Directed Hamilton cycles in digraphs and matching alternating Hamilton cycles in bipartite graphs *

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Abstract

In 1972, Woodall raised the following Ore type condition for directed Hamilton cycles in digraphs: Let D be a digraph. If for every vertex pair u and v, where there is no arc from u to v, we have $d^+(u) + d^-(v) \ge |D|$, then D has a directed Hamilton cycle. By a correspondence between bipartite graphs and digraphs, the above result is equivalent to the following result of Las Vergnas: Let G = (B, W) be a balanced bipartite graph. If for any $b \in B$ and $w \in W$, where b and w are nonadjacent, we have $d(w) + d(b) \ge |G|/2 + 1$, then every perfect matching of G is contained in a Hamilton cycle.

The lower bounds in both results are tight. In this paper, we reduce both bounds by 1, and prove that the conclusions still hold, with only a few exceptional cases that can be clearly characterized.

Key words: degree sum, Matching alternating Hamilton cycle, Hamilton cycle.

1 Introduction

Hamiltonian problems, and their many variations, have been studied extensively for more than half a century. The readers could refer to the surveys of Gould ([17] and [18]), Kawarabayashi ([22]) and Broersma ([11]) to trace the development in this field. Recently, approximate solutions of many traditional Hamiltonian problems and conjectures in digraphs came forth ([24], [23], [12] and [26]), which are surveyed by Kühn and Osthus ([25]).

Hamiltonicity and related properties are also important in practical applications. For example, in network design, the existence of Hamilton cycles in the underlying topology of an interconnection network provide advantage for the routing algorithm to make use of a ring structure, while the existence of a hamiltonian decomposition allows the load to be equally distributed, making network robust ([9]).

There are lots of degree or degree sum conditions for hamiltonicity. Often, the lower bounds in such conditions are best possible. However, we could still reduce the bounds and try to identify all exceptional graphs, that is, the extremal graphs for the conditions. Such kind of research often leads to the discovery of interesting topology structures. In this paper, we apply this idea to Woodall's condition for the existence of directed Hamilton cycles in digraphs.

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2 Terminology, notations and preliminary results

In this paper we consider finite, simple and connected graphs, and finite and simple digraphs. For the terminology not defined in this paper, the reader is referred to [10] and [3].

Let G be a graph with vertex set V(G) and edge set E(G). We denote by ν or |G| the order of V(G). For $u \in V(G)$, we denote by d(u) the degree of u, and N(u) or $N_G(u)$ the set of neighbors of u in G. For a subgraph H of G and a vertex $u \in V(G - H)$, we also denote by $N_H(u)$ the set of neighbors of u in H. For any two disjoint vertex sets X, Y of G we denote by e(X, Y) the number of edges of G from X to Y. For $u, v \in V(G)$, we denote by d(u, v) the distance between u and v, that is, the length of the shortest path connecting u and v. By uv+(uv-) we mean the vertices u and v are adjacent (nonadjacent). If a vertex u sends (no) edges to X, where X is a subgraph or a vertex subset of G, we write $u \to X$ $(u \not\to X)$. By nK_2 , we denote a graph consisting of n independent edges.

Let D be a digraph with vertex set V(D) and arc set A(D), u, v and w distinct vertices of D. We denote by |D| the order of V(D), $d^+(u)$ and $d^-(u)$ the out-degree and in-degree of u, respectively. The degree of u is the sum of its out-degree and in-degree. The minimum out-degree and in-degree of the vertices in D, is denoted by $\delta^+(D)$ and $\delta^-(D)$. We let $\delta^0(D) =$ $min\{\delta^+(D), \ \delta^-(D)\}$. Let (u, v) denote an arc from u to v. If $(u, v) \in A(D)$ or $(v, u) \in A(D)$, we say that u and v are adjacent. If $(w, u) \in A(D)$ and $(w, v) \in A(D)$, then we say that the pair $\{u, v\}$ is dominated, if $(u, w) \in A(D)$ and $(v, w) \in A(D)$, then we say that the pair $\{u, v\}$ is dominating. The complete digraph on $n \ge 1$ vertices, denoted by K_n , is obtained from the complete graph K_n by replacing every edge xy with two arcs (x, y) and (y, x). Without causing ambiguity, we use I_n to denote a graph or a digraph consisting of n independent vertices. A *transitive tournament* is an orientation of complete graph for which the vertices can be numbered in such a way that (i, j) is an edge if and only if i < j.

Let $C = u_0 u_1 \dots u_{m-1} u_0$ be a cycle in a graph G. Throughout this paper, the subscript of u_i is reduced modulo m. We always orient C such that u_{i+1} is the successor of u_i . For $0 \le i, j \le m-1$, the path $u_i u_{i+1} \dots u_j$ is denoted by $u_i C^+ u_j$, while the path $u_i u_{i-1} \dots u_j$ is denoted by $u_i C^- u_j$. For a path $P = v_0 v_1 \dots v_{p-1}$ and $0 \le i, j \le p-1$, the segment of P from v_i to v_j is denoted by $v_i P v_j$.

A matching M of G is a subset of E(G) in which no two elements are adjacent. If every $v \in V(G)$ is covered by an edge in M then M is said to be a perfect matching of G. For a matching M, an M-alternating path (M-alternating cycle) is a path (cycle) of which the edges appear alternately in M and $E(G) \setminus M$. We call an edge in M or an M-alternating path starting and ending with edges in M a closed M-alternating path, while an edge in $E(G) \setminus M$ or an M-alternating path starting and ending with edges in $E(G) \setminus M$ an open M-alternating path.

The following results of Dirac and Ore for the existence of Hamilton cycles in graphs are basic and famous.

Theorem 2.1. (Dirac, 1952 [15]) If G is a simple graph with $|G| \ge 3$ and every vertex of G has degree at least |G|/2, then G has a Hamilton cycle.

Theorem 2.2. (Ore, 1960 [32]) Let G be a simple graph. If for every distinct nonadjacent vertices u, v of G, we have $d(u) + d(v) \ge |G|$, then G has a Hamilton cycle.

Below are some of their digraph versions.

Theorem 2.3. (Ghouila-Houri, 1960 [16]) Let D be a strong digraph. If the degree of every vertex of D is at least |D|, then D has a directed Hamilton cycle.

Theorem 2.4. ([3], Corollary 5.6.3) If D is a digraph with $\delta^0(D) \ge |D|/2$, then D has a directed Hamilton cycle.

Theorem 2.5. (Woodall, 1972 [36]) Let D be a digraph. If for every vertex pair u and v, where there is no arc from u to v, we have $d^+(u) + d^-(v) \ge |D|$, then D has a directed Hamilton cycle.

It is not hard to verify that the bounds in above theorems are tight. Nash-Williams [31] raised the problem of describing all the extremal digraphs in Theorem 2.3, that is, all digraphs with minimum degree at least |D| - 1, who do not have a directed Hamilton cycle. As a partial solution to this problem, Thomassen proved a structural theorem on the extremal graphs.

Theorem 2.6. (Thomassen, 1981 [34]) Let D be a strong non-Hamiltonian digraph, with minimum degree |D| - 1. Let C be a longest directed cycle in D. Then any two vertices of D - Care adjacent, every vertex of D - C has degree |D| - 1 (in D), and every component of D - Cis complete. Furthermore, if D is strongly 2-connected, then C can be chosen such that D - Cis a transitive tournament.

Darbinyan characterized the digraphs of even order that are extremal for both Theorem 2.3 and Theorem 2.4.

Theorem 2.7. (Darbinyan, 1986 [13]) Let D be a digraph of even order such that the degree of every vertex of D is at least |D| - 1 and $\delta^0(D) \ge |D|/2 - 1$. Then either D is hamiltonian or D belongs to a non-empty finite family of non-hamiltonian digraphs.

We study the extremal graphs of Theorem 2.5 in this paper. Compared with Theorem 2.6 and Theorem 2.7, we can completely determine all the extremal graphs.

For other results on degree sum conditions for the existence of Hamilton cycles in digraphs see [4], [5], [6], [13], [14], [29], [30], [37], [38], and a good summary in chapter 5 of [3].

