Convergence of the Gauss-Newton method for convex composite optimization under a majorant condition

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Abstract

Under the hypothesis that an initial point is a quasi-regular point, we use a majorant condition to present a new semi-local convergence analysis of an extension of the Gauss-Newton method for solving convex composite optimization problems. In this analysis the conditions and proof of convergence are simplified by using a simple majorant condition to define regions where a Gauss-Newton sequence is "well behaved".

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1 Introduction

Consider the convex composite optimation problem

$$\min\ h(F(x)), \tag{1}$$

where $h : \mathbb{R}^m \to \mathbb{R}$ is a real-valued convex and $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable. As it is well known, see [1, 7, 8] and references therein, a wide variety of applications with this formulation can be found in mathematical programming literature, e.g., nonlinear inclusions, penalization methods, minimax, and goal programming. Besides its practical applications,

this model provides a convenient tool for the study of first and second order optimality conditions in constrained optimization.

The basic algorithm considered in [1, 7, 8], which is an extension of the Gauss-Newton method for solving nonlinear least square problem, will be considered in this paper. The study of (1) is related to the convex inclusion problem

$$F(x) \in C := \{z \in \mathbb{R}^m : h(z) \leq h(x), \, x \in \mathbb{R}^m\}, \tag{2}$$

because if $x_* \in \mathbb{R}^n$ satisfies the convex inclusion (2) then x_* is a solution of (1), but if $x_* \in \mathbb{R}^n$ is a solution of (1) it does not necessarily satisfy the inclusion convex (2). Although a priori, our goal is to give criteria that ensure the convergence of the sequence generated by the Gauss-Newton algorithm for a solution of (1), we will give a criteria that ensure the convergence of that sequence for some $x_* \in \mathbb{R}^n$ satisfying $F(x_*) \in C$ which, in particular, solves (1).

In this paper, we are interested in the semi-local convergence analysis, i.e., based on the information at an initial point, criteria are given that ensure the convergence of the sequence generated by the Gauss-Newton algorithm for some $x_* \in \mathbb{R}^n$ with $F(x_*) \in C$. Under the hypothesis that the initial point is a quasi-regular point of the inclusion (2), we use a majorant condition similar to the one used in [3, 4, 5] to present a new semi-local convergence analysis for the sequence generated by the Gauss-Newton algorithm. The convergence analysis presented here communicates the conditions and proof in quite a simple manner. This is possible thanks to our majorant condition and a demonstration techinique in which instead of only looking to the generated sequence, we identify regions where the Gauss-Newton sequence (for the convex composite optimation problem) is well behaved, as compared with Newton method applied to an auxiliary function associated with the majorant function. This technique was introduced in [4].

The convergence of the sequence generated by the Gauss-Newton algorithm was also studied in [1, 7, 8]. Among these, the criterion introduced by Li and Ng in [7] is the best. Besides the technique used in the demonstration, the main difference from our analysis regarding [7] is that they used Wang's condition, introduced in [13], in place of our majorant condition. But, the formulation using the majorant condition provides a clear relationship between the majorant function and the nonlinear function F under consideration. Besides this, the majorant condition simplifies the proof of convergence.

The organization of our paper is as follows. In section 1.1, we list some notation and one basic result. The Gauss-Newton algorithm is discussed in Section 2, in Section 2.1 we present some regularity properties, and an analysis of the majorant and auxiliary functions is established in Section 2.2. In Section 3 the main result is stated and in Section 3.1 it is proved. Some applications of this result are given in Section 4.

1.1 Notation and auxiliary result

The following notation and result are used throughout our presentation. Let $\mathbb{R}^n$ be with a norm $\|\cdot\|$. The open and closed ball in $\mathbb{R}^n$ with center x and radius r are denoted, respectively by $B(x,r)$ and $B[x,r]$. The polar of a closed convex $W \subset \mathbb{R}^n$ is the set $W^o := \{z \in \mathbb{R}^n : \langle z, w \rangle \leq 0, \forall w \in W\}$. The distance from a point x to a set $W \subset \mathbb{R}^n$ is given by $d(x,W) := \inf\{\|x - w\| : w \in W\}$. The set of all subsets of $\mathbb{R}^n$ is denoted by $P(\mathbb{R}^n)$ and $Ker(A)$ represents the kernel of the linear map A. Finally, the sum of a point $x \in \mathbb{R}^n$ with a set $X \in P(\mathbb{R}^n)$ is the set given by $y + X = \{y + x : x \in X\}$.

The following auxiliary result of elementary convex analysis will be needed:

Proposition 1. *Let $I \subset \mathbb{R}$ be an interval, and $\varphi : I \to \mathbb{R}$ be convex. If $u, v, w \in I$, $u < w$, and $u \leq v \leq w$ then*

$$\varphi(v) - \varphi(u) \leq [\varphi(w) - \varphi(u)] \frac{v - u}{w - u}.$$

Proof. See Theorem 4.1.1 on p.21 of [6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2 Preliminary

In this section we present the algorithm to solve problem (1), a brief study of regularity, and an analysis of our majorant and auxiliary functions. The results of this section are the main tools used in the proof of convergence of the sequence generated by the Gauss-Newton algorithm.

In order to state the Gauss-Newton algorithm, for solving problem (1), we need the following definition: For $\Delta \in (0, +\infty)$ and $x \in \mathbb{R}^n$ define

$$D_\Delta(x) := argmin\left\{h(F(x) + F'(x)d) : d \in \mathbb{R}^n, \|d\| \leq \Delta\right\}, \tag{3}$$

that is, $D_\Delta(x)$ is the solution set for the following problem

$$\min\left\{h(F(x) + F'(x)d) : d \in \mathbb{R}^n, \|d\| \leq \Delta\right\}. \tag{4}$$

Given that $\Delta \in (0, +\infty]$, $\eta \in [1, +\infty)$ and a point $x_0 \in \mathbb{R}^n$, *the Gauss-Newton type algorithm* associated with (Δ, η, x_0) as defined in [1] (see also, [7, 8]) is as follows:

Algorithm 1.

INITIALIZATION. *Take $\Delta \in (0, +\infty]$, $\eta \in [1, +\infty)$ and $x_0 \in \mathbb{R}^n$. Set $k = 0$.*
STOP CRITERION. *Compute $D_\Delta(x_k)$. If $0 \in D_\Delta(x_k)$, STOP. Otherwise.*
ITERATIVE STEP. *Compute d_k satisfying*

$$d_k \in D_\Delta(x_k), \qquad \|d_k\| \leq \eta d(0, D_\Delta(x_k)),$$

and set

$$x_{k+1} = x_k + d_k,$$

$k = k + 1$ *and GO TO* STOP CRITERION.

Note that, since (4) is a convex optimization problem in a compact set, it follows that the set $D_\Delta(x)$ is nonempty, for all $x \in \mathbb{R}^n$. Therefore, the sequence $\{x_k\}$ generated by **Algorithm 1** is well defined.

2.1 Regularity

In this section we state the hypothesis on the starting point of the sequence generated by **Algorithm 1**, which we need in our analysis, as well as some related concepts.

Let C be as defined in (2), that is, C is the set of all minimum points of h. For each $x \in \mathbb{R}^n$, we define the set $D_C(x)$ associated to C as

$$D_C(x) := \{d \in \mathbb{R}^n : F(x) + F'(x)d \in C\}.$$

In the next proposition we state a relation between the sets $D_\Delta(x)$ and $D_C(x)$.

Proposition 2. *Let $x \in \mathbb{R}^n$. If $D_C(x) \neq \emptyset$ and $d(0, D_C(x)) \leq \Delta$, then*

$$D_\Delta(x) = \{d \in \mathbb{R}^n : \|d\| \leq \Delta, F(x) + F'(x)d \in C\} \subset D_C(x).$$

As a consequence, $d(0, D_\Delta(x)) = d(0, D_C(x))$.

Proof. By definition of C in (2) and $D_\Delta(x)$ in (3) it can be seen that

$$\{d \in \mathbb{R}^n : \|d\| \leq \Delta, F(x) + F'(x)d \in C\} \subset D_\Delta(x).$$

Let $d \in D_\Delta(x)$. Since $D_C(x) \neq \emptyset$ and $d(0, D_C(x)) \leq \Delta$, there exists $\bar{d} \in D_C(x)$ such that $\|\bar{d}\| \leq \Delta$ and $F(x) + F'(x)\bar{d} \in C$. Hence, from the definition of C in (2) and $D_\Delta(x)$ in (3) we obtain $\bar{d} \in D_\Delta(x)$. Therefore, as $\bar{d}, d \in D_\Delta(x)$, and using again the definition of $D_\Delta(x)$ in (3), we have

$$h(F(x) + F'(x)d) = h(F(x) + F'(x)\bar{d}).$$

Now, using $F(x) + F'(x)\bar{d} \in C$, the last equality and definition of C, we obtain $F(x) + F'(x)d \in C$, which proves the first statement. The second statement, i.e., $D_\Delta(x) \subset D_C(x)$ can be seen by definition of $D_C(x)$. To conclude the proof, first note that the inclusion $D_\Delta(x) \subset D_C(x)$ implies that

$$d(0, D_\Delta(x)) \geq d(0, D_C(x)). \tag{5}$$

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Since $D_C(x) \neq \emptyset$ and $d(0, D_C(x)) \leq \Delta$, there exists $\bar{d} \in D_C(x)$ such that

$$\|\bar{d}\| = d(0, D_C(x)) \leq \Delta.$$

Hence, from definition of C in (2) and $D_\Delta(x)$ in (3) we conclude that $\bar{d} \in D_\Delta(x)$. Therefore,

$$d(0, D_\Delta(x)) \leq \|\bar{d}\| = d(0, D_C(x))$$

and taking into account (5), the proof is concluded. $\qquad\qquad\qquad\qquad\qquad$ $\square$

