Inverse boundary value problem for Schrödinger equation in two dimensions

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Abstract

We relax the regularity condition on potentials of the Schrödinger equation in uniqueness results on the inverse boundary value problem which were recently proved in [11] and [5].

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain with $\partial \Omega = \bigcup_{j=0}^K \Sigma_j$ where Σ_j are smooth contours and Σ_0 is the external contour. Let $\nu = (\nu_1, \nu_2)$ be the unit outer normal to $\partial \Omega$ and let $\frac{\partial}{\partial \nu} = \nabla \cdot \nu$.

In this domain we consider the Schrödinger equation with some potential q:

$$(\Delta + q)u = 0 \quad \text{in } \Omega. \tag{1}$$

Let $\widetilde{\Gamma}$ be a non-empty arbitrary fixed relatively open subset of $\partial\Omega$. Denote $\Gamma_0 = Int(\partial\Omega \setminus \widetilde{\Gamma})$. Consider the partial Cauchy data

$$\mathcal{C}_{q} = \left\{ \left(u, \frac{\partial u}{\partial \nu} \right) \Big|_{\widetilde{\Gamma}}; \ (\Delta + q)u = 0 \quad \text{in } \Omega, \ u|_{\Gamma_{0}} = 0, u|_{\widetilde{\Gamma}} = f \right\}.$$

$$(2)$$

The goal of this article is to improve the regularity assumption on the potential q in the case of arbitrary subboundary $\tilde{\Gamma}$ for the uniqueness result in the inverse problem of recovery of potential from the partial data (2). In the case of $\tilde{\Gamma} = \partial \Omega$, this inverse problem was formulated by Calderón in [7]. Under the assumption $q \in C^{4+\alpha}(\overline{\Omega})$ the result was proved in Imanuvilov, Uhlmann and Yamamoto [11]. In Guillarmou and Tzou [10], the assumption on potentials was improved up to $C^{2+\alpha}(\overline{\Omega})$.

In particular, in the two-dimensional full Cauchy data case of $\Gamma = \partial \Omega$, we refer to Astala and Päivärinta [1], Blasten [2], Brown and Uhlmann [4], Bukhgeim [5], Nachman [14]. In [2], the full Cauchy data uniquely determine the potential within $W_p^1(\Omega)$ with p > 2. As for the related problem of recovery of the conductivity, [1] proved the uniqueness result for conductivities from $L^{\infty}(\Omega)$, improving the result of [14]. We also mention that for the case of full Cauchy data a relaxed regularity assumption on potential was claimed in [5] but the proof itself is missing some details.

In three or higher dimensions, for the full Cauchy data, Sylvester and Uhlmann [16] proved the uniqueness of recovery of conductivity in $C^2(\overline{\Omega})$, and later the regularity assumption was relaxed up to $C^{\frac{3}{2}}(\overline{\Omega})$ in Päivärinta, Panchenko and Uhlmann [15] and up to $W_p^{\frac{3}{2}}(\Omega)$ with p > 2n in Brown and Torres [3]. For the case of partial Cauchy data, uniqueness theorems were proved under assumption that a potential of the Schrödinger equation belongs to $L^{\infty}(\Omega)$ (see Bukhgeim and Uhlmann [6], Kenig, Sjöstrand and Uhlmann [13]).

Our main result is as follows

Theorem 1 Let $q_1, q_2 \in C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ if $\widetilde{\Gamma} = \partial \Omega$ and $q_1, q_2 \in W_p^1(\Omega)$ for some p > 2 otherwise. If $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ then $q_1 = q_2$.

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The rest part of the paper is devoted to the proof of the theorem. Throughout the article, we use the following notations.

Notations. $i = \sqrt{-1}$, $x_1, x_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$, \overline{z} denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$, $D = (\frac{1}{i}\partial_{x_1}, \frac{1}{i}\partial_{x_2})$. The tangential derivative on the boundary is given by $\partial_{\overline{\tau}} = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, where $\nu = (\nu_1, \nu_2)$ is the unit outer normal to $\partial\Omega$.