Another interesting aspect of directed Hamilton cycle problems is their connection with the problem of matching alternating Hamilton cycles in bipartite graphs. Given a bipartite graph G with a perfect matching M, if we orient the edges of G towards the same part, then contracting all edges in M, we get a digraph D. An M-alternating Hamilton cycle of G corresponds to a directed Hamilton cycle of D, and vice versa. Hence, Theorem 2.5 is equivalent to the following theorem.

Theorem 2.8. (Las Vergnas, 1972 [27]) Let G = (B, W) be a balanced bipartite graph of order ν . If for any $b \in B$ and $w \in W$, where b and w are nonadjacent, we have $d(w) + d(b) \ge \nu/2 + 2$, then for every perfect matching M of G, there is an M-alternating Hamilton cycle.

Hence, we also determine the extremal graphs for the result of Las Vergnas in this paper.

Theorem 2.8 is an instance of the problem of cycles containing matchings, which studies the conditions that enforce certain matchings to be contained in certain cycles. Some related works can be found in [1], [2], [8], [19], [20], [21], [33] and [35]. In particular, Berman proved the following.

Theorem 2.9. (Berman, 1983 [8]) Let G be a graph on $\nu \geq 3$ vertices. If for any pair of independent vertices $x, y \in V(G)$, we have $d(x) + d(y) \geq \nu + 1$, then every matching lies in a cycle.

Similarly to the above-mentioned works, Jackson and Wormald determined all the extremal graphs of a generalized version of Berman's result.

Theorem 2.10. (Jackson and Wormald, 1990 [20]) Let G be a graph on ν vertices and M be a matching of G such that (1) $d(x) + d(y) \ge \nu$ for all pairs of independent vertices x, y that are incident with M. Then M is contained in a cycle of G unless equality holds in (1) and several exceptional cases happen.

We will state our main results and their proofs in the following sections.

3 Main results

Let $m, n \ge 1$ be integers. Let \mathcal{D}_1 be the set of all digraphs obtained by identifying one vertex of K_{n+1} with one vertex of K_{m+1} . Let D_2 be an arbitrary digraph on n vertices, and take a copy of I_{n+1} . Let \mathcal{D}_2 be the set of all digraphs obtained by adding arcs of two directions between every vertex of I_{n+1} and every vertex of D_2 . Let D_3 be as shown in Figure 1, and take a copy of K_n . Let \mathcal{D}_3 be the set of all graphs constructed by adding arcs of two directions between v_i , i = 0, 1, and every vertex of K_n , and possibly, adding any of the arcs (v_0, v_1) and (v_1, v_0) , or both. Finally, let D_4 be the digraph showed in Figure 2. Our main result is as below.

Theorem 3.1. Let D be a digraph. For every vertex pair u and v, where there is no arc from u to v, we have $d^+(u) + d^-(v) \ge |D| - 1$, then D has a directed Hamilton cycle, unless $D \in \mathcal{D}_1$, \mathcal{D}_2 or \mathcal{D}_3 , or $D = D_4$.

Let \mathcal{G}_1 be the class of graphs G constructed by identifying an edge of one $K_{m+1,m+1}$ and one $K_{n+1,n+1}$, and \mathcal{M}_1 be the set of all perfect matchings of G containing the identified edge. Let \mathcal{G}_2 be the class of graphs G, constructed by taking a copy of $(n+1)K_2$ with bipartition (B, W), and an arbitrary bipartite graph G_2 with bipartition (B_1, W_1) , where $|B_1| = |W_1| = n$, which has at least one perfect matching, then connecting every vertex in B to every vertex in W_1 , and every vertex in W to every vertex in B_1 . Furthermore, let \mathcal{M}_2 be the set of all perfect matchings of G, containing all the edges in $(n+1)K_2$ (shown thick in Figure 3). Let G_3 be as shown in Figure 4, and \mathcal{G}_3 the set of the graphs G constructed by taking one copy of $K_{n,n}$ with bipartition (B, W), and connecting every vertex in B to w_0 and w_1 , every vertex in W to b_0 and b_1 , and possibly, adding any of the edges w_0b_1 , w_1b_0 , or both. Let \mathcal{M}_3 be the set of perfect matchings of G, containing the thick edges in G_3 . Finally, we let graph G_4 be the graph in Figure 5, and \mathcal{M}_4 the perfect matching of it, consisting of the thick edges. We have the following version of our main theorem.

Theorem 3.2. Let G = (W, B) be a bipartite graph with a perfect matching M, for every vertex pair $w \in W$ and $b \in B$, where wb-, we have $d(w)+d(b) \ge \nu/2+1$. Then G has an M-alternating Hamilton cycle, unless one of the following holds.

(1) $G \in \mathcal{G}_1$, and $M \in \mathcal{M}_1$. (2) $G \in \mathcal{G}_2$, and $M \in \mathcal{M}_2$. (3) $G \in \mathcal{G}_3$, and $M \in \mathcal{M}_3$. (4) $G = G_4$ and $M = M_4$.

Since the two results are equivalent, we only prove Theorem 3.2 in the next section. Before that, let's say a few words on the non-existence of M-alternating Hamilton cycles in the four exceptional cases. In Case (1), an M-alternating cycle of G must contain the identified edge, whose endvertices form a vertex cut of G, so G does not have an M-alternating Hamilton cycle. In Case (2), if there is an M-alternating Hamilton cycle C of G, then the edges on C that belong to M must be in $(n + 1)K_2$ and G_2 alternately, but there is one more such edge in $(n + 1)K_2$, a contradiction. In Case (3), we can not have an M-alternating Hamilton cycle containing both e_0 and e_1 . Finally in Case (4), the non-existence of any M-alternating Hamilton cycle can be verified directly.

4 Proof of Theorem 3.2

Let G = (W, B) be a bipartite graph satisfying the condition of the theorem, M a perfect matching of G. Suppose that G does not have an M-alternating Hamilton cycle. We prove the theorem by characterizing G.

The following two lemmas will be used in our proof.

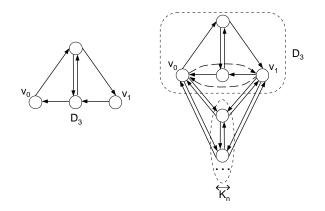


Figure 1: Exceptional graph family: \mathcal{D}_3

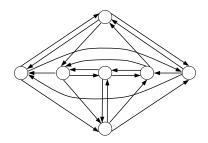


Figure 2: Exceptional graph D_4

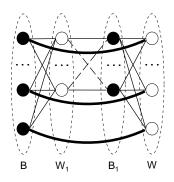


Figure 3: Exceptional graph family: \mathcal{G}_2

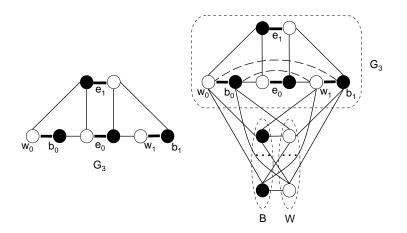


Figure 4: Exceptional graph family: \mathcal{G}_3

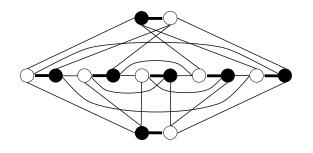


Figure 5: Exceptional graph G_4

Lemma 4.1. Let G = (W, B) be a bipartite graph with a perfect matching M. Let $C = u_0u_1 \ldots u_{2m-1}u_0$ be a longest M-alternating cycle in G, where $u_{2i} \in W$, $u_{2i+1} \in B$, and $u_{2i}u_{2i+1} \in M$, $0 \le i \le m-1$. Let $b \in B$, $w \in W$ be the ending vertices of a closed M-alternating path P in G - C. Then, for every $0 \le i \le m-1$, either $u_{2i}b$ - or $u_{2i-1}w$ -. Furthermore, if $b \to C$ and $w \to C$, then $|N_C(b)| + |N_C(w)| \le m - |P|/2 + 1$.

Proof. If there exists $0 \le k \le m - 1$, such that $u_{2k}b_+$ and $u_{2k-1}w_+$, then $u_{2k}C^+u_{2k-1}w_{2k}b_{2k}$ is an *M*-alternating cycle longer than *C*, a contradiction. Thus, for $0 \le i \le m - 1$, either $u_{2i}b_-$ or $u_{2i-1}w_-$.