Definition 1. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function and let $h : \mathbb{R}^m \to \mathbb{R}$ be a real-valued convex function. A point $x_0 \in \mathbb{R}^n$ is called a quasi-regular point of the inclusion (2), that is, of the inclusion*

$$F(x) \in C := \{z \in \mathbb{R}^m : h(z) \leq h(x), \ x \in \mathbb{R}^m\},$$

if $r \in (0, +\infty)$ exists as well as an increasing positive-valued function $\beta : [0, r) \to (0, +\infty)$ such that

$$D_C(x) \neq \emptyset, \qquad d(0, D_C(x)) \leq \beta(\|x - x_0\|) d(F(x), C), \qquad \forall x \in B(x_0, r). \qquad (6)$$

Let $x_0 \in \mathbb{R}^n$ be a quasi-regular point of the inclusion (2). We denote r_{x_0} the supremum of r such that (6) holds for some increasing positive-valued function β on $[0, r)$, that is,

$$r_{x_0} := \sup \{r : \exists \beta : [0, r) \to (0, +\infty) \text{ satisfying } (6)\}. \qquad (7)$$

Let $r \in [0, r_{x_0})$. The set $\mathcal{B}_r(x_0)$ denotes the set of all increasing positive-valued functions β on $[0, r)$ such that (6) holds, that is,

$$\mathcal{B}_r(x_0) := \{\beta : [0, r) \to (0, +\infty) : \beta \text{ satisfying } (6)\}.$$

Define

$$\beta_{x_0}(t) := \inf \left\{\beta(t) : \beta \in \mathcal{B}_{r_{x_0}}(x_0)\right\}, \qquad t \in [0, r_{x_0}). \qquad (8)$$

The number r_{x_0} and the function β_{x_0} are called, respectively, the *quasi-regular radius* and the *quasi-regular bound function* of the quasi-regular point x_0.

Remark 1. *Note that from the definition of r_{x_0} and β_{x_0} it is easy to prove that for all $r \leq r_{x_0}$ such that $\lim_{t \to r^-} \beta(t) < +\infty$ it holds that*

$$\beta_{x_0}(t) = \inf \{\beta(t) : \beta \in \mathcal{B}_r(x_0)\}, \qquad t \in [0, r).$$

2.2 The majorant condition

In this section, we define the majorant condition for the nonlinear function F, which relaxes the assumption of Lipschitz continuity to F', used in our analysis. We present an analysis of the behavior of the majorant function and of a certain associated auxiliary function - more details about the majorant condition can be found in [3, 4, 5].

Definition 2. *Let $R > 0$, $x_0 \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable. A twice-differentiable function $f : [0, R) \to \mathbb{R}$ is a majorant function for the function F on $B(x_0, R)$ if it satisfies*

$$\|F'(y) - F'(x)\| \leq f'(\|y - x\| + \|x - x_0\|) - f'(\|x - x_0\|), \tag{9}$$

for any $x, y \in B(x_0, R)$, $\|x - x_0\| + \|y - x\| < R$, and moreover,

h1) $f(0) = 0$, $f'(0) = -1$;

h2) f' *is convex and strictly increasing.*

In the next result we bound the linearization error of the function F by the error in the linearization on the majorant function.

Lemma 3. *Take*

$$x, y \in B(x_0, R) \quad and \quad 0 \leq t < v < R.$$

If $\|x - x_0\| \leqslant t$ and $\|y - x\| \leqslant v - t$, then

$$\|F(y) - [F(x) + F'(x)(y - x)]\| \leqslant f(v) - [f(t) + f'(t)(v - t)] \left(\frac{\|y - x\|}{v - t}\right)^2.$$

Proof. The proof follows the same pattern as Lemma 7 from [5]. $\qquad\square$

To state our main theorem we need a certain auxiliary function associated with the majorant function. We shall see later that the sequence generated by **Algorithm 1** will be "majorized" by the Newton sequence associated with this auxiliary function.

Let $f : [0, R) \to \mathbb{R}$ be a majorant function for the function F on $B(x_0, R)$. Take $\xi > 0$, $\alpha > 0$ and define the *auxiliary function*

$$\begin{aligned} f_{\xi,\alpha} : [0, R) &\to \mathbb{R} \\ t &\mapsto \xi + (\alpha - 1)t + \alpha f(t). \end{aligned} \tag{10}$$

Now, consider the following conditions on the auxiliary function $f_{\xi,\alpha}$:

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h3) there exists $t_* \in (0, R)$ such that $f_{\xi,\alpha}(t) > 0$ for all $t \in (0, t_*)$ and $f_{\xi,\alpha}(t_*) = 0$;

h4) $f'_{\xi,\alpha}(t_*) < 0$.

From now on, we assume that $f : [0, \ R) \to \mathbb{R}$ is a majorant function for the function F on $B(x_0, R)$ and that **h3** holds. The assumption **h4** will be considered to hold only when explicitly stated.

Proposition 4. *The following statements hold:*

   i) $f_{\xi,\alpha}(0) = \xi > 0$, $f'_{\xi,\alpha}(0) = -1$;

   ii) $f'_{\xi,\alpha}$ *is convex and strictly increasing.*

Proof. Seen from the definition in (10) and assumptions **h1** and **h2**. $\qquad\square$

Proposition 5. *The function $f_{\xi,\alpha}$ is strictly convex, and*

$$f_{\xi,\alpha}(t) > 0, \quad f'_{\xi,\alpha}(t) < 0, \qquad t < t - f_{\xi,\alpha}(t)/f'_{\xi,\alpha}(t) < t_*, \qquad\qquad \forall\, t \in [0, t_*). \qquad (11)$$

Moreover, $f'_{\xi,\alpha}(t_*) \leq 0$.

Proof. Using Proposition 4, the proof follows the same pattern as Proposition 3 from [5]. $\qquad\square$

In view of the second inequality in (11), the Newton iteration map is well defined in $[0, t_*)$. Let us call it

$$\begin{aligned} n_{f_{\xi,\alpha}} : [0, t_*) &\to \mathbb{R} \\ t &\mapsto t - f_{\xi,\alpha}(t)/f'_{\xi,\alpha}(t). \end{aligned} \qquad (12)$$

Proposition 6. *For each $t \in [0, t^*)$ it holds that $\xi \leq n_{f_{\xi,\alpha}}(t)$.*

Proof. Proposition 5 implies that $f_{\xi,\alpha}$ is convex. Hence, using the first item of Proposition 4 it is easy to see, by using convexity properties, that $t - \xi \geq -f_{\xi,\alpha}(t)$. So, the above definition implies that

$$n_{f_{\xi,\alpha}}(t) - \xi = t - \frac{f_{\xi,\alpha}(t)}{f'_{\xi,\alpha}(t)} - \xi \geq -f_{\xi,\alpha}(t) - \frac{f_{\xi,\alpha}(t)}{f'_{\xi,\alpha}(t)} = \frac{f_{\xi,\alpha}(t)}{-f'_{\xi,\alpha}(t)}[f'_{\xi,\alpha}(t) + 1], \qquad \forall\, t \in [0, t_*).$$

Proposition 4 implies that $f'_{\xi,\alpha}(0) = -1$ and $f'_{\xi,\alpha}$ is strictly increasing. Thus, we obtain $f'_{\xi,\alpha}(t) + 1 \geq 0$, for all $t \in [0, t_*)$. Therefore, combining the above inequality with the first two inequalities in Proposition 5 the desired result follows. $\qquad\square$

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Proposition 7. *Newton iteration map $n_{f_{\xi,\alpha}}$ maps $[0, t^*)$ in $[0, t^*)$, and it holds that*

$$t < n_{f_{\xi,\alpha}}(t), \qquad t_* - n_{f_{\xi,\alpha}}(t) \leqslant \frac{1}{2}(t_* - t), \qquad \forall\, t \in [0, t_*).$$

If $f_{\xi,\alpha}$ also satisfies **h4***, i.e., $f'_{\xi,\alpha}(t_*) < 0$, then*

$$t_* - n_{f_{\xi,\alpha}}(t) \leq \frac{f''_{\xi,\alpha}(t_*)}{-2f'_{\xi,\alpha}(t_*)}(t_* - t)^2, \qquad \forall\, t \in [0, t_*).$$

Proof. The proof follows the same pattern as Proposition 4 of [5]. □

The Newton sequence $\{t_k\}$ for solving the equation $f_{\xi,\alpha}(t) = 0$ with starting point $t_0 = 0$ is defined as

$$t_0 = 0, \quad t_{k+1} = n_{f_{\xi,\alpha}}(t_k), \qquad k = 0, 1, \dots . \tag{13}$$

