DISTRIBUTIONS OF DEMMEL AND RELATED CONDITION NUMBERS*

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Abstract. Consider a random matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ $(m \ge n)$ containing independent complex Gaussian entries with zero mean and unit variance, and let $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n < \infty$ denote the eigenvalues of $\mathbf{A}^* \mathbf{A}$ where $(\cdot)^*$ represents conjugate-transpose. This paper investigates the distribution of the random variables $\frac{\sum_{j=1}^n \lambda_j}{\lambda_k}$, for k = 1 and k = 2. These two variables are related to certain condition number metrics, including the so-called Demmel condition number, which have been shown to arise in a variety of applications. For both cases, we derive new exact expressions for the probability densities, and establish the asymptotic behavior as the matrix dimensions grow large. In particular, it is shown that as n and m tend to infinity with their difference fixed, both densities scale on the order of n^3 . After suitable transformations, we establish exact expressions for the asymptotic densities, obtaining simple closed-form expressions in some cases. Our results generalize the work of Edelman on the Demmel condition number for the case m = n.

Key words. Demmel condition number, Eigenvalues, Random matrix, Wishart distribution

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1. Introduction. Understanding the sensitivity of outcomes to initial conditions in certain iterative algorithms is important for the design and analysis of many physical systems. Example algorithms include iterative methods used in linear algebra, interior-point methods of convex optimization, and polynomial zero finding [6]. Various condition numbers have been defined as a measure of the sensitivity of the solutions with respect to small perturbations of the input. One of the earliest studies by Turin [45] on iterative algorithms related to matrix inversion and the solution of large systems of linear equations defined two such condition numbers as a measure of the degree of ill-conditioning in a matrix. In particular, for a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, the first was referred to as the N-condition number, defined as $N(\mathbf{A})N(\mathbf{A}^{-1})/n$ with $N(\cdot)$ representing the Frobenius norm, whilst the second was referred to as the *M*-condition number, defined as $M(\mathbf{A})M(\mathbf{A}^{-1})/n$ with $M(\cdot)$ denoting the operator returning the largest absolute entry of the matrix. However, probably the best known condition number, introduced in [46], takes the form $\kappa(\mathbf{A}) = ||\mathbf{A}||_2 ||\mathbf{A}^{-1}||_2$, where $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $|| \cdot ||_2$ denotes the 2-norm. This condition number, similar to the N-condition number and *M*-condition number, is important in problems involving matrix inversion and solutions of linear equations.

As conjectured in [38], the calculation of a condition number pertaining to a problem with a certain deterministic data set is as hard as solving the problem itself with the data set. Whilst very difficult to prove in general, a proof of this conjecture has been established for *conic* condition numbers in [12]. To gain further insights into the behavior of the condition number, probabilistic analysis based on certain probability measures on the data has been employed; see [5, 13, 14, 20, 22, 39, 40, 47]

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for a partial list. In general, most existing literature dealing with the probabilistic analysis of condition numbers has focused on one of two main perspectives. The first is to establish bounds on the tail of the distribution of $\kappa(\mathbf{A})$, which is important for the geometrical characterization of the condition number [17]. Central to the geometric characterization is the fact that $\kappa(\mathbf{A})$ is proportional to the reciprocal of the distance to the set of ill-conditioned matrices [17]. The second has been to establish bounds on the expectation of $\ln(\kappa(\mathbf{A}))$, which is important for characterizing the average loss of numerical precision and average running time [6, 22] of iterative algorithms. In general, the "randomness" of the condition number is introduced by selecting the elements of \mathbf{A} to be standard independent normal real/complex random variables [20, 22, 40].

Whilst in this paper we focus primarily on a probabilistic analysis, it is worth noting that a deterministic version of the problems considered above, related to the inversion of positive-definite Hankel (or moment) matrices appearing naturally in random matrix theory, involves the determination of the smallest eigenvalues of Hankel matrices of order n. See [3] and [4, 7, 9, 43, 50] for contributions dealing with this problem.

Demmel, in his seminal paper [17], introduced a new condition number of *conic* type, defined for square $n \times n$ matrix **A** as

$$\kappa_D(\mathbf{A}) = ||\mathbf{A}||_F ||\mathbf{A}^{-1}||_2,$$
(1.1)

where $|| \cdot ||_F$ is the Frobenius norm. This, along with a theorem due to Eckart and Young [19], permits a geometrical characterization [6, 17], which has enabled the calculation of bounds on the tail distribution of $\kappa_D(\mathbf{A})$ with the help of integral geometry. A random variable similar to $\kappa_D(\mathbf{A})$ also arises in problems dealing with the entanglement of a bi-partite quantum system [2, 8]. As demonstrated in [16, 17], the condition numbers arising in various contexts, including matrix inversion, eigenvalue calculation, polynomial zero finding, as well as pole assignment in linear control systems, can be bounded by $\kappa_D(\mathbf{A})$. The definition (1.1) extends naturally to rectangular matrices by replacing \mathbf{A}^{-1} in (1.1) with \mathbf{A}^{\dagger} (also known as the Moore-Penrose inverse or the pseudo-inverse) to yield

$$\kappa_D(\mathbf{A}) = ||\mathbf{A}||_F ||\mathbf{A}^{\dagger}||_2. \tag{1.2}$$

For $\mathbf{A} \in \mathbb{C}^{m \times n}$ with rank $(\mathbf{A}) = r \ (\leq \min(m, n)), \ (1.2)$ simplifies to

$$\kappa_D(\mathbf{A}) = \sqrt{\frac{\sum_{j=1}^r \lambda_j}{\lambda_1}} , \qquad (1.3)$$

where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r$ are the non-zero eigenvalues of $\mathbf{A}^*\mathbf{A}$ (or $\mathbf{A}\mathbf{A}^*$) with $(\cdot)^*$ denoting conjugate-transpose. In this paper, we deal exclusively with Gaussian random matrices $\mathbf{A} \in \mathbb{C}^{m \times n}$ $(m \geq n)$ having independent complex standard normal entries¹. Therefore, (1.3) can be written as

$$\kappa_D(\mathbf{A}) = \sqrt{\frac{\sum_{j=1}^n \lambda_j}{\lambda_1}}, \qquad (1.4)$$

¹Henceforth, when referring to "Gaussian matrices", we will implicitly assume that the matrices have independent standard complex normal entries; i.e., this will not be explicitly stated.

which follows from the fact that the matrix $\mathbf{A}^* \mathbf{A}$ is positive-definite with probability one [18, 30]. This form of the Demmel condition number was used by Krishnaiah *et. al.* in a sequence of papers [27, 28, 29] to derive the inferences on certain sub-hypotheses when the total hypothesis, which is composed of various component sub-hypotheses, is rejected. Recently, it was shown in [6] that the condition number of the form (1.2) is useful in analyzing the loss of precision in the computation of the solution to the classical linear least squares problem. Also, the statistical properties of the Demmel condition number $\kappa_D(\mathbf{A})$ (for arbitrary m and n) have recently found important applications in the design and analysis of contemporary wireless communication systems. Specific examples include the design of adaptive multi-antenna transmission techniques [24], and the modeling of physical multi-antenna transmission channels [33].

From the discussion given above, there is clear motivation for studying the statistical properties of the Demmel condition number of random matrices. Here we review some of the key existing contributions dealing with this problem. These contributions include [6, 17, 20, 21, 22], which investigated the exact distributions as well as bounds on the tail probabilities over various regimes depending on the size of the random matrix. For instance, Demmel showed that for Gaussian matrices $\mathbf{A} \in \mathbb{C}^{n \times n}$ [17],

$$\frac{(1-1/x)^{2n^2-2}}{2n^4x^2} < \Pr\left(\kappa_D(\mathbf{A}) > x\right) < \frac{e^2n^5(1+n^2/x)^{2n^2-2}}{x^2}$$

while Edelman concluded that as $n \to \infty$ [21],

$$\Pr\left(\frac{2\kappa_D(\mathbf{A})}{n^{\frac{3}{2}}} < x\right) \to \exp\left(-\frac{4}{x^2}\right) \ .$$

For Gaussian matrices $\mathbf{A} \in \mathbb{C}^{m \times n}$ with $m \ge n$, one could also exploit the bound $\kappa_D(\mathbf{A}) \leq \sqrt{n}\kappa(\mathbf{A})$ (following from the fact that $||\mathbf{A}||_F \leq \sqrt{n}||\mathbf{A}||_2$) along with the upper bound on the distribution tail of $\kappa(\mathbf{A})$ given in [11, Theorem 4.6] to derive an upper bound on the distribution tail of $\kappa_D(\mathbf{A})$. This, in turn, reveals that the $n^{3/2}$ asymptotic scaling order observed in [21] for square Gaussian matrices also serves as an upper bound on the scaling order for rectangular Gaussian matrices. The exact scaling order, however, has yet to be determined. In addition to these results, more recently, various exact closed-form expressions for the distribution of the Demmel condition number for Gaussian matrices $\mathbf{A} \in \mathbb{C}^{m \times n}$ have been reported in [31, 52, 49]. These results, however, are rather complicated and they become unwieldy computationally when the matrix dimensions are not small. Moreover, the expressions in [31, 52, 49] are not suitable for understanding the behavior of the scaled Demmel condition number as the matrix dimensions grow large. In this respect, only for the case m = n given in [31, 52, 49], the asymptotic behavior of the exact probability density function (p.d.f.) of the Demmel condition number is known. Establishing the asymptotic properties for more general $m \times n$ (m < n) matrices is one of the key objectives to be addressed in this paper.

