi = 0$ at one of the vertices of T . Assume that $\varphi(A) = 0$ with A being a vertex point of T . Then, we have from $\varphi(x) = (x - A) \cdot \nabla \varphi$ that

$$\int_T |\varphi|^2 dT \leq h_T^2 \int_T |\nabla \varphi|^2 dT,$$

where h_T is the diameter of the element T . Substituting the above into (3.23) we obtain

$$(3.24) \quad \sum_{T \in D_k} \int_T |\nabla \varphi|^2 dT \leq \nu^2 \sum_{T \in D_k} h_T^2 \int_T |\nabla \varphi|^2 dT \leq \nu^2 h^2 \sum_{T \in D_k} \int_T |\nabla \varphi|^2 dT,$$

which leads to

$$1 \leq h \nu.$$

The above inequality is an obvious contradiction to the assumption of $h \nu < 1$ as given in (3.18). This completes the proof. \square

4. Nodal Basis and Geometry of Finite Elements. On each triangle or tetrahedron $T \in \mathcal{T}_h$, the finite element function $v \in S_h$ is a linear function and can be represented by local shape functions $\ell_i = \ell_i(x)$ defined as follows: (1) ℓ_i is linear on T , (2) $\ell_i(A(j)) = \delta_{ij}$ where δ_{ij} is the usual Kronecker symbol (see Fig. 4.1). The local representative property asserts that

$$(4.1) \quad v(x) = \sum_{i=1}^{d+1} v(A(i)) \ell_i(x), \quad \forall x \in T.$$

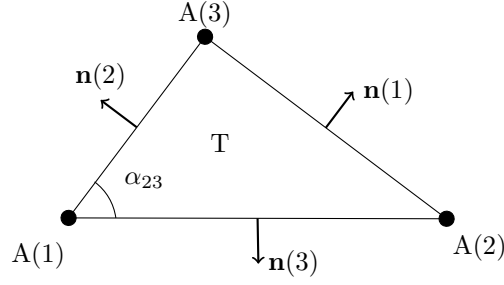


FIG. 4.1. A triangular element with acute angles

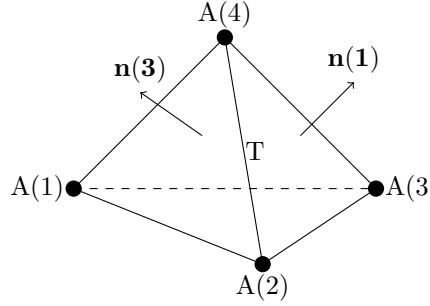


FIG. 4.2. A tetrahedron with acute interior dihedral angles

Note that the gradient of a function $\psi = \psi(x)$ is a vector along which the function ψ increases the most. Thus, the gradient of the shape function ℓ_i would be parallel to the outward normal direction of the edge/face opposite to the vertex $A(i)$; i.e.,

$$\nabla \ell_i = \alpha_i \mathbf{n}(i),$$

where $\mathbf{n}(i)$ represents the outward normal direction to the edge/face opposite to the vertex $A(i)$ (see Fig. 4.1 and Fig. 4.2). Denote by $\|\xi\|$ the ℓ^2 -length of any vector $\xi \in \mathbb{R}^d$. It follows that

$$\alpha_i = -\|\nabla \ell_i\|.$$

Thus, we have

$$(4.2) \quad \nabla \ell_i = -\|\nabla \ell_i\| \mathbf{n}(i).$$

The angles of the triangle $\Delta A(1)A(2)A(3)$ (see Fig. 4.1) can be characterized by using the outward normal directions $\mathbf{n}(i)$. For example, the angle α_{23} is related to the angle of the two normal vectors $\mathbf{n}(2)$ and $\mathbf{n}(3)$ as follows:

$$\alpha_{23} = \pi - \angle(\mathbf{n}(2), \mathbf{n}(3)),$$

where $\angle(\mathbf{n}(2), \mathbf{n}(3))$ stands for the angle between $\mathbf{n}(2)$ and $\mathbf{n}(3)$. Likewise, for the tetrahedron T as depicted in Fig. 4.2, the interior angle between the two planes $P(A(1), A(2), A(4))$ and $P(A(2), A(3), A(4))$ can be defined as

$$\theta = \pi - \angle(\mathbf{n}(1), \mathbf{n}(3)).$$

The angle θ is known as an interior dihedral angle. The definition of other five interior dihedral angles for T can be defined similarly. For simplicity, we introduce the following notation:

$$(4.3) \quad \alpha_{ij} := \pi - \angle(\mathbf{n}(i), \mathbf{n}(j)).$$

It follows from (4.2) that

$$(4.4) \quad \alpha_{ij} = \pi - \angle(\nabla \ell_i, \nabla \ell_j).$$

The triangle T is called non-obtuse if all the angles satisfy $\alpha_{ij} \leq \pi/2$. It is said to be acute if $\alpha_{ij} < \pi/2$. Likewise, a tetrahedron T is called acute if each of its six interior dihedral angles is less than $\pi/2$ in radian; T is said to be non-obtuse if all six interior dihedral angles are no more than $\pi/2$ in radian. For the purpose of the maximum principles for finite element approximations, we introduce the following concept.

DEFINITION 4.1. *The finite element partition \mathcal{T}_h is called $\mathcal{O}(h^\alpha)$ -acute if there exists a parameter $\gamma > 0$ such that for each element $T \in \mathcal{T}_h$ we have $\alpha_{ij} \leq \frac{\pi}{2} - \gamma h^\alpha$, where $\alpha \geq 0$ and h is the meshsize of \mathcal{T}_h .*

5. Verification of the Key Assumption for DMP. Recall that the validity of DMPs as shown in Theorems 3.2 and 3.3 is based on the Assumption 1 which states that

$$(5.1) \quad \mathfrak{Q}(u_h; (u_h - k)_-, (u_h - k)_+) \geq 0$$

for all $k \geq k_*$. The goal of this section is to verify the above assumption under certain conditions for the finite element partition \mathcal{T}_h .

5.1. An Element-Based Approach. By an element-wise approach, we mean a representation of the form $\mathfrak{Q}(u_h; (u_h - k)_-, (u_h - k)_+)$ as integrals over each element $T \in \mathcal{T}_h$. To verify the assumption (5.1), we shall explore conditions that make each element integral be non-negative. To this end, on each element $T \in \mathcal{T}_h$, we use the local shape functions ℓ_j to represent both $(u_h - k)_-$ and $(u_h - k)_+$ as follows

$$\begin{aligned} (u_h - k)_-(x) &= \sum_{i=1}^{d+1} (u_h(A(i)) - k)_- \ell_i(x), \\ (u_h - k)_+(x) &= \sum_{j=1}^{d+1} (u_h(A(j)) - k)_+ \ell_j(x). \end{aligned}$$

Denote by $\varphi = (u_h - k)_+$ and $\psi = (u_h - k)_-$. It follows that

$$\begin{aligned} \mathfrak{Q}(u_h; (u_h - k)_-, (u_h - k)_+) &= \mathfrak{Q}(u_h; \psi, \varphi) \\ &= \sum_{T \in \mathcal{T}_h} \{ (a \nabla \psi, \nabla \varphi)_T + (\mathbf{b} \cdot \nabla \psi, \varphi)_T + (c \psi, \varphi)_T \} \end{aligned}$$

On each element T , we have

$$\begin{aligned} & (a \nabla \psi, \nabla \varphi)_T + (\mathbf{b} \cdot \nabla \psi, \varphi)_T + (c \psi, \varphi)_T \\ &= \sum_{i,j=1}^{d+1} (u_h(A(i)) - k)_- (u_h(A(j)) - k)_+ \int_T \{ a \nabla \ell_i \cdot \nabla \ell_j + \mathbf{b} \cdot (\nabla \ell_i) \ell_j + c \ell_i \ell_j \} dx. \end{aligned}$$

Using the angle relation (4.4) we obtain

$$\begin{aligned}\nabla \ell_i \cdot \nabla \ell_j &= \|\nabla \ell_i\| \|\nabla \ell_j\| \cos(\angle(\nabla \ell_i, \nabla \ell_j)) \\ &= \|\nabla \ell_i\| \|\nabla \ell_j\| \cos(\pi - \alpha_{ij}) \\ &= -\|\nabla \ell_i\| \|\nabla \ell_j\| \cos(\alpha_{ij}).\end{aligned}$$

Thus, it follows from the boundedness (1.3) that

$$\begin{aligned}& - \int_T \{a \nabla \ell_i \cdot \nabla \ell_j + \mathbf{b} \cdot (\nabla \ell_i) \ell_j + c \ell_i \ell_j\} dx \\ &= \int_T \{a \|\nabla \ell_i\| \|\nabla \ell_j\| \cos(\alpha_{ij}) - \mathbf{b} \cdot (\nabla \ell_i) \ell_j - c \ell_i \ell_j\} dx \\ &\geq \int_T \{a \|\nabla \ell_i\| \|\nabla \ell_j\| \cos(\alpha_{ij}) - \|\mathbf{b}\| \|\nabla \ell_i\| - |c|\} dx \\ &\geq \int_T \{a \|\nabla \ell_i\| \|\nabla \ell_j\| \cos(\alpha_{ij}) - \lambda \nu \|\nabla \ell_i\| - \lambda \nu\} dx.\end{aligned}$$