If $b \to C$ and $w \to C$, let $u_{2r} \in N_C(b)$ and $u_{2s-1} \in N_C(w)$ be such that $P' = u_{2s}C^+u_{2r-1}$ is the shortest. Then, there is no neighbor of w and b on P'. Since C is the longest, we have $|P'| \ge |P|$. So $|N_C(w)| + |N_C(b)| \le 2 + (|C| - |P'| - 2)/2 = m - |P'|/2 + 1 \le m - |P|/2 + 1$. \Box

Lemma 4.2. Let G be a bipartite graph with a perfect matching M. Let $C = u_0u_1 \dots u_{2m-1}u_0$ be a longest M-alternating cycle in G, where $u_{2i}u_{2i+1} \in M$, $0 \le i \le m-1$. Let C_1 be an M-alternating cycle in G-C. For any vertex set $\{u_{2i-1}, u_{2i}\}, 0 \le i \le m-1$, either $u_{2i-1} \nleftrightarrow C_1$ or $u_{2i} \nleftrightarrow C_1$.

Proof. Suppose there exists $0 \le k \le m-1$ such that $u_{2k-1} \to C_1$ and $u_{2k} \to C_1$. Let $b \in N_{C_1}(u_{2k})$ and $w \in N_{C_1}(u_{2k-1})$. We can always find a closed *M*-alternating path, *P*, as a segment of C_1 , connecting *b* and *w*. Then $u_{2k}C^+u_{2k-1}wPbu_{2k}$ is an *M*-alternating cycle longer than *C*, contradicting our condition.

In our proof, some important intermediate results are shown as claims.

Claim 1. There is an M-alternating cycle in G whose length is at least $\nu/2 + 1$.

Proof. Let $P = u_0 u_1 \dots u_{2p-1}$ be a longest closed *M*-alternating path in *G*, then, all neighbors of u_0 and u_{2p-1} in *G* should be on *P*.

If u_0u_{2p-1} +, then we obtain a cycle $C = u_0u_1 \dots u_{2p-1}u_0$. Since P is the longest, e(V(C), V(G-C)) = 0. However, G is connected, so C must be an M-alternating Hamilton cycle and the claim holds.

If u_0u_{2p-1} , by our condition, $d(u_0) + d(u_{2p-1}) \ge \nu/2 + 1$. Without lost of generality, assume that $d(u_0) \ge d(u_{2p-1})$ and let u_{2i-1} be the neighbor of u_0 with the maximum $i, 1 \le i \le p$. Then, $i \ge (\nu/2 + 1)/2$ and $u_0Pu_{2i-1}u_0$ is an *M*-alternating cycle with length at least $2i \ge \nu/2 + 1$. This proves our claim.

Now let $C = u_0 u_1 \dots u_{2m-1} u_0$ be a longest *M*-alternating cycle in *G*, where $u_{2i} \in W$, $u_{2i-1} \in B$ and $u_{2i} u_{2i+1} \in M$. Let $G_1 = G - C$. Denote the neighborhood and degree of $v \in V(G_1)$ in G_1 by $N_1(v)$ and $d_1(v)$. By Claim 1, $|G_1| \leq \nu/2 - 1$.

Let $P_1 = v_0 v_1 \dots v_{2p_1-1}$ be a longest closed *M*-alternating path in G_1 , where $v_{2i} \in W$ and $v_{2i+1} \in B$, $0 \leq i \leq p_1 - 1$. Then $N_1(v_0)$, $N_1(v_{2p_1-1}) \subseteq V(P_1)$, and $d_1(v_0)$, $d_1(v_{2p_1-1}) \leq p_1$. Firstly, we prove that $v_0 \to C$ and $v_{2p_1-1} \to C$.

If $v_0 \not\rightarrow C$ and $v_{2p_1-1} \not\rightarrow C$, then $d(v_0) + d(v_{2p_1-1}) \leq 2p_1 \leq |G_1| \leq \nu/2 - 1$. By the condition of our theorem, $v_0v_{2p_1-1}+$, and we get a cycle $C_1 = v_0v_1 \dots v_{2p_1-1}v_0$ in G_1 . By Lemma 4.2, for any two vertices u_{2i-1} and u_{2i} on C, at least one of them, say $u_{2i} \not\rightarrow C_1$. Then $d(u_{2i}) \leq \nu/2 - p_1$. But then $d(u_{2i}) + d(v_{2p_1-1}) \leq \nu/2$, contradicting the condition of the theorem.

If only one of v_0 and v_{2p_1-1} , say $v_0 \to C$. Let a neighbor of v_0 on C be u_{2j-1} , by Lemma 4.1, u_{2j} sends no edge to P_1 , so $d(u_{2j}) \leq \nu/2 - p_1$, and $d(u_{2j}) + d(v_{2p_1-1}) \leq \nu/2$, again contradicting the condition of the theorem.

Therefore $v_0 \to C$ and $v_{2p_1-1} \to C$.

By Lemma 4.1, $|N_C(v_0)| + |N_C(v_{2p_1-1})| \le m - p_1 + 1$. Therefore,

$$d(v_0) + d(v_{2p_1-1}) \leq 2p_1 + (m - p_1 + 1)$$

= $m + p_1 + 1$
 $\leq m + |G_1|/2 + 1$
= $\nu/2 + 1.$ (1)

If $v_0v_{2p_1-1}-$, then by our condition, $d(v_0) + d(v_{2p_1-1}) \ge \nu/2 + 1$ and hence equalities in (1) hold. But then we must have $v_0v_{2p_1-1}+$, a contradiction. So $v_0v_{2p_1-1}+$, and we get a cycle $C_1 = v_0v_1 \dots v_{2p_1-1}v_0$.

If $G_1 - C_1$ is nonempty, then there exists an edge $wb \in M \cap E(G_1 - C_1)$, where $w \in W$ and $b \in B$. By the choice of P_1 , $e(V(C_1), V(G_1 - C_1)) = 0$. By our condition, $d(w) + d(b) + d(v_0) + d(v_{2p_1-1}) \ge 2(\nu/2 + 1) = \nu + 2$. However, by Lemma 4.1, $|N_C(w)| + |N_C(b)| \le m$, and hence $d(w) + d(b) \le |G_1| - 2p_1 + m$, while $d(v_0) + d(v_{2p_1-1}) \le m + p_1 + 1$ by (1), therefore $d(w) + d(b) + d(v_0) + d(v_{2p_1-1}) \le |G_1| + 2m - p_1 + 1 = \nu - p_1 + 1 < \nu + 1$, a contradiction. Hence, $G_1 - C_1$ must be empty, then $|G_1| = 2p_1$ and C_1 is an *M*-alternating Hamilton cycle of G_1 .

We claim that every vertex of G_1 sends some edges to C. Let v be any vertex in G_1 . Since G_1 has an M-alternating Hamilton cycle C_1 , we can choose a closed M-alternating Hamilton path P_1 of G_1 starting from v. By above discussion, v sends some edges to C.

For a longest *M*-alternating cycle *C* in *G*, we call the graph $G_1 = G - C$ a critical graph (with respect to *C*) and a closed *M*-alternating Hamilton path of G_1 , $P_1 = v_0v_1 \dots v_{2p_1-1}$, where $v_{2i} \in W$ and $v_{2i+1} \in B$, a critical path, or a critical edge if $|P_1| = 2$. For a critical path P_1 , we can always find $u_{2s-1} \in N_C(v_0)$ and $u_{2r} \in N_C(v_{2p_1-1})$, such that $P_2 = u_{2s}C^+u_{2r-1}$ is the shortest. We let $R = u_{2r}C^+u_{2s-1}$.