Proof.

First Step.

Let $\Phi = \varphi + i\psi$ be a holomorphic function on Ω such that φ, ψ are real-valued and

$$\Phi \in C^2(\overline{\Omega}), \quad \operatorname{Im} \Phi|_{\Gamma_0} = 0. \tag{3}$$

Denote by \mathcal{H} the set of the critical points of the function Φ . Suppose that this set is not empty, each critical point is nondegenerate, $\mathcal{H} \cap \overline{\Gamma}_0 = \emptyset$ and

$$\operatorname{mes}\left(\mathcal{J}\right) = 0, \quad \mathcal{J} = \{x; \, \partial_{\vec{\tau}}\psi(x) = 0, x \in \Gamma\}.$$

$$\tag{4}$$

Here $\vec{\tau}$ is an unit tangential vector to $\partial\Omega$. Consider the operator $L_q(x, D) = -\sum_{j=1}^2 (D_j + \tau i \varphi_{x_j})^2 + q$. It is known (see [12] Proposition 2.5) that there exists a constant τ_0 such that for $|\tau| \ge \tau_0$ and any $f \in L^2(\Omega)$, there exists a solution to the boundary value problem

$$L_q(x, D)u = f \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0 \tag{5}$$

such that

$$\|u\|_{H^{1,\tau}(\Omega)} / \sqrt{|\tau|} \le C \|f\|_{L^2(\Omega)}.$$
(6)

Moreover if $f/\partial_z \Phi \in L^2(\Omega)$, then for any $|\tau| \ge \tau_0$ there exists a solution to the boundary value problem (5) such that

$$\|u\|_{H^{1,\tau}(\Omega)} \le C \|f/\partial_z \Phi\|_{L^2(\Omega)}.$$
 (7)

The constants C in (6) and (7) are independent of τ . Here and henceforth we set

$$||u||_{H^{1,\tau}(\Omega)} = (||u||_{H^{1}(\Omega)}^{2} + |\tau|^{2} ||u||_{L^{2}(\Omega)}^{2})^{\frac{1}{2}}.$$

Second Step.

Here we will construct complex geometrical optics solutions. Henceforth by $o_{L^2(\Omega)}(\frac{1}{\tau})$, we mean a function $f(\epsilon, \tau, \cdot) \in L^2(\Omega)$ such that $\lim_{\tau \to \infty} |\tau| \| f(\epsilon, \tau, \cdot) \|_{L^2(\Omega)} = 0$ for all small $\epsilon > 0$, and by $o(\frac{1}{\tau})$, we mean $a(\epsilon, \tau)$ such that $\lim_{\tau \to \infty} |\tau| \| a(\epsilon, \tau) \| = 0$ for all small $\epsilon > 0$.

Let $\{q_{1,\epsilon}\}_{\epsilon \in (0,1)}$ be a sequence of smooth functions converging to q_1 in $W_p^1(\Omega)$ or $C^{\alpha}(\overline{\Omega})$ (depending on the assumption on the regularity of q_1) such that $q_{1,\epsilon} = q_1$ on \mathcal{H} . Let p_{ϵ} be the complex geometrical optics solution to the Schrödinger operator $\Delta + q_{1,\epsilon}$ which we constructed in [11]. The function p_{ϵ} can be written in the form:

$$p_{\epsilon}(x) = e^{\tau \Phi} (a + a_{0,\epsilon}/\tau) + e^{\tau \overline{\Phi}} \overline{(a + b_{1,\epsilon}/\tau)} - \left(e^{\tau \Phi} \frac{(\partial_{\overline{z}}^{-1}(aq_{1,\epsilon}) - M_{1,\epsilon})}{4\tau \partial_z \Phi} + e^{\tau \overline{\Phi}} \frac{(\partial_z^{-1}(\overline{a}q_{1,\epsilon}) - M_{3,\epsilon})}{4\tau \overline{\partial_z} \overline{\Phi}} \right) + e^{\tau \varphi} o_{L^2(\Omega)}(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty,$$
(8)