In addition to $\kappa_D(\mathbf{A})$, other related metrics of the form $\lambda_k / \sum_{j=1}^n \lambda_j$, $k = 1, 2, \ldots, n$ have been considered by Krishnaiah *et. al.* [27, 28, 29] in certain hypothesis testing problems. Therein, they employed a result due to Davis [15], which gives an expression for the p.d.f. of $\lambda_k / \sum_{j=1}^n \lambda_j$ by establishing a Laplace transform relationship with respect to the p.d.f. of λ_k . However, the expressions obtained are very complicated, both analytically and numerically. This complexity, in turn, does not easily facilitate the understanding of the asymptotic behavior of the p.d.f. as the matrix dimensions

become large. In this paper, we tackle this problem by adopting a different approach, making use of techniques developed in [32]. We focus our analysis on the quantities, $\kappa_D^2(\mathbf{A})$ and

$$\kappa_E^2(\mathbf{A}) = \frac{\sum_{j=1}^n \lambda_j}{\lambda_2}$$

whilst noting that our approach may also pave the way for studying more generalized metrics of the form $\frac{\sum_{j=1}^{n} \lambda_j}{\lambda_k}$, k = 1, 2, ..., n. We derive new expressions for the exact and asymptotic distributions of $\kappa_D^2(\mathbf{A})$ and $\kappa_E^2(\mathbf{A})$ for Gaussian matrices $\mathbf{A} \in \mathbb{C}^{m \times n}$ $(m \ge n)$ by adopting a moment generating function (m.g.f.) based approach. We show the interesting result that both $\kappa_D^2(\mathbf{A})$ and $\kappa_E^2(\mathbf{A})$ scale on the order of n^3 when m and n tend to infinity in such a way that m - n remains a fixed integer. These results agree with and generalize the scaling behavior obtained previously by Edelman in [21] for $n \times n$ Gaussian matrices. The scaled asymptotic p.d.f. which we derive for $\kappa_D^2(\mathbf{A})$ is expressed in closed form for arbitrary $m \ge n$, whilst for $\kappa_E^2(\mathbf{A})$ it involves a single finite-range integral for the general case and a closed-form solution for the scenario m = n.

2. Preliminaries. To facilitate our main derivations, we will require the following preliminary results and definitions.

DEFINITION 2.1. Let the elements of $\mathbf{A} \in \mathbb{C}^{m \times n}$ $(m \geq n)$ be independent and identically distributed complex standard normal variables. Then the matrix $\mathbf{W} = \mathbf{A}^* \mathbf{A}$ is said to follow a complex Wishart distribution, i.e., $\mathbf{W} \sim \mathcal{W}_n(m, \mathbf{I}_n)$.

THEOREM 2.2. The joint density of the ordered eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq$ $\lambda_n < \infty$ of **W** is given by [25]

$$f(\lambda_1, \lambda_2, \dots, \lambda_n) = K_{n,\alpha} \,\Delta_n^2(\boldsymbol{\lambda}) \prod_{j=1}^n \lambda_j^{\alpha} e^{-\lambda_j}$$
(2.1)

where, for $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \dots, \lambda_n\}, \ \Delta_n(\boldsymbol{\lambda}) := \prod_{1 \leq j < k \leq n} (\lambda_k - \lambda_j), \ \alpha = m - n, \ and K_{n,\alpha} = n! \left(\prod_{j=0}^{n-1} (j+1)! (j+\alpha)!\right)^{-1}.$ LEMMA 2.3. For $\rho > -1$, the generalized Laguerre polynomial of degree N,

 $L_N^{(\rho)}(z)$, is defined by [44]:

$$L_N^{(\rho)}(z) = \frac{(\rho+1)_N}{N!} {}_1F_1(-N,\rho+1,z) = \frac{(\rho+1)_N}{N!} \sum_{j=0}^N \frac{(-N)_j}{(\rho+1)_j} \frac{z^j}{j!} , \qquad (2.2)$$

with k^{th} derivative

$$\frac{d^k}{dz^k} L_N^{(\rho)}(z) = (-1)^k L_{N-k}^{(\rho+k)}(z) , \qquad (2.3)$$

where $(a)_j = a(a+1)\dots(a+j-1)$ with $(a)_0 = 1$ is the Pochhammer symbol and $_1F_1(a;c;z)$ is the confluent hypergeometric function of the first kind.

LEMMA 2.4. The monomial z^n can be expanded in terms of the Stirling number of the second kind, $S_n^{(m)}$, as follows [1]:

$$z^{n} = \sum_{j=0}^{n} \mathbf{S}_{n}^{(j)} \ z(z-1)(z-2)\cdots(z-j+1),$$
(2.4)

where

$$\mathbf{S}_{n}^{(m)} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^{n}, \quad \mathbf{S}_{0}^{(0)} = \mathbf{S}_{n}^{(n)} = \mathbf{S}_{n}^{(1)} = 1,$$

and $\binom{m}{j} = \frac{m!}{j!(m-j)!}$. Finally, we use the following notation to compactly represent the determinant of an $N \times N$ block matrix:

$$\det \begin{bmatrix} a_{i,j} & b_{i,k-2} \end{bmatrix}_{\substack{i=1,2,\dots,N\\j=1,2\\k=3,4,\dots,N}} = \begin{vmatrix} a_{1,1} & a_{1,2} & b_{1,1} & b_{1,2} & \dots & b_{1,N-2}\\ a_{2,1} & a_{2,2} & b_{2,1} & b_{2,2} & \dots & b_{2,N-2}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{N,1} & a_{N,2} & b_{N,1} & b_{N,2} & \dots & b_{N,N-2} \end{vmatrix} .$$
(2.5)

3. M.g.f. and p.d.f. of $\kappa_D^2(\mathbf{A})$. In this section we derive new expressions for the m.g.f. and the p.d.f. of $\kappa_D^2(\mathbf{A})$. Our approach follows along similar lines to [32, Chap. 22].

The m.g.f. of $\kappa_D^2(\mathbf{A})$ is given by

$$\mathcal{M}_{\kappa_D^2(\mathbf{A})}(s) = E\left[e^{-s\kappa_D^2(\mathbf{A})}\right]$$
$$= \int_0^\infty \int_{\mathcal{R}_1} e^{-s\frac{\sum_{j=1}^n \lambda_j}{\lambda_1}} f(\lambda_1, \lambda_2, ..., \lambda_n) d\lambda_2 ... d\lambda_n d\lambda_1,$$

where $\mathcal{R}_1 = \{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < \infty\}$. Let $\lambda_1 = x$. An easy manipulation gives

$$\mathcal{M}_{\kappa_D^2(\mathbf{A})}(s) = e^{-s} \int_0^\infty \int_{\mathcal{R}_1} e^{-s\frac{\sum_{j=2}^n \lambda_j}{x}} f(x, \lambda_2, \dots, \lambda_n) \, d\lambda_2 \dots d\lambda_n \, dx.$$
(3.1)

The theorem below gives a closed-form representation for the m.g.f.

THEOREM 3.1. The m.g.f. of $\kappa_D^2(\mathbf{A})$ is given by

$$\mathcal{M}_{\kappa_D^2(\mathbf{A})}(s) = \frac{n! \ e^{-ns}}{(m-1)!} \int_0^\infty \frac{x^{mn-1} e^{-nx}}{(s+x)^{mn-\alpha-1}} \det \left[L_{n+k-l-1}^{(l+1)}(-s-x) \right]_{k,l=1,2,\dots,\alpha} dx.$$
(3.2)

Proof. Substituting (2.1) into (3.1) we find, keeping the x integration last,

$$\mathcal{M}_{\kappa_D^2(\mathbf{A})}(s) = K_{n,\alpha} \ e^{-s} \int_0^\infty x^\alpha e^{-x} \left\{ \int_{\mathcal{R}_1} \prod_{j=2}^n (\lambda_j - x)^2 \lambda_j^\alpha e^{-\lambda_j \left(1 + \frac{s}{x}\right)} \right. \\ \left. \times \prod_{2 \le i < j \le n} (\lambda_j - \lambda_i)^2 \ d\lambda_2 \dots d\lambda_n \right\} dx.$$

Relabeling variables as $\lambda_j = x_{j-1}, j = 2, 3, ..., n$, and exploiting symmetry to remove the ordered region of integration (i.e., achieved by dividing through by (n-1)!) gives

$$\mathcal{M}_{\kappa_{D}^{2}(\mathbf{A})}(s) = \frac{K_{n,\alpha} e^{-s}}{(n-1)!} \int_{0}^{\infty} x^{\alpha} e^{-x} \left\{ \int_{[x,\infty)^{n-1}} \prod_{j=1}^{n-1} (x_{j} - x)^{2} x_{j}^{\alpha} e^{-x_{j} \left(1 + \frac{s}{x}\right)} \times \Delta_{n-1}^{2} (\mathbf{x}) \, dx_{1} dx_{2} \dots dx_{n-1} \right\} dx. \quad (3.3)$$

Now we apply the change of variables $y_j = \frac{(x+s)}{x} (x_j - x)$, j = 1, 2, ..., n-1 to the inner (n-1)-fold integral in (3.3) with some algebraic manipulation to obtain

$$\mathcal{M}_{\kappa_D^2(\mathbf{A})}(s) = \frac{K_{n,\alpha} \ e^{-ns}}{(n-1)!} \int_0^\infty \frac{x^{mn-1} e^{-nx}}{(x+s)^{mn-\alpha-1}} (-1)^{\alpha(n-1)} Q(n-1,\alpha,-x-s) \ dx,$$
(3.4)

where we have defined

$$Q(n,\alpha,z) := \int_{[0,\infty)^n} \Delta_n^2(\mathbf{y}) \prod_{j=1}^n y_j^2 e^{-y_j} (z-y_j)^\alpha \, dy_1 dy_2 \dots dy_n \,. \tag{3.5}$$

The remaining task is to obtain a closed-form solution to $Q(n, \alpha, z)$. We point out that a solution to a generalization of this integral, not restricting α to an integer, has been derived in [10, 35] as a solution to a Painlevé V equation. Here, for α an integer, we establish a much simpler closed-form algebraic solution. For this purpose, we employ a result from random matrix theory [32, Section 22.2.2], which gives

$$Q(n,\alpha,z) = \tilde{b} Q(n,0,z) \det \left[\frac{d^l}{dz^l} C_{n+k}(z)\right]_{k,l=0,1,\dots,\alpha-1} ,$$

where $\tilde{b} = \left(\prod_{j=0}^{\alpha-1} j!\right)^{-1}$. For our problem $C_j(x)$ are monic polynomials orthogonal with respect to the weight $x^2 e^{-x}$, over $0 \leq x < \infty$. We see that $C_j(x) = (-1)^j j! L_j^{(2)}(x)$. Hence (3.5) becomes

$$Q(n,\alpha,z) = \tilde{b} Q(n,0,z) \det \left[(-1)^{n+k} (n+k)! \frac{d^l}{dz^l} L_{n+k}^{(2)}(z) \right]_{k,l=0,1,\dots,\alpha-1}$$

= $\tilde{b} Q(n,0,z) (-1)^{n\alpha} \prod_{j=0}^{\alpha-1} (n+j)! \det \left[L_{n+k-l}^{(2+l)}(z) \right]_{k,l=0,1,\dots,\alpha-1}.$ (3.6)

Moreover, we have $Q(n, 0, z) = \prod_{j=0}^{n-1} (1+j)! (2+j)!$, which when used with (3.6) in (3.4), followed by translating the indices from $k, l = 0, 1, \ldots, \alpha - 1$ to $k, l = 1, 2, \ldots, \alpha$ gives (3.2). \Box

Now we take the inverse Laplace transform of (3.2) to arrive at the p.d.f. of $\kappa_D^2(\mathbf{A})$, which is given below.