By Lemma 4.2, $u_{2s} \nleftrightarrow G_1$ and $u_{2r-1} \nleftrightarrow G_1$. Further, for any edge $u_{2i-1}u_{2i}$ on R, we must have $e(\{u_{2i-1}, u_{2i}\}, \{u_{2s}, u_{2r-1}\}) \leq 1$, or we get an *M*-alternating Hamilton cycle

$$u_{2r}C^+u_{2i-1}u_{2s}C^+u_{2r-1}u_{2i}C^+u_{2s-1}v_0P_1v_{2p_1-1}u_{2r}.$$

Hence,

$$d(u_{2s}) + d(u_{2r-1}) \le |P_2| + 2 + (|R| - 2)/2 = |P_2| + |R|/2 + 1.$$
(2)

Moreover,

$$d(v_0) + d(v_{2p_1-1}) \le 2p_1 + 2 + (|R| - 2)/2 = 2p_1 + |R|/2 + 1.$$
(3)

So,

$$d(u_{2s}) + d(u_{2r-1}) + d(v_0) + d(v_{2p_1-1}) \le 2p_1 + |P_2| + |R| + 2 = \nu + 2.$$
(4)

However $v_0 u_{2r-1}$ - and $v_{2p_1-1} u_{2s}$ -, by our condition,

$$d(u_{2s}) + d(u_{2r-1}) + d(v_0) + d(v_{2p_1-1}) \ge 2(\nu/2 + 1) = \nu + 2.$$
(5)

So all equalities in (2), (3), (4) and (5) must hold. To get equality in (3), v_0 (respectively v_{2p_1-1}) must be adjacent to all vertices in $V(G_1) \cap B$ (respectively $V(G_1) \cap W$). and for any edge $u_{2i-1}u_{2i}$ on R, $e(\{u_{2i-1}, u_{2i}\}, \{v_0, v_{2p_1-1}\}) = 1$. Therefore, for a critical path $P_1 = v_0v_1 \dots v_{2p_1-1}$, we find two closed M-alternating paths R and P_2 as segments of C, such that $V(C) = V(R) \cup V(P_2)$, where the ending vertices of R is adjacent to v_0 and v_{2p_1-1} , respectively, and for any edge $u_{2i-1}u_{2i} \notin M$ on R, $e(\{u_{2i-1}, u_{2i}\}, \{v_0, v_{2p_1-1}\}) = 1$, while $e(V(P_2), \{v_0, v_{2p_1-1}\}) = 0$. We call P_2 the opposite path, and R the central path for P_1 .

Furthermore, to get equality in (2), u_{2s} (respectively u_{2r-1}) must be adjacent to all vertices in $V(P_2) \cap B$ (respectively $V(P_2) \cap W$). In particular $u_{2s}u_{2r-1}$ +.

Claim 2. A critical graph G_1 is complete bipartite.

Proof. Since C_1 is an *M*-alternating Hamilton cycle of G_1 , for any vertex $v \in V(G_1)$, P_1 can be chosen so that it is starting from v. By the equality of (3), v sends edges to every vertex in the opposite part of G_1 .

Let $G_2 = G[V(P_2)]$. We call G_2 the opposite graph. We choose C, G_1 and P_1 so that the opposite path P_2 is the shortest.

Claim 3. $e(V(G_1), V(G_2)) = 0$, and u_{2s-1} (respectively u_{2r}) is adjacent to every vertex in $V(G_1) \cap W$ (respectively $V(G_1) \cap B$).

Proof. If $|G_1| = 2$ the conclusion holds. We assume that $|G_1| \ge 4$.

For any closed *M*-alternating Hamilton path P'_1 of G_1 with ending vertices $w \in W$ and $b \in B$, we can find an opposite path P'_2 and a central path R' for P'_1 . Since P_2 is chosen as the shortest, $|P'_2| \geq |P_2|$ and $|R'| \leq |R|$. Similar to (3) we have

$$d(w) + d(b) \le 2p_1 + |R'|/2 + 1 \le 2p_1 + |R|/2 + 1.$$
(6)

Together with (2), we have

$$d(u_{2s}) + d(u_{2r-1}) + d(w) + d(b) \le \nu + 2.$$
(7)

Since u_{2r} and u_{2s-1} send edges to G_1 , which has an *M*-alternating Hamilton cycle, by Lemma 4.2, $u_{2r-1} \rightarrow G_1$ and $u_{2s} \rightarrow G_1$, and hence wu_{2r-1} - and bu_{2s} -. By the condition given,

$$d(u_{2s}) + d(u_{2r-1}) + d(w) + d(b) \ge 2(\nu/2 + 1) = \nu + 2.$$
(8)

Hence all equalities in (6), (7) and (8) must hold. Therefore |R| = |R'|, $|P'_2| = |P_2|$, $d(w) = d(v_0) = \nu/2 + 1 - d(u_{2r-1})$ and $d(b) = d(v_{2p_1-1}) = \nu/2 + 1 - d(u_{2s})$. In other words, all opposite paths (respectively all central paths) have the same length. Since any vertex in G_1 can be an ending vertex of an *M*-alternating Hamilton path, all vertices in $V(G_1) \cap W$ have the same degree $\nu/2 + 1 - d(u_{2r-1})$, and all vertices in $V(G_1) \cap B$ have the same degree $\nu/2 + 1 - d(u_{2s})$.

Let $b \neq v_{2p_1-1}$ be a vertex in $V(G_1) \cap B$, assume that b has a neighbor $u_{2r'}$ on P_2 . Since G_1 is complete bipartite we can always find a closed *M*-alternating path P''_1 connecting v_0 and b in G_1 . (Note that P''_1 need not to be Hamilton. If $b = v_1$, P''_1 can only be the edge v_0v_1 .) Let $P''_2 = u_{2s}C^+u_{2r'-1}$ and $R'' = u_{2r'}C^+u_{2s-1}$. For any vertex pair $\{u_{2i-1}, u_{2i}\}$ on the path R'', we have $e(\{u_{2i-1}, u_{2i}\}, \{u_{2s}, u_{2r'-1}\}) \leq 1$, or we get an *M*-alternating cycle

$$u_{2r'}C^+u_{2i-1}u_{2s}C^+u_{2r'-1}u_{2i}C^+u_{2s-1}v_0P_1''bu_{2r'},$$

which is longer than C. Therefore,

$$d(u_{2s}) + d(u_{2r'-1}) \le |P_2''| + 2 + (|R''| - 2)/2 = |P_2''| + |R''|/2 + 1 < |P_2| + |R|/2 + 1.$$

By $d(v_0) + d(b) = d(v_0) + d(v_{2p_1-1}) = 2p_1 + |R|/2 + 1$, we have $d(u_{2s}) + d(u_{2r'-1}) + d(v_0) + d(b) < (|P_2| + |R|/2 + 1) + (2p_1 + |R|/2 + 1) = \nu + 2$. However, since $u_{2s}b$ - and $u_{2r'-1}v_0$ -, by our condition, $d(u_{2s}) + d(u_{2r'-1}) + d(v_0) + d(b) \ge \nu + 2$, a contradiction. Hence b, and similarly any $w \in V(G_1) \cap W$, must not have any neighbor on P_2 . That is, $e(V(G_1), V(G_2)) = 0$.

For any closed *M*-alternating Hamilton path P'_1 of G_1 with ending vertices $w \in W$ and $b \in B$, let P'_2 be an opposite path of it. Since w and b send no edges to P_2 , P_2 must be part of P'_2 . However, all opposite paths have the same length, so $|P'_2| = |P_2|$, and therefore $P'_2 = P_2$. Then, wu_{2s-1} + and bu_{2r} +. Since any vertex in G_1 can be an ending vertex of a closed *M*-alternating Hamilton path of G_1 , we prove the second part of the claim.

Claim 4. G_2 is complete bipartite, and u_{2s-1} (respectively u_{2r}) is adjacent to every vertex in $V(G_2) \cap W$ (respectively $V(G_2) \cap B$).

Proof. By above discussions, $u_{2s}u_{2r-1}$ and we have a cycle $C_2 = u_{2s}C^+u_{2r-1}u_{2s}$. Since $e(V(G_1), V(G_2)) = 0$, for every edge $u_{2j-1}u_{2j}$ on P_2 , where $s + 1 \leq j \leq r - 1$, we can replace u_{2r-1} with u_{2j-1} and u_{2s} with u_{2j} in (2), (4) and (5), and all equalities must hold. So, u_{2j-1} (respectively u_{2j}) must be adjacent to all vertices in $V(P_2) \cap W$ (respectively $V(P_2) \cap B$), $u_{2j-1}u_{2r}$ and $u_{2j}u_{2s-1}$, therefore the claim holds.