Therefore, by also using Proposition 7 it is easy to prove that

Corollary 8. *The sequence $\{t_k\}$ is well defined, is strictly increasing, is contained in $[0, t_*)$, and converges Q-linearly to t_* as follows*

$$t_* - t_{k+1} \leq \frac{1}{2}(t_* - t_k), \qquad k = 0, 1, \dots .$$

If $f_{\xi,\alpha}$ also satisfies assumption **h4***, then $\{t_k\}$ converges Q-quadratically to t_* as follows*

$$t_* - t_{k+1} \leq \frac{f''_{\xi,\alpha}(t_*)}{-2f'_{\xi,\alpha}(t_*)}(t_* - t_k)^2, \quad k = 0, 1, \dots .$$

Proposition 9. *The map $[0, t_*) \ni t \to -f_{\xi,\alpha}(t)/f'_{\xi,\alpha}(t)$ is decreasing.*

Proof. Proposition 5 implies that $f'_{\xi,\alpha}(t) \neq 0$ for all $t \in [0, t_*)$. So, the function in the proposition is well defined. As $f_{\xi,\alpha}$ is twice-differentiable we have

$$\left(\frac{-f_{\xi,\alpha}(t)}{f'_{\xi,\alpha}(t)}\right)' = \frac{f_{\xi,\alpha}(t)f''_{\xi,\alpha}(t) - (f'_{\xi,\alpha}(t))^2}{(f'_{\xi,\alpha}(t))^2}, \qquad \forall\, t \in [0, t_*).$$

Hence, it suffices to show that

$$f_{\xi,\alpha}(t)f''_{\xi,\alpha}(t) - (f'_{\xi,\alpha}(t))^2 \leq 0, \qquad \forall\, t \in [0, t_*). \tag{14}$$

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Since $f_{\xi,\alpha}$ is strictly convex (Proposition 5) and $f'_{\xi,\alpha}$ is convex (Proposition 4), we have

$$0 > f_{\xi,\alpha}(t) + f'_{\xi,\alpha}(t)(t_* - t), \quad f''_{\xi,\alpha}(t) \geq 0, \quad f'_{\xi,\alpha}(t_*) \geq f'_{\xi,\alpha}(t) + f''_{\xi,\alpha}(t)(t_* - t), \quad \forall\, t \in [0, t_*).$$

Using these inequalities and the second inequality in (11), we obtain

$$f_{\xi,\alpha}(t)f''_{\xi,\alpha}(t) - (f'_{\xi,\alpha}(t))^2 \leq f'_{\xi,\alpha}(t)(t - t_*)f''_{\xi,\alpha}(t) - (f'_{\xi,\alpha}(t))^2 \leq -f'_{\xi,\alpha}(t)f'_{\xi,\alpha}(t_*),$$

which combined with Proposition 5 yields the inequality in (14). Therefore, the proposition is fulfilled. $\square$

Proposition 10. *It holds that $\xi < t_*$. Moreover, if*

$$\alpha \geq \frac{\eta\beta_{x_0}(t)}{\eta\beta_{x_0}(t)[f'(t) + 1] + 1}, \qquad \forall\, \xi \leq t < t_*, \tag{15}$$

then

$$\eta\beta_{x_0}(t)/\alpha \leq -1/f'_{\xi,\alpha}(t), \qquad \forall\, \xi \leq t < t_*.$$

Proof. Proposition 5 implies that $f_{\xi,\alpha}$ is strictly convex, which combined with the definition of t_* in **h3** and item **i** of Proposition 4 gives

$$0 = f_{\xi,\alpha}(t_*) > f_{\xi,\alpha}(0) + f'_{\xi,\alpha}(0)(t_* - 0) = \xi - t_*,$$

which proves the first statement.

Combining the assumption in (15), as well as **h1** and **h2**, we obtain after simple calculus that

$$\alpha\eta\beta_{x_0}(t)(f'(t) + 1) + \alpha \geq \eta\beta_{x_0}(t), \qquad \forall\, \xi \leq t < t_*.$$

Hence, using $f'_{\xi,\alpha}(t) = (\alpha - 1) + \alpha f'(t)$ and some algebraic manipulations, the last inequality becomes

$$\eta\beta_{x_0}(t)f'_{\xi,\alpha}(t) \geq -\alpha, \qquad \forall\, \xi \leq t < t_*,$$

which combined with the second inequality in (11) yields the desired inequality. $\square$

Proposition 11. *Let $0 < \bar\alpha < \alpha$ for the corresponding auxiliary functions $f_{\xi,\bar\alpha}$ and $f_{\xi,\alpha}$, as well as $\bar{t}_*$ and t_*, its smallest zeros, respectively. Then the following assertions hold:*

i) $f_{\xi,\bar\alpha} < f_{\xi,\alpha}$ *on* $(0, R)$;

ii) $f'_{\xi,\bar\alpha} < f'_{\xi,\alpha}$ *on* $(0, R)$;

iii) $\bar{t}_* < t_*$.

Proof. From **h2** it follows that f' is strictly increasing which implies that f is strictly convex. Thus, using **h1** we conclude that $f(t) + t > 0$, for all $t \in (0, R)$ and hence the assumption $\alpha > \bar{\alpha}$ implies

$$\bar{\alpha}(t + f(t)) < \alpha(t + f(t)), \qquad \forall\, t \in [0, R).$$

To conclude the proof of item **i**, add $\xi - t$ on both sides of the last inequality and use the definition in (10).

To prove item **ii**, we first use that f' is strictly increasing (**h2**), as well as the assumption $\alpha > \bar{\alpha}$ to obtain that $(\alpha - \bar{\alpha})(f'(t) - f'(0)) > 0$, for all $t \in (0, R)$. Hence, from **h1** and some algebraic manipulation, we obtain

$$(\bar{\alpha} - 1) + \bar{\alpha} f'(t) < (\alpha - 1) + \alpha f'(t), \qquad \forall\, t \in [0, R).$$

So, by using the definition in (10), the statement holds true.

To establish item **iii**, use item **i** and the definition of $\bar{t}_*$ and t_* in **h3**. $\qquad\square$

3   Semi-local analysis for the Gauss-Newton method

In this section our goal is to state and prove a semi-local theorem for the sequence generated by **Algorithm 1** in order to solve problem (1). Under the hypothesis that the initial point is a quasi-regular point of the inclusion (2) and the nonlinear function F satisfies the majorant condition in Definition 2, we will prove convergence of the sequence to a point $x_* \in B[x_0, t_*]$ such that $F(x_*) \in C$, and in particular that x_* solves (1). The statement of the theorem is:

Theorem 12. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function. Assume that $R > 0$, $x_0 \in \mathbb{R}^n$ and $f : [0,\ R) \to \mathbb{R}$ is a majorant function for F on $B(x_0, R)$. Take the constants $\alpha > 0$ and $\xi > 0$ and consider the auxiliary function $f_{\xi,\alpha} : [0, R) \to \mathbb{R}$,*

$$f_{\xi,\alpha}(t) := \xi + (\alpha - 1)t + \alpha f(t).$$

*If $f_{\xi,\alpha}$ satisfies **h3**, i.e., t_* is the smallest zero of $f_{\xi,\alpha}$, then the sequence generated by Newton's method for solving $f_{\xi,\alpha}(t) = 0$, with starting point $t_0 = 0$,*

$$t_{k+1} = t_k - f'_{\xi,\alpha}(t_k)^{-1} f_{\xi,\alpha}(t_k), \quad k = 0, 1, \dots, \tag{16}$$

is well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$, and converges Q-linearly to t_*. Let $\eta \in [1, \infty)$, $\Delta \in (0, \infty]$ and $h : \mathbb{R}^m \to \mathbb{R}$ a real-valued convex function with minimizer set C nonempty. Suppose that $x_0 \in \mathbb{R}^n$ is a quasi-regular point of the inclusion

$$F(x) \in C,$$

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with the quasi-regular radius r_{x_0} and the quasi-regular bound function β_{x_0} as defined in (7) and (8), respectively. If $d(F(x_0), C) > 0$, $t_* \leq r_{x_0}$,

$$\Delta \geq \xi \geq \eta \beta_{x_0}(0) d(F(x_0), C), \qquad \alpha \geq \sup \left\{ \frac{\eta \beta_{x_0}(t)}{\eta \beta_{x_0}(t)[f'(t) + 1] + 1} : \xi \leq t < t_* \right\}, \qquad (17)$$

then the sequence generated by **Algorithm 1**, denoted by $\{x_k\}$, is contained in $B(x_0, t_*)$,

$$F(x_k) + F'(x_k)(x_{k+1} - x_k) \in C, \quad k = 0, 1, \ldots, \qquad (18)$$

satisfies the inequalities

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \qquad \|x_{k+1} - x_k\| \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} \|x_k - x_{k-1}\|^2, \qquad (19)$$

$k = 0, 1, \ldots$, and $k = 1, 2, \ldots$, respectively, converge to a point $x_* \in B[x_0, t_*]$ such that $F(x_*) \in C$,

$$\|x_* - x_k\| \leq t_* - t_k, \qquad k = 0, 1, \ldots \qquad (20)$$

and the convergence is R-linear. If, additionally, $f_{\xi, \alpha}$ satisfies **h4** then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q-quadratically and R-quadratically to t_* and x_*, respectively.