Assume that the element T is non-obtuse (i.e., $0 \leq \alpha_{ij} \leq \pi/2$). Then we have from the above inequality and the ellipticity (1.2) that

$$\begin{aligned}& - \int_T \{a \nabla \ell_i \cdot \nabla \ell_j + \mathbf{b} \cdot (\nabla \ell_i) \ell_j + c \ell_i \ell_j\} dx \\ &\geq \lambda \int_T \{\|\nabla \ell_i\| \|\nabla \ell_j\| \cos(\alpha_{ij}) - \nu(\|\nabla \ell_i\| + 1)\} dx.\end{aligned}$$

Next, we see from Taylor expansion, for $\alpha_{ij} \in [\rho_0, \pi/2]$ with $\rho_0 > 0$ being a fixed angle, there is a constant $\gamma^* > 0$ such that

$$\cos(\alpha_{ij}) \geq \gamma^* \left(\frac{\pi}{2} - \alpha_{ij} \right).$$

Observe that both $\|\nabla \ell_i\|$ and $\|\nabla \ell_j\|$ are of size $\mathcal{O}(h_T^{-1})$ where h_T is the size of T . Thus, with $|T|$ being the measure of T , we have

$$\begin{aligned}& - \int_T \{a \nabla \ell_i \cdot \nabla \ell_j + \mathbf{b} \cdot (\nabla \ell_i) \ell_j + c \ell_i \ell_j\} dx \\ &\geq \lambda \gamma^* \int_T \{\|\nabla \ell_i\| \|\nabla \ell_j\| (\pi/2 - \alpha_{ij}) - \nu(\|\nabla \ell_i\| + 1)\} dx \\ &\geq \lambda^* \|\nabla \ell_i\| \|\nabla \ell_j\| |T|\end{aligned}$$

for some $\lambda^* > 0$ when the size of T is sufficiently small and $\pi/2 - \alpha_{ij} \geq \gamma h$ for a large, but fixed constant γ . In the case of $\mathbf{b} = 0$, the angle requirement can be weakened to $\pi/2 - \alpha_{ij} \geq \gamma h^2$. The very same argument holds true if u_h is replaced by any finite element function $v \in S_h$. The result can be summarized into a lemma as follows.

LEMMA 5.1. *Let $v \in S_h$ be any finite element function and k any real number. Assume that the ellipticity (1.2) and the boundedness (1.3) hold true. Assume also that the partition \mathcal{T}_h is $\mathcal{O}(h^\alpha)$ -acute. Then, the following results hold true:*

(i) For general \mathbf{b} and $c \geq 0$, with $\alpha = 1$, we have

$$(5.2) \quad \mathfrak{Q}(v; (v-k)_-, (v-k)_+) \geq \lambda^* \sum_{T \in \mathcal{T}_h} \sum_{i \neq j} |(v(A(i)) - k)_-| |(v(A(j)) - k)_+| \|\nabla \ell_i\| \|\nabla \ell_j\| |T|,$$

provided that the meshsize h for the partition \mathcal{T}_h is sufficiently small. Here λ^* is a positive number smaller than λ and $|T|$ stands for the area or volume of the element T .

(ii) For the case $\mathbf{b} = 0$ and $c \geq 0$, with $\alpha = 2$, we have

$$(5.3) \quad \mathfrak{Q}(v; (v-k)_-, (v-k)_+) \geq \lambda^* \sum_{T \in \mathcal{T}_h} \sum_{i \neq j} |(v(A(i)) - k)_-| |(v(A(j)) - k)_+| \|\nabla \ell_i\| \|\nabla \ell_j\| |T|,$$

provided that h is sufficiently small.

(iii) For the case of $\mathbf{b} = 0$ and $c = 0$, we have

$$(5.4) \quad \mathfrak{Q}(v; (v-k)_-, (v-k)_+) \geq \lambda \sum_{T \in \mathcal{T}_h} \sum_{i \neq j} |(v(A(i)) - k)_-| |(v(A(j)) - k)_+| \|\nabla \ell_i\| \|\nabla \ell_j\| \cos(\alpha_{ij}) |T|,$$

as long as each $T \in \mathcal{T}_h$ is non-obtuse.

