# Improved Bounds for Geometric Permutations* 

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#### Abstract

We show that the number of geometric permutations of an arbitrary collection of $n$ pairwise disjoint convex sets in $\mathbb{R}^{d}$, for $d \geq 3$, is $O\left(n^{2 d-3} \log n\right)$, improving Wenger's 20 years old bound of $O\left(n^{2 d-2}\right)$.


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## 1 Introduction

Let $\mathcal{K}$ be a collection of $n$ convex sets in $\mathbb{R}^{d}$. A line $\ell$ is a transversal of $\mathcal{K}$ if it intersects all the sets in $\mathcal{K}$. If the objects in $\mathcal{K}$ are pairwise disjoint, an oriented line transversal meets them in a well-defined order, called a geometric permutation. The study of geometric permutations plays a central role in geometric transversal theory; see [8, 20] for compherensive surveys.
Previous work. In 1985, Katchalski et al. [11] initiated the study of the maximum possible number $g_{d}(n)$ of geometric permutations induced by a set $\mathcal{K}$ of $n$ pairwise disjoint convex objects in $\mathbb{R}^{d}$. They constructed, for any $n \geq 4$, a family of $n$ pairwise disjoint convex sets in $\mathbb{R}^{2}$ that admits $2 n-2$ geometric permutations. Edelsbrunner and Sharir [5] showed, five years later, that this bound is tight in the worst case, implying that $g_{2}(n)=2 n-2$. Wenger [19 proved, also in 1990, that $g_{d}(n)=O\left(n^{2 d-2}\right)$ in any dimension $d \geq 3$. In 1992, Katchalski et al. 12 generalized their lower bound construction and showed that there exist collections of $n$ pairwise disjoint convex sets in $\mathbb{R}^{d}$, for any $d \geq 3$, which admit $\Omega\left(n^{d-1}\right)$ geometric permutations. Since then, closing (or even reducing) the fairly large gap between these upper and lower bounds on $g_{d}(n)$, in any dimension $d \geq 3$, has remained one of the major long standing open problems in geometric transversal theory.

Several partial steps towards this goal were made in the past decade. Most of them deal with geometric permutations of certain restricted families of pairwise disjoint convex bodies in $\mathbb{R}^{d}$. For example, Smorodinsky et al. [17] derived a tight upper bound of $\Theta\left(n^{d-1}\right)$ on the number of geometric permutations induced by an arbitrary collection of $n$ pairwise disjoint balls in $\mathbb{R}^{d}$. Katz and Varadarajan [14] generalized this result to arbitrary collections of $n$ pairwise disjoint fat convex bodies. Other recent works [3, 9, 13, 21] show that the maximum possible number of geometric permutations induced by pairwise disjoint unit balls (or, more generally, balls of bounded size disparity) is constant in any dimension.

Other studies bound the number of geometric permutations induced by arbitrary collections of pairwise disjoint convex sets, whose realizing transversal lines belong to some restricted subfamily of lines in $\mathbb{R}^{d}$. For example, Aronov and Smorodinsky [2] derive a tight bound of $\Theta\left(n^{d-1}\right)$ on the maximum number of geometric permutations realized by lines that pass through a fixed point in $\mathbb{R}^{d}$. A recent paper [10 by the authors studies line transversals of arbitrary convex polyhedra in $\mathbb{R}^{3}$ and derives (as a byproduct) an improved upper bound of $O\left(n^{3+\varepsilon}\right)$, for any $\varepsilon>0$, on the number of geometric permutations realized by lines which pass through a fixed line in $\mathbb{R}^{3}$.

The space of line transversals. Lines in $\mathbb{R}^{d}$ have $2 d-2$ degrees of freedom, and are naturally represented in a real projective space (so-called the Grassmannian manifold; see [8]). However, for the purpose of combinatorial analysis, we can represent them (with the exclusion of some "negligible" subset which we may ignore) by points in the real Euclidean space $\mathbb{R}^{2 d-2}$; see [8] for more details.

Let $\mathcal{K}$ be a collection of $n$ convex sets in $\mathbb{R}^{d}$, not necessarily pairwise disjoint. The transversal space $\mathcal{T}(\mathcal{K})$ of $\mathcal{K}$ is the set in $\mathbb{R}^{2 d-2}$ of all (points representing) the transversal lines of $\mathcal{K}$.

If the sets of $\mathcal{K}$ are pairwise disjoint then any two lines in the same connected component of $\mathcal{T}(\mathcal{K})$ induce the same geometric permutation, so the number of geometric permutations is upper bounded by the number of components of $\mathcal{T}(\mathcal{K})$. In two dimensions, the converse property also holds. That is, lines that stab $\mathcal{K}$ in a fixed order form a single connected component of $\mathcal{T}(\mathcal{K})$; see, e.g., [7]. Thus, according to [5], the transversal space $\mathcal{T}(\mathcal{K})$, of any family $\mathcal{K}$ of $n$ pairwise disjoint convex sets in $\mathbb{R}^{2}$ has at most $2 n-2$ connected components.

The situation becomes considerably more complicated already in $\mathbb{R}^{3}$ : There exist collections of
four (pairwise disjoint) convex sets whose transversal space consists of an arbitrarily large number of connected components [7, 10]. This is a simple instance of the phenomenon that the shape of $\mathcal{T}(\mathcal{K})$ depends on the shape of the sets in $\mathcal{K}$, and may grow out of control if we do not impose any restrictions on the sets of $\mathcal{K}$. This might explain (in part) the difficulty of extending the relatively simple analysis of the number of geometric permutations in $\mathbb{R}^{2}$ to higher dimensions.

In three dimensions, if the sets in $\mathcal{K}$ have constant description complexity (i.e., each set can be described as a Boolean combination of a constant number of polynomial equalities and inequalities of constant maximum degree) then one can obtain sharp bounds on the combinatorial complexity of $\mathcal{T}(\mathcal{K})$ (see [15, 20] for a precise definition). Specifically, the analysis of Koltun and Sharir [15] yields an improved bound of $O\left(n^{3+\varepsilon}\right)$, for any $\varepsilon>0$, on the combinatorial complexity, and thus also on the number of connected components of $\mathcal{T}(\mathcal{K})$, for collections $\mathcal{K}$ of this kind. (If $\mathcal{K}$ is a collection of $n$ triangles in $\mathbb{R}^{3}$, an improved bound of $O\left(n^{3} \log n\right)$ holds, see [1.) Hence, this also serves as an upper bound on the number of geometric permutations induced by any such collection $\mathcal{K}$. Using this approach, and continuing to assume that the sets in $\mathcal{K}$ have constant description complexity, one can strengthen Wenger's bound of $O\left(n^{2 d-2}\right)$ [19] to apply to the combinatorial complexity of $T(\mathcal{K})$, and not just to the number of geometric permutations. The strength (and beauty) of Wenger's analysis is that it yields this bound without making any assumptions whatsoever on the shape of the sets in $\mathcal{K}$ (other than being convex and pairwise disjoint).

Our results. We first show that the number of geometric permutations admitted by any collection of $n$ pairwise disjoint convex sets in $\mathbb{R}^{3}$ is $O\left(n^{3} \log n\right)$, thus improving Wenger's previous upper bound on $g_{3}(n)$ roughly by a factor of $n$. Our approach can be generalized to higher dimensions, and yields an improved upper bound of $O\left(n^{2 d-3} \log n\right)$ on $g_{d}(n)$, for any $d \geq 3$. (In three dimensions, our bound is also a slight improvement of the bound $O\left(n^{3+\varepsilon}\right)$, for any $\varepsilon>0$, of [15] for the case where the sets in $\mathcal{K}$ have constant description complexity.)

Here is a brief overview of our solution in $\mathbb{R}^{3}$. Following the approach of Wenger [19], we represent the directions of transversal lines by points on the unit 2 -sphere $\mathbb{S}^{2}$, separate every pair of objects in $\mathcal{K}$ by a plane, and associate with each such plane the great circle on $\mathbb{S}^{2}$ parallel to it. We then consider the arrangement $\mathcal{A}$ of the resulting $\binom{n}{2}$ great circles on $\mathbb{S}^{2}$, which consists of $O\left(n^{4}\right)$ 2 -faces. The crucial observation made in [19] is that all transversal lines, whose directions belong to the same 2 -face of $\mathcal{A}$, stab the sets of $\mathcal{K}$ in the same order (if the face contains such directions at all). Hence, the number of geometric permutations is upper bounded by the total number of 2 -faces of $\mathcal{A}$, implying that $g_{3}(n)=O\left(n^{4}\right)$.

We improve this bound by showing that the actual number of faces which contain at least one direction of a transversal line (so-called permutation faces) is only $O\left(n^{3} \log n\right)$. Moreover, we show that the overall number of edges and vertices on the boundaries of these faces is also at most $O\left(n^{3} \log n\right)$.

The analysis proceeds in two steps. First, we use a direct geometric analysis to show that the number of vertices whose four incident faces are all permutation faces is $O\left(n^{3}\right)$. We refer to such vertices as popular vertices. Informally, we associate with each popular vertex $v$ (with the possible exception of $O\left(n^{3}\right)$ "degenerate" ones) the intersection line $\lambda_{v}$ of the two separating planes $h, h^{\prime}$ that correspond to the two circles incident to $v$, and show that $\lambda_{v}$ stabs exactly $n-4$ sets of $\mathcal{K}$ (all but the sets in the two pairs separated by $h$ and $h^{\prime}$, respectively). We then apply, within each of the $\binom{n}{2}$ separating planes, the linear bound on the number of geometric permutations in $\mathbb{R}^{2}$, due to Edelsbrunner and Sharir [5] combined with a simple application of the Clarkson-Shor probabilistic analysis technique [4, and thereby obtain the overall $O\left(n^{3}\right)$ asserted bound on the
number of popular vertices.
We then use this bound to analyze the overall number of vertices incident to permutation faces. This is achieved by a refined (and simplified) variant of the charging scheme of Tagansky [18].

The analysis can be extended to any dimension $d \geq 4$, but its technical details become somewhat more involved.

The paper is organized as follows. We first derive the nearly-cubic upper bound on $g_{3}(n)$. To this end, we begin in Section 2 by introducing some notations and the infrastructure, and then establish this bound in Section 3. In Section [4, we extend the analysis to any dimension $d \geq 4$.

## 2 Preliminaries

The setup in $\mathbb{R}^{3}$. Let $\mathcal{K}$ be a collection of $n$ arbitrary pairwise disjoint convex sets in $\mathbb{R}^{3}$. We may also assume, without loss of generality, that the elements of $\mathcal{K}$ are compact. Indeed, let $g_{d}(n)$ be the maximum possible number of geometric permutations induced by a collection of $n$ pairwise disjoint compact convex sets in $\mathbb{R}^{3}$. Let $\mathcal{K}=\left\{K_{1}, \ldots, K_{n}\right\}$ be a collection of $n$ arbitrary pairwise disjoint convex sets in $\mathbb{R}^{3}$, which induces $m$ geometric permutations, realized by $m$ respective lines $\ell_{1}, \ldots, \ell_{m}$. For each $1 \leq i \leq n$ and $1 \leq j \leq m$, let $p_{i j}$ denote an arbitrary point in $K_{i} \cap \ell_{j}$. For each $1 \leq i \leq n$, let $K_{i}^{\prime}$ denote the convex hull of $\left\{p_{i j} \mid 1 \leq j \leq m\right\}$, and observe that $K_{i}^{\prime}$ is a compact convex subset of $K_{i}$. Hence $\mathcal{K}^{\prime}=\left\{K_{1}^{\prime}, \ldots, K_{n}^{\prime}\right\}$ is a collection of $n$ pairwise disjoint compact convex sets, which induces (at least) $m$ geometric permutations (realized by the same lines $\ell_{1}, \ldots, \ell_{m}$ ), so $m \leq g_{d}(n)$.

We use the following setup, introduced by Wenger [19] and briefly mentioned in the introduction, to analyze geometric permutations of $\mathcal{K}$. Enumerate the elements of $\mathcal{K}$ as $K_{1}, K_{2}, \ldots, K_{n}$. For each $1 \leq i<j \leq n$ we fix some plane $h_{i j}$ which strictly separates $K_{i}$ and $K_{j}$. We orient $h_{i j}$ so that $K_{i}$ lies in the open negative halfspace $h_{i j}^{-}$that it bounds, and $K_{j}$ lies in the open positive halfspace $h_{i j}^{+}$. We represent directions of (oriented) lines in $\mathbb{R}^{3}$ by points on the unit 2-sphere $\mathbb{S}^{2}$. Without loss of generality we may assume that the planes $h_{i j}$ are in general position, meaning that every triple of them intersect at a single point, and no four meet at a common point.