COROLLARY 3.2. The p.d.f. of $\kappa_D^2(\mathbf{A})$ is given by

$$f_{\kappa_D^2(\mathbf{A})}^{(\alpha)}(y) = \frac{n!}{(n+\alpha-1)!} \frac{\Gamma(mn)}{y^{mn}} \mathcal{L}^{-1} \left\{ \frac{e^{-ns}}{s^{mn-\alpha-1}} \det \left[L_{n+k-l-1}^{(l+1)}(-s) \right]_{k,l=1,2,\dots,\alpha} \right\}$$
(3.7)

where $\mathcal{L}^{-1}(\cdot)$ denotes the inverse Laplace transform.

Interestingly, by noting that the p.d.f. of the minimum eigenvalue λ_1 takes the form

$$f_{\min}(x) = \frac{K_{n,\alpha}}{(n-1)!} \int_{[x,\infty)^{n-1}} f(x,\lambda_2,\dots,\lambda_n) d\lambda_2\dots d\lambda_n$$

= $\frac{K_{n,\alpha}}{(n-1)!} x^{\alpha} e^{-nx} (-1)^{\alpha(n-1)} Q(n-1,\alpha,-x)$
= $\frac{n!}{(n+\alpha-1)!} x^{\alpha} e^{-nx} \det \left[L_{n+k-l-1}^{(l+1)}(-x) \right]_{k,l=1,2,\dots,\alpha},$ (3.8)

we can obtain the following alternative representation for the p.d.f. of $\kappa_D^2(\mathbf{A})$:

$$f_{\kappa_D^2(\mathbf{A})}^{(\alpha)}(y) = \frac{\Gamma(mn)}{y^{mn}} \mathcal{L}^{-1} \left\{ \frac{f_{\min}(s)}{s^{mn-1}} \right\}.$$

This turns out to be a simpler representation of an equivalent relation given previously in [29] (obtaining one relation from the other, however, appears to be non-trivial). This simplified form results as a consequence of the m.g.f. derivation approach, in contrast to the p.d.f.-based approach in [15]. We also mention that the minimum eigenvalue p.d.f. (3.8) fixes a sign problem with a result given in [23, Eq. 3.12].

The new expression (3.7) facilitates the exact evaluation of the p.d.f. of $\kappa_D^2(\mathbf{A})$ in closed-form, for any value of α . This is given in the following key theorem:

THEOREM 3.3. The exact p.d.f. of $\kappa_D^2(\mathbf{A})$ is given by

$$f_{\kappa_{D}^{2}(\mathbf{A})}^{(\alpha)}(y) = \Gamma(mn) \left(\prod_{k=0}^{\alpha} \frac{n+k}{(k+1)!} \right) (y-n)^{mn-\alpha-2} y^{-mn} \\ \times \sum_{j_{1}=0}^{n+\alpha-2} \dots \sum_{j_{\alpha}=0}^{n-1} \left(\prod_{k=1}^{\alpha} (-1)^{j_{k}} \frac{(-n-\alpha+k+1)_{j_{k}}}{(k+2)_{j_{k}} j_{k}!} (y-n)^{-j_{k}} \right)$$
(3.9)
$$\times \frac{\Delta_{\alpha}(\mathbf{c})}{\Gamma(mn-\alpha-1-\sum_{k=1}^{\alpha} j_{k})} H(y-n) ,$$

where $\mathbf{c} = \{c_1(j_1), c_2(j_2), \dots, c_{\alpha}(j_{\alpha})\}$ with $c_l(j_l) = l + j_l$, and H(z) denote the Heaviside unit step function, i.e., H(z) = 1, $z \ge 0$, and H(z) = 0, z < 0.

Proof. We use (2.2) to write the determinant term in (3.7) as

$$\det \left[L_{n+k-l-1}^{(l+1)}(-s) \right]_{k,l=1,2,\dots,\alpha} = \prod_{k=1}^{\alpha} \frac{(n+k)!}{(k+1)!} \det \left[\frac{1}{(n+k-l-1)!} \sum_{j_l=0}^{n+k-l-1} \frac{(-n-k+l+1)_{j_l}}{(l+2)_{j_l}} \frac{(-s)^{j_l}}{j_l!} \right]_{k,l=1,2,\dots,\alpha}$$
(3.10)

Further manipulation in this form is difficult due to the dependence of the summation upper limits on k and l. To circumvent this problem, we use the factorization

$$\frac{(-n-k+l+1)_{j_l}}{(l+2)_{j_l}} = \frac{(-n-k+l+1)_{j_l}}{(-n-\alpha+l+1)_{j_l}} \frac{(-n-\alpha+l+1)_{j_l}}{(l+2)_{j_l}}$$
$$= \frac{(n+k-l-1)!}{(n+\alpha-l-1)!} \frac{(-n-\alpha+l+1)_{j_l}}{(l+2)_{j_l}} \prod_{i=0}^{\alpha-k-1} (\tilde{c}_l-i)$$

where $\tilde{c}_l = n + \alpha - 1 - j_l - l$, in (3.10) with some algebraic manipulation to obtain

$$\det \left[L_{n+k-l-1}^{(l+1)}(-s) \right]_{k,l=1,2,\dots,\alpha} = \frac{(n+\alpha)!(n+\alpha-1)!}{n! (n-1)! \prod_{k=1}^{\alpha} (k+1)!} \times \sum_{j_1=0}^{n+\alpha-2} \dots \sum_{j_{\alpha}=0}^{n-1} \left(\prod_{k=1}^{\alpha} \frac{(-n-\alpha+k+1)_{j_k}}{(k+2)_{j_k} j_k!} (-s)^{j_k} \right) \det \left[\prod_{i=0}^{\alpha-k-1} (\tilde{c}_l-i) \right]_{k,l=1,2,\dots,\alpha}$$
(3.11)

Now, invoking Lemma A.1 in the Appendix, then substituting (3.11) into (3.7) yields

$$f_{\kappa_{D}^{2}(\mathbf{A})}^{(\alpha)}(y) = \Gamma(mn) \left(\prod_{k=0}^{\alpha} \frac{n+k}{(k+1)!} \right) y^{-mn} \sum_{j_{1}=0}^{n+\alpha-2} \dots \sum_{j_{\alpha}=0}^{n-1} \left(\prod_{k=1}^{\alpha} (-1)^{j_{k}} \frac{(-n-\alpha+k+1)_{j_{k}}}{(k+2)_{j_{k}} j_{k}!} \right) \times \Delta_{\alpha}(\mathbf{c}) \mathcal{L}^{-1} \left\{ \frac{e^{-ns}}{s^{mn-1-\alpha-\sum_{k=1}^{\alpha} j_{k}}} \right\}.$$

Finally, the result (3.9) follows upon carrying out the remaining Laplace inversion using [37, Eq. 1.1.2.1]. \Box

For some small values of α , (3.9) admits the following simple forms.

COROLLARY 3.4. The exact p.d.f.s of $\kappa_D^2(\mathbf{A})$ corresponding to $\alpha = 0$ and $\alpha = 1$ are given, respectively, by

The expression for $\alpha = 0$ agrees with a previous result given in [21].

REMARK 1. Whilst previous equivalent expressions have been derived in [31, 52, 49], the exact p.d.f. of $\kappa_D^2(\mathbf{A})$ given in (3.9) is a generalized and/or simpler representation. Indeed, noting that the number of nested summations depends only on α , this formula provides an efficient way of evaluating the p.d.f. of $\kappa_D^2(\mathbf{A})$, particularly for small values of α . Moreover, since the algebraic complexity depends only on n and the difference of m and n, this in turn makes our result (3.9) very useful for conducting an asymptotic analysis of $\kappa_D^2(\mathbf{A})$ as m and n grow large, but their difference does not (something which appears infeasible with previous expressions in [31, 52, 49]). This is the objective of the next section.

4. Asymptotic Characterization of $\kappa_D^2(\mathbf{A})$. In this section, we employ the exact p.d.f. representation (3.9) to investigate the distribution of $\kappa_D^2(\mathbf{A})$, suitably scaled, for fixed α when $m, n \to \infty$. We have the following key result:

THEOREM 4.1. As m and n tend to ∞ such that $\alpha = m - n$ is fixed, $\kappa_D^2(\mathbf{A})$ scales on the order of n^3 . More specifically, the scaled random variable $V = \kappa_D^2(\mathbf{A})/(\mu n^3)$, with $\mu \in \mathbb{R}^+$ an arbitrary constant, has the following asymptotic p.d.f. as m and n tend to ∞ with $\alpha = m - n$ fixed:

$$f_{V}^{(\alpha)}(v) = \frac{e^{-\frac{1}{\mu v}}}{\mu v^{2}} \det \left[\sum_{i=0}^{k-1} \sum_{j=0}^{i} \mathbf{S}_{i}^{(j)} \binom{k-1}{i} l^{k-1-i} \frac{I_{l+j+1}\left(\frac{2}{\sqrt{\mu v}}\right)}{(\mu v)^{\frac{j+1-l}{2}}} \right]_{k,l=1,2,\dots,\alpha} H(v) , \quad (4.1)$$

where

$$I_n(z) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left(\frac{z}{2}\right)^{n+2k}$$

is the modified Bessel function of the first kind of order n.