For convenience we change some notations henceforth. We let $|G_2| = 2p_2$ and the vertices of G_2 be $v'_0, v'_1, \ldots, v'_{2p_2-1}$, where $v'_{2j}v'_{2j+1} \in M$, for $0 \le j \le p_2 - 1$, and let $R = u_0u_1 \ldots u_{2r-1}$.

Now we discuss the situations case by case, with respect to the length of R and the distribution of edges between R and G_i , i = 1, 2.

Case 1. |R| = 2.

Then $R = u_0 u_1$. By Claim 3 and Claim 4, For any $0 \le i \le p_1 - 1$ and $0 \le j \le p_2 - 1$, $u_0 v_{2i+1} +, u_0 v'_{2j+1} +, u_1 v_{2i} +$ and $u_1 v'_{2j} +$. Therefore $G \in \mathcal{G}_1$ and $M \in \mathcal{M}_1$. *Case 2.* $|R| \ge 4$.

Claim 5. For j = 1, 2, and every edge $u_{2i-1}u_{2i}, 1 \leq i \leq r-1$, exactly one of $u_{2i-1} \rightarrow G_j$ and $u_{2i} \rightarrow G_j$ holds. Furthermore, if $u_{2i-1} \rightarrow G_j$ (respectively $u_{2i} \rightarrow G_j$), it is adjacent to all vertices in $V(G_j) \cap W$ (respectively $V(G_j) \cap B$).

Proof. Firstly, we prove that for j = 1, 2 and every edge $u_{2i-1}u_{2i}$, $1 \le i \le r-1$, $u_{2i-1} \nrightarrow G_j$ or $u_{2i} \nrightarrow G_j$. By Lemma 4.2, the conclusion holds for G_1 . Now we prove it for G_2 . Suppose to the contrary that there exists $1 \le l \le r-1$ such that $u_{2l-1} \rightarrow G_2$ and $u_{2l} \rightarrow G_2$, and let $v'_{2s} \in N_{G_2}(u_{2l-1})$ and $v'_{2t+1} \in N_{G_2}(u_{2l})$. If $|G_2| = 2$ or $t \ne s$, We can find a closed *M*-alternating Hamilton path *Q* of G_2 connecting v'_{2s} and v'_{2t-1} , and hence we have an *M*-alternating Hamilton cycle

$$u_0 R u_{2l-1} v'_{2s} Q v'_{2t-1} u_{2l} R u_{2r-1} v_0 P_1 v_{2p_1-1} u_0$$

of G, contradicting our assumption. If $|G_2| \ge 4$ and t = s, let P'_2 be a closed M-alternating Hamilton path of $G_2 - \{v'_{2s}, v'_{2s+1}\}$. Then P'_2 is an opposite path for P_1 , with the central path $u_0Ru_{2l-1}v'_{2s}v'_{2s+1}u_{2l}Ru_{2r-1}$, which is shorter than P_2 , contradicting our choice of P_2 . Hence $u_{2i-1} \nrightarrow G_2$ or $u_{2i} \nrightarrow G_2$, for $1 \le i \le r-1$.

Arbitrarily choose $0 \le l \le p_1 - 1$ and $0 \le k \le p_2 - 1$. We have $d(v_{2l}) + d(v_{2l+1}) \le 2p_1 + 2 + (|R| - 2)/2 = 2p_1 + r + 1$ and similarly $d(v'_{2k}) + d(v'_{2k+1}) \le 2p_2 + r + 1$. So

$$d(v_{2l}) + d(v_{2l+1}) + d(v'_{2k}) + d(v'_{2k+1}) \le 2p_1 + 2p_2 + 2r + 2 = \nu + 2.$$
(9)

However $v_{2l}v'_{2k+1}$ - and $v_{2l+1}v'_{2k}$ -, by the condition of the theorem,

$$d(v_{2l}) + d(v'_{2k+1}) + d(v_{2l+1}) + d(v'_{2k}) \ge 2(\nu/2 + 1) = \nu + 2, \tag{10}$$

and all equalities must hold. To obtain equalities, for j = 1, 2, and every edge $u_{2i-1}u_{2i}$, $1 \leq i \leq r-1$, exactly one of $u_{2i-1} \to G_j$ and $u_{2i} \to G_j$ must hold. Furthermore, since l and k are arbitrarily chosen, we prove that if $u_{2i-1} \to G_j$ (respectively $u_{2i} \to G_j$), it is adjacent to all vertices in $V(G_j) \cap W$ (respectively $V(G_j) \cap B$).

Let $1 \le i \le r-1$. We define $E_1(E'_1)$ to be the set of edges $u_{2i-1}u_{2i}$, where $u_{2i-1}v_{2j}+$, for every $0 \le j \le p_1 - 1$ $(u_{2i-1}v'_{2k}+$, for every $0 \le k \le p_2 - 1)$, and $E_2(E'_2)$ to be the set of edges $u_{2i-1}u_{2i}$, where $u_{2i}v_{2j+1}+$, for every $0 \le j \le p_1 - 1$ $(u_{2i}v'_{2k+1}+,$ for every $0 \le k \le p_2 - 1)$.

By Claim 5, for every $1 \le i \le r-1$, $u_{2i-1}u_{2i} \in E_1 \cap E'_1$, $E_1 \cap E'_2$, $E_2 \cap E'_1$ or $E_2 \cap E'_2$. Accordingly, we say that $u_{2i-1}u_{2i}$ is an edge of type I, II, III or IV for G_1 , G_2 and R. Let the number of edges $u_{2i-1}u_{2i}$ belonging to $E_1 \cap E'_1$, $E_1 \cap E'_2$, $E_2 \cap E'_1$ and $E_2 \cap E'_2$ be t_{11} , t_{12} , t_{21} and t_{22} , respectively. We have $d(v_0) = t_{11} + t_{12} + p_1 + 1$, $d(v_1) = t_{22} + t_{21} + p_1 + 1$, $d(v'_0) = t_{11} + t_{21} + p_2 + 1$ and $d(v'_1) = t_{22} + t_{12} + p_2 + 1$.

Since equalities hold in (9) and (10), we have $d(v_{2l}) + d(v'_{2k+1}) = d(v_{2l+1}) + d(v'_{2k}) = \nu/2 + 1$ for any $0 \le l \le p_1 - 1$ and $0 \le k \le p_2 - 1$, Hence

$$t_{11} + t_{22} + 2t_{12} + p_1 + p_2 + 2 = d(v_0) + d(v'_1)$$

= $\nu/2 + 1$
= $d(v_1) + d(v'_0)$
= $t_{11} + t_{22} + 2t_{21} + p_1 + p_2 + 2.$ (11)

So $t_{12} = t_{21}$.

We let $t_1 = t_{11}$, $t_2 = t_{22}$ and $t_0 = t_{12} = t_{21}$, then $t_1 + t_2 + 2t_0 = r - 1$.

We summarise some structural results in the form of observations.

Observation 1. If there exists $1 \le j < i \le r-1$, such that $u_{2i-1}u_{2i} \in E_1$ (E'_1) and $u_{2j-1}u_{2j} \in E'_2$ (E_2) . Then $u_{2j-1}u_{2i}-.$

Proof. If $u_{2i-1}u_{2i}+$, we obtain an *M*-alternating Hamilton cycle

$$u_0 R u_{2j-1} u_{2i} R u_{2r-1} v'_0 P_2 v'_{2p_2-1} u_{2j} R u_{2i-1} v_0 P_1 v_{2p_1-1} u_0$$

($u_0 R u_{2j-1} u_{2i} R u_{2r-1} v_0 P_1 v_{2p_1-1} u_{2j} R u_{2i-1} v'_0 P_2 v'_{2p_2-1} u_0$),

contradicting our assumption.

Observation 2. If there exists $1 \le i \le r-2$, such that $u_{2i-1}u_{2i} \in E_1$ and $u_{2i+1}u_{2i+2} \in E_2$, then $u_{2i}u_{2i+1}$ is a critical edge, $|G_1| = |G_2| = 2$, and exactly one of $u_{2i}v'_1 + and u_{2i+1}u_0 + (u_{2i+1}v'_0 + and u_{2i}u_{2r-1} +)$ holds.