Each separating plane $h_{i j}$ induces a great circle $C_{i j}$ on $\mathbb{S}^{2}$, formed by the intersection of $\mathbb{S}^{2}$ with the plane parallel to $h_{i j}$ through the origin. Equivalently, $C_{i j}$ is the locus of the directions of all lines parallel to $h_{i j}$. $C_{i j}$ partitions $\mathbb{S}^{2}$ into two open hemispheres $C_{i j}^{+}, C_{i j}^{-}$, so that $C_{i j}^{+}$(resp., $C_{i j}^{-}$) consists of the directions of lines which cross $h_{i j}$ from $h_{i j}^{-}$to $h_{i j}^{+}$(resp., from $h_{i j}^{+}$to $h_{i j}^{-}$). Note that lines whose directions lie in $C_{i j}$ cannot stab both $K_{i}$ and $K_{j}$. Thus, any oriented common transversal line of $K_{i}$ and $K_{j}$ intersects $K_{j}$ after (resp., before) $K_{i}$ if and only if its direction lies in $C_{i j}^{+}$(resp., $C_{i j}^{-}$).

Put $\mathcal{C}(\mathcal{K})=\left\{C_{i j} \mid 1 \leq i<j \leq n\right\}$, and consider the arrangement $\mathcal{A}(\mathcal{K})$ of the $\binom{n}{2}$ great circles of $\mathcal{C}(\mathcal{K})$. The assumption that the planes $h_{i j}$ are in general position is easily seen to imply that the circles in $\mathcal{C}(\mathcal{K})$ are also in general position, in the sense that no pair of them coincide and no three have a common point. Each 2-face $f$ of $\mathcal{A}(\mathcal{K})$ induces a relation $\prec_{f}$ on $\mathcal{K}$, in which $K_{i} \prec_{f} K_{j}$ (resp., $K_{j} \prec_{f} K_{i}$ ) if $f \subseteq C_{i j}^{+}$(resp., $f \subseteq C_{i j}^{-}$). Clearly, the direction of each oriented line transversal $\lambda$ of $\mathcal{K}$ belongs to the unique 2-face $f$ of $\mathcal{A}(\mathcal{K})$ whose relation $\prec_{f}$ coincides with the order in which $\lambda$ visits the sets of $\mathcal{K}$ (as noted above, the direction of $\lambda$ cannot lie on an edge or at a vertex of $\mathcal{A}(\mathcal{K})$ ). In particular, the number of geometric permutations is bounded by the number of 2 -faces of $\mathcal{A}(\mathcal{K})$, which is $O\left(n^{4}\right)$.

This is the way in which Wenger established this upper bound (in three dimensions) 20 years ago [19]. Moreover, this approach can be extended to any dimension $d \geq 3$, and yields the upper bound $O\left(n^{2 d-2}\right)$ on $g_{d}(n)$; see [19] and Section 4 below. The main weakness of this argument (as follows from the analysis in this paper) is that most faces of $\mathcal{A}(\mathcal{K})$ do not induce a geometric permutation of $\mathcal{K}$. Specifically, for some faces $f$ the relation $\prec_{f}$ might have cycles, in which case $f$ clearly cannot contain the direction of a transversal of $\mathcal{K}$. But even if $\prec_{f}$ is acyclic (and thus a total order) there need not exist any line transversal with direction in $f$.

More definitions. We need a few more notations. We call a 2 -face of $\mathcal{A}(\mathcal{K})$ a permutation face if there is at least one line transversal of $\mathcal{K}$ whose direction belongs to $f$. Note, however, that the directions of the line transversals of $\mathcal{K}$ within a fixed permutation face $f$ is only a subset of $f$, which need not even be connected; see, e.g., a construction in [7] and the introduction.

Each pair of great circles of $\mathcal{C}(\mathcal{K})$ intersect at exactly two antipodal points of $\mathbb{S}^{2}$. By the general position assumption, all the circles are distinct, and each vertex $v$ of $\mathcal{A}(\mathcal{K})$ is incident to exactly two great circles. Hence, each vertex is incident to exactly four (distinct) faces of $\mathcal{A}(\mathcal{K})$. Assuming that $|\mathcal{K}| \geq 3, \mathcal{C}(\mathcal{K})$ contains at least three great circles, so the boundary of each cell of $\mathcal{A}(\mathcal{K})$ contains at least three vertices. This, and the fact that each vertex is incident to four faces, imply that the number of permutation faces in $\mathcal{A}(\mathcal{K})$ is at most proportional to the overall number of their vertices. It is this latter quantity that we proceed to bound.

We say that vertex $v$ in $\mathcal{A}(\mathcal{K})$ is regular if the two great circles $C_{i j}, C_{k \ell}$ incident to $v$ are defined by four distinct sets of $\mathcal{K}$; otherwise, when only three of the indices $i, j, k, \ell$ are distinct, we call $v$ a degenerate vertex. Clearly, the number of degenerate vertices is $O\left(n^{3}\right)$, so it suffices to bound the number of regular vertices of permutation faces.

In the forthcoming analysis we will use subcollections $\mathcal{K}^{\prime}$ of $\mathcal{K}$, typically obtained by removing one set, say $K_{q}$, from $\mathcal{K}$. Doing so eliminates all separating planes $h_{i q}$, for $i=1, \ldots, q-1$, and $h_{q i}$, for $i=q+1, \ldots, n$. Accordingly, the corresponding circles $C_{i q}, C_{q i}$ are also eliminated from $\mathcal{C}\left(\mathcal{K}^{\prime}\right)$, and $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ is constructed only from the remaining circles. In particular, a regular vertex $v$ of $\mathcal{A}(\mathcal{K})$, formed by the intersection of $C_{i j}$ and $C_{k \ell}$, remains a vertex of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ if and only if $q \neq i, j, k, \ell$. An edge (resp., face) of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ may contain several edges (resp., faces) of $\mathcal{A}(\mathcal{K})$. Note that if $f^{\prime}$ is a face of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ which contains a permutation face $f$ of $\mathcal{A}(\mathcal{K})$ then $f^{\prime}$ is a permutation face in $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$; the permutation that it induces is the permutation of $f$ with $K_{q}$ removed.

## 3 The Number of Geometric Permutations in $\mathbb{R}^{3}$

Our analysis uses the setup of Tagansky [18], somewhat adapted to our context. To make this paper more self-contained, we will spell out many of the details of the technique as it applies in our context.
Popular vertices and edges. We say that an edge $e$ of $\mathcal{A}(\mathcal{K})$ is popular if its two incident faces are both permutation faces. We say that a vertex $v$ of $\mathcal{A}(\mathcal{K})$ is popular if its four incident faces are all permutation faces. We establish the upper bound $O\left(n^{3}\right)$ on the number of popular vertices, using a direct geometric argument. The analysis then proceeds by applying two charging schemes. The first scheme results in a recurrence which expresses the number of popular edges in terms of the number of popular vertices. The second scheme leads to a recurrence which expresses the number of vertices of permutation faces in terms of the number of popular edges. The solutions of both recurrences are nearly cubic. Naive (and simpler) implementation of both schemes incurs an extra
logarithmic factor in each recurrence, resulting in the overall bound $g_{3}(n)=O\left(n^{3} \log ^{2} n\right)$. With a more careful analysis of the second scheme, we are able to eliminate one of these factors, and thus obtain the bound $g_{3}(n)=O\left(n^{3} \log n\right)$.

### 3.1 The number of popular vertices

For a regular vertex $v$ of $\mathcal{A}(\mathcal{K})$, formed by the intersection of $C_{i j}, C_{k \ell} \in \mathcal{C}(\mathcal{K})$, we denote by $\mathcal{K}_{v}$ the collection $\left\{K_{i}, K_{j}, K_{k}, K_{\ell}\right\}$ of the four sets defining (the circles meeting at) $v$.

Lemma 3.1. Let $v$ be a regular popular vertex of $\mathcal{A}(\mathcal{K})$, incident to $C_{i j}, C_{k \ell} \in \mathcal{C}(\mathcal{K})$.
(i) Each pair of sets $K_{a} \in \mathcal{K}_{v}$ and $K_{b} \in \mathcal{K} \backslash \mathcal{K}_{v}$ appear in the same order in all four permutations induced by the faces incident to $v$.
(ii) The elements of each pair $K_{i}, K_{j}$ and $K_{k}, K_{\ell}$ are consecutive in all four permutations induced by the faces incident to $v$.

Proof. Any two distinct faces $f, g$ incident to $v$ are separated only by one or two great circles from $\left\{C_{i j}, C_{k \ell}\right\}$, so the orders $\prec_{f}$ and $\prec_{g}$ may disagree only over the pairs $\left(K_{i}, K_{j}\right)$ and ( $K_{k}, K_{\ell}$ ). As a matter of fact, the four permutations are obtained from each other only by swapping $K_{i}$ and $K_{j}$ and/or swapping $K_{k}$ and $K_{\ell}$. This is easily seen to imply both parts of the lemma.

Lemma 3.2. Let $v$ be a regular popular vertex in $\mathcal{A}(\mathcal{K})$, incident to $C_{i j}, C_{k \ell} \in \mathcal{C}(\mathcal{K})$. Then the line $\lambda_{v}=h_{i j} \cap h_{k \ell}$ stabs all the $n-4$ sets in $\mathcal{K} \backslash \mathcal{K}_{v}$, and misses all four sets in $\mathcal{K}_{v}$.

Proof. By definition, $\lambda_{v}$ misses every set $K \in \mathcal{K}_{v}$, because it is contained in a plane separating $K$ from another set in $\mathcal{K}_{v}$. Hence, it suffices to show that $\lambda_{v}$ is a transversal of $\mathcal{K} \backslash \mathcal{K}_{v}$.

To show this, we fix a set $K_{a} \in \mathcal{K} \backslash \mathcal{K}_{v}$ and show that each of the four dihedral wedges determined by $h_{i j}$ and $h_{k \ell}$ meets $K_{a}$. The convexity of $K_{a}$ then implies that $\lambda_{v}$ intersects $K_{a}$; see Figure $\mathbb{1}$ (left).


Figure 1: Left: $K_{a}$ must cross $\lambda_{v}=h_{i j} \cap h_{k \ell}$ since it meets each of the four incident wedges (one of which is highlighted). Right: The transversal line $\mu$ crosses $K_{a}$ after $K_{i}, K_{j}, K_{k}, K_{\ell}$, so the segment $K_{a} \cap \mu$ (highlighted) is contained in $h_{i j}^{+} \cap h_{k \ell}^{+}$.

Lemma 3.1 implies that $K_{a}$ lies at the same position in each of the four permutations induced by the faces incident to $v$. Without loss of generality, assume that the consecutive pair $K_{i}, K_{j}$ appears in these permutations before the consecutive pair $K_{k}, K_{\ell}$. Then either $K_{a}$ precedes both pairs in all four permutations, or appears in between them, or succeeds both of them. In what follows we assume that $K_{a}$ succeeds both pairs in all the permutations, but similar arguments handle the other two cases too.

Consider the permutation $\pi_{1}$ induced by the face $f_{1}$ incident to $v$ and lying in $C_{i j}^{+} \cap C_{k \ell}^{+}$, and let $\mu$ be a line transversal which induces $\pi_{1}$. Since the direction of $\mu$ lies in $C_{i j}^{+} \cap C_{k \ell}^{+}$, it follows that $\mu$ crosses $h_{i j}$ from the side containing $K_{i}$ to the side containing $K_{j}$, and it crosses $h_{k \ell}$ from
the side containing $K_{k}$ to the side containing $K_{\ell}$. Hence $K_{i}$ precedes $K_{j}$ and $K_{k}$ precedes $K_{\ell}$ in $\pi_{1}$. Moreover, $\mu$ crosses $h_{i j}$ in between its intersections with $K_{i}$ and $K_{j}$, and it crosses $h_{k \ell}$ in between its intersections with $K_{k}$ and $K_{\ell}$. Thus, $\mu \cap K_{a}$ lies in $h_{i j}^{+} \cap h_{k \ell}^{+}$; see Figure 1 (right). That is, $K_{a}$ intersects the dihedral wedge $h_{i j}^{+} \cap h_{k \ell}^{+}$. Fully symmetric arguments, applied to the permutations induced by the three other faces $f_{2}, f_{3}, f_{4}$ incident to $v$, show that $K_{a}$ intersects each of the three other dihedral wedges determined by $h_{i j}$ and $h_{k \ell}$, which, as argued above, implies that $\lambda_{v}$ stabs $K_{a}$. As promised, slightly modified variants of this argument (with different correspondences between the wedges around $\lambda_{v}$ and the faces around $v$ ) handle the cases where $K_{a}$ precedes both pairs $K_{i}, K_{j}$ and $K_{k}, K_{\ell}$ in all four permutations, or appears in between these pairs.

Theorem 3.3. Let $\mathcal{K}$ be a collection of $n$ pairwise disjoint compact convex sets in $\mathbb{R}^{3}$. Then the number of popular vertices in $\mathcal{A}(\mathcal{K})$ is $O\left(n^{3}\right)$.

Proof. Note first that each popular vertex must be regular. Indeed, if $v$ is a degenerate popular vertex incident to, say, $C_{i j}, C_{i k} \in \mathcal{C}(\mathcal{K})$, then, arguing as in Lemma 3.1, each of the two pairs $K_{i}, K_{j}$ and $K_{i}, K_{k}$ appears consecutively in each of the four permutations near $v$. Let $f$ be one of the four permutation faces incident to $v$, and assume, without loss of generality, that $K_{k} \prec_{f} K_{i} \prec_{f} K_{j}$. Let $g$ be the permutation face neighboring to $f$ and separated from it only by the circle $C_{i k}$. Then we must have $K_{i} \prec_{g} K_{k} \prec_{g} K_{j}$, contradicting the fact that $K_{i}, K_{j}$ are consecutive also under $\prec_{g}$.