Proof. We arrange the terms in (3.9), noting that $m = n + \alpha$, to obtain

$$f_{\kappa_{D}^{2}(\mathbf{A})}^{(\alpha)}(y) = \Gamma(n(n+\alpha)) \left(\prod_{k=0}^{\alpha} \frac{n+k}{(k+1)!} \right) \left(1 - \frac{n}{y} \right)^{n(n+\alpha)-\alpha-2} y^{-\alpha-2} \\ \times \sum_{j_{1}=0}^{n+\alpha-2} \dots \sum_{j_{\alpha}=0}^{n-1} \left(\prod_{k=1}^{\alpha} (-1)^{j_{k}} \frac{(-n-\alpha+k+1)_{j_{k}}}{(k+2)_{j_{k}} j_{k}!} \left(1 - \frac{n}{y} \right)^{-j_{k}} y^{-j_{k}} \right) \\ \times \frac{\Delta_{\alpha}(\mathbf{c})}{\Gamma(n(n+\alpha)-\alpha-1-\sum_{k=1}^{\alpha} j_{k})} H(y-n).$$
(4.2)

Now we have to choose a suitable scaling for the variable $\kappa_D^2(\mathbf{A})$ in terms of n so that the above p.d.f. converges as $n \to \infty$. Careful thought reveals that the term $\left(1-\frac{n}{y}\right)^{n(n+\alpha)}$ converges to a finite non-zero limit as $n \to \infty$ if we scale $\kappa_D^2(\mathbf{A})$ proportional to n^3 . For this reason, we introduce the scaled random variable $V = \kappa_D^2(\mathbf{A})/(\mu n^3)$, and focus on the function $\mu n^3 f_{\mu n^3 V}^{(3)}(\mu n^3 v)$ as $n \to \infty$. To this end, with $y = \mu n^3 v$, by using elementary limiting arguments it can be shown that

$$\lim_{n \to \infty} \mu n^3 f_{\mu n^3 V}^{(\alpha)}(\mu n^3 v) = \left(\prod_{k=1}^{\alpha} \frac{1}{(k+1)!}\right) \frac{e^{-\frac{1}{\mu v}}}{\mu^{\alpha+1} v^{\alpha+2}} \sum_{j_1=0}^{\infty} \cdots \sum_{j_{\alpha}=0}^{\infty} \left(\prod_{k=1}^{\alpha} \frac{1}{(k+2)_{j_k}} \frac{1}{j_k!} \frac{1}{(\mu v)^{j_k}}\right) \Delta_{\alpha}(\mathbf{c}) H(v) \\
= \left(\prod_{k=1}^{\alpha} \frac{1}{(k+1)!}\right) \frac{e^{-\frac{1}{\mu v}}}{\mu^{\alpha+1} v^{\alpha+2}} \det \left[\sum_{j_l=0}^{\infty} \frac{(l+j_l)^{k-1}}{(l+2)_{j_l}} \frac{1}{j_l!} \frac{1}{(\mu v)^{j_l}}\right]_{k,l=1,2,\dots,\alpha} H(v). \tag{4.3}$$

We now focus on simplifying the determinant. To this end, we use the binomial theorem and the definition of the Stirling number (2.4) to arrive at

$$\begin{split} \det \left[\sum_{j_l=0}^{\infty} \frac{(l+j_l)^{k-1}}{(l+2)_{j_l} \ j_l!} \frac{1}{(\mu v)^{j_l}} \right]_{k,l=1,2,\dots,\alpha} \\ &= \det \left[\sum_{i=0}^{k-1} \sum_{j=0}^{i} \binom{k-1}{i} l^{k-1-i} \mathbf{S}_i^{(j)} \ \sum_{j_l=j}^{\infty} \frac{1}{(j_l-j)!(l+2)_{j_l}} \frac{1}{(\mu v)^{j_l}} \right]_{k,l=1,2,\dots,\alpha}, \end{split}$$

which can be simplified upon noting the relations [48]

$$\sum_{j_{l}=j}^{\infty} \frac{1}{(j_{l}-j)!(l+2)_{j_{l}}} \frac{1}{(\mu v)^{j_{l}}} = \left(\frac{1}{\mu v}\right)^{j} \frac{(l+1)!}{(l+j+1)!} \,_{0}F_{1}\left(-;l+j+2;\frac{1}{\mu v}\right),$$

$$_{0}F_{1}\left(-;l+j+2;\frac{1}{\mu v}\right) = (l+j+1)!(\mu v)^{\frac{l+j+1}{2}} I_{l+j+1}\left(\frac{2}{\sqrt{\mu v}}\right),$$

with $_0F_1(-;\rho;z) = \sum_{k=0}^{\infty} z^k/(\rho)_k k!$ denoting the generalized hypergeometric function, to obtain

$$\det \left[\sum_{j_l=0}^{\infty} \frac{(l+j_l)^{k-1}}{(l+2)_{j_l} \ j_l!} \frac{1}{(\mu v)^{j_l}} \right]_{k,l=1,2,\dots,\alpha}$$
$$= \prod_{k=1}^{\alpha} (k+1)! \det \left[\sum_{i=0}^{k-1} \sum_{j=0}^{i} \binom{k-1}{i} l^{k-1-i} \mathbf{S}_i^{(j)} \ (\mu v)^{\frac{l-j+1}{2}} I_{l+j+1} \left(\frac{2}{\sqrt{\mu v}}\right) \right]_{k,l=1,2,\dots,\alpha}$$

Finally, using this result in (4.3) with some algebraic manipulation, and noting that $\mu n^3 f^{(\alpha)}_{\mu n^3 V}(\mu n^3 v)$ denotes the p.d.f. of the new variable V, concludes the proof. \Box

REMARK 2. Clearly, the above approach of deriving the p.d.f. of the asymptotic scaled version of $\kappa_D^2(\mathbf{A})$ explicitly depends on the availability of a closed-form expression for the p.d.f. of $\kappa_D^2(\mathbf{A})$. Interestingly, one can directly manipulate the m.g.f. of $\kappa_D^2(\mathbf{A})$ instead of the p.d.f. to yield the same asymptotic density. Although we do not demonstrate it here, we exploit this technique in deriving the scaled asymptotic p.d.f. of $\kappa_E^2(\mathbf{A})$ in section 6.

The exact asymptotic p.d.f. of the scaled $\kappa_D^2(\mathbf{A})/(\mu n^3)$ takes the following simple forms for small values of α .

COROLLARY 4.2.

For $\alpha = 0, 1, 2$, the result (4.1) becomes:

$$\begin{aligned} f_V^{(0)}(v) &= \frac{1}{\mu v^2} e^{-\frac{1}{\mu v}} H(v), \qquad f_V^{(1)}(v) = \frac{1}{\mu v^2} e^{-\frac{1}{\mu v}} I_2\left(\frac{2}{\sqrt{\mu v}}\right) H(v) \\ f_V^{(2)}(v) &= \frac{1}{\mu v^2} e^{-\frac{1}{\mu v}} \left\{ I_2\left(\frac{2}{\sqrt{\mu v}}\right) I_4\left(\frac{2}{\sqrt{\mu v}}\right) - \left[I_3\left(\frac{2}{\sqrt{\mu v}}\right)\right]^2 + \sqrt{\mu v} I_2\left(\frac{2}{\sqrt{\mu v}}\right) I_3\left(\frac{2}{\sqrt{\mu v}}\right) \right\} H(v). \end{aligned}$$

The asymptotic p.d.f. corresponding to the case $\alpha = 0$ and $\mu = 1/4$ is given in [21]

The advantage of the asymptotic formula given in Theorem 4.1 is that it provides an easy to use expression which compares favorably with finite n results. To further highlight this fact, in Fig. 4.1, we compare the analytical asymptotic p.d.f. derived in Theorem 4.1 with simulated data points corresponding to $\alpha = 1, n = 50, \mu = 4$ and $\alpha = 2, n = 50, \mu = 4$.

Having characterized $\kappa_D^2(\mathbf{A})$, we now focus on $\kappa_E^2(\mathbf{A})$.

5. M.g.f. and p.d.f. of $\kappa_E^2(\mathbf{A})$. Here we give new expressions for the m.g.f and the p.d.f. of $\kappa_E^2(\mathbf{A})$. We point out that the key derivation steps developed in this section may also be useful for characterizing the distributional properties of more general condition number metrics, given in [27, 28, 29].

Let $\lambda_2 = x$. By definition, the m.g.f. of $\kappa_E^2(\mathbf{A})$ for $n \geq 3$, is

$$\mathcal{M}_{\kappa_{E}^{2}(\mathbf{A})}(s) = e^{-s} \int_{0}^{\infty} \left\{ \int_{\mathcal{R}_{2}}^{\infty} \left\{ \int_{0}^{x} e^{-\frac{\lambda_{1}}{x}s - \frac{\sum_{j=3}^{n}\lambda_{j}}{x}s} f\left(\lambda_{1}, x, \dots, \lambda_{n}\right) d\lambda_{1} \right\} d\lambda_{3} d\lambda_{4} \dots d\lambda_{n} \right\} dx,$$
(5.1)

where $\mathcal{R}_2 = \{x \leq \lambda_3 \leq \lambda_4 \leq \ldots \leq \lambda_n\}$. Analogous to the result given in (3.2), the following theorem provides an exact simple solution for this m.g.f.:



FIG. 4.1. Comparison of simulated data points and the analytical p.d.f. $f_V^{(\alpha)}(v)$ for n = 50 with $\mu = 4$.