If there exists $1 \le i \le r-2$, such that $u_{2i-1}u_{2i} \in E'_1$ and $u_{2i+1}u_{2i+2} \in E'_2$, then $u_{2i}u_{2i+1}$ is a critical edge, $|G_1| = 2$, and exactly one of $u_{2i}v_{1+}$ and $u_{2i+1}u_0 + (u_{2i+1}v_0 + and u_{2i}u_{2r-1} +)$ holds.

Proof. Suppose there exists $1 \leq i \leq r-2$, such that $u_{2i-1}u_{2i} \in E_1$ and $u_{2i+1}u_{2i+2} \in E_2$, then $u_{2i}u_{2i+1}$ is a critical edge with respect to the *M*-alternating cycle

$$u_0 R u_{2i-1} v_0 P_1 v_{2p_1-1} u_{2i+2} R u_{2r-1} v'_0 P_2 v'_{2p_2-1} u_{0p_2} v'_{2p_2-1} v'_{2p_2-1$$

where P_1 is an opposite path. Since G_1 is critical, $|G_1| = 2$. Since $|P_1| = 2$, and P_2 is the shortest opposite path, $|G_2| = 2$. Since $u_0v'_1(u_{2r-1}v'_0)$ are on a central path for the critical edge $u_{2i}u_{2i+1}$ and the opposite path v_0v_1 , exactly one of $u_{2i+1}u_0$ + and $u_{2i}v'_1 + (u_{2i+1}v'_0 + and u_{2i}u'_{2r-1} +)$ holds.

Now suppose there exists $1 \leq i \leq r-2$, such that $u_{2i-1}u_{2i} \in E'_1$ and $u_{2i+1}u_{2i+2} \in E'_2$. Then $u_{2i}u_{2i+1}$ is a critical edge with respect to the *M*-alternating cycle

$$u_0 R u_{2i-1} v'_0 P_2 v'_{2p_2-1} u_{2i+2} R u_{2r-1} v_0 P_1 v_{2p_1-1} u_0,$$

where P_2 is an opposite path. Since G_1 is critical, $|G_1| = 2$. Since u_0v_1 $(u_{2r-1}v_0)$ are on a central path for the critical edge $u_{2i}u_{2i+1}$ and the opposite path P_2 , exactly one of $u_{2i+1}u_0 +$ and $u_{2i}v_1 + (u_{2i+1}v_0 + \text{ and } u_{2i}u_{2r-1} +)$ holds.

Observation 3. If there exists $1 \leq i < k < j \leq r-1$, such that $u_{2i-1}u_{2i} \in E_1$ (E'_1) , $u_{2j-1}u_{2j} \in E_1$ E_2 (E'_2) , $u_{2k-1}u_{2k} \in E'_2$ (E_2) and $u_{2k-1}u_0+$, then $u_{2i}u_{2j-1}-$.

Proof. If $u_{2i}u_{2i-1}+$, we obtain an *M*-alternating Hamilton cycle

 $u_0 R u_{2i-1} v_0 P_1 v_{2p_1-1} u_{2j} R u_{2r-1} v'_0 P_2 v'_{2p_2-1} u_{2k} R u_{2j-1} u_{2i} R u_{2k-1} u_0,$

contradicting our assumption.

By symmetry, the claim holds under the other situation.

Claim 6. $|G_1| = 2$.

Proof. Suppose $|G_1| \geq 4$. By Observation 2, there does not exist $1 \leq i \leq r-1$, such that $u_{2i-1}u_{2i} \in E_1$ (E'_1) and $u_{2i+1}u_{2i+2} \in E_2$ (E'_2) . Therefore, there can not exist i < j, such that $u_{2i-1}u_{2i} \in E_1$ (E'_1) and $u_{2i-1}u_{2i} \in E_2$ (E'_2) . In other words, there exits an integer $0 \le k_1 \le r-1$ $(0 \le k_2 \le r-1)$, such that for all $i \le k_1$ $(j \le k_2)$, $u_{2i-1}u_{2i} \in E_2$ $(u_{2j-1}u_{2j} \in E'_2)$ and for all $i > k_1 \ (j > k_2), \ u_{2i-1}u_{2i} \in E_1 \ (u_{2j-1}u_{2j} \in E'_1)$. It is easily seen that $t_0 = 0$ and $k_1 = k_2$. We let $k = k_1 = k_2.$

Suppose that $t_1, t_2 \neq 0$, or equally, $1 \leq k \leq r-2$. Consider the vertices u_{2k-1} and u_{2k+2} . By Observation 1, for all $j \ge k+1$, $u_{2k-1}u_{2j}$, and for all $j \le k$, $u_{2k+2}u_{2j-1}$. Particularly, $u_{2k-1}u_{2k+2}$. But then we have $d(u_{2k-1}) \leq k+1$, $d(u_{2k+2}) \leq r-k$ and $d(u_{2k-1}) + d(u_{2k+2}) \leq r-k$ $r+1 < \nu/2 + 1$, contradicting our condition.

Suppose one of t_1 and t_2 , say $t_1 = 0$. Then for $1 \le i \le r - 1$, $d(u_{2i-1}) \le r$. Moreover $d(v_0) = p_1 + 1$, so $d(u_{2i-1}) + d(v_0) \le r + p_1 + 1 < \nu/2 + 1$ but $v_0 u_{2i-1} - n$, a contradiction. So we must have $|G_1| = 2$.

Claim 7. Either $t_0 = 0$, or $t_1 = t_2 = 0$.

Proof. Suppose that $t_0 > 0$, and one of t_1 and t_2 is greater than 0. Without lost of generality, we may assume that $t_1 \ge t_2$, and so $t_1 > 0$.

Let $u_{2i-1}u_{2i} \in E_1 \cap E'_1$, $1 \leq i \leq r-1$, be such that i is the maximum. Then by our condition, $d(u_{2i}) + d(v_1) \ge \nu/2 + 1$. Hence, $d(u_{2i}) \ge \nu/2 + 1 - d(v_1) = \nu/2 + 1 - (t_2 + 1)$ t_0+2 = $t_1+t_0+\nu/2-r$. By Observation 1, u_{2i} can not be adjacent to any u_{2i-1} , where $u_{2i-1}u_{2i} \in E_2 \cup E'_2$ and j < i. Hence u_{2i} sends at least $t_1 + t_0 + \nu/2 - r - (t_1 + 1) = t_0 + \nu/2 - r - 1$ edges to $\{u_{2r-1}\} \cup \{u_{2j-1} : u_{2j-1}u_{2j} \in E_2 \cup E'_2, j > i+1\}$. Since $t_0 > 0$ and $\nu/2 - r \ge 2$, $u_{2i} \to \{u_{2j-1} : u_{2j-1}u_{2j} \in E_2 \cup E'_2, j > i+1\}$, so there exists at least one $u_{2j-1}u_{2j}$ such that j > i + 1 and $u_{2j-1}u_{2j} \in E_2 \cup E'_2$.

By our choice of $u_{2i-1}u_{2i}$, $u_{2i+1}u_{2i+2} \in E_2 \cup E'_2$. If $u_{2i+1}u_{2i+2} \in E_2$, then by Observation 2, $u_{2i}u_{2i+1}$ is a critical edge, and exactly one of $u_{2i}v'_1$ + and $u_{2i+1}u_0$ + holds. By $u_{2i-1}u_{2i} \in E'_1$ we have $u_{2i}v'_1$, therefore $u_{2i+1}u_0$. If $u_{2i+1}u_{2i+2} \in E'_2$, then again by Observation 2, $u_{2i}u_{2i+1}$ is a critical edge, and exactly one of $u_{2i}v_1$ + and $u_{2i+1}u_0$ + holds. By $u_{2i-1}u_{2i} \in E_1$ we have $u_{2i}v_1$ -, hence $u_{2i+1}u_0+$.

Now we discuss different situations of $u_{2i+1}u_{2i+2}$.

