# Directed Hamilton cycles in digraphs and matching alternating Hamilton cycles in bipartite graphs * 

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#### Abstract

In 1972, Woodall raised the following Ore type condition for directed Hamilton cycles in digraphs: Let $D$ be a digraph. If for every vertex pair $u$ and $v$, where there is no arc from $u$ to $v$, we have $d^{+}(u)+d^{-}(v) \geq|D|$, then $D$ has a directed Hamilton cycle. By a correspondence between bipartite graphs and digraphs, the above result is equivalent to the following result of Las Vergnas: Let $G=(B, W)$ be a balanced bipartite graph. If for any $b \in B$ and $w \in W$, where $b$ and $w$ are nonadjacent, we have $d(w)+d(b) \geq|G| / 2+1$, then every perfect matching of $G$ is contained in a Hamilton cycle.

The lower bounds in both results are tight. In this paper, we reduce both bounds by 1 , and prove that the conclusions still hold, with only a few exceptional cases that can be clearly characterized.


Key words: degree sum, Matching alternating Hamilton cycle, Hamilton cycle.

## 1 Introduction

Hamiltonian problems, and their many variations, have been studied extensively for more than half a century. The readers could refer to the surveys of Gould ([17] and [18]), Kawarabayashi ([22]) and Broersma ([11]) to trace the development in this field. Recently, approximate solutions of many traditional Hamiltonian problems and conjectures in digraphs came forth ([24], [23], [12] and [26]), which are surveyed by Kühn and Osthus ([25]).

Hamiltonicity and related properties are also important in practical applications. For example, in network design, the existence of Hamilton cycles in the underlying topology of an interconnection network provide advantage for the routing algorithm to make use of a ring structure, while the existence of a hamiltonian decomposition allows the load to be equally distributed, making network robust (9).

There are lots of degree or degree sum conditions for hamiltonicity. Often, the lower bounds in such conditions are best possible. However, we could still reduce the bounds and try to identify all exceptional graphs, that is, the extremal graphs for the conditions. Such kind of research often leads to the discovery of interesting topology structures. In this paper, we apply this idea to Woodall's condition for the existence of directed Hamilton cycles in digraphs.

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## 2 Terminology, notations and preliminary results

In this paper we consider finite, simple and connected graphs, and finite and simple digraphs. For the terminology not defined in this paper, the reader is referred to [10] and [3].

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $\nu$ or $|G|$ the order of $V(G)$. For $u \in V(G)$, we denote by $d(u)$ the degree of $u$, and $N(u)$ or $N_{G}(u)$ the set of neighbors of $u$ in $G$. For a subgraph $H$ of $G$ and a vertex $u \in V(G-H)$, we also denote by $N_{H}(u)$ the set of neighbors of $u$ in $H$. For any two disjoint vertex sets $X, Y$ of $G$ we denote by $e(X, Y)$ the number of edges of $G$ from $X$ to $Y$. For $u, v \in V(G)$, we denote by $d(u, v)$ the distance between $u$ and $v$, that is, the length of the shortest path connecting $u$ and $v$. By $u v+$ (uv-) we mean the vertices $u$ and $v$ are adjacent (nonadjacent). If a vertex $u$ sends (no) edges to $X$, where $X$ is a subgraph or a vertex subset of $G$, we write $u \rightarrow X(u \nrightarrow X)$. By $n K_{2}$, we denote a graph consisting of $n$ independent edges.

Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D), u, v$ and $w$ distinct vertices of $D$. We denote by $|D|$ the order of $V(D), d^{+}(u)$ and $d^{-}(u)$ the out-degree and in-degree of $u$, respectively. The degree of $u$ is the sum of its out-degree and in-degree. The minimum out-degree and in-degree of the vertices in $D$, is denoted by $\delta^{+}(D)$ and $\delta^{-}(D)$. We let $\delta^{0}(D)=$ $\min \left\{\delta^{+}(D), \delta^{-}(D)\right\}$. Let $(u, v)$ denote an arc from $u$ to $v$. If $(u, v) \in A(D)$ or $(v, u) \in A(D)$, we say that $u$ and $v$ are adjacent. If $(w, u) \in A(D)$ and $(w, v) \in A(D)$, then we say that the pair $\{u, v\}$ is dominated, if $(u, w) \in A(D)$ and $(v, w) \in A(D)$, then we say that the pair $\{u, v\}$ is dominating. The complete digraph on $n \geq 1$ vertices, denoted by $\overleftrightarrow{K}_{n}$, is obtained from the complete graph $K_{n}$ by replacing every edge $x y$ with two $\operatorname{arcs}(x, y)$ and $(y, x)$. Without causing ambiguity, we use $I_{n}$ to denote a graph or a digraph consisting of $n$ independent vertices. A transitive tournament is an orientation of complete graph for which the vertices can be numbered in such a way that $(i, j)$ is an edge if and only if $i<j$.

Let $C=u_{0} u_{1} \ldots u_{m-1} u_{0}$ be a cycle in a graph $G$. Throughout this paper, the subscript of $u_{i}$ is reduced modulo $m$. We always orient $C$ such that $u_{i+1}$ is the successor of $u_{i}$. For $0 \leq i, j \leq m-1$, the path $u_{i} u_{i+1} \ldots u_{j}$ is denoted by $u_{i} C^{+} u_{j}$, while the path $u_{i} u_{i-1} \ldots u_{j}$ is denoted by $u_{i} C^{-} u_{j}$. For a path $P=v_{0} v_{1} \ldots v_{p-1}$ and $0 \leq i, j \leq p-1$, the segment of $P$ from $v_{i}$ to $v_{j}$ is denoted by $v_{i} P v_{j}$.

A matching $M$ of $G$ is a subset of $E(G)$ in which no two elements are adjacent. If every $v \in V(G)$ is covered by an edge in $M$ then $M$ is said to be a perfect matching of $G$. For a matching $M$, an $M$-alternating path ( $M$-alternating cycle) is a path (cycle) of which the edges appear alternately in $M$ and $E(G) \backslash M$. We call an edge in $M$ or an $M$-alternating path starting and ending with edges in $M$ a closed $M$-alternating path, while an edge in $E(G) \backslash M$ or an $M$-alternating path starting and ending with edges in $E(G) \backslash M$ an open $M$-alternating path.

The following results of Dirac and Ore for the existence of Hamilton cycles in graphs are basic and famous.
Theorem 2.1. (Dirac, 1952 [15]) If $G$ is a simple graph with $|G| \geq 3$ and every vertex of $G$ has degree at least $|G| / 2$, then $G$ has a Hamilton cycle.
Theorem 2.2. (Ore, 1960 [32]) Let $G$ be a simple graph. If for every distinct nonadjacent vertices $u$, $v$ of $G$, we have $d(u)+d(v) \geq|G|$, then $G$ has a Hamilton cycle.

Below are some of their digraph versions.
Theorem 2.3. (Ghouila-Houri, 1960 [16]) Let $D$ be a strong digraph. If the degree of every vertex of $D$ is at least $|D|$, then $D$ has a directed Hamilton cycle.

Theorem 2.4. ([3], Corollary 5.6.3) If $D$ is a digraph with $\delta^{0}(D) \geq|D| / 2$, then $D$ has a directed Hamilton cycle.
Theorem 2.5. (Woodall, 1972 [36]) Let $D$ be a digraph. If for every vertex pair $u$ and $v$, where there is no arc from $u$ to $v$, we have $d^{+}(u)+d^{-}(v) \geq|D|$, then $D$ has a directed Hamilton cycle.

It is not hard to verify that the bounds in above theorems are tight. Nash-Williams 31 raised the problem of describing all the extremal digraphs in Theorem 2.3, that is, all digraphs with minimum degree at least $|D|-1$, who do not have a directed Hamilton cycle. As a partial solution to this problem, Thomassen proved a structural theorem on the extremal graphs.

Theorem 2.6. (Thomassen, 1981 [34]) Let $D$ be a strong non-Hamiltonian digraph, with minimum degree $|D|-1$. Let $C$ be a longest directed cycle in $D$. Then any two vertices of $D-C$ are adjacent, every vertex of $D-C$ has degree $|D|-1$ (in $D$ ), and every component of $D-C$ is complete. Furthermore, if $D$ is strongly 2-connected, then $C$ can be chosen such that $D-C$ is a transitive tournament.

Darbinyan characterized the digraphs of even order that are extremal for both Theorem 2.3 and Theorem 2.4.

Theorem 2.7. (Darbinyan, 1986 [13]) Let $D$ be a digraph of even order such that the degree of every vertex of $D$ is at least $|D|-1$ and $\delta^{0}(D) \geq|D| / 2-1$. Then either $D$ is hamiltonian or $D$ belongs to a non-empty finite family of non-hamiltonian digraphs.

We study the extremal graphs of Theorem 2.5 in this paper. Compared with Theorem 2.6 and Theorem 2.7, we can completely determine all the extremal graphs.

For other results on degree sum conditions for the existence of Hamilton cycles in digraphs see [4], [5], [6], [13], [14], [29], 30], 37], [38], and a good summary in chapter 5 of [3].

Another interesting aspect of directed Hamilton cycle problems is their connection with the problem of matching alternating Hamilton cycles in bipartite graphs. Given a bipartite graph $G$ with a perfect matching $M$, if we orient the edges of $G$ towards the same part, then contracting all edges in $M$, we get a digraph $D$. An $M$-alternating Hamilton cycle of $G$ corresponds to a directed Hamilton cycle of $D$, and vice versa. Hence, Theorem 2.5 is equivalent to the following theorem.