where $a \in C^6(\overline{\Omega})$ is some holomorphic function on Ω such that $\operatorname{Re} a|_{\Gamma_0} = 0$. The operators ∂_z^{-1} and $\partial_{\overline{z}}^{-1}$ are given by

$$\partial_{\overline{z}}^{-1}g = -\frac{1}{\pi} \int_{\Omega} \frac{g(\zeta,\overline{\zeta})}{\zeta-z} d\xi_2 d\xi_1, \quad \partial_{\overline{z}}^{-1}g = \overline{\partial_{\overline{z}}^{-1}\overline{g}},$$

Moreover for some $\tilde{x} \in \mathcal{H}$, we assume that $a(\tilde{x}) \neq 0$ and a(x) = 0 for $x \in \mathcal{H} \setminus {\{\tilde{x}\}}$, and the polynomials $M_{1,\epsilon}(z)$ and $M_{3,\epsilon}(\overline{z})$ satisfy

$$\partial_z^j(\partial_{\overline{z}}^{-1}(aq_{1,\epsilon}) - M_{1,\epsilon})(x) = 0, \qquad \partial_{\overline{z}}^j(\partial_z^{-1}(\overline{a}q_{1,\epsilon}) - M_{3,\epsilon})(x) = 0, \quad x \in \mathcal{H},$$

 $a_{0,\epsilon}, a_{1,\epsilon} \in C^6(\overline{\Omega})$ are holomorphic functions such that

$$(a_{0,\epsilon} + \overline{a}_{1,\epsilon})|_{\Gamma_0} = \frac{\left(\partial_{\overline{z}}^{-1}(aq_{1,\epsilon}) - M_{1,\epsilon}\right)}{4\partial_z \Phi} + \frac{\left(\partial_{\overline{z}}^{-1}(\overline{a}q_{1,\epsilon}) - M_{3,\epsilon}\right)}{4\overline{\partial_z \Phi}}.$$

We look for a solution u_1 in the form $u_1 = p_{\epsilon} + m_{\epsilon}$. Consider the equation

$$L_{q_1}(x,D)u_1 = L_{q_{1,\epsilon}}(x,D)(p_{\epsilon}+m_{\epsilon}) + (q_1-q_{1,\epsilon})(p_{\epsilon}+m_{\epsilon}) = L_{q_1}(x,D)m_{\epsilon} + (q_1-q_{1,\epsilon})p_{\epsilon} = 0.$$

By (7) there exists a solution to the boundary value problem

$$L_{q_1}(x,D)m_{\epsilon} + (q_1 - q_{1,\epsilon})p_{\epsilon} = 0 \quad \text{in} \quad \Omega, \quad m_{\epsilon}|_{\Gamma_0} = 0$$

such that

$$\|m_{\epsilon}\|_{H^{1,\tau}(\Omega)} \le C(\epsilon) \quad \forall \tau > \tau_0(\epsilon), \tag{9}$$

where $C(\epsilon)$ is independent of τ and

$$C(\epsilon) \to 0$$
 as $\epsilon \to 0$.

Since the Cauchy data (2) for potentials q_1 and q_2 , are equal, there exists a solution u_2 to the Schrödinger equation with the potential q_2 such that $u_1 = u_2$ on $\partial\Omega$ and $\frac{\partial u_1}{\partial\nu} = \frac{\partial u_2}{\partial\nu}$ on $\tilde{\Gamma}$. Setting $u = u_1 - u_2$, we obtain

$$(\Delta + q_2)u = (q_2 - q_1)u_1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}|_{\widetilde{\Gamma}} = 0.$$
(10)

In a way similar to the construction of u_1 , we construct the complex geometrical optics solution v for the Schrödinger equation with the potential q_2 . The construction of v repeats the corresponding steps of the construction of u_1 . The only difference is that instead of $q_{1,\epsilon}$ and τ , we use $q_{2,\epsilon}$ and $-\tau$ respectively. We provide details of the construction of v for the sake of completeness.