Remark 2. *If,*

$$\alpha > \bar{\alpha} := \sup \left\{ \frac{\eta \beta_{x_0}(t)}{\eta \beta_{x_0}(t)[f'(t) + 1] + 1} : \xi \leq t < t_* \right\},$$

*then the sequence $\{x_k\}$ converges R-quadratically to x_*. To prove this assertion, note that through item **iii** of Proposition 11, we have $\bar{t}_* < t_*$. Hence, using that $f'_{\xi, \bar{\alpha}}$ strictly increasing and item **ii** of Proposition 11, we obtain*

$$f'_{\xi, \bar{\alpha}}(\bar{t}_*) < f'_{\xi, \bar{\alpha}}(t_*) < f'_{\xi, \alpha}(t_*),$$

which, combined with Proposition 5 implies that $f'_{\xi, \bar{\alpha}}(t_{\bar{*}}) < 0$. So, the statement is correct if $f_{\xi, \alpha}$ is replaced by $f_{\xi, \bar{\alpha}}$ in Theorem 12.

Remember that all statements made in Theorem 12 for the sequence t_k were proven in Corollary 8.

From now on, we assume that the hypotheses of Theorem 12 hold, with the exception of **h4**, which will be considered to hold only when explicitly stated.

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3.1 Proof of convergence

In this section we prove convergence of the sequence $\{x_k\}$ generated by **Algorithm 1** for solving (1), based on the assumptions stated in Theorem 12.

As we saw in Section 2, $D_\Delta(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$, therefore the sequence $\{x_k\}$ is well defined. But this is not enough to prove the convergence of sequence $\{x_k\}$ to some point $x_* \in \mathbb{R}^n$ such that $F(x_*) \in C$, because we have no relationship between the set of search directions $D_\Delta(x)$ to the set of solutions of the linearized inclusion

$$F(x) + F'(x)d \in C, \qquad \|d\| \leq \Delta.$$

Now, if we prove that $D_\Delta(x) \subset D_C(x)$ for suitable points, then we can use the results of regularity to relate the sets mentioned above. First, we define some subsets of $B(x_0, t_*)$ in which, as we shall prove, the desired inclusion holds for all points in these subsets.

$$K(t) := \left\{ x \in \mathbb{R}^n \ : \ \|x - x_0\| \leq t, \ \eta d(0, D_C(x)) \leqslant -\frac{f_{\xi,\alpha}(t)}{f'_{\xi,\alpha}(t)} \right\}, \qquad t \in [0, t_*), \qquad (21)$$

$$K := \bigcup_{t \in [0, t_*)} K(t). \qquad (22)$$

In (21) we assume that $0 \leqslant t < t_*$, therefore it follows from Proposition 5 that $f'_{\xi,\alpha}(t) \neq 0$. So, the above definitions are consistent.

Proposition 13. *If $x \in K$, then*

$$D_\Delta(x) = \{d \in \mathbb{R}^n \ : \ F(x) + F'(x)d \in C, \ \|d\| \leq \Delta\} \subset D_C(x),$$

and

$$d(0, D_\Delta(x)) = d(0, D_C(x)).$$

Proof. From Proposition 2 it is sufficient to prove that $D_C(x) \neq \emptyset$ and $d(0, D_C(x)) \leq \Delta$ for all $x \in K$. Let $x \in K$, then $x \in K(t)$ for some $t \in [0, t_*)$ which implies that $x \in B(x_0, t_*)$. Since $t_* \leq r_{x_0}$ and x_0 is a quasi-regular point, it follows from Definition 1 and the definition of the quasi-regular radius in (7) that $D_C(x) \neq \emptyset$.

By hypothesis $\eta \geq 1$ and $\xi \leqslant \Delta$. Thus, as $x \in K(t)$, by using the definition in (21), Proposition 9, and Proposition 4 we obtain

$$d(0, D_C(x)) \leqslant \eta d(0, D_C(x)) \leqslant -\frac{f_{\xi,\alpha}(t)}{f'_{\xi,\alpha}(t)} \leqslant -\frac{f_{\xi,\alpha}(0)}{f'_{\xi,\alpha}(0)} = \xi \leqslant \Delta,$$

which proves the desired result. $\qquad \square$

For each $x \in \mathbb{R}^n$, we define the set $\bar{D}_\Delta(x)$ as

$$\bar{D}_\Delta(x) := \left\{ d \in D_\Delta(x) \; : \; \|d\| \leq \eta d(0, D_\Delta(x)) \right\}. \tag{23}$$

As $D_\Delta(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$, we have $\bar{D}_\Delta(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$ and consequently, the Gauss-Newton iteration multifunction is well defined. Let us call G_F the Gauss-Newton iteration multifunction for F in $B(x_0, t_*)$:

$$\begin{aligned} G_F : B(x_0, t_*) &\to P(\mathbb{R}^n) \\ x &\mapsto x + \bar{D}_\Delta(x). \end{aligned} \tag{24}$$

We shall prove that the Gauss-Newton iteration multifunction is "well behaved" on the subsets defined in (21), but first we need the following technical result:

Lemma 14. *For each $t \in [0, t_*)$, $x \in K(t)$ and $y \in G_F(x)$ it holds that:*

i) $\|y - x\| \leq n_{f_{\xi,\alpha}}(t) - t$;

ii) $\|y - x_0\| \leq n_{f_{\xi,\alpha}}(t) < t_*$;

iii) $\eta d(0, D_C(y)) \leq -\dfrac{f_{\xi,\alpha}(n_{f_{\xi,\alpha}}(t))}{f'_{\xi,\alpha}(n_{f_{\xi,\alpha}}(t))} \left(\dfrac{\|y - x\|}{n_{f_{\xi,\alpha}}(t) - t} \right)^2.$

Proof. Since $t \in [0, t_*)$ and $x \in K(t)$, by using the definition in (21), Proposition 13 and the first two statements in Proposition 7, we obtain

$$\|x - x_0\| \leq t, \qquad \eta d(0, D_\Delta(x)) = \eta d(0, D_C(x)) \leqslant -\frac{f_{\xi,\alpha}(t)}{f'_{\xi,\alpha}(t)}, \qquad t < n_{f_{\xi,\alpha}}(t) < t_*. \tag{25}$$

Now, as $y \in G_F(x)$ there exists $d \in \bar{D}_\Delta(x)$ such that $y = x + d$. Using the definition of the set $\bar{D}_\Delta(x)$ in (23) and the second inequality in (25) it follows that

$$\|d\| \leq \eta d(0, D_\Delta(x)) = \eta d(0, D_C(x)) \leq -f_{\xi,\alpha}(t)/f'_{\xi,\alpha}(t).$$

Since $d = y - x$, the last inequality together with the definition in (12) implies item **i**.

Triangular inequality combined with the first inequality in (25), item **i**, and the last inequality in (25) yields

$$\|y - x_0\| \leq \|y - x\| + \|x - x_0\| \leq n_{f_{\xi,\alpha}}(t) < t_*, \tag{26}$$

which proves item **ii**.

Since $\|y - x_0\| < t_*$ and $t_* \le r_{x_0}$ we obtain by the quasi regularity assumption

$$D_C(y) \ne \emptyset, \qquad d(0, D_C(y)) \le \beta_{x_0}(\|y - x_0\|)d(F(y), C).$$

As $x \in K(t) \subset K$ and $y - x = d \in D_\Delta(x)$, it follows from Proposition 13 that

$$F(x) + F'(x)(y - x) \in C.$$

Therefore, taking into account that $\eta \ge 1$, by using the above inequality and the last inclusion it is easy to conclude that

$$\eta d(0, D_C(y)) \le \eta \beta_{x_0}(\|y - x_0\|)\|F(y) - F(x) - F'(x)(y - x)\|.$$

On the other hand, from item **i** we have $\|y - x\| \le n_{f_{\xi,\alpha}}(t) - t$ and, as $\|x - x_0\| \le t$, by using Lemma 3 we have

$$\|F(y) - F(x) - F'(x)(y - x)\| \le [f(n_{f_{\xi,\alpha}}(t)) - f(t) - f'(t)(n_{f_{\xi,\alpha}}(t) - t)]\left(\frac{\|y - x\|}{n_{f_{\xi,\alpha}}(t) - t}\right)^2.$$

Hence, combining the two above inequalities we conclude that

$$\eta d(0, D_C(y)) \le \eta \beta_{x_0}(\|y - x_0\|)[f(n_{f_{\xi,\alpha}}(t)) - f(t) - f'(t)(n_{f_{\xi,\alpha}}(t) - t)]\left(\frac{\|y - x\|}{n_{f_{\xi,\alpha}}(t) - t}\right)^2. \quad (27)$$

Now, the definition in (12) implies that $f_{\xi,\alpha}(t) + f'_{\xi,\alpha}(t)(n_{f_{\xi,\alpha}}(t) - t) = 0$. So, we have

$$f_{\xi,\alpha}(n_{f_{\xi,\alpha}}(t)) = f_{\xi,\alpha}(n_{f_{\xi,\alpha}}(t)) - f_{\xi,\alpha}(t) - f'_{\xi,\alpha}(t)(n_{f_{\xi,\alpha}}(t) - t)$$

By using the definition in (10) and after simple algebraic manipulation, the last equality becomes

$$f_{\xi,\alpha}(n_{f_{\xi,\alpha}}(t)) = \alpha \left(f(n_{f_{\xi,\alpha}}(t)) - f(t) - f'(t)(n_{f_{\xi,\alpha}}(t) - t)\right).$$

So, as β_{x_0} is an increasing function, by a simple combination of (26), (27) and the last equality, we obtain

$$\eta d(0, D_C(y)) \le \frac{\eta \beta_{x_0}(n_{f_{\xi,\alpha}}(t))}{\alpha} f_{\xi,\alpha}(n_{f_{\xi,\alpha}}(t))\left(\frac{\|y - x\|}{n_{f_{\xi,\alpha}}(t) - t}\right)^2.$$

From Proposition 6 and the first statement in Proposition 7 we have $\xi \le n_{f_{\xi,\alpha}}(t) < t_*$. Thus, by using the last inequality and Proposition 10, the last inequality of the lemma follows. $\quad\square$

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In the next result we prove the desired result, namely, that the Gauss-Newton iteration multifunction is "well behaved" on the subsets defined in (21).