Theorem 2.8. (Las Vergnas, 1972 [27]) Let $G=(B, W)$ be a balanced bipartite graph of order $\nu$. If for any $b \in B$ and $w \in W$, where $b$ and $w$ are nonadjacent, we have $d(w)+d(b) \geq \nu / 2+2$, then for every perfect matching $M$ of $G$, there is an $M$-alternating Hamilton cycle.

Hence, we also determine the extremal graphs for the result of Las Vergnas in this paper.
Theorem 2.8 is an instance of the problem of cycles containing matchings, which studies the conditions that enforce certain matchings to be contained in certain cycles. Some related works can be found in [1], [2], [8], [19], [20], [21], [33] and [35]. In particular, Berman proved the following.

Theorem 2.9. (Berman, 1983 [8]) Let $G$ be a graph on $\nu \geq 3$ vertices. If for any pair of independent vertices $x, y \in V(G)$, we have $d(x)+d(y) \geq \nu+1$, then every matching lies in a cycle.

Similarly to the above-mentioned works, Jackson and Wormald determined all the extremal graphs of a generalized version of Berman's result.

Theorem 2.10. (Jackson and Wormald, 1990 [20]) Let $G$ be a graph on $\nu$ vertices and $M$ be a matching of $G$ such that (1) $d(x)+d(y) \geq \nu$ for all pairs of independent vertices $x, y$ that are incident with $M$. Then $M$ is contained in a cycle of $G$ unless equality holds in (1) and several exceptional cases happen.

We will state our main results and their proofs in the following sections.

## 3 Main results

Let $m, n \geq 1$ be integers. Let $\mathcal{D}_{1}$ be the set of all digraphs obtained by identifying one vertex of $\overleftrightarrow{K}_{n+1}$ with one vertex of $\overleftrightarrow{K}_{m+1}$. Let $D_{2}$ be an arbitrary digraph on $n$ vertices, and take a copy of $I_{n+1}$. Let $\mathcal{D}_{2}$ be the set of all digraphs obtained by adding arcs of two directions between every vertex of $I_{n+1}$ and every vertex of $D_{2}$. Let $D_{3}$ be as shown in Figure 1, and take a copy of $\overleftrightarrow{K}_{n}$. Let $\mathcal{D}_{3}$ be the set of all graphs constructed by adding arcs of two directions between $v_{i}$, $i=0,1$, and every vertex of $\overleftrightarrow{K}_{n}$, and possibly, adding any of the $\operatorname{arcs}\left(v_{0}, v_{1}\right)$ and $\left(v_{1}, v_{0}\right)$, or both. Finally, let $D_{4}$ be the digraph showed in Figure 2. Our main result is as below.

Theorem 3.1. Let $D$ be a digraph. For every vertex pair $u$ and $v$, where there is no arc from u to $v$, we have $d^{+}(u)+d^{-}(v) \geq|D|-1$, then $D$ has a directed Hamilton cycle, unless $D \in \mathcal{D}_{1}$, $\mathcal{D}_{2}$ or $\mathcal{D}_{3}$, or $D=D_{4}$.

Let $\mathcal{G}_{1}$ be the class of graphs $G$ constructed by identifying an edge of one $K_{m+1, m+1}$ and one $K_{n+1, n+1}$, and $\mathcal{M}_{1}$ be the set of all perfect matchings of $G$ containing the identified edge. Let $\mathcal{G}_{2}$ be the class of graphs $G$, constructed by taking a copy of $(n+1) K_{2}$ with bipartition ( $B, W$ ), and an arbitrary bipartite graph $G_{2}$ with bipartition $\left(B_{1}, W_{1}\right)$, where $\left|B_{1}\right|=\left|W_{1}\right|=n$, which has at least one perfect matching, then connecting every vertex in $B$ to every vertex in $W_{1}$, and every vertex in $W$ to every vertex in $B_{1}$. Furthermore, let $\mathcal{M}_{2}$ be the set of all perfect matchings of $G$, containing all the edges in $(n+1) K_{2}$ (shown thick in Figure 3). Let $G_{3}$ be as shown in Figure 4 , and $\mathcal{G}_{3}$ the set of the graphs $G$ constructed by taking one copy of $K_{n, n}$ with bipartition $(B, W)$, and connecting every vertex in $B$ to $w_{0}$ and $w_{1}$, every vertex in $W$ to $b_{0}$ and $b_{1}$, and possibly, adding any of the edges $w_{0} b_{1}, w_{1} b_{0}$, or both. Let $\mathcal{M}_{3}$ be the set of perfect matchings of $G$, containing the thick edges in $G_{3}$. Finally, we let graph $G_{4}$ be the graph in Figure 5 , and $M_{4}$ the perfect matching of it, consisting of the thick edges. We have the following version of our main theorem.

Theorem 3.2. Let $G=(W, B)$ be a bipartite graph with a perfect matching $M$, for every vertex pair $w \in W$ and $b \in B$, where $w b-$, we have $d(w)+d(b) \geq \nu / 2+1$. Then $G$ has an $M$-alternating Hamilton cycle, unless one of the following holds.
(1) $G \in \mathcal{G}_{1}$, and $M \in \mathcal{M}_{1}$.
(2) $G \in \mathcal{G}_{2}$, and $M \in \mathcal{M}_{2}$.
(3) $G \in \mathcal{G}_{3}$, and $M \in \mathcal{M}_{3}$.
(4) $G=G_{4}$ and $M=M_{4}$.

Since the two results are equivalent, we only prove Theorem 3.2 in the next section. Before that, let's say a few words on the non-existence of $M$-alternating Hamilton cycles in the four exceptional cases. In Case (1), an $M$-alternating cycle of $G$ must contain the identified edge, whose endvertices form a vertex cut of $G$, so $G$ does not have an $M$-alternating Hamilton cycle. In Case (2), if there is an $M$-alternating Hamilton cycle $C$ of $G$, then the edges on $C$ that belong to $M$ must be in $(n+1) K_{2}$ and $G_{2}$ alternately, but there is one more such edge in $(n+1) K_{2}$, a contradiction. In Case (3), we can not have an $M$-alternating Hamilton cycle containing both $e_{0}$ and $e_{1}$. Finally in Case (4), the non-existence of any $M$-alternating Hamilton cycle can be verified directly.

## 4 Proof of Theorem 3.2

Let $G=(W, B)$ be a bipartite graph satisfying the condition of the theorem, $M$ a perfect matching of $G$. Suppose that $G$ does not have an $M$-alternating Hamilton cycle. We prove the theorem by characterizing $G$.

The following two lemmas will be used in our proof.


Figure 1: Exceptional graph family: $\mathcal{D}_{3}$


Figure 2: Exceptional graph $D_{4}$


Figure 3: Exceptional graph family: $\mathcal{G}_{2}$


Figure 4: Exceptional graph family: $\mathcal{G}_{3}$


Figure 5: Exceptional graph $G_{4}$

Lemma 4.1. Let $G=(W, B)$ be a bipartite graph with a perfect matching $M$. Let $C=$ $u_{0} u_{1} \ldots u_{2 m-1} u_{0}$ be a longest $M$-alternating cycle in $G$, where $u_{2 i} \in W, u_{2 i+1} \in B$, and $u_{2 i} u_{2 i+1} \in M, 0 \leq i \leq m-1$. Let $b \in B, w \in W$ be the ending vertices of a closed $M$-alternating path $P$ in $G-C$. Then, for every $0 \leq i \leq m-1$, either $u_{2 i} b-$ or $u_{2 i-1} w-$. Furthermore, if $b \rightarrow C$ and $w \rightarrow C$, then $\left|N_{C}(b)\right|+\left|N_{C}(w)\right| \leq m-|P| / 2+1$.

Proof. If there exists $0 \leq k \leq m-1$, such that $u_{2 k} b+$ and $u_{2 k-1} w+$, then $u_{2 k} C^{+} u_{2 k-1} w P b u_{2 k}$ is an $M$-alternating cycle longer than $C$, a contradiction. Thus, for $0 \leq i \leq m-1$, either $u_{2 i} b-$ or $u_{2 i-1} w-$.
If $b \rightarrow C$ and $w \rightarrow C$, let $u_{2 r} \in N_{C}(b)$ and $u_{2 s-1} \in N_{C}(w)$ be such that $P^{\prime}=u_{2 s} C^{+} u_{2 r-1}$ is the shortest. Then, there is no neighbor of $w$ and $b$ on $P^{\prime}$. Since $C$ is the longest, we have $\left|P^{\prime}\right| \geq|P|$. So $\left|N_{C}(w)\right|+\left|N_{C}(b)\right| \leq 2+\left(|C|-\left|P^{\prime}\right|-2\right) / 2=m-\left|P^{\prime}\right| / 2+1 \leq m-|P| / 2+1$.

Lemma 4.2. Let $G$ be a bipartite graph with a perfect matching $M$. Let $C=u_{0} u_{1} \ldots u_{2 m-1} u_{0}$ be a longest $M$-alternating cycle in $G$, where $u_{2 i} u_{2 i+1} \in M, 0 \leq i \leq m-1$. Let $C_{1}$ be an $M$-alternating cycle in $G-C$. For any vertex set $\left\{u_{2 i-1}, u_{2 i}\right\}, 0 \leq i \leq m-1$, either $u_{2 i-1} \nrightarrow C_{1}$ or $u_{2 i} \nrightarrow C_{1}$.

Proof. Suppose there exists $0 \leq k \leq m-1$ such that $u_{2 k-1} \rightarrow C_{1}$ and $u_{2 k} \rightarrow C_{1}$. Let $b \in$ $N_{C_{1}}\left(u_{2 k}\right)$ and $w \in N_{C_{1}}\left(u_{2 k-1}\right)$. We can always find a closed $M$-alternating path, $P$, as a segment of $C_{1}$, connecting $b$ and $w$. Then $u_{2 k} C^{+} u_{2 k-1} w P b u_{2 k}$ is an $M$-alternating cycle longer than $C$, contradicting our condition.

