

ON THE MULTI-DIMENSIONAL CONTROLLER-AND-STOPPER GAMES

ERHAN BAYRAKTAR AND YU-JUI HUANG

ABSTRACT. We consider a zero-sum stochastic differential controller-and-stopper game in which the state process is a controlled diffusion evolving in a multi-dimensional Euclidean space. In this game, the controller affects both the drift and diffusion terms of the state process, and the diffusion term can be degenerate. Under appropriate conditions, we show that the game has a value and the value function is the unique viscosity solution to an obstacle problem for a Hamilton-Jacobi-Bellman equation.

Key Words: Controller-stopper games, weak dynamic programming principle, viscosity solutions, robust optimal stopping.

1. INTRODUCTION

We consider a zero-sum stochastic differential game of control and stopping under a fixed time horizon $T > 0$. There are two players, the “controller” and the “stopper,” and a state process X^α which can be manipulated by the controller through the selection of the control α . Suppose the game starts at time $t \in [0, T]$. While the stopper has the right to choose the duration of this game (in the form of a random time τ), she incurs the running cost $f(s, X_s^\alpha, \alpha_s)$ at every moment $t \leq s < \tau$, and the terminal cost $g(X_\tau^\alpha)$ at the time the game stops. Given the instantaneous discount rate $c(s, X_s^\alpha)$, the stopper would like to minimize her expected discounted cost

$$\mathbb{E} \left[\int_t^\tau e^{-\int_t^s c(u, X_u^\alpha) du} f(s, X_s^\alpha, \alpha_s) ds + e^{-\int_t^\tau c(u, X_u^\alpha) du} g(X_\tau^\alpha) \right] \quad (1.1)$$

over all choices of τ . At the same time, however, the controller plays against her by maximizing (1.1) over all choices of α .

Ever since the game of control and stopping was introduced by Maitra & Sudderth [25], it has been known to be closely related to some common problems in mathematical finance, such as pricing American contingent claims (see e.g. [17, 21, 22]) and minimizing the probability of lifetime ruin (see [5]). The game itself, however, has not been studied to a great extent except certain particular cases. Karatzas and Sudderth [20] study a zero-sum controller-and-stopper game in which the state process X^α is a one-dimensional diffusion along a given interval on \mathbb{R} . Under appropriate conditions they prove that this game has a value and describe fairly explicitly a saddle point of

Date: January 6, 2013.

We would like to thank Mihai Sirbu for his thoughtful suggestions. We also would like to thank the two anonymous referees whose suggestions helped us improve our paper.

This research is supported in part by the National Science Foundation under an applied mathematics research grants DMS-0906257 and DMS-1118673, and a Career grant, and in part by the Susan M. Smith Professorship.

optimal choices. It turns out, however, difficult to extend their results to multi-dimensional cases, as their techniques rely heavily on theorems of optimal stopping for one-dimensional diffusions. To deal with zero-sum multi-dimensional games of control and stopping, Karatzas and Zamfirescu [23] develop a martingale approach; also see [2], [4] and [3]. Again, it is shown that the game has a value, and a saddle point of optimal choices is constructed. However, it is assumed to be that the controller can affect only the drift term of X^α .

There is yet another subtle discrepancy between the one-dimensional game in [20] and the multi-dimensional game in [23]: the use of “strategies”. Typically, in a two-player game, the player who acts first would not choose a fixed static action. Instead, she prefers to employ a strategy, which will give different responses to different future actions the other player will take. This additional flexibility enables the player to further decrease (increase) the expected cost, if she is the minimizer (maximizer). For example, in a game with two controllers (see e.g. [13, 12, 14, 9, 7]), the controller who acts first employs a strategy, which is a function that takes the other controller’s latter decision as input and generates a control. Note that the use of strategies is preserved in the one-dimensional controller-and-stopper game in [20]: what the stopper employs is not simply a stopping time, but a strategy in the form of a random time which depends on the controller’s decision. This kind of dynamic interaction is missing, however, in the multi-dimensional case: in [23], the stopper is restricted to use stopping times, which give the same response to any choice the controller makes.

Zero-sum multi-dimensional controller-and-stopper games are also covered in Hamadène & Lepeltier [16] and Hamadène [15], as a special case of mixed games introduced there. The main tool used in these papers is the theory of backward differential equations with two reflecting barriers. Interestingly, even though the method in [16, 15] differs largely from that in [23], these two papers also require a diffusion coefficient which is not affected by the controller, and do not allow the use of strategies. This is in contrast with the one-dimensional case in [20], where everything works out fine without any of the above restrictions. It is therefore of interest to see whether we can construct a new methodology under which multi-dimensional controller-and-stopper games can be analyzed even when the conditions required in [23, 16, 15] fail to hold.

In this paper, such a methodology is built, under a Markovian framework. On the one hand, we allow both the drift and diffusion terms of the state process X^α to be controlled. On the other hand, we allow the players to use strategies. Specifically, we first define *non-anticipating* strategies in Definition 3.1. Then, in contrast to two-controller games where both players use strategies, only the stopper chooses to use strategies in our case (which coincides with the set-up in [20]). This is because by the nature of a controller-and-stopper game, the controller cannot benefit from using non-anticipating strategies; see Remark 3.2. With this observation in mind, we give appropriate definitions of the upper value function U and the lower value function V in (3.6) and (3.7) respectively. Under this set-up, one presumably could construct a saddle point of optimal choices by imposing suitable assumptions on the cost functions, the dynamics of X^α , the associated Hamiltonian, or the control set (as is done in [20, 23, 15, 16]; see Remark 3.4). However, we have no plan to impose assumptions for constructing a saddle point. Instead, we intend to work under a

rather general framework, and determine under what conditions the game has a value (i.e. $U = V$) and how we can derive a PDE characterization for this value when it exists.

Our method is motivated by Bouchard & Touzi [8], where the weak dynamic programming principle for stochastic control problems was first introduced. By generalizing the weak dynamic programming principle in [8] to the context of controller-and-stopper games, we show that V is a viscosity supersolution and U^* is a viscosity subsolution to an obstacle problem for a Hamilton-Jacobi-Bellman equation, where U^* denotes the upper semicontinuous envelope of U defined as in (1.3). More specifically, we first prove a continuity result for an optimal stopping problem embedded in V (Lemma 4.1), which enables us to follow the arguments in [8, Theorem 3.5] even under the current context of controller-and-stopper games. We obtain, accordingly, a weak dynamic programming principle for V (Proposition 4.1), which is the key to proving the supersolution property of V (Propositions 4.3). On the other hand, by generalizing the arguments in Chapter 3 of Krylov [24], we derive a continuity result for an optimal control problem embedded in U (Lemma 5.4). This leads to a weak dynamic programming principle for U (Proposition 5.1), from which the subsolution property of U^* follows (Proposition 5.2). Finally, under appropriate conditions, we prove a comparison result for the associated obstacle problem. Since V is a viscosity supersolution and U^* is a viscosity subsolution, the comparison result implies $U^* \leq V$. Recalling that U^* is actually larger than V by definition, we conclude that $U^* = V$. This in particular implies $U = V$, i.e. the game has a value, and the value function is the unique viscosity solution to the associated obstacle problem. This is the main result of this paper; see Theorem 6.1. Note that once we have this PDE characterization, we can compute the value of the game using a stochastic numerical scheme proposed in Bayraktar & Fahim [1].

Another important advantage of our method is that it does not require any non-degeneracy condition on the diffusion term of X^α . For the multi-dimensional case in [23, 16, 15], Girsanov's theorem plays a crucial role, which entails non-degeneracy of the diffusion term. Even for the one-dimensional case in [20], this non-degeneracy is needed to ensure the existence of the state process (in the weak sense). Note that Weerasinghe [32] actually follows the one-dimensional model in [20] and extends it to the case with degenerate diffusion term; but at the same time, she assumes boundedness of the diffusion term, and some specific conditions including twice differentiability of the drift term and concavity of the cost function.

It is worth noting that while [23, 16, 15] do not allow the use of strategies and require the diffusion coefficient be control-independent and non-degenerate, they allow for non-Markovian dynamics and cost structures, as well as for non-Lipschitz drift coefficients. As a first step to allowing the use of strategies and incorporating controlled, and possibly degenerate, diffusion coefficients in a zero-sum multi-dimensional controller-and-stopper game, this paper focuses on proving the existence and characterization of the value of the game under a Markovian framework with Lipschitz coefficients. We leave the general non-Markovian and non-Lipschitz case for future research.

The structure of this paper is as follows: in Section 2, we set up the framework of our study. In Section 3, we define strategies, and give appropriate definitions of the upper value function U and

the lower value function V . In Sections 4 and 5, the supersolution property of V and the subsolution property U^* are derived, respectively. In Section 6, we prove a comparison theorem, which leads to the existence of the value of the game and the viscosity solution property of the value function.

1.1. Notation. We collect some notation and definitions here for readers' convenience.

- Given a probability space (E, \mathcal{I}, P) , we denote by $L^0(E, \mathcal{I})$ the set of real-valued random variables on (E, \mathcal{I}) ; for $p \in [1, \infty)$, let $L_n^p(E, \mathcal{I}, P)$ denote the set of \mathbb{R}^n -valued random variables R on (E, \mathcal{I}) s.t. $\mathbb{E}_P[|R|^p] < \infty$. For the “ $n = 1$ ” case, we simply write L_1^p as L^p .
- $\mathbb{R}_+ := [0, \infty)$ and $\mathcal{S} := \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$.
- \mathbb{M}^d denotes the set of $d \times d$ real matrices.
- Given $E \subseteq \mathbb{R}^n$, $\text{LSC}(E)$ denotes the set of lower semicontinuous functions defined on E , and $\text{USC}(E)$ denotes the set of upper semicontinuous functions defined on E .
- Let E be a normed space. For any $(t, x) \in [0, T] \times E$, we define two types of balls centered at (t, x) with radius $r > 0$ as follows

$$\begin{aligned} B_r(t, x) &:= \{(t', x') \in [0, T] \times E \mid |t' - t| < r, |x' - x| < r\}; \\ B(t, x; r) &:= \{(t', x') \in [0, T] \times E \mid t' \in (t - r, t], |x' - x| < r\}. \end{aligned} \quad (1.2)$$

We denote by $\bar{B}_r(t, x)$ and $\bar{B}(t, x; r)$ the closures of $B_r(t, x)$ and $B(t, x; r)$, respectively. Moreover, given $w : [0, T] \times E \mapsto \mathbb{R}$, we define the upper and lower semicontinuous envelopes of w , respectively, by

$$\begin{aligned} w^*(t, x) &:= \limsup_{\delta \downarrow 0} \{w(t', x') \mid (t', x') \in [0, T] \times E \text{ with } (t', x') \in B_\delta(t, x)\}; \\ w_*(t, x) &:= \liminf_{\delta \downarrow 0} \{w(t', x') \mid (t', x') \in [0, T] \times E \text{ with } (t', x') \in B_\delta(t, x)\}. \end{aligned} \quad (1.3)$$

2. PRELIMINARIES

2.1. The Set-up. Fix $T > 0$ and $d \in \mathbb{N}$. For any $t \in [0, T]$, let $\Omega^t := C([t, T]; \mathbb{R}^d)$ be the canonical space of continuous paths equipped with the uniform norm $\|\tilde{\omega}\|_{t, T} := \sup_{s \in [t, T]} |\tilde{\omega}_s|$, $\tilde{\omega} \in \Omega^t$. Let W^t denote the canonical process on Ω^t , and $\mathbb{G}^t = \{\mathcal{G}_s^t\}_{s \in [t, T]}$ denote the natural filtration generated by W^t . Let \mathbb{P}^t be the Wiener measure on $(\Omega^t, \mathcal{G}_T^t)$, and consider the collection of \mathbb{P}^t -null sets $\mathcal{N}^t := \{N \in \mathcal{G}_T^t \mid \mathbb{P}^t(N) = 0\}$ and its completion $\bar{\mathcal{N}}^t := \{A \subseteq \Omega^t \mid A \subseteq N \text{ for some } N \in \mathcal{N}^t\}$. Now, define $\bar{\mathcal{G}}^t = \{\bar{\mathcal{G}}_s^t\}_{s \in [t, T]}$ as the augmentation of \mathcal{G}^t by the sets in $\bar{\mathcal{N}}^t$, i.e. $\bar{\mathcal{G}}_s^t := \sigma(\mathcal{G}_s^t \cup \bar{\mathcal{N}}^t)$, $s \in [t, T]$. For any $x \in \mathbb{R}^d$, we also consider $\mathcal{G}_s^{t, x} := \mathcal{G}_s^t \cap \{W_s^t = x\}$, $\forall s \in [t, T]$. For $\Omega^t, W^t, \mathcal{N}^t, \bar{\mathcal{N}}^t, \mathcal{G}_s^t, \bar{\mathcal{G}}_s^t$ and $\mathcal{G}_s^{t, x}$, we drop the superscript t whenever $t = 0$.

Given $x \in \mathbb{R}^d$, we define for any $\tilde{\omega} \in \Omega^t$ the shifted path $(\tilde{\omega} + x)_\cdot := \tilde{\omega}_\cdot + x$, and for any $A \subseteq \Omega^t$ the shifted set $A + x := \{\tilde{\omega} \in \Omega^t \mid \tilde{\omega} - x \in A\}$. Then, we define the shifted Wiener measure $\mathbb{P}^{t, x}$ by $\mathbb{P}^{t, x}(F) := \mathbb{P}^t(F - x)$, $F \in \mathcal{G}_T^t$, and let $\bar{\mathbb{P}}^{t, x}$ denote the extension of $\mathbb{P}^{t, x}$ on $(\Omega^t, \bar{\mathcal{G}}_T^t)$. For $\mathbb{P}^{t, x}$ and $\bar{\mathbb{P}}^{t, x}$, we drop the superscripts t and x whenever $t = 0$ and $x = 0$. We let \mathbb{E} denote the expectation taken under $\bar{\mathbb{P}}$.

Fix $t \in [0, T]$ and $\omega \in \Omega$. For any $\tilde{\omega} \in \Omega^t$, we define the concatenation of ω and $\tilde{\omega}$ at t as

$$(\omega \otimes_t \tilde{\omega})_r := \omega_r 1_{[0, t]}(r) + (\tilde{\omega}_r - \tilde{\omega}_t + \omega_t) 1_{(t, T]}(r), \quad r \in [0, T].$$

Note that $\omega \otimes_t \tilde{\omega}$ lies in Ω . Consider the shift operator in space $\psi_t : \Omega^t \mapsto \Omega^t$ defined by $\psi_t(\tilde{\omega}) := \tilde{\omega} - \tilde{\omega}_t$, and the shift operator in time $\phi_t : \Omega \mapsto \Omega^t$ defined by $\phi_t(\omega) := \omega|_{[t,T]}$, the restriction of $\omega \in \Omega$ on $[t, T]$. For any $r \in [t, T]$, since ψ_t and ϕ_t are by definition continuous under the norms $\|\cdot\|_{t,r}$ and $\|\cdot\|_{0,r}$ respectively, $\psi_t : (\Omega^t, \mathcal{G}_r^t) \mapsto (\Omega^t, \mathcal{G}_r^t)$ and $\phi_t : (\Omega, \mathcal{G}_r) \mapsto (\Omega^t, \mathcal{G}_r^t)$ are Borel measurable. Then, for any $\xi : \Omega \mapsto \mathbb{R}$, we define the shifted functions $\xi^{t,\omega} : \Omega \mapsto \mathbb{R}$ by

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \phi_t(\omega')) \text{ for } \omega' \in \Omega.$$

Given a random time $\tau : \Omega \mapsto [0, \infty]$, whenever $\omega \in \Omega$ is fixed, we simplify our notation as

$$\omega \otimes_\tau \tilde{\omega} = \omega \otimes_{\tau(\omega)} \tilde{\omega}, \quad \xi^{\tau,\omega} = \xi^{\tau(\omega),\omega}, \quad \phi_\tau = \phi_{\tau(\omega)}, \quad \psi_\tau = \psi_{\tau(\omega)}.$$

Definition 2.1. On the space Ω , we define, for each $t \in [0, T]$, the filtration $\mathbb{F}^t = \{\mathcal{F}_s^t\}_{s \in [0, T]}$ by

$$\mathcal{F}_s^t := \mathcal{J}_{s+}^t, \text{ where } \mathcal{J}_s^t := \begin{cases} \{\emptyset, \Omega\}, & \text{if } s \in [0, t], \\ \sigma\left(\phi_t^{-1}\psi_t^{-1}\mathcal{G}_s^{t,0} \cup \overline{\mathcal{N}}\right), & \text{if } s \in [t, T]. \end{cases}$$

We drop the superscript t whenever $t = 0$.

Remark 2.1. Given $t \in [0, T]$, note that \mathcal{F}_s^t is a collection of subsets of Ω for each $s \in [0, T]$, whereas \mathcal{G}_s^t , $\overline{\mathcal{G}}_s^t$ and $\mathcal{G}_s^{t,x}$ are collections of subsets of Ω^t for each $s \in [t, T]$.

Remark 2.2. By definition, $\mathcal{J}_s = \overline{\mathcal{G}}_s \forall s \in [0, T]$; then the right continuity of $\overline{\mathcal{G}}$ implies $\mathcal{F}_s = \overline{\mathcal{G}}_s \forall s \in [0, T]$ i.e. $\mathbb{F} = \overline{\mathcal{G}}$. Moreover, from Lemma A.1 (iii) in Appendix A and the right continuity of $\overline{\mathcal{G}}$, we see that $\mathcal{F}_s^t \subseteq \overline{\mathcal{G}}_s = \mathcal{F}_s \forall s \in [0, T]$, i.e. $\mathbb{F}^t \subseteq \mathbb{F}$.

Remark 2.3. Intuitively, \mathbb{F}^t represents the information structure one would have if one starts observing at time $t \in [0, T]$. More precisely, for any $s \in [t, T]$, $\mathcal{G}_s^{t,0}$ represents the information structure one obtains after making observations on W^t in the period $[t, s]$. One could then deduce from $\mathcal{G}_s^{t,0}$ the information structure $\phi_t^{-1}\psi_t^{-1}\mathcal{G}_s^{t,0}$ for W on the interval $[0, s]$.

We define \mathcal{T}^t as the set of all \mathbb{F}^t -stopping times which take values in $[0, T]$ $\overline{\mathbb{P}}$ -a.s., and \mathcal{A}_t as the set of all \mathbb{F}^t -progressively measurable M -valued processes, where M is a separable metric space. Also, for any \mathbb{F} -stopping times τ_1, τ_2 with $\tau_1 \leq \tau_2$ $\overline{\mathbb{P}}$ -a.s., we denote by $\mathcal{T}_{\tau_1, \tau_2}^t$ the set of all $\tau \in \mathcal{T}^t$ which take values in $[\tau_1, \tau_2]$ $\overline{\mathbb{P}}$ -a.s. Again, we drop the sub- or superscript t whenever $t = 0$.

2.2. The State Process. Given $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, let $X^{t,x,\alpha}$ denote a \mathbb{R}^d -valued process satisfying the following SDE:

$$dX_s^{t,x,\alpha} = b(s, X_s^{t,x,\alpha}, \alpha_s)ds + \sigma(s, X_s^{t,x,\alpha}, \alpha_s)dW_s, \quad s \in [t, T], \quad (2.1)$$

with the initial condition $X_t^{t,x,\alpha} = x$. Let \mathbb{M}^d be the set of $d \times d$ real matrices. We assume that $b : [0, T] \times \mathbb{R}^d \times M \mapsto \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \times M \mapsto \mathbb{M}^d$ are deterministic Borel functions, and $b(t, x, u)$ and $\sigma(t, x, u)$ are continuous in (x, u) ; moreover, there exists $K > 0$ such that for any $t \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$,

$$|b(t, x, u) - b(t, y, u)| + |\sigma(t, x, u) - \sigma(t, y, u)| \leq K|x - y|, \quad (2.2)$$

$$|b(t, x, u)| + |\sigma(t, x, u)| \leq K(1 + |x|). \quad (2.3)$$

The conditions above imply that: for any initial condition $(t, x) \in [0, T] \times \mathbb{R}^d$ and control $\alpha \in \mathcal{A}$, (2.1) admits a unique strong solution $X^{t,x,\alpha}$. Moreover, without loss of generality, we define

$$X_s^{t,x,\alpha} := x \quad \text{for } s < t. \quad (2.4)$$

Remark 2.4. Fix $\alpha \in \mathcal{A}$. Under (2.2) and (2.3), the same calculations in [28, Appendix] and [6, Proposition 1.2.1] yield the following estimates: for each $p \geq 1$, there exists $C_p(\alpha) > 0$ such that for any $(t, x), (t', x') \in [0, T] \times \mathbb{R}^d$, and $h \in [0, T - t]$,

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |X_s^{t,x,\alpha}|^p \right] \leq C_p(1 + |x|^p); \quad (2.5)$$

$$\mathbb{E} \left[\sup_{0 \leq s \leq t+h} |X_s^{t,x,\alpha} - x|^p \right] \leq C_p h^{\frac{p}{2}}(1 + |x|^p); \quad (2.6)$$

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} |X_s^{t',x',\alpha} - X_s^{t,x,\alpha}|^p \right] \leq C_p \left[|x' - x|^p + |t' - t|^{\frac{p}{2}}(1 + |x|^p) \right]. \quad (2.7)$$

Remark 2.5 (flow property). By pathwise uniqueness of the solution to (2.1), for any $0 \leq t \leq s \leq T$, $x \in \mathbb{R}^d$, and $\alpha \in \mathcal{A}$, we have the following two properties:

- (i) $X_r^{t,x,\alpha}(\omega) = X_r^{s,X_s^{t,x,\alpha},\alpha}(\omega) \quad \forall r \in [s, T]$, for \mathbb{P} -a.e. $\omega \in \Omega$; see [6, Chapter 2] and [29, p.41].
- (ii) By (1.16) in [14] and the discussion below it, for \mathbb{P} -a.e. $\omega \in \Omega$, we have

$$X_r^{t,x,\alpha}(\omega \otimes_s \phi_s(\omega')) = X_r^{s,X_s^{t,x,\alpha}(\omega),\alpha^{s,\omega}}(\omega') \quad \forall r \in [s, T], \text{ for } \mathbb{P}\text{-a.e. } \omega' \in \Omega;$$

see also [27, Lemma 3.3].

2.3. Properties of Shifted Objects. Let us first derive some properties of \mathcal{F}_T^t -measurable random variables.

Proposition 2.1. Fix $t \in [0, T]$ and $\xi \in L^0(\Omega, \mathcal{F}_T^t)$.

- (i) \mathcal{F}_T^t and \mathcal{F}_t are independent. This in particular implies that ξ is independent of \mathcal{F}_t .
- (ii) There exist $\overline{N}, \overline{M} \in \overline{\mathcal{N}}$ such that: for any fixed $\omega \in \Omega \setminus \overline{N}$, $\xi^{t,\omega}(\omega') = \xi(\omega') \quad \forall \omega' \in \Omega \setminus \overline{M}$.

Proof. See Appendix A.1. □

Fix $\theta \in \mathcal{T}$. Given $\alpha \in \mathcal{A}$, we can define, for \mathbb{P} -a.e. $\omega \in \Omega$, a control $\alpha^{\theta,\omega} \in \mathcal{A}_{\theta(\omega)}$ by

$$\alpha^{\theta,\omega}(\omega') := \{\alpha_r^{\theta,\omega}(\omega')\}_{r \in [0, T]} = \{\alpha_r(\omega \otimes_{\theta} \phi_{\theta}(\omega'))\}_{r \in [0, T]}, \quad \omega' \in \Omega;$$

see [8, proof of Proposition 5.4]. Here, we state a similar result for stopping times in \mathcal{T} .