THEOREM 5.1. The m.g.f. of $\kappa_E^2(\mathbf{A})$, for $n \geq 3$, is given by

$$\mathcal{M}_{\kappa_{E}^{2}(\mathbf{A})}(s) = e^{-s(n-1)} \int_{0}^{\infty} \frac{e^{-(n-1)x} x^{mn-1}}{(x+s)^{mn-4}} \Biggl\{ \int_{0}^{1} \frac{z^{2} e^{-(1-z)(s+x)}}{(1-z)^{\alpha}}$$

$$\times \det \left[L_{n+i-j-2}^{(j+1)} (-z(x+s)) L_{n+i-k}^{(k-1)} (-(x+s)) \right]_{\substack{i=1,2,\dots,\alpha+2\\ j=1,2\\ k=3,4,\dots,\alpha+2}} dz \Biggr\} dx.$$
(5.2)

Proof. We use (2.1) in (5.1) with some manipulation to obtain

$$\mathcal{M}_{\kappa_{E}^{2}(\mathbf{A})}(s) = K_{n,\alpha} \ e^{-s} \int_{0}^{\infty} e^{-x} x^{\alpha} \Biggl\{ \int_{\mathcal{R}_{2}} e^{-\frac{\sum_{j=3}^{n} \lambda_{j}}{x}s} \\ \times \prod_{j=3}^{n} \lambda_{j}^{\alpha} e^{-\lambda_{j}} \prod_{j=3}^{n} (\lambda_{j} - x)^{2} \prod_{3 \le k < j \le n} (\lambda_{j} - \lambda_{k})^{2} \\ \times \left[\int_{0}^{x} (x - \lambda_{1})^{2} \prod_{j=3}^{n} (\lambda_{j} - \lambda_{1})^{2} \lambda_{1}^{\alpha} e^{-\lambda_{1} \left(1 + \frac{s}{x}\right)} d\lambda_{1} \right] d\lambda_{3} \dots d\lambda_{n} \Biggr\} dx ,$$

where we have used $\Delta_n^2(\boldsymbol{\lambda}) = (x - \lambda_1)^2 \prod_{j=3}^n (\lambda_j - \lambda_1)^2 \prod_{j=3}^n (\lambda_j - x)^2 \prod_{3 \le k < j \le n} (\lambda_j - \lambda_k)^2$, which is valid for $n \ge 3$. Applying the variable transformation $\lambda_1 = xz$ to the innermost integral yields

$$\mathcal{M}_{\kappa_{E}^{2}(\mathbf{A})}(s) = K_{n,\alpha}e^{-s} \int_{0}^{\infty} e^{-x}x^{2m-1} \left\{ \int_{\mathcal{R}_{2}} e^{-\frac{\sum_{j=3}^{n}\lambda_{j}}{x}s} \prod_{j=3}^{n} \lambda_{j}^{\alpha}e^{-\lambda_{j}} \prod_{j=3}^{n} (\lambda_{j} - x)^{2} \right. \\ \left. \times \prod_{3 \le k < j \le n} (\lambda_{j} - \lambda_{k})^{2} \varphi\left(\frac{\lambda_{3}}{x}, \frac{\lambda_{4}}{x}, \dots, \frac{\lambda_{n}}{x}, x\right) d\lambda_{3} \dots d\lambda_{n} \right\} dx,$$

where we have defined:

$$\varphi\left(\frac{\lambda_3}{x},\frac{\lambda_4}{x},\ldots,\frac{\lambda_n}{x},x\right) := \int_0^1 z^\alpha (1-z)^2 \prod_{j=3}^n \left(\frac{\lambda_j}{x}-z\right)^2 e^{-z(s+x)} dz \quad .$$

By symmetry, we convert the ordered region of integration to an unordered region, and subsequently introduce the variable transformations $y_j = \frac{(x+s)}{x} (\lambda_j - x)$, $j = 3, 4, \ldots, n$, to obtain

$$\mathcal{M}_{\kappa_{E}^{2}(\mathbf{A})}(s) = \frac{K_{n,\alpha}}{(n-2)!} e^{-s(n-1)} \int_{0}^{\infty} \frac{e^{-(n-1)x} x^{mn-1}}{(x+s)^{mn-2m}} \\ \times \left\{ \int_{[0,\infty)^{n-2}} \prod_{j=3}^{n} y_{j}^{2} (y_{j}+x+s)^{\alpha} e^{-y_{j}} \prod_{3 \le k < j \le n} (y_{j}-y_{k})^{2} \\ \times \left[\int_{0}^{1} z^{\alpha} (1-z)^{2} \prod_{j=3}^{n} \left(\frac{y_{j}}{x+s}+1-z\right)^{2} e^{-z(s+x)} dz \right] dy_{3} \dots dy_{n} \right\} dx.$$

Now, changing the order of the innermost integrals and then relabeling the variables according to $y_j = x_{j-2}, j = 3, 4, ..., n$, yields

$$\mathcal{M}_{\kappa_{E}^{2}(\mathbf{A})}(s) = \frac{K_{n,\alpha}}{(n-2)!} e^{-s(n-1)} \int_{0}^{\infty} \frac{e^{-(n-1)x} x^{mn-1}}{(x+s)^{mn-2\alpha-4}} \left[\int_{0}^{1} z^{\alpha} (1-z)^{2} e^{-z(s+x)} \right] \times (-1)^{n\alpha} R(n-2, -(1-z)(x+s), -(x+s), \alpha) dz dx, \quad (5.3)$$

where we have defined

$$R(n,a,b,\alpha) := \int_{[0,\infty)^n} \prod_{j=1}^n x_j^2 e^{-x_j} (a-x_j)^2 (b-x_j)^{\alpha} \Delta_n^2(\mathbf{x}) \, dx_1 \dots dx_n \, .$$

The remainder of the proof is focused on evaluating $R(n, a, b, \alpha)$. Following [32, Eqs. 22.4.2, 22.4.11], we start with the related integral

$$\int_{[0,\infty)^n} \prod_{j=1}^n x_j^2 e^{-x_j} \prod_{i=1}^{\alpha+2} (r_i - x_j) \Delta_n^2(\mathbf{x}) \, dx_1 dx_2 \dots dx_n$$
$$= \frac{n!}{K_{n,2}} \Delta_{\alpha+2}^{-1}(\mathbf{r}) \det \left[C_{n+i-1}(r_j) \right]_{i,j=1,2,\dots,\alpha+2}, \quad (5.4)$$

where $C_k(x)$ are monic polynomials orthogonal with respect to $x^2 e^{-x}$, over $0 \le x < \infty$. As such, $C_k(x) = (-1)^k k! L_k^{(2)}(x)$, which upon substituting into (5.4) gives

$$\int_{[0,\infty)^n} \prod_{j=1}^n x_j^2 e^{-x_j} \prod_{i=1}^{\alpha+2} (r_i - x_j) \Delta_n^2(\mathbf{x}) \, dx_1 dx_2 \dots dx_n$$

= $\frac{n!}{K_{n,2}} (-1)^{(n-1)\alpha} \prod_{j=0}^{\alpha+1} (-1)^{j+1} (n+j)! \frac{\det \left[L_{n+i-1}^{(2)}(r_j) \right]_{i,j=1,2,\dots,\alpha+2}}{\Delta_{\alpha+2}(\mathbf{r})} .$ (5.5)

In the above formula, the r_i s are, in general, distinct parameters. However, if we can choose r_i such that

$$r_i = \begin{cases} a & \text{if } i = 1, 2 \\ b & \text{if } i = 3, 4, \dots, \alpha + 2, \end{cases}$$

then the left side of (5.5) becomes precisely $R(n, a, b, \alpha)$. Under direct substitution, however, it turns out that the right-hand side of (5.5) then gives a $\frac{0}{0}$ indeterminate form. Therefore, the task is to evaluate the following limits:

$$R(n, a, b, \alpha) = \frac{n!}{K_{n,2}} (-1)^{(n-1)\alpha} \prod_{j=0}^{\alpha+1} (-1)^{j+1} (n+j)!$$

$$\times \lim_{\substack{r_1, r_2 \to a \\ r_3, r_4, \dots, r_{\alpha+2} \to b}} \frac{\det \left[L_{n+i-1}^{(2)}(r_j) \right]_{i,j=1,2,\dots,\alpha+2}}{\Delta_{\alpha+2}(\mathbf{r})}.$$
(5.6)

This can be solved by capitalizing on an approach outlined in [41] which gives

$$\lim_{\substack{r_1, r_2 \to a \\ r_3, r_4, \dots, r_{\alpha+2} \to b}} \frac{\det \left[L_{n+i-1}^{(2)}(r_j) \right]_{i,j=1,2,\dots,\alpha+2}}{\Delta_{\alpha+2}(\mathbf{r})} = \frac{\det \left[\frac{d^{j-1}}{da^{j-1}} L_{n+i-1}^{(2)}(a) - \frac{d^{k-3}}{db^{k-3}} L_{n+i-1}^{(2)}(b) \right]_{\substack{i=1,2,\dots,\alpha+2 \\ j=1,2 \\ k=3,4,\dots,\alpha+2}}}{\det \left[\frac{d^{j-1}}{da^{j-1}} a^{i-1} - \frac{d^{k-3}}{db^{k-3}} b^{i-1} \right]_{\substack{i=1,2,\dots,\alpha+2 \\ k=3,4,\dots,\alpha+2}}}.$$
(5.7)

The determinant in the denominator of (5.7) can be written as

$$\det\left[\frac{d^{j-1}}{da^{j-1}}a^{i-1} \quad \frac{d^{k-3}}{db^{k-3}}b^{i-1}\right]_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}} = \prod_{j=0}^{\alpha-1} j!(a-b)^{2\alpha}.$$

The numerator can also be simplified using (2.3) to yield

$$\det \left[\frac{d^{j-1}}{da^{j-1}} L_{n+i-1}^{(2)}(a) \quad \frac{d^{k-3}}{db^{k-3}} L_{n+i-1}^{(2)}(b) \right]_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}} = (-1)^{-\alpha} \prod_{j=0}^{\alpha+1} (-1)^{j+1} \det \left[L_{n+i-j}^{(j+1)}(a) \quad L_{n+i-k+2}^{(k-1)}(b) \right]_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}}$$