In our proof, some important intermediate results are shown as claims.
Claim 1. There is an $M$-alternating cycle in $G$ whose length is at least $\nu / 2+1$.
Proof. Let $P=u_{0} u_{1} \ldots u_{2 p-1}$ be a longest closed $M$-alternating path in $G$, then, all neighbors of $u_{0}$ and $u_{2 p-1}$ in $G$ should be on $P$.
If $u_{0} u_{2 p-1}+$, then we obtain a cycle $C=u_{0} u_{1} \ldots u_{2 p-1} u_{0}$. Since $P$ is the longest, $e(V(C), V(G-$ $C))=0$. However, $G$ is connected, so $C$ must be an $M$-alternating Hamilton cycle and the claim holds.
If $u_{0} u_{2 p-1}-$, by our condition, $d\left(u_{0}\right)+d\left(u_{2 p-1}\right) \geq \nu / 2+1$. Without lost of generality, assume that $d\left(u_{0}\right) \geq d\left(u_{2 p-1}\right)$ and let $u_{2 i-1}$ be the neighbor of $u_{0}$ with the maximum $i, 1 \leq i \leq p$. Then, $i \geq(\nu / 2+1) / 2$ and $u_{0} P u_{2 i-1} u_{0}$ is an $M$-alternating cycle with length at least $2 i \geq \nu / 2+1$. This proves our claim.

Now let $C=u_{0} u_{1} \ldots u_{2 m-1} u_{0}$ be a longest $M$-alternating cycle in $G$, where $u_{2 i} \in W, u_{2 i-1} \in$ $B$ and $u_{2 i} u_{2 i+1} \in M$. Let $G_{1}=G-C$. Denote the neighborhood and degree of $v \in V\left(G_{1}\right)$ in $G_{1}$ by $N_{1}(v)$ and $d_{1}(v)$. By Claim 1, $\left|G_{1}\right| \leq \nu / 2-1$.

Let $P_{1}=v_{0} v_{1} \ldots v_{2 p_{1}-1}$ be a longest closed $M$-alternating path in $G_{1}$, where $v_{2 i} \in W$ and $v_{2 i+1} \in B, 0 \leq i \leq p_{1}-1$. Then $N_{1}\left(v_{0}\right), N_{1}\left(v_{2 p_{1}-1}\right) \subseteq V\left(P_{1}\right)$, and $d_{1}\left(v_{0}\right), d_{1}\left(v_{2 p_{1}-1}\right) \leq p_{1}$. Firstly, we prove that $v_{0} \rightarrow C$ and $v_{2 p_{1}-1} \rightarrow C$.

If $v_{0} \nrightarrow C$ and $v_{2 p_{1}-1} \nrightarrow C$, then $d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right) \leq 2 p_{1} \leq\left|G_{1}\right| \leq \nu / 2-1$. By the condition of our theorem, $v_{0} v_{2 p_{1}-1}+$, and we get a cycle $C_{1}=v_{0} v_{1} \ldots v_{2 p_{1}-1} v_{0}$ in $G_{1}$. By Lemma 4.2, for any two vertices $u_{2 i-1}$ and $u_{2 i}$ on $C$, at least one of them, say $u_{2 i} \nrightarrow C_{1}$. Then $d\left(u_{2 i}\right) \leq \nu / 2-p_{1}$. But then $d\left(u_{2 i}\right)+d\left(v_{2 p_{1}-1}\right) \leq \nu / 2$, contradicting the condition of the theorem.

If only one of $v_{0}$ and $v_{2 p_{1}-1}$, say $v_{0} \rightarrow C$. Let a neighbor of $v_{0}$ on $C$ be $u_{2 j-1}$, by Lemma 4.1, $u_{2 j}$ sends no edge to $P_{1}$, so $d\left(u_{2 j}\right) \leq \nu / 2-p_{1}$, and $d\left(u_{2 j}\right)+d\left(v_{2 p_{1}-1}\right) \leq \nu / 2$, again contradicting the condition of the theorem.

Therefore $v_{0} \rightarrow C$ and $v_{2 p_{1}-1} \rightarrow C$.
By Lemma 4.1, $\left|N_{C}\left(v_{0}\right)\right|+\left|N_{C}\left(v_{2 p_{1}-1}\right)\right| \leq m-p_{1}+1$. Therefore,

$$
\begin{align*}
d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right) & \leq 2 p_{1}+\left(m-p_{1}+1\right) \\
& =m+p_{1}+1 \\
& \leq m+\left|G_{1}\right| / 2+1 \\
& =\nu / 2+1 \tag{1}
\end{align*}
$$

If $v_{0} v_{2 p_{1}-1}-$, then by our condition, $d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right) \geq \nu / 2+1$ and hence equalities in (1) hold. But then we must have $v_{0} v_{2 p_{1}-1}+$, a contradiction. So $v_{0} v_{2 p_{1}-1}+$, and we get a cycle $C_{1}=v_{0} v_{1} \ldots v_{2 p_{1}-1} v_{0}$.

If $G_{1}-C_{1}$ is nonempty, then there exists an edge $w b \in M \cap E\left(G_{1}-C_{1}\right)$, where $w \in W$ and $b \in B$. By the choice of $P_{1}, e\left(V\left(C_{1}\right), V\left(G_{1}-C_{1}\right)\right)=0$. By our condition, $d(w)+d(b)+$ $d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right) \geq 2(\nu / 2+1)=\nu+2$. However, by Lemma 4.1, $\left|N_{C}(w)\right|+\left|N_{C}(b)\right| \leq m$, and hence $d(w)+d(b) \leq\left|G_{1}\right|-2 p_{1}+m$, while $d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right) \leq m+p_{1}+1$ by (11), therefore $d(w)+d(b)+d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right) \leq\left|G_{1}\right|+2 m-p_{1}+1=\nu-p_{1}+1<\nu+1$, a contradiction. Hence, $G_{1}-C_{1}$ must be empty, then $\left|G_{1}\right|=2 p_{1}$ and $C_{1}$ is an $M$-alternating Hamilton cycle of $G_{1}$.

We claim that every vertex of $G_{1}$ sends some edges to $C$. Let $v$ be any vertex in $G_{1}$. Since $G_{1}$ has an $M$-alternating Hamilton cycle $C_{1}$, we can choose a closed $M$-alternating Hamilton path $P_{1}$ of $G_{1}$ starting from $v$. By above discussion, $v$ sends some edges to $C$.

For a longest $M$-alternating cycle $C$ in $G$, we call the graph $G_{1}=G-C$ a critical graph (with respect to $C$ ) and a closed $M$-alternating Hamilton path of $G_{1}, P_{1}=v_{0} v_{1} \ldots v_{2 p_{1}-1}$, where $v_{2 i} \in W$ and $v_{2 i+1} \in B$, a critical path, or a critical edge if $\left|P_{1}\right|=2$. For a critical path $P_{1}$, we can always find $u_{2 s-1} \in N_{C}\left(v_{0}\right)$ and $u_{2 r} \in N_{C}\left(v_{2 p_{1}-1}\right)$, such that $P_{2}=u_{2 s} C^{+} u_{2 r-1}$ is the shortest. We let $R=u_{2 r} C^{+} u_{2 s-1}$.

By Lemma 4.2, $u_{2 s} \nrightarrow G_{1}$ and $u_{2 r-1} \nrightarrow G_{1}$. Further, for any edge $u_{2 i-1} u_{2 i}$ on $R$, we must have $e\left(\left\{u_{2 i-1}, u_{2 i}\right\},\left\{u_{2 s}, u_{2 r-1}\right\}\right) \leq 1$, or we get an $M$-alternating Hamilton cycle

$$
u_{2 r} C^{+} u_{2 i-1} u_{2 s} C^{+} u_{2 r-1} u_{2 i} C^{+} u_{2 s-1} v_{0} P_{1} v_{2 p_{1}-1} u_{2 r}
$$

Hence,

$$
\begin{equation*}
d\left(u_{2 s}\right)+d\left(u_{2 r-1}\right) \leq\left|P_{2}\right|+2+(|R|-2) / 2=\left|P_{2}\right|+|R| / 2+1 . \tag{2}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right) \leq 2 p_{1}+2+(|R|-2) / 2=2 p_{1}+|R| / 2+1 . \tag{3}
\end{equation*}
$$

So,

$$
\begin{equation*}
d\left(u_{2 s}\right)+d\left(u_{2 r-1}\right)+d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right) \leq 2 p_{1}+\left|P_{2}\right|+|R|+2=\nu+2 . \tag{4}
\end{equation*}
$$

However $v_{0} u_{2 r-1}-$ and $v_{2 p_{1}-1} u_{2 s}-$, by our condition,

$$
\begin{equation*}
d\left(u_{2 s}\right)+d\left(u_{2 r-1}\right)+d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right) \geq 2(\nu / 2+1)=\nu+2 . \tag{5}
\end{equation*}
$$

So all equalities in (2), (3), (4) and (5) must hold. To get equality in (3), $v_{0}$ (respectively $v_{2 p_{1}-1}$ ) must be adjacent to all vertices in $V\left(G_{1}\right) \cap B$ (respectively $V\left(G_{1}\right) \cap W$ ). and for any edge $u_{2 i-1} u_{2 i}$ on $R, e\left(\left\{u_{2 i-1}, u_{2 i}\right\},\left\{v_{0}, v_{2 p_{1}-1}\right\}\right)=1$. Therefore, for a critical path $P_{1}=v_{0} v_{1} \ldots v_{2 p_{1}-1}$, we find two closed $M$-alternating paths $R$ and $P_{2}$ as segments of $C$, such that $V(C)=V(R) \cup V\left(P_{2}\right)$, where the ending vertices of $R$ is adjacent to $v_{0}$ and $v_{2 p_{1}-1}$, respectively, and for any edge $u_{2 i-1} u_{2 i} \notin M$ on $R, e\left(\left\{u_{2 i-1}, u_{2 i}\right\},\left\{v_{0}, v_{2 p_{1}-1}\right\}\right)=1$, while $e\left(V\left(P_{2}\right),\left\{v_{0}, v_{2 p_{1}-1}\right\}\right)=0$. We call $P_{2}$ the opposite path, and $R$ the central path for $P_{1}$.
Furthermore, to get equality in (2), $u_{2 s}$ (respectively $u_{2 r-1}$ ) must be adjacent to all vertices in $V\left(P_{2}\right) \cap B$ (respectively $V\left(P_{2}\right) \cap W$ ). In particular $u_{2 s} u_{2 r-1}+$.