Proposition 2.2. Fix $\theta \in \mathcal{T}$. For any $\tau \in \mathcal{T}_{\theta, T}$, we have $\tau^{\theta,\omega} \in \mathcal{T}_{\theta(\omega), T}^{\theta(\omega)}$ for \mathbb{P} -a.e. $\omega \in \Omega$.

Proof. See Appendix A.2. □

Let $\rho : M \times M \mapsto \mathbb{R}$ be any given metric on M . By [24, p.142], $\rho'(u, v) := \frac{2}{\pi} \arctan \rho(u, v) < 1$ for $u, v \in M$ is a metric equivalent to ρ , from which we can construct a metric on \mathcal{A} by

$$\tilde{\rho}(\alpha, \beta) := \mathbb{E} \left[\int_0^T \rho'(\alpha_t, \beta_t) dt \right] \quad \text{for } \alpha, \beta \in \mathcal{A}. \quad (2.8)$$

Now, we state a generalized version of Proposition 2.1 (ii) for controls $\alpha \in \mathcal{A}$.

Proposition 2.3. Fix $t \in [0, T]$ and $\alpha \in \mathcal{A}_t$. There exists $\bar{N} \in \bar{\mathcal{N}}$ such that: for any $\omega \in \Omega \setminus \bar{N}$, $\tilde{\rho}(\alpha^{t,\omega}, \alpha) = 0$. Furthermore, for any $(s, x) \in [0, T] \times \mathbb{R}^d$, $X_r^{s,x,\alpha^{t,\omega}}(\omega') = X_r^{s,x,\alpha}(\omega')$, $r \in [s, T]$, for $\bar{\mathbb{P}}$ -a.e. $\omega' \in \Omega$.

Proof. See Appendix A.3. □

3. PROBLEM FORMULATION

We consider a controller-and-stopper game under the finite time horizon $T > 0$. While the controller has the ability to affect the state process X^α through the selection of the control α , the stopper has the right to choose the duration of this game, in the form of a random time τ . Suppose the game starts at time $t \in [0, T]$. The stopper incurs the running cost $f(s, X_s^\alpha, \alpha_s)$ at every moment $t \leq s < \tau$, and the terminal cost $g(X_\tau^\alpha)$ at the time the game stops, where f and g are some given deterministic functions. According to the instantaneous discount rate $c(s, X_s^\alpha)$ for some given deterministic function c , the two players interact as follows: the stopper would like to stop optimally so that her expected discounted cost could be minimized, whereas the controller intends to act adversely against her by manipulating the state process X^α in a way that frustrates the effort of the stopper.

For any $t \in [0, T]$, there are two possible scenarios for this game. In the first scenario, the stopper acts first. At time t , while the stopper is allowed to use the information of the path of W up to time t for her decision making, the controller has advantage: she has access to not only the path of W up to t but also the stopper's decision. Choosing one single stopping time, as a result, might not be optimal for the stopper. Instead, she would like to employ a stopping *strategy* which will give different responses to different future actions the controller will take.

Definition 3.1. Given $t \in [0, T]$, we say a function $\pi : \mathcal{A} \mapsto \mathcal{T}_{t,T}$ is an *admissible stopping strategy* on the horizon $[t, T]$ if it satisfies the following conditions:

(i) for any $\alpha, \beta \in \mathcal{A}$, it holds for $\bar{\mathbb{P}}$ -a.e. $\omega \in \Omega$ that

$$\text{if } \min\{\pi[\alpha](\omega), \pi[\beta](\omega)\} \leq \inf \left\{ s \geq t \mid \int_t^s \rho'(\alpha_r(\omega), \beta_r(\omega)) dr \neq 0 \right\}, \quad (3.1)$$

$$\text{then } \pi[\alpha](\omega) = \pi[\beta](\omega).$$

Recall that ρ' is a metric on M defined right above (2.8).

(ii) for any $s \in [0, t]$, if $\alpha \in \mathcal{A}_s$, then $\pi[\alpha] \in \mathcal{T}_{t,T}^s$.

(iii) for any $\alpha \in \mathcal{A}$ and $\theta \in \mathcal{T}$ with $\{\theta \leq t\} \notin \bar{\mathcal{N}}$, it holds for $\bar{\mathbb{P}}$ -a.e. $\omega \in \{\theta \leq t\}$ that

$$\pi[\alpha]^{\theta,\omega}(\omega') = \pi[\alpha^{\theta,\omega}](\omega'), \text{ for } \bar{\mathbb{P}}\text{-a.e. } \omega' \in \Omega.$$

We denote by $\Pi_{t,T}$ the set of all admissible stopping strategies on the horizon $[t, T]$.

Remark 3.1. Definition 3.1 (i) serves as the non-anticipativity condition for the stopping strategies. The intuition behind it should be clear: Suppose we begin our observation at time t , and employ a strategy $\pi \in \Pi_{t,T}$. By taking the control α and following the path ω , we decide to stop at the

moment $\pi[\alpha](\omega)$. If, up to this moment, we actually cannot distinguish between the controls α and β , then we should stop at the same moment if we were taking the control β .

Moreover, as shown in Proposition 3.1 below, (3.1) is equivalent to the following statement:

$$\text{For any } \alpha, \beta \in \mathcal{A} \text{ and } s \in [t, T], \quad 1_{\{\pi[\alpha] \leq s\}} = 1_{\{\pi[\beta] \leq s\}} \text{ for } \bar{\mathbb{P}}\text{-a.e. } \omega \in \{\alpha =_{[t,s]} \beta\}, \quad (3.2)$$

where $\{\alpha =_{[t,s]} \beta\} := \{\omega \in \Omega \mid \alpha_r(\omega) = \beta_r(\omega) \text{ for a.e. } r \in [t, s]\}$. This shows that Definition 3.1 (i) extends the non-anticipativity of strategies from two-controller games (see e.g. [9]) to current context of controller-and-stopper games.

Also notice that (3.2) is similar to, yet a bit weaker than, Assumption (C5) in [7]. This is because in the definition of $\{\alpha =_{[t,s]} \beta\}$, [7] requires $\alpha_r = \beta_r$ for all, instead of almost every, $r \in [t, s]$.

Proposition 3.1. Fix $t \in [0, T]$. For any function $\pi : \mathcal{A} \mapsto \mathcal{T}_{t,T}$, (3.1) holds iff (3.2) holds.

Proof. For any $\alpha, \beta \in \mathcal{A}$, we set $\theta(\omega) := \inf\{s \geq t \mid \int_t^s \rho'(\alpha_r(\omega), \beta_r(\omega)) dr \neq 0\}$.

Step 1: Suppose π satisfies (3.1). For any $\alpha, \beta \in \mathcal{A}$, take some $N \in \bar{\mathcal{N}}$ such that (3.1) holds for $\omega \in \Omega \setminus N$. Fix $s \in [t, T]$. Given $\omega \in \{\alpha =_{[t,s]} \beta\} \setminus N$, we have $s \leq \theta(\omega)$. If $\pi[\alpha](\omega) \leq \theta(\omega)$, then (3.1) implies $\pi[\alpha](\omega) = \pi[\beta](\omega)$, and thus $1_{\{\pi[\alpha] \leq s\}}(\omega) = 1_{\{\pi[\beta] \leq s\}}(\omega)$. If $\pi[\alpha](\omega) > \theta(\omega)$, then (3.1) implies $\pi[\beta](\omega) > \theta(\omega)$ too. It follows that $1_{\{\pi[\alpha] \leq s\}}(\omega) = 0 = 1_{\{\pi[\beta] \leq s\}}(\omega)$, since $s \leq \theta(\omega)$. This already proves (3.2).

Step 2: Suppose (3.2) holds. Fix $\alpha, \beta \in \mathcal{A}$. By (3.2), there exists some $N \in \bar{\mathcal{N}}$ such that

$$\text{for any } s \in \mathbb{Q} \cap [t, T], \quad 1_{\{\pi[\alpha] \leq s\}} = 1_{\{\pi[\beta] \leq s\}} \text{ for } \omega \in \{\alpha =_{[t,s]} \beta\} \setminus N. \quad (3.3)$$

Fix $\omega \in \Omega \setminus N$. For any $s \in \mathbb{Q} \cap [t, \theta(\omega)]$, we have $\omega \in \{\alpha =_{[t,s]} \beta\}$. Then (3.3) yields

$$1_{\{\pi[\alpha] \leq s\}}(\omega) = 1_{\{\pi[\beta] \leq s\}}(\omega), \text{ for all } s \in \mathbb{Q} \cap [t, \theta(\omega)]. \quad (3.4)$$

If $\pi[\alpha](\omega) \leq \theta(\omega)$, take an increasing sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{Q} \cap [t, \theta(\omega)]$ such that $s_n \uparrow \pi[\alpha](\omega)$. Then (3.4) implies $\pi[\beta](\omega) > s_n$ for all n , and thus $\pi[\beta](\omega) \geq \pi[\alpha](\omega)$. Similarly, by taking a decreasing sequence $\{r_n\}_{n \in \mathbb{N}} \subset \mathbb{Q} \cap [t, \theta(\omega)]$ such that $r_n \downarrow \pi[\alpha](\omega)$, we see from (3.4) that $\pi[\beta] \leq r_n$ for all n , and thus $\pi[\beta](\omega) \leq \pi[\alpha](\omega)$. We therefore conclude $\pi[\beta](\omega) = \pi[\alpha](\omega)$. Now, if $\pi[\beta](\omega) \leq \theta(\omega)$, we may argue as above to show that $\pi[\alpha](\omega) = \pi[\beta](\omega)$. This proves (3.1). \square

Next, we give concrete examples of strategies under Definition 3.1.

Example 1. Given $t \in [0, T]$, define $\lambda_t : \Omega \mapsto \Omega$ by $(\lambda_t(\omega)). := \omega_{\cdot \wedge t}$. Recall the space $C([t, T]; \mathbb{R}^d)$ of continuous functions mapping $[t, T]$ into \mathbb{R}^d . For any $x \in \mathbb{R}^d$, we define $\pi : \mathcal{A} \mapsto \mathcal{T}_{t,T}$ by

$$\pi[\alpha](\omega) := S(\{X_r^{t,x,\alpha}(\omega)\}_{r \in [t,T]}), \quad (3.5)$$

for some function $S : C([t, T]; \mathbb{R}^d) \mapsto [t, T]$ satisfying $\{\xi \mid S(\xi) \leq s\} \in \lambda_s^{-1} \mathcal{X}_T^t \forall s \in [t, T]$, where \mathcal{X}_T^t denotes the Borel σ -algebra generated by $C([t, T]; \mathbb{R}^d)$. Note that the formulation (3.5) is similar to the stopping rules introduced in the one-dimensional controller-and-stopper game in [20], and it covers concrete examples such as exit strategies of a Borel set (see e.g. (5.14) below). We claim that Definition 3.1 readily includes the formulation (3.5).

Let the function $\pi : \mathcal{A} \mapsto \mathcal{T}_{t,T}$ be given as in (3.5). First, for any $\alpha, \beta \in \mathcal{A}$, set $\theta := \inf\{s \geq t \mid \int_t^s \rho'(\alpha_r(\omega), \beta_r(\omega)) dr \neq 0\}$. Observing that the strong solutions $X^{t,x,\alpha}$ and $X^{t,x,\beta}$ coincide on the interval $[t, \theta]$ $\bar{\mathbb{P}}$ -a.s., we conclude that π satisfies Definition 3.1 (i). Next, for any $s \in [0, t]$, since $X^{t,x,\alpha}$ depends on \mathcal{F}_s only through the control α , Definition 3.1 (ii) also holds for π . To check Definition 3.1 (iii), let us introduce, for any $\theta \in \mathcal{T}$ with $\{\theta \leq t\} \notin \bar{\mathcal{N}}$, the strong solution \tilde{X} to the SDE (2.1) with the drift coefficient $\tilde{b}(s, x, u) := 1_{\{s < t\}}0 + 1_{\{s \geq t\}}b(s, x, u)$ and the diffusion coefficient $\tilde{\sigma}(s, x, u) := 1_{\{s < t\}}0 + 1_{\{s \geq t\}}\sigma(s, x, u)$. Then, by using the pathwise uniqueness of strong solutions and Remark 2.5 (ii), for $\bar{\mathbb{P}}$ -a.e. $\omega \in \{\theta \leq t\}$,

$$X_r^{t,x,\alpha}(\omega \otimes_\theta \phi_\theta(\omega')) = \tilde{X}_r^{0,x,\alpha}(\omega \otimes_\theta \phi_\theta(\omega')) = \tilde{X}_r^{\theta(\omega), \tilde{X}_\theta^{0,x,\alpha}(\omega), \alpha^{\theta,\omega}}(\omega') = \tilde{X}_r^{\theta(\omega), x, \alpha^{\theta,\omega}}(\omega') = X_r^{t,x,\alpha^{\theta,\omega}}(\omega'),$$

$\forall r \in [t, T]$, for $\bar{\mathbb{P}}$ -a.e. $\omega' \in \Omega$. This implies

$$\pi[\alpha]^{\theta,\omega}(\omega') = S(\{X_r^{t,x,\alpha}(\omega \otimes_\theta \phi_\theta(\omega'))\}_{r \in [t,T]}) = S(\{X_r^{t,x,\alpha^{\theta,\omega}}(\omega')\}_{r \in [t,T]}) = \pi[\alpha^{\theta,\omega}](\omega'),$$

for $\bar{\mathbb{P}}$ -a.e. $\omega' \in \Omega$, which is Definition 3.1 (iii).

Let us now look at the second scenario in which the controller acts first. In this case, the stopper has access to not only the path of W up to time t but also the controller's decision. The controller, however, does not use strategies as an attempt to offset the advantage held by the stopper. As the next remark explains, the controller merely chooses one single control because she would not benefit from using non-anticipating strategies.

Remark 3.2. Fix $t \in [0, T]$. Let $\gamma : \mathcal{T} \mapsto \mathcal{A}_t$ satisfy the following non-anticipativity condition: for any $\tau_1, \tau_2 \in \mathcal{T}$ and $s \in [t, T]$, it holds for $\bar{\mathbb{P}}$ -a.e. $\omega \in \Omega$ that

$$\text{if } \min\{\tau_1(\omega), \tau_2(\omega)\} > s, \text{ then } (\gamma[\tau_1])_r(\omega) = (\gamma[\tau_2])_r(\omega) \text{ for } r \in [t, s].$$

Then, observe that $\gamma[\tau](\omega) = \gamma[T](\omega)$ on $[t, \tau(\omega))$ $\bar{\mathbb{P}}$ -a.s. for any $\tau \in \mathcal{T}$. This implies that employing the strategy γ has the same effect as employing the control $\gamma[T]$. In other words, the controller would not benefit from using non-anticipating strategies.

Now, we are ready to introduce the upper and lower value functions of the game of control and stopping. For $(t, x) \in [0, T] \times \mathbb{R}^d$, if the stopper acts first, the associated value function is given by

$$U(t, x) := \inf_{\pi \in \Pi_{t,T}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E} \left[\int_t^{\pi[\alpha]} e^{-\int_t^s c(u, X_u^{t,x,\alpha}) du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^{\pi[\alpha]} c(u, X_u^{t,x,\alpha}) du} g(X_{\pi[\alpha]}^{t,x,\alpha}) \right]. \quad (3.6)$$

On the other hand, if the controller acts first, the associated value function is given by

$$V(t, x) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E} \left[\int_t^\tau e^{-\int_t^s c(u, X_u^{t,x,\alpha}) du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^\tau c(u, X_u^{t,x,\alpha}) du} g(X_\tau^{t,x,\alpha}) \right]. \quad (3.7)$$

By definition, we have $U \geq V$. We therefore call U the upper value function, and V the lower value function. We say the game has a value if these two functions coincide.

Remark 3.3. In a game with two controllers (see e.g. [13, 12, 14, 9]), upper and lower value functions are also introduced. However, since both of the controllers use strategies, it is difficult to

tell, just from the definitions, whether one of the value functions is larger than the other (despite their names). In contrast, in a controller-stopper game, only the stopper uses strategies, thanks to Remark 3.2. We therefore get $U \geq V$ for free, which turns out to be a crucial relation in the PDE characterization for the value of the game.

We assume that the cost functions f, g and the discount rate c satisfy the following conditions: $f : [0, T] \times \mathbb{R}^d \times M \mapsto \mathbb{R}_+$ is Borel measurable, and $f(t, x, u)$ is continuous in (x, u) , and continuous in x uniformly in $u \in M$ for each t ; $g : \mathbb{R}^d \mapsto \mathbb{R}_+$ is continuous; $c : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}_+$ is continuous and bounded above by some real number $\bar{c} > 0$. Moreover, we impose the following polynomial growth condition on f and g

$$|f(t, x, u)| + |g(x)| \leq K(1 + |x|^{\bar{p}}) \text{ for some } \bar{p} \geq 1. \quad (3.8)$$

Remark 3.4. Presumably, by imposing additional assumptions, one could construct a saddle point of optimal choices for a controller-and-stopper game. For example, in the one-dimensional game in [20], a saddle point is constructed under additional assumptions on the cost function and the dynamics of the state process (see (6.1)-(6.3) in [20]). For the multi-dimensional case, in order to find a saddle point, [23] assumes that the cost function and the drift coefficient are continuous with respect to the control variable, and the associated Hamiltonian always attains its infimum (see (71)-(73) in [23]); whereas [15] and [16] require compactness of the control set.

In this paper, we have no plan to impose additional assumptions for constructing saddle points. Instead, we intend to investigate, under a rather general set-up, whether the game has a value and how we can characterize this value if it exists.

Remark 3.5. For any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, the polynomial growth condition (3.8) and (2.5) imply that

$$\mathbb{E} \left[\sup_{t \leq r \leq T} \left(\int_t^r e^{-\int_t^s c(u, X_u^{t,x,\alpha}) du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^r c(u, X_u^{t,x,\alpha}) du} g(X_r^{t,x,\alpha}) \right) \right] < \infty. \quad (3.9)$$

Lemma 3.1. Fix $\alpha \in \mathcal{A}$ and $(s, x) \in [0, T] \times \mathbb{R}^d$. For any $\{(s_n, x_n)\}_{n \in \mathbb{N}} \subset [0, T] \times \mathbb{R}^d$ such that $(s_n, x_n) \rightarrow (s, x)$, we have

$$\mathbb{E} \left[\sup_{0 \leq r \leq T} |g(X_r^{s_n, x_n, \alpha}) - g(X_r^{s, x, \alpha})| \right] \rightarrow 0; \quad (3.10)$$

$$\mathbb{E} \int_0^T |1_{[s_n, T]}(r) f(r, X_r^{s_n, x_n, \alpha}, \alpha_r) - 1_{[s, T]}(r) f(r, X_r^{s, x, \alpha}, \alpha_r)| dr \rightarrow 0. \quad (3.11)$$

Proof. In view of (2.7), we have, for any $p \geq 1$,

$$\mathbb{E} \left[\sup_{0 \leq r \leq T} |X_r^{s_n, x_n, \alpha} - X_r^{s, x, \alpha}|^p \right] \rightarrow 0. \quad (3.12)$$

Thanks to the above convergence and the polynomial growth condition (3.8) on f , we observe that (3.11) is a consequence of [24, Lemma 2.7.6].

It remains to prove (3.10). Fix $\varepsilon, \eta > 0$. Take $a > 0$ large enough such that $\frac{2C_1 T(2+|x|)}{a} < \frac{\eta}{3}$, where $C_1 > 0$ is given as in Remark 2.4. Since g is continuous, it is uniformly continuous on

$\bar{B}_a(x) := \{y \in \mathbb{R}^d \mid |y - x| \leq a\}$. Thus, there exists some $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ for all $x, y \in \bar{B}_a(x)$ with $|x - y| < \delta$. Define

$$A := \left\{ \sup_{0 \leq r \leq T} |X_r^{s,x,\alpha} - x| > a \right\}, \quad B_n := \left\{ \sup_{0 \leq r \leq T} |X_r^{s_n, x_n, \alpha} - x| > a \right\},$$

$$B'_n := \left\{ \sup_{0 \leq r \leq T} |X_r^{s_n, x_n, \alpha} - x_n| > \frac{a}{2} \right\}, \quad D_n := \left\{ \sup_{0 \leq r \leq T} |X_r^{s_n, x_n, \alpha} - X_r^{s,x,\alpha}| \geq \delta \right\}.$$

By the Markov inequality and (2.6),

$$\bar{\mathbb{P}}(A) \leq \frac{C_1 \sqrt{T}(1 + |x|)}{a} < \frac{\eta}{3}, \quad \bar{\mathbb{P}}(B'_n) \leq \frac{2C_1 \sqrt{T}(1 + |x_n|)}{a} < \frac{\eta}{3} \text{ for } n \text{ large enough.}$$

On the other hand, (3.12) implies that $\bar{\mathbb{P}}(D_n) < \frac{\eta}{3}$ for n large enough. Noting that $(B'_n)^c \subseteq B_n^c$ for n large enough, we obtain

$$\begin{aligned} \bar{\mathbb{P}}\left(\sup_{0 \leq r \leq T} |g(X_r^{s_n, x_n, \alpha}) - g(X_r^{s,x,\alpha})| > \varepsilon\right) &\leq 1 - \bar{\mathbb{P}}(A^c \cap B_n^c \cap D_n^c) = \bar{\mathbb{P}}(A \cup B_n \cup D_n) \\ &\leq \bar{\mathbb{P}}(A \cup B'_n \cup D_n) < \eta, \quad \text{for } n \text{ large enough.} \end{aligned}$$

Thus, we have $h_n := \sup_{0 \leq r \leq T} |g(X_r^{s_n, x_n, \alpha}) - g(X_r^{s,x,\alpha})| \rightarrow 0$ in probability. Finally, observing that the polynomial growth condition (3.8) on g and (2.5) imply that $\{h_n\}_{n \in \mathbb{N}}$ is L^2 -bounded, we conclude that $h_n \rightarrow 0$ in L^1 , which gives (3.10). \square

3.1. The Associated Hamiltonian. For $(t, x, p, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M}^d$, we associate the following Hamiltonian with our mixed control/stopping problem:

$$H(t, x, p, A) := \inf_{a \in M} H^a(t, x, p, A), \quad (3.13)$$

where

$$H^a(t, x, p, A) := -b(t, x, a) \cdot p - \frac{1}{2} \text{Tr}[\sigma \sigma'(t, x, a) A] - f(t, x, a).$$

Since b , σ , and f are assumed to be continuous only in (x, a) , and M is a separable metric space without any compactness assumption, the operator H may be neither upper nor lower semicontinuous. As a result, we will need to consider an upper semicontinuous version of H defined by

$$\bar{H}(t, x, p, A) := \inf_{a \in M} (H^a)^*(t, x, p, A), \quad (3.14)$$

where $(H^a)^*$ is the upper semicontinuous envelope of H^a , defined as in (1.3); see Proposition 4.3. On the other hand, we will need to consider the lower semicontinuous envelope H_* , defined as in (1.3), in Proposition 5.2. Notice that \bar{H} is different from the upper semicontinuous envelope H^* , defined as in (1.3) (in fact, $\bar{H} \geq H^*$). See Remark 4.2 for our choice of \bar{H} over H^* .