Now let $v$ be a regular popular vertex in $\mathcal{A}(\mathcal{K})$, incident to $C_{i j}, C_{k \ell} \in \mathcal{C}(\mathcal{K})$, and let $\lambda_{v}=h_{i j} \cap h_{k \ell}$ be the line considered in Lemma 3.2. Put $K_{q}^{*}=K_{q} \cap h_{i j}$, for each index $q \neq i, j$, and denote by $\mathcal{K}^{*}$ the collection of these $n-2$ planar cross-sections within $h_{i j}$. Clearly, all sets in $\mathcal{K}^{*}$ are pairwise disjoint, compact, and convex.


Figure 2: View inside $h_{i j}$ : The line $\lambda_{v}=h_{i j} \cap h_{k \ell}$ misses $K_{k}^{*}, K_{\ell}^{*}$ but stabs all other sets in $\mathcal{K}^{*}$. The line $\mu_{v}$ is tangent to $K_{a}^{*}=K_{k}^{*}$ and to $K_{b}^{*}$, so it misses only $K_{\ell}^{*}$.

By Lemma 3.2, $\lambda_{v}$ lies in $h_{i j}$, stabs all the sets in $\mathcal{K}^{*} \backslash\left\{K_{k}^{*}, K_{\ell}^{*}\right\}$ (so they are all nonempty) and misses the two sets $K_{k}^{*}, K_{\ell}^{*}$. (As can be easily verified, both of $K_{k}^{*}, K_{\ell}^{*}$ are also nonempty, although our analysis does not rely on this property.)

Translate $\lambda_{v}$ within $h_{i j}$ until it becomes tangent to some set $K_{a}^{*} \in \mathcal{K}^{*}$, and then rotate the resulting line around $K_{a}^{*}$, say counterclockwise, keeping it tangent to that set, until it becomes tangent to another set $K_{b}^{*} \in \mathcal{K}^{*} \backslash\left\{K_{a}^{*}\right\}$. The sets $K_{k}^{*}, K_{\ell}^{*}, K_{a}^{*}, K_{b}^{*}$ need not all be distinct, so the resulting extremal tangent $\mu_{v}$ misses at most two sets of $\mathcal{K}^{*}$ and intersects all the other sets; see Figure 2

We charge $\lambda_{v}$ to $\mu_{v}$, and argue that each extremal line $\mu$ in $h_{i j}$, which is tangent to two sets of $\mathcal{K}^{*}$ and misses at most two other sets of $\mathcal{K}^{*}$, is charged in this manner at most twice. Indeed, by the general position assumption, $\mu$ lies in a single plane $h_{i j}$. Within that plane, if $\mu$ misses two sets of $\mathcal{K}^{*}$ then these must be the sets $K_{k}^{*}, K_{\ell}^{*}$. If $\mu$ misses only one set of $\mathcal{K}^{*}$ then this set must be one of the sets $K_{k}^{*}, K_{\ell}^{*}$, and the other set is one of the two sets $\mu$ is tangent to. Finally, if $\mu$ does
not miss any set of $\mathcal{K}^{*}$ then $K_{k}^{*}, K_{\ell}^{*}$ are the two sets $\mu$ is tangent to. Hence $\mu$ determines at most two quadruples $K_{i}, K_{j}, K_{k}, K_{\ell}$ whose lines $\lambda_{v}$ can charge $\mu$, and the claim follows.

It therefore suffices to bound the number of extremal lines $\mu$ charged in this manner. This can be done using the Clarkson-Shor technique [4, by observing that each such line $\mu$ is defined by two sets of $\mathcal{K}^{*}$ (those it is tangent to; any such pair of sets determine four common tangents) and is "in conflict" with at most two other sets (those that it misses). Thus, the Clarkson-Shor technique implies that the number of lines $\mu_{v}$ is $O\left(L_{0}(n / 2)\right)$, where $L_{0}(r)$ is the (expected) number of extremal lines which are transversals to a (random) sample of $r$ sets of $\mathcal{K}^{*}$. Edelsbrunner and Sharir 5 establish an upper bound of $O(r)$ on the complexity of the space of line transversals to a collection of $r$ pairwise-disjoint compact convex sets in the plane, implying that $L_{0}(r)=O(r)$. Hence the number of charged lines $\mu$ in a single plane $h_{i j}$ is $O(n)$, for a total of $O\left(\binom{n}{2} \cdot n\right)=O\left(n^{3}\right)$. Since, as noted above, each line is charged at most twice in its plane, this also bounds the number of popular vertices.

### 3.2 The number of popular edges

We next bound the number of popular edges in $\mathcal{A}(\mathcal{K})$, using the bound on popular vertices just derived. We define an edge border in $\mathcal{A}(\mathcal{K})$ to be a pair $(v, Q)$, where $v$ is a vertex of $\mathcal{A}(\mathcal{K})$, incident to two great circles $C_{i j}, C_{k \ell}$, and $Q$ is one of the four open hemispheres $C_{i j}^{-}, C_{i j}^{+}, C_{k \ell}^{-}, C_{k \ell}^{+}$determined by one of these circles. See Figure 3 (left). Note that $Q$ determines a unique edge $e$ of $\mathcal{A}(\mathcal{K})$ which is incident to $v$ and is contained in $Q$. If, in addition, $e$ is a popular edge, we say that $(v, Q)$ is a popular edge border. For the purpose of the analysis, we will also refer to $(v, Q)$ as a 0 -level edge border.


Figure 3: Left: Charging a 0 -level edge border $(v, Q)$ to a 1 -level edge border $\left(v_{1}, Q_{1}\right)$. Right: If the edges $e, e_{1}$ are both popular then $v$ is a popular vertex.

One useful feature of the border notation is that if $(v, Q)$ is an edge border in $\mathcal{A}(\mathcal{K})$ and $\mathcal{K}^{\prime}$ is a subcollection of $\mathcal{K}$ so that $v$ is still a vertex of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$, then $(v, Q)$ is also an edge border in $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$. The edge $e^{\prime}$ of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ associated with $(v, Q)$ in $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ either is equal to $e$, or strictly contains $e$ (in the latter case both $e$ and $e^{\prime}$ have $v$ as a common endpoint).

If an edge border $(v, Q)$, which is not a 0 -level edge border, becomes a 0 -level edge border after removing from $\mathcal{K}$ some single set $K_{a} \in \mathcal{K}$, we call it a 1-level edge border. In this case we say that $(v, Q)$ is in conflict with $K_{a}$. Note that the set $K_{a}$, whose removal makes $(v, Q)$ a 0 -level edge border, need not be unique; see Section 3.3 for further discussion.

Clearly, to bound the number of popular edges it suffices to bound the number of 0-level edge borders, which is twice the number of popular edges in $\mathcal{A}(\mathcal{K})$ (each edge is counted once at each of its endpoints).

Since each vertex of $\mathcal{A}(\mathcal{K})$ participates in exactly four edge borders, the number of edge borders which are incident to a degenerate vertex is $O\left(n^{3}\right)$. We bound the number of remaining 0 -level edge borders using the following charging scheme.

Let $(v, Q)$ be a 0 -level edge border, where $v$ is incident to $C_{i j}$ and $C_{k \ell}$, so that $Q=C_{i j}^{+}$, say. Let $e$ be the popular edge associated with $(v, Q)$. Trace $C_{k \ell}$ from $v$ away from $e$ (into $C_{i j}^{-}$), and let $v_{1}$ be the next encountered vertex. Let $e_{1}$ be the edge connecting $v$ and $v_{1}$. Let $C_{p q}$ be the other circle incident to $v_{1}$ and assume, without loss of generality, that $v$ lies in $C_{p q}^{+}$. See Figure 3 (left). Note that, assuming $|\mathcal{K}| \geq 3$, we have $C_{p q} \neq C_{i j}$ (i.e., $v_{1}$ is not antipodal to $v$ ), because otherwise $C_{i j}$ would have intersected only $C_{k \ell}$. One of the following cases must arise:
(i) $v_{1}$ is degenerate.
(ii) The edge $e_{1}$ is also popular, so $v$ is a popular vertex; see Figure 3 (right).
(iii) $e_{1}$ is not popular. Since $C_{p q} \neq C_{i j}$, one of $i, j$, say $i$, is different from both $p$ and $q$. This (and the fact that $i \neq k, \ell$ ) implies that removing $K_{i}$ from $\mathcal{K}$ also removes $C_{i j}$ from $\mathcal{A}$, keeps $v_{1}$ intact, and makes the appropriate extension of $e$ reach (and terminate at) $v_{1}$, thereby making $\left(v_{1}, Q_{1}\right)$ a 0 -level edge border in $\mathcal{A}\left(\mathcal{K} \backslash\left\{K_{i}\right\}\right)$, where $Q_{1}=C_{p q}^{+}$. See Figure 3 (left).

In case (i) we charge $(v, Q)$ to $v_{1}$. The number of degenerate vertices is $O\left(n^{3}\right)$ and each of them can be charged only $O(1)$ times in this manner. Hence, the number of 0 -level edge borders that fall into this subcase is $O\left(n^{3}\right)$.

In case (ii) we can charge $(v, Q)$ to $v$. Since a popular vertex participates in exactly four 0 -level edge borders, the number of 0-level edge borders that fall into this subcase is $O\left(n^{3}\right)$, by Theorem 3.3,

In case (iii) we charge $(v, Q)$ to the 1-level edge border $\left(v_{1}, Q_{1}\right)$. Note that $\left(v_{1}, Q_{1}\right)$ is charged in this manner only by $(v, Q)$.

Let us denote by $E_{0}(\mathcal{K})$ (resp., $E_{1}(\mathcal{K})$ ) the number of 0-level edge borders (resp., 1-level edge borders) in $\mathcal{A}(\mathcal{K})$. Then we have the following recurrence:

$$
\begin{equation*}
E_{0}(\mathcal{K}) \leq E_{1}(\mathcal{K})+O\left(n^{3}\right) \tag{1}
\end{equation*}
$$

To solve this recurrence, we apply the technique of Tagansky [18]. Specifically, we remove from $\mathcal{K}$ a randomly chosen set $K \in \mathcal{K}$, and denote by $\mathcal{R}$ the collection of the $n-1$ remaining sets. A 0 -level edge border $(v, Q)$ in $\mathcal{A}(\mathcal{K})$, where $v$ is an intersection point of $C_{i j}$ and $C_{k \ell}$ and is regular, appears as a 0 -level edge border in $\mathcal{A}(\mathcal{R})$ if and only if $K$ is different from each of the four sets $K_{i}, K_{j}, K_{k}, K_{\ell}$ defining $v$, which happens with probability $\frac{n-4}{n}$. A 1-level edge border $(v, Q)$ in $\mathcal{A}(\mathcal{K})$ becomes a 0 -level edge border in $\mathcal{A}(\mathcal{R})$ if and only if $K$ is in conflict with $(v, Q)$, which happens with probability at least $\frac{1}{n}$. No other edge border in $\mathcal{A}(\mathcal{K})$ can appear as a 0 -level edge border in $\mathcal{A}(\mathcal{R})$. Hence, we obtain

$$
\begin{equation*}
\mathbf{E}\left\{E_{0}(\mathcal{R})\right\} \geq \frac{n-4}{n} E_{0}(\mathcal{K})+\frac{1}{n} E_{1}(\mathcal{K}), \tag{2}
\end{equation*}
$$

where $\mathbf{E}$ denotes expectation with respect to the random sample $\mathcal{R}$, as constructed above. Combining (11) and (2) yields

$$
\frac{1}{n} E_{0}(\mathcal{K}) \leq \frac{1}{n} E_{1}(\mathcal{K})+O\left(n^{2}\right) \leq \mathbf{E}\left\{E_{0}(\mathcal{R})\right\}-\frac{n-4}{n} E_{0}(\mathcal{K})+O\left(n^{2}\right)
$$

Denoting by $E_{0}(n)$ the maximum number of 0 -level edge borders in $\mathcal{A}(\mathcal{K})$, for any collection $\mathcal{K}$ of size $n$ with the assumed properties, we get the recurrence

$$
\frac{n-3}{n} E_{0}(n) \leq E_{0}(n-1)+O\left(n^{2}\right),
$$

whose solution is easily seen to be $E_{0}(n)=O\left(n^{3} \log n\right)$ (see, e.g., [18, Proposition 3.1]).

### 3.3 The number of permutation faces

Finally, we bound the number of vertices of permutation faces using the bound on popular edges just derived. This will also serve as an upper bound on the number of permutation faces, and thus also on $g_{3}(n)$. We present the analysis in two stages. The first stage derives the slightly weaker upper bound $O\left(n^{3} \log ^{2} n\right)$, but is considerably simpler. The second stage involves a more careful examination of the possible charging scenarios, and leads to a sharper recurrence, whose soution is only $O\left(n^{3} \log n\right)$.