Claim 2. A critical graph $G_{1}$ is complete bipartite.
Proof. Since $C_{1}$ is an $M$-alternating Hamilton cycle of $G_{1}$, for any vertex $v \in V\left(G_{1}\right), P_{1}$ can be chosen so that it is starting from $v$. By the equality of (3), $v$ sends edges to every vertex in the opposite part of $G_{1}$.

Let $G_{2}=G\left[V\left(P_{2}\right)\right]$. We call $G_{2}$ the opposite graph. We choose $C, G_{1}$ and $P_{1}$ so that the opposite path $P_{2}$ is the shortest.

Claim 3. $e\left(V\left(G_{1}\right), V\left(G_{2}\right)\right)=0$, and $u_{2 s-1}$ (respectively $u_{2 r}$ ) is adjacent to every vertex in $V\left(G_{1}\right) \cap W$ (respectively $\left.V\left(G_{1}\right) \cap B\right)$.

Proof. If $\left|G_{1}\right|=2$ the conclusion holds. We assume that $\left|G_{1}\right| \geq 4$.
For any closed $M$-alternating Hamilton path $P_{1}^{\prime}$ of $G_{1}$ with ending vertices $w \in W$ and $b \in B$, we can find an opposite path $P_{2}^{\prime}$ and a central path $R^{\prime}$ for $P_{1}^{\prime}$. Since $P_{2}$ is chosen as the shortest, $\left|P_{2}^{\prime}\right| \geq\left|P_{2}\right|$ and $\left|R^{\prime}\right| \leq|R|$. Similar to (3) we have

$$
\begin{equation*}
d(w)+d(b) \leq 2 p_{1}+\left|R^{\prime}\right| / 2+1 \leq 2 p_{1}+|R| / 2+1 . \tag{6}
\end{equation*}
$$

Together with (2), we have

$$
\begin{equation*}
d\left(u_{2 s}\right)+d\left(u_{2 r-1}\right)+d(w)+d(b) \leq \nu+2 . \tag{7}
\end{equation*}
$$

Since $u_{2 r}$ and $u_{2 s-1}$ send edges to $G_{1}$, which has an $M$-alternating Hamilton cycle, by Lemma 4.2. $u_{2 r-1} \nrightarrow G_{1}$ and $u_{2 s} \nrightarrow G_{1}$, and hence $w u_{2 r-1}-$ and $b u_{2 s}{ }^{-}$. By the condition given,

$$
\begin{equation*}
d\left(u_{2 s}\right)+d\left(u_{2 r-1}\right)+d(w)+d(b) \geq 2(\nu / 2+1)=\nu+2 . \tag{8}
\end{equation*}
$$

Hence all equalities in (6), (7) and (8) must hold. Therefore $|R|=\left|R^{\prime}\right|,\left|P_{2}^{\prime}\right|=\left|P_{2}\right|, d(w)=$ $d\left(v_{0}\right)=\nu / 2+1-d\left(u_{2 r-1}\right)$ and $d(b)=d\left(v_{2 p_{1}-1}\right)=\nu / 2+1-d\left(u_{2 s}\right)$. In other words, all opposite paths (respectively all central paths) have the same length. Since any vertex in $G_{1}$ can be an ending vertex of an $M$-alternating Hamilton path, all vertices in $V\left(G_{1}\right) \cap W$ have the same degree $\nu / 2+1-d\left(u_{2 r-1}\right)$, and all vertices in $V\left(G_{1}\right) \cap B$ have the same degree $\nu / 2+1-d\left(u_{2 s}\right)$.

Let $b \neq v_{2 p_{1}-1}$ be a vertex in $V\left(G_{1}\right) \cap B$, assume that $b$ has a neighbor $u_{2 r^{\prime}}$ on $P_{2}$. Since $G_{1}$ is complete bipartite we can always find a closed $M$-alternating path $P_{1}^{\prime \prime}$ connecting $v_{0}$ and $b$ in $G_{1}$. (Note that $P_{1}^{\prime \prime}$ need not to be Hamilton. If $b=v_{1}, P_{1}^{\prime \prime}$ can only be the edge $v_{0} v_{1}$.) Let $P_{2}^{\prime \prime}=u_{2 s} C^{+} u_{2 r^{\prime}-1}$ and $R^{\prime \prime}=u_{2 r^{\prime}} C^{+} u_{2 s-1}$. For any vertex pair $\left\{u_{2 i-1}, u_{2 i}\right\}$ on the path $R^{\prime \prime}$, we have $e\left(\left\{u_{2 i-1}, u_{2 i}\right\},\left\{u_{2 s}, u_{2 r^{\prime}-1}\right\}\right) \leq 1$, or we get an $M$-alternating cycle

$$
u_{2 r^{\prime}} C^{+} u_{2 i-1} u_{2 s} C^{+} u_{2 r^{\prime}-1} u_{2 i} C^{+} u_{2 s-1} v_{0} P_{1}^{\prime \prime} b u_{2 r^{\prime}}
$$

which is longer than $C$. Therefore,

$$
d\left(u_{2 s}\right)+d\left(u_{2 r^{\prime}-1}\right) \leq\left|P_{2}^{\prime \prime}\right|+2+\left(\left|R^{\prime \prime}\right|-2\right) / 2=\left|P_{2}^{\prime \prime}\right|+\left|R^{\prime \prime}\right| / 2+1<\left|P_{2}\right|+|R| / 2+1 .
$$

By $d\left(v_{0}\right)+d(b)=d\left(v_{0}\right)+d\left(v_{2 p_{1}-1}\right)=2 p_{1}+|R| / 2+1$, we have $d\left(u_{2 s}\right)+d\left(u_{2 r^{\prime}-1}\right)+d\left(v_{0}\right)+d(b)<$ $\left(\left|P_{2}\right|+|R| / 2+1\right)+\left(2 p_{1}+|R| / 2+1\right)=\nu+2$. However, since $u_{2 s} b-$ and $u_{2 r^{\prime}-1} v_{0}-$, by our condition, $d\left(u_{2 s}\right)+d\left(u_{2 r^{\prime}-1}\right)+d\left(v_{0}\right)+d(b) \geq \nu+2$, a contradiction. Hence $b$, and similarly any $w \in V\left(G_{1}\right) \cap W$, must not have any neighbor on $P_{2}$. That is, $e\left(V\left(G_{1}\right), V\left(G_{2}\right)\right)=0$.

For any closed $M$-alternating Hamilton path $P_{1}^{\prime}$ of $G_{1}$ with ending vertices $w \in W$ and $b \in B$, let $P_{2}^{\prime}$ be an opposite path of it. Since $w$ and $b$ send no edges to $P_{2}, P_{2}$ must be part of $P_{2}^{\prime}$. However, all opposite paths have the same length, so $\left|P_{2}^{\prime}\right|=\left|P_{2}\right|$, and therefore $P_{2}^{\prime}=P_{2}$. Then, $w u_{2 s-1}+$ and $b u_{2 r}+$. Since any vertex in $G_{1}$ can be an ending vertex of a closed $M$-alternating Hamilton path of $G_{1}$, we prove the second part of the claim.

Claim 4. $G_{2}$ is complete bipartite, and $u_{2 s-1}$ (respectively $u_{2 r}$ ) is adjacent to every vertex in $V\left(G_{2}\right) \cap W$ (respectively $\left.V\left(G_{2}\right) \cap B\right)$.

Proof. By above discussions, $u_{2 s} u_{2 r-1}+$ and we have a cycle $C_{2}=u_{2 s} C^{+} u_{2 r-1} u_{2 s}$. Since $e\left(V\left(G_{1}\right), V\left(G_{2}\right)\right)=0$, for every edge $u_{2 j-1} u_{2 j}$ on $P_{2}$, where $s+1 \leq j \leq r-1$, we can replace $u_{2 r-1}$ with $u_{2 j-1}$ and $u_{2 s}$ with $u_{2 j}$ in (2), (4) and (5), and all equalities must hold. So, $u_{2 j-1}\left(\right.$ respectively $\left.u_{2 j}\right)$ must be adjacent to all vertices in $V\left(P_{2}\right) \cap W$ (respectively $\left.V\left(P_{2}\right) \cap B\right)$, $u_{2 j-1} u_{2 r}+$ and $u_{2 j} u_{2 s-1}+$, therefore the claim holds.

For convenience we change some notations henceforth. We let $\left|G_{2}\right|=2 p_{2}$ and the vertices of $G_{2}$ be $v_{0}^{\prime}, v_{1}^{\prime}, \ldots, v_{2 p_{2}-1}^{\prime}$, where $v_{2 j}^{\prime} v_{2 j+1}^{\prime} \in M$, for $0 \leq j \leq p_{2}-1$, and let $R=u_{0} u_{1} \ldots u_{2 r-1}$.