3.2. Reduction to the Mayer Form. Given $t \in [0, T]$ and $\alpha \in \mathcal{A}_t$, let us increase the state process to (X, Y, Z) , where

$$\begin{aligned} dY_s^{t,x,y,\alpha} &= -Y_s^{t,x,y,\alpha} c(s, X_s^{t,x,\alpha}) ds, \quad s \in [t, T], \text{ with } Y_t^{t,x,y,\alpha} = y \geq 0; \\ Z_s^{t,x,y,z,\alpha} &:= z + \int_t^s Y_r^{t,x,y,\alpha} f(r, X_r^{t,x,\alpha}, \alpha_r) dr, \quad \text{for some } z \geq 0. \end{aligned}$$

Set $\mathcal{S} := \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$. For any $\mathbf{x} := (x, y, z) \in \mathcal{S}$, we define

$$\mathbf{X}_s^{t,\mathbf{x},\alpha} := \begin{pmatrix} X_s^{t,x,\alpha} \\ Y_s^{t,x,y,\alpha} \\ Z_s^{t,x,y,z,\alpha} \end{pmatrix},$$

and consider the function $F : \mathcal{S} \mapsto \mathbb{R}_+$ defined by

$$F(x, y, z) := z + yg(x).$$

Now, we introduce the functions $\bar{U}, \bar{V} : [0, T] \times \mathcal{S} \mapsto \mathbb{R}$ defined by

$$\begin{aligned} \bar{U}(t, x, y, z) &:= \inf_{\pi \in \Pi_{t,T}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E} \left[F(X_{\pi[\alpha]}^{t,x,\alpha}, Y_{\pi[\alpha]}^{t,x,y,\alpha}, Z_{\pi[\alpha]}^{t,x,y,z,\alpha}) \right] = \inf_{\pi \in \Pi_{t,T}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E} \left[F(\mathbf{X}_{\pi[\alpha]}^{t,\mathbf{x},\alpha}) \right], \\ \bar{V}(t, x, y, z) &:= \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E} \left[F(X_{\tau}^{t,x,\alpha}, Y_{\tau}^{t,x,y,\alpha}, Z_{\tau}^{t,x,y,z,\alpha}) \right] = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E} [F(\mathbf{X}_{\tau}^{t,\mathbf{x},\alpha})]. \end{aligned}$$

Given $\tau \in \mathcal{T}_{t,T}$, consider the function

$$J(t, \mathbf{x}; \alpha, \tau) := \mathbb{E}[F(\mathbf{X}_{\tau}^{t,\mathbf{x},\alpha})]. \quad (3.15)$$

Observing that $F(\mathbf{X}_{\tau}^{t,\mathbf{x},\alpha}) = z + yF(\mathbf{X}_{\tau}^{t,x,1,0,\alpha})$, we have

$$J(t, \mathbf{x}; \alpha, \tau) = z + yJ(t, (x, 1, 0); \alpha, \tau), \quad (3.16)$$

which in particular implies

$$\bar{U}(t, x, y, z) = z + yU(t, x) \quad \bar{V}(t, x, y, z) = z + yV(t, x). \quad (3.17)$$

Thus, we can express the value functions U and V as

$$U(t, x) = \inf_{\pi \in \Pi_{t,T}} \sup_{\alpha \in \mathcal{A}_t} J(t, (x, 1, 0); \alpha, \pi[\alpha]), \quad V(t, x) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} J(t, (x, 1, 0); \alpha, \tau).$$

The following result will be useful throughout this paper.

Lemma 3.2. *Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\alpha \in \mathcal{A}$. For any $\theta \in \mathcal{T}_{t,T}$ and $\tau \in \mathcal{T}_{\theta,T}$, we have*

$$\mathbb{E}[F(\mathbf{X}_{\tau}^{t,\mathbf{x},\alpha}) \mid \mathcal{F}_{\theta}](\omega) = J\left(\theta(\omega), \mathbf{X}_{\theta}^{t,\mathbf{x},\alpha}(\omega); \alpha^{\theta,\omega}, \tau^{\theta,\omega}\right), \text{ for } \bar{\mathbb{P}}\text{-a.e. } \omega \in \Omega.$$

Proof. See Appendix A.4. □

4. SUPERSOLUTION PROPERTY OF V

In this section, we will first study the following two functions

$$G^{\alpha}(s, \mathbf{x}) := \inf_{\tau \in \mathcal{T}_{s,T}^s} J(s, \mathbf{x}; \alpha, \tau), \quad \tilde{G}^{\alpha}(s, \mathbf{x}) := \inf_{\tau \in \mathcal{T}_{s,T}^s} J(s, \mathbf{x}; \alpha, \tau), \quad \text{for } (s, \mathbf{x}) \in [0, T] \times \mathcal{S}, \quad (4.1)$$

where $\alpha \in \mathcal{A}$ is being fixed. A continuity result of G^{α} enables us to adapt the arguments in [8] to current context. We therefore obtain a weak dynamic programming principle (WDPP) for the function V (Proposition 4.1), which in turn leads to the supersolution property of V (Proposition 4.3).

Lemma 4.1. *Fix $\alpha \in \mathcal{A}$.*

- (i) \tilde{G}^{α} is continuous on $[0, T] \times \mathcal{S}$.

(ii) Suppose $\alpha \in \mathcal{A}_t$ for some $t \in [0, T]$. Then $G^\alpha = \tilde{G}^\alpha$ on $[0, t] \times \mathcal{S}$. As a result, G^α is continuous on $[0, t] \times \mathcal{S}$.

Proof. (i) For any $s \in [0, T]$ and $\mathbf{x} = (x, y, z) \in \mathcal{S}$, observe from (3.16) that $\tilde{G}^\alpha(s, \mathbf{x}) = z + y\tilde{G}^\alpha(s, (x, 1, 0))$. Thus, it is enough to prove that $\tilde{G}^\alpha(s, (x, 1, 0))$ is continuous on $[0, T] \times \mathbb{R}^d$. Also note that under (2.4), we have

$$\tilde{G}^\alpha(s, \mathbf{x}) = \inf_{\tau \in \mathcal{T}_{s,T}} J(s, \mathbf{x}; \alpha, \tau) = \inf_{\tau \in \mathcal{T}_{0,T}} J(s, \mathbf{x}; \alpha, \tau).$$

Now, for any $(s, x) \in [0, T] \times \mathbb{R}^d$, take an arbitrary sequence $\{(s_n, x_n)\}_{n \in \mathbb{N}} \subset [0, T] \times \mathbb{R}^d$ such that $(s_n, x_n) \rightarrow (s, x)$. Then the continuity of $\tilde{G}^\alpha(s, (x, 1, 0))$ can be seen from the following estimation

$$\begin{aligned} \left| \tilde{G}^\alpha(s_n, (x_n, 1, 0)) - \tilde{G}^\alpha(s, (x, 1, 0)) \right| &= \left| \inf_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[F(\mathbf{X}_\tau^{s_n, x_n, 1, 0, \alpha})] - \inf_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[F(\mathbf{X}_\tau^{s, x, 1, 0, \alpha})] \right| \\ &\leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E} \left[|F(\mathbf{X}_\tau^{s_n, x_n, 1, 0, \alpha}) - F(\mathbf{X}_\tau^{s, x, 1, 0, \alpha})| \right] \leq \mathbb{E} \left[\sup_{0 \leq r \leq T} |F(\mathbf{X}_r^{s_n, x_n, 1, 0, \alpha}) - F(\mathbf{X}_r^{s, x, 1, 0, \alpha})| \right] \rightarrow 0, \end{aligned}$$

where the convergence follows from Lemma 3.1.

(ii) Suppose $\alpha \in \mathcal{A}_t$ for some $t \in [0, T]$. For any $(s, \mathbf{x}) \in [0, t] \times \mathcal{S}$ and $\tau \in \mathcal{T}_{s,T}$, by taking $\theta = s$ in Lemma 3.2, we have

$$J(s, \mathbf{x}; \alpha, \tau) = \mathbb{E}[\mathbb{E}[F(\mathbf{X}_\tau^{s, \mathbf{x}, \alpha}) \mid \mathcal{F}_s](\omega)] = \mathbb{E}[J(s, \mathbf{x}; \alpha, \tau^{s, \omega})] \geq \inf_{\tau \in \mathcal{T}_{s,T}^s} J(s, \mathbf{x}; \alpha, \tau), \quad (4.2)$$

where in the second equality we replace $\alpha^{s, \omega}$ by α , thanks to Proposition 2.3. We then conclude

$$\inf_{\tau \in \mathcal{T}_{s,T}} J(s, \mathbf{x}; \alpha, \tau) = \inf_{\tau \in \mathcal{T}_{s,T}^s} J(s, \mathbf{x}; \alpha, \tau), \quad (4.3)$$

as the “ \leq ” relation is trivial. That is, $\tilde{G}^\alpha(s, \mathbf{x}) = G^\alpha(s, \mathbf{x})$. \square

Now, we want to modify the arguments in the proof of [8, Theorem 3.5] to get a weak dynamic programming principle for V . Given $w : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$, we mimic the relation between V and \bar{V} in (3.17) and define $\bar{w} : [0, T] \times \mathcal{S} \mapsto \mathbb{R}$ by

$$\bar{w}(t, x, y, z) := z + yw(t, x), \quad (t, x, y, z) \in [0, T] \times \mathcal{S}. \quad (4.4)$$

Proposition 4.1. Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\varepsilon > 0$. Take arbitrary $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}_{t,T}^t$ and $\varphi \in \text{USC}([0, T] \times \mathbb{R}^d)$ with $\varphi \leq V$. We have the following:

- (i) $\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha})] < \infty$;
- (ii) If, moreover, $\mathbb{E}[\bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha})] < \infty$, then there exists $\alpha^* \in \mathcal{A}_t$ with $\alpha_s^* = \alpha_s$ for $s \in [0, \theta]$ such that

$$\mathbb{E}[F(\mathbf{X}_\tau^{t, \mathbf{x}, \alpha^*})] \geq \mathbb{E}[Y_{\tau \wedge \theta}^{t, x, y, \alpha} \varphi(\tau \wedge \theta, X_{\tau \wedge \theta}^{t, x, \alpha}) + Z_{\tau \wedge \theta}^{t, x, y, z, \alpha}] - 4\varepsilon, \quad \forall \tau \in \mathcal{T}_{t,T}^t.$$

Proof. (i) First, observe that for any $\mathbf{x} = (x, y, z) \in \mathcal{S}$, $\bar{\varphi}(t, \mathbf{x}) = y\varphi(t, x) + z \leq yV(t, x) + z \leq yg(x) + z$, which implies $\bar{\varphi}^+(t, \mathbf{x}) \leq yg(x) + z$. It follows that

$$\begin{aligned} \bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) &\leq Y_\theta^{t, x, y, \alpha} g(X_\theta^{t, x, \alpha}) + Z_\theta^{t, x, y, z, \alpha} \\ &\leq Y_\theta^{t, x, y, \alpha} g(X_\theta^{t, x, \alpha}) + z + \int_t^\theta Y_s^{t, x, y, \alpha} f(s, X_s^{t, x, \alpha}, \alpha_s) ds, \end{aligned}$$

the right-hand-side is integrable as a result of (3.9).

(ii) For each $(s, \eta) \in [0, T] \times \mathcal{S}$, by the definition of \bar{V} , there exists $\alpha^{(s, \eta), \varepsilon} \in \mathcal{A}_s$ such that

$$\inf_{\tau \in \mathcal{T}_{s, T}^s} J(s, \eta; \alpha^{(s, \eta), \varepsilon}, \tau) \geq \bar{V}(s, \eta) - \varepsilon. \quad (4.5)$$

Note that $\varphi \in \text{USC}([0, T] \times \mathbb{R}^d)$ implies $\bar{\varphi} \in \text{USC}([0, T] \times \mathcal{S})$. Then by the upper semicontinuity of $\bar{\varphi}$ on $[0, T] \times \mathcal{S}$ and the lower semicontinuity of $G^{\alpha^{(s, \eta), \varepsilon}}$ on $[0, s] \times \mathcal{S}$ (from Lemma 4.1 (ii)), there must exist $r^{(s, \eta)} > 0$ such that

$$\bar{\varphi}(t', x') - \bar{\varphi}(s, \eta) \leq \varepsilon \text{ and } G^{\alpha^{(s, \eta), \varepsilon}}(s, \eta) - G^{\alpha^{(s, \eta), \varepsilon}}(t', x') \leq \varepsilon \text{ for all } (t', x') \in B(s, \eta; r^{(s, \eta)}),$$

where $B(s, \eta; r) = \{(t', x') \in [0, T] \times \mathcal{S} \mid t' \in (s - r, s], |x' - \eta| < r\}$, defined as in (1.2). It follows that if $(t', x') \in B(s, \eta; r^{(s, \eta)})$, we have

$$G^{\alpha^{(s, \eta), \varepsilon}}(t', x') \geq G^{\alpha^{(s, \eta), \varepsilon}}(s, \eta) - \varepsilon \geq \bar{V}(s, \eta) - 2\varepsilon \geq \bar{\varphi}(s, \eta) - 2\varepsilon \geq \bar{\varphi}(t', x') - 3\varepsilon,$$

where the second inequality is due to (4.5). Here, we do not use the usual topology induced by balls of the form $B_r(s, \eta) = \{(t', x') \in [0, T] \times \mathcal{S} \mid |t' - s| < r, |x' - \eta| < r\}$; instead, for the time variable, we consider the topology induced by half-closed intervals on $[0, T]$, i.e. the so-called upper limit topology (see e.g. [11, Ex.4 on p.66]). Note from [11, Ex.3 on p.174] and [26, Ex.3 on p.192] that $(0, T]$ is a Lindelöf space under this topology. It follows that, under this setting, $\{B(s, \eta; r) \mid (s, \eta) \in [0, T] \times \mathcal{S}, 0 < r \leq r^{(s, \eta)}\}$ forms an open covering of $(0, T] \times \mathcal{S}$, and there exists a countable subcovering $\{B(t_i, x_i; r_i)\}_{i \in \mathbb{N}}$ of $(0, T] \times \mathcal{S}$. Now set $A_0 := \{T\} \times \mathcal{S}$, $C_{-1} := \emptyset$ and define for all $i \in \mathbb{N} \cup \{0\}$

$$A_{i+1} := B(t_{i+1}, x_{i+1}; r_{i+1}) \setminus C_i, \text{ where } C_i := C_{i-1} \cup A_i.$$

Under this construction, we have

$$(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) \in \cup_{i \in \mathbb{N} \cup \{0\}} A_i \text{ } \mathbb{P}\text{-a.s.}, \quad A_i \cap A_j = \emptyset \text{ for } i \neq j, \text{ and } G^{\alpha^{i, \varepsilon}}(t', x') \geq \bar{\varphi}(t', x') - 3\varepsilon \text{ for } (t', x') \in A_i, \quad (4.6)$$

where $\alpha^{i, \varepsilon} := \alpha^{(t_i, x_i), \varepsilon}$.

For any $n \in \mathbb{N}$, set $A^n := \cup_{0 \leq i \leq n} A_i$ and define

$$\alpha^{\varepsilon, n} := \alpha 1_{[0, \theta)} + \left(\alpha 1_{(A^n)^c}(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) + \sum_{i=0}^n \alpha^{i, \varepsilon} 1_{A_i}(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) \right) 1_{[\theta, T]} \in \mathcal{A}_t.$$

Note that $\alpha_s^{\varepsilon, n} = \alpha_s$ for $s \in [0, \theta)$. Whenever $\omega \in \{(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) \in A_i\}$, observe that $(\alpha^{\varepsilon, n})^{\theta, \omega}(\omega') = \alpha^{\varepsilon, n}(\omega \otimes_\theta \phi_\theta(\omega')) = \alpha^{i, \varepsilon}(\omega \otimes_\theta \phi_\theta(\omega')) = (\alpha^{i, \varepsilon})^{\theta, \omega}(\omega')$; also, we have $\alpha^{i, \varepsilon} \in \mathcal{A}_{\theta(\omega)}$, as $\alpha^{i, \varepsilon} \in \mathcal{A}_{t_i}$ and $\theta(\omega) \leq t_i$ on A_i . We then deduce from Lemma 3.2, Proposition 2.3, and (4.6) that for $\bar{\mathbb{P}}$ -a.e. $\omega \in \Omega$

$$\begin{aligned} \mathbb{E}[F(\mathbf{X}_\tau^{t, \mathbf{x}, \alpha^{\varepsilon, n}}) 1_{\{\tau \geq \theta\}} | \mathcal{F}_\theta] 1_{A^n}(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) &= 1_{\{\tau \geq \theta\}} \sum_{i=0}^n J(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}; \alpha^{i, \varepsilon}, \tau^{\theta, \omega}) 1_{A_i}(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) \\ &\geq 1_{\{\tau \geq \theta\}} \sum_{i=0}^n G^{\alpha^{i, \varepsilon}}(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) 1_{A_i}(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) \\ &\geq 1_{\{\tau \geq \theta\}} [\bar{\varphi}(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}) - 3\varepsilon] 1_{A^n}(\theta, \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}). \end{aligned} \quad (4.7)$$

Hence, we have

$$\begin{aligned}
\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{\varepsilon,n}})] &= \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})1_{\{\tau < \theta\}}] + \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{\varepsilon,n}})1_{\{\tau \geq \theta\}}] \\
&= \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})1_{\{\tau < \theta\}}] + \mathbb{E}\left[\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{\varepsilon,n}})1_{\{\tau \geq \theta\}}|\mathcal{F}_\theta]1_{A^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})\right] \\
&\quad + \mathbb{E}\left[\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^{\varepsilon,n}})1_{\{\tau \geq \theta\}}|\mathcal{F}_\theta]1_{(A^n)^c}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})\right] \\
&\geq \mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha})1_{\{\tau < \theta\}}] + \mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{A^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - 3\varepsilon \\
&\geq \mathbb{E}[1_{\{\tau < \theta\}}\bar{\varphi}(\tau, \mathbf{X}_\tau^{t,\mathbf{x},\alpha})] + \mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{A^n}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - 3\varepsilon,
\end{aligned} \tag{4.8}$$

where the first inequality comes from (4.7), and the second inequality is due to the observation that

$$\begin{aligned}
F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) &= Y_\tau^{t,x,y,\alpha}g(X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha} \geq Y_\tau^{t,x,y,\alpha}V(\tau, X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha} \\
&\geq Y_\tau^{t,x,y,\alpha}\varphi(\tau, X_\tau^{t,x,\alpha}) + Z_\tau^{t,x,y,z,\alpha}.
\end{aligned}$$

Since $\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$ (by part (i)), there exists $n^* \in \mathbb{N}$ such that

$$\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - \mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \varepsilon.$$

We observe the following holds for any $\tau \in \mathcal{T}_{t,T}^t$

$$\begin{aligned}
\mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - \mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] \\
\leq \mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - \mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \varepsilon.
\end{aligned} \tag{4.9}$$

Suppose $\mathbb{E}[\bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$, then we can conclude from (4.9) that for any $\tau \in \mathcal{T}_{t,T}^t$

$$\begin{aligned}
\mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] &= \mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - \mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] \\
&\leq \mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] + \varepsilon - \mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] \\
&= \mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})1_{A^{n^*}}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] + \varepsilon.
\end{aligned} \tag{4.10}$$

Taking $\alpha^* = \alpha^{\varepsilon,n^*}$, we now conclude from (4.8) and (4.10) that

$$\begin{aligned}
\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^*})] &\geq \mathbb{E}[1_{\{\tau < \theta\}}\bar{\varphi}(\tau, \mathbf{X}_\tau^{t,\mathbf{x},\alpha})] + \mathbb{E}[1_{\{\tau \geq \theta\}}\bar{\varphi}(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] - 4\varepsilon \\
&= \mathbb{E}[\bar{\varphi}(\tau \wedge \theta, \mathbf{X}_{\tau \wedge \theta}^{t,\mathbf{x},\alpha})] - 4\varepsilon \\
&= \mathbb{E}[Y_{\tau \wedge \theta}^{t,x,y,\alpha}\varphi(\tau \wedge \theta, X_{\tau \wedge \theta}^{t,x,\alpha}) + Z_{\tau \wedge \theta}^{t,x,y,z,\alpha}] - 4\varepsilon.
\end{aligned}$$

□

We still need the following property of V in order to obtain the supersolution property.

Proposition 4.2. *For any $(t, x) \in [0, T] \times \mathbb{R}^d$, $V(t, x) = \sup_{\alpha \in \mathcal{A}} \tilde{G}^\alpha(t, (x, 1, 0))$.*

Proof. Thanks to Lemma 4.1 (ii), we immediately have

$$V(t, x) = \sup_{\alpha \in \mathcal{A}_t} G^\alpha(t, (x, 1, 0)) = \sup_{\alpha \in \mathcal{A}_t} \tilde{G}^\alpha(t, (x, 1, 0)) \leq \sup_{\alpha \in \mathcal{A}} \tilde{G}^\alpha(t, (x, 1, 0)).$$

For the reverse inequality, fix $\alpha \in \mathcal{A}$ and $\mathbf{x} \in \mathcal{S}$. By a calculation similar to (4.2), we have $J(t, \mathbf{x}; \alpha, \tau) = \mathbb{E}[J(t, \mathbf{x}; \alpha^{t,\omega}, \tau^{t,\omega})]$, for any $\tau \in \mathcal{T}_{t,T}$. Observing that $\tau^{t,\omega} \in \mathcal{T}_{t,T}^t$ for all $\tau \in \mathcal{T}_{t,T}$

(by Proposition 2.2), and that $\mathbb{E}[J(t, \mathbf{x}; \alpha^{t,\omega}, \tau^{t,\omega})] = \mathbb{E}[J(t, \mathbf{x}; \alpha^{t,\omega}, \tau)]$ for all $\tau \in \mathcal{T}_{t,T}^t$ (by Proposition 2.1), we obtain

$$\begin{aligned} \inf_{\tau \in \mathcal{T}_{t,T}^t} J(t, \mathbf{x}; \alpha, \tau) &= \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}[J(t, \mathbf{x}; \alpha^{t,\omega}, \tau^{t,\omega})] = \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}[J(t, \mathbf{x}; \alpha^{t,\omega}, \tau)] \\ &\leq \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} \mathbb{E}[J(t, \mathbf{x}; \alpha, \tau)] = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}_{t,T}^t} J(t, \mathbf{x}; \alpha, \tau), \end{aligned}$$

where the inequality is due to the fact that $\alpha^{t,\omega} \in \mathcal{A}_t$. By setting $\mathbf{x} := (x, 1, 0)$ and taking supremum over $\alpha \in \mathcal{A}$, we get $\sup_{\alpha \in \mathcal{A}} \tilde{G}^\alpha(t, (x, 1, 0)) \leq V(t, x)$. \square

Corollary 4.1. $V \in \text{LSC}([0, T] \times \mathbb{R}^d)$.

Proof. By Proposition 4.2 and Lemma 4.1 (i), V is a supremum of a collection of continuous functions defined on $[0, T] \times \mathbb{R}^d$, and thus has to be lower semicontinuous on the same space. \square

Now, we are ready to present the main result of this section. Recall that the operator \overline{H} is defined in (3.14).

Proposition 4.3. *The function V is a lower semicontinuous viscosity supersolution to the obstacle problem of a Hamilton-Jacobi-Bellman equation*

$$\max \left\{ c(t, x)w - \frac{\partial w}{\partial t} + \overline{H}(t, x, D_x w, D_x^2 w), w - g(x) \right\} = 0 \text{ on } [0, T] \times \mathbb{R}^d, \quad (4.11)$$

and satisfies the polynomial growth condition: there exists $N > 0$ such that

$$|V(t, x)| \leq N(1 + |x|^{\bar{p}}), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d. \quad (4.12)$$

Proof. The lower semicontinuity of V was shown in Corollary 4.1. Observe that $0 \leq V(t, x) \leq \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}[F(\mathbf{X}_T^{t,x,1,0,\alpha})] \leq \sup_{\alpha \in \mathcal{A}} \mathbb{E}[F(\mathbf{X}_T^{t,x,1,0,\alpha})] =: v(t, x)$. Since v satisfies (4.12) as a result of [24, Theorem 3.1.5], so does V .