Lemma 15. *For each $t \in [0, t_*)$, the following inclusions hold: $K(t) \subset B(x_0, t_*)$ and*

$$G_F\left(K(t)\right) \subset K\left(n_{f_{\xi,\alpha}}(t)\right).$$

As a consequence, $K \subset B(x_0, t_*)$ and $G_F(K) \subset K$.

Proof. The first inclusion follows trivially from the definition of $K(t)$. Take $x \in K(t)$ and $y \in G_F(x)$. Combining items **i** and **iii** of Lemma 14 we have

$$\eta d(0, D_C(y)) \leq -\frac{f_{\xi,\alpha}(n_{f_{\xi,\alpha}}(t))}{f'_{\xi,\alpha}(n_{f_{\xi,\alpha}}(t))}.$$

The last inequality together with item **ii** of Lemma 14 and the definition in (21) show us that $y \in K(n_{f_{\xi,\alpha}}(t))$, which proves the second inclusion.

The next inclusion, first on the second sentence, follows trivially from definitions (21) and (22). To verify the last inclusion, take $x \in K$. Therefore, $x \in K(t)$ for some $t \in [0, t_*)$. Using the first part of the lemma, we conclude that $G_F(x) \subset K(n_{f_{\xi,\alpha}}(t))$. To end the proof, note that $n_{f_{\xi,\alpha}}(t) \in [0, t_*)$ and use the definition of K. $\square$

Finally, we are ready to prove the main result of this section which is an immediate consequence of the latter results. First, note that definitions (23) and (24) imply that the sequence $\{x_k\}$ satisfies

$$x_{k+1} \in G_F(x_k), \qquad k = 0, 1, \ldots, \tag{28}$$

which is indeed an equivalent definition of this sequence.

Corollary 16. *The sequence $\{x_k\}$ which is contained in $B(x_0, t_*)$, converges to a point $x_* \in B[x_0, t_*]$ such that $F(x_*) \in C$. Moreover, $\{x_k\}$ and $\{t_k\}$ satisfy (18), (19) and (20). Furthermore, if $f_{\xi,\alpha}$ also satisfies assumption **h4** then $\{x_k\}$ converges R-quadratically to x_*.*

Proof. Since $x_0 \in B(x_0, t_*) \subseteq B(x_0, r_{x_0})$; by using the quasi regularity assumption, $\eta \geq 1$, the first inequality in (17), and Proposition 4; we obtain

$$D_C(x_0) \neq \emptyset, \qquad \eta d(0, D_C(x_0)) \leq \eta \beta_{x_0}(0) d(F(x_0), C) \leq \xi = -\frac{f_{\xi,\alpha}(0)}{f'_{\xi,\alpha}(0)}.$$

Therefore,

$$x_0 \in K(0) \subset K,$$

where the second inclusion follows trivially from (22). Using the above inclusion, the inclusions $G_F(K) \subset K$ (Lemma 15) and (28), we conclude that the sequence $\{x_k\}$ rests in K and, in particular, we have $\{x_k\}$ contained in $B(x_0, t_*)$. Since $\{x_k\} \subset K$, by combining Proposition 13 and **Algorithm 1**, the inclusion in (18) follows. Now, we prove by induction that

$$x_k \in K(t_k), \qquad k = 0, 1, \dots . \tag{29}$$

The above inclusion, for $k = 0$, is the first result in this proof. Assume that $x_k \in K(t_k)$. From (13) we have $t_{k+1} = \eta_{f_{\xi,\alpha}}(t_k)$ and, as $x_k \in K(t_k)$, Lemma 15 implies that $G_F(x_k) \subset K(t_{k+1})$, which taking into account (28) completes the induction proof.

Simple combination of **Algorithm 1** with (29), Proposition 13, and (21) yields

$$\|x_{k+1} - x_k\| \leq \eta d(0, D_\Delta(x_k)) = \eta d(0, D_C(x_k)) \leqslant -\frac{f_{\xi,\alpha}(t_k)}{f'_{\xi,\alpha}(t_k)}, \qquad k = 0, 1, \dots , \tag{30}$$

which, using (16) becomes

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \qquad k = 0, 1, \dots .$$

So, the first inequality in (19) holds. On the other hand, as $\{t_k\}$ converges to t_*, the above inequalities imply that

$$\sum_{k=k_0}^{\infty} \|x_{k+1} - x_k\| \leq \sum_{k=k_0}^{\infty} t_{k+1} - t_k = t_* - t_{k_0} < +\infty,$$

for any $k_0 \in \mathbb{N}$. Hence, $\{x_k\}$ is a Cauchy sequence in $B(x_0, t_*)$ and so converges to some $x_* \in B[x_0, t_*]$. Moreover, the above inequality also implies (20), i.e., $\|x_* - x_k\| \leq t_* - t_k$, for any k. As C is closed, $\{x_k\}$ converges to x_*,

$$F(x_k) + F'(x_k)(x_{k+1} - x_k) \in C,$$

and F is a continuously differentiable function; therefore, we have $F(x_*) \in C$.

In order to prove the second inequality in (19), first note that $x_k \in K(t_k)$ and $t_{k+1} = n_{f_{\xi,\alpha}}(t_k)$, for all $k = 0, 1, \dots .$ Thus, take an arbitrary k and apply item **iii** of Lemma 14 with $y = x_k$, $x = x_{k-1}$ and $t = t_{k-1}$ to obtain

$$\eta d(0, D_C(x_k)) \leq -\frac{f_{\xi,\alpha}(t_k)}{f'_{\xi,\alpha}(t_k)} \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}} \right)^2 ,$$

which, using (16) and the first inequality in (30) yields the desired inequality.

To end the proof, combine (20) with the last inequality in Corollary 8. $\square$

Therefore, it follows from Corollaries 8 and 16 that all statements in Theorem 12 are valid.

4   Special cases

In this section, we present special cases for Theorem 12. They include the case where x_0 is a regular point of the inclusion (2), and the case where x_0 satisfies the Robinson condition. Moreover, we present the result of convergence under the Lipschitz and Smale conditions.

4.1   Convergence result for regular starting point

In this section we present a correspondent theorem to Theorem 12, namely, we assume that x_0 is a regular point of the inclusion (2), see [1] and references therein. We also present results of convergence under the Lipschitz and Smale condition. We start by defining regularity.

Definition 3. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function and let $h : \mathbb{R}^m \to \mathbb{R}$ be a real-valued convex function with minimizer set C nonempty. A point $x_0 \in \mathbb{R}^n$ is a regular point of the inclusion $F(x) \in C$ if*

$$Ker(F'(x_0)^T) \cap (C - F(x_0))^o = \{0\},$$

As we know (see [7]) the definition of a quasi-regular point extends the definition of a regular point. The following proposition relates these two concepts, where the existence of constants r and β is due to Burke and Ferris in [1], and the second assertion then follows from Remark 1.

Proposition 17. *Let $x_0 \in \mathbb{R}^n$ be a regular point of the inclusion $F(x) \in C$. Then there exist constants $r > 0$ and $\beta > 0$ such that*

$$D_C(x) \neq \emptyset \quad e \quad d(0, D_C(x)) \leq \beta d(F(x), C), \qquad \forall x \in B(x_0, r).$$

Consequently, x_0 is a quasi-regular point with the quasi-regular radius $r_{x_0} \geq r$ and the quasi-regular bound function $\beta_{x_0}(\cdot) \leq \beta$ on $[0, r)$, as defined in (7) and (8), respectively.

From now on, for each regular point $x_0 \in \mathbb{R}^n$ of the inclusion $F(x) \in C$ we will denote by $r > 0$ and $\beta > 0$, the associated constants given by the last proposition.