Now we discuss the situations case by case, with respect to the length of $R$ and the distribution of edges between $R$ and $G_{i}, i=1,2$.
Case 1. $|R|=2$.
Then $R=u_{0} u_{1}$. By Claim 3 and Claim 4, For any $0 \leq i \leq p_{1}-1$ and $0 \leq j \leq p_{2}-1$, $u_{0} v_{2 i+1}+, u_{0} v_{2 j+1}^{\prime}+, u_{1} v_{2 i}+$ and $u_{1} v_{2 j}^{\prime}+$. Therefore $G \in \mathcal{G}_{1}$ and $M \in \mathcal{M}_{1}$.
Case 2. $|R| \geq 4$.
Claim 5. For $j=1,2$, and every edge $u_{2 i-1} u_{2 i}, 1 \leq i \leq r-1$, exactly one of $u_{2 i-1} \rightarrow G_{j}$ and $u_{2 i} \rightarrow G_{j}$ holds. Furthermore, if $u_{2 i-1} \rightarrow G_{j}$ (respectively $u_{2 i} \rightarrow G_{j}$ ), it is adjacent to all vertices in $V\left(G_{j}\right) \cap W$ (respectively $\left.V\left(G_{j}\right) \cap B\right)$.
Proof. Firstly, we prove that for $j=1,2$ and every edge $u_{2 i-1} u_{2 i}, 1 \leq i \leq r-1, u_{2 i-1} \nrightarrow G_{j}$ or $u_{2 i} \nrightarrow G_{j}$. By Lemma 4.2, the conclusion holds for $G_{1}$. Now we prove it for $G_{2}$. Suppose to the contrary that there exists $1 \leq l \leq r-1$ such that $u_{2 l-1} \rightarrow G_{2}$ and $u_{2 l} \rightarrow G_{2}$, and let $v_{2 s}^{\prime} \in N_{G_{2}}\left(u_{2 l-1}\right)$ and $v_{2 t+1}^{\prime} \in N_{G_{2}}\left(u_{2 l}\right)$. If $\left|G_{2}\right|=2$ or $t \neq s$, We can find a closed $M$-alternating Hamilton path $Q$ of $G_{2}$ connecting $v_{2 s}^{\prime}$ and $v_{2 t-1}^{\prime}$, and hence we have an $M$-alternating Hamilton cycle

$$
u_{0} R u_{2 l-1} v_{2 s}^{\prime} Q v_{2 t-1}^{\prime} u_{2 l} R u_{2 r-1} v_{0} P_{1} v_{2 p_{1}-1} u_{0}
$$

of $G$, contradicting our assumption. If $\left|G_{2}\right| \geq 4$ and $t=s$, let $P_{2}^{\prime}$ be a closed $M$-alternating Hamilton path of $G_{2}-\left\{v_{2 s}^{\prime}, v_{2 s+1}^{\prime}\right\}$. Then $P_{2}^{\prime}$ is an opposite path for $P_{1}$, with the central path $u_{0} R u_{2 l-1} v_{2 s}^{\prime} v_{2 s+1}^{\prime} u_{2 l} R u_{2 r-1}$, which is shorter than $P_{2}$, contradicting our choice of $P_{2}$. Hence $u_{2 i-1} \nrightarrow G_{2}$ or $u_{2 i} \nrightarrow G_{2}$, for $1 \leq i \leq r-1$.

Arbitrarily choose $0 \leq l \leq p_{1}-1$ and $0 \leq k \leq p_{2}-1$. We have $d\left(v_{2 l}\right)+d\left(v_{2 l+1}\right) \leq$ $2 p_{1}+2+(|R|-2) / 2=2 p_{1}+r+1$ and similarly $d\left(v_{2 k}^{\prime}\right)+d\left(v_{2 k+1}^{\prime}\right) \leq 2 p_{2}+r+1$. So

$$
\begin{equation*}
d\left(v_{2 l}\right)+d\left(v_{2 l+1}\right)+d\left(v_{2 k}^{\prime}\right)+d\left(v_{2 k+1}^{\prime}\right) \leq 2 p_{1}+2 p_{2}+2 r+2=\nu+2 \tag{9}
\end{equation*}
$$

However $v_{2 l} v_{2 k+1}^{\prime}-$ and $v_{2 l+1} v_{2 k}^{\prime}-$, by the condition of the theorem,

$$
\begin{equation*}
d\left(v_{2 l}\right)+d\left(v_{2 k+1}^{\prime}\right)+d\left(v_{2 l+1}\right)+d\left(v_{2 k}^{\prime}\right) \geq 2(\nu / 2+1)=\nu+2 \tag{10}
\end{equation*}
$$

and all equalities must hold. To obtain equalities, for $j=1,2$, and every edge $u_{2 i-1} u_{2 i}$, $1 \leq i \leq r-1$, exactly one of $u_{2 i-1} \rightarrow G_{j}$ and $u_{2 i} \rightarrow G_{j}$ must hold. Furthermore, since $l$ and $k$ are arbitrarily chosen, we prove that if $u_{2 i-1} \rightarrow G_{j}$ (respectively $u_{2 i} \rightarrow G_{j}$ ), it is adjacent to all vertices in $V\left(G_{j}\right) \cap W$ (respectively $V\left(G_{j}\right) \cap B$ ).

Let $1 \leq i \leq r-1$. We define $E_{1}\left(E_{1}^{\prime}\right)$ to be the set of edges $u_{2 i-1} u_{2 i}$, where $u_{2 i-1} v_{2 j}+$, for every $0 \leq j \leq p_{1}-1\left(u_{2 i-1} v_{2 k}^{\prime}+\right.$, for every $\left.0 \leq k \leq p_{2}-1\right)$, and $E_{2}\left(E_{2}^{\prime}\right)$ to be the set of edges $u_{2 i-1} u_{2 i}$, where $u_{2 i} v_{2 j+1}+$, for every $0 \leq j \leq p_{1}-1\left(u_{2 i} v_{2 k+1}^{\prime}+\right.$, for every $\left.0 \leq k \leq p_{2}-1\right)$.

By Claim 5, for every $1 \leq i \leq r-1, u_{2 i-1} u_{2 i} \in E_{1} \cap E_{1}^{\prime}, E_{1} \cap E_{2}^{\prime}, E_{2} \cap E_{1}^{\prime}$ or $E_{2} \cap E_{2}^{\prime}$. Accordingly, we say that $u_{2 i-1} u_{2 i}$ is an edge of type I, II, III or IV for $G_{1}, G_{2}$ and $R$. Let the number of edges $u_{2 i-1} u_{2 i}$ belonging to $E_{1} \cap E_{1}^{\prime}, E_{1} \cap E_{2}^{\prime}, E_{2} \cap E_{1}^{\prime}$ and $E_{2} \cap E_{2}^{\prime}$ be $t_{11}$, $t_{12}, t_{21}$ and $t_{22}$, respectively. We have $d\left(v_{0}\right)=t_{11}+t_{12}+p_{1}+1, d\left(v_{1}\right)=t_{22}+t_{21}+p_{1}+1$, $d\left(v_{0}^{\prime}\right)=t_{11}+t_{21}+p_{2}+1$ and $d\left(v_{1}^{\prime}\right)=t_{22}+t_{12}+p_{2}+1$.

Since equalities hold in (9) and 10 , we have $d\left(v_{2 l}\right)+d\left(v_{2 k+1}^{\prime}\right)=d\left(v_{2 l+1}\right)+d\left(v_{2 k}^{\prime}\right)=\nu / 2+1$ for any $0 \leq l \leq p_{1}-1$ and $0 \leq k \leq p_{2}-1$, Hence

$$
\begin{align*}
t_{11}+t_{22}+2 t_{12}+p_{1}+p_{2}+2 & =d\left(v_{0}\right)+d\left(v_{1}^{\prime}\right) \\
& =\nu / 2+1 \\
& =d\left(v_{1}\right)+d\left(v_{0}^{\prime}\right) \\
& =t_{11}+t_{22}+2 t_{21}+p_{1}+p_{2}+2 \tag{11}
\end{align*}
$$

So $t_{12}=t_{21}$.
We let $t_{1}=t_{11}, t_{2}=t_{22}$ and $t_{0}=t_{12}=t_{21}$, then $t_{1}+t_{2}+2 t_{0}=r-1$.

We summarise some structural results in the form of observations.
Observation 1. If there exists $1 \leq j<i \leq r-1$, such that $u_{2 i-1} u_{2 i} \in E_{1}$ ( $E_{1}^{\prime}$ ) and $u_{2 j-1} u_{2 j} \in$ $E_{2}^{\prime}\left(E_{2}\right)$. Then $u_{2 j-1} u_{2 i}-$.

Proof. If $u_{2 j-1} u_{2 i}+$, we obtain an $M$-alternating Hamilton cycle

$$
\begin{gathered}
u_{0} R u_{2 j-1} u_{2 i} R u_{2 r-1} v_{0}^{\prime} P_{2} v_{2 p_{2}-1}^{\prime} u_{2 j} R u_{2 i-1} v_{0} P_{1} v_{2 p_{1}-1} u_{0} \\
\left(u_{0} R u_{2 j-1} u_{2 i} R u_{2 r-1} v_{0} P_{1} v_{2 p_{1}-1} u_{2 j} R u_{2 i-1} v_{0}^{\prime} P_{2} v_{2 p_{2}-1}^{\prime} u_{0}\right)
\end{gathered}
$$

contradicting our assumption.
Observation 2. If there exists $1 \leq i \leq r-2$, such that $u_{2 i-1} u_{2 i} \in E_{1}$ and $u_{2 i+1} u_{2 i+2} \in E_{2}$, then $u_{2 i} u_{2 i+1}$ is a critical edge, $\left|G_{1}\right|=\left|G_{2}\right|=2$, and exactly one of $u_{2 i} v_{1}^{\prime}+$ and $u_{2 i+1} u_{0}+\left(u_{2 i+1} v_{0}^{\prime}+\right.$ and $u_{2 i} u_{2 r-1}+$ ) holds.