To prove the supersolution property, let $h \in C^{1,2}([0, T] \times \mathbb{R}^d)$ be such that

$$0 = (V - h)(t_0, x_0) < (V - h)(t, x), \text{ for any } (t, x) \in [0, T] \times \mathbb{R}^d, (t, x) \neq (t_0, x_0), \quad (4.13)$$

for some $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$. If $V(t_0, x_0) = g(x_0)$, then there is nothing to prove. We, therefore, assume that $V(t_0, x_0) < g(x_0)$. For such (t_0, x_0) it is enough to prove the following inequality:

$$0 \leq c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + \overline{H}(\cdot, D_x h, D_x^2 h)(t_0, x_0).$$

Assume the contrary. Then, by the definition of \overline{H} in (3.14), there must exist $\zeta_0 \in M$ such that

$$0 > c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + (H^{\zeta_0})^*(\cdot, D_x h, D_x^2 h)(t_0, x_0).$$

Moreover, from the upper semicontinuity of $(H^{\zeta_0})^*$ and the fact that $(H^{\zeta_0})^* \geq H^{\zeta_0}$, we can choose some $r > 0$ with $t_0 + r < T$ such that

$$0 > c(t, x)h(t, x) - \frac{\partial h}{\partial t}(t, x) + H^{\zeta_0}(\cdot, D_x h, D_x^2 h)(t, x), \text{ for all } (t, x) \in \bar{B}_r(t_0, x_0). \quad (4.14)$$

Define $\zeta \in \mathcal{A}$ by setting $\zeta_t = \zeta_0$ for all $t \geq 0$, and introduce the stopping time

$$\theta := \inf \left\{ s \geq t_0 \mid (s, X_s^{t_0, x_0, \zeta}) \notin B_r(t_0, x_0) \right\} \in \mathcal{T}_{t_0, T}^{t_0}.$$

Note that we have $\theta \in \mathcal{T}_{t_0, T}^{t_0}$ as the control ζ is by definition independent of \mathcal{F}_{t_0} . Now, by applying the product rule of stochastic calculus to $Y_s^{t_0, x_0, 1, \zeta} h(s, X_s^{t_0, x_0, \zeta})$ and recalling (4.14) and $c \leq \bar{c}$, we obtain that for any $\tau \in \mathcal{T}_{t_0, T}^{t_0}$,

$$\begin{aligned} V(t_0, x_0) = h(t_0, x_0) &= \mathbb{E} \left[Y_{\theta \wedge \tau}^{t_0, x_0, 1, \zeta} h(\theta \wedge \tau, X_{\theta \wedge \tau}^{t_0, x_0, \zeta}) \right. \\ &\quad \left. + \int_{t_0}^{\theta \wedge \tau} Y_s^{t_0, x_0, 1, \zeta} \left(ch - \frac{\partial h}{\partial t} + H^{\zeta_0}(\cdot, D_x h, D_x^2 h) + f \right) (s, X_s^{t_0, x_0, \zeta}, \zeta_0) ds \right] \\ &< \mathbb{E} \left[Y_{\theta \wedge \tau}^{t_0, x_0, 1, \zeta} h(\theta \wedge \tau, X_{\theta \wedge \tau}^{t_0, x_0, \zeta}) + \int_{t_0}^{\theta \wedge \tau} Y_s^{t_0, x_0, 1, \zeta} f(s, X_s^{t_0, x_0, \zeta}, \zeta_0) ds \right]. \end{aligned} \quad (4.15)$$

In the following, we will work towards a contradiction to (4.15). First, define

$$\bar{h}(\theta, \mathbf{X}_{\theta}^{t_0, x_0, 1, 0, \zeta}) := Y_{\theta}^{t_0, x_0, 1, \zeta} h(\theta, X_{\theta}^{t_0, x_0, \zeta}) + \int_{t_0}^{\theta} Y_s^{t_0, x_0, 1, \zeta} f(s, X_s^{t_0, x_0, \zeta}, \zeta_0) ds.$$

Note from (4.15) that $\mathbb{E}[\bar{h}(\theta, \mathbf{X}_{\theta}^{t_0, x_0, 1, 0, \zeta})]$ is bounded from below. It follows from this fact that $\mathbb{E}[\bar{h}^-(\theta, \mathbf{X}_{\theta}^{t_0, x_0, 1, 0, \zeta})] < \infty$, as we already have $\mathbb{E}[\bar{h}^+(\theta, \mathbf{X}_{\theta}^{t_0, x_0, 1, 0, \zeta})] < \infty$ from Proposition 4.1 (i). For each $n \in \mathbb{N}$, we can therefore apply Proposition 4.1 (ii) and conclude that there exists $\alpha^{*, n} \in \mathcal{A}_{t_0}$, with $\alpha_s^{*, n} = \zeta_s$ for all $s \leq \theta$, such that for any $\tau \in \mathcal{T}_{t_0, T}^{t_0}$,

$$\mathbb{E}[F(\mathbf{X}_{\tau}^{t_0, x_0, 1, 0, \alpha^{*, n}})] \geq \mathbb{E} \left[Y_{\theta \wedge \tau}^{t_0, x_0, 1, \zeta} h(\theta \wedge \tau, X_{\theta \wedge \tau}^{t_0, x_0, \zeta}) + \int_{t_0}^{\theta \wedge \tau} Y_s^{t_0, x_0, 1, \zeta} f(s, X_s^{t_0, x_0, \zeta}, \zeta_0) ds \right] - \frac{1}{n}. \quad (4.16)$$

Next, thanks to the definition of V and the classical theory of Snell envelopes (see e.g. Appendix D, and especially Theorem D.12, in [19]), we have

$$V(t_0, x_0) \geq G^{\alpha^{*, n}}(t_0, (x_0, 1, 0)) = \mathbb{E}[F(\mathbf{X}_{\tau^n}^{t_0, x_0, 1, 0, \alpha^{*, n}})], \quad (4.17)$$

where

$$\tau^n := \inf \left\{ s \geq t_0 \mid G^{\alpha^{*, n}}(s, \mathbf{X}_s^{t_0, x_0, 1, 0, \alpha^{*, n}}) = g(X_s^{t_0, x_0, \alpha^{*, n}}) \right\} \in \mathcal{T}_{t_0, T}^{t_0}.$$

Note that we may apply [19, Theorem D.12] because (3.9) holds. Combining (4.17) and (4.16), we obtain

$$V(t_0, x_0) \geq \mathbb{E} \left[Y_{\theta \wedge \tau^n}^{t_0, x_0, 1, \zeta} h(\theta \wedge \tau^n, X_{\theta \wedge \tau^n}^{t_0, x_0, \zeta}) + \int_{t_0}^{\theta \wedge \tau^n} Y_s^{t_0, x_0, 1, \zeta} f(s, X_s^{t_0, x_0, \zeta}, \zeta_0) ds \right] - \frac{1}{n}.$$

By sending n to infinity and using Fatou's Lemma, we conclude that

$$V(t_0, x_0) \geq \mathbb{E} \left[Y_{\theta \wedge \tau^*}^{t_0, x_0, 1, \zeta} h(\theta \wedge \tau^*, X_{\theta \wedge \tau^*}^{t_0, x_0, \zeta}) + \int_{t_0}^{\theta \wedge \tau^*} Y_s^{t_0, x_0, 1, \zeta} f(s, X_s^{t_0, x_0, \zeta}, \zeta_0) ds \right],$$

where $\tau^* := \liminf_{n \rightarrow \infty} \tau^n$ is a stopping time in $\mathcal{T}_{t_0, T}^{t_0}$, thanks to the right continuity of the filtration \mathbb{F}^{t_0} . The above inequality, however, contradicts (4.15). \square

Remark 4.1. The lower semicontinuity of V is needed for the proof of Proposition 4.3. To see this, suppose V is not lower semicontinuous. Then V should be replaced by V_* in (4.13) and (4.15). The last inequality in the proof and (4.15) would then yield $V_*(t_0, x_0) < V(t_0, x_0)$, which is not a contradiction.

Remark 4.2. Due to the lack of continuity in t of the functions b , σ , and f , we use \overline{H} , instead of H^* , in (4.11). If we were using H^* , we in general would not be able to find a $\zeta_0 \in M$ such that (4.14) holds (due to the lack of continuity in t). If b , σ , and f are actually continuous in t , then we see from (3.13) and (3.14) that $\overline{H} = H = H^*$.

5. SUBSOLUTION PROPERTY OF U^*

As in Section 4, we will first prove a continuity result (Lemma 5.4), which leads to a weak dynamic programming principle for U (Proposition 5.1). Then, we will show that the subsolution property of U^* follows from this weak dynamic programming principle (Proposition 5.2). Remember that U^* is the upper semicontinuous envelope of U defined as in (1.3).

Fix $s \in [0, T]$ and $\xi \in L_d^p(\Omega, \mathcal{F}_s)$ for some $p \in [1, \infty)$. For any $\alpha \in \mathcal{A}$ and $\pi_1, \pi_2 \in \Pi_{s,T}$ with $\pi_1[\beta] \leq \pi_2[\beta]$ $\overline{\mathbb{P}}$ -a.s. for all $\beta \in \mathcal{A}$, we define

$$\mathcal{B}_{\pi_1}^{s,\xi,\alpha} := \left\{ \beta \in \mathcal{A} \mid \int_s^{\pi_1[\alpha]} \rho'(\beta_u, \alpha_u) du = 0 \text{ } \overline{\mathbb{P}}\text{-a.s.} \right\}, \quad (5.1)$$

and introduce the random variable

$$K^{s,\xi,\alpha}(\pi_1, \pi_2) := \operatorname{esssup}_{\beta \in \mathcal{B}_{\pi_1}^{s,\xi,\alpha}} \mathbb{E} \left[\int_{\pi_1[\alpha]}^{\pi_2[\beta]} Y_u^{\pi_1[\alpha], X_{\pi_1[\alpha]}^{s,\xi,\beta}, 1, \beta} f(u, X_u^{s,\xi,\beta}, \beta_u) du + Y_{\pi_2[\beta]}^{\pi_1[\alpha], X_{\pi_1[\alpha]}^{s,\xi,\beta}, 1, \beta} g(X_{\pi_2[\beta]}^{s,\xi,\beta}) \mid \mathcal{F}_{\pi_1[\alpha]} \right]. \quad (5.2)$$

Observe from the definition of $\mathcal{B}_{\pi_1}^{s,\xi,\alpha}$ and Definition 3.1 (i) that

$$\pi_1[\beta] = \pi_1[\alpha] \text{ } \overline{\mathbb{P}}\text{-a.s. } \forall \beta \in \mathcal{B}_{\pi_1}^{s,\xi,\alpha}. \quad (5.3)$$

This in particular implies $\pi_2[\beta] \geq \pi_1[\beta] = \pi_1[\alpha]$ $\overline{\mathbb{P}}$ -a.s. $\forall \beta \in \mathcal{B}_{\pi_1}^{s,\xi,\alpha}$, which shows that $K^{s,\xi,\alpha}(\pi_1, \pi_2)$ is well-defined. Given any constant strategies $\pi_1[\cdot] \equiv \tau_1 \in \mathcal{T}_{s,T}^s$ and $\pi_2[\cdot] \equiv \tau_2 \in \mathcal{T}_{s,T}^s$, we will simply write $K^{s,\xi,\alpha}(\pi_1, \pi_2)$ as $K^{s,\xi,\alpha}(\tau_1, \tau_2)$. For the particular case where $\xi = x \in \mathbb{R}^d$, we also consider

$$\Gamma^{s,x,\alpha}(\pi_1, \pi_2) := \int_s^{\pi_1[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u) du + Y_{\pi_1[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_1, \pi_2).$$

Remark 5.1. Let us write $K^{s,x,\alpha}(\pi_1, \pi_2) = \operatorname{esssup}_{\beta \in \mathcal{B}_{\pi_1}^{s,x,\alpha}} \mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta) \mid \mathcal{F}_{\pi_1[\alpha]}]$ for simplicity. Note that the set of random variables $\{\mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta) \mid \mathcal{F}_{\pi_1[\alpha]}] : \beta \in \mathcal{B}_{\pi_1}^{s,x,\alpha}\}$ is closed under pairwise maximization. Indeed, given $\beta_1, \beta_2 \in \mathcal{B}_{\pi_1}^{s,x,\alpha}$, set $A := \{\mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta_1) \mid \mathcal{F}_{\pi_1[\alpha]}] \geq \mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta_2) \mid \mathcal{F}_{\pi_1[\alpha]}]\} \in \mathcal{F}_{\pi_1[\alpha]}$ and define $\beta_3 := \beta_1 1_{[0, \pi_1[\alpha]]} + (\beta_1 1_A + \beta_2 1_{A^c}) 1_{[\pi_1[\alpha], T]} \in \mathcal{B}_{\pi_1}^{s,x,\alpha}$. Then, observe that

$$\begin{aligned} \mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta_3) \mid \mathcal{F}_{\pi_1[\alpha]}] &= \mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta_1) \mid \mathcal{F}_{\pi_1[\alpha]}] 1_A + \mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta_2) \mid \mathcal{F}_{\pi_1[\alpha]}] 1_{A^c} \\ &= \mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta_1) \mid \mathcal{F}_{\pi_1[\alpha]}] \vee \mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta_2) \mid \mathcal{F}_{\pi_1[\alpha]}]. \end{aligned}$$

Thus, we conclude from Theorem A.3 in [19, Appendix A] that there exists a sequence $\{\beta^n\}_{n \in \mathbb{N}}$ in $\mathcal{B}_{\pi_1}^{s,x,\alpha}$ such that $K^{s,x,\alpha}(\pi_1, \pi_2) = \uparrow \lim_{n \rightarrow \infty} \mathbb{E}[R_{\pi_1, \pi_2}^{s,x,\alpha}(\beta^n) \mid \mathcal{F}_{\pi_1[\alpha]}] \bar{\mathbb{P}}\text{-a.s.}$

Lemma 5.1. Fix $(s, x) \in [0, T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$. For any $r \in [s, T]$ and $\pi \in \Pi_{r,T}$, we have

$$K^{s,x,\alpha}(r, \pi) = K^{r, X_r^{s,x,\alpha}, \alpha}(r, \pi) \bar{\mathbb{P}}\text{-a.s.}$$

Proof. For any $\beta \in \mathcal{B}_r^{s,x,\alpha}$, we see from Remark 2.5 (i) that $X_u^{s,x,\beta} = X_u^{r, X_r^{s,x,\alpha}, \beta}$ for $u \in [r, T]$ $\bar{\mathbb{P}}\text{-a.s.}$ It follows from (5.2) that

$$K^{s,x,\alpha}(r, \pi) = \text{esssup}_{\beta \in \mathcal{B}_r^{s,x,\alpha}} \mathbb{E} \left[\int_r^{\pi[\beta]} Y_u^{r, X_r^{s,x,\alpha}, 1, \beta} f(u, X_u^{r, X_r^{s,x,\alpha}, \beta}, \beta_u) du + Y_{\pi[\beta]}^{r, X_r^{s,x,\alpha}, 1, \beta} g(X_{\pi[\beta]}^{r, X_r^{s,x,\alpha}, \beta}) \mid \mathcal{F}_r \right].$$

Observing from (5.1) that $\mathcal{B}_r^{s,x,\alpha} \subseteq \mathcal{A} = \mathcal{B}_r^{r, X_r^{s,x,\alpha}, \alpha}$, we conclude $K^{s,x,\alpha}(r, \pi) \leq K^{r, X_r^{s,x,\alpha}, \alpha}(r, \pi)$. On the other hand, for any $\beta \in \mathcal{A}$, define $\bar{\beta} := \alpha 1_{[0,r)} + \beta 1_{[r,T]} \in \mathcal{B}_r^{s,x,\alpha}$. Then, by Remark 2.5 (i) again, we have $X_u^{s,x,\bar{\beta}} = X_u^{r, X_r^{s,x,\alpha}, \beta}$ for $u \in [r, T]$ $\bar{\mathbb{P}}\text{-a.s.}$ Also, we have $\pi[\bar{\beta}] = \pi[\beta]$, thanks to Definition 3.1 (i). Therefore,

$$\begin{aligned} & \mathbb{E} \left[\int_r^{\pi[\beta]} Y_u^{r, X_r^{s,x,\alpha}, 1, \beta} f(u, X_u^{r, X_r^{s,x,\alpha}, \beta}, \beta_u) du + Y_{\pi[\beta]}^{r, X_r^{s,x,\alpha}, 1, \beta} g(X_{\pi[\beta]}^{r, X_r^{s,x,\alpha}, \beta}) \mid \mathcal{F}_r \right] \\ &= \mathbb{E} \left[\int_r^{\pi[\bar{\beta}]} Y_u^{r, X_r^{s,x,\bar{\beta}}, 1, \bar{\beta}} f(u, X_u^{s,x,\bar{\beta}}, \bar{\beta}_u) du + Y_{\pi[\bar{\beta}]}^{r, X_r^{s,x,\bar{\beta}}, 1, \bar{\beta}} g(X_{\pi[\bar{\beta}]}^{s,x,\bar{\beta}}) \mid \mathcal{F}_r \right]. \end{aligned}$$

In view of (5.2), this implies $K^{r, X_r^{s,x,\alpha}, \alpha}(r, \pi) \leq K^{s,x,\alpha}(r, \pi)$. \square

Lemma 5.2. Fix $(s, x) \in [0, T] \times \mathbb{R}^d$. Given $\alpha \in \mathcal{A}$ and $\pi_1, \pi_2, \pi_3 \in \Pi_{s,T}$ with $\pi_1[\beta] \leq \pi_2[\beta] \leq \pi_3[\beta]$ $\bar{\mathbb{P}}\text{-a.s.}$ for all $\beta \in \mathcal{A}$, it holds $\bar{\mathbb{P}}\text{-a.s.}$ that

$$\mathbb{E} \left[\int_{\pi_1[\alpha]}^{\pi_2[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u) du + Y_{\pi_2[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_2, \pi_3) \mid \mathcal{F}_{\pi_1[\alpha]} \right] \leq Y_{\pi_1[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_1, \pi_3).$$

Moreover, we have the following supermartingale property:

$$\mathbb{E}[\Gamma^{s,x,\alpha}(\pi_2, \pi_3) \mid \mathcal{F}_{\pi_1[\alpha]}] \leq \Gamma^{s,x,\alpha}(\pi_1, \pi_3) \bar{\mathbb{P}}\text{-a.s.}$$

Proof. By Remark 5.1, there exists a sequence $\{\beta^n\}_{n \in \mathbb{N}}$ in $\mathcal{B}_{\pi_2}^{s,x,\alpha}$ such that $K^{s,x,\alpha}(\pi_2, \pi_3) = \uparrow \lim_{n \rightarrow \infty} \mathbb{E}[R_{\pi_2, \pi_3}^{s,x,\alpha}(\beta^n) \mid \mathcal{F}_{\pi_2[\alpha]}] \bar{\mathbb{P}}\text{-a.s.}$ From the definition of $\mathcal{B}_{\pi_2}^{s,x,\alpha}$ in (5.1), $\beta_u^n = \alpha_u$ for a.e. $u \in [s, \pi_2[\alpha]]$ $\bar{\mathbb{P}}\text{-a.s.}$ We can then compute as follows:

$$\begin{aligned} & \mathbb{E} \left[Y_{\pi_2[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_2, \pi_3) \mid \mathcal{F}_{\pi_1[\alpha]} \right] \\ &= \mathbb{E} \left\{ Y_{\pi_2[\alpha]}^{s,x,1,\alpha} \cdot \right. \\ & \quad \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{\pi_2[\alpha]}^{\pi_3[\beta^n]} Y_u^{\pi_2[\alpha], X_{\pi_2[\alpha]}^{s,x,\beta^n}, 1, \beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n) du + Y_{\pi_3[\beta^n]}^{\pi_2[\alpha], X_{\pi_2[\alpha]}^{s,x,\beta^n}, 1, \beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n}) \mid \mathcal{F}_{\pi_2[\alpha]} \right] \mid \mathcal{F}_{\pi_1[\alpha]} \Big\} \\ &= \mathbb{E} \left\{ \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{\pi_2[\alpha]}^{\pi_3[\beta^n]} Y_u^{s,x,1,\beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n) du + Y_{\pi_3[\beta^n]}^{s,x,1,\beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n}) \mid \mathcal{F}_{\pi_2[\alpha]} \right] \mid \mathcal{F}_{\pi_1[\alpha]} \right\} \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{\pi_2[\alpha]}^{\pi_3[\beta^n]} Y_u^{s,x,1,\beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n) du + Y_{\pi_3[\beta^n]}^{s,x,1,\beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n}) \mid \mathcal{F}_{\pi_1[\alpha]} \right], \end{aligned}$$

where the last line follows from the monotone convergence theorem and the tower property for conditional expectations. We therefore conclude that

$$\begin{aligned}
& \mathbb{E} \left[\int_{\pi_1[\alpha]}^{\pi_2[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u) du + Y_{\pi_2[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_2, \pi_3) \mid \mathcal{F}_{\pi_1[\alpha]} \right] \\
&= \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{\pi_1[\alpha]}^{\pi_3[\beta^n]} Y_u^{s,x,1,\beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n) du + Y_{\pi_3[\beta^n]}^{s,x,1,\beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n}) \mid \mathcal{F}_{\pi_1[\alpha]} \right] \\
&= Y_{\pi_1[\alpha]}^{s,x,1,\alpha} \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{\pi_1[\alpha]}^{\pi_3[\beta^n]} Y_u^{\pi_1[\alpha], X_{\pi_1[\alpha]}^{s,x,\beta^n}, 1, \beta^n} f(u, X_u^{s,x,\beta^n}, \beta_u^n) du + Y_{\pi_3[\beta^n]}^{\pi_1[\alpha], X_{\pi_1[\alpha]}^{s,x,\beta^n}, 1, \beta^n} g(X_{\pi_3[\beta^n]}^{s,x,\beta^n}) \mid \mathcal{F}_{\pi_1[\alpha]} \right] \\
&\leq Y_{\pi_1[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_1, \pi_3),
\end{aligned}$$

where the inequality follows from the fact that $\beta^n \in \mathcal{B}_{\pi_2}^{s,x,\alpha} \subseteq \mathcal{B}_{\pi_1}^{s,x,\alpha}$. It then follows that

$$\begin{aligned}
\mathbb{E}[\Gamma^{s,x,\alpha}(\pi_2, \pi_3) \mid \mathcal{F}_{\pi_1[\alpha]}] &= \int_s^{\pi_1[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u) du \\
&\quad + \mathbb{E} \left[\int_{\pi_1[\alpha]}^{\pi_2[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u) du + Y_{\pi_2[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_2, \pi_3) \mid \mathcal{F}_{\pi_1[\alpha]} \right] \\
&\leq \int_s^{\pi_1[\alpha]} Y_u^{s,x,1,\alpha} f(u, X_u^{s,x,\alpha}, \alpha_u) du + Y_{\pi_1[\alpha]}^{s,x,1,\alpha} K^{s,x,\alpha}(\pi_1, \pi_3) = \Gamma^{s,x,\alpha}(\pi_1, \pi_3).
\end{aligned}$$

□

Lemma 5.3. *For any $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\pi \in \Pi_{t,T}$,*

$$\sup_{\alpha \in \mathcal{A}} J(t, \mathbf{x}; \alpha, \pi[\alpha]) = \sup_{\alpha \in \mathcal{A}_t} J(t, \mathbf{x}; \alpha, \pi[\alpha]).$$

Proof. Fix $\alpha \in \mathcal{A}$ and $\mathbf{x} \in \mathcal{S}$. For any $\pi \in \Pi_{t,T}$, by taking $\theta = t$ in Lemma 3.2, we have

$$J(t, \mathbf{x}; \alpha, \pi[\alpha]) = \mathbb{E} \left[\mathbb{E}[F(\mathbf{X}_{\pi[\alpha]}^{t,\mathbf{x},\alpha}) \mid \mathcal{F}_t](\omega) \right] = \mathbb{E} [J(t, \mathbf{x}; \alpha^{t,\omega}, \pi[\alpha^{t,\omega}])] \leq \sup_{\alpha \in \mathcal{A}_t} J(t, \mathbf{x}; \alpha, \pi[\alpha]).$$

Note that in the second equality we replace $\pi[\alpha]^{t,\omega}$ by $\pi[\alpha^{t,\omega}]$, thanks to Definition 3.1 (iii). Then, the last inequality holds as $\alpha^{t,\omega} \in \mathcal{A}_t$ for \mathbb{P} -a.e. $\omega \in \Omega$. Now, by taking supremum over $\alpha \in \mathcal{A}$, we have $\sup_{\alpha \in \mathcal{A}} J(t, \mathbf{x}; \alpha, \pi[\alpha]) \leq \sup_{\alpha \in \mathcal{A}_t} J(t, \mathbf{x}; \alpha, \pi[\alpha])$. Since the reverse inequality is trivial, this lemma follows. □

Now, we are ready to state a continuity result for an optimal control problem.