Substituting these expressions into (5.7) and then the result into (5.6) gives

$$R(n,a,b,\alpha) = \frac{(-1)^{-n\alpha}}{(a-b)^{2\alpha}} \frac{n!}{K_{n,2}} \frac{\prod_{j=0}^{\alpha+1} (n+j)!}{\prod_{j=0}^{\alpha-1} j!} \det \left[L_{n+i-j}^{(j+1)}(a) \ L_{n+i-k+2}^{(k-1)}(b) \right]_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}}_{\substack{k=3,4,\dots,\alpha+2\\(5.8)}}$$

Now we use this result in (5.3), apply the relation $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ in the inner integral (i.e., with respect to z), along with some basic manipulations which concludes the proof. \Box

A straightforward Laplace inversion of the m.g.f. of $\kappa_E^2(\mathbf{A})$ now gives the p.d.f. of $\kappa_E^2(\mathbf{A})$, which is presented in the following corollary: COROLLARY 5.2. The p.d.f. of $\kappa_E^2(\mathbf{A})$, for $n \ge 3$, is given by

$$f_{\kappa_{E}^{2}(\mathbf{A})}^{(\alpha)}(y) = \frac{\Gamma(mn)}{y^{mn}} \mathcal{L}^{-1} \left\{ \frac{e^{-s(n-1)}}{s^{mn-4}} \int_{0}^{1} z^{2} (1-z)^{-\alpha} e^{-(1-z)s} \right. \\ \left. \times \det \left[L_{n+i-j-2}^{(j+1)}(-sz) \ L_{n+i-k}^{(k-1)}(-s) \right]_{\substack{i=1,2,\dots,\alpha+2\\ j=1,2\\ k=3,4,\dots,\alpha+2}} dz \right\}.$$
(5.9)

Although further simplification of (5.9) seems intractable for general matrix dimensions m and n, we can obtain a closed-form solution in the important case of square Gaussian matrices (i.e., m = n). This is given as follows:

COROLLARY 5.3. For $\alpha = 0$, (5.9) becomes

$$f_{\kappa_{E}^{2}(\mathbf{A})}^{(0)}(y) = \frac{\Gamma(n^{2})}{12}n^{2}(n^{2}-1)\sum_{i=0}^{n-1}\sum_{j=0}^{n-2}\frac{(-n+1)_{i}(-n+2)_{j}}{(3)_{i}(4)_{j}i!j!}(j+1-i)(i+j+2)!$$

$$\times \left(\sum_{k=0}^{i+j+2}\frac{(-1)^{i+j+k}}{(i+j+2-k)!}\frac{y^{-n^{2}}(y-n+1)^{n^{2}+k-i-j-4}}{\Gamma(n^{2}+k-i-j-3)}H(y-n+1)\right)$$

$$-\frac{y^{-n^{2}}(y-n)^{n^{2}-2}}{\Gamma(n^{2}-1)}H(y-n)\right).$$
(5.10)

It turns out that there is also an interesting connection between the density of $\kappa_E^2(\mathbf{A})$ and the density of the second smallest eigenvalue, λ_2 . To see this, we start by writing the p.d.f. of λ_2 as

$$f_{\lambda_2}(x) = \int_{\mathcal{R}_2} \int_0^x f(\lambda_1, x, \dots, \lambda_n) \ d\lambda_1 d\lambda_3 d\lambda_4 \dots d\lambda_n,$$

by definition. This can be further simplified by using very similar techniques to those used in the above m.g.f. derivation (we omit the specific details) to yield

$$f_{\lambda_2}(x) = \frac{K_{n,\alpha}}{(n-2)!} x^{2\alpha+3} e^{-(n-1)x} \int_0^1 z^{\alpha} (1-z)^2 e^{-zx} (-1)^{n\alpha} R(n-2, -(1-z)x, -x, \alpha) dz.$$

Now, applying (5.8), we obtain the following expression for the p.d.f. of λ_2 :

$$f_{\lambda_2}(x) = x^3 e^{-(n-1)x} \int_0^1 z^2 (1-z)^{-\alpha} e^{-(1-z)x} \\ \times \det \left[L_{n+i-j-2}^{(j+1)}(-xz) \ L_{n+i-k}^{(k-1)}(-x) \right]_{\substack{i=1,2,\dots,\alpha+2\\k=3,4,\dots,\alpha+2}} dz.$$
(5.11)

An equivalent expression can also be obtained by starting with the joint density of λ_1 and λ_2 given in [23].

REMARK 3. We remark that the integrands corresponding to variable z in (5.2), (5.9) and (5.11) are well defined in the vicinity of z = 1.

Combining (5.9) and (5.11), we obtain the following interesting connection between the densities of $\kappa_E^2(\mathbf{A})$ and λ_2 :

$$f_{\kappa_E^2(\mathbf{A})}^{(\alpha)}(y) = \frac{\Gamma(mn)}{y^{mn}} \mathcal{L}^{-1}\left\{\frac{f_{\lambda_2}(s)}{s^{mn-1}}\right\}.$$
(5.12)

A previous connection between these two densities was also established in [29], but the result obtained was more complicated. Moreover, it appears difficult to establish our simplified connection (5.12) as a consequence of the representation in [29]. Our simplified representation here was made possible by adopting an alternative derivation approach based on the m.g.f., in contrast to the p.d.f.-based approach used in [29].

It is also worth contrasting the expression (5.11) with prior equivalent results in the literature, particularly those presented in [34, 51]. First, the result (5.11) is more compact, and in contrast to prior results in [34, 51] it does not involve summations of functions over combinatorial partitions of integers. Moreover, whilst both (5.11)and the previous expressions in [34, 51] involve determinants, the representations are markedly different—in (5.11), the size of the determinant depends on the *difference* between m and n (i.e., α), whilst in [34, 51] it depends on m explicitly. (Determinantal expressions having a similar m-dependence have also been derived for various more complicated random matrix ensembles; for example, complex non-central Wishart matrices [26] and generalized-F matrices [42].) This, in turn, gives remarkably simplified expressions in our case when m and n are of roughly the same order. For example, when $\alpha = 0$ (i.e., m = n), we have the following simple closed-form expression:

COROLLARY 5.4. For $\alpha = 0$, (5.11) evaluates to

$$f_{\lambda_2}(x) = \frac{1}{12} n^2 (n^2 - 1) e^{-(n-1)x} \sum_{i=0}^{n-1} \sum_{j=0}^{n-2} \frac{(-n+1)_i (-n+2)_j}{(3)_i (4)_j \ i! j!} (j+1-i)(i+j+2)! \\ \times \left(\sum_{k=0}^{i+j+2} \frac{(-1)^{i+j+k}}{(i+j+2-k)!} x^{i+j+2-k} - e^{-x} \right).$$
(5.13)

Having analyzed $\kappa_E^2(\mathbf{A})$ for finite values of m and n, we now focus our attention on the asymptotic behavior of the scaled version of $\kappa_E^2(\mathbf{A})$. 6. Asymptotic characterization of $\kappa_E^2(\mathbf{A})$. Here we investigate how $\kappa_E^2(\mathbf{A})$ scales as m and n tend to ∞ with $m - n = \alpha$ being fixed. Unlike in the previous asymptotic analysis, in this case it is most convenient to manipulate the m.g.f. of $\kappa_E^2(\mathbf{A})$, rather than the p.d.f. We have the following key result:

THEOREM 6.1. As m and n tend to ∞ such that $\alpha = m - n$ is fixed, $\kappa_E^2(\mathbf{A})$ scales on the order of n^3 . More specifically, the scaled random variable $V = \kappa_E^2(\mathbf{A})/(\mu n^3)$, with $\mu \in \mathbb{R}^+$ being an arbitrary constant, has the following asymptotic p.d.f. as m and n tend to ∞ with $\alpha = m - n$ fixed:

$$f_V^{(\alpha)}(v) = \frac{e^{-\frac{1}{\mu v}}}{\mu^4 v^5} \int_0^1 \frac{z^2}{(1-z)^{\alpha}} \det \left[g_{i,j}(z,v) \quad g_{i,k-2}(1,v)\right]_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}} dz, \tag{6.1}$$

where, for $i, j = 1, 2, ..., \alpha + 2$,

$$g_{i,j}(z,v) = \sum_{p=0}^{i-1} \sum_{q=0}^{p} \mathbf{S}_p^{(q)} \binom{i-1}{p} j^{i-1-p} \left(\frac{z}{\mu v}\right)^{\frac{q-j-1}{2}} I_{q+j+1} \left(2\sqrt{\frac{z}{\mu v}}\right) .$$

Proof. Our strategy is to derive the m.g.f. of V using the m.g.f. of $\kappa_E^2(\mathbf{A})$ given in (5.2). Subsequent application of the limits on m and n followed by the Laplace inversion will then yield the desired asymptotic p.d.f. As such, we can write

$$\mathcal{M}_{\frac{\kappa_E^2(\mathbf{A})}{\mu n^3}}(s) = \mathcal{M}_{\kappa_E^2(\mathbf{A})}\left(\frac{s}{\mu n^3}\right).$$
(6.2)