Let $\{q_{2,\epsilon}\}_{\epsilon \in (0,1)}$ be a sequence of smooth functions converging to sufficiently close to q_2 in $W_p^1(\Omega)$ or $C^{\alpha}(\overline{\Omega})$ such that $q_{2,\epsilon} = q_2$ on \mathcal{H} . Let \widetilde{p}_{ϵ} be the complex geometrical optics solution to the Schrödinger operator $\Delta + q_{2,\epsilon}$ constructed in [11]:

$$\widetilde{p}_{\epsilon}(x) = e^{-\tau\Phi}(a+b_{0,\epsilon}/\tau) + e^{-\tau\overline{\Phi}}\overline{(a+b_{1,\epsilon}/\tau)} + \left(e^{-\tau\Phi}\frac{(\partial_{\overline{z}}^{-1}(aq_{2,\epsilon}) - M_{2,\epsilon})}{4\tau\partial_{z}\Phi} + e^{-\tau\overline{\Phi}}\frac{(\partial_{\overline{z}}^{-1}(\overline{a}q_{2,\epsilon}) - M_{4,\epsilon})}{4\tau\overline{\partial_{z}\Phi}}\right) + e^{-\tau\varphi}o_{L^{2}(\Omega)}(\frac{1}{\tau}),$$
(11)

where $M_{2,\epsilon}(z)$ and $M_{4,\epsilon}(\overline{z})$ satisfy

$$\partial_z^j(\partial_{\overline{z}}^{-1}(aq_{1,\epsilon}) - M_{2,\epsilon})(x) = 0, \qquad \partial_{\overline{z}}^j(\partial_{\overline{z}}^{-1}(\overline{a}q_{1,\epsilon}) - M_{4,\epsilon})(x) = 0, \quad x \in \mathcal{H}.$$

and $b_{0,\epsilon}, b_{1,\epsilon}$ are holomorphic functions such that

$$(b_{0,\epsilon} + \overline{b}_{1,\epsilon})|_{\Gamma_0} = -\frac{(\partial_{\overline{z}}^{-1}(aq_{2,\epsilon}) - M_{2,\epsilon})}{4\partial_z \Phi} - \frac{(\partial_{\overline{z}}^{-1}(\overline{a}q_{2,\epsilon}) - M_{4,\epsilon})}{4\overline{\partial_z}\Phi}$$

We look for a solution v in the form $v = \tilde{p}_{\epsilon} + \tilde{m}_{\epsilon}$. Consider the operator

$$L_{q_2}(x,D)v = L_{q_{2,\epsilon}}(x,D)(\widetilde{p}_{\epsilon} + \widetilde{m}_{\epsilon}) + (q_2 - q_{2,\epsilon})(\widetilde{p}_{\epsilon} + \widetilde{m}_{\epsilon}) = L_{q_2}(x,D)\widetilde{m}_{\epsilon} + (q_2 - q_{2,\epsilon})\widetilde{p}_{\epsilon} = 0.$$

By (7) there exists a solution to the boundary value problem

$$L_{q_2}(x,D)\widetilde{m}_{\epsilon} + (q_2 - q_{2,\epsilon})\widetilde{p}_{\epsilon} = 0 \quad \text{in} \quad \Omega, \quad \widetilde{m}_{\epsilon}|_{\Gamma_0} = 0$$

such that

$$\|\widetilde{m}_{\epsilon}\|_{H^{1,\tau}(\Omega)} \le C(\epsilon) \quad \forall \tau > \tau_0(\epsilon),$$
(12)

where $C(\epsilon)$ is independent of τ and

$$C(\epsilon) \to 0$$
 as $\epsilon \to 0$.