Lemma 5.4. *Fix $t \in [0, T]$. For any $\pi \in \Pi_{t,T}$, the function $L^\pi : [0, t] \times \mathcal{S}$ defined by*

$$L^\pi(s, \mathbf{x}) := \sup_{\alpha \in \mathcal{A}_s} J(s, \mathbf{x}; \alpha, \pi[\alpha]) \tag{5.4}$$

is continuous.

Proof. Observing from (3.16) that $L^\pi(s, \mathbf{x}) = yL^\pi(s, (x, 1, 0)) + z$, it is enough to show the continuity of $L^\pi(s, (x, 1, 0))$ in (s, x) on $[0, t] \times \mathbb{R}^d$. By [24, Theorem 3.2.2], we know that $J(s, (x, 1, 0); \alpha, \tau)$ is continuous in x uniformly with respect to $s \in [0, t]$, $\alpha \in \mathcal{A}$, and $\tau \in \mathcal{T}_{t,T}$. This shows that the map

$(s, x, \alpha) \mapsto J(s, (x, 1, 0); \alpha, \pi[\alpha])$ is continuous in x uniformly with respect to $s \in [0, t]$ and $\alpha \in \mathcal{A}$. Then, we see from the following estimation

$$\sup_{s \in [0, t]} |L^\pi(s, (x, 1, 0)) - L^\pi(s, (x', 1, 0))| \leq \sup_{s \in [0, t]} \sup_{\alpha \in \mathcal{A}_s} |J(s, (x, 1, 0); \alpha, \pi[\alpha]) - J(s, (x', 1, 0); \alpha, \pi[\alpha])|$$

that $L^\pi(s, (x, 1, 0))$ is continuous in x uniformly with respect to $s \in [0, t]$. Thus, it suffices to prove that $L^\pi(s, (x, 1, 0))$ is continuous in s for each fixed x . To this end, we will first derive a dynamic programming principle for $L^\pi(s, (x, 1, 0))$, which corresponds to [24, Theorem 3.3.6]; the rest of the proof will then follow from the same argument in [24, Lemma 3.3.7].

Fix $(s, x) \in [0, t] \times \mathbb{R}^d$. Observe from (5.1) that $\mathcal{B}_s^{s, x, \alpha} = \mathcal{A}$ for all $\alpha \in \mathcal{A}$. In view of (5.2), this implies that $K^{s, x, \alpha}(s, \pi) = \text{esssup}_{\beta \in \mathcal{A}} \mathbb{E}[F(\mathbf{X}_{\pi[\beta]}^{s, x, 1, 0, \beta}) \mid \mathcal{F}_s]$, which is independent of $\alpha \in \mathcal{A}$. We will therefore drop the superscript α in the rest of the proof. Now, we claim that $K^{s, x}(s, \pi)$ is deterministic and equal to $L^\pi(s, (x, 1, 0))$. First, since $\pi[\alpha] \in \mathcal{T}_{t, T}^s$ for all $\alpha \in \mathcal{A}_s$ (by Definition 3.1 (ii)), we observe from Lemma 3.2, Proposition 2.1 (ii), and Proposition 2.3 that

$$\begin{aligned} K^{s, x}(s, \pi) &\geq \text{esssup}_{\alpha \in \mathcal{A}_s} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha]}^{s, x, 1, 0, \alpha}) \mid \mathcal{F}_s](\cdot) = \text{esssup}_{\alpha \in \mathcal{A}_s} J(s, (x, 1, 0); \alpha^{s, \cdot}, \pi[\alpha]^{s, \cdot}) \\ &= \sup_{\alpha \in \mathcal{A}_s} J(s, (x, 1, 0); \alpha, \pi[\alpha]) = L^\pi(s, (x, 1, 0)). \end{aligned} \quad (5.5)$$

On the other hand, in view of Remark 5.1, there exists a sequence $\{\alpha^n\}_{n \in \mathbb{N}}$ in \mathcal{A} such that $K^{s, x}(s, \pi) = \uparrow \lim_{n \rightarrow \infty} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha^n]}^{s, x, 1, 0, \alpha^n}) \mid \mathcal{F}_s] \bar{\mathbb{P}}\text{-a.s.}$ By the monotone convergence theorem,

$$\begin{aligned} \mathbb{E}[K^{s, x}(s, \pi)] &= \mathbb{E} \left[\lim_{n \rightarrow \infty} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha^n]}^{s, x, 1, 0, \alpha^n}) \mid \mathcal{F}_s] \right] = \lim_{n \rightarrow \infty} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha^n]}^{s, x, 1, 0, \alpha^n})] \\ &\leq \sup_{\alpha \in \mathcal{A}} \mathbb{E}[F(\mathbf{X}_{\pi[\alpha]}^{s, x, 1, 0, \alpha})] = L^\pi(s, (x, 1, 0)), \end{aligned} \quad (5.6)$$

where the last equality is due to Lemma 5.3. From (5.5) and (5.6), we get $K^{s, x}(s, \pi) = L^\pi(s, (x, 1, 0))$. Then, for any $\alpha \in \mathcal{A}$, thanks to the supermartingale property introduced in Lemma 5.2, we have for all $r \in [s, t]$ that

$$L^\pi(s, (x, 1, 0)) = K^{s, x}(s, \pi) = \Gamma^{s, x, \alpha}(s, \pi) \geq \mathbb{E}[\Gamma^{s, x, \alpha}(r, \pi)] \geq \mathbb{E}[\Gamma^{s, x, \alpha}(\pi, \pi)] \geq \mathbb{E}[F(\mathbf{X}_{\pi[\alpha]}^{s, x, 1, 0, \alpha})],$$

where the last equality follows from the fact that $K^{s, x, \alpha}(\pi, \pi) = \text{esssup}_{\beta \in \mathcal{B}_\pi^{s, x, \alpha}} g(X_{\pi[\beta]}^{s, x, \beta}) \geq g(X_{\pi[\alpha]}^{s, x, \alpha}) \bar{\mathbb{P}}\text{-a.s.}$; see (5.2). By taking supremum over $\alpha \in \mathcal{A}$ and using Lemma 5.3, we obtain the following dynamic programming principle for $L^\pi(s, (x, 1, 0))$: for all $r \in [s, t]$,

$$\begin{aligned} L^\pi(s, (x, 1, 0)) &= \sup_{\alpha \in \mathcal{A}} \mathbb{E}[\Gamma^{s, x, \alpha}(r, \pi)] \\ &= \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_s^r Y_u^{s, x, 1, \alpha} f(u, X_u^{s, x, \alpha}, \alpha_u) du + Y_r^{s, x, 1, \alpha} L^\pi(r, (X_r^{s, x, \alpha}, 1, 0)) \right], \end{aligned}$$

where the second equality follows from the fact $K^{s, x, \alpha}(r, \pi) = K^{r, X_r^{s, x, \alpha}, \alpha}(r, \pi) = L^\pi(r, (X_r^{s, x, \alpha}, 1, 0)) \bar{\mathbb{P}}\text{-a.s.}$, as a consequence of Lemma 5.1. Now, we may apply the same argument in [24, Lemma 3.3.7] to show that $L^\pi(s, (x, 1, 0))$ is continuous in s on $[0, t]$. \square

Proposition 5.1. Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\varepsilon > 0$. For any $\pi \in \Pi_{t,T}$ and $\varphi \in \text{LSC}([0, T] \times \mathbb{R}^d)$ with $\varphi \geq U$, there exists $\pi^* \in \Pi_{t,T}$ such that

$$\mathbb{E} \left[F(\mathbf{X}_{\pi^*[\alpha]}^{t, \mathbf{x}, \alpha}) \right] \leq \mathbb{E} \left[Y_{\pi[\alpha]}^{t, x, y, \alpha} \varphi \left(\pi[\alpha], X_{\pi[\alpha]}^{t, x, \alpha} \right) + Z_{\pi[\alpha]}^{t, x, y, z, \alpha} \right] + 3\varepsilon, \quad \forall \alpha \in \mathcal{A}.$$

Proof. For each $(s, \eta) \in [0, T] \times \mathcal{S}$, by the definition of \bar{U} , there exists $\pi^{(s, \eta), \varepsilon} \in \Pi_{s,T}$ such that

$$\sup_{\alpha \in \mathcal{A}_s} J \left(s, \eta; \alpha, \pi^{(s, \eta), \varepsilon}[\alpha] \right) \leq \bar{U}(s, \eta) + \varepsilon. \quad (5.7)$$

Recall the definition of $\bar{\varphi}$ in (4.4) and note that $\varphi \in \text{LSC}([0, T] \times \mathbb{R}^d)$ implies $\bar{\varphi} \in \text{LSC}([0, T] \times \mathcal{S})$. Then, by the lower semicontinuity of $\bar{\varphi}$ on $[0, T] \times \mathcal{S}$ and the upper semicontinuity of $L^{\pi^{(s, \eta), \varepsilon}}$ on $[0, s] \times \mathcal{S}$ (from Lemma 5.4), there must exist $r^{(s, \eta)} > 0$ such that

$$\bar{\varphi}(t', x') - \bar{\varphi}(s, \eta) \geq -\varepsilon \text{ and } L^{\pi^{(s, \eta), \varepsilon}}(t', x') - L^{\pi^{(s, \eta), \varepsilon}}(s, \eta) \leq \varepsilon,$$

for any (t', x') contained in the ball $B(s, \eta; r^{(s, \eta)})$, defined as in (1.2). It follows that if $(t', x') \in B(s, \eta; r^{(s, \eta)})$, we have

$$L^{\pi^{(s, \eta), \varepsilon}}(t', x') \leq L^{\pi^{(s, \eta), \varepsilon}}(s, \eta) + \varepsilon \leq \bar{U}(s, \eta) + 2\varepsilon \leq \bar{\varphi}(s, \eta) + 2\varepsilon \leq \bar{\varphi}(t', x') + 3\varepsilon,$$

where the second inequality is due to (5.7). By the same construction in the proof of Proposition 4.1, there exists a countable covering $\{B(t_i, x_i; r_i)\}_{i \in \mathbb{N}}$ of $(0, T] \times \mathcal{S}$, from which we can take a countable disjoint covering $\{A_i\}_{i \in \mathbb{N} \cup \{0\}}$ of $(0, T] \times \mathcal{S}$ such that

$$\begin{aligned} (\pi[\alpha], \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}) &\in \cup_{i=1}^{\ell} A_i \text{ } \bar{\mathbb{P}}\text{-a.s. } \forall \alpha \in \mathcal{A}, \\ L^{\pi^{i, \varepsilon}}(t', x') &\leq \bar{\varphi}(t', x') + 3\varepsilon \text{ for } (t', x') \in A_i, \text{ where } \pi^{i, \varepsilon} := \pi^{(t_i, x_i), \varepsilon}. \end{aligned} \quad (5.8)$$

Now, define $\pi^* \in \Pi_{t,T}$ by

$$\pi^*[\alpha] := \sum_{i \geq 1} \pi^{i, \varepsilon}[\alpha] 1_{A_i}(\pi[\alpha], \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}), \quad \forall \alpha \in \mathcal{A}.$$

Fix $\alpha \in \mathcal{A}_t$. Observe that for $\bar{\mathbb{P}}$ -a.e. $\omega \in \left\{ (\pi[\alpha], \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}) \in A_i \right\} \subseteq \{ \pi[\alpha] \leq t_i \}$, Definition 3.1 (iii) gives

$$(\pi^{i, \varepsilon}[\alpha])^{\pi[\alpha], \omega}(\omega') = \pi^{i, \varepsilon}[\alpha^{\pi[\alpha], \omega}](\omega') \text{ for } \bar{\mathbb{P}}\text{-a.e. } \omega' \in \Omega. \quad (5.9)$$

We then deduce from Lemma 3.2, (5.9), (5.4), and (5.8) that for $\bar{\mathbb{P}}$ -a.e. $\omega \in \Omega$,

$$\begin{aligned} &\mathbb{E} \left[F(\mathbf{X}_{\pi^*[\alpha]}^{t, \mathbf{x}, \alpha}) \mid \mathcal{F}_{\pi[\alpha]} \right] (\omega) 1_{A_i}(\pi[\alpha](\omega), \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}(\omega)) \\ &= J \left(\pi[\alpha](\omega), \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}(\omega); \alpha^{\pi[\alpha], \omega}, \pi^{i, \varepsilon}[\alpha^{\pi[\alpha], \omega}] \right) 1_{A_i}(\pi[\alpha](\omega), \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}(\omega)) \\ &\leq L^{\pi^{i, \varepsilon}} \left(\pi[\alpha](\omega), \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}(\omega) \right) 1_{A_i}(\pi[\alpha](\omega), \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}(\omega)) \\ &\leq \left[\bar{\varphi} \left(\pi[\alpha](\omega), \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}(\omega) \right) + 3\varepsilon \right] 1_{A_i}(\pi[\alpha](\omega), \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}(\omega)). \end{aligned}$$

It follows from the monotone convergence theorem that

$$\mathbb{E} \left[F(\mathbf{X}_{\pi^*[\alpha]}^{t, \mathbf{x}, \alpha}) \right] = \sum_{i \geq 1} \mathbb{E} \left[\mathbb{E} \left[F(\mathbf{X}_{\pi^*[\alpha]}^{t, \mathbf{x}, \alpha}) \mid \mathcal{F}_{\pi[\alpha]} \right] 1_{A_i}(\pi[\alpha], \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}) \right] \leq \mathbb{E} \left[\bar{\varphi}(\pi[\alpha], \mathbf{X}_{\pi[\alpha]}^{t, \mathbf{x}, \alpha}) \right] + 3\varepsilon,$$

which is the desired result by recalling again the definition of $\bar{\varphi}$ in (4.4). \square

The following is the main result of this section. Recall that the operator H is defined in (3.13), and H_* denotes the lower semicontinuous envelope of H defined as in (1.3).

Proposition 5.2. *The function U^* is a viscosity subsolution to the obstacle problem of a Hamilton-Jacobi-Bellman equation*

$$\max \left\{ c(t, x)w - \frac{\partial w}{\partial t} + H_*(t, x, D_x w, D_x^2 w), w - g(x) \right\} = 0 \text{ on } [0, T) \times \mathbb{R}^d,$$

and satisfies the polynomial growth condition (4.12).

Proof. We may argue as in the proof of Proposition 4.3 to show that U^* satisfies (4.12). To prove the subsolution property, we assume the contrary that there exist $h \in C^{1,2}([0, T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ satisfying

$$0 = (U^* - h)(t_0, x_0) > (U^* - h)(t, x), \text{ for any } (t, x) \in [0, T) \times \mathbb{R}^d, (t, x) \neq (t_0, x_0),$$

such that

$$\max \left\{ c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + H_*(\cdot, D_x h, D_x^2 h)(t_0, x_0), h(t_0, x_0) - g(x_0) \right\} > 0. \quad (5.10)$$

Since $U^*(t_0, x_0) = h(t_0, x_0)$ and $U \leq g$ by definition, continuity of g implies that $h(t_0, x_0) = U^*(t_0, x_0) \leq g(x_0)$. Therefore, we can conclude from (5.10) that

$$c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + H_*(\cdot, D_x h, D_x^2 h)(t_0, x_0) > 0. \quad (5.11)$$

Define the function \tilde{h} by

$$\tilde{h}(t, x) := h(t, x) + \varepsilon(|t - t_0|^2 + |x - x_0|^4).$$

Note that $(\tilde{h}, \partial_t \tilde{h}, D_x \tilde{h}, D_x^2 \tilde{h})(t_0, x_0) = (h, \partial_t h, D_x h, D_x^2 h)(t_0, x_0)$. Then, by the lower semicontinuity of H_* , there exists $r > 0$ with $t_0 + r < T$ such that

$$c(t, x)\tilde{h}(t, x) - \frac{\partial \tilde{h}}{\partial t}(t, x) + H^a(\cdot, D_x \tilde{h}, D_x^2 \tilde{h})(t, x) > 0, \quad \forall a \in M \text{ and } (t, x) \in \bar{B}_r(t_0, x_0). \quad (5.12)$$

Now, define $\eta > 0$ by

$$\eta e^{\bar{c}T} := \min_{\partial B_r(t_0, x_0)} (\tilde{h} - h) > 0. \quad (5.13)$$

Take $(\hat{t}, \hat{x}) \in B_r(t_0, x_0)$ such that $|(U - \tilde{h})(\hat{t}, \hat{x})| < \eta/2$, and define $\pi \in \Pi_{\hat{t}, T}$ by

$$\pi[\alpha] := \inf \left\{ s \geq \hat{t} \mid (s, X_s^{\hat{t}, \hat{x}, \alpha}) \notin B_r(t_0, x_0) \right\}, \quad \forall \alpha \in \mathcal{A}. \quad (5.14)$$

For any $\alpha \in \mathcal{A}_{\hat{t}}$, applying the product rule of stochastic calculus to $Y_s^{\hat{t}, \hat{x}, 1, \alpha} \tilde{h}(s, X_s^{\hat{t}, \hat{x}, \alpha})$, we get

$$\begin{aligned} \tilde{h}(\hat{t}, \hat{x}) &= \mathbb{E} \left[Y_{\pi[\alpha]}^{\hat{t}, \hat{x}, 1, \alpha} \tilde{h}(\pi[\alpha], X_{\pi[\alpha]}^{\hat{t}, \hat{x}, \alpha}) \right. \\ &\quad \left. + \int_{\hat{t}}^{\pi[\alpha]} Y_s^{\hat{t}, \hat{x}, 1, \alpha} \left(c\tilde{h} - \frac{\partial \tilde{h}}{\partial t} + H^{\alpha_s}(\cdot, D_x \tilde{h}, D_x^2 \tilde{h}) + f \right)(s, X_s^{\hat{t}, \hat{x}, \alpha}, \alpha_s) ds \right] \\ &> \mathbb{E} \left[Y_{\pi[\alpha]}^{\hat{t}, \hat{x}, 1, \alpha} h(\pi[\alpha], X_{\pi[\alpha]}^{\hat{t}, \hat{x}, \alpha}) + \int_{\hat{t}}^{\pi[\alpha]} Y_s^{\hat{t}, \hat{x}, 1, \alpha} f(s, X_s^{\hat{t}, \hat{x}, \alpha}, \alpha_s) ds \right] + \eta, \end{aligned}$$

where the inequality follows from (5.13), (5.12) and $c \leq \bar{c}$. Moreover, by our choice of (\hat{t}, \hat{x}) , we have $U(\hat{t}, \hat{x}) + \eta/2 > \tilde{h}(\hat{t}, \hat{x})$. It follows that

$$U(\hat{t}, \hat{x}) > \mathbb{E} \left[Y_{\pi[\alpha]}^{\hat{t}, \hat{x}, 1, \alpha} h(\pi[\alpha], X_{\pi[\alpha]}^{\hat{t}, \hat{x}, \alpha}) + \int_{\hat{t}}^{\pi[\alpha]} Y_s^{\hat{t}, \hat{x}, 1, \alpha} f(s, X_s^{\hat{t}, \hat{x}, \alpha}, \alpha_s) ds \right] + \frac{\eta}{2}, \text{ for any } \alpha \in \mathcal{A}_{\hat{t}}. \quad (5.15)$$

Finally, we conclude from the definition of U and Proposition 5.1 that there exist $\pi^* \in \Pi_{\hat{t}, T}$ and $\hat{\alpha} \in \mathcal{A}_{\hat{t}}$ such that

$$\begin{aligned} U(\hat{t}, \hat{x}) = \bar{U}(\hat{t}, \hat{x}, 1, 0) &\leq \sup_{\alpha \in \mathcal{A}_{\hat{t}}} \mathbb{E} \left[F \left(\mathbf{X}_{\pi^*[\alpha]}^{\hat{t}, \hat{x}, 1, 0, \alpha} \right) \right] \leq \mathbb{E} \left[F \left(\mathbf{X}_{\pi^*[\hat{\alpha}]}^{\hat{t}, \hat{x}, 1, 0, \hat{\alpha}} \right) \right] + \frac{\eta}{4} \\ &\leq \mathbb{E} \left[Y_{\pi[\hat{\alpha}]}^{\hat{t}, \hat{x}, 1, \hat{\alpha}} h(\pi[\hat{\alpha}], X_{\pi[\hat{\alpha}]}^{\hat{t}, \hat{x}, \hat{\alpha}}) + Z_{\pi[\hat{\alpha}]}^{\hat{t}, \hat{x}, 1, 0, \hat{\alpha}} \right] + \frac{\eta}{2}, \end{aligned}$$

which contradicts (5.15). \square

6. COMPARISON

In this section, to state an appropriate comparison result, we assume a stronger version of (2.2) as follows: there exists $K > 0$ such that for any $t, s \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$,

$$|b(t, x, u) - b(s, y, u)| + |\sigma(t, x, u) - \sigma(s, y, u)| \leq K(|t - s| + |x - y|). \quad (6.1)$$

Moreover, we impose an additional condition on f :

$$f(t, x, u) \text{ is uniformly continuous in } (t, x), \text{ uniformly in } u \in M. \quad (6.2)$$

Note that the conditions (6.1) and (6.2), together with the linear growth condition (2.3) on b and σ , imply that the operator H defined in (3.13) is continuous, and $\overline{H} = H = H_*$.

Proposition 6.1. *Assume (6.1) and (6.2). Let u (resp. v) be an upper semicontinuous viscosity subsolution (resp. a lower semicontinuous viscosity supersolution), with polynomial growth in x , to*

$$\max \left\{ c(t, x)w - \frac{\partial w}{\partial t} + H(t, x, D_x w, D_x^2 w), w - g(x) \right\} = 0 \text{ on } [0, T) \times \mathbb{R}^d, \quad (6.3)$$

and $u(T, x) \leq v(T, x)$ for all $x \in \mathbb{R}^d$. Then $u \leq v$ on $[0, T) \times \mathbb{R}^d$.

Proof. For $\lambda > 0$, define $u^\lambda := e^{\lambda t} u(t, x)$, $v^\lambda := e^{\lambda t} v(t, x)$, and

$$H_\lambda(t, x, p, A) := \inf_{a \in M} \left\{ -b(t, x, a) \cdot p - \frac{1}{2} \text{Tr}[\sigma \sigma'(t, x, a) A] - e^{\lambda t} f(t, x, a) \right\}.$$

Note that the conditions (6.1) and (6.2), together with the linear growth condition (2.3) on b and σ and the polynomial growth condition (3.8) on f , imply that H_λ is continuous. By definition, u (resp. v) is upper semicontinuous (resp. lower semicontinuous) and has polynomial growth. Moreover, by direct calculations, the subsolution property of u (resp. supersolution property of v) implies that u^λ (resp. v^λ) is a viscosity subsolution (resp. viscosity supersolution) to

$$\max \left\{ (c(t, x) + \lambda) w - \frac{\partial w}{\partial t} + H_\lambda(t, x, D_x w, D_x^2 w), w - e^{\lambda t} g(x) \right\} = 0 \text{ on } [0, T) \times \mathbb{R}^d. \quad (6.4)$$

For any $(t, x, r, q, p, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}^d$, define

$$F_1(t, x, r, q, p, A) := (c(t, x) + \lambda)r - q + H_\lambda(t, x, p, A) \text{ and } F_2(t, x, r) := r - e^{\lambda t}g(x).$$

Since F_1 and F_2 are by definition continuous, so is $F_3 := \max\{F_1, F_2\}$. We can then write (6.4) as $F_3(t, x, w, \frac{\partial w}{\partial t}, D_x w, D_x^2 w) = 0$.