We first employ (2.2), and use similar arguments as in the derivation of (3.11), to write the determinant in (5.2) as

$$\det \left[L_{n+i-j-2}^{(j+1)}(-z(x+s)) \ L_{n+i-k}^{(k-1)}(-(x+s)) \right]_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}} = \frac{\prod_{j=1}^{\alpha+2}(n+j-1)! \left(\prod_{j=3}^{\alpha+2}(j-1)!\right)^{-1}}{2!3!(n+\alpha-1)!(n+\alpha-2)! \prod_{j=3}^{\alpha+2}(n+\alpha+2-j)!} \\ \times \sum_{l_1=0}^{n+\alpha-1} \sum_{l_2=0}^{n+\alpha-1} \sum_{l_3=0}^{n+\alpha-2} \sum_{l_4=0}^{n+\alpha-2} \sum_{l_5=0}^{n+\alpha-3} \dots \sum_{l_{\alpha+2}=0}^{n} \frac{(-n-\alpha+1)_{l_1}(-n-\alpha+2)_{l_2}}{(3)_{l_1}(4)_{l_2}l_1!l_2!} \\ \times z^{l_1+l_2}(-(x+s))^{l_1+l_2} \left(\prod_{k=1}^{\alpha} \frac{(-n-\alpha+k)_{l_{k+2}}}{(k+2)_{l_{k+2}}l_{k+2}!}(-(x+s))^{l_{k+2}}\right) \\ \times \det \left[\prod_{l=0}^{\alpha+1-i} (\tilde{z}_j-l) \prod_{l=0}^{\alpha+1-i} (\tilde{w}_k-l)\right]_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}}$$
(6.3)

where $\tilde{z}_j = n + \alpha - j - l_j$ and $\tilde{w}_k = n + \alpha + 2 - k - l_k$. Now, invoking Lemma A.2 to compute the remaining determinant, we use the resulting expression along with (5.2)

in (6.2) to obtain the m.g.f. of $\frac{\kappa_E^2(\mathbf{A})}{\mu n^3}$ as

$$\mathcal{M}_{\frac{\kappa_{E}^{2}(\mathbf{A})}{\mu n^{3}}}(s) = \frac{(n+\alpha)^{2}(n+\alpha+1)(n+\alpha-1)}{2!3!\prod_{j=3}^{\alpha+2}(j-1)!}e^{-\frac{s}{\mu n^{3}}(n-1)}$$

$$\times \int_{0}^{\infty} \frac{e^{-(n-1)x}x^{n(n+\alpha)-1}}{\left(x+\frac{s}{\mu n^{3}}\right)^{n(n+\alpha)-4}} \left\{ \int_{0}^{1} (1-z)^{-\alpha}z^{2}e^{-(1-z)\left(\frac{s}{\mu n^{3}}+x\right)} \right.$$

$$\times \sum_{l_{1}=0}^{n+\alpha-1} \sum_{l_{2}=0}^{n+\alpha-1} \sum_{l_{3}=0}^{n+\alpha-2} \sum_{l_{4}=0}^{n+\alpha-2} \sum_{l_{5}=0}^{n+\alpha-3} \cdots \sum_{l_{\alpha+2}=0}^{n} \frac{(-n-\alpha+1)_{l_{1}}(-n-\alpha+2)_{l_{2}}}{(3)_{l_{1}}(4)_{l_{2}}l_{1}!l_{2}!} z^{l_{1}+l_{2}}$$

$$\times \left(\prod_{k=1}^{\alpha} \frac{(-n-\alpha+k)_{l_{k+2}}}{(k+2)_{l_{k+2}}l_{k+2}!} \left(-\left(x+\frac{s}{\mu n^{3}}\right) \right)^{l_{k+2}} \right) \left(-\left(x+\frac{s}{\mu n^{3}}\right) \right)^{l_{1}+l_{2}} \Delta_{\alpha+2}(\mathbf{w}) \, dz \right\} dx$$

Then, we use the variable transformation t = xn, change the order of integration, and apply elementary limiting arguments as $n \to \infty$ to obtain

$$\lim_{n \to \infty} \mathcal{M}_{\frac{\kappa_E^2(\mathbf{A})}{\mu n^3}}(s) = \frac{1}{2!3! \prod_{j=3}^{\alpha+2} (j-1)!} \int_0^1 (1-z)^{-\alpha} z^2 \sum_{l_1=0}^\infty \sum_{l_2=0}^\infty \dots \sum_{l_{\alpha+2}=0}^\infty \frac{z^{l_1+l_2} \Delta_{\alpha+2}(\mathbf{w})}{(3)_{l_1}(4)_{l_2} l_1! l_2!} \\ \times \left(\prod_{k=1}^\alpha \frac{1}{(k+2)_{l_{k+2}} l_{k+2}!} \right) \left\{ \int_0^\infty e^{-t - \frac{s}{\mu t}} t^{3+\sum_{j=1}^{\alpha+2} l_j} dt \right\} dz.$$

Using the variable transformation $x = \frac{1}{\mu t}$ we compute the inner integral, and subsequently perform a Laplace inversion to yield the limiting p.d.f. of $V = \frac{\kappa_E^2(\mathbf{A})}{\mu n^3}$:

$$\begin{split} f_{V}^{(\alpha)}(v) &= \frac{1}{2!3! \prod_{j=3}^{\alpha+2} (j-1)!} \frac{e^{-\frac{1}{\mu v}}}{\mu^{4} v^{5}} \int_{0}^{1} (1-z)^{-\alpha} z^{2} \sum_{l_{1}=0}^{\infty} \sum_{l_{2}=0}^{\infty} \dots \sum_{l_{\alpha+2}=0}^{\infty} \frac{z^{l_{1}+l_{2}} \Delta_{\alpha+2}(\mathbf{w})}{(3)_{l_{1}}(4)_{l_{2}} l_{1}! l_{2}! (\mu v)^{l_{1}+l_{2}}} \\ &\times \left(\prod_{k=1}^{\alpha} \frac{1}{(k+2)_{l_{k+2}} l_{k+2}! (\mu v)^{l_{k+2}}} \right) dz \\ &= \frac{1}{2!3! \prod_{j=3}^{\alpha+2} (j-1)!} \frac{e^{-\frac{1}{\mu v}}}{\mu^{4} v^{5}} \int_{0}^{1} (1-z)^{-\alpha} z^{2} \\ &\times \det \left[\sum_{l_{j}=0}^{\infty} \frac{z_{j}^{i-1}}{(j+2)_{l_{j}} l_{j}!} \frac{z^{l_{j}}}{(\mu v)^{l_{j}}} - \sum_{l_{k}=0}^{\infty} \frac{w_{k}^{i-1}}{(k)_{l_{k}} l_{k}!} \frac{1}{(\mu v)^{l_{k}}} \right]_{\substack{i=1,2,\dots,\alpha+2\\ k=3,4,\dots,\alpha+2}} dz. \end{split}$$

Recalling the relations

$$z_j = j + l_j,$$
 $j = 1, 2,$
 $w_j = j + l_j - 2,$ $j = 3, 4, \dots, \alpha + 2,$

with some manipulation then gives

$$f_V^{(\alpha)}(v) = \frac{1}{2!3! \prod_{j=3}^{\alpha+2} (j-1)!} \frac{e^{-\frac{1}{\mu v}}}{\mu^4 v^5} \int_0^1 (1-z)^{-\alpha} z^2 \det\left[\tilde{g}_{i,j}(z,v) \quad \tilde{g}_{i,k-2}(1,v)\right]_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2\\(6.4)}} dz,$$

where

$$\tilde{g}_{i,j}(z,v) = \sum_{p=0}^{i-1} \binom{i-1}{p} j^{i-1-p} \sum_{l_j=0}^{\infty} \frac{l_j^p}{(j+2)_{l_j} l_j!} \left(\frac{z}{\mu v}\right)^{l_j}.$$

Noting the fact that $l_j^p = \sum_{q=0}^p \mathbf{S}_p^{(q)} l_j (l_j - 1) \cdots (l_j - q + 1)$ and making the identification

$$\sum_{k=0}^{\infty} \frac{x^k}{(a)_k (k-N)!} = (a-1)! x^{\frac{N+1-a}{2}} I_{a+N-1}(2\sqrt{x})$$

we arrive at

$$\tilde{g}_{i,j}(z,v) = (j+1)! \sum_{p=0}^{i-1} \sum_{q=0}^{p} {\binom{i-1}{p}} j^{i-1-p} \mathbf{S}_{p}^{(q)} \left(\frac{z}{\mu v}\right)^{\frac{q-j-1}{2}} I_{q+j+1} \left(2\sqrt{\frac{z}{\mu v}}\right).$$
(6.5)

Finally, noting the fact that $\tilde{g}_{i,j}(z,v) = (j+1)! g_{i,j}(z,v)$, $i, j = 1, 2, ..., \alpha + 2$, and using (6.5) in (6.4) with some algebraic manipulation concludes the proof. \Box

For $\alpha = 0$, we have the following remarkably simple closed-form solution: COROLLARY 6.2. For $\alpha = 0$, (6.1) becomes

$$f_V^{(0)}(v) = \frac{1}{576} \frac{e^{-\frac{1}{\mu v}}}{\mu^4 v^5} \left(16 \ _2F_3\left(3, \frac{7}{2}; 4, 4, 6; \frac{4}{\mu v}\right) - \frac{4}{\mu v} \ _1F_2\left(\frac{7}{2}; 5, 7; \frac{4}{\mu v}\right) \right. \\ \left. + \frac{3}{\mu v} \ _3F_4\left(\frac{7}{2}, 4, 4; 3, 5, 5, 7; \frac{4}{\mu v}\right) \right)$$

where ${}_{p}F_{q}(a_{1}, a_{2}, \ldots, a_{p}; b_{1}, b_{2}, \ldots, b_{q}; z)$ is the generalized hypergeometric function. This is easily obtained by setting $\alpha = 0$ and integrating using [36, Eq. 2.15.19.1] with some basic algebraic manipulations.

In Fig. 6.1, we illustrate the applicability of the analytical asymptotic results in the finite context. Specifically, we compare the analytical asymptotic p.d.f. given in Theorem 6.1 with that of simulated points corresponding to $\alpha = 0, n = 10, \mu = 4$ and $\alpha = 1, n = 50, \mu = 4$. The close agreement is clearly apparent.

Appendix A. Some Useful Determinant Results.

The following results are useful for the proofs of Theorems 3.3 and 6.1. For notational convenience we denote $\Delta_N(\mathbf{x}) = \prod_{1 \le l < k \le N} (x_k - x_l)$ where $\mathbf{x} = \{x_1, x_2, \ldots, x_N\}$ with $\Delta_1(\mathbf{x}) = 1$.