Third Step.

We will prove $q_1(\tilde{x}) = q_2(\tilde{x})$ where $a(\tilde{x}) \neq 0$ and a(x) = 0 for $x \in \mathcal{H} \setminus {\tilde{x}}$ in the case where $q_1, q_2 \in W_p^1(\Omega)$. Denote $q = q_1 - q_2$. Taking the scalar product of equation (10) and the function v, we have:

$$\int_{\Omega} q u_1 v dx = 0. \tag{13}$$

By (9) and (12)

$$0 = \int_{\Omega} q u_1 v dx = \int_{\Omega} q p_{\epsilon} \tilde{p}_{\epsilon} dx + K(\epsilon, \tau), \qquad (14)$$

where

$$\overline{\lim_{\tau \to +\infty}} \tau |K(\epsilon, \tau)| \le C(\epsilon), \quad C(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0.$$
(15)

From (14), (15) and the explicit formulae (8), (11) for the construction of complex geometrical optics solutions, we have

$$\int_{\Omega} q(a^2 + \overline{a}^2) dx = 0$$

Computing the remaining terms, we have:

$$K(\epsilon,\tau) + \frac{1}{\tau} \int_{\Omega} q(a(a_{0,\epsilon} + b_{0,\epsilon}) + \overline{a(a_{1,\epsilon} + b_{1,\epsilon})}) dx + \int_{\Omega} q(a\overline{a}e^{2\tau i\psi} + a\overline{a}e^{-2\tau i\psi}) dx + \frac{1}{4\tau} \int_{\Omega} \left(qa \frac{\partial_{\overline{z}}^{-1}(aq_{2,\epsilon}) - M_{2,\epsilon}}{\partial_{z}\Phi} + q\overline{a} \frac{\partial_{\overline{z}}^{-1}(q_{2,\epsilon}\overline{a}) - M_{4,\epsilon}}{\partial_{z}\overline{\Phi}} \right) dx - \frac{1}{4\tau} \int_{\Omega} \left(qa \frac{\partial_{\overline{z}}^{-1}(q_{1,\epsilon}a) - M_{1,\epsilon}}{\partial_{z}\Phi} + q\overline{a} \frac{\partial_{\overline{z}}^{-1}(q_{1,\epsilon}\overline{a}) - M_{3,\epsilon}}{\partial_{z}\overline{\Phi}} \right) dx + o(\frac{1}{\tau}) = 0 \quad \text{as } \tau \to +\infty.$$
(16)

Since the functions q_j are not supposed to be from $C^2(\overline{\Omega})$, we can not directly use the stationary phase argument (e.g., Evans [8]). Consider two cases. Assume that $q \in W_p^1(\Omega)$ with p > 2. We have

$$\int_{\Omega} q \operatorname{Re}\left(a\overline{a}e^{2\tau i\psi}\right) dx = \int_{\Omega} q_{\epsilon} \operatorname{Re}\left(a\overline{a}e^{2\tau i\psi}\right) dx + \int_{\Omega} (q - q_{\epsilon}) \operatorname{Re}\left(a\overline{a}e^{2\tau i\psi}\right) dx.$$
(17)

We set $q_{\epsilon} = q_{1,\epsilon} - q_{2,\epsilon}$. Taking into account that $q_{j,\epsilon} = q_j$ on \mathcal{H} , j = 1, 2, (4) and using the stationary phase argument, similar to [11], we compute

$$\int_{\Omega} q_{\epsilon} (a\overline{a}e^{2\tau i\psi} + a\overline{a}e^{-2\tau i\psi}) dx = \frac{2\pi (q|a|^2)(\widetilde{x})\operatorname{Re} e^{2\tau i\operatorname{Im} \Phi(\widetilde{x})}}{\tau |(\det\operatorname{Im} \Phi'')(\widetilde{x})|^{\frac{1}{2}}} + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.$$
(18)