From the polynomial growth condition on u^λ and v^λ , there exists some $p > 0$ such that

$$\sup_{[0, T] \times \mathbb{R}^d} \frac{|u^\lambda(t, x)| + |v^\lambda(t, x)|}{1 + |x|^p} < \infty.$$

Define $\gamma(x) := 1 + |x|^{2p}$ and set $\varphi(t, x) := e^{-\lambda t}\gamma(x)$. From the linear growth condition (2.3) on b and σ , a direct calculation shows that $|b(t, x, a) \cdot D_x \gamma + \frac{1}{2}Tr[\sigma\sigma'(t, x, a)D_x^2 \gamma]| \leq C\gamma(x)$ for some $C > 0$. It follows that

$$\begin{aligned} & (c(t, x) + \lambda)\varphi - \frac{\partial \varphi}{\partial t} + \inf_{a \in M} \left\{ -b(t, x, a)D_x \varphi - \frac{1}{2}Tr[\sigma\sigma'(t, x, a)D_x^2 \varphi] \right\} \\ &= e^{-\lambda t} \left([c(t, x) + 2\lambda]\gamma + \inf_{a \in M} \left\{ -b(t, x, a)D_x \gamma - \frac{1}{2}Tr[\sigma\sigma'(t, x, a)D_x^2 \gamma] \right\} \right) \\ &\geq e^{-\lambda t}[c(t, x) + 2\lambda - C]\gamma \geq 0, \text{ if } \lambda \geq \frac{C}{2}. \end{aligned} \quad (6.5)$$

Now, take $\lambda \geq \frac{C}{2}$ and define $v_\varepsilon^\lambda := v^\lambda + \varepsilon\varphi$ for all $\varepsilon > 0$. By definition, v_ε^λ is lower semi-continuous. Given any $h \in C^{1,2}([0, T] \times \mathbb{R}^d)$ and $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ such that $v_\varepsilon^\lambda - h$ attains a local minimum, which equals 0, at (t_0, x_0) , the supersolution property of v^λ implies either $F_1(\cdot, h(\cdot), \frac{\partial h}{\partial t}(\cdot), D_x h(\cdot), D_x^2 h(\cdot))(t_0, x_0) \geq 0$ or $F_2(\cdot, h(\cdot))(t_0, x_0) \geq 0$. If the former holds true, we see from (6.5) that

$$F_1\left(\cdot, v_\varepsilon^\lambda(\cdot), \frac{\partial v_\varepsilon^\lambda}{\partial t}(\cdot), D_x v_\varepsilon^\lambda(\cdot), D_x^2 v_\varepsilon^\lambda(\cdot)\right)(t_0, x_0) \geq 0;$$

if the latter holds true, then $F_2(\cdot, v_\varepsilon^\lambda(\cdot))(t_0, x_0) = v_\varepsilon^\lambda(t_0, x_0) - e^{\lambda t_0}g(x_0) = F_2(\cdot, v^\lambda(\cdot))(t_0, x_0) + \varepsilon\varphi(t_0, x_0) = F_2(\cdot, h(\cdot))(t_0, x_0) + \varepsilon\varphi(t_0, x_0) \geq 0$. Therefore, v_ε^λ is a lower semicontinuous viscosity supersolution to (6.4).

We would like to show $u^\lambda \leq v_\varepsilon^\lambda$ on $[0, T] \times \mathbb{R}^d$ for all $\varepsilon > 0$; then by sending ε to 0, we can conclude $u \leq v$ on $[0, T] \times \mathbb{R}^d$, as desired. We will argue by contradiction, and thus assume that

$$N := \sup_{[0, T] \times \mathbb{R}^d} (u^\lambda - v_\varepsilon^\lambda)(t, x) > 0$$

From the polynomial growth condition on u^λ and v^λ and the definition of φ , we have

$$\lim_{|x| \rightarrow \infty} \sup_{[0, T]} (u^\lambda - v_\varepsilon^\lambda)(t, x) = -\infty.$$

It follows that there exists some bounded open set $\mathcal{O} \subset \mathbb{R}^d$ such that the maximum N is attained at some point contained in $[0, T] \times \mathcal{O}$. For each $\delta > 0$, define the functions

$$\Phi_\delta(t, s, x, y) := u^\lambda(t, x) - v_\varepsilon^\lambda(s, y) - \eta_\delta(t, s, x, y), \text{ with } \eta_\delta(t, s, x, y) := \frac{1}{2\delta}[|t - s|^2 + |x - y|^2].$$

Since Φ_δ is upper semicontinuous, it attains its maximum, denoted by N_δ , on the compact set $[0, T]^2 \times \overline{\mathcal{O}}^2$ at some point $(t_\delta, s_\delta, x_\delta, y_\delta)$. Then, the upper semicontinuity of $u^\lambda(t, x) - v_\varepsilon^\lambda(s, y)$ implies that $(u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta))_\delta$ is bounded above; moreover, it is also bounded below as

$$N \leq N_\delta = u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta) - \eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta) \leq u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta). \quad (6.6)$$

Then we see from (6.6) and the boundedness of $(u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta))_\delta$ that $(\eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta))_\delta$ is also bounded. Now, note that the bounded sequence $(t_\delta, s_\delta, x_\delta, y_\delta)_\delta$ converges, up to a subsequence, to some point $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y}) \in [0, T]^2 \times \overline{\mathcal{O}}^2$. Then the definition of η_δ and the boundedness of $(\eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta))_\delta$ imply that $\tilde{t} = \tilde{s}$ and $\tilde{x} = \tilde{y}$. Then, by sending δ to 0 in (6.6), we see that the last expression becomes $(u^\lambda - v_\varepsilon^\lambda)(\tilde{t}, \tilde{x}) \leq N$, which implies that

$$N_\delta \rightarrow N \text{ and } \eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta) \rightarrow 0. \quad (6.7)$$

In view of Ishii's Lemma (see e.g. [29, Lemma 4.4.6]) and [29, Remark 4.4.9], for each $\delta > 0$, there exist $A_\delta, B_\delta \in \mathbb{M}^d$ such that

$$Tr(CC'A_\delta - DD'B_\delta) \leq \frac{3}{\delta}|C - D|^2 \text{ for all } C, D \in \mathbb{M}^d, \quad (6.8)$$

and

$$\left(\frac{1}{\delta}(t_\delta - s_\delta), \frac{1}{\delta}(x_\delta - y_\delta), A_\delta\right) \in \bar{\mathcal{P}}^{2,+}u^\lambda(t_\delta, x_\delta), \quad \left(\frac{1}{\delta}(t_\delta - s_\delta), \frac{1}{\delta}(x_\delta - y_\delta), B_\delta\right) \in \bar{\mathcal{P}}^{2,-}v_\varepsilon^\lambda(s_\delta, y_\delta),$$

where $\bar{\mathcal{P}}^{2,+}w(t, x)$ (resp. $\bar{\mathcal{P}}^{2,-}w(t, x)$) denotes the superjet (resp. subjet) of an upper semicontinuous (resp. a lower semicontinuous) function w at $(t, x) \in [0, T] \times \mathbb{R}^d$; for the definition of these notions, see e.g. [10] and [29]. Since the function $F_3 = \max\{F_1, F_2\}$ is continuous, we may apply [29, Lemma 4.4.5] and obtain that

$$\begin{aligned} \max \left\{ (c(t_\delta, x_\delta) + \lambda) u^\lambda(t_\delta, x_\delta) - \frac{1}{\delta}(t_\delta - s_\delta) + H_\lambda(t_\delta, x_\delta, \frac{1}{\delta}(x_\delta - y_\delta), A_\delta), u^\lambda(t_\delta, x_\delta) - e^{\lambda t_\delta} g(x_\delta) \right\} &\leq 0, \\ \max \left\{ (c(s_\delta, y_\delta) + \lambda) v_\varepsilon^\lambda(s_\delta, y_\delta) - \frac{1}{\delta}(t_\delta - s_\delta) + H_\lambda(s_\delta, y_\delta, \frac{1}{\delta}(x_\delta - y_\delta), B_\delta), v_\varepsilon^\lambda(s_\delta, y_\delta) - e^{\lambda s_\delta} g(y_\delta) \right\} &\geq 0. \end{aligned}$$

Noting that $\max\{a, b\} - \max\{c, d\} \geq \min\{a - c, b - d\}$ for any $a, b, c, d \in \mathbb{R}$, we then have

$$\begin{aligned} \min \left\{ (c(t_\delta, x_\delta) + \lambda) u^\lambda(t_\delta, x_\delta) - (c(s_\delta, y_\delta) + \lambda) v_\varepsilon^\lambda(s_\delta, y_\delta) + H_\lambda(t_\delta, x_\delta, \frac{1}{\delta}(x_\delta - y_\delta), A_\delta) \right. \\ \left. - H_\lambda(s_\delta, y_\delta, \frac{1}{\delta}(x_\delta - y_\delta), B_\delta), u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta) + e^{\lambda s_\delta} g(y_\delta) - e^{\lambda t_\delta} g(x_\delta) \right\} &\leq 0. \end{aligned} \quad (6.9)$$

Since $u^\lambda(t_\delta, x_\delta) - v_\varepsilon^\lambda(s_\delta, y_\delta) + e^{\lambda s_\delta} g(y_\delta) - e^{\lambda t_\delta} g(x_\delta) = N_\delta + \eta_\delta(t_\delta, s_\delta, x_\delta, y_\delta) + e^{\lambda s_\delta} g(y_\delta) - e^{\lambda t_\delta} g(x_\delta) \rightarrow N > 0$, we conclude from (6.9) that as δ small enough, we must have

$$\begin{aligned} (c(t_\delta, x_\delta) + \lambda) u^\lambda(t_\delta, x_\delta) - (c(s_\delta, y_\delta) + \lambda) v_\varepsilon^\lambda(s_\delta, y_\delta) \\ \leq H_\lambda(s_\delta, y_\delta, \frac{1}{\delta}(x_\delta - y_\delta), B_\delta) - H_\lambda(t_\delta, x_\delta, \frac{1}{\delta}(x_\delta - y_\delta), A_\delta) \leq \mu(|t_\delta - s_\delta| + |x_\delta - y_\delta| + \frac{3}{\delta}|x_\delta - y_\delta|^2), \end{aligned}$$

for some function μ such that $\mu(z) \rightarrow 0$ as $z \rightarrow 0$; note that the second inequality follows from (6.1), (6.2), and (6.8). Finally, by sending δ to 0 and using (6.7), we get $(c(\tilde{t}, \tilde{x}) + \lambda)N \leq 0$, a contradiction. \square

Now, we turn to the behavior of V_* , the lower semicontinuous envelope of V defined as in (1.3), at terminal time T .

Lemma 6.1. *For all $x \in \mathbb{R}^d$, $V_*(T, x) \geq g(x)$.*

Proof. Fix $\alpha \in \mathcal{A}$. Take an arbitrary sequence $(t_m, x_m) \rightarrow (T, x)$ with $t_m < T$ for all $m \in \mathbb{N}$. By the definition of V , we can choose for each $m \in \mathbb{N}$ a stopping time $\tau_m \in \mathcal{T}_{t_m, T}^{t_m}$ such that

$$\begin{aligned} V(t_m, x_m) &\geq \inf_{\tau \in \mathcal{T}_{t_m, T}^{t_m}} \mathbb{E} \left[\int_{t_m}^{\tau} Y^{t_m, x_m, 1, \alpha} f(s, X^{t_m, x_m, \alpha}, \alpha_s) ds + Y_{\tau}^{t_m, x_m, 1, \alpha} g(X_{\tau}^{t_m, x_m, \alpha}) \right] \\ &\geq \mathbb{E} \left[\int_{t_m}^{\tau_m} Y^{t_m, x_m, 1, \alpha} f(s, X^{t_m, x_m, \alpha}, \alpha_s) ds + Y_{\tau_m}^{t_m, x_m, 1, \alpha} g(X_{\tau_m}^{t_m, x_m, \alpha}) \right] - \frac{1}{m}. \end{aligned}$$

Note that $\tau_m \rightarrow T$ as $\tau_m \in \mathcal{T}_{t_m, T}^{t_m}$ and $t_m \rightarrow T$. Then it follows from Fatou's lemma that $\liminf_{m \rightarrow \infty} V(t_m, x_m) \geq g(x)$. Since (t_m, x_m) is arbitrarily chosen, we conclude $V_*(T, x) \geq g(x)$. \square

Theorem 6.1. *Assume (6.1) and (6.2). Then $U^* = V$ on $[0, T] \times \mathbb{R}^d$. In particular, $U = V$ on $[0, T] \times \mathbb{R}^d$, i.e. the game has a value, which is the unique viscosity solution to (4.11) with terminal condition $w(T, x) = g(x)$ for $x \in \mathbb{R}^d$.*

Proof. Since by definition $U(t, x) \leq g(x)$ on $[0, T] \times \mathbb{R}^d$, we have $U^*(t, x) \leq g(x)$ on $[0, T] \times \mathbb{R}^d$ by the continuity of g . Then by Lemma 6.1 and the fact that $U^* \geq U \geq V$, we have $U^*(T, x) = V(T, x) = g(x)$ for all $x \in \mathbb{R}^d$. Recall that under (6.1) and (6.2), the function H is continuous, and $\overline{H} = H = H_*$. Now, in view of Propositions 4.3 and 5.2, and the fact that $U^*(T, \cdot) = V(T, \cdot)$ and $\overline{H} = H = H_*$, we conclude from Proposition 6.1 that $U^* = V$ on $[0, T] \times \mathbb{R}^d$, which in particular implies $U = V$ on $[0, T] \times \mathbb{R}^d$. \square

APPENDIX A. PROOFS FOR SECTIONS 2 AND 3

This Appendix is devoted to rigorous proofs of Propositions 2.1, 2.2, 2.3, and Lemma 3.2. To this end, we will first derive several auxiliary results.

Recall the definitions introduced in Subsection 2.1. Fix $t \in [0, T]$. For any $A \subseteq \Omega$, $\tilde{A} \subseteq \Omega^t$, and $x \in \mathbb{R}^d$, we set

$$\tilde{A}_x := \{\tilde{\omega} \in \tilde{A} \mid \tilde{\omega}_t = x\},$$

and define

$$A^{t, \omega} := \{\tilde{\omega} \in \Omega^t \mid \omega \otimes_t \tilde{\omega} \in A\}, \quad A_x^{t, \omega} := (A^{t, \omega})_x, \quad \omega \otimes_t \tilde{A} := \{\omega \otimes_t \tilde{\omega} \mid \tilde{\omega} \in \tilde{A}\}.$$

Given a random time $\tau : \Omega \mapsto [0, \infty]$, whenever $\omega \in \Omega$ is fixed, we simplify our notation as $A^{\tau, \omega} = A^{\tau(\omega), \omega}$. We also consider

$$\mathcal{H}_s^t := \psi_t^{-1} \mathcal{G}_s^{t, 0} \subseteq \mathcal{G}_s^t, \quad \forall s \in [t, T]. \quad (\text{A.1})$$

Note that the inclusion follows from the Borel measurability of ψ_t . Finally, while \mathbb{E} denotes the expectation taken under $\overline{\mathbb{P}}$, in this appendix we also consider $\mathbb{E}_{\mathbb{P}}$, the expectation taken under \mathbb{P} .

Lemma A.1. *Fix $t \in [0, T]$ and $\omega \in \Omega$. For any $r \in [t, T]$, $A \in \mathcal{G}_r$, $\tilde{A} \in \mathcal{G}_r^t$, and $\xi \in L^0(\Omega, \mathcal{G}_r)$,*

- (i) $A_x^{t,\omega} = A_0^{t,\omega} + x$ and $A_x^{t,\omega} \in \mathcal{G}_r^{t,x}$, $\forall x \in \mathbb{R}^d$.
- (ii) $A^{t,\omega} = \psi_t^{-1} A_0^{t,\omega} \in \mathcal{H}_r^t \subseteq \mathcal{G}_r^t$ and $\mathbb{P}^t(A^{t,\omega}) = \mathbb{P}^{t,x}(A_x^{t,\omega}) = \mathbb{P}^{t,x}(A^{t,\omega})$, $\forall x \in \mathbb{R}^d$.
- (iii) $\phi_t^{-1} A^{t,\omega} \in \phi_t^{-1} \mathcal{H}_r^t \subseteq \mathcal{G}_r$ and $\mathbb{P}(\phi_t^{-1} A^{t,\omega}) = \mathbb{P}^t(A^{t,\omega})$.
- (iv) $\omega \otimes_t \tilde{A}_{\omega_t} \in \mathcal{G}_r$. Hence, $\omega \otimes_t A_{\omega_t}^{t,\omega} \in \mathcal{G}_r$.
- (v) For any Borel subset \mathcal{E} of \mathbb{R} , $(\xi^{t,\omega})^{-1}(\mathcal{E}) \in \phi_t^{-1} \mathcal{H}_r^t \subseteq \mathcal{G}_r$. Hence, $\xi^{t,\omega} \in L^0(\Omega, \mathcal{G}_r)$.

Proof. (i) Fix $x \in \mathbb{R}^d$. Since $\tilde{\omega} \in A_0^{t,\omega} \Leftrightarrow \omega \otimes_t \tilde{\omega} \in A$ and $\tilde{\omega}_t = 0 \Leftrightarrow (\omega \otimes_t (\tilde{\omega} + x))_t = \omega \cdot 1_{[0,t]}(\cdot) + ((\tilde{\omega} + x) - (\tilde{\omega}_t + x) + \omega_t)1_{(t,T]}(\cdot) = (\omega \otimes_t \tilde{\omega})_t \in A$ and $(\tilde{\omega} + x)_t = x \Leftrightarrow \tilde{\omega} + x \in A_x^{t,\omega}$, we conclude $A_x^{t,\omega} = A_0^{t,\omega} + x$.

Set $\Lambda := \{A \subseteq \Omega \mid A_x^{t,\omega} \in \mathcal{G}_r^{t,x}\}$. Note that $\Omega \in \Lambda$ since $\Omega_x^{t,\omega} = \{\tilde{\omega} \in \Omega^t \mid \omega \otimes_t \tilde{\omega} \in \Omega, \tilde{\omega}_t = x\} = (\Omega^t)_x \in \mathcal{G}_r^{t,x}$. Given $A \in \Lambda$, we have $(A^c)_x^{t,\omega} = (\Omega^t)_x \setminus \{\tilde{\omega} \in \Omega^t \mid \omega \otimes_t \tilde{\omega} \in A, \tilde{\omega}_t = x\} = (\Omega^t)_x \setminus A_x^{t,\omega} \in \mathcal{G}_r^{t,x}$, which shows $A^c \in \Lambda$. Given $\{A_i\}_{i \in \mathbb{N}} \subset \Lambda$, we have $(\bigcup_{i \in \mathbb{N}} A_i)_x^{t,\omega} = \bigcup_{i \in \mathbb{N}} \{\tilde{\omega} \in \Omega^t \mid \omega \otimes_t \tilde{\omega} \in A_i, \tilde{\omega}_t = x\} = \bigcup_{i \in \mathbb{N}} (A_i)_x^{t,\omega} \in \mathcal{G}_r^{t,x}$, which shows $\bigcup_{i \in \mathbb{N}} A_i \in \Lambda$. Thus, we conclude Λ is a σ -algebra of Ω . For any $x \in \mathbb{Q}^d$ and $\lambda \in \mathbb{Q}_+$, the set of positive rationals, let $O_\lambda(x)$ denote the open ball in \mathbb{R}^d centered at x with radius λ . Note from [18, p.307] that for each $s \in [0, T]$, \mathcal{G}_r^s is countably generated by

$$\mathcal{C}_r^s := \left\{ \bigcap_{i=1}^m (W_{t_i}^s)^{-1}(O_{\lambda_i}(x_i)) \mid m \in \mathbb{N}, t_i \in \mathbb{Q}, s \leq t_1 < \dots < t_m \leq r, x_i \in \mathbb{Q}^d, \lambda_i \in \mathbb{Q}_+ \right\}. \quad (\text{A.2})$$

Given $C = \bigcap_{i=1}^m (W_{t_i})^{-1}(O_{\lambda_i}(x_i))$ in $\mathcal{C}_r = \mathcal{C}_r^0$, if $t_m \geq t$, set $k = \min\{i = 1, \dots, m \mid t_i \geq t\}$; otherwise, set $k = m + 1$. Then, if $\omega_{t_i} \notin O_{\lambda_i}(x_i)$ for some $i = 1, \dots, k - 1$, we have $C_x^{t,\omega} = \emptyset \in \mathcal{G}_r^{t,x}$; if $k = m + 1$ and $\omega_{t_i} \in O_{\lambda_i}(x_i) \forall i = 1, \dots, m$, we have $C_x^{t,\omega} = (\Omega^t)_x \in \mathcal{G}_r^{t,x}$; for all other cases,

$$C_x^{t,\omega} = \{W_t^t = x\} \cap \bigcap_{i=k}^m (W_{t_i}^t)^{-1}(O_{\lambda_i}(x_i - \omega_t + x)) \in \mathcal{G}_r^{t,x}. \quad (\text{A.3})$$

Thus, $\mathcal{C}_r \subseteq \Lambda$, which implies $\mathcal{G}_r = \sigma(\mathcal{C}_r) \subseteq \Lambda$. Now, for any $A \in \mathcal{G}_r$, $A_x^{t,\omega} \in \mathcal{G}_r^{t,x} \subseteq \mathcal{G}_r^t$.

(ii) Observe from part (i) that $\tilde{\omega} \in A^{t,\omega} \Leftrightarrow \tilde{\omega} \in A_{\tilde{\omega}_t}^{t,\omega} \Leftrightarrow \tilde{\omega} - \tilde{\omega}_t \in A_0^{t,\omega}$ i.e. $\psi_t(\tilde{\omega}) \in A_0^{t,\omega} \Leftrightarrow \tilde{\omega} \in \psi_t^{-1}(A_0^{t,\omega})$. Thus, $A^{t,\omega} = \psi_t^{-1}(A_0^{t,\omega}) \in \psi_t^{-1}(\mathcal{G}_r^{t,0}) = \mathcal{H}_r^t \subseteq \mathcal{G}_r^t$, thanks to part (i) and (A.1). Then, using part (i) again, $\mathbb{P}^t(A^{t,\omega}) = \mathbb{P}^t(A_0^{t,\omega}) = \mathbb{P}^{t,x}(A_0^{t,\omega} + x) = \mathbb{P}^{t,x}(A_x^{t,\omega}) = \mathbb{P}^{t,x}(A^{t,\omega})$, $\forall x \in \mathbb{R}^d$.

(iii) By part (ii) and the Borel measurability of $\phi_t : (\Omega, \mathcal{G}_r) \mapsto (\Omega^t, \mathcal{G}_r^t)$, we immediately have $\phi_t^{-1} A^{t,\omega} \in \phi_t^{-1} \mathcal{H}_r^t \subseteq \mathcal{G}_r$. Now, by property (e") in [18, p.84] and part (ii),

$$\mathbb{P}[\phi_t^{-1} A^{t,\omega} \mid \mathcal{G}_{t+}](\omega') = \mathbb{P}^{t,\omega'_t}(A^{t,\omega}) = \mathbb{P}^t(A^{t,\omega}) \text{ for } \mathbb{P}\text{-a.e. } \omega' \in \Omega,$$

which implies $\mathbb{P}[\phi_t^{-1} A^{t,\omega}] = \mathbb{P}^t(A^{t,\omega})$.