Theorem 18. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function. Assume that $R > 0$, $x_0 \in \mathbb{R}^n$ and $f : [0, R) \to \mathbb{R}$ is a majorant function for F on $B(x_0, R)$. Take the constants $\alpha > 0$ and $\xi > 0$ and consider the auxiliary function $f_{\xi,\alpha} : [0, R) \to \mathbb{R}$,*

$$f_{\xi,\alpha}(t) = \xi + (\alpha - 1)t + \alpha f(t).$$

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*If, $f_{\xi,\alpha}$ satisfies **h3**, i.e., t_* is the smallest zero of $f_{\xi,\alpha}$, then the sequence generated by Newton's Method for solving $f_{\xi,\alpha}(t) = 0$, with starting point $t_0 = 0$,*

$$t_{k+1} = t_k - f_{\xi,\alpha}'(t_k)^{-1}f_{\xi,\alpha}(t_k), \quad k = 0, 1, \ldots,$$

is well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$, and converges Q-linearly to t_*. Let $\eta \in [1, \infty)$, $\Delta \in (0, \infty]$ and $h : \mathbb{R}^m \to \mathbb{R}$ be a real-valued convex function with minimizer set C nonempty. Suppose that $x_0 \in \mathbb{R}^n$ is a regular point of the inclusion $F(x) \in C$ with associated constants $r > 0$ and $\beta > 0$. If $d(F(x_0), C) > 0$, $t_* \leq r$,

$$\Delta \geq \xi \geq \eta\beta d(F(x_0), C), \qquad \alpha \geq \eta\beta/(\eta\beta[f'(\xi) + 1] + 1),$$

*then the sequence generated by **Algorithm 1**, denoted by $\{x_k\}$, is contained in $B(x_0, t_*)$,*

$$F(x_k) + F'(x_k)(x_{k+1} - x_k) \in C, \quad k = 0, 1, \ldots,$$

satisfies the inequalities

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \qquad \|x_{k+1} - x_k\| \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2}\|x_k - x_{k-1}\|^2,$$

for $k = 0, 1, \ldots$, and $k = 1, 2, \ldots$, respectively, converging to a point $x_* \in B[x_0, t_*]$ such that $F(x_*) \in C$,

$$\|x_* - x_k\| \leq t_* - t_k, \qquad k = 0, 1, \ldots$$

*and the convergence is R-linear. If, additionally, $f_{\xi,\alpha}$ satisfies **h4** then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q-quadratically and R-quadratically to t_* and x_*, respectively.*

Proof. Since x_0 is a regular point for the inclusion, we have from Proposition 17 that x_0 is a quasi-regular point for the inclusion $F(x) \in C$ with the quasi-regular radius $r_{x_0} \geq r$. So, taking into account the assumption $t_* \leq r$ we obtain

$$t_* < r_{x_0}.$$

Moreover, Proposition 17 also implies that the quasi-regular bound function

$$\beta_{x_0}(t) \leq \beta, \qquad \forall \, t \, \in [0, r). \tag{31}$$

Since $\Delta \geq \xi \geq \eta\beta d(F(x_0), C)$ and the last inequality implies that $\beta_{x_0}(0) \leq \beta$, we have

$$\Delta \geq \xi \geq \eta\beta_{x_0}(0)d(F(x_0), C).$$

Now, combining the assumptions $0 < \xi$ and $t_* \leq r$ with the first statement in Proposition 10 we conclude that $0 < \xi < t_* \leq r$. So, using (31), $f'(0) = -1$, f' as strictly increasing and $\eta \geq 1$; after simple algebraic manipulation we obtain

$$\frac{\eta\beta}{\eta\beta[f'(\xi)+1]+1} \geq \frac{\eta\beta_{x_0}(t)}{\eta\beta_{x_0}(t)[f'(t)+1]+1}, \qquad \forall\, t \in [\xi, t_*).$$

Hence, the assumption $\alpha \geq \eta\beta/(\eta\beta[f'(\xi)+1]+1)$ and the last inequality imply that

$$\alpha \geq \sup\left\{\frac{\eta\beta_{x_0}(t)}{\eta\beta_{x_0}(t)[f'(t)+1]+1} : \xi \leq t < t_*\right\}.$$

Therefore, F and x_0 satisfy all assumptions in Theorem 12 and consequently the statements of the theorem are satisfied. □

Under the Lipschitz condition, Theorem 18 becomes:

Theorem 19. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function. Assume that $x_0 \in \mathbb{R}^n$, $R > 0$ and $K > 0$, such that*

$$\|F'(y) - F'(x)\| \leq K\|y - x\|, \qquad x, y \in B(x_0, R).$$

Take the constants $\alpha > 0$ and $\xi > 0$ and consider the auxiliary function $f_{\xi,\alpha} : [0, R) \to \mathbb{R}$,

$$f_{\xi,\alpha}(t) = \xi - t + (\alpha K t^2)/2.$$

If $2\alpha K\xi \leq 1$, then $t_* = (1 - \sqrt{1 - 2\alpha K\xi})/(\alpha K)$ is the smallest zero of $f_{\xi,\alpha}$, the sequence generated by Newton's Method for solving $f_{\xi,\alpha}(t) = 0$, with starting point $t_0 = 0$,

$$t_{k+1} = t_k - f'_{\xi,\alpha}(t_k)^{-1} f_{\xi,\alpha}(t_k), \quad k = 0, 1, \dots,$$

is well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$, and converges Q-linearly to t_*. Let $\eta \in [1, \infty)$, $\Delta \in (0, \infty]$ and $h : \mathbb{R}^m \to \mathbb{R}$ be a real-valued convex function with minimizer set C nonempty. Suppose that $x_0 \in \mathbb{R}^n$ is a regular point of the inclusion $F(x) \in C$ with associated constants $r > 0$ and $\beta > 0$. If $d(F(x_0), C) > 0$, $t_* \leq r$,

$$\Delta \geq \xi \geq \eta\beta d(F(x_0), C), \qquad \alpha \geq \eta\beta/(K\eta\beta\xi + 1),$$

*then the sequence generated by **Algorithm 1**, denoted by $\{x_k\}$, is contained in $B(x_0, t_*)$,*

$$F(x_k) + F'(x_k)(x_{k+1} - x_k) \in C, \quad k = 0, 1, \dots,$$

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satisfies the inequalities

$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k, \qquad \|x_{k+1} - x_k\| \le \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2}\|x_k - x_{k-1}\|^2,$$

for $k = 0, 1, \dots$, and $k = 1, 2, \dots$, respectively, converging to a point $x_* \in B[x_0, t_*]$ such that $F(x_*) \in C$,

$$\|x_* - x_k\| \le t_* - t_k, \qquad k = 0, 1, \dots$$

and the convergence is R-linear. If, additionally, $2\alpha K\xi < 1$ then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q-quadratically and R-quadratically to t_* and x_*, respectively.

Proof. It is promptly proved that $f : [0, R) \to \mathbb{R}$ defined by $f(t) = Kt^2/2 - t$ is a majorant function for the function F on $B(x_0, R)$. Hence,

$$f_{\xi,\alpha}(t) = \xi - t + (\alpha K t^2)/2 = \xi + (\alpha - 1)t + \alpha f(t),$$

and, since $2\alpha K\xi \le 1$, we conclude that $f_{\xi,\alpha}$ satisfies **h3** and $t_* = (1 - \sqrt{1 - 2\alpha K\xi})/(\alpha K)$ is its smallest root. In this case, the constant α satisfies

$$\alpha \ge \frac{\eta\beta}{1 + K\eta\beta\xi} = \frac{\eta\beta}{\eta\beta[f'(\xi) + 1] + 1}.$$

Therefore, taking α, $f_{\xi,\alpha}$ and t_* as defined above, all the statements of the first part of the theorem follow from Theorem 18. For proving the second part, it is sufficient to note that the assumption $2\alpha K\xi < 1$ implies that $f_{\xi,\alpha}$ satisfies **h4**. $\square$

Under the Smale condition, see [12], Theorem 18 becomes:

Theorem 20. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be an analytic function. Assume that $x_0 \in \mathbb{R}^n$ and*

$$\gamma := \sup_{n>1} \left\| \frac{F^{(n)}(x_0)}{n!} \right\|^{1/(n-1)} < +\infty. \tag{32}$$

Take the constants $\alpha > 0$ and $\xi > 0$ and consider the auxiliary function $f_{\xi,\alpha} : [0, 1/\gamma) \to \mathbb{R}$,

$$f_{\xi,\alpha}(t) = \frac{\alpha\gamma}{1 - \gamma t}t^2 - t + \xi.$$

If $\xi\gamma \le 1 + 2\alpha - 2\sqrt{\alpha(1+\alpha)}$ then

$$t_* = \frac{1 + \gamma\xi - \sqrt{(1+\gamma\xi)^2 - 4(1+\alpha)\gamma\xi}}{2(1+\alpha)\gamma},$$

is the smallest zero of $f_{\xi,\alpha}$, the sequence generated by Newton's Method for solving $f_{\xi,\alpha}(t) = 0$, with starting point $t_0 = 0$,

$$t_{k+1} = t_k - f_{\xi,\alpha}'(t_k)^{-1} f_{\xi,\alpha}(t_k), \quad k = 0, 1, \dots,$$

is well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$, and converges Q-linearly to t_*. Let $\eta \in [1, \infty)$, $\Delta \in (0, \infty]$ and $h : \mathbb{R}^m \to \mathbb{R}$ be a real-valued convex function with minimizer set C nonempty. Suppose that $x_0 \in \mathbb{R}^n$ is a regular point of the inclusion $F(x) \in C$ with associated constants $r > 0$ and $\beta > 0$. If $d(F(x_0), C) > 0$, $t_* \leq r$,

$$\Delta \geq \xi \geq \eta\beta d(F(x_0), C), \qquad \alpha \geq \frac{\eta\beta(1 - \gamma\xi)^2}{\eta\beta + (1 - \eta\beta)(1 - \gamma\xi)^2},$$

then the sequence generated by **Algorithm 1**, denoted by $\{x_k\}$, is contained in $B(x_0, t_*)$,

$$F(x_k) + F'(x_k)(x_{k+1} - x_k) \in C, \quad k = 0, 1, \dots,$$

satisfies the inequalities

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \qquad \|x_{k+1} - x_k\| \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} \|x_k - x_{k-1}\|^2,$$

for $k = 0, 1, \dots$, and $k = 1, 2, \dots$, respectively, converging to a point $x_* \in B[x_0, t_*]$ such that $F(x_*) \in C$,

$$\|x_* - x_k\| \leq t_* - t_k, \qquad k = 0, 1, \dots$$

and the convergence is R-linear. If, additionally, $\xi\gamma < 1 + 2\alpha - 2\sqrt{\alpha(1 + \alpha)}$ then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q-quadratically and R-quadratically to t_* and x_*, respectively.