Each vertex $v$ is incident to exactly four faces of $\mathcal{A}(\mathcal{K})$, so we need to count $v$ with multiplicity of at most 4-once for each permutation face incident to $v$. For this we extend the notion of borders as follows. The two great circles passing through $v$ partition $\mathbb{S}^{2}$ into four wedges, or rather slices. Each such slice $R$ contains a unique face $f$ incident to $v$, and defines, together with $v$, a border $(v, R)$. We call $f$ the face associated with $(v, R)$. Similarly to the notation involving edge borders in Section 3.2, we call $(v, R)$ a popular border, or a 0 -level border, if the face associated with $(v, R)$ is a permutation face. It thus suffices to bound the number of 0 -level borders in $\mathcal{A}(\mathcal{K})$.

If $(v, R)$ is a border in $\mathcal{A}(\mathcal{K})$ with an associated face $f$, and $\mathcal{K}^{\prime}$ is a subcollection of $\mathcal{K}$, so that $v$ is still a vertex of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$, then $(v, R)$ is also a border in $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$, except that the face $f^{\prime}$ of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ associated with $(v, R)$ may be different from $f$ (or, more precisely, properly contain $f$ ).

If a border $(v, R)$, which is not a 0 -level border in $\mathcal{A}(\mathcal{K})$, becomes a 0 -level border after removing from $\mathcal{K}$ some set $K$, we call it a 1 -level border. The set $K$ is said to be in conflict with $(v, R)$. Note that $K$ cannot be one of the (at most) four sets defining $v$, and that a 1 -level border may be in conflict with more than one set of $\mathcal{K}$. See Figure 4 (left).


Figure 4: Left: A non-permutation face $f$, associated with the 1 -level border $(v, R)$, is separated from permutation faces $f_{1}, f_{2}, f_{3}$ by the respective edges $e_{1} \subset C_{p_{1} q_{1}}, e_{2} \subset C_{p_{2} q_{2}}, e_{3} \subset C_{p_{3} q_{3}}$. If none of $p_{1}, q_{1}, p_{2}, q_{2}, p_{3}, q_{3}$ belongs to $\{i, j, k, \ell\}$ then $(v, R)$ is a 1-level border in conflict with each of $K_{p_{1}}, K_{q_{1}}, K_{p_{2}}, K_{q_{2}}, K_{p_{3}}, K_{q_{3}}$. Right: Charging a 0 -level border $(v, R)$ to the two 1 -level borders $\left(v_{1}, R_{1}\right),\left(v_{2}, R_{2}\right)$, along the two edges $e_{1}, e_{2}$ emanating from $v$ away from $R$.

We bound the number of 0 -level borders using a charging scheme similar to that in Section 3.2, Let $(v, R)$ be a 0 -level border, and let $f$ be the permutation face associated with it. Note that the
number of borders incident to degenerate vertices is $O\left(n^{3}\right)$. We may therefore assume that $v$ is regular, and let $C_{i j}$ and $C_{k \ell}$ denote the two great circles incident to $v$ (so $i, j, k, \ell$ are all distinct). Without loss of generality, assume that $R=C_{i j}^{+} \cap C_{k \ell}^{+}$.

Let $e_{1}$ and $e_{2}$ be the two edges incident to $v$ and emanating from it away from $R$, where $e_{1} \subset C_{i j} \cap C_{k \ell}^{-}$and $e_{2} \subset C_{k \ell} \cap C_{i j}^{-}$; see Figure团 (right). Let $v_{1}$ (resp., $v_{2}$ ) be the other endpoint of $e_{1}$ (resp., of $e_{2}$ ).

Our charging scheme is based on the following case analysis:
(i) If one of the two edges incident to $v$ and bounding $R$ is popular, we charge $(v, R)$ to this edge. Since the number of popular edges is $O\left(n^{3} \log n\right)$, and each of them is charged by at most four 0 -level borders (twice for each of its endpoints), the number of 0 -level borders that fall into this subcase is also $O\left(n^{3} \log n\right)$.
(ii) If no edge incident to $v$ and bounding $R$ is popular, we charge $(v, R)$ to two 1-level borders, one incident to $v_{1}$ and one to $v_{2}$. Specifically, consider $v_{1}$, say, and let $C_{p q}$ be the circle whose intersection with $C_{i j}$ forms $v_{1}$, and assume, again without loss of generality, that $v$ lies in $C_{p q}^{+}$. We then charge $(v, R)$ to $\left(v_{1}, R_{1}\right)$, where $R_{1}=C_{p q}^{+} \cap C_{i j}^{+}$. Let $f_{1}$ be the face of $\mathcal{A}(\mathcal{K})$ associated with $\left(v_{1}, R_{1}\right)$ (this is the face whose boundary we trace from $v$ to $v_{1}$ along $e_{1}$, and it is also incident to $v$ ). Since the edge incident to $f, f_{1}$ (and to $v$ ) is not popular, $f_{1}$ is not a permutation face. Clearly, one of the indices $k, \ell$, say $k$, is different from both $p, q$. Thus, removing $K_{k}$ keeps $v_{1}$ as a vertex in the new spherical arrangement, and makes $C_{k \ell}$ disappear, so both faces $f, f_{1}$ fuse into a single larger permutation face contained in $R_{1}$. Hence, $\left(v_{1}, R_{1}\right)$ is a 1 -level border which is in conflict with $K_{k}$. A fully symmmetic argument applies to $v_{2}$. We say that the 1 -level borders ( $v_{1}, R_{1}$ ) and $\left(v_{2}, R_{2}\right)$, which we charge, are the neighbors of $(v, R)$ in $\mathcal{A}(\mathcal{K})$.

Note that each 1-level border $\left(v^{\prime}, R^{\prime}\right)$ is charged by at most two 0 -level borders in this manner (at most once along each of the two edges incident to $v^{\prime}$ and bounding the face associated with the border).

Let $V_{0}(\mathcal{K})$ and $V_{1}(\mathcal{K})$ denote, respectively, the number of 0 -level borders and the number of 1-level borders in $\mathcal{A}(\mathcal{K})$ (where we also include degenerate vertices in both counts). Then we have the following recurrence:

$$
\begin{equation*}
V_{0}(\mathcal{K}) \leq V_{1}(\mathcal{K})+O\left(n^{3} \log n\right) \tag{3}
\end{equation*}
$$

Indeed, each 0 -level border which falls into case (ii) charges two 1 -level borders, and each 1 -level border is charged at most twice. The number of all other 0-level borders is $O\left(n^{3} \log n\right)$, as argued above. Combining this inequality with the random sampling technique of Tagansky [18], as in Section 3.2, results in the recurrence

$$
\frac{n-3}{n} V_{0}(n) \leq V_{0}(n-1)+O\left(n^{2} \log n\right)
$$

where $V_{0}(n)$ is the maximum value of $V_{0}(\mathcal{K})$, over all collections $\mathcal{K}$ of $n$ pairwise disjoint compact convex sets in $\mathbb{R}^{3}$. The solution of this recurrence is $V_{0}(n)=O\left(n^{3} \log ^{2} n\right)$, which yields the same upper bound on the number of geometric permutations induced by $\mathcal{K}$.

An improved bound. We next improve the bound by replacing the recurrence (3) by a refined recurrence. Let $(v, R)$ be a 1 -level border which is in conflict with $w \geq 1$ sets of $\mathcal{K}$. Then $(v, R)$ becomes a 0 -level border in $\mathcal{A}(K \backslash\{K\}$ ), after removing a random set $K \in \mathcal{K}$, with probability exactly $\frac{w}{n}$. Namely, this happens if and only if $K$ is one of the $w$ sets in conflict with $(v, R)$. We refer to $w$ as the weight of $(v, R)$.

In the refined setting, $V_{1}(\mathcal{K})$ counts the total weight of all the 1 -level borders in $\mathcal{A}(\mathcal{K})$, so now the contribution of each 1-level border to $V_{1}(\mathcal{K})$ is equal to its weight. By an appropriate adaptation of the argument in Section 3.2, we obtain the following equality:

$$
\begin{equation*}
\mathbf{E}\left\{V_{0}(\mathcal{R})\right\}=\frac{n-4}{n} V_{0}(\mathcal{K})+\frac{1}{n} V_{1}(\mathcal{K}), \tag{4}
\end{equation*}
$$

where $\mathcal{R}$ denotes a random sample of $n-1$ sets of $\mathcal{K}$. This follows by noting that the probability of a 1-level border of weight $w$ to be counted in $V_{0}(\mathcal{R})$ is $\frac{w}{n}$, and it contributes $w$ to $V_{1}(\mathcal{K})$.

In the refined charging scheme, each 1-level border $(v, R)$ of weight $w \geq 1$ gets a supply of $w$ units of charge, which it can give to its charging neighboring 0-level borders. Hence, as long as the number of these charging 0 -level borders, which is at most two, does not exceed $w,(v, R)$ can pay each of its neighbors 1 unit. Hence, the only problematic case is when $w=1$ and $(v, R)$ is charged twice. The following technical lemma takes care of this case.

Lemma 3.4. The number of 1 -level borders having weight 1 and charged by two 0 -level borders is $O\left(n^{3} \log n\right)$.

Before proving Lemma [3.4, we show how to use it to replace 3 by a better recurrence, and thereby establish an improved bound on the number of geometric permutations in $\mathbb{R}^{3}$.

If a 1-level border $(v, R)$ has only one neighboring 0 -level border $\left(v^{\prime}, R^{\prime}\right)$ then $\left(v^{\prime}, R^{\prime}\right)$ can receive one unit of charge from $(v, R)$, regardless of what the weight of $(v, R)$ is. Similarly, if $(v, R)$ has weight at least 2 , and it has two neighboring 0 -level borders, each of these 0 -level borders can receive one unit of charge from $(v, R)$. The number of remaining 1-level borders, namely the 1-level borders of weight 1 with two neighboring 0 -level borders, is $O\left(n^{3} \log n\right)$, by Lemma 3.4.

To recap, each 0 -level border, except possibly for $O\left(n^{3} \log n\right)$ ones, receives 1 unit of charge from each of its two neighboring 1-level borders. Moreover, the number of charges made to each of the remaining 1 -level borders, by its neighboring 0 -level borders, does not exceed its weight. Thus, we can replace (3) by the following inequality:

$$
2 V_{0}(\mathcal{K}) \leq V_{1}(\mathcal{K})+O\left(n^{3} \log n\right)
$$

Combining this with (4) we get

$$
\frac{2}{n} V_{0}(\mathcal{K}) \leq \frac{1}{n} V_{1}(\mathcal{K})+O\left(n^{2} \log n\right) \leq \mathbf{E}\left\{V_{0}(\mathcal{R})\right\}-\frac{n-4}{n} V_{0}(\mathcal{K})+O\left(n^{2} \log n\right)
$$

or

$$
\frac{n-2}{n} V_{0}(\mathcal{K}) \leq \mathbf{E}\left\{V_{0}(\mathcal{R})\right\}+O\left(n^{2} \log n\right)
$$

Replacing $V_{0}(\mathcal{K}), V_{0}(\mathcal{R})$ by their respective maximum values $V_{0}(n), V_{0}(n-1)$, we thus obtain the recurrence

$$
\frac{n-2}{n} V_{0}(n) \leq V_{0}(n-1)+O\left(n^{2} \log n\right),
$$

whose solution is easily seen to be $V_{0}(n)=O\left(n^{3} \log n\right)$ (again, see [18, Proposition 3.1]).
As mentioned earlier, $V_{0}(n)$ serves as an upper bound on the number of geometric permutations induced by $\mathcal{K}$. We thus conclude with the following main result of this section.

Theorem 3.5. Any collection $\mathcal{K}$ of $n$ pairwise disjoint convex sets in $\mathbb{R}^{3}$ admits at most $O\left(n^{3} \log n\right)$ geometric permutations.

Proof of Lemma 3.4. Consider a 1 -level border $(v, R)$ of weight 1 , where $v$ is incident to two great circles $C_{i j}, C_{k \ell}$, which is charged twice. We may assume that $v$ is regular (i.e., the four indices $i, j, k, \ell$ are distinct), since the number of remaining borders is $O\left(n^{3}\right)$. Let ( $v_{1}, R_{1}$ ) be the 0 -level border that charges $(v, R)$ along $C_{i j}$, and let $\left(v_{2}, R_{2}\right)$ be the 0 -level border that charges $(v, R)$ along $C_{k \ell}$. By construction, both $v_{1}$ and $v_{2}$ are regular (otherwise they do not charge $v$ ). Let $C_{p_{1} q_{1}}$ denote the other circle incident to $v_{1}$, and let $C_{p_{2} q_{2}}$ denote the other circle incident to $v_{2}$. Clearly, each index in $\left\{p_{1}, q_{1}, p_{2}, q_{2}\right\}$ which does not belong to $\{i, j, k, \ell\}$ contributes to the weight of $(v, R)$, so, by assumption, there is only one such index, call it $q$. Since $v_{1}$ is regular, neither $p_{1}$ nor $q_{1}$ belongs to $\{i, j\}$, so (exactly) one of them must belong to $\{k, \ell\}$, say $p_{1}=k$ and then $q_{1}=q$. Symmetrically, we may assume that $p_{2}=i$, say, and then $q_{2}=q$. Since $v$ is regular and $q \notin\{i, j, k, \ell\}$, the two circles $C_{p_{1} q_{1}}, C_{p_{2} q_{2}}$ (i.e., $C_{k q}, C_{i q}$ ) are distinct. See Figure 5 ,


Figure 5: Two scenarios depicting a 1-level border $(v, R)$ of weight 1 that is charged by two 0-level borders $\left(v_{1}, R_{1}\right),\left(v_{2}, R_{2}\right)$.