If there exists $1 \leq i \leq r-2$, such that $u_{2 i-1} u_{2 i} \in E_{1}^{\prime}$ and $u_{2 i+1} u_{2 i+2} \in E_{2}^{\prime}$, then $u_{2 i} u_{2 i+1}$ is a critical edge, $\left|G_{1}\right|=2$, and exactly one of $u_{2 i} v_{1}+$ and $u_{2 i+1} u_{0}+\left(u_{2 i+1} v_{0}+\right.$ and $\left.u_{2 i} u_{2 r-1}+\right)$ holds.

Proof. Suppose there exists $1 \leq i \leq r-2$, such that $u_{2 i-1} u_{2 i} \in E_{1}$ and $u_{2 i+1} u_{2 i+2} \in E_{2}$, then $u_{2 i} u_{2 i+1}$ is a critical edge with respect to the $M$-alternating cycle

$$
u_{0} R u_{2 i-1} v_{0} P_{1} v_{2 p_{1}-1} u_{2 i+2} R u_{2 r-1} v_{0}^{\prime} P_{2} v_{2 p_{2}-1}^{\prime} u_{0}
$$

where $P_{1}$ is an opposite path. Since $G_{1}$ is critical, $\left|G_{1}\right|=2$. Since $\left|P_{1}\right|=2$, and $P_{2}$ is the shortest opposite path, $\left|G_{2}\right|=2$. Since $u_{0} v_{1}^{\prime}\left(u_{2 r-1} v_{0}^{\prime}\right)$ are on a central path for the critical edge $u_{2 i} u_{2 i+1}$ and the opposite path $v_{0} v_{1}$, exactly one of $u_{2 i+1} u_{0}+$ and $u_{2 i} v_{1}^{\prime}+\left(u_{2 i+1} v_{0}^{\prime}+\right.$ and $\left.u_{2 i} u_{2 r-1}+\right)$ holds.

Now suppose there exists $1 \leq i \leq r-2$, such that $u_{2 i-1} u_{2 i} \in E_{1}^{\prime}$ and $u_{2 i+1} u_{2 i+2} \in E_{2}^{\prime}$. Then $u_{2 i} u_{2 i+1}$ is a critical edge with respect to the $M$-alternating cycle

$$
u_{0} R u_{2 i-1} v_{0}^{\prime} P_{2} v_{2 p_{2}-1}^{\prime} u_{2 i+2} R u_{2 r-1} v_{0} P_{1} v_{2 p_{1}-1} u_{0}
$$

where $P_{2}$ is an opposite path. Since $G_{1}$ is critical, $\left|G_{1}\right|=2$. Since $u_{0} v_{1}\left(u_{2 r-1} v_{0}\right)$ are on a central path for the critical edge $u_{2 i} u_{2 i+1}$ and the opposite path $P_{2}$, exactly one of $u_{2 i+1} u_{0}+$ and $u_{2 i} v_{1}+\left(u_{2 i+1} v_{0}+\right.$ and $\left.u_{2 i} u_{2 r-1}+\right)$ holds.

Observation 3. If there exists $1 \leq i<k<j \leq r-1$, such that $u_{2 i-1} u_{2 i} \in E_{1}\left(E_{1}^{\prime}\right), u_{2 j-1} u_{2 j} \in$ $E_{2}\left(E_{2}^{\prime}\right), u_{2 k-1} u_{2 k} \in E_{2}^{\prime}\left(E_{2}\right)$ and $u_{2 k-1} u_{0}+$, then $u_{2 i} u_{2 j-1}-$.
Proof. If $u_{2 i} u_{2 j-1}+$, we obtain an $M$-alternating Hamilton cycle

$$
u_{0} R u_{2 i-1} v_{0} P_{1} v_{2 p_{1}-1} u_{2 j} R u_{2 r-1} v_{0}^{\prime} P_{2} v_{2 p_{2}-1}^{\prime} u_{2 k} R u_{2 j-1} u_{2 i} R u_{2 k-1} u_{0},
$$

contradicting our assumption.
By symmetry, the claim holds under the other situation.
Claim 6. $\left|G_{1}\right|=2$.
Proof. Suppose $\left|G_{1}\right| \geq 4$. By Observation 2, there does not exist $1 \leq i \leq r-1$, such that $u_{2 i-1} u_{2 i} \in E_{1}\left(E_{1}^{\prime}\right)$ and $u_{2 i+1} u_{2 i+2} \in E_{2}\left(E_{2}^{\prime}\right)$. Therefore, there can not exist $i<j$, such that $u_{2 i-1} u_{2 i} \in E_{1}\left(E_{1}^{\prime}\right)$ and $u_{2 j-1} u_{2 j} \in E_{2}\left(E_{2}^{\prime}\right)$. In other words, there exits an integer $0 \leq k_{1} \leq r-1$ $\left(0 \leq k_{2} \leq r-1\right)$, such that for all $i \leq k_{1}\left(j \leq k_{2}\right), u_{2 i-1} u_{2 i} \in E_{2}\left(u_{2 j-1} u_{2 j} \in E_{2}^{\prime}\right)$ and for all $i>k_{1}\left(j>k_{2}\right), u_{2 i-1} u_{2 i} \in E_{1}\left(u_{2 j-1} u_{2 j} \in E_{1}^{\prime}\right)$. It is easily seen that $t_{0}=0$ and $k_{1}=k_{2}$. We let $k=k_{1}=k_{2}$.
Suppose that $t_{1}, t_{2} \neq 0$, or equally, $1 \leq k \leq r-2$. Consider the vertices $u_{2 k-1}$ and $u_{2 k+2}$. By Observation 1, for all $j \geq k+1, u_{2 k-1} u_{2 j-}$, and for all $j \leq k, u_{2 k+2} u_{2 j-1}-$. Particularly, $u_{2 k-1} u_{2 k+2}-$. But then we have $d\left(u_{2 k-1}\right) \leq k+1, d\left(u_{2 k+2}\right) \leq r-k$ and $d\left(u_{2 k-1}\right)+d\left(u_{2 k+2}\right) \leq$ $r+1<\nu / 2+1$, contradicting our condition.

Suppose one of $t_{1}$ and $t_{2}$, say $t_{1}=0$. Then for $1 \leq i \leq r-1, d\left(u_{2 i-1}\right) \leq r$. Moreover $d\left(v_{0}\right)=p_{1}+1$, so $d\left(u_{2 i-1}\right)+d\left(v_{0}\right) \leq r+p_{1}+1<\nu / 2+1$ but $v_{0} u_{2 i-1}-$, a contradiction.

So we must have $\left|G_{1}\right|=2$.
Claim 7. Either $t_{0}=0$, or $t_{1}=t_{2}=0$.
Proof. Suppose that $t_{0}>0$, and one of $t_{1}$ and $t_{2}$ is greater than 0 . Without lost of generality, we may assume that $t_{1} \geq t_{2}$, and so $t_{1}>0$.

Let $u_{2 i-1} u_{2 i} \in E_{1} \cap E_{1}^{\prime}, 1 \leq i \leq r-1$, be such that $i$ is the maximum. Then by our condition, $d\left(u_{2 i}\right)+d\left(v_{1}\right) \geq \nu / 2+1$. Hence, $d\left(u_{2 i}\right) \geq \nu / 2+1-d\left(v_{1}\right)=\nu / 2+1-\left(t_{2}+\right.$ $\left.t_{0}+2\right)=t_{1}+t_{0}+\nu / 2-r$. By Observation 1, $u_{2 i}$ can not be adjacent to any $u_{2 j-1}$, where $u_{2 j-1} u_{2 j} \in E_{2} \cup E_{2}^{\prime}$ and $j<i$. Hence $u_{2 i}$ sends at least $t_{1}+t_{0}+\nu / 2-r-\left(t_{1}+1\right)=t_{0}+\nu / 2-r-1$ edges to $\left\{u_{2 r-1}\right\} \cup\left\{u_{2 j-1}: u_{2 j-1} u_{2 j} \in E_{2} \cup E_{2}^{\prime}, j>i+1\right\}$. Since $t_{0}>0$ and $\nu / 2-r \geq 2$, $u_{2 i} \rightarrow\left\{u_{2 j-1}: u_{2 j-1} u_{2 j} \in E_{2} \cup E_{2}^{\prime}, j>i+1\right\}$, so there exists at least one $u_{2 j-1} u_{2 j}$ such that $j>i+1$ and $u_{2 j-1} u_{2 j} \in E_{2} \cup E_{2}^{\prime}$.

By our choice of $u_{2 i-1} u_{2 i}, u_{2 i+1} u_{2 i+2} \in E_{2} \cup E_{2}^{\prime}$. If $u_{2 i+1} u_{2 i+2} \in E_{2}$, then by Observation 2 , $u_{2 i} u_{2 i+1}$ is a critical edge, and exactly one of $u_{2 i} v_{1}^{\prime}+$ and $u_{2 i+1} u_{0}+$ holds. By $u_{2 i-1} u_{2 i} \in E_{1}^{\prime}$ we have $u_{2 i} v_{1}^{\prime}-$, therefore $u_{2 i+1} u_{0}+$. If $u_{2 i+1} u_{2 i+2} \in E_{2}^{\prime}$, then again by Observation $2, u_{2 i} u_{2 i+1}$ is a critical edge, and exactly one of $u_{2 i} v_{1}+$ and $u_{2 i+1} u_{0}+$ holds. By $u_{2 i-1} u_{2 i} \in E_{1}$ we have $u_{2 i} v_{1}-$, hence $u_{2 i+1} u_{0}+$.