For the second integral in (17) we obtain

$$\int_{\Omega} (q - q_{\epsilon}) (a\overline{a}e^{2\tau i\psi} + a\overline{a}e^{-2\tau i\psi}) dx = \int_{\Omega} (q - q_{\epsilon}) \left(a\overline{a} \frac{(\nabla\psi, \nabla)e^{2\tau i\psi}}{2\tau i|\nabla\psi|^2} - a\overline{a} \frac{(\nabla\psi, \nabla)e^{-2\tau i\psi}}{2\tau i|\nabla\psi|^2} \right) dx$$
$$= \int_{\partial\Omega} (q - q_{\epsilon}) \left(a\overline{a} \frac{(\nabla\psi, \nu)e^{2\tau i\psi}}{2\tau i|\nabla\psi|^2} - a\overline{a} \frac{(\nabla\psi, \nu)e^{-2\tau i\psi}}{2\tau i|\nabla\psi|^2} \right) d\sigma$$
$$- \frac{1}{2\tau i} \int_{\Omega} \left\{ e^{2\tau i\psi} \operatorname{div} \left((q - q_{\epsilon})a\overline{a} \frac{\nabla\psi}{|\nabla\psi|^2} \right) - e^{-2\tau i\psi} \operatorname{div} \left((q - q_{\epsilon})a\overline{a} \frac{\nabla\psi}{|\nabla\psi|^2} \right) \right\} dx. \tag{19}$$

Since $\psi|_{\Gamma_0} = 0$ we have

$$\int_{\partial\Omega} (q-q_{\epsilon}) a\overline{a} \left(\frac{(\nabla\psi,\nu)e^{2\tau i\psi}}{2\tau i |\nabla\psi|^2} - \frac{(\nabla\psi,\nu)e^{-2\tau i\psi}}{2\tau i |\nabla\psi|^2} \right) d\sigma = \int_{\widetilde{\Gamma}} \frac{(q-q_{\epsilon})a\overline{a}}{2\tau i |\nabla\psi|^2} (\nabla\psi,\nu)(e^{2\tau i\psi} - e^{-2\tau i\psi}) d\sigma.$$

By (4) and Proposition 2.4 in [11] we have that

$$\int_{\partial\Omega} (q - q_{\epsilon}) a\overline{a} \left(\frac{(\nabla\psi, \nu) e^{2\tau i\psi}}{2\tau i |\nabla\psi|^2} - \frac{(\nabla\psi, \nu) e^{-2\tau i\psi}}{2\tau i |\nabla\psi|^2} \right) d\sigma = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty.$$

The last integral over Ω in formula (19) is $o(\frac{1}{\tau})$ and so

$$\int_{\Omega} (q - q_{\epsilon}) (a\overline{a}e^{2\tau i\psi} + a\overline{a}e^{-2\tau i\psi}) dx = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty.$$
⁽²⁰⁾

Taking into account that $\psi(\tilde{x}) \neq 0$ and using (26), (20) we have from (16) that

$$\frac{2\pi(q|a|^2)(\widetilde{x})}{|(\det\operatorname{Im}\Phi'')(\widetilde{x})|^{\frac{1}{2}}} + \widetilde{C}(\epsilon) = 0,$$
(21)

where $\widetilde{C}(\epsilon) \to +0$ as $\epsilon \to 0$. Hence

$$q(\widetilde{x}) = 0 \quad \text{if } a(\widetilde{x}) \neq 0 \text{ and } a(x) = 0 \text{ for } x \in \mathcal{H} \setminus \{\widetilde{x}\}.$$
(22)

Since a point \tilde{x} can be chosen arbitrarily close to any given point in Ω (see [11]), we have $q \equiv 0$, that is, the proof of the theorem is completed if $q_1, q_2 \in W_p^1(\Omega)$.

Fourth Step.