In this special scenario we have two distinct permutation faces $f_{1}$ and $f_{2}$, where $f_{1}$ is the face associated with $\left(v_{1}, R_{1}\right)$ and $f_{2}$ is the face associated with $\left(v_{2}, R_{2}\right)$.

There are two possible subcases: Assume first that the face $f$ associated with $(v, R)$ is just the quadrangle bounded by $C_{i j}, C_{k \ell}, C_{i q}$ and $C_{k q}$. In this case the fourth vertex of $f$, formed by intersection of $C_{i q}$ and $C_{k q}$, is degenerate. Since each degenerate vertex is incident to at most four faces, the number of 1-level borders falling into this subcase is $O\left(n^{3}\right)$.

Suppose then that $f$ has additional edges and vertices. Consider, for example, the vertex $u$ which is the other endpoint (other than $v_{1}$ ) of the edge $e$ of $f$ lying on $C_{k q}$. Let $C_{a b}$ denote the other circle incident to $u$. Assume with no loss of generality that $v$ lies in the hemisphere $C_{a b}^{+}$. We may also assume that neither $a$ nor $b$ is in $\{k, q\}$, for otherwise $u$ is a degenerate vertex, so we can argue similarly to the previous subcase.

Suppose first that neither $a$ nor $b$ is equal to $i$. Then removing $K_{i}$ keeps $u$ as a vertex of $\mathcal{A}\left(\mathcal{K} \backslash\left\{K_{i}\right\}\right)$. The edge $e$ extends at its other end into a longer popular edge (it bounds on one side an extension of $f \cup f_{2}$ and on the other side an extension of $f_{1}$, both of which are now permutation faces; see Figure 5 (left)), so $\left(u, C_{a b}^{+}\right)$is a 1-level edge border. We charge the 1-level border $(v, R)$ to $\left(u, C_{a b}^{+}\right)$. By construction, such an edge border is charged only once, as is easily checked.

The number of 1-level edge borders can be bounded using the Clarkson-Shor analysis technique [4], similar to the way it was used in the proof of Theorem 3.3. That is, since each 1-level edge border is defined by at most four sets of $\mathcal{K}$ and becomes a 0 -level edge border when we remove (at
least) one set from $\mathcal{K}$, the number of 1-level edge borders is $O\left(\mathbf{E}\left\{E_{0}\left(\mathcal{K}^{\prime}\right)\right\}\right)$, where $\mathcal{K}^{\prime}$ is a random sample of $n / 2$ sets of $\mathcal{K}$. Hence, the analysis in the preceding subsection implies that the number of 1-level edge borders in $\mathcal{A}(\mathcal{K})$ is $O\left(n^{3} \log n\right)$, and therefore the same bound holds for the number of 1-level borders $(v, R)$ under consideration.

We are therefore left with the situation where, say, $b=i$. Applying a fully symmetric argument to the edge of $f$ lying on $C_{i q}$, we conclude that the only problematic case is where $f$ is at least pentagonal, with five consecutive vertices $u, v_{1}, v, v_{2}, w$, so that $u$ is incident to $C_{a i}$ and $C_{k q}, v_{1}$ is incident to $C_{k q}$ and $C_{i j}, v$ is incident to $C_{i j}$ and $C_{k \ell}, v_{2}$ is incident to $C_{k \ell}$ and $C_{i q}$, and $w$ is incident to $C_{i q}$ and $C_{k b}$; here $a$ and $b$ are two indices, neither of which belongs to $\{i, j, k, \ell, q\} ; a$ and $b$ may be equal. See Figure 5 (right).

Let $\mathcal{A}_{i}$ be the arrangement of the $n-1$ great circles of the form $C_{i r}$ or $C_{r i}$, for $r \neq i$. Let $f_{0}$ be the face of $\mathcal{A}_{i}$ containing $f$. By assumption, the boundary of $f$ touches three distinct boundary edges of $f_{0}$. We charge the 1 -level border $(v, R)$ to the triple ( $f_{0}, e_{0}, e_{1}$ ), where $e_{0} \subset C_{i j}$ and $e_{1} \subset C_{i q}$ are the two boundary edges of $f_{0}$ which contain the respective edges of $\partial f$. To complete the proof of Lemma 3.4, we need the following two lemmas.

Lemma 3.6. Let $1 \leq i \leq n$, and let $\mathcal{A}_{i}$ be the arrangement of the $n-1$ great circles $C_{i r}$ or $C_{r i}$, for $r \neq i$. Let $f_{0}$ be a face in $\mathcal{A}_{i}$, and let $e_{0}, e_{1}$ be two edges of $f_{0}$. Then there exist at most two faces of $\mathcal{A}$ which are contained in $f_{0}$ and are bounded by a portion of $e_{0}$, by a portion of $e_{1}$, and by a portion of some other edge of $f_{0}$.

Proof. The edges $e_{0}$ and $e_{1}$ partition $\partial f_{0}$ into up to four connected portions, $e_{0}, \gamma^{-}, e_{1}, \gamma^{+}$. We claim that there can be at most one face $f$ of $\mathcal{A}$ which is contained in $f_{0}$ and which is bounded by a portion of $e_{0}$, a portion of $e_{1}$, and a portion of $\gamma^{+}$. A symmetric claim holds if we replace $\gamma^{+}$by $\gamma^{-}$, and the lemma follows. The latter claim follows by observing that the existence of two distinct faces $f_{1}, f_{2}$ of $\mathcal{A}$ contained in $f_{0}$ and touching $e_{0}, e_{1}$ and $\gamma^{+}$would lead to an impossible planar drawing of $K_{3,3}$, as illustrated in Figure 6. See, e.g., [6] for a similar argument.


Figure 6: A face $f_{0}$ of $\mathcal{A}_{i}$ cannot contain two distinct faces $f_{1}, f_{2}$ of $\mathcal{A}(\mathcal{K})$ that touch $e_{0}, e_{1}$ and $\gamma^{+}$.
Lemma 3.7. The number of triples $\left(f_{0}, e_{0}, e_{1}\right)$, where $f_{0}$ is a face in $\mathcal{A}_{i}$, as defined in Lemma 3.6, and $e_{0}, e_{1}$ are two edges of $f_{0}$, summed over all $i$, is $O\left(n^{3}\right)$.

Proof. This follows from the well known result that the sum of the squares of the face complexities in an arrangement of $n$ lines in the plane is $O\left(n^{2}\right)$; see, e.g., [16]. The same analysis applies to an arrangement of great circles on the unit sphere. Summing this bound over all $i$, the lemma follows.

Lemma 3.6 implies that any triple $\left(f_{0}, e_{0}, e_{1}\right)$, as above, is charged by at most four 1 -level borders $(v, R)$. Indeed, the triple determines at most two possible faces $f$ of $\mathcal{A}$, and the edge $e_{0}$ determines a unique edge of $f$ with $v$ as one of its endpoints. By Lemma 3.7, the overall number of charged triples $\left(f_{0}, e_{0}, e_{1}\right)$ is $O\left(n^{3}\right)$, so the overall number of 1-level borders $(v, R)$ falling into the last subcase is $O\left(n^{3}\right)$. This completes the proof of Lemma 3.4.

## 4 Geometric Permutations in Higher Dimensions

In this section we generalize Theorem 3.5 by showing that the number of geometric permutations induced by any collection $\mathcal{K}=\left\{K_{1}, \ldots, K_{n}\right\}$ of $n$ pairwise disjoint convex sets in $\mathbb{R}^{d}$ is $O\left(n^{2 d-3} \log n\right)$, for any $d \geq 3$.

Setup. The basic setup is similar to that in three dimensions, but we repeat it here for the sake of readability. Specifically, we may assume, using the same reasoning as before, that the sets of $\mathcal{K}$ are compact (in addition to being pairwise disjoint and convex). For each $1 \leq i<j \leq n$ we fix some hyperplane $h_{i j}$ which strictly separates $K_{i}$ and $K_{j}$. We orient $h_{i j}$ so that $K_{i}$ lies in the negative open halfspace $h_{i j}^{-}$that it bounds, and $K_{j}$ lies in the positive open halfspace $h_{i j}^{+}$. We represent directions of lines in $\mathbb{R}^{d}$ by points on the unit $(d-1)$-sphere $\mathbb{S}^{d-1}$. We may assume that the separating hyperplanes $h_{i j}$ are in general position, so that every $d$ of them intersect in a unique point, and no $d+1$ of them have a point in common.

Each separating hyperplane $h_{i j}$ induces a great $(d-2)$-sphere $C_{i j}$ on $\mathbb{S}^{d-1}$, which is the locus of the directions of all lines parallel to $h_{i j}$. $C_{i j}$ partitions $\mathbb{S}^{d-1}$ into two open hemispheres $C_{i j}^{+}, C_{i j}^{-}$, so that $C_{i j}^{+}$(resp., $C_{i j}^{-}$), consists of the directions of lines which cross $h_{i j}$ from $h_{i j}^{-}$to $h_{i j}^{+}$(resp., from $h_{i j}^{+}$to $h_{i j}^{-}$). Any oriented common transversal line of $K_{i}$ and $K_{j}$ visits $K_{j}$ after (resp., before) $K_{i}$ if and only if its direction lies in $C_{i j}^{+}$(resp., in $C_{i j}^{-}$).

Put $\mathcal{C}(\mathcal{K})=\left\{C_{i j} \mid 1 \leq i<j \leq n\right\}$, and consider the arrangement $\mathcal{A}(\mathcal{K})$ of these $\binom{n}{2}(d-2)$ spheres on $\mathbb{S}^{d-1}$. It partitions $\mathbb{S}^{d-1}$ into relatively open cells of dimensions $0,1, \ldots, d-1$; we refer to an $s$-dimensional cell of $\mathcal{A}(\mathcal{K})$ simply as an $s$-cell. The assumption that the hyperplanes $h_{i j}$ are in general position implies that the $(d-2)$-spheres of $\mathcal{C}(\mathcal{K})$ are also in general position, in the sense that the intersection of any $s$ distinct spheres of $\mathcal{C}(\mathcal{K})$, for $1 \leq s \leq d-1$, is a ( $d-s-1$ )-sphere, and the intersection of any $d$ distinct spheres of $\mathcal{C}(\mathcal{K})$ is empty. Each $(d-1)$-cell $f$ of $\mathcal{A}(\mathcal{K})$ induces a relation $\prec_{f}$ on $\mathcal{K}$, in which $K_{i} \prec_{f} K_{j}$ (resp., $K_{j} \prec_{f} K_{i}$ ) if $f \subseteq C_{i j}^{+}$(resp., $f \subseteq C_{i j}^{-}$). The direction of each oriented line transversal $\lambda$ of $\mathcal{K}$ belongs to the unique $(d-1)$-cell $f$ of $\mathcal{A}(\mathcal{K})$ whose relation $\prec_{f}$ coincides with the linear order in which $\lambda$ visits the sets of $\mathcal{K}$. In particular, as noted by Wenger [19], the number of geometric permutations is bounded by the number of ( $d-1$ )-cells of $\mathcal{A}(\mathcal{K})$, which is $O\left(n^{2 d-2}\right)$.

We call a $(d-1)$-cell $f$ of $\mathcal{A}(\mathcal{K})$ a permutation cell if there is at least one line transversal of $\mathcal{K}$ whose direction belongs to $f$. As in the three-dimensional case, we improve the above bound by showing that the number of permutation cells in $\mathcal{A}(\mathcal{K})$ is $O\left(n^{2 d-3} \log n\right)$, which also bounds the number of geometric permutations induced by $\mathcal{K}$.

We refer to 0 -cells in $\mathcal{A}(\mathcal{K})$ as vertices, and to 1 -cells as edges. We say that a vertex $v$ of $\mathcal{A}(\mathcal{K})$ is regular if the $d-1(d-2)$-spheres of $\mathcal{C}(\mathcal{K})$ that are incident to $v$ are defined by $2 d-2$ distinct sets of $\mathcal{K}$; otherwise $v$ is a degenerate vertex. Clearly, the number of degenerate vertices is $O\left(n^{2 d-3}\right)$, so it suffices to bound the number of regular vertices of permutation cells.

As in the three-dimensional case, we will also consider subcollections $\mathcal{K}^{\prime}$ of $\mathcal{K}$, typically obtained
by removing one set, say $K_{q}$, from $\mathcal{K}$. Doing so eliminates all separating hyperplanes $h_{i q}, h_{q i}$, as well as all the corresponding $(d-2)$-spheres $C_{i q}, C_{q i}$, and $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ is constructed only from the remaining spheres. In particular, a vertex ${ }^{11} v$ of the intersection $C_{i_{1} j_{1}} \cap C_{i_{2} j_{2}} \cap \cdots \cap C_{i_{d-1} j_{d-1}}$ of $\mathcal{A}(\mathcal{K})$ remains a vertex of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ if and only if $q \notin\left\{i_{1}, j_{1}, \ldots, i_{d-1}, j_{d-1}\right\}$. A cell of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$, of any dimension $s \geq 1$, may contain several cells of $\mathcal{A}(\mathcal{K})$. As before, if $f^{\prime}$ is a $(d-1)$-cell of $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$ which contains a permutation cell $f$ of $\mathcal{A}(\mathcal{K})$ then $f^{\prime}$ is a permutation cell in $\mathcal{A}\left(\mathcal{K}^{\prime}\right)$; the permutation that it induces is the permutation of $f$ with $K_{q}$ removed.