Now we discuss different situations of $u_{2 i+1} u_{2 i+2}$.
If $u_{2 i+1} u_{2 i+2} \in E_{2} \cap E_{2}^{\prime}$, let $j>i+1$ be such that $u_{2 i} u_{2 j-1}+, u_{2 j-1} u_{2 j} \in E_{2} \cup E_{2}^{\prime}$. By Observation 3, $u_{2 i} u_{2 j-1}-$, a contradiction.
If $u_{2 i+1} u_{2 i+2} \in E_{1} \cap E_{2}^{\prime}$ or $E_{2} \cap E_{1}^{\prime}$, without lost of generality, we may assume that $u_{2 i+1} u_{2 i+2} \in$ $E_{1} \cap E_{2}^{\prime}$. Since $u_{2 i} u_{2 i+1}$ is a critical edge and $u_{2 i+1} v_{0}+$, by Observation 2, we have $u_{2 i} u_{2 r-1}-$. For $j>i+1$, where $u_{2 j-1} u_{2 j} \in E_{2}$, by Observation 3, $u_{2 i} u_{2 j-1}-$. Therefore $u_{2 i}$ sends at least $t_{0}+\nu / 2-r-1 \geq t_{0}+1$ edges to $\left\{u_{2 j-1}: u_{2 j-1} u_{2 j} \in E_{1} \cap E_{2}^{\prime}, j>i+1\right\}$. However, the number of such $u_{2 j-1}$ is at most $t_{0}$, a contradiction.

Case 2.1. $t_{0}=0$.
Without lost of generality, we may assume that $t_{1}>0$, and let $u_{2 i-1} u_{2 i} \in E_{1} \cap E_{1}^{\prime}$.
If there exists $u_{2 j-1} u_{2 j}, j<i$, such that $u_{2 j-1} u_{2 j} \in E_{2} \cap E_{2}^{\prime}$, then $u_{2 j-1} u_{2 i}$ - by Observation (1).

If there exists $u_{2 j-1} u_{2 j}, j>i+1$, such that $u_{2 j-1} u_{2 j} \in E_{2} \cap E_{2}^{\prime}$, then there exists $i \leq k \leq j-1$, such that $u_{2 k-1} u_{2 k} \in E_{1} \cap E_{1}^{\prime}$ and $u_{2 k+1} u_{2 k+2} \in E_{2} \cap E_{2}^{\prime}$. By Observation 2, $u_{2 k} u_{2 k+1}$ is a critical edge, and since $u_{2 k+1} v_{0}-$ and $u_{2 k} v_{1}-$, we have $u_{2 k} u_{2 r-1}+$ and $u_{2 k+1} u_{0}+$. By Observation 3. $u_{2 i} u_{2 j-1}-$.

Hence, for all $u_{2 j-1} u_{2 j} \in E_{2} \cap E_{2}^{\prime}, j \neq i+1, u_{2 i} u_{2 j-1}-$. So, $d\left(u_{2 i}\right) \leq t_{1}+2$. But then

$$
\begin{equation*}
\nu / 2+1 \leq d\left(u_{2 i}\right)+d\left(v_{1}\right) \leq t_{1}+2+t_{2}+2=\left(\nu-2 p_{2}-2-2\right) / 2+4=\nu / 2-p_{2}+2 . \tag{12}
\end{equation*}
$$

Since $p_{2} \geq 1$, all equalities must hold, hence $p_{2}=1$ and $2 r-1=\nu-5$. Furthermore, to get $d\left(u_{2 i}\right)=t_{1}+2$, we must have the following.
(a) $u_{2 i+1} u_{2 i+2} \in E_{2} \cap E_{2}^{\prime}$, hence $u_{2 i-1} u_{2 i} \neq u_{\nu-7} u_{\nu-6}$.
(b) $u_{2 i} u_{2 j-1}+$, for all $u_{2 j-1} u_{2 j} \in E_{1} \cap E_{1}^{\prime}$.
(c) $u_{2 i} u_{\nu-5}+$.

By (a), $t_{2} \geq 0$, and similarly, for any $u_{2 i-1} u_{2 i} \in E_{2} \cap E_{2}^{\prime}$, we can prove the following.
(d) $u_{2 i-3} u_{2 i-2} \in E_{1} \cap E_{1}^{\prime}$, hence $u_{2 i-1} u_{2 i} \neq u_{1} u_{2}$.
(e) $u_{2 i-1} u_{2 j}+$, for all $u_{2 j-1} u_{2 j} \in E_{2} \cap E_{2}^{\prime}$.
(f) $u_{2 i-1} u_{0}+$.

So, the edges $u_{2 i-1} u_{2 i}, 1 \leq i \leq \nu / 2-3$, belong to $E_{1} \cap E_{1}^{\prime}$ and $E_{2} \cap E_{2}^{\prime}$ alternatively. Moreover, $u_{1} u_{2} \in E_{1} \cap E_{1}^{\prime}$ and $u_{\nu-7} u_{\nu-6} \in E_{2} \cap E_{2}^{\prime}$. Hence we must have $\nu=4 n+2$, for some integer $n \geq 2$, $u_{4 j+1} u_{4 j+2} \in E_{1} \cap E_{1}^{\prime}$ and $u_{4 j+3} u_{4 j+4} \in E_{2} \cap E_{2}^{\prime}$ for $0 \leq j \leq n-2$. The vertex set $\left\{u_{4 j+1}, u_{4 j+2}\right.$ : $0 \leq j \leq n-2\} \cup\left\{v_{0}, v_{0}^{\prime}, u_{4 n-3}\right\}$, as well as $\left\{u_{4 j+3}, u_{4 j+4}: 0 \leq j \leq n-2\right\} \cup\left\{v_{1}, v_{1}^{\prime}, u_{0}\right\}$, induce complete bipartite subgraphs, respectively.

Let $B_{1}=\left\{u_{4 j+1}: 0 \leq j \leq n-1\right\}, W=\left\{u_{4 j+2}: 0 \leq j \leq n-2\right\} \cup\left\{v_{0}, v_{0} \prime\right\}, B=\left\{u_{4 j+3}: 0 \leq\right.$ $j \leq n-2\} \cup\left\{v_{1}, v_{1}^{\prime}\right\}$ and $W_{1}=\left\{u_{4 j}: 0 \leq j \leq n-1\right\}$. By above discussion, there can be no more edge between $B$ and $W$. But we can add edges between $B_{1}$ and $W_{1}$ freely, to obtain all graphs $G \in \mathcal{G}_{2}$, with $M \in \mathcal{M}_{2}$.

Case 2.2. $t_{1}=t_{2}=0$. Since $t_{1}+t_{2}+2 t_{0}=r-1$, we have $r=2 t_{0}+1$ and $r$ must be odd.
If there exists $1 \leq i \leq r-2$, such that $u_{2 i-1} u_{2 i} \in E_{1} \cap E_{2}^{\prime}$ and $u_{2 i+1} u_{2 i+2} \in E_{2} \cap E_{1}^{\prime}$ $\left(u_{2 i-1} u_{2 i} \in E_{2} \cap E_{1}^{\prime}\right.$ and $\left.u_{2 i+1} u_{2 i+2} \in E_{1} \cap E_{2}^{\prime}\right)$, we say that an A-change (B-change) occurs at $u_{2 i-1}$. If there exist $i$ and $j$, such that $2 \leq i+1<j \leq r-2$, and there is an A-change (B-change) occurs at $u_{2 i-1}$ and a B-change (A-change) occurs at $u_{2 j-1}$, we say that a change couple occurs at $\left(u_{2 i-1}, u_{2 j-1}\right)$.
Case 2.2.1. $\left|G_{2}\right| \geq 4$.
There can not be any A-change, or by Observation $2,\left|G_{1}\right|=\left|G_{2}\right|=2$. To avoid any A-change, for $1 \leq i \leq(r-1) / 2, u_{2 i-1} u_{2 i} \in E_{2} \cap E_{1}^{\prime}$ and for $(r+1) / 2 \leq i \leq r-1, u_{2 i-1} u_{2 i} \in E_{1} \cap E_{2}^{\prime}$.

Suppose that $r=3$. It is not hard to see that $u_{0} u_{3}-$ and $u_{2} u_{5}-$, while each of $u_{0} u_{5}$ and $u_{1} u_{4}$ can be exist or not. Hence we obtain all the graph in class $\mathcal{G}_{3}$, except those with $n=1$.

If $r \geq 5$, then $u_{r-1} u_{r}$ is a critical edge, with central path $u_{r+1} R u_{2 r-1} v_{0} v_{1} u_{0} R u_{r-2}$ and opposite graph $G_{2}$ (Figure 6). Consider the edge $v_{1} u_{0}$ and $u_{1} u_{2}$ on the central path. We have $v_{1} u_{r-1}+$, $u_{0} \rightarrow G_{2}, u_{1} \rightarrow G_{2}$, and by Claim[7, $u_{2} u_{r}+$. But then an A-change occurs at $v_{1}$, a contradiction. Case 2.2.2. $\left|G_{2}\right|=2$.

Then $\nu=4 n+6$, for some $n \geq 1$. For $n=1$, it is not hard to verify that $G \in \mathcal{G}_{3}, M \in \mathcal{M}_{3}$, and we obtain all graphs in $\mathcal{G}_{3}$ together with Case 2.2.1. For $n=2$, it can be checked that $G=G_{4}, M=M_{4}$. Henceforth we assume that $n \geq 3$, and then $r=2 n+1 \geq 7$.