Now let $q \in C^{\alpha}(\overline{\Omega})$ with some $\alpha \in (0, 1)$ and $\partial \Omega = \widetilde{\Gamma}$.

We recall the following classical result of Hörmander [9]. Consider the "oscillatory integral operator"

$$T_{\tau}f(x) = \int_{\Omega} e^{-\tau i\psi(x,y)} a(x,y)f(y)dy,$$

where $\psi \in C^{\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$ and $a(\cdot, \cdot) \in C_0^{\infty}(\mathbb{R}^2 \times \mathbb{R}^2)$. We introduce the following matrix

$$H_{\psi} = \{\partial_{x_i y_j}^2 \psi\}.$$

Theorem 2 Suppose that det $H_{\psi} \neq 0$ on suppa. Then

$$||T_{\tau}||_{L^2 \to L^2} \le \frac{C}{\tau}.$$

Consider our holomorphic function $\Phi(x, y) = (x_1 + ix_2 - (y_1 + iy_2))^2 + i$. We set $\psi(x, y) = 2(x_1 - y_1)(x_2 - y_2) - 1$. Then

$$H_{\psi}(x,y) = \left(\begin{array}{cc} 0 & -2\\ -2 & 0 \end{array}\right)$$

and $det H_{\psi}(x, y) = -4$. Then the condition in Theorem 2 holds true.

We set $a(x, y) = \chi(x)\chi(y)$ where $\chi \in C_0^{\infty}(\mathbb{R}^n)$ and $\chi|_{\Omega} \equiv 1$. Then, by Theorem 2, there exists a constant C independent of τ such that

$$||T_{\tau}||_{L^2 \to L^2} + ||T_{-\tau}||_{L^2 \to L^2} \le C/\tau.$$
(23)

Setting $f = (q - q_{\epsilon})a\overline{a}\chi_{\Omega}$ by (23) we have

$$||T_{\tau}f||_{L^{2}(\Omega)} + ||T_{-\tau}f||_{L^{2}(\Omega)} \le C(\epsilon)/\tau, \quad C(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to +0.$$

$$\tag{24}$$

Therefore, by (24), in the ball $B(\tilde{x}, \delta) \equiv \{x; |x - \tilde{x}| < \delta\}$, there exists a sequence of points $y(\tau)$ such that

$$|(T_{\tau})f(y(\tau))| + |(T_{-\tau})f(y(\tau))| \le \frac{C\epsilon}{\tau\delta^2}.$$
(25)

Let $y(\tau) = (y_1(\tau), y_2(\tau)) \rightarrow \hat{y}(\epsilon)$ as $\tau \rightarrow +\infty$. By the stationary phase argument taking into account that $\psi(\tilde{x}, \tilde{x}) = -1$, we have

$$\int_{\Omega} (q_{\epsilon} - (q_{\epsilon} - q)(y(\tau)) \operatorname{Re}\left\{a\overline{a}e^{-2\tau i\psi(y(\tau),x)}\right\} dx = \frac{2\pi (q|a|^2)(\hat{y}(\epsilon)) \operatorname{Re}e^{2\tau i}}{\tau} + o\left(\frac{1}{\tau}\right).$$
(26)

From (16), (26), (25) we obtain

$$2\pi (q|a|^2)(\hat{y}(\epsilon))\operatorname{Re} e^{2\tau i} + \widetilde{C}(\epsilon) = 0, \qquad (27)$$

where $\overline{\lim}_{\tau \to +\infty} |\tilde{C}(\epsilon)| \to +0$ as $\epsilon \to 0$. Therefore as ϵ goes to zero, we have

 $q(\hat{x}) = 0.$

Here $\hat{x} \in B(\tilde{x}, \delta)$ such that $\hat{y}(\epsilon) \to \hat{x}$ as $\epsilon \to +0$. Since $\delta > 0$ and \tilde{x} are chosen arbitrarily, we conclude that $q \equiv 0$ in Ω . Thus the proof of the theorem is completed. \Box

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