Each $s$-cell $f$ of $\mathcal{A}(\mathcal{K})$ is incident to $2^{d-s-1}(d-1)$-cells of $\mathcal{A}(\mathcal{K})$. If all these cells are permutation cells, $f$ is called popular. In particular, a popular vertex is incident to $2^{d-1}$ permutation cells, a popular edge is incident to $2^{d-2}$ permutation cells, and a popular $(d-1)$-cell is a permutation cell.
Overview. We show that the number of popular vertices is $O\left(n^{2 d-3}\right)$ by a straightforward generalization of the analysis in Section 3.1. The analysis then proceeds by applying, for each $1 \leq s \leq d-1$, a charging scheme, which expresses the number of popular $s$-cells in terms of the number of popular ( $s-1$ )-cells (and degenerate vertices). A naive charging scheme produces a recurrence whose solution incurs an additional logarithmic factor for each $s$, resulting in the weaker bound $g_{d}(n)=O\left(n^{2 d-3} \log ^{d-1} n\right)$. A more careful analysis, as in the three-dimensional case, leads to refined recurrences, whose solution yields the improved bound $g_{d}(n)=O\left(n^{2 d-3} \log n\right)$. (We lose a logarithmic factor only when passing from vertices to edges, as in the three-dimensional case.)

### 4.1 The number of popular vertices

For a regular vertex $v \in \bigcap_{q=1}^{d-1} C_{i_{q} j_{q}}$ of $\mathcal{A}(\mathcal{K})$, we denote by $\mathcal{K}_{v}$ the collection $\left\{K_{i_{q}}, K_{j_{q}} \mid 1 \leq q \leq\right.$ $d-1\}$ of the $2 d-2$ sets defining $v$.

Lemma 4.1. Let $v \in \bigcap_{q=1}^{d-1} C_{i_{q} j_{q}}$ be a regular popular vertex of $\mathcal{A}(\mathcal{K})$.
(i) Each pair of sets $K_{a} \in \mathcal{K}_{v}$ and $K_{b} \in \mathcal{K} \backslash \mathcal{K}_{v}$ appear in the same order in all the $2^{d-1}$ permutations induced by the $(d-1)$-cells incident to $v$.
(ii) The elements of each pair $K_{i_{q}}, K_{j_{q}} \in \mathcal{K}_{v}$, for $1 \leq q \leq d-1$, are consecutive in all these $2^{d-1}$ permutations.

Proof. Each pair of distinct ( $d-1$ )-cells $f, g$ incident to $v$ are separated by at most $d-1(d-2)$ spheres from $\left\{C_{i_{1} j_{1}}, \ldots, C_{i_{d-1} j_{d-1}}\right\}$, and only by these spheres. Hence the orders $\prec_{f}$ and $\prec_{g}$ may disagree only over the pairs $\left(K_{i_{q}}, K_{j_{q}}\right)$, for $1 \leq q \leq d-1$. As in the proof of Lemma 3.1, this is easily seen to imply both parts of the lemma.

Lemma 4.2. Let $v \in \bigcap_{q=1}^{d-1} C_{i_{q} j_{q}}$ be a regular popular vertex in $\mathcal{A}(\mathcal{K})$. Then the line $\lambda_{v}=\bigcap_{q=1}^{d-1} h_{i_{q} j_{q}}$ stabs all the $n-2 d+2$ sets in $\mathcal{K} \backslash \mathcal{K}_{v}$, and misses all the $2 d-2$ sets in $\mathcal{K}_{v}$.

Proof. By definition, $\lambda_{v}$ misses every set $K \in \mathcal{K}_{v}$, because it is contained in a hyperplane separating $K$ from another set in $\mathcal{K}_{v}$. Hence, it suffices to show that $\lambda_{v}$ is a transversal of $\mathcal{K} \backslash \mathcal{K}_{v}$.

To show this, we fix a set $K_{a} \in \mathcal{K} \backslash \mathcal{K}_{v}$ and show that each of the $2^{d-1}$ wedges determined by $\left\{h_{i_{q} j_{q}} \mid 1 \leq q \leq d-1\right\}$ meets $K_{a}$. Each of these wedges is the intersection of $d-1$ halfspaces, where the $q$-th halfspace is either $h_{i_{q} j_{q}}^{+}$or $h_{i_{q} j_{q}}^{-}$, for $q=1, \ldots, d-1$. All these wedges have $\lambda_{q}$ on their boundary, and the convexity of $K_{a}$ then implies, exactly as in the three-dimensional case, that $\lambda_{v}$ intersects $K_{a}$.

[^1]For specificity, we show that $K_{a}$ intersects the wedge $\bigcap_{q=1}^{d-1} h_{i_{q} j_{q}}^{+}$; the proof for the other wedges is essentially the same. Lemma 4.1 implies that $K_{a}$ lies at the same position in each of the $2^{d-1}$ permutations induced by the cells incident to $v$. For each index $q$, if $K_{i_{q}}, K_{j_{q}}$ appear before $K_{a}$ (resp., after $K_{a}$ ) in all permutations induced by the cells incident to $v$, put $C_{q}=C_{i_{q} j_{q}}^{+}$(resp., $\left.C_{q}=C_{i_{q} j_{q}}^{-}\right)$.

Let $f$ be the cell incident to $v$ and contained in $\bigcap_{q=1}^{d-1} C_{q}$, and let $\mu_{f}$ be a transversal line stabbing $\mathcal{K}$ in the order $\prec_{f}$ (so its direction lies in $f$ ). By the choice of $f$ and by our assumption, we have either $K_{i_{q}} \prec_{f} K_{j_{q}} \prec_{f} K_{a}$, or $K_{a} \prec_{f} K_{j_{q}} \prec_{f} K_{i_{q}}$. This implies in the former case that $\mu_{f}$ visits $K_{a}$ after crossing $h_{i_{q} j_{q}}$ from $h_{i_{q} j_{q}}^{-}$(the side containing $K_{i_{q}}$ ) to $h_{i_{q} j_{q}}^{+}$(the side containing $K_{j_{q}}$ ). In the latter case, $\mu_{f}$ first visits $K_{a}$ and then crosses $h_{i_{q} j_{q}}$ from $h_{i_{q} j_{q}}^{+}$to $h_{i_{q} j_{q}}^{-}$. Thus, in either case, the segment $\lambda_{f} \cap K_{a}$ lies in $h_{i_{q} j_{q}}^{+}$, and this holds for every $1 \leq q \leq d-1$. Hence $\lambda_{f} \cap K_{a} \subset \bigcap_{q=1}^{d-1} h_{i_{q} j_{q}}^{+}$, and the claim follows.

Theorem 4.3. Let $\mathcal{K}$ be a collection of $n$ pairwise disjoint compact convex sets in $\mathbb{R}^{d}$. Then the number of popular vertices in $\mathcal{A}(\mathcal{K})$ is $O\left(n^{2 d-3}\right)$.

Proof. As in the three-dimensional case, it is easily checked that a popular vertex must be regular. Let $v \in \bigcap_{q=1}^{d-1} C_{i_{q} j_{q}}$ be a (regular) popular vertex in $\mathcal{A}(\mathcal{K})$, and let $\lambda_{v}=\bigcap_{q=1}^{d-1} h_{i_{q} j_{q}}$ be the intersection line of the corresponding hyperplanes. Consider the plane $H=\bigcap_{q=1}^{d-2} h_{i_{q} j_{q}}$, put $K_{a}^{*}=K_{a} \cap H$, for each index $a \notin\left\{i_{q}, j_{q} \mid 1 \leq q \leq d-2\right\}$, and denote by $\mathcal{K}^{*}$ the collection of these $n-2 d+4$ planar cross-sections. Clearly, all sets in $\mathcal{K}^{*}$ are pairwise disjoint, compact, and convex.

By Lemma 4.2, $\lambda_{v}$ lies in $H$, stabs all the sets in $\mathcal{K}^{*} \backslash\left\{K_{i_{d-1}}^{*}, K_{j_{d-1}}^{*}\right\}$, and misses the two sets $K_{i_{d-1}}^{*}, K_{j_{d-1}}^{*}$. As in Theorem [3.3, we charge $\lambda_{v}$ to an extremal line $\mu=\mu_{v}$ within $H$ which is tangent to two sets of $\mathcal{K}^{*}$, and misses only the sets among $K_{i_{d-1}}^{*}, K_{j_{d-1}}^{*}$ that it does not touch. As in the preceding analysis, each extremal line $\mu$ of this kind is charged at most twice. Applying the Clarkson-Shor analysis 4, similarly to Theorem 3.3, the number of lines $\mu$, charged within $H$, is $O(n)$. Summing over all possible choices of the 2-planes $H$, namely over all choices of $d-2$ of the hyperplanes $h_{i j}$, the number of lines $\lambda_{v}$, and thus the number of popular vertices, is $O\left(n \cdot n^{2 d-4}\right)=O\left(n^{2 d-3}\right)$.

### 4.2 The number of permutation cells

We next generalize the analysis of Section 3.3 to higher dimensions. We first extend the notion of borders. Let $v$ be a vertex of $\mathcal{A}(\mathcal{K})$, so that $v \in \bigcap_{1 \leq q \leq d-1} C_{i_{q} j_{q}}$. For any subset $J$ of $\{1, \ldots, d-1\}$, let $R \subseteq \mathbb{S}^{d-1}$ be a connected component of $\mathbb{S}^{d-1} \backslash \bigcup_{q \in J} C_{i_{q} j_{q}}$. Equivalently, it is the intersection of $|J|$ hemispheres, where the $q$-th hemisphere, for $q \in J$, is either $C_{i_{q} j_{q}}^{+}$or $C_{i_{q} j_{q}}^{-}$. Note that there are $2^{|J|}$ such regions (for any fixed $J$ ). We call $(v, R)$ an $s$-border, where $s=|J|$. Given $v$ and $R$, for $s \geq 1$, there is a unique $s$-dimensional cell $f$ of $\mathcal{A}(\mathcal{K})$ which is incident to $v$ and is contained in the interior of $R$. This cell $f$ lies in the intersection of the interior of $R$ with $\bigcap_{q \in J^{c}} C_{i_{q} j_{q}}$, where $J^{c}=\{1, \ldots, d-1\} \backslash J$. We refer to $f$ as the $s$-cell of $\mathcal{A}(\mathcal{K})$ associated with $(v, R)$. For $s=0$ we define the $s$-cell of $\mathcal{A}(\mathcal{K})$ associated with $(v, R)$ to be $v$ itself, and for $s=d-1$ we define the $s$-cell of $\mathcal{A}(\mathcal{K})$ associated with $(v, R)$ to be the unique $(d-1)$-cell incident to $v$ and contained in $R$. The reader is invited to check that, for $d=3$, a 0 -border, in the new definition, is a vertex of $\mathcal{A}(\mathcal{K})$, a 1 -border is an edge border, and a 2-border is what we simply called a border.

Let $(v, R)$ be an $s$-border and let $f$ be the $s$-cell associated with $(v, R)$. If $f$ is a popular cell, we say that $(v, R)$ is a 0 -level $s$-border of $\mathcal{A}(\mathcal{K})$. An $s$-border $(v, R)$ is a 1 -level $s$-border in $\mathcal{A}(\mathcal{K})$ if it is not a 0 -level $s$-border, but becomes such a border after removing from $\mathcal{K}$ some single set $K$. In this case we say that $K$ is in conflict with $(v, R)$. As in the three-dimensional case, $K$ need not be unique.

For each $t=0,1$ and $0 \leq s \leq d-1$, let $N_{t}^{(s)}(\mathcal{K})$ be the number of $t$-level $s$-borders in $\mathcal{A}(\mathcal{K})$, and let $N_{t}^{(s)}(n)$ denote the maximum value of $N_{t}^{(s)}(\mathcal{K})$, over all collections $\mathcal{K}$ of $n$ pairwise disjoint compact convex sets in $\mathbb{R}^{d}$.

Note that $N_{0}^{(0)}(\mathcal{K})$ is the number of popular vertices in $\mathcal{A}(\mathcal{K})$, so we have $N_{0}^{(0)}(n)=O\left(n^{2 d-3}\right)$. The term $N_{0}^{(d-1)}(\mathcal{K})$ counts the overall number of vertices incident to permutation cells, where each vertex is counted once for each permutation cell incident to it. Assuming $n \geq d$, each permutation cell in $\mathcal{A}(\mathcal{K})$ is incident to at least one vertex (and each vertex is incident to at most $2^{d-1}$ permutation cells). Thus, the number of geometric permutations of $\mathcal{K}$ is at most $N_{0}^{(d-1)}(\mathcal{K})$.