If $u_{2i+1}u_{2i+2} \in E_2 \cap E'_2$, let j > i+1 be such that $u_{2i}u_{2j-1}+, u_{2j-1}u_{2j} \in E_2 \cup E'_2$. By Observation 3, $u_{2i}u_{2j-1}$, a contradiction.

If $u_{2i+1}u_{2i+2} \in E_1 \cap E'_2$ or $E_2 \cap E'_1$, without lost of generality, we may assume that $u_{2i+1}u_{2i+2} \in E_1 \cap E'_2$ $E_1 \cap E'_2$. Since $u_{2i}u_{2i+1}$ is a critical edge and $u_{2i+1}v_0+$, by Observation 2, we have $u_{2i}u_{2r-1}-$. For j > i + 1, where $u_{2j-1}u_{2j} \in E_2$, by Observation 3, $u_{2i}u_{2j-1}$. Therefore u_{2i} sends at least $t_0 + \nu/2 - r - 1 \ge t_0 + 1$ edges to $\{u_{2j-1} : u_{2j-1}u_{2j} \in E_1 \cap E'_2, j > i+1\}$. However, the number of such u_{2j-1} is at most t_0 , a contradiction.

Case 2.1. $t_0 = 0$.

Without lost of generality, we may assume that $t_1 > 0$, and let $u_{2i-1}u_{2i} \in E_1 \cap E'_1$.

If there exists $u_{2j-1}u_{2j}$, j < i, such that $u_{2j-1}u_{2j} \in E_2 \cap E'_2$, then $u_{2j-1}u_{2i}$ by Observation 1.

If there exists $u_{2j-1}u_{2j}$, j > i+1, such that $u_{2j-1}u_{2j} \in E_2 \cap E'_2$, then there exists $i \le k \le j-1$, such that $u_{2k-1}u_{2k} \in E_1 \cap E'_1$ and $u_{2k+1}u_{2k+2} \in E_2 \cap E'_2$. By Observation 2, $u_{2k}u_{2k+1}$ is a critical edge, and since $u_{2k+1}v_0$ and $u_{2k}v_1$, we have $u_{2k}u_{2r-1}$ and $u_{2k+1}u_0$. By Observation 3, $u_{2i}u_{2j-1}$.

Hence, for all $u_{2j-1}u_{2j} \in E_2 \cap E'_2$, $j \neq i+1$, $u_{2i}u_{2j-1}-i$. So, $d(u_{2i}) \leq t_1+2$. But then

$$\nu/2 + 1 \le d(u_{2i}) + d(v_1) \le t_1 + 2 + t_2 + 2 = (\nu - 2p_2 - 2 - 2)/2 + 4 = \nu/2 - p_2 + 2.$$
 (12)

Since $p_2 \ge 1$, all equalities must hold, hence $p_2 = 1$ and $2r - 1 = \nu - 5$. Furthermore, to get $d(u_{2i}) = t_1 + 2$, we must have the following.

- (a) $u_{2i+1}u_{2i+2} \in E_2 \cap E'_2$, hence $u_{2i-1}u_{2i} \neq u_{\nu-7}u_{\nu-6}$.
- (b) $u_{2i}u_{2j-1}+$, for all $u_{2j-1}u_{2j} \in E_1 \cap E'_1$.
- (c) $u_{2i}u_{\nu-5}+$.

By (a), $t_2 \ge 0$, and similarly, for any $u_{2i-1}u_{2i} \in E_2 \cap E'_2$, we can prove the following.

- (d) $u_{2i-3}u_{2i-2} \in E_1 \cap E'_1$, hence $u_{2i-1}u_{2i} \neq u_1u_2$.
- (e) $u_{2i-1}u_{2j}+$, for all $u_{2j-1}u_{2j} \in E_2 \cap E'_2$.
- (f) $u_{2i-1}u_0+$.

So, the edges $u_{2i-1}u_{2i}$, $1 \le i \le \nu/2-3$, belong to $E_1 \cap E'_1$ and $E_2 \cap E'_2$ alternatively. Moreover, $u_1u_2 \in E_1 \cap E'_1$ and $u_{\nu-7}u_{\nu-6} \in E_2 \cap E'_2$. Hence we must have $\nu = 4n+2$, for some integer $n \ge 2$, $u_{4j+1}u_{4j+2} \in E_1 \cap E'_1$ and $u_{4j+3}u_{4j+4} \in E_2 \cap E'_2$ for $0 \le j \le n-2$. The vertex set $\{u_{4j+1}, u_{4j+2} : 0 \le j \le n-2\} \cup \{v_0, v'_0, u_{4n-3}\}$, as well as $\{u_{4j+3}, u_{4j+4} : 0 \le j \le n-2\} \cup \{v_1, v'_1, u_0\}$, induce complete bipartite subgraphs, respectively.

Let $B_1 = \{u_{4j+1} : 0 \leq j \leq n-1\}$, $W = \{u_{4j+2} : 0 \leq j \leq n-2\} \cup \{v_0, v_0'\}$, $B = \{u_{4j+3} : 0 \leq j \leq n-2\} \cup \{v_1, v_1'\}$ and $W_1 = \{u_{4j} : 0 \leq j \leq n-1\}$. By above discussion, there can be no more edge between B and W. But we can add edges between B_1 and W_1 freely, to obtain all graphs $G \in \mathcal{G}_2$, with $M \in \mathcal{M}_2$.

Case 2.2. $t_1 = t_2 = 0$. Since $t_1 + t_2 + 2t_0 = r - 1$, we have $r = 2t_0 + 1$ and r must be odd.

If there exists $1 \leq i \leq r-2$, such that $u_{2i-1}u_{2i} \in E_1 \cap E'_2$ and $u_{2i+1}u_{2i+2} \in E_2 \cap E'_1$ $(u_{2i-1}u_{2i} \in E_2 \cap E'_1 \text{ and } u_{2i+1}u_{2i+2} \in E_1 \cap E'_2)$, we say that an A-change (B-change) occurs at u_{2i-1} . If there exist *i* and *j*, such that $2 \leq i+1 < j \leq r-2$, and there is an A-change (B-change) occurs at u_{2i-1} and a B-change (A-change) occurs at u_{2j-1} , we say that a change couple occurs at (u_{2i-1}, u_{2j-1}) .

Case 2.2.1. $|G_2| \ge 4$.

There can not be any A-change, or by Observation 2, $|G_1| = |G_2| = 2$. To avoid any A-change, for $1 \le i \le (r-1)/2$, $u_{2i-1}u_{2i} \in E_2 \cap E'_1$ and for $(r+1)/2 \le i \le r-1$, $u_{2i-1}u_{2i} \in E_1 \cap E'_2$.

Suppose that r = 3. It is not hard to see that $u_0u_3 -$ and $u_2u_5 -$, while each of u_0u_5 and u_1u_4 can be exist or not. Hence we obtain all the graph in class \mathcal{G}_3 , except those with n = 1.

If $r \ge 5$, then $u_{r-1}u_r$ is a critical edge, with central path $u_{r+1}Ru_{2r-1}v_0v_1u_0Ru_{r-2}$ and opposite graph G_2 (Figure 6). Consider the edge v_1u_0 and u_1u_2 on the central path. We have $v_1u_{r-1}+$, $u_0 \to G_2$, $u_1 \to G_2$, and by Claim 7, u_2u_r+ . But then an A-change occurs at v_1 , a contradiction. **Case 2.2.2.** $|G_2| = 2$.

Then $\nu = 4n + 6$, for some $n \ge 1$. For n = 1, it is not hard to verify that $G \in \mathcal{G}_3$, $M \in \mathcal{M}_3$, and we obtain all graphs in \mathcal{G}_3 together with Case 2.2.1. For n = 2, it can be checked that $G = G_4$, $M = M_4$. Henceforth we assume that $n \ge 3$, and then $r = 2n + 1 \ge 7$.

We call G_1 and G_2 a critical edge pair with central path R. Since we have discussed all other cases, we may assume that for every critical edge pair and the central path, every edge of the central path that is not in M is of type II or III.