We call $G_{1}$ and $G_{2}$ a critical edge pair with central path $R$. Since we have discussed all other cases, we may assume that for every critical edge pair and the central path, every edge of the central path that is not in $M$ is of type II or III.

Let there be a change couple occurs at $\left(u_{2 i-1}, u_{2 j-1}\right)$. Without lost of generality, suppose that an A-change occurs at $u_{2 i-1}$ and a B-change occurs at $u_{2 j-1}$, then $u_{2 i} u_{2 i+1}$ and $u_{2 j} u_{2 j+1}$ are critical edges. Since $u_{2 i} u_{2 i+1}$ and $v_{1} v_{0}$ is a critical edge pair, with the central path $u_{2 i+2} R u_{2 r-1} v_{0}^{\prime} v_{1}^{\prime} u_{0} R u_{2 i-1}$, by our assumption, $u_{2 j-1} u_{2 j}$ and $u_{2 j+1} u_{2 j+2}$ are of type II or III. By


Figure 6: Contradiction in Case 2.2.1
$u_{2 j} v_{1}+$ and $u_{2 j+1} v_{0}+$, we have $u_{2 j-1} u_{2 i}+$ and $u_{2 j+2} u_{2 i+1}+$. Similarly, we have $u_{2 i-1} u_{2 j}+$ and $u_{2 i+2} u_{2 j+1}+$. However, we get an $M$-alternating Hamilton cycle

$$
u_{0} R u_{2 i-1} u_{2 j} u_{2 j+1} u_{2 i+2} R u_{2 j-1} u_{2 i} u_{2 i+1} v_{0}^{\prime} v_{1}^{\prime} u_{2 j+2} R u_{2 r-1} v_{0} v_{1} u_{0}
$$

then, a contradiction. Therefore, there must not be any change couple.
By symmetry, we may assume that $u_{1} u_{2} \in E_{1} \cap E_{2}^{\prime}$, and let $r_{0}>0, r_{1}>r_{0}$ and $r_{2} \geq r_{1}$ be such that $u_{1} u_{2}, \ldots, u_{2 r_{0}-1} u_{2 r_{0}} \in E_{1} \cap E_{2}^{\prime}, u_{2 r_{0}+1} u_{2 r_{0}+2}, \ldots, u_{2 r_{1}-1} u_{2 r_{1}} \in E_{2} \cap E_{1}^{\prime}, u_{2 r_{1}+1} u_{2 r_{1}+2}$, $\ldots, u_{2 r_{2}-1} u_{2 r_{2}} \in E_{1} \cap E_{2}^{\prime}$ and if $u_{2 r_{2}+1} u_{2 r_{2}+2}$ exists, $u_{2 r_{2}+1} u_{2 r_{2}+2} \in E_{2} \cap E_{1}^{\prime}$.

If $r_{1}-r_{0} \geq 2$ and $r_{2}-r_{1} \geq 1$, then a change couple occurs at ( $u_{2 r_{0}-1}, u_{2 r_{1}-1}$ ), a contradiction. Hence, $r_{1}-r_{0}=1$ or $r_{2}=r_{1}$.

If $r_{1}-r_{0}=1$, then $r_{2}>r_{1}$, and the edge $u_{2 r_{2}+1} u_{2 r_{2}+2}$ exits. If $r_{2}-r_{1} \geq 2$, a change couple occurs at ( $u_{2 r_{1}-1}, u_{2 r_{2}-1}$ ), a contradiction. Therefore $r_{2}=r_{1}+1$. Moreover, if any B-change occurs at $u_{2 j-1}$ where $j \geq r_{2}+1$, we obtain a change couple ( $u_{2 r_{0}-1}, u_{2 j-1}$ ), again leading to a contradiction. Hence, we must have $u_{2 r_{2}+1} u_{2 r_{2}+2}, \ldots, u_{2 r-3} u_{2 r-2} \in E_{2} \cap E_{1}^{\prime}$, and then $r_{0}=(r-3) / 2, r_{1}=(r-1) / 2$ and $r_{2}=(r+1) / 2$.

Then $u_{r+1} u_{r+2}$ and $v_{1} v_{0}$ is a critical edge pair, with the central path $u_{r+3} R u_{2 r-1} v_{0}^{\prime} v_{1}^{\prime} u_{0} R u_{r}$. Again we may assume that the edge of the central path not in $M$ are of type II or III. Consider the edges $u_{r-4} u_{r-3}$ and $u_{r-2} u_{r-1}$, Since $u_{r-4} v_{0}+$ and $u_{r-1} v_{1}+$, we must have $u_{r-3} u_{r+2}+$ and $u_{r-2} u_{r+1}+$. Since $r \geq 7,2 r-3>r+3$. Consider the edges $u_{2 r-3} u_{2 r-2}$. Since $v_{1} u_{2 r-2}+$, we must have $u_{2 r-3} u_{r+1}+$. But then we find a change couple occur at ( $u_{2 r-3}, u_{r-4}$ ), a contradiction (Figure 7).


Figure 7: Critical pair $u_{r+1} u_{r+2}$ and $v_{1} v_{0}$
If $r_{2}=r_{1}$, then $u_{1} u_{2}, \ldots, u_{r-2} u_{r-1} \in E_{1} \cap E_{2}^{\prime}$ and $u_{r} u_{r+1}, \ldots, u_{2 r-3} u_{2 r-2} \in E_{2} \cap E_{1}^{\prime}$. Then, $u_{r-1} u_{r}$ and $v_{0} v_{1}$ is a critical pair, with the central path $u_{r+1} R u_{2 r-1} v_{0}^{\prime} v_{1}^{\prime} u_{0} R u_{r-2}$. For the edges
$u_{2 i-1} u_{2 i}$ with $(r+3) / 2 \leq i \leq r-1, v_{1} u_{2 i}+$, so we must have $u_{2 i-1} u_{r-1}+$. For the edges $u_{2 i-1} u_{2 i}$ with $1 \leq i \leq(r-3) / 2, v_{0} u_{2 i-1}+$, so we must have $u_{2 i} u_{r}+$. For the edge $u_{2 r-1} v_{0}^{\prime}$ and $v_{1}^{\prime} u_{0}$, we have $u_{2 r-1} v_{0}+, v_{0}^{\prime} u_{r}+, v_{1}^{\prime} u_{r-1}+$ and $u_{0} v_{1}+$. Thus we reach a same config with the case that $r_{1}-r_{0}=1$.

## 5 Final Remarks

Most of the degree sum conditions for Hamilton problems care about independent vertex sets. In our work, we try to strengthen the condition of our main theorem, by replacing "for every vertex pair $u$ and $v$, where there is not arc from $u$ to $v$ " with "for every vertex pair $u$ and $v$ ". Naturally, if the former condition guarantees hamiltonicity without exception, then such a strengthening brings nothing. But in the case where there are exceptions, we do find some differences. Let $\mathcal{D}_{1}^{\prime}$ be a subset of $\mathcal{D}_{1}$, in which $n=m$. Let $\mathcal{D}_{3}^{\prime}$ be a subset of $\mathcal{D}_{3}$, where $n=1$. We have the following result.

Theorem 5.1. Let $D$ be a digraph. If for every vertex pair $u$ and $v$, we have $d^{+}(u)+d^{-}(v) \geq$ $|D|-1$, then $D$ has a directed Hamilton cycle, unless $D \in \mathcal{D}_{1}^{\prime}, \mathcal{D}_{2}$ or $\mathcal{D}_{3}^{\prime}$, or $D=D_{4}$.

As a corollary, we can improve the Ore condition as well. Given a (undirected) graph $G$, if we replace every edge $u v \in E(G)$ with two arcs $u v$ and $v u$, we have a digraph $D$. Applying Theorem 3.1 on $D$, we obtain the following result.
Let $n, m \geq 1$, and $\mathcal{G}_{5}$ be the set of graphs obtained by identify one vertex of a complete graph $K_{m+1}$ and one vertex of a complete graph $K_{n+1}$, where $n, m \geq 1$. Let $\mathcal{G}_{6}$ be the set of all graphs obtained by joining every vertex of a graph $I_{n+1}$ to every vertex of an arbitrary graph on $n$ vertices.

Corollary 5.2. Let $G$ be a graph. If for every distinct nonadjacent vertex pair $u$ and $v$, we have $d(u)+d(v) \geq|G|-1$, then $G$ has a Hamilton cycle, unless $G \in \mathcal{G}_{5}$, or $G \in \mathcal{G}_{6}$.

A slightly stronger result can be found in [28]. There is only one exceptional class, for it considers only 2 -connected graphs.

Theorem 5.3. (Li, Li and Feng, 2007) Let $G$ be a 2-connected graph with $|G| \geq 3$. If $d(u)+$ $d(v) \geq|G|-1$ for every pair of vertices $u$ and $v$ with $d(u, v)=2$, then $G$ has a Hamilton cycle, unless $|G|$ is odd and $G \in \mathcal{G}_{6}$.

Stimulated by above results, we conjecture that the lower bound of degree sum in the following result can be reduced by 1 , with some exceptional cases.

Theorem 5.4. (Bang-Jensen, Gutin and Li, 1996 [5]) Let $D$ be a strong digraph such that for every pair of dominating non-adjacent and every pair of dominated non-adjacent vertices $\{u, v\}$, we have $\min \left\{d^{+}(u)+d^{-}(v), d^{-}(u)+d^{+}(v)\right\} \geq|D|$. Then $D$ has a directed Hamilton cycle.

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