For each $1 \leq s \leq d-1$, we apply a charging scheme, which results in a recurrence which expresses $N_{0}^{(s)}(\mathcal{K})$ in terms of $N_{0}^{(s-1)}(\mathcal{K})$ and $N_{1}^{(s)}(\mathcal{K})$.

Fix $1 \leq s \leq d-1$. Let $(v, R)$ be a 0 -level $s$-border in $\mathcal{A}(\mathcal{K})$, and let $f$ be the popular $s$-cell associated with $(v, R)$. Let $C_{i_{1} j_{1}}, C_{i_{2} j_{2}}, \ldots, C_{i_{d-1} j_{d-1}}$ be the $d-1(d-2)$-spheres of $\mathcal{C}(\mathcal{K})$ incident to $v$, and assume, with no loss of generality, that $R=\bigcap_{q=1}^{s} C_{i_{q} j_{q}}^{+}$. Moreover, we may assume that $v$ is regular, since the number of $s$-borders incident to degenerate vertices is clearly $O\left(n^{2 d-3}\right)$. To simplify the notation, we refer to borders incident to a regular (resp., degenerate) vertex as regular borders (resp., degenerate borders).

For each $1 \leq q \leq s$ there exists a unique edge $e_{q}^{+}$of $f$ which is incident to $v$ and not contained in $C_{i_{q} j_{q}}$. Indeed, by construction, $f$ lies in the intersection $s$-sphere $\bigcap_{q=s+1}^{d-1} C_{i_{q} j_{q}}$ (for $s=d-1$, this is the entire $\mathbb{S}^{d-1}$ ), and each edge of $f$ incident to $v$ is formed by further intersecting this sphere with $s-1$ additional spheres from $C_{i_{1} j_{1}}, \ldots, C_{i_{s} j_{s}}$. The claim follows since only one side of the resulting intersection circle lies (near $v$ ) in the closure of $R$. Let $e_{q}^{-}$denote the other edge of $\mathcal{A}(\mathcal{K})$ which is incident to $v$ and lies on the same intersection circle $\gamma$ as $e_{q}^{+}$, so $e_{q}^{-}$emanates from $v$ away from $R$. Let $v_{q}$ denote the other endpoint of $e_{q}^{-}$, and let $g_{q}$ denote the (unique) $(s-1)$-cell which bounds $f$, lies in $C_{i_{q} j_{q}} \cap\left(\bigcap_{t=s+1}^{d-1} C_{i_{t} j_{t}}\right)$, is incident to $v$ and is contained in $R_{q}=\bigcap_{1 \leq t \leq s, t \neq q} C_{i_{t} j_{t}}^{+}$. Also, $g_{q}$ is the $(s-1)$-cell associated with $\left(v, R_{q}\right)$. See Figure 7 . There are two possible cases:


Figure 7: Charging a 0 -level $s$-border along the edge $e_{q}$.
(i) If one of the $(s-1)$-cells $g_{1}, g_{2}, \ldots, g_{s}$, say $g_{s}$, is popular, we charge $(v, R)$ to the 0 -level
( $s-1$ )-border $\left(v, R_{s}\right)$, noting, as above, that $g_{s}$ is the $(s-1)$-cell associated with this border. By construction, each 0 -level $(s-1)$-border $(v, R)$ is charged at most $2(d-s)$ times in this manner, once from each $s$-border associated with an $s$-cell which is bounded by the ( $s-1$ )-cell associated with this border (there are $d-s$ choices for the great sphere $C_{i_{t} j_{t}}$ that participates in the definition of $(v, R)$ but is absent in the $s$-border, and two choices of the corresponding hemisphere $\left.C_{i_{t} j_{t}}^{+}, C_{i_{t} j_{t}}^{-}\right)$. Hence, the number of 0 -level $s$-borders falling into subcase (i) is $O\left(N_{0}^{(s-1)}(\mathcal{K})\right)$.
(ii) None of the $(s-1)$-cells $g_{1}, g_{2}, \ldots, g_{s}$ is popular. For each $1 \leq q \leq s$, let $C_{k_{q} \ell_{q}}$ be the additional great sphere incident to $v_{q}$, and suppose, for specificity, that $v \in C_{k_{q} \ell_{q}}^{+}$. The vertex $v_{q}$ participates in the 1-level $s$-border $\left(v_{q}, R_{q}^{\prime}\right)$, where $R_{q}^{\prime}=C_{k_{q} \ell_{q}}^{+} \cap\left(\bigcap_{1 \leq t \leq s, t \neq q} C_{i_{t} j_{t}}^{+}\right)$.

Since $g_{q}$ is not popular, $\left(v_{q}, R_{q}^{\prime}\right)$ is not a 0 -level $s$-border. Let $f_{q}$ be the $s$-cell associated with $\left(v_{q}, R_{q}^{\prime}\right)$. Clearly, at least one of $i_{q}, j_{q}$ does not belong to $\left\{k_{q}, \ell_{q}\right\}$; say it is $i_{q}$. Thus, and since $v$ is regular, removing $K_{i_{q}}$ keeps $v_{q}$ (and hence $\left(v_{q}, R_{q}^{\prime}\right)$ ) intact, and makes $f$ and $f_{q}$ fuse into a larger $s$-cell $f^{\prime}$ containing both of them. Clearly, $f^{\prime}$ is the cell associated with $\left(v_{q}, R_{q}^{\prime}\right)$ in $\mathcal{A}\left(\mathcal{K} \backslash\left\{K_{i_{q}}\right\}\right)$, and it is popular there because $f \subset f^{\prime}$ was popular in $\mathcal{A}(\mathcal{K})$. We say that the borders $(v, R)$, $\left(v_{q}, R_{q}^{\prime}\right)$ are neighbors in $\mathcal{A}(\mathcal{K})$.

We then charge $(v, R)$ to its $s$ neighboring 1-level $s$-borders $\left(v_{q}, R_{q}^{\prime}\right)$, for $q=1, \ldots, s$. Note that each 1-level $s$-border $(v, R)$ is charged at most $s$ times, once along each of the $s$ edges, incident to $v$, of the $s$-cell associated with it. We thus obtain the following recurrence.

$$
\begin{equation*}
N_{0}^{(s)}(\mathcal{K}) \leq N_{1}^{(s)}(\mathcal{K})+O\left(N_{0}^{(s-1)}(\mathcal{K})+n^{2 d-3}\right) \tag{5}
\end{equation*}
$$

where the first term in the right hand side bounds the number of 0 -level $s$-borders falling into case (ii), and the second term bounds the number of the remaining 0 -level $s$-borders.

Similarly to the three-dimensional case, we combine the system (5) of recurrences with the analysis technique of Tagansky, and solve the resulting recurrences to obtain a slightly inferior bound (involving a larger polylogarithmic factor). We then refine the recurrences, using a more careful analysis, similar to the one in Section 3, and thereby obtain the improved bound $O\left(n^{2 d-3} \log n\right)$.
Applying Tagansky's technique: The simpler variant. We prove that $N_{0}^{(s)}(n)=O\left(n^{2 d-3} \log ^{s} n\right)$ by induction on $s$. For the base case $s=0$, we have $N_{0}^{(0)}(n)=O\left(n^{2 d-3}\right)$ by Theorem 4.3. Consider a fixed $s \geq 1$ and assume that the bound holds for $s-1$, so (5) becomes

$$
\begin{equation*}
N_{0}^{(s)}(\mathcal{K}) \leq N_{1}^{(s)}(\mathcal{K})+O\left(n^{2 d-3} \log ^{s-1} n\right) . \tag{6}
\end{equation*}
$$

Let $\mathcal{R}$ be a random sample of $n-1$ sets of $\mathcal{K}$, obtained by removing a random set $K$ from $\mathcal{K}$. The expected number of 0 -level popular $s$-borders in $\mathcal{A}(\mathcal{R})$ satisfies

$$
\begin{equation*}
\mathbf{E}\left\{N_{0}^{(s)}(\mathcal{R})\right\} \geq \frac{n-2 d+2}{n} N_{0}^{(s)}(\mathcal{K})+\frac{1}{n} N_{1}^{(s)}(\mathcal{K}) . \tag{7}
\end{equation*}
$$

This follows since a 0 -level $s$-border $(v, R)$ (with $v$ regular) survives after removing $K$ if and only if $K \notin \mathcal{K}_{v}$, and a 1 -level $s$-border becomes a 0 -level $s$-border if and only if it is in conflict with $K$. Combining this inequality with (6), we get

$$
\begin{aligned}
& \frac{1}{n} N_{0}^{(s)}(\mathcal{K}) \leq \frac{1}{n} N_{1}^{(s)}(\mathcal{K})+O\left(n^{2 d-4} \log ^{s-1} n\right) \leq \\
& \quad \mathbf{E}\left\{N_{0}^{(s)}(\mathcal{R})\right\}-\frac{n-2 d+2}{n} N_{0}^{(s)}(\mathcal{K})+O\left(n^{2 d-4} \log ^{s-1} n\right),
\end{aligned}
$$

or

$$
\frac{n-2 d+3}{n} N_{0}^{(s)}(\mathcal{K}) \leq \mathbf{E}\left\{N_{0}^{(s)}(\mathcal{R})\right\}+O\left(n^{2 d-4} \log ^{s-1} n\right)
$$

Replacing $N_{0}^{(s)}(\mathcal{K})$ and $N_{0}^{(s)}(\mathcal{R})$ by their respective maximum possible values $N_{0}^{(s)}(n)$ and $N_{0}^{(s)}(n-$ 1 ), we get the recurrence

$$
\frac{n-2 d+3}{n} N_{0}^{(s)}(n) \leq N_{0}^{(s)}(n-1)+O\left(n^{2 d-4} \log ^{s-1} n\right),
$$

whose solution is easily seen to be $N_{0}^{(s)}(n)=O\left(n^{2 d-3} \log ^{s} n\right)$. This establishes the induction step and thus proves the asserted bound. In particular, we have so far

$$
g_{d}(n)=O\left(n^{2 d-3} \log ^{d-1} n\right) .
$$

Improved bounds for $s \geq 2$. As promised, we next refine the analysis, and show that

$$
\begin{equation*}
N_{0}^{(s)}(\mathcal{K})=O\left(n^{2 d-3} \log n\right), \tag{8}
\end{equation*}
$$

for any $1 \leq s \leq d-1$, by establishing a sharper variant of (55).
As in the three-dimensional case, the weakness of the preceding analysis lies in the random sampling inequality (7), or, more precisely, in the term $N_{1}^{(s)}(\mathcal{K}) / n$ thereof.

Specifically, if a 1 -level $s$-border $(v, R)$ is in conflict with $w>1$ sets of $\mathcal{K}$ then removing any one of these sets will make $(v, R)$ a 0 -level $s$-border, so the probability of this to happen is $w / n$, which is significantly larger than the bound $1 / n$ used in (77). As above, we refer to $w$ as the weight of $(v, R)$. We can therefore modify the definition of $N_{1}^{(s)}(\mathcal{K})$ so a border of weight $w$ is counted $w$ times. The preceeding discussion ensures that (7) still holds in the new setting.

We proceed to prove 8 by induction on $s$. The base case $s=1$ has already been analyzed, and we have shown that $N_{0}^{(1)}(n)=O\left(n^{2 d-3} \log n\right)$. Fix $2 \leq s \leq d-1$, and suppose that we have already proved that $N_{0}^{\left(s^{\prime}\right)}(\mathcal{K})=O\left(n^{2 d-3} \log n\right)$, for all $1 \leq s^{\prime}<s$.

The following lemma generalizes Lemma 3.4 to arbitrary dimension $d \geq 4$.
Lemma 4.4. (i) The number of 1 -level 2 -borders, having weight 1 and charged by two 0 -level neighboring 2 -borders, is $O\left(N_{1}^{(1)}(\mathcal{K})+n^{2 d-3}\right)$.
(ii) For $s \geq 3$, there are no 1 -level $s$-borders incident to a regular vertex, having weight 1 , and charged by s 0 -level neighboring s-borders.

Proof of Lemma 4.4. The proof of (i) is very similar to the proof of Lemma 3.4, and will be briefly presented later, after we prove (ii).

So we assume that $s \geq 3$. Let $(v, R)$ be a 1 -level $s$-border which has weight 1 and is charged by $s 0$-level neighboring $s$-borders, so that $v$ is regular. Let $C_{i_{1} j_{1}}, C_{i_{2} j_{2}}, \ldots, C_{i_{d-1} j_{d-1}}$ be the ( $d-2$ )spheres incident to $v$. Without loss of generality, assume that $R=\bigcap_{q=1}^{s} C_{i_{q} j_{q}}^{+}$. Let $f$ be the $s$-cell associated with $(v, R)$, and let $e_{1}, \ldots, e_{s}$ be the $s$ edges of $f$ incident to $v$, so that, for each $k=1, \ldots, s$, the edge $e_{k}$ lies on the circle $\bigcap \bigcap_{1 \leq q \leq d-1, q \neq k} C_{i_{q} j_{q}}$. For $k=1, \ldots, s$, let $v_{k}$ denote the other endpoint of $e_{k}$, and let $C_{a_{k} b_{k}}$ denote the (unique) great sphere incident to $v_{k}$ and not containing $e_{k}$. Assume, without loss of generality, that $v$ lies in $C_{a_{k} b_{k}}^{-}$, and put $R_{k}=C_{a_{k} b_{k}}^{+} \cap \bigcap_{1 \leq q \leq s, q \neq k} C_{i_{q} j_{q}}^{+}$. By construction, the $s s$-borders $\left(v_{k}, R_{k}\right)$, for $k=1, \ldots, s$, are precisely those that charge $(v, R)$, so they are all regular 0 -level $s$-borders.