We need the following results to prove the above theorem.

Lemma 21. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be an analytic function. Suppose that $x_0 \in \mathbb{R}^n$ and γ is defined in (40). Then, for all $x \in B(x_0, 1/\gamma)$ it holds that*

$$\|F''(x)\| \leqslant (2\gamma)/(1 - \gamma\|x - x_0\|)^3.$$

Proof. The proof follows the same pattern as Lemma 21 from [3]. □

Lemma 22. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be twice continuously differentiable. If there exists a $f : [0, R) \to \mathbb{R}$ twice continuously differentiable and satisfying*

$$\|F''(x)\| \leqslant f''(\|x - x_0\|),$$

for all $x \in \mathbb{R}^n$ such that $\|x - x_0\| < R$, then F and f satisfy (9).

Proof. The proof follows the same pattern as Lemma 22 from [3]. $\qquad\square$

Proof of Theorem 20. Consider the real function $f : [0, 1/\gamma) \to \mathbb{R}$ defined by

$$f(t) = \frac{t}{1 - \gamma t} - 2t.$$

It is straightforward to show that f is analytic and that

$$f(0) = 0, \quad f'(t) = 1/(1-\gamma t)^2 - 2, \quad f'(0) = -1, \quad f''(t) = (2\gamma)/(1-\gamma t)^3, \quad f^n(0) = n!\,\gamma^{n-1},$$

for $n \geq 2$. It follows from the last equalities that f satisfies **h1** and **h2** in Definition 2. Now, as $f''(t) = (2\gamma)/(1 - \gamma t)^3$, combining the Lemmas 21 and 22, we have F and f satisfy (9) with $R = 1/\gamma$. Therefore, f is a majorant function for F on $B(x_0, 1/\gamma)$. Hence,

$$f_{\xi,\alpha}(t) = \frac{\alpha\gamma}{1 - \gamma t}t^2 - t + \xi = \xi + (\alpha - 1)t + \alpha f(t),$$

and, since $\xi\gamma \leq 1 + 2\alpha - 2\sqrt{\alpha(1+\alpha)}$, we conclude that $f_{\xi,\alpha}$ satisfies **h3** and

$$t_* = \frac{1 + \gamma\xi - \sqrt{(1 + \gamma\xi)^2 - 4(1+\alpha)\gamma\xi}}{2(1+\alpha)\gamma}$$

is its smallest root. In this case, the constant α satisfies

$$\alpha \geq \frac{\eta\beta(1 - \gamma\xi)^2}{\eta\beta + (1 - \eta\beta)(1 - \gamma\xi)^2} = \frac{\eta\beta}{\eta\beta[f'(\xi) + 1] + 1}.$$

Therefore, taking α, $f_{\xi,\alpha}$ and t_* as defined above, all the statements of the first part of the theorem follow from Theorem 18. For proving the second part, it is sufficient to note that the assumption $\xi\gamma < 1 + 2\alpha - 2\sqrt{\alpha(1+\alpha)}$ implies that $f_{\xi,\alpha}$ satisfies **h4**. $\qquad\square$

4.2 Convergence result under the Robinson condition

In this section we present a correspondent theorem to Theorem 12, namely, we assume that x_0 satisfies the Robinson condition, see [7] and [9]. Under the Robinson condition, we also present results of convergence for the Lipschitz and Smale conditions. We start by defining the Robinson condition.

Let $C \subset \mathbb{R}^m$ be a nonempty closed convex cone, $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function and $x \in \mathbb{R}^x$. Define the multifunction $T_x : \mathbb{R}^n \to P(\mathbb{R}^m)$ as

$$T_x d = F'(x)d - C. \tag{33}$$

The multifunction T_x is a convex process from $\mathbb{R}^n$ to $\mathbb{R}^m$. Convex process has been extensively studied in [10, 11]. As usual, the domain, norm and inverse of T_x are defined, respectively, by

$$\mathcal{D}(T_x) := \{d \in \mathbb{R}^n : T_x d \neq \emptyset\}, \qquad \|T_x\| := sup\,\{\|T_x d\| : x \in \mathcal{D}(T_x),\ \|d\| \leq 1\},$$
$$T_x^{-1} y := \{d \in \mathbb{R}^n : F'(x)d \in y + C\}, \qquad y \in \mathbb{R}^m.$$

where $\|T_x d\| := \inf\{\|v\| \ : \ v \in T_x d\}$.

The point $x_0 \in \mathbb{R}^n$ satisfies the *Robinson condition* if the multifunction T_{x_0} carries $\mathbb{R}^n$ onto $\mathbb{R}^m$, that is,

$$\forall\ y \in \mathbb{R}^m \quad \exists\ d \in \mathbb{R}^n,\ \exists\ c \in C; \quad y = F'(x_0)d - c. \qquad (34)$$

Theorem 23. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function. Assume that $R > 0$, $x_0 \in \mathbb{R}^n$ and $f : [0,\ R) \to \mathbb{R}$ is a majorant function for F on $B(x_0, R)$. Take the constants $\alpha > 0$ and $\xi > 0$ and consider the auxiliary function $f_{\xi,\alpha} : [0, R) \to \mathbb{R}$,*

$$f_{\xi,\alpha}(t) = \xi + (\alpha - 1)t + \alpha f(t).$$

*If, $f_{\xi,\alpha}$ satisfies **h3**, i.e., t_* is the smallest zero of $f_{\xi,\alpha}$, then the sequence generated by Newton's Method for solving $f_{\xi,\alpha}(t) = 0$, with starting point $t_0 = 0$,*

$$t_{k+1} = t_k - f'_{\xi,\alpha}(t_k)^{-1} f_{\xi,\alpha}(t_k), \quad k = 0, 1, \dots,$$

is well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$, and converges Q-linearly to t_*. Let $\eta \in [1, \infty)$, $\Delta \in (0, \infty]$ and $h : \mathbb{R}^m \to \mathbb{R}$ be a real-valued convex function with minimizer set C nonempty. Suppose that C is a cone and $x_0 \in \mathbb{R}^n$ satisfies the Robinson condition. Let $\beta_0 = \|T_{x_0}^{-1}\|$. If $d(F(x_0), C) > 0$, $t_* \leq r_{\beta_0} := \{t \in [0, R) \ : \ \beta_0 - 1 + \beta_0 f'(t) < 0\}$,

$$\Delta \geq \xi \geq \eta \beta_0 d(F(x_0), C), \qquad \alpha \geq \frac{\eta \beta_0}{1 + (\eta - 1)\beta_0[f'(\xi) + 1]},$$

*then the sequence generated by **Algorithm 1**, denoted by $\{x_k\}$, is contained in $B(x_0, t_*)$,*

$$F(x_k) + F'(x_k)(x_{k+1} - x_k) \in C, \quad k = 0, 1, \dots,$$

satisfies the inequalities

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \qquad \|x_{k+1} - x_k\| \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2}\|x_k - x_{k-1}\|^2,$$

for $k = 0, 1, \dots$, and $k = 1, 2, \dots$, respectively, converging to a point $x_* \in B[x_0, t_*]$ such that $F(x_*) \in C$,

$$\|x_* - x_k\| \leq t_* - t_k, \qquad k = 0, 1, \dots$$

*and the convergence is R-linear. If, additionally, $f_{\xi,\alpha}$ satisfies **h4** then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q-quadratically and R-quadratically to t_* and x_*, respectively.*

We need the following two results to prove the above theorem.

Lemma 24. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function and C a nonempty closed convex cone. Suppose that $x_0 \in \mathbb{R}^n$ satisfies the Robinson condition. Then*

$$\|T_{x_0}^{-1}\| < +\infty.$$

Moreover, if S is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that $\|T_{x_0}^{-1}\|\|S\| < 1$, then the convex process $\bar{T}$, defined by $\bar{T} := T_{x_0} + S$, carries $\mathbb{R}^n$ onto $\mathbb{R}^m$, $\|\bar{T}^{-1}\| < +\infty$ and

$$\|\bar{T}^{-1}\| \leq \frac{\|T_{x_0}^{-1}\|}{1 - \|T_{x_0}^{-1}\|\|S\|}.$$

Proof. See Theorem 1 on p.342 of [9]. □

Lemma 25. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function and let $h : \mathbb{R}^m \to \mathbb{R}$ be a real-valued convex function with minimizer set C nonempty. Suppose that $x_0 \in \mathbb{R}^n$ satisfies the Robinson condition. Then x_0 is a regular point of the inclusion $F(x) \in C$, and in particular, x_0 is a quasi-regular point of the inclusion $F(x) \in C$. Moreover, assume C is a cone, $R > 0$ and $f : [0, R) \to \mathbb{R}$ is a majorant function for F on $B(x_0, R)$. Let $\xi > 0$, $\beta_0 = \|T_{x_0}^{-1}\|$, the auxiliary function $f_{\xi,\beta_0} : [0, R) \to \mathbb{R}$,*

$$f_{\xi,\beta_0}(t) := \xi + (\beta_0 - 1)t + \beta_0 f(t),$$

and $r_{\beta_0} := \sup\{t \in [0, R) : f'_{\xi,\beta_0}(t) < 0\}$. If r_{x_0} is the quasi-regular radius and $\beta_{x_0}(\cdot)$ is the quasi-regular bound function for the quasi-regular point x_0, then

$$r_{x_0} \geq r_{\beta_0}, \qquad \beta_{x_0}(t) \leq \frac{\beta_0}{1 - \beta_0[f'(t) + 1]}, \qquad \forall\, t \in [0,\ r_{\beta_0}).$$