(iv) Set $\Lambda := \{\tilde{A} \subseteq \Omega^t \mid \omega \otimes_t \tilde{A}_{\omega_t} \in \mathcal{G}_r\}$. Let \mathcal{C}_r^t be given as in (A.2). For any $C = \bigcap_{i=1}^m (W_{t_i}^t)^{-1}(O_{\lambda_i}(x_i))$ in \mathcal{C}_r^t , we deduce from the continuity of paths in Ω that

$$\begin{aligned} \omega \otimes_t C_{\omega_t} &= \{\omega' \in \Omega \mid \omega'_s = \omega_s \forall s \in \mathbb{Q} \cap [0, t) \text{ and } \omega'_{t_i} \in O_{\lambda_i}(x_i) \text{ for } i = 1, \dots, m\} \\ &= \left(\bigcap_{s \in \mathbb{Q} \cap [0, t)} (W_s)^{-1}(\omega_s) \right) \cap \left(\bigcap_{i=1}^m (W_{t_i})^{-1}(O_{\lambda_i}(x_i)) \right) \in \mathcal{G}_r. \end{aligned}$$

Thus, we have $\mathcal{C}_r^t \subseteq \Lambda$. Given $\{\tilde{A}_i\}_{i \in \mathbb{N}} \subset \Lambda$, we have $\omega \otimes_t (\bigcup_{i \in \mathbb{N}} \tilde{A}_i)_{\omega_t} = \bigcup_{i \in \mathbb{N}} (\omega \otimes_t (\tilde{A}_i)_{\omega_t}) \in \mathcal{G}_r$, which shows $\bigcup_{i \in \mathbb{N}} \tilde{A}_i \in \Lambda$; this in particular implies $\Omega^t = \bigcup_{n \in \mathbb{N}} (W_r^t)^{-1}(O_n(0)) \in \Lambda$. Given $\tilde{A} \in \Lambda$, we have $\omega \otimes_t (\tilde{A}^c)_{\omega_t} = (\omega \otimes_t (\Omega^t)_{\omega_t}) \setminus (\omega \otimes_t \tilde{A}_{\omega_t}) \in \mathcal{G}_r$, which shows $\tilde{A}^c \in \Lambda$. Hence, Λ is a σ -algebra of Ω^t , which implies $\mathcal{G}_r^t = \sigma(\mathcal{C}_r^t) \subseteq \Lambda$. Now, by part (i), we must have $\omega \otimes_t A_{\omega_t}^{t,\omega} \in \mathcal{G}_r$.

(v) Since $\xi^{-1}(\mathcal{E}) \in \mathcal{G}_r$, $(\xi^{t,\omega})^{-1}(\mathcal{E}) = \{\omega' \in \Omega \mid \xi(\omega \otimes_t \phi_t(\omega')) \in \mathcal{E}\} = \{\omega' \in \Omega \mid \omega \otimes_t \phi_t(\omega') \in \xi^{-1}(\mathcal{E})\} = \phi_t^{-1}(\xi^{-1}(\mathcal{E}))^{t,\omega} \in \phi_t^{-1}\mathcal{H}_r^t \subseteq \mathcal{G}_r$, thanks to part (iii). \square

In light of Theorem 1.3.4 and equation (1.3.15) in [31], for any \mathbb{G} -stopping time τ , there exists a family $\{Q_\tau^\omega\}_{\omega \in \Omega}$ of probability measures on (Ω, \mathcal{G}_T) , called a regular conditional probability distribution (r.c.p.d.) of \mathbb{P} given \mathcal{G}_τ , such that

- (i) for each $A \in \mathcal{G}_T$, the mapping $\omega \mapsto Q_\tau^\omega(A)$ is \mathcal{G}_τ -measurable.
- (ii) for each $A \in \mathcal{G}_T$, it holds for \mathbb{P} -a.e. $\omega \in \Omega$ that $\mathbb{P}[A \mid \mathcal{G}_\tau](\omega) = Q_\tau^\omega(A)$.
- (iii) for each $\omega \in \Omega$, $Q_\tau^\omega(\omega \otimes_\tau (\Omega^{\tau(\omega)})_{\omega_\tau}) = 1$.

By property (iii) above and Lemma A.1 (iv), for any fixed $\omega \in \Omega$, we can define a probability measure $Q^{\tau,\omega}$ on $(\Omega^{\tau(\omega)}, \mathcal{G}_T^{\tau(\omega)})$ by

$$Q^{\tau,\omega}(\tilde{A}) := Q_\tau^\omega(\omega \otimes_\tau \tilde{A}_{\omega_\tau}), \quad \forall \tilde{A} \in \mathcal{G}_T^{\tau(\omega)}.$$

Then, combining properties (ii) and (iii) above, we have: for $A \in \mathcal{G}_T$, it holds for \mathbb{P} -a.e. $\omega \in \Omega$ that

$$\mathbb{P}[A \mid \mathcal{G}_\tau](\omega) = Q_\tau^\omega((\omega \otimes_\tau (\Omega^{\tau(\omega)})_{\omega_\tau}) \cap A) = Q_\tau^\omega(\omega \otimes_\tau A_{\omega_\tau}^{\tau,\omega}) = Q^{\tau,\omega}(A^{\tau,\omega}). \quad (\text{A.4})$$

Note that the r.c.p.d. $\{Q_\tau^\omega\}_{\omega \in \Omega}$ is generally not unique. For each $(t, x) \in [0, T] \times \mathbb{R}^d$, observe that the shifted Wiener measure $\mathbb{P}^{t,x}$ can be characterized as the unique solution to the martingale problem for the operator $L := \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j}$ starting from time t with initial value x (see [30, Remark 7.1.23] and [31, Exercise 6.7.3]). Then, thanks to the strong Markov property of solutions to the martingale problem (see e.g. [31, Theorem 6.2.2]), there exists a particular r.c.p.d. $\{Q_\tau^\omega\}_{\omega \in \Omega}$ such that $Q^{\tau,\omega} = \mathbb{P}^{\tau(\omega), \omega_{\tau(\omega)}}$. Now, by (A.4) and Lemma A.1 (ii), we have: for $A \in \mathcal{G}_T$,

$$\mathbb{P}[A \mid \mathcal{G}_\tau](\omega) = \mathbb{P}^{\tau(\omega), \omega_{\tau(\omega)}}(A^{\tau,\omega}) = \mathbb{P}^{\tau(\omega)}(A^{\tau,\omega}), \quad \mathbb{P}\text{-a.s.} \quad (\text{A.5})$$

So far, we have restricted ourselves to \mathbb{G} -stopping times. We say a random variable $\tau : \Omega \mapsto [0, \infty]$ is a \mathbb{G} -optional time if $\{\tau < t\} \in \mathcal{G}_t$ for all $t \in [0, T]$. In the following, we obtain a generalized version of (A.5) for \mathbb{G} -optional times.

Lemma A.2. *Fix a \mathbb{G} -optional time $\tau \leq T$. For any $A \in \mathcal{G}_T$,*

$$\mathbb{P}[A \mid \mathcal{G}_{\tau+}](\omega) = \mathbb{P}^{\tau(\omega)}(A^{\tau,\omega}) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Proof. Step 1: By [18, Problem 1.2.24], we can take a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of \mathbb{G} -stopping times such that $\tau_n(\omega) \downarrow \tau(\omega)$ for all $\omega \in \Omega$. Fix $A \in \mathcal{G}_T$. For each $n \in \mathbb{N}$, (A.5) implies that for any $B \in \mathcal{G}_{\tau_n}$,

$$\mathbb{E}_{\mathbb{P}}[1_A 1_B] = \mathbb{E}_{\mathbb{P}}[\mathbb{P}^{\tau_n(\omega)}(A^{\tau_n,\omega}) 1_B]. \quad (\text{A.6})$$

Then, for any $B \in \mathcal{G}_{\tau+}$, we must have (A.6) for all $n \in \mathbb{N}$, since $\mathcal{G}_{\tau+} = \bigcap_{n \in \mathbb{N}} \mathcal{G}_{\tau_n}$. Now, by taking the limit in n and assuming that for each $\omega \in \Omega$

$$\lim_{n \rightarrow \infty} \mathbb{P}^{\tau_n(\omega)}(A^{\tau_n, \omega}) = \mathbb{P}^{\tau(\omega)}(A^{\tau, \omega}), \quad (\text{A.7})$$

we obtain from the dominated convergence theorem that $\mathbb{E}_{\mathbb{P}}[1_A 1_B] = \mathbb{E}_{\mathbb{P}}[\mathbb{P}^{\tau(\omega)}(A^{\tau, \omega}) 1_B]$. Since $B \in \mathcal{G}_{\tau+}$ is arbitrary, we conclude $\mathbb{P}[A \mid \mathcal{G}_{\tau+}](\omega) = \mathbb{P}^{\tau(\omega)}(A^{\tau, \omega})$ for \mathbb{P} -a.e. $\omega \in \Omega$.

Step 2: It remains to prove (A.7). Fix $\omega \in \Omega$ and set $\Lambda := \{A \subseteq \Omega \mid (\text{A.7}) \text{ holds}\}$. Since $\Omega^{s, \omega} = \Omega^s$, $\forall s \in [0, T]$, (A.7) holds for Ω and thus $\Omega \in \Lambda$. Given $A \in \Lambda$, we have $\mathbb{P}^{\tau_n(\omega)}[(A^c)^{\tau_n, \omega}] = \mathbb{P}^{\tau_n(\omega)}[(A^{\tau_n, \omega})^c] = 1 - \mathbb{P}^{\tau_n(\omega)}(A^{\tau_n, \omega}) \rightarrow 1 - \mathbb{P}^{\tau(\omega)}(A^{\tau, \omega}) = \mathbb{P}^{\tau(\omega)}[(A^{\tau, \omega})^c] = \mathbb{P}^{\tau(\omega)}[(A^c)^{\tau, \omega}]$, which shows $A^c \in \Lambda$. Given a sequence $\{A_i\}_{i \in \mathbb{N}}$ of disjoint sets in Λ , observe that $\{A_i^{s, \omega}\}_{i \in \mathbb{N}}$ is a sequence of disjoint sets in Ω^s for any $s \in [0, T]$. Then we have $\mathbb{P}^{\tau_n(\omega)}[(\bigcup_{i \in \mathbb{N}} A_i)^{\tau_n, \omega}] = \mathbb{P}^{\tau_n(\omega)}[\bigcup_{i \in \mathbb{N}} A_i^{\tau_n, \omega}] = \sum_{i \in \mathbb{N}} \mathbb{P}^{\tau_n(\omega)}(A_i^{\tau_n, \omega}) \rightarrow \sum_{i \in \mathbb{N}} \mathbb{P}^{\tau(\omega)}(A_i^{\tau, \omega}) = \mathbb{P}^{\tau(\omega)}[\bigcup_{i \in \mathbb{N}} A_i^{\tau, \omega}] = \mathbb{P}^{\tau(\omega)}[(\bigcup_{i \in \mathbb{N}} A_i)^{\tau, \omega}]$, which shows $\bigcup_{i \in \mathbb{N}} A_i \in \Lambda$. Thus, we conclude that Λ is a σ -algebra of Ω .

As mentioned in the proof of Lemma A.1 (i), \mathcal{G}_T is countably generated by $\mathcal{C}_T = \mathcal{C}_T^0$ given in (A.2). Given $C = \bigcap_{i=1}^m (W_{t_i})^{-1}(O_{\lambda_i}(x_i))$ in \mathcal{C}_T , if $t_m \geq \tau(\omega)$ we set $k := \min\{i = 1, \dots, m \mid t_i \geq \tau(\omega)\}$; otherwise, set $k := m + 1$. We see that: **1.** If $\omega_{t_i} \notin O_{\lambda_i}(x_i)$ for some $i = 1, \dots, k-1$, then $C^{s, \omega} = \emptyset$ $\forall s \in [\tau(\omega), T]$ and thus (A.7) holds for C . **2.** If $k = m + 1$ and $\omega_{t_i} \in O_{\lambda_i}(x_i)$ for all $i = 1, \dots, m$, we have $C^{s, \omega} = \Omega^s$ $\forall s \in [\tau(\omega), T]$ and thus (A.7) still holds for C . **3.** For all other cases, $C_{\omega_s}^{s, \omega}$ is of the form in (A.3) $\forall s \in [\tau(\omega), T]$. Let B be a d -dimensional Brownian motion defined on any given filtered probability space $(E, \mathcal{I}, \{\mathcal{I}_s\}_{s \geq 0}, P)$. Then by Lemma A.1 (ii),

$$\begin{aligned} \mathbb{P}^{\tau_n(\omega)}[C^{\tau_n, \omega}] &= \mathbb{P}^{\tau_n(\omega), \omega_{\tau_n(\omega)}}[C_{\omega_{\tau_n(\omega)}}^{\tau_n, \omega}] = P[B_{t_i - \tau_n(\omega)} \in O_{\lambda_i}(x_i - \omega_{\tau_n(\omega)}), i = k \dots, m] \\ &\rightarrow P[B_{t_i - \tau(\omega)} \in O_{\lambda_i}(x_i - \omega_{\tau(\omega)}), i = k \dots, m] = \mathbb{P}^{\tau(\omega), \omega_{\tau(\omega)}}[C_{\omega_{\tau(\omega)}}^{\tau, \omega}] = \mathbb{P}^{\tau(\omega)}[C^{\tau, \omega}]. \end{aligned}$$

Hence, we conclude that $\mathcal{C}_T \subseteq \Lambda$ and therefore $\mathcal{G}_T = \sigma(\mathcal{C}_T) \subseteq \Lambda$. \square

Now, we want to generalize Lemma A.1 to incorporate \mathbb{F} -stopping times.

Lemma A.3. Fix $\theta \in \mathcal{T}$. We have

(i) For any $\overline{N} \in \overline{\mathcal{N}}$, $\overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}^{\theta(\omega)}$ and $\phi_{\theta}^{-1} \overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}$ for $\overline{\mathbb{P}}$ -a.e. $\omega \in \Omega$.

(ii) For any $r \in [0, T]$ and $A \in \mathcal{F}_r$, it holds for $\overline{\mathbb{P}}$ -a.e. $\omega \in \Omega$ that

$$\text{if } \theta(\omega) \leq r, \quad A^{\theta, \omega} \in \mathcal{H}_r^{\theta(\omega)} \cup \overline{\mathcal{N}}^{\theta(\omega)} \subseteq \overline{\mathcal{G}}_r^{\theta(\omega)} \quad \text{and} \quad \phi_{\theta}^{-1} A^{\theta, \omega} \in \mathcal{F}_r^{\theta(\omega)}.$$

(iii) For any $r \in [0, T]$ and $\xi \in L^0(\Omega, \mathcal{F}_r)$, it holds for $\overline{\mathbb{P}}$ -a.e. $\omega \in \Omega$ that

$$\text{if } \theta(\omega) \leq r, \quad \xi^{\theta, \omega} \in L^0(\Omega, \mathcal{F}_r^{\theta(\omega)}).$$

Proof. (i) Take $N \in \mathcal{N}$ such that $\overline{N} \subseteq N$. By [18, Exercise 2.7.11], there exists a \mathbb{G} -optional time τ such that $\overline{N}_1 := \{\theta \neq \tau\} \in \overline{\mathcal{N}}$. By Lemma A.2, there exists $\overline{N}_2 \in \mathcal{N} \subset \overline{\mathcal{N}}$ such that $0 = \mathbb{P}[N \mid \mathcal{G}_{\tau+}](\omega) = \mathbb{P}^{\tau(\omega)}(N^{\tau, \omega})$, for $\omega \in \Omega \setminus \overline{N}_2$. Thus, for $\omega \in \Omega \setminus (\overline{N}_1 \cup \overline{N}_2)$, we have $0 = \mathbb{P}^{\tau(\omega)}(N^{\tau, \omega}) = \mathbb{P}^{\theta(\omega)}(N^{\theta, \omega})$, i.e. $N^{\theta, \omega} \in \mathcal{N}^{\theta(\omega)}$. Since $\overline{N}^{\theta, \omega} \subseteq N^{\theta, \omega}$, we have $\overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}^{\theta(\omega)}$ $\overline{\mathbb{P}}$ -a.s.

On the other hand, from Lemma A.1 (iii), $\mathbb{P}(\phi_{\theta}^{-1} N^{\theta, \omega}) = \mathbb{P}^{\theta(\omega)}(N^{\theta, \omega}) = 0$ for $\omega \in \Omega \setminus (\overline{N}_1 \cup \overline{N}_2)$, which shows $\phi_{\theta}^{-1} N^{\theta, \omega} \in \mathcal{N}$ $\overline{\mathbb{P}}$ -a.s. Since $\phi_{\theta}^{-1} \overline{N}^{\theta, \omega} \subseteq \phi_{\theta}^{-1} N^{\theta, \omega}$, we conclude $\phi_{\theta}^{-1} \overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}$ $\overline{\mathbb{P}}$ -a.s.

(ii) By [18, Problem 2.7.3], there exist $\tilde{A} \in \mathcal{G}_r$ and $\overline{N} \in \overline{\mathcal{N}}$ such that $A = \tilde{A} \cup \overline{N}$ and $\tilde{A} \cap \overline{N} = \emptyset$. From Lemma A.1 (ii), we know that for any $\omega \in \Omega$, if $\theta(\omega) \leq r$ then $\tilde{A}^{\theta, \omega} \in \mathcal{H}_r^{\theta(\omega)} \subseteq \mathcal{G}_r^{\theta(\omega)}$. Also, from part (i) we have $\overline{N}^{\theta, \omega} \in \overline{\mathcal{N}}^{\theta(\omega)}$ \mathbb{P} -a.s. We therefore conclude that for \mathbb{P} -a.e. $\omega \in \Omega$, if $\theta(\omega) \leq r$, then $A^{\theta, \omega} = \tilde{A}^{\theta, \omega} \cup \overline{N}^{\theta, \omega} \in \mathcal{H}_r^{\theta(\omega)} \cup \overline{\mathcal{N}}^{\theta(\omega)} \subseteq \overline{\mathcal{G}}_r^{\theta(\omega)}$. Then, thanks to part (i) and Definition 2.1, it holds \mathbb{P} -a.s. that $\phi_\theta^{-1} A^{\theta, \omega} = \phi_\theta^{-1} \tilde{A}^{\theta, \omega} \cup \phi_\theta^{-1} \overline{N}^{\theta, \omega} \in \phi_\theta^{-1} \mathcal{H}_r^{\theta(\omega)} \cup \overline{\mathcal{N}} \subseteq \mathcal{F}_r^{\theta(\omega)}$ if $\theta(\omega) \leq r$.

(iii) Let \mathcal{E} be a Borel subset of \mathbb{R} . Since $\xi^{-1}(\mathcal{E}) \in \mathcal{F}_r$, we see from part (ii) that, for \mathbb{P} -a.e. $\omega \in \Omega$, $(\xi^{\theta, \omega})^{-1}(\mathcal{E}) = \{\omega' \in \Omega \mid \xi(\omega \otimes_\theta \phi_\theta(\omega')) \in \mathcal{E}\} = \{\omega' \in \Omega \mid \omega \otimes_\theta \phi_\theta(\omega') \in \xi^{-1}(\mathcal{E})\} = \phi_\theta^{-1}(\xi^{-1}(\mathcal{E}))^{\theta, \omega} \in \mathcal{F}_r^{\theta(\omega)}$ if $\theta(\omega) \leq r$. \square

Now, we generalize Lemma A.2 to incorporate \mathbb{F} -stopping times.

Lemma A.4. *Fix $\theta \in \mathcal{T}$. For any $A \in \mathcal{F}_T$, $\overline{\mathbb{P}}[A \mid \mathcal{F}_\theta](\omega) = \overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta, \omega})$, for \mathbb{P} -a.e. $\omega \in \Omega$.*

Proof. Thanks again to [18, Exercise 2.7.11], we may take a \mathbb{G} -optional time τ such that $\overline{N}_1 := \{\theta \neq \tau\} \in \overline{\mathcal{N}}$ and $\mathcal{F}_\tau = \mathcal{F}_\theta$. Moreover, we have $A = \tilde{A} \cup \overline{N}$ for some $\tilde{A} \in \mathcal{G}_T$ and $\overline{N} \in \overline{\mathcal{N}}$ with $\tilde{A} \cap \overline{N} = \emptyset$, by using [18, Exercise 2.7.3]. Then, in view of Lemma A.1 (ii), Lemma A.3 (i), and Lemma A.2, we can take some $\overline{N}_2 \in \overline{\mathcal{N}}$ such that for $\omega \in \Omega \setminus (\overline{N}_1 \cup \overline{N}_2)$,

$$\begin{aligned} \overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta, \omega}) &= \overline{\mathbb{P}}^{\tau(\omega)}(A^{\tau, \omega}) = \overline{\mathbb{P}}^{\tau(\omega)}(\tilde{A}^{\tau, \omega}) + \overline{\mathbb{P}}^{\tau(\omega)}(\overline{N}^{\tau, \omega}) = \mathbb{P}^{\tau(\omega)}(\tilde{A}^{\tau, \omega}) \\ &= \mathbb{P}[\tilde{A} \mid \mathcal{G}_{\tau+}](\omega) = \overline{\mathbb{P}}[\tilde{A} \mid \mathcal{G}_{\tau+}](\omega) = \overline{\mathbb{P}}[A \mid \mathcal{G}_{\tau+}](\omega). \end{aligned} \quad (\text{A.8})$$

For any $B \in \mathcal{F}_\tau$, $B = \tilde{B} \cup \overline{N}'$ for some $\tilde{B} \in \mathcal{G}_\tau \subseteq \mathcal{G}_{\tau+}$ and $\overline{N}' \in \overline{\mathcal{N}}$ with $\tilde{B} \cap \overline{N}' = \emptyset$, thanks again to [18, Exercise 2.7.3]. We then deduce from (A.8) that $\mathbb{E}[1_{\tilde{A}} 1_B] = \mathbb{E}[1_{\tilde{A}} 1_{\tilde{B}}] = \mathbb{E}[\overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta, \omega}) 1_{\tilde{B}}] = \mathbb{E}[\overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta, \omega}) 1_B]$. Hence, we conclude $\overline{\mathbb{P}}^{\theta(\omega)}(A^{\theta, \omega}) = \overline{\mathbb{P}}[A \mid \mathcal{F}_\tau](\omega) = \overline{\mathbb{P}}[A \mid \mathcal{F}_\theta](\omega)$, for $\omega \in \Omega \setminus (\overline{N}_1 \cup \overline{N}_2)$. \square

Finally, we are able to generalize Lemma A.1 (iii) to incorporate \mathbb{F} -stopping times.

Proposition A.1. *Fix $\theta \in \mathcal{T}$. We have*

- (i) *for any $A \in \mathcal{F}_T$, $\overline{\mathbb{P}}[A \mid \mathcal{F}_\theta](\omega) = \overline{\mathbb{P}}[\phi_\theta^{-1} A^{\theta, \omega}]$, for \mathbb{P} -a.e. $\omega \in \Omega$.*
- (ii) *for any $\xi \in L^1(\Omega, \mathcal{F}_T, \overline{\mathbb{P}})$, $\mathbb{E}[\xi \mid \mathcal{F}_\theta](\omega) = \mathbb{E}[\xi^{\theta, \omega}]$ for \mathbb{P} -a.e. $\omega \in \Omega$.*

Proof. (i) By Lemma A.3 (i) and Lemma A.1 (iii), it holds \mathbb{P} -a.s. that

$$\overline{\mathbb{P}}[\phi_\theta^{-1} A^{\theta, \omega}] = \overline{\mathbb{P}}[\phi_\theta^{-1} \tilde{A}^{\theta, \omega}] + \overline{\mathbb{P}}[\phi_\theta^{-1} \overline{N}^{\theta, \omega}] = \mathbb{P}[\phi_\theta^{-1} \tilde{A}^{\theta, \omega}] = \mathbb{P}^{\theta(\omega)}[\tilde{A}^{\theta, \omega}] = \overline{\mathbb{P}}^{\theta(\omega)}[\tilde{A}^{\theta, \omega}] = \overline{\mathbb{P}}^{\theta(\omega)}[A^{\theta, \omega}].$$

The desired result then follows from the above equality and Lemma A.4.