Note that $(v, R)$ is in conflict with each of the sets $K_{a_{1}}, K_{b_{1}}, \ldots, K_{a_{s}}, K_{b_{s}}$ for which the corresponding index $a_{k}$ or $b_{k}$ is not one of $i_{1}, j_{1}, \ldots, i_{d-1}, j_{d-1}$. Indeed, removing such a set $K_{a_{k}}$, say, eliminates the sphere $C_{a_{k} b_{k}}$ and thereby exposes $v$ to the extended $2^{d-1-s}$ permutation cells that surround $v_{k}$, so that they are all now contained in $R$, so $(v, R)$ becomes a 0 -level $s$-border. However, since the weight of $(v, R)$ is 1 , only one of these sets, call it $K_{b}$, can be in conflict with $(v, R)$ (so $\left.b \notin\left\{i_{1}, j_{1}, \ldots, i_{d-1}, j_{d-1}\right\}\right)$. This, and the fact that each of the $v_{k}$ 's is regular, is easily seen to imply the following property: For each $k$, one of $a_{k}, b_{k}$, say $a_{k}$, belongs to $\left\{i_{k}, j_{k}\right\}$, and the other index $b_{k}$ is $b$.

Fix a pair of distinct vertices $v_{k}, v_{\ell}$, and denote by $\Pi_{k}$ (resp., $\Pi_{\ell}$ ) the collection of the $2^{d-1-s}$ permutations induced by the permutation cells that surround $v_{k}$ (resp., $v_{\ell}$ ) and are contained in $R_{k}$ (resp., $R_{\ell}$ ). Any pair of permutations in $\Pi_{k}$ differ from each other only by swaps of some of the pairs $\left(i_{q}, j_{q}\right)$, for $q=s+1, \ldots, d-1$. Hence the indices of each of these pairs appear consecutively in any of these permutations, and the locations of these pairs are fixed for all permutations. The set $K_{b}$ appears, somewhere in between these pairs, in a fixed location in all permutations. A similar property holds for the permutations in $\Pi_{\ell}$.

Fix a permutation $\pi \in \Pi_{k}$. It has a "twin" permutation $\pi^{\prime}$ in $\Pi_{\ell}$, in which the order of the two indices in each of the pairs $\left(i_{q}, j_{q}\right)$, for $q=s+1, \ldots, d-1$, is the same as their order in $\pi$. To gain more insight into the structure of $\pi$ and $\pi^{\prime}$, let $\varphi$ and $\varphi^{\prime}$ denote, respectively, the permutation cells of $\mathcal{A}(\mathcal{K})$ in which $\pi$ and $\pi^{\prime}$ are generated. We can get from $\varphi$ to $\varphi^{\prime}$ by first crossing $C_{i_{k} b}$ into a corresponding $(d-1)$-cell $\varphi_{0}$ surrounding $f$ and then cross $C_{i_{\ell} b}$ into $\varphi^{\prime}$. This means that $\prec_{\varphi}$ and $\prec_{\varphi^{\prime}}$ (i.e., $\pi$ and $\pi^{\prime}$ ) are obtained from each other by first swapping $K_{b}$ with $K_{i_{k}}$ and then by swapping $K_{b}$ with $K_{i_{\ell}}$. As is easily checked, this implies that $K_{i_{k}}$ and $K_{i_{\ell}}$ must be adjacent in $\pi$ and in $\pi^{\prime}$. This however cannot hold for every pair of distinct indices in $\left\{i_{1}, \ldots, i_{s}\right\}$ if $s \geq 3$. This contradiction shows that for $s \geq 3$ there are no 1 -level $s$-borders which satisfy the assumptions in the lemma. This completes the proof of part (ii).

We now consider the case $s=2$, which, as noted above, can be handled in a manner that is very similar to the analysis in Lemma 3.4. Specifically, let $(v, R)$ be a regular 1-level 2-border of weight 1 which is charged by two 0 -level 2 -borders $\left(v_{1}, R_{1}\right),\left(v_{2}, R_{2}\right)$. (The number of degenerate 1-level 2-borders is $O\left(n^{2 d-3}\right)$.) As in the proof of part (i), we may assume that both $v_{1}$ and $v_{2}$ are regular (for otherwise they would not charge $(v, R)$ ). Let $C_{i_{1} j_{1}}, C_{i_{2} j_{2}}, \ldots, C_{i_{d-1} j_{d-1}} \in \mathcal{C}(\mathcal{K})$ be the ( $d-2$ )-spheres incident to $v$, and assume that $R=C_{i_{1} j_{1}}^{+} \cap C_{i_{2} j_{2}}^{+}$. Let $f$ be the 2 -face associated with $(v, R)$. For $k=1,2$, let $e_{k}$ denote the edge of $f$ incident to $v$ and contained in $C_{i_{k} j_{k}}$, and assume that $v_{k}$ is the other endpoint of $e_{k}$. Let $C_{a_{k} b_{k}}$ be the (unique) great sphere passing through $v_{k}$ and not containing $e_{k}$.

As in the three-dimensional case, and similar to the preceding analysis, since $(v, R)$ has weight 1 , the only case to be considered, up to symmetry, is where $a_{1}=i_{2}, a_{2}=i_{1}$, and $b_{1}=b_{2}=b$, where $b \neq\left\{i_{1}, j_{1}, \ldots, i_{d-1}, j_{d-1}\right\}$.

The proof now continues as in the three-dimensional case, and we only provide a brief sketch of it. For $k=1,2$, we consider the other edge $e_{k}^{\prime}$ of $f$ incident to $v_{k}$, denote by $u_{k}$ the other endpoint of $e_{k}$, and assume that neither of $u_{1}, u_{2}$ is degenerate. See Figure 8, We then consider the other great sphere $C_{r_{k} s_{k}}$ incident to $u_{k}$, for $k=1,2$, and distinguish between the following two cases: (a) $i_{1} \neq r_{1}, s_{1}$ or $i_{2} \neq r_{2}, s_{2}$. In the former case, removing $K_{i_{1}}$ leaves $u_{1}$ intact and extends $e_{1}^{\prime}$ into a 0 -level 1 -border; the proof is argued exactly as in the three-dimensional case. The latter case is handled symmetrically, and we conclude that the number of 2-borders of this kind is $O\left(N_{1}^{(1)}(\mathcal{K})\right)$. (b) $i_{1}=r_{1}$ and $i_{2}=r_{2}$ (or any of the symmetric pairs of equalities). In this case we consider the


Figure 8: The setup in the proof of Theorem 4.4 for $s=2$ : View within the sphere $\sigma_{0}=\bigcap_{q=3}^{d-1} C_{i_{q} j_{q}}$.

2-sphere $\sigma_{0}=\bigcap_{q=3}^{d-1} C_{i_{q} j_{q}}$ which contains $f$, and construct in it the arrangement $\mathcal{A}_{i_{1}}^{\left(\sigma_{0}\right)}$, formed by the circles $C_{i_{1} x} \cap \sigma_{0}$, for $x \notin\left\{i_{1}\right\} \cup\left\{i_{3}, j_{3}, \ldots, i_{d-1}, j_{d-1}\right\}$. We note that $f$ is contained in a face $f_{0}$ of $\mathcal{A}_{i_{1}}^{\left(\sigma_{0}\right)}$ and touches its boundary at three distinct edges. This allows us to bound the number of 2 -borders under consideration by $O\left(n^{3}\right)$, for a fixed choice of $i_{3}, j_{3}, \ldots, i_{d-1}, j_{d-1}$, arguing exactly as in the three-dimensional case. In total, the number of these 2 -borders is $O\left(n^{3} \cdot n^{2 d-6}\right)=O\left(n^{2 d-3}\right)$. This completes the proof of the lemma.

First, for $s=2$, we bound the quantity $N_{1}^{(1)}(\mathcal{K})$ using the Clarkson-Shor analysis technique [4], as we did in the proof of Lemma 3.4. That is, since each 1-level 1-border is defined by at most $2 d-2$ sets of $\mathcal{K}$ and becomes a 0-level 1 -border when we remove (at least) one set from $\mathcal{K}$, the number of 1-level 1-borders is $O\left(\mathbf{E}\left\{N_{0}^{(1)}\left(\mathcal{K}^{\prime}\right)\right\}\right)$, where $\mathcal{K}^{\prime}$ is a random sample of $n / 2$ sets of $\mathcal{K}$. Thus, combining this with the bound already established for $s=1$, we have

$$
\begin{equation*}
N_{1}^{(1)}(\mathcal{K})=O\left(\mathbf{E}\left\{N_{0}^{(1)}\left(\mathcal{K}^{\prime}\right)\right\}\right)=O\left(N_{0}^{(1)}(n / 2)\right)=O\left(n^{2 d-3} \log n\right) \tag{9}
\end{equation*}
$$

With these preparations, we are now ready to complete the induction step for $s$.
Let $N_{1,1}^{(s)}(\mathcal{K})$ denote the number of 1-level $s$-borders having weight 1 , and let $N_{1,2}^{(s)}(\mathcal{K})$ denote the number of 1-level $s$-borders having weight at least 2 . Since a 1 -level $s$-border of weight $w_{i}$ contributes to $N_{1}^{(s)}(\mathcal{K}) w_{i}$ units, we have

$$
\begin{equation*}
N_{1}^{(s)}(\mathcal{K}) \geq N_{1,1}^{(s)}(\mathcal{K})+2 N_{1,2}^{(s)}(\mathcal{K}) \tag{10}
\end{equation*}
$$

Recall that we charge every 0-level $s$-border (falling into subcase (ii)) to $s$ neighboring 1-level $s$-borders. By Lemma 4.4 (and (91)), all but $O\left(n^{2 d-3} \log n\right)$ 1-level $s$-borders, that have weight 1 , are charged by at most $s-1$ neighboring 0 -level $s$-borders. (This is the situation for $s=2$; the bound drops to $O\left(n^{2 d-3}\right)$ for $s \geq 3$.) Thus, we obtain the following refinement of 5:

$$
\begin{equation*}
s N_{0}^{(s)}(\mathcal{K}) \leq(s-1) N_{1,1}^{(s)}(\mathcal{K})+s N_{1,2}^{(s)}(\mathcal{K})+O\left(n^{2 d-3} \log n\right) \tag{11}
\end{equation*}
$$

The combination of (10) and (11), and the assumption that $s \geq 2$ (so $s /(s-1) \geq 2)$ imply that,

$$
\frac{s}{s-1} N_{0}^{(s)}(\mathcal{K}) \leq N_{1}^{(s)}(\mathcal{K})+O\left(n^{2 d-3} \log n\right)
$$

Substituting $t=\frac{s}{s-1}-1=\frac{1}{s-1}>0$ and combining this with (7), we get

$$
\begin{gathered}
\frac{1+t}{n} N_{0}^{(s)}(\mathcal{K}) \leq \frac{1}{n} N_{1}^{(s)}(\mathcal{K})+O\left(n^{2 d-4} \log n\right) \\
\leq \mathbf{E}\left\{N_{0}^{(s)}(\mathcal{R})\right\}-\frac{n-2 d+2}{n} N_{0}^{(s)}(\mathcal{K})+O\left(n^{2 d-4} \log n\right),
\end{gathered}
$$

or

$$
\frac{n-2 d+3+t}{n} N_{0}^{(s)}(\mathcal{K}) \leq \mathbf{E}\left\{N_{0}^{(s)}(\mathcal{R})\right\}+O\left(n^{2 d-4} \log n\right) .
$$

Thus, as above, we get the following recurrence

$$
\frac{n-2 d+3+t}{n} N_{0}^{(s)}(n) \leq N_{0}^{(s)}(n-1)+O\left(n^{2 d-4} \log n\right),
$$

whose solution is easily seen to be

$$
N_{0}^{(s)}(n)=O\left(n^{2 d-3} \log n\right)
$$

(see, e.g., [18, Proposition 3.1]), which readily implies Theorem 4.5, This completes the induction step and thus establishes 8 for all $s$. We thus obtain the main result of the paper.

Theorem 4.5. Any collection $\mathcal{K}$ of $n$ pairwise disjoint convex sets in $\mathbb{R}^{d}$, for any $d \geq 3$, admits at most $O\left(n^{2 d-3} \log n\right)$ geometric permutations.

## 5 Discussion

Although the improvement presented in this paper is significant, especially since no progress was made on the problem during the past 20 years, it is far from satisfactory, since we strongly believe (and tend to conjecture) that the correct upper bounds are close to $O\left(n^{d-1}\right)$, for any $d \geq 3$. Improving further the bounds is the main open problem left by this study. A modest subgoal is to get rid of the logarithmic factor in our bounds, and show, e.g., that $g_{3}(n)=O\left(n^{3}\right)$.

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[^1]:    ${ }^{1}$ As in the three-dimensional case, the intersection consists of two antipodal points, so there are two choices for $v$.