Proof. Take $y \in Ker(F'(x_0)^T) \cap (C - F(x_0))^o$. Hence,

$$0 = \langle F'(x_0)^T y, d \rangle = \langle y, F'(x_0)d \rangle, \quad \forall\, d \in \mathbb{R}^n, \qquad \langle y, c - F(x_0) \rangle \leq 0, \quad \forall\, c \in C.$$

Since x_0 satisfies the Robinson condition, $d \in \mathbb{R}^n$ and $c \in C$ exist, such that $-y - F(x_0) = F'(x_0)d - c$, which combining with the above inequalities gives

$$\langle y, y \rangle = \langle y, c - F(x_0) - F'(x_0)d \rangle = \langle d, c - F(x_0) \rangle \leq 0.$$

So $y = 0$, and we obtain from Definition 3 that x_0 is a regular point of the inclusion $F(x) \in C$.

To establish the second part, first take $x \in \mathbb{R}^n$ such that $\|x - x_0\| \leq r_{\beta_0}$. Using f as a majorant function of F on $B(x_0, R)$, as well as the definitions of β_0, f_{ξ,β_0} and r_{β_0}, we obtain

$$\|T_{x_0}^{-1}\| \|F'(x) - F'(x_0)\| \leq \beta_0[f'(\|x - x_0\|) - f'(0)] = f'_{\xi,\beta_0}(\|x - x_0\|) + 1 < 1. \qquad (35)$$

Using that x_0 satisfies the Robinson condition and the last inequality, it follows from Lemma 24 that the convex process

$$T_x d = F'(x)d - C = T_{x_0}d + [F'(x) - F'(x_0)]d, \qquad \forall \, d \in \mathbb{R}^n,$$

carries $\mathbb{R}^n$ onto $\mathbb{R}^m$ and

$$\|T_x^{-1}\| \leq \frac{\|T_{x_0}^{-1}\|}{1 - \|T_{x_0}^{-1}\| \|F'(x) - F'(x_0)\|} \leq \frac{\beta_0}{1 - \beta_0[f'(\|x - x_0\|) - f'(0)]}, \qquad (36)$$

where the last inequality follows the definition of β_0 and (35). Moreover, as T_x carries $\mathbb{R}^n$ onto $\mathbb{R}^m$, we also have

$$D_C(x) = \{d \in \mathbb{R}^n : F(x) + F'(x)d \in C\} \neq \emptyset, \qquad \forall \, x \in B(x_0, r_{\beta_0}). \qquad (37)$$

Now, let $d \in T_x^{-1}(c - F(x))$. Using the definition of T_x^{-1} it follows that

$$F'(x)d \in c - F(x) + C = C - F(x),$$

hence we conclude that $F(x) + F'(x)d \in C$, which combining with the definition of $D_C(x)$ yields

$$T_x^{-1}(c - F(x)) \subset D_C(x).$$

Therefore,

$$d(0, D_C(x)) \leq \|T_x^{-1}(c - F(x))\| \leq \|T_x^{-1}\| \|c - F(x)\|, \qquad \forall c \in C.$$

The last inequality together with (36) imply

$$d(0, D_C(x)) \leq \|T_x^{-1}\| d(F(x), C) \leq \frac{\beta_0}{1 - \beta_0[f'(\|x - x_0\|) - f'(0)]} d(F(x), C),$$

which combined with (37), as well as definitions of r_{x_0} and $\beta_{x_0}(\cdot)$ in (7) and (8), respectively, yields the desired inequalities. $\qquad \square$

[**Proof of Theorem 23**] Since $x_0 \in \mathbb{R}^n$ satisfies the Robinson condition, we have from Lemma 25 that x_0 is a quasi-regular point of the inclusion $F(x) \in C$ with the quasi-regular radius $r_{x_0} \geq r_{\beta_0}$. So, taking into account the assumption $t_* \leq r_{\beta_0}$ we obtain

$$t_* < r_{x_0}.$$

Moreover, Lemma 25 also implies that the quasi-regular bound function $\beta_{x_0}(\cdot)$ satisfies

$$\beta_{x_0}(t) \leq \frac{\beta_0}{1 - \beta_0[f'(t) + 1]}, \qquad \forall\, t \in [0,\ r_{\beta_0}). \tag{38}$$

Since $\Delta \geq \xi \geq \eta\beta_0 d(F(x_0), C)$ and the last inequality implies that $\beta_{x_0}(0) \leq \beta_0$, we have

$$\Delta \geq \xi \geq \eta\beta_{x_0}(0)d(F(x_0), C).$$

Now, combining the assumptions $0 < \xi$ and $t_* \leq r_{\beta_0}$ with the first statement in Proposition 10 we conclude that $0 < \xi < t_* \leq r_{\beta_0}$. So, using (38), $f'(0) = -1$, f' as strictly increasing and $\eta \geq 1$; after simple algebraic manipulation we obtain

$$\eta[f'(t) + 1] + \frac{1}{\beta_{x_0}(t)} \geq \frac{1}{\beta_0} + (\eta - 1)[f'(t) + 1] \geq \frac{1}{\beta_0} + (\eta - 1)[f'(\xi) + 1], \qquad \forall\, t \in [\xi,\ t_*),$$

or equivalently,

$$\frac{\eta\beta_0}{1 + (\eta - 1)\beta_0[f'(\xi) + 1]} \geq \frac{\eta\beta_{x_0}(t)}{\eta\beta_{x_0}(t)[f'(t) + 1] + 1}, \qquad \forall\, t \in [\xi,\ t_*). \tag{39}$$

Hence, the assumption $\alpha \geq \eta\beta_0/[1 + (\eta - 1)\beta_0(f'(\xi) + 1)]$ and the last inequality imply that

$$\alpha \geq \sup\left\{\frac{\eta\beta_{x_0}(t)}{\eta\beta_{x_0}(t)[f'(t) + 1] + 1} : \xi \leq t < t_*\right\}.$$

Therefore, F and x_0 satisfy all assumptions in Theorem 12 and so statements of the theorem follows. $\square$

Remark 3. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function. Assume that $x_0 \in \mathbb{R}^n$, $R > 0$ and $K > 0$ exists, such that*

$$\|F'(y) - F'(x)\| \leq K\|y - x\|, \qquad x, y\, \in B(x_0, R).$$

*Note that $f : [0, R) \to \mathbb{R}$ defined by $f(t) = Kt^2/2 - t$ is a majorant function for the function F on $B(x_0, R)$. In this case, it is easy to see that **h3**, **h4** and t_* in Theorem 23 become*

$$2\alpha K\xi \leq 1, \qquad 2\alpha K\xi < 1, \qquad t_* = (1 - \sqrt{1 - 2\alpha K\xi})/(\alpha K),$$

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and α satisfies

$$\alpha \geq \frac{\eta\beta_0}{1 + (\eta - 1)K\beta_0\xi}.$$

In particular, if $C = \{0\}$ and $n = m$, the Robinson condition is equivalent to the condition that $F'(x_0)^{-1}$ is non-singular. Hence, for $\eta = 1$ we obtain the semi-local convergence for the Newton method under the Lipschitz condition, see [2].

Remark 4. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be an analytic function. Assume that $x_0 \in \mathbb{R}^n$ and*

$$\gamma := \sup_{n>1} \left\| \frac{F^{(n)}(x_0)}{n!} \right\|^{1/(n-1)} < +\infty. \tag{40}$$

*Note that the real function $f : [0, 1/\gamma) \to \mathbb{R}$ defined by $f(t) = t/(1 - \gamma t) - 2t$ is a majorant function for the function F on $B(x_0, 1/\gamma)$. In this case, it is easy see that **h3**, **h4** and t_* in Theorem 23 become*

$$\xi\gamma \leq 1 + 2\alpha - 2\sqrt{\alpha(1 + \alpha)}, \qquad \xi\gamma < 1 + 2\alpha - 2\sqrt{\alpha(1 + \alpha)},$$

$$t_* = \frac{1 + \gamma\xi - \sqrt{(1 + \gamma\xi)^2 - 4(1 + \alpha)\gamma\xi}}{2(1 + \alpha)\gamma},$$

and α satisfies

$$\alpha \geq \frac{\eta\beta_0(1 - \gamma\xi)^2}{(\eta - 1)\beta_0 + [1 - \beta_0(\eta - 1)](1 - \gamma\xi)^2}.$$

In particular, if $C = \{0\}$ and $n = m$, the Robinson condition is equivalent to the condition that $F'(x_0)^{-1}$ is non-singular. Hence, for $\eta = 1$ we obtain the semi-local convergence for the Newton method under the Smale condition, see [12].

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