Let there be a change couple occurs at (u_{2i-1}, u_{2j-1}) . Without lost of generality, suppose that an A-change occurs at u_{2i-1} and a B-change occurs at u_{2j-1} , then $u_{2i}u_{2i+1}$ and $u_{2j}u_{2j+1}$ are critical edges. Since $u_{2i}u_{2i+1}$ and v_1v_0 is a critical edge pair, with the central path $u_{2i+2}Ru_{2r-1}v'_0v'_1u_0Ru_{2i-1}$, by our assumption, $u_{2j-1}u_{2j}$ and $u_{2j+1}u_{2j+2}$ are of type II or III. By

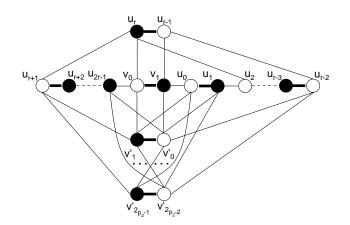


Figure 6: Contradiction in Case 2.2.1

 $u_{2j}v_1$ + and $u_{2j+1}v_0$ +, we have $u_{2j-1}u_{2i}$ + and $u_{2j+2}u_{2i+1}$ +. Similarly, we have $u_{2i-1}u_{2j}$ + and $u_{2i+2}u_{2j+1}$ +. However, we get an *M*-alternating Hamilton cycle

 $u_0 R u_{2i-1} u_{2j} u_{2j+1} u_{2i+2} R u_{2j-1} u_{2i} u_{2i+1} v'_0 v'_1 u_{2j+2} R u_{2r-1} v_0 v_1 u_0$

then, a contradiction. Therefore, there must not be any change couple.

By symmetry, we may assume that $u_1u_2 \in E_1 \cap E'_2$, and let $r_0 > 0$, $r_1 > r_0$ and $r_2 \ge r_1$ be such that $u_1u_2, \ldots, u_{2r_0-1}u_{2r_0} \in E_1 \cap E'_2, u_{2r_0+1}u_{2r_0+2}, \ldots, u_{2r_1-1}u_{2r_1} \in E_2 \cap E'_1, u_{2r_1+1}u_{2r_1+2}, \ldots, u_{2r_2-1}u_{2r_2} \in E_1 \cap E'_2$ and if $u_{2r_2+1}u_{2r_2+2}$ exists, $u_{2r_2+1}u_{2r_2+2} \in E_2 \cap E'_1$.

If $r_1 - r_0 \ge 2$ and $r_2 - r_1 \ge 1$, then a change couple occurs at (u_{2r_0-1}, u_{2r_1-1}) , a contradiction. Hence, $r_1 - r_0 = 1$ or $r_2 = r_1$.

If $r_1 - r_0 = 1$, then $r_2 > r_1$, and the edge $u_{2r_2+1}u_{2r_2+2}$ exits. If $r_2 - r_1 \ge 2$, a change couple occurs at (u_{2r_1-1}, u_{2r_2-1}) , a contradiction. Therefore $r_2 = r_1 + 1$. Moreover, if any B-change occurs at u_{2j-1} where $j \ge r_2 + 1$, we obtain a change couple (u_{2r_0-1}, u_{2j-1}) , again leading to a contradiction. Hence, we must have $u_{2r_2+1}u_{2r_2+2}, \ldots, u_{2r-3}u_{2r-2} \in E_2 \cap E'_1$, and then $r_0 = (r-3)/2, r_1 = (r-1)/2$ and $r_2 = (r+1)/2$.

Then $u_{r+1}u_{r+2}$ and v_1v_0 is a critical edge pair, with the central path $u_{r+3}Ru_{2r-1}v'_0v'_1u_0Ru_r$. Again we may assume that the edge of the central path not in M are of type II or III. Consider the edges $u_{r-4}u_{r-3}$ and $u_{r-2}u_{r-1}$, Since $u_{r-4}v_0$ + and $u_{r-1}v_1$ +, we must have $u_{r-3}u_{r+2}$ + and $u_{r-2}u_{r+1}$ +. Since $r \ge 7$, 2r - 3 > r + 3. Consider the edges $u_{2r-3}u_{2r-2}$. Since v_1u_{2r-2} +, we must have $u_{2r-3}u_{r+1}$ +. But then we find a change couple occur at (u_{2r-3}, u_{r-4}) , a contradiction (Figure 7).

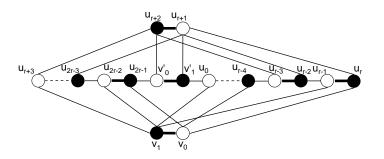


Figure 7: Critical pair $u_{r+1}u_{r+2}$ and v_1v_0

If $r_2 = r_1$, then $u_1 u_2, \ldots, u_{r-2} u_{r-1} \in E_1 \cap E'_2$ and $u_r u_{r+1}, \ldots, u_{2r-3} u_{2r-2} \in E_2 \cap E'_1$. Then, $u_{r-1} u_r$ and $v_0 v_1$ is a critical pair, with the central path $u_{r+1} R u_{2r-1} v'_0 v'_1 u_0 R u_{r-2}$. For the edges

 $u_{2i-1}u_{2i}$ with $(r+3)/2 \le i \le r-1$, $v_1u_{2i}+$, so we must have $u_{2i-1}u_{r-1}+$. For the edges $u_{2i-1}u_{2i}$ with $1 \le i \le (r-3)/2$, $v_0u_{2i-1}+$, so we must have $u_{2i}u_r+$. For the edge $u_{2r-1}v'_0$ and v'_1u_0 , we have $u_{2r-1}v_0+$, v'_0u_r+ , $v'_1u_{r-1}+$ and u_0v_1+ . Thus we reach a same config with the case that $r_1 - r_0 = 1$.

5 Final Remarks

Most of the degree sum conditions for Hamilton problems care about independent vertex sets. In our work, we try to strengthen the condition of our main theorem, by replacing "for every vertex pair u and v, where there is not arc from u to v" with "for every vertex pair u and v". Naturally, if the former condition guarantees hamiltonicity without exception, then such a strengthening brings nothing. But in the case where there are exceptions, we do find some differences. Let \mathcal{D}'_1 be a subset of \mathcal{D}_1 , in which n = m. Let \mathcal{D}'_3 be a subset of \mathcal{D}_3 , where n = 1. We have the following result.

Theorem 5.1. Let D be a digraph. If for every vertex pair u and v, we have $d^+(u) + d^-(v) \ge |D| - 1$, then D has a directed Hamilton cycle, unless $D \in \mathcal{D}'_1$, \mathcal{D}_2 or \mathcal{D}'_3 , or $D = D_4$.

As a corollary, we can improve the Ore condition as well. Given a (undirected) graph G, if we replace every edge $uv \in E(G)$ with two arcs uv and vu, we have a digraph D. Applying Theorem 3.1 on D, we obtain the following result.

Let $n, m \geq 1$, and \mathcal{G}_5 be the set of graphs obtained by identify one vertex of a complete graph K_{m+1} and one vertex of a complete graph K_{n+1} , where $n, m \geq 1$. Let \mathcal{G}_6 be the set of all graphs obtained by joining every vertex of a graph I_{n+1} to every vertex of an arbitrary graph on n vertices.

Corollary 5.2. Let G be a graph. If for every distinct nonadjacent vertex pair u and v, we have $d(u) + d(v) \ge |G| - 1$, then G has a Hamilton cycle, unless $G \in \mathcal{G}_5$, or $G \in \mathcal{G}_6$.

A slightly stronger result can be found in [28]. There is only one exceptional class, for it considers only 2-connected graphs.

Theorem 5.3. (Li, Li and Feng, 2007) Let G be a 2-connected graph with $|G| \ge 3$. If $d(u) + d(v) \ge |G| - 1$ for every pair of vertices u and v with d(u, v) = 2, then G has a Hamilton cycle, unless |G| is odd and $G \in \mathcal{G}_6$.

Stimulated by above results, we conjecture that the lower bound of degree sum in the following result can be reduced by 1, with some exceptional cases.

Theorem 5.4. (Bang-Jensen, Gutin and Li, 1996 [5]) Let D be a strong digraph such that for every pair of dominating non-adjacent and every pair of dominated non-adjacent vertices $\{u, v\}$, we have $min\{d^+(u) + d^-(v), d^-(u) + d^+(v)\} \ge |D|$. Then D has a directed Hamilton cycle.

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