LEMMA A.1. Let $n, \alpha \in \mathbb{Z}^+$ with $n, \alpha \geq 1$. Then

$$\det\left[\prod_{i=0}^{\alpha-k-1} (\tilde{c}_l - i)\right]_{k,l=1,2,\dots,\alpha} = \Delta_{\alpha}(\mathbf{c})$$

where $\tilde{c}_l = n + \alpha - 1 - j_l - l$, $\mathbf{c} = \{c_1(j_1), c_2(j_2), \dots, c_{\alpha}(j_{\alpha})\}$ and $c_l(j_l) = l + j_l$ with $j_l = 0, 1, \dots, n + \alpha - 1 - l$.



FIG. 6.1. Comparison of simulated data points and the analytical p.d.f. $f_V^{(\alpha)}(v)$ for two different n, α settings with $\mu = 4$.

Proof. Let us rewrite the determinant as

$$\det \begin{bmatrix} \alpha^{-k-1} \\ \prod_{i=0}^{\alpha-k-1} (\tilde{c}_{l}-i) \end{bmatrix}_{k,l=1,2,\dots,\alpha} = \begin{vmatrix} \prod_{i=0}^{\alpha-2} (\tilde{c}_{1}-i) & \prod_{i=0}^{\alpha-2} (\tilde{c}_{2}-i) & \dots & \prod_{i=0}^{\alpha-2} (\tilde{c}_{\alpha}-i) \\ \prod_{i=0}^{\alpha-3} (\tilde{c}_{1}-i) & \prod_{i=0}^{\alpha-3} (\tilde{c}_{2}-i) & \dots & \prod_{i=0}^{\alpha-3} (\tilde{c}_{\alpha}-i) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{1} & \tilde{c}_{2} & \dots & \tilde{c}_{\alpha} \\ 1 & 1 & \dots & 1 \end{vmatrix}$$

Now the repeated application of the following row operations

kth row $\rightarrow k$ th row $+ S_{\alpha-k}^{(\alpha-k-i)} \times (k+i)$ th row $, i = 1, 2, \dots, \alpha - 1 - k$

on each row, from k = 1 to $k = \alpha - 2$, of the above determinant gives

$$\det \left[\prod_{i=0}^{\alpha-k-1} (\tilde{c}_{l}-i) \right]_{k,l=1,2,\dots,\alpha}$$

$$= \left| \begin{array}{ccc} \sum_{j=0}^{\alpha-1} \mathbf{S}_{\alpha-1}^{(j)} \prod_{i=0}^{j-1} (\tilde{c}_{1}-i) & \sum_{j=0}^{\alpha-1} \mathbf{S}_{\alpha-1}^{(j)} \prod_{i=0}^{j-1} (\tilde{c}_{2}-i) & \dots & \sum_{j=0}^{\alpha-1} \mathbf{S}_{\alpha-1}^{(j)} \prod_{i=0}^{j-1} (\tilde{c}_{\alpha}-i) \\ \sum_{j=0}^{\alpha-2} \mathbf{S}_{\alpha-2}^{(j)} \prod_{i=0}^{j-1} (\tilde{c}_{1}-i) & \sum_{j=0}^{\alpha-2} \mathbf{S}_{\alpha-2}^{(j)} \prod_{i=0}^{j-1} (\tilde{c}_{2}-i) & \dots & \sum_{j=0}^{\alpha-2} \mathbf{S}_{\alpha-2}^{(j)} \prod_{i=0}^{j-1} (\tilde{c}_{\alpha}-i) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{1} & \tilde{c}_{2} & \dots & \tilde{c}_{\alpha} \\ 1 & 1 & \dots & 1 \end{array} \right|$$

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We may use Lemma 2.4 to simplify each entry of the above determinant to obtain

$$\det \begin{bmatrix} \alpha^{-k-1} \\ \prod_{i=0}^{\alpha-k-1} (\tilde{c}_l - i) \end{bmatrix}_{k,l=1,2,\dots,\alpha} = \begin{vmatrix} \tilde{c}_1^{\alpha-1} & \tilde{c}_2^{\alpha-1} & \dots & \tilde{c}_{\alpha}^{\alpha-1} \\ \tilde{c}_1^{\alpha-2} & \tilde{c}_2^{\alpha-2} & \dots & \tilde{c}_{\alpha}^{\alpha-2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_1 & \tilde{c}_2 & \dots & \tilde{c}_{\alpha} \\ 1 & 1 & \dots & 1 \end{vmatrix}$$
$$= (-1)^{\lfloor \frac{\alpha}{2} \rfloor} \det \begin{bmatrix} \tilde{c}_l^{k-1} \end{bmatrix}_{k,l=1,2,\dots,\alpha}$$

where $\lfloor \cdot \rfloor$ is the floor function. Noting the relation

$$\tilde{c}_k = n + \alpha - 1 - c_k(j_k) , \ k = 1, 2, \dots, \alpha ,$$

we obtain

$$\det \left[\prod_{i=0}^{\alpha-k-1} (\tilde{c}_l - i) \right]_{k,l=1,2,\dots,\alpha} = (-1)^{\lfloor \frac{\alpha}{2} \rfloor} \det \left[\tilde{c}_l^{k-1} \right]_{k,l=1,2,\dots,\alpha}$$
$$= (-1)^{\lfloor \frac{\alpha}{2} \rfloor} \prod_{1 \le l < k \le \alpha} (\tilde{c}_k - \tilde{c}_l)$$
$$= (-1)^{\lfloor \frac{\alpha^2}{2} \rfloor} \prod_{1 \le l < k \le \alpha} (c_k(j_k) - c_l(j_l))$$
$$= \Delta_{\alpha}(\mathbf{c}),$$

where we have used the fact that $\lfloor \frac{\alpha^2}{2} \rfloor$ is an even number for $\alpha = 1, 2, ..., \square$ LEMMA A.2. Let $n \in \mathbb{Z}^+$ and $\alpha \in \mathbb{Z}^+ \cup \{0\}$ with $n \ge 2$. Then

$$\det \left[\prod_{l=0}^{\alpha+1-i} (\tilde{z}_j - l) \quad \prod_{l=0}^{\alpha+1-i} (\tilde{w}_k - l)\right]_{\substack{i=1,2,...,\alpha+2\\j=1,2\\k=3,4,...,\alpha+2}} = \Delta_{\alpha+2}(\mathbf{w})$$

where $\mathbf{w} = (z_1, z_2, w_3, \dots, w_{\alpha+2})$ and

$$z_j = j + l_j,$$
 $\tilde{z}_j = n + \alpha - z_j, \quad j = 1, 2,$
 $w_k = k + l_k - 2,$ $\tilde{w}_k = n + \alpha - w_k, \quad k = 3, 4, \dots, \alpha + 2,$

with

$$l_j = 0, 1, \dots, n + \alpha - j, \qquad j = 1, 2,$$

 $l_k = 0, 1, \dots, n + \alpha + 2 - k, \ k = 3, 4, \dots, \alpha + 2.$

Proof.

$$\det \begin{bmatrix} \prod_{l=0}^{\alpha+1-i} (\tilde{z}_{j}-l) & \prod_{l=0}^{\alpha+1-i} (\tilde{w}_{k}-l) \end{bmatrix}_{\substack{i=1,2,\dots,\alpha+2\\ j=1,2\\ k=3,4,\dots,\alpha+2}} \\ = \begin{vmatrix} \prod_{i=0}^{\alpha} (\tilde{z}_{1}-i) & \prod_{i=0}^{\alpha} (\tilde{z}_{2}-i) & \prod_{i=0}^{\alpha} (\tilde{w}_{3}-i) & \dots & \prod_{i=0}^{\alpha} (\tilde{w}_{\alpha+2}-i) \\ \prod_{i=0}^{\alpha-1} (\tilde{z}_{1}-i) & \prod_{i=0}^{\alpha-1} (\tilde{z}_{2}-i) & \prod_{i=0}^{\alpha-1} (\tilde{w}_{3}-i) & \dots & \prod_{i=0}^{\alpha} (\tilde{w}_{\alpha+2}-i) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{z}_{1} & \tilde{z}_{2} & \tilde{w}_{3} & \dots & \tilde{w}_{\alpha+2} \\ 1 & 1 & 1 & \dots & 1 \end{vmatrix} \end{vmatrix}.$$

Now, repeatedly applying the following row operations

kth row \rightarrow kth row + $\mathbf{S}_{\alpha+2-k}^{(\alpha+2-k-i)}(k+i)$ th row, $i = 1, 2, \dots, \alpha+1-k$

on each row (i.e., $k = 1, 2, ..., \alpha$) and using Lemma 2.4 gives

$$\det \begin{bmatrix} \prod_{l=0}^{\alpha+1-i} (\tilde{z}_j - l) & \prod_{l=0}^{\alpha+1-i} (\tilde{w}_k - l) \end{bmatrix}_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}} \\ = \begin{vmatrix} \tilde{z}_1^{\alpha+1} & \tilde{z}_2^{\alpha+1} & \tilde{w}_3^{\alpha+1} & \dots & \tilde{w}_{\alpha+2}^{\alpha+1} \\ \tilde{z}_1^{\alpha} & \tilde{z}_2^{\alpha} & \tilde{w}_3^{\alpha} & \dots & \tilde{w}_{\alpha+2}^{\alpha+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{z}_1 & \tilde{z}_2 & \tilde{w}_3 & \dots & \tilde{w}_{\alpha+2} \\ 1 & 1 & 1 & \dots & 1 \end{vmatrix} \\ = (-1)^{\lfloor \frac{\alpha+2}{2} \rfloor} \det \begin{bmatrix} \tilde{z}_j^{i-1} & \tilde{w}_k^{i-1} \end{bmatrix}_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}}$$

Interestingly, the resultant simplified determinant is of Vandermonde type. This in turn gives

$$(-1)^{\lfloor \frac{\alpha+2}{2} \rfloor} \det \begin{bmatrix} \tilde{z}_j^{i-1} & \tilde{w}_k^{i-1} \end{bmatrix}_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}} = \det \begin{bmatrix} z_j^{i-1} & w_k^{i-1} \end{bmatrix}_{\substack{i=1,2,\dots,\alpha+2\\j=1,2\\k=3,4,\dots,\alpha+2}} = \Delta_{\alpha+2}(\mathbf{w}).$$

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