(ii) Given $A \in \mathcal{F}_T$, observe that for any fixed $\omega \in \Omega$, $(1_A)^{\theta, \omega}(\omega') = 1_A(\omega \otimes_\theta \phi_\theta(\omega')) = 1_{\phi_\theta^{-1} A^{\theta, \omega}}(\omega')$. Then we see immediately from part (i) that part (ii) is true for $\xi = 1_A$. It follows that part (ii) also holds true for any \mathcal{F}_T -measurable simple function ξ . For any positive $\xi \in L^1(\Omega, \mathcal{F}_T, \overline{\mathbb{P}})$, we can take a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ of \mathcal{F}_T -measurable simple functions such that $\xi_n(\omega) \uparrow \xi(\omega) \forall \omega \in \Omega$. By the monotone convergence theorem, there exists $\overline{N} \in \overline{\mathcal{N}}$ such that $\mathbb{E}[\xi_n \mid \mathcal{F}_\theta](\omega) \uparrow \mathbb{E}[\xi \mid \mathcal{F}_\theta](\omega)$, for $\omega \in \Omega \setminus \overline{N}$. For each $n \in \mathbb{N}$, since ξ_n is an \mathcal{F}_T -measurable simple function, there exists $\overline{N}_n \in \overline{\mathcal{N}}$ such that $\mathbb{E}[\xi_n \mid \mathcal{F}_\theta](\omega) = \mathbb{E}[(\xi_n)^{\theta, \omega}]$, for $\omega \in \Omega \setminus \overline{N}_n$. Finally, noting that there exists $\overline{N}' \in \overline{\mathcal{N}}$ such that $\xi^{\theta, \omega}$ is \mathcal{F}_T -measurable for $\omega \in \Omega \setminus \overline{N}'$ (from Lemma A.3 (iii))

and that $(\xi_n)^{\theta, \omega}(\omega') \uparrow \xi^{\theta, \omega}(\omega') \forall \omega' \in \Omega$ (from the everywhere convergence $\xi_n \uparrow \xi$), we obtain from the monotone convergence theorem again that for $\omega \in \Omega \setminus \left(\left(\bigcup_{n \in \mathbb{N}} \overline{N}_n \right) \cup \overline{N} \cup \overline{N}' \right)$,

$$\mathbb{E}[\xi \mid \mathcal{F}_\theta](\omega) = \lim_{n \rightarrow \infty} \mathbb{E}[\xi_n \mid \mathcal{F}_\theta](\omega) = \lim_{n \rightarrow \infty} \mathbb{E}[(\xi_n)^{\theta, \omega}] = \mathbb{E}[\xi^{\theta, \omega}].$$

The same result holds true for any general $\xi \in L^1(\Omega, \mathcal{F}_T, \overline{\mathbb{P}})$ as $\xi = \xi^+ - \xi^-$. \square

A.1. Proof of Proposition 2.1.

Proof. (i) Set $\Lambda := \{A \subseteq \Omega \mid \overline{\mathbb{P}}(A \cap B) = \overline{\mathbb{P}}(A)\overline{\mathbb{P}}(B) \forall B \in \mathcal{F}_t\}$. It can be checked that Λ is a σ -algebra of Ω . Take $A \in \phi_t^{-1}\mathcal{H}_T^t \cup \overline{N}$. If $A \in \overline{N}$, it is trivial that $A \in \Lambda$; if $A = \phi_t^{-1}C$ with $C \in \mathcal{H}_T^t$, then for any $B \in \mathcal{F}_t$,

$$\overline{\mathbb{P}}(A \cap B) = \overline{\mathbb{P}}(B \cap \phi_t^{-1}C) = \mathbb{E}[\overline{\mathbb{P}}(B \cap \phi_t^{-1}C \mid \mathcal{F}_t)] = \mathbb{E}[\overline{\mathbb{P}}(B \cap \phi_t^{-1}C \mid \mathcal{F}_t)(\omega)1_B(\omega)].$$

By Proposition A.1 (i), for $\overline{\mathbb{P}}$ -a.e. $\omega \in \Omega$, $\overline{\mathbb{P}}(B \cap \phi_t^{-1}C \mid \mathcal{F}_t)(\omega) = \overline{\mathbb{P}}[\phi_t^{-1}(B \cap \phi_t^{-1}C)^{t, \omega}] = \overline{\mathbb{P}}[\phi_t^{-1}C] = \overline{\mathbb{P}}(A)$ if $\omega \in B$. We therefore have $\overline{\mathbb{P}}(A \cap B) = \overline{\mathbb{P}}(A)\overline{\mathbb{P}}(B)$, and conclude $A \in \Lambda$. It follows that $\phi_t^{-1}\mathcal{H}_T^t \cup \overline{N} \subseteq \Lambda$, which implies $\mathcal{F}_T^t = \sigma(\phi_t^{-1}\mathcal{H}_T^t \cup \overline{N}) \subseteq \Lambda$. Thus, \mathcal{F}_T^t and \mathcal{F}_t are independent.

(ii) Let Δ denote the set operation of symmetric difference. Set $\Lambda := \{A \subseteq \Omega \mid (\phi_t^{-1}A^{t, \omega})\Delta A \in \overline{N} \text{ for } \overline{\mathbb{P}}\text{-a.e. } \omega \in \Omega\}$. It can be checked that Λ is a σ -algebra of Ω . Take $A \in \phi_t^{-1}\mathcal{H}_T^t \cup \overline{N}$. If $A \in \overline{N}$, we see from Lemma A.3 (i) that $A \in \Lambda$; if $A = \phi_t^{-1}C$ with $C \in \mathcal{H}_T^t$, then $\phi_t^{-1}A^{t, \omega} = \phi_t^{-1}C = A$ for all $\omega \in \Omega$, and thus $A \in \Lambda$. We then conclude that $\mathcal{F}_T^t = \sigma(\phi_t^{-1}\mathcal{H}_T^t \cup \overline{N}) \subseteq \Lambda$.

Take a sequence $\{\xi_n\}$ of random variables in $L^0(\Omega, \mathcal{F}_T^t)$ taking countably many values $\{r_i\}_{i \in \mathbb{N}}$ such that $\xi_n(\omega) \rightarrow \xi(\omega)$ for all $\omega \in \Omega$. This everywhere convergence implies that for any fixed $\omega \in \Omega$, $(\xi_n)^{t, \omega}(\omega') \rightarrow \xi^{t, \omega}(\omega')$ for all $\omega' \in \Omega$. Now, fix $n \in \mathbb{N}$. For each $i \in \mathbb{N}$, since $(\xi_n)^{-1}\{r_i\} \in \mathcal{F}_T^t \subseteq \Lambda$, there exists $\overline{N}_i^n \in \overline{N}$ such that for $\omega \in \Omega \setminus \overline{N}_i^n$,

$$\left([(\xi_n)^{t, \omega}]^{-1}\{r_i\} \right) \Delta (\xi_n)^{-1}\{r_i\} = \left[\phi_t^{-1}((\xi_n)^{-1}\{r_i\})^{t, \omega} \right] \Delta (\xi_n)^{-1}\{r_i\} =: \overline{M}_i^n \in \overline{N}, \quad (\text{A.9})$$

where the first equality follows from the calculation in the proof of Lemma A.3 (iii). Then, we deduce from (A.9) that: for any fixed $\omega \in \Omega \setminus \bigcup_{i \in \mathbb{N}} \overline{N}_i^n$, $(\xi_n)^{t, \omega}(\omega') = \xi_n(\omega')$ for all $\omega' \in \Omega \setminus \bigcup_{i \in \mathbb{N}} \overline{M}_i^n$. It follows that: for any fixed $\omega \in \Omega \setminus \bigcup_{i, n \in \mathbb{N}} \overline{N}_i^n$, $(\xi_n)^{t, \omega}(\omega') = \xi_n(\omega')$ for all $\omega' \in \Omega \setminus \bigcup_{i, n \in \mathbb{N}} \overline{M}_i^n$ and $n \in \mathbb{N}$. Setting $\overline{N} = \bigcup_{i, n \in \mathbb{N}} \overline{N}_i^n$ and $\overline{M} = \bigcup_{i, n \in \mathbb{N}} \overline{M}_i^n$, we obtain that for any $\omega \in \Omega \setminus \overline{N}$,

$$\xi(\omega') = \lim_{n \rightarrow \infty} \xi_n(\omega') = \lim_{n \rightarrow \infty} (\xi_n)^{t, \omega}(\omega') = \xi^{t, \omega}(\omega'), \text{ for } \omega' \in \Omega \setminus \overline{M}.$$

\square

A.2. Proof of Proposition 2.2.

Proof. Take a sequence of stopping times $\{\tau_i\}_{i \in \mathbb{N}} \subset \mathcal{T}$ such that τ_i takes values in $\{m/2^i \mid m \in \mathbb{N}\}$ for each $i \in \mathbb{N}$ and $\tau_i(\omega) \downarrow \tau(\omega)$ for all $\omega \in \Omega$ (thanks to [18, Problem 1.2.24]). Set $\overline{N} := \{\tau < \theta\} \in \overline{N}$. Since $\tau_i(\omega) \downarrow \tau(\omega)$ for all $\omega \in \Omega$, we have $\tau_i \geq \theta$ on $\Omega \setminus \overline{N}$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $r_m^i := m/2^i$, $m \in \mathbb{N}$. Since $\{\tau_i \leq r_m^i\} \in \mathcal{F}_{r_m^i}$ for all $m \in \mathbb{N}$, we deduce from Lemma A.3 (ii) and the countability of $\{r_m^i\}_{m \in \mathbb{N}}$ that there exists $\overline{N}^i \in \overline{N}$ such that for $\omega \in \Omega \setminus \overline{N}^i$,

$$\text{if } \theta(\omega) \leq r_m^i, \phi_\theta^{-1}\{\tau_i \leq r_m^i\}^{\theta, \omega} \in \mathcal{F}_{r_m^i}^{\theta(\omega)} \text{ for all } m \in \mathbb{N}. \quad (\text{A.10})$$

Fix $r \in [0, T]$. For any $\omega \in \Omega \setminus (\overline{N} \cup \overline{N}^i)$, if $\theta(\omega) > r$, then $\tau_i(\omega) \geq \theta(\omega) > r$ and thus $\phi_\theta^{-1}\{\tau_i \leq r\}^{\theta, \omega} = \phi_\theta^{-1}\emptyset = \emptyset \in \mathcal{F}_r^{\theta(\omega)}$; if $\theta(\omega) \leq r$, there are two cases: **1.** $\exists m^* \in \mathbb{N}$ s.t. $r_{m^*}^i \in [\theta(\omega), r]$ and $r_{m^*+1}^i > r$. Then, by (A.10), $\phi_\theta^{-1}\{\tau_i \leq r\}^{\theta, \omega} = \phi_\theta^{-1}\{\tau_i \leq r_{m^*}^i\}^{\theta, \omega} \in \mathcal{F}_{r_{m^*}^i}^{\theta(\omega)} \subset \mathcal{F}_r^{\theta(\omega)}$; **2.** $\exists m^* \in \mathbb{N}$ s.t. $r_{m^*}^i < \theta(\omega)$ and $r_{m^*+1}^i > r$. Since $\tau_i(\omega) \geq \theta(\omega) > r_{m^*}^i$, $\phi_\theta^{-1}\{\tau_i \leq r\}^{\theta, \omega} = \phi_\theta^{-1}\{\tau_i \leq r_{m^*}^i\}^{\theta, \omega} = \phi_\theta^{-1}\emptyset = \emptyset \in \mathcal{F}_r^{\theta(\omega)}$. Thus, for $\omega \in \Omega \setminus (\overline{N} \cup \overline{N}^i)$, we have $\phi_\theta^{-1}\{\tau_i \leq r\}^{\theta, \omega} \in \mathcal{F}_r^{\theta(\omega)}$, and therefore

$$\{\tau_i^{\theta, \omega} \leq r\} = \{\tau_i(\omega \otimes_\theta \phi_\theta(\omega')) \leq r\} = \phi_\theta^{-1}\{\tau_i \leq r\}^{\theta, \omega} \in \mathcal{F}_r^{\theta(\omega)}, \quad \forall r \in [0, T].$$

This shows that $\tau_i^{\theta, \omega} \in \mathcal{T}_{\theta(\omega), T}^{\theta(\omega)}$ for $\omega \in \Omega \setminus (\overline{N} \cup \overline{N}^i)$. Hence, for $\omega \in \Omega \setminus (\overline{N} \cup (\bigcup_{i \in \mathbb{N}} \overline{N}^i))$, we have $\tau_i^{\theta, \omega} \in \mathcal{T}_{\theta(\omega), T}^{\theta(\omega)} \quad \forall i \in \mathbb{N}$. Finally, since the filtration $\mathbb{F}^{\theta(\omega)}$ is right-continuous, $\tau^{\theta, \omega}(\omega') = \downarrow \lim_{i \rightarrow \infty} \tau_i^{\theta, \omega}(\omega')$ (this is true since $\tau_i \downarrow \tau$ everywhere) must also be a stopping time in $\mathcal{T}_{\theta(\omega), T}^{\theta(\omega)}$. \square

A.3. Proof of Proposition 2.3. Recall the metric $\tilde{\rho}$ on \mathcal{A} defined in (2.8). We say $\beta \in \mathcal{A}$ is a step control if there exists a subdivision $0 = t_0 < t_1 < \dots < t_m = T$, $m \in \mathbb{N}$, of the interval $[0, T]$ such that $\beta_t = \beta_{t_i}$ for $t \in [t_i, t_{i+1})$ for $i = 0, 1, \dots, m-1$.

Proof. By [24, Lemma 3.2.6], there exist a sequence $\{\alpha^n\}$ of step controls such that $\alpha^n \rightarrow \alpha$. For each $n \in \mathbb{N}$, in view of Proposition 2.1 (ii), there exist $\overline{N}_n, \overline{M}_n \in \overline{\mathcal{N}}$ such that: for any fixed $\omega \in \Omega \setminus \overline{N}_n$, $(\alpha_r^n)^{t, \omega}(\omega') = \alpha_r^n(\omega')$ for $(r, \omega') \in [0, T] \times (\Omega \setminus \overline{M}_n)$. It follows that: for any fixed $\omega \in \Omega \setminus \bigcup_{n \in \mathbb{N}} \overline{N}_n$, $(\alpha_r^n)^{t, \omega}(\omega') = \alpha_r^n(\omega')$ for all $(r, \omega') \in [0, T] \times (\Omega \setminus \bigcup_{n \in \mathbb{N}} \overline{M}_n)$ and $n \in \mathbb{N}$. With the aid of Proposition A.1 (ii), we obtain

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \tilde{\rho}(\alpha^n, \alpha) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \rho'(\alpha_r^n, \alpha_r) dr \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left(\mathbb{E} \left[\int_0^T \rho'(\alpha_r^n, \alpha_r) dr \mid \mathcal{F}_t \right] (\omega) \right) \\ &= \lim_{n \rightarrow \infty} \int \int \left(\int_0^T \rho'(\alpha_r^n, \alpha_r) dr \right)^{t, \omega} (\omega') d\overline{\mathbb{P}}(\omega') d\overline{\mathbb{P}}(\omega) \\ &= \lim_{n \rightarrow \infty} \int \int \int_0^T \rho' \left((\alpha_r^n)^{t, \omega}(\omega'), \alpha_r^{t, \omega}(\omega') \right) dr d\overline{\mathbb{P}}(\omega') d\overline{\mathbb{P}}(\omega) \\ &= \lim_{n \rightarrow \infty} \int \tilde{\rho}((\alpha^n)^{t, \omega}, \alpha^{t, \omega}) d\overline{\mathbb{P}}(\omega) = \lim_{n \rightarrow \infty} \int \tilde{\rho}(\alpha^n, \alpha^{t, \omega}) d\overline{\mathbb{P}}(\omega) = \int \lim_{n \rightarrow \infty} \tilde{\rho}(\alpha^n, \alpha^{t, \omega}) d\overline{\mathbb{P}}(\omega), \end{aligned}$$

where the last equality is due to the dominated convergence theorem. This implies that $0 = \lim_{n \rightarrow \infty} \tilde{\rho}(\alpha^n, \alpha^{t, \omega})$, for $\overline{\mathbb{P}}$ -a.e. $\omega \in \Omega$. Recalling that $\alpha^n \rightarrow \alpha$, we conclude that $\tilde{\rho}(\alpha^{t, \omega}, \alpha) = 0$ for $\overline{\mathbb{P}}$ -a.e. $\omega \in \Omega$. The second assertion follows immediately from [24, Exercise 3.2.4]. \square

A.4. Proof of Lemma 3.2.

Proof. By taking $\xi = F(\mathbf{X}_\tau^{t, \mathbf{x}, \alpha})$ in Proposition A.1 (ii) and using Remark 2.5 (ii),

$$\begin{aligned} \mathbb{E}[F(\mathbf{X}_\tau^{t, \mathbf{x}, \alpha}) \mid \mathcal{F}_\theta](\omega) &= \mathbb{E} \left[F(\mathbf{X}_\tau^{t, \mathbf{x}, \alpha})^{\theta, \omega} \right] = \int F(\mathbf{X}_\tau^{t, \mathbf{x}, \alpha}(\omega \otimes_\theta \phi_\theta(\omega'))) d\overline{\mathbb{P}}(\omega') \\ &= \int F \left(\mathbf{X}_{\tau, \omega}^{\theta(\omega), \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}(\omega), \alpha^{\theta, \omega}}(\omega') \right) d\overline{\mathbb{P}}(\omega') = J \left(\theta(\omega), \mathbf{X}_\theta^{t, \mathbf{x}, \alpha}(\omega); \alpha^{\theta, \omega}, \tau^{\theta, \omega} \right), \text{ for } \overline{\mathbb{P}}\text{-a.e. } \omega \in \Omega. \end{aligned}$$

\square

REFERENCES

- [1] E. BAYRAKTAR AND A. FAHIM, *A stochastic approximation for fully nonlinear free boundary problems*, tech. rep., University of Michigan, 2011. Available at <http://arxiv.org/abs/1109.5752>.
- [2] E. BAYRAKTAR, I. KARATZAS, AND S. YAO, *Optimal stopping for dynamic convex risk measures*, Illinois Journal of Mathematics, A special volume in honor of Donald Burkholder, 54 (2010), pp. 1025–1067.
- [3] E. BAYRAKTAR AND S. YAO, *Optimal stopping for non-linear expectations—Part I*, Stochastic Process. Appl., 121 (2011), pp. 185–211.
- [4] ———, *Optimal stopping for non-linear expectations—Part II*, Stochastic Process. Appl., 121 (2011), pp. 212–264.
- [5] E. BAYRAKTAR AND V. R. YOUNG, *Proving regularity of the minimal probability of ruin via a game of stopping and control*, Finance and Stochastics, 15 (2011), pp. 785–818.
- [6] B. BOUCHARD, *Introduction to stochastic control of mixed diffusion processes, viscosity solutions and applications in finance and insurance*, Lecture Notes Preprint, 2007. Available at <http://www.ceremade.dauphine.fr/~bouchard/pdf/PolyContSto.pdf>.
- [7] B. BOUCHARD, L. MOREAU, AND M. NUTZ, *Stochastic target games with controlled loss*, tech. rep., Université Paris Dauphine and Columbia University, 2012. Available at <http://arxiv.org/abs/1206.6325>.
- [8] B. BOUCHARD AND N. TOUZI, *Weak dynamic programming principle for viscosity solutions*, SIAM Journal on Control and Optimization, 49 (2011), pp. 948–962.
- [9] R. BUCKDAHN AND J. LI, *Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations*, SIAM Journal on Control and Optimization, 47 (2008), pp. 444–475.
- [10] M. G. CRANDALL, H. ISHII, AND P.-L. LIONS, *User’s guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.
- [11] J. DUGUNDJI, *Topology*, Allyn and Bacon Inc., Boston, Mass., 1966.
- [12] R. J. ELLIOTT AND N. J. KALTON, *Boundary value problems for nonlinear partial differential operators*, Journal of Mathematical Analysis and Applications, 46 (1974), pp. 228–241.
- [13] ———, *Cauchy problems for certain Isaacs-Bellman equations and games of survival*, Transactions of the American Mathematical Society, 198 (1974), pp. 45–72.
- [14] W. H. FLEMING AND P. E. SOUGANIDIS, *On the existence of value functions of two-player, zero-sum stochastic differential games*, Indiana Univ. Math. J., 38 (1989), pp. 293–314.
- [15] S. HAMADÈNE, *Mixed zero-sum stochastic differential game and American game options*, SIAM Journal on Control and Optimization, 45 (2006), pp. 496–518.
- [16] S. HAMADÈNE AND J.-P. LEPELTIER, *Reflected BSDEs and mixed game problem*, Stochastic Processes and their Applications, 85 (2000), pp. 177–188.
- [17] I. KARATZAS AND S. G. KOU, *Hedging American contingent claims with constrained portfolios*, Finance Stoch., 2 (1998), pp. 215–258.
- [18] I. KARATZAS AND S. E. SHREVE, *Brownian motion and stochastic calculus*, vol. 113 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 1991.
- [19] ———, *Methods of mathematical finance*, vol. 39 of Applications of Mathematics (New York), Springer-Verlag, New York, 1998.
- [20] I. KARATZAS AND W. D. SUDDERTH, *The controller-and-stopper game for a linear diffusion*, Ann. Probab., 29 (2001), pp. 1111–1127.
- [21] I. KARATZAS AND H. WANG, *A barrier option of American type*, Appl. Math. Optim., 42 (2000), pp. 259–279.
- [22] I. KARATZAS AND I.-M. ZAMFIRESCU, *Game approach to the optimal stopping problem*, Stochastics, 77 (2005), pp. 401–435.
- [23] ———, *Martingale approach to stochastic differential games of control and stopping*, Ann. Probab., 36 (2008), pp. 1495–1527.

- [24] N. V. KRYLOV, *Controlled diffusion processes*, vol. 14 of Applications of Mathematics, Springer-Verlag, New York, 1980. Translated from the Russian by A. B. Aries.
- [25] A. P. MAITRA AND W. D. SUDDERTH, *The gambler and the stopper*, in Statistics, probability and game theory, vol. 30 of IMS Lecture Notes Monogr. Ser., Inst. Math. Statist., Hayward, CA, 1996, pp. 191–208.
- [26] J. R. MUNKRES, *Topology: a first course*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.
- [27] M. NUTZ, *A quasi-sure approach to the control of non-markovian stochastic differential equations*, Electronic Journal of Probability, 17 (2012), pp. 1–23.
- [28] H. PHAM, *Optimal stopping of controlled jump diffusion processes: a viscosity solution approach*, Journal of Mathematical Systems, Estimation, and Control, 8 (1998), p. 27 pp. (electronic).
- [29] H. PHAM, *Continuous-time stochastic control and optimization with financial applications*, vol. 61 of Stochastic Modelling and Applied Probability, Springer-Verlag, Berlin, 2009.
- [30] D. W. STROOCK, *Probability theory*, Cambridge University Press, Cambridge, second ed., 2011.
- [31] D. W. STROOCK AND S. R. S. VARADHAN, *Multidimensional diffusion processes*, Classics in Mathematics, Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.
- [32] A. WEERASINGHE, *A controller and a stopper game with degenerate variance control*, Electronic Communications in Probability, 11 (2006), pp. 89–99 (electronic).

(Erhan Bayraktar) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 530 CHURCH STREET, ANN ARBOR, MI 48109, USA

E-mail address: `erhan@umich.edu`

(Yu-Jui Huang) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 530 CHURCH STREET, ANN ARBOR, MI 48109, USA

E-mail address: `jayhuang@umich.edu`