In other words, the Assumption 1 is satisfied if the finite element partition \mathcal{T}_h satisfies certain angle conditions.

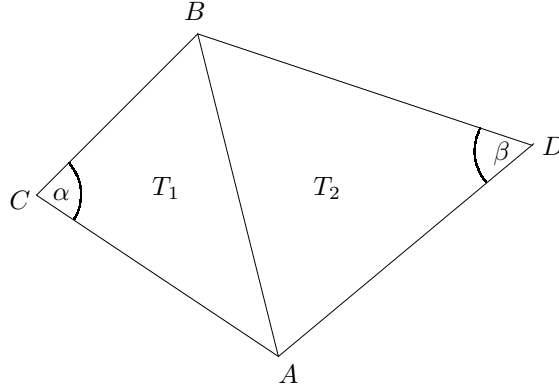


FIG. 5.1. An interior edge shared by two elements T_1 and T_2 .

5.2. An Edge-Based Approach. By an edge-wise approach, we mean a representation of the form $\mathfrak{Q}(u_h; (u_h - k)_-, (u_h - k)_+)$ as integrals over macro-elements that share a common edge. To verify the assumption (5.1), we shall explore conditions that make each integral on macro-elements be non-negative. To this end, we use the

notation $\varphi = (u_h - k)_+$ and $\psi = (u_h - k)_-$ to arrive at

$$\begin{aligned} & \mathfrak{Q}(u_h; (u_h - k)_-, (u_h - k)_+) = \mathfrak{Q}(u_h; \psi, \varphi) \\ &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{d+1} \psi(A(i)) \varphi(A(j)) \int_T \{a \nabla \ell_i \cdot \nabla \ell_j + \mathbf{b} \cdot (\nabla \ell_i) \ell_j + c \ell_i \ell_j\} dx \\ &= \sum_{e_{mn} \in \mathcal{E}_h^0} \psi(A_m) \varphi(A_n) \sum_{s=1}^2 \int_{T_s} \left\{ a \nabla \ell_m^{(s)} \cdot \nabla \ell_n^{(s)} + \mathbf{b} \cdot (\nabla \ell_m^{(s)}) \ell_n^{(s)} + c \ell_m^{(s)} \ell_n^{(s)} \right\} dx, \end{aligned}$$

where \mathcal{E}_h^0 denotes the set of all interior edges, A_m and A_n are two end points of the edge e_{mn} , T_1 and T_2 share e_{mn} as a common edge. In Fig. 5.1, one may identify A_m with A , and A_n with B . Here $\ell_m^{(s)}$ is the shape function on the element T_s associated with the vertex point A_m . Thus, the validity of various DMPs can be derived if the following holds true

$$(5.5) \quad \sum_{s=1}^2 \int_{T_s} \left\{ a \nabla \ell_m^{(s)} \cdot \nabla \ell_n^{(s)} + \mathbf{b} \cdot (\nabla \ell_m^{(s)}) \ell_n^{(s)} + c \ell_m^{(s)} \ell_n^{(s)} \right\} dx \leq 0.$$

In the case of Poisson problem, one has $a \equiv 1$, $\mathbf{b} \equiv 0$, and $c \equiv 0$. Thus, it suffices to have

$$(5.6) \quad \sum_{s=1}^2 \int_{T_s} \nabla \ell_m^{(s)} \cdot \nabla \ell_n^{(s)} dx \leq 0.$$

It was known that (see for example [8])

$$\int_{T_1} \nabla \ell_m^{(1)} \cdot \nabla \ell_n^{(1)} dx = -\frac{\cot(\alpha)}{2},$$

and

$$\int_{T_2} \nabla \ell_m^{(2)} \cdot \nabla \ell_n^{(2)} dx = -\frac{\cot(\beta)}{2}.$$

It follows that

$$\begin{aligned} \sum_{s=1}^2 \int_{T_s} \nabla \ell_m^{(s)} \cdot \nabla \ell_n^{(s)} dx &= -\frac{\cot(\alpha)}{2} - \frac{\cot(\beta)}{2} \\ &= -\frac{\sin(\alpha + \beta)}{2 \sin \alpha \sin \beta}, \end{aligned}$$

and (5.6) holds true if and only if $\alpha + \beta \leq \pi$.

A similar, but more complicated, analysis can be conducted for tetrahedral elements; this is left to readers with interest and curiosity on DMPs for Poisson problems in 3D.

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