The Multi-player Nonzero-sum Dynkin Game in Continuous Time

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Abstract

In this paper we study the N-player nonzero-sum Dynkin game $(N \ge 3)$ in continuous time, which is a non-cooperative game where the strategies are stopping times. We show that the game has a Nash equilibrium point for general payoff processes.

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1 Introduction

A Dynkin game is a game where the controllers make use of stopping times as control actions. Actually assume one has N players denoted by $\pi_1, ..., \pi_N$ and each of which is allowed, according to its advantages, to stop the evolution of a system. The system can be for example an option contract which binds several agents (players) in a financial market. So for i = 1, ..., N, assume that the player π_i makes the decision to stop the system at τ_i , then its corresponding yield is given by:

$$J_i(\tau_1, \cdots, \tau_N) := \mathbf{E}[X^i_{\tau_i} \mathbf{1}_{\{\tau_i < R_i\}} + Q^i_{\tau_i} \mathbf{1}_{\{\tau_i = R_i\}} + Y^i_{R_i} \mathbf{1}_{\{\tau_i > R_i\}}]$$
(1.1)

where $R_i := \min\{\tau_j, j \neq i\}$ and X^i, Q^i, Y^i are stochastic processes described precisely below. This yield depends actually on whether π_i is the first to stop the evolution of the

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system or not. So the main problem we are interested in is to find a Nash equilibrium point (hereafter *NEP* for short) for the game, i.e., an *N*-uplet of stopping times $(\tau_1^*, ..., \tau_N^*)$ such that for any i = 1, ..., N, for any τ_i ,

$$J_i(\tau_1^*, \cdots, \tau_N^*) \ge J_i(\tau_1^*, \dots, \tau_{i-1}^*, \tau_i, \tau_{i+1}^*, \dots, \tau_N^*).$$

A NEP is a collective strategy of stopping for the players which has the feature that if one of them decides unilaterally to change a strategy of stopping then it is penalized.

In the case when N = 2 and $J_1 + J_2 = 0$, the game is called of *zero-sum* type and the corresponding NEP is called a *saddle-point* for the game. Otherwise it is called of *nonzero-sum* type.

The first works related to Dynkin games, which concern mainly the zero-sum setting, go back to several decades before (see e.g. [2, 3, 8, 11, 16, 17, 20, 25, 21, 27], etc. and the references given there). This latter setting was revisited several years later by many authors especially in connection with reflected backward equations [6], the pricing of American game contingent claims introduced by Y. Kifer (see e.g. [15, 12, 14, 18], etc.), convertible bonds (see e.g. [5, 24]) or other reasons ([19, 26]), etc.

In comparison, nonzero-sum Dynkin games in continuous time, introduced by Bensoussan and Friedman in [1] have attracted few research activities (see e.g. [1, 4, 10, 13, 23], etc.). Moreover, on the one hand, those papers deal with the case of two players (even in [1]) and, on the other hand, the assumptions on the data of the problem are rather tough since, except a recent paper by S.Hamadène and J.Zhang [13], authors assume that e.g. the processes Y^i of (1.1) are supermartingales. Even in the discrete time setting, this latter assumption is supposed (e.g. in [22]).

So the main objective of this paper is to study the continuous time nonzero-sum Dynkin game in the case when there are more than two players and general stochastic processes X^i , Y^i and Q^i , i = 1, ..., N. The processes Y^i are no longer supposed being supermartingales. We actually show this game has a Nash equilibrium point under minimal assumptions. We should point out that according to our best knowledge this problem has been not considered yet and the generalization from the two-player setting to the multi-player one, even with respect to the work by Hamadène-Zhang [13], is not formal and raises questions which are far from to be obvious, especially the construction of the approximating scheme and the study of its properties.

This paper is organized as follows. In Section 2, we set accurately the problem, recall the Snell envelope notion and provide a result (Prop. 2.1) which is in a way the streamline in the construction of the of the NEP of the nonzero-sum Dynkin game. The approximating scheme and its main properties are introduced in Section 3. Finally in Section 4, we show that the limit of the approximating scheme is a NEP for the game.

2 Setting of the problem and hypotheses

Throughout this paper T is a positive real constant which stands for the horizon of the problem and $(\Omega, \mathcal{F}, \mathcal{P})$ is a fixed probability space on which is defined a filtration $\mathbf{F} := (\mathcal{F}_t)_{t \leq T}$ which satisfies the usual conditions, *i.e.*, it is complete and right continuous.

Next for any stopping time $\theta \leq T$, let us denote by:

(i) \mathcal{T}_{θ} the set of **F**-stopping times τ such that P-a.s. $\tau \in [\theta, T]$;

(ii) $E_{\theta}[.]$ the conditional expectation w.r.t. \mathcal{F}_{θ} , i.e., $\mathbf{E}_{\theta}[X] := \mathbf{E}[X|\mathcal{F}_{\theta}]$, for any integrable random variable X;

(iii) $\mathcal{J} := \{1, ..., N\}.$

Next an **F**-progressively measurable \mathbb{R} -valued stochastic process $(\zeta_t)_{t\leq T}$ is called of class [D] if the set of random variables $\{\zeta_{\tau}, \tau \in \mathcal{T}_0\}$ is uniformly integrable. Now for i = 1, ..., N, let us introduce **F**-progressively measurable and \mathbb{R} -valued stochastic processes of class [D], $X^i := (X^i_t)_{t\leq T}, Q^i := (Q^i_t)_{t\leq T}$ and $Y^i := (Y^i_t)_{t\leq T}$, which moreover satisfy the following hypotheses:

Assumption 2.1 : For any $i = 1, \ldots, N$,

(A1): X^i is right continuous with left limits (RCLL for short) and does not have negative predictable jumps;

(A2): The process Y^i is right continuous, i.e., P-a.s., for any t < T, $Y^i_t = \lim_{s \searrow t} Y^i_s$ and of class [D]. As a consequence if $(\gamma_n)_{n\ge 0}$ is a decreasing sequence of **F**-stopping times then $\mathbf{E}[Y^i_{\lim n \gamma_n}] = \lim_n \mathbf{E}[Y^i_{\gamma_n}]$;

(A3): The processes X^i , Y^i and Q^i verify:

$$P-a.s. \ \forall t \leq T, X_t^i \leq Q_t^i \leq Y_t^i;$$

(A4): For all $\tau \in \mathcal{T}_0$, P-a.s.

$$(Q^i_\tau < Y^i_\tau; \tau < T) \subseteq \bigcap_{j=1}^N (X^j_\tau < Y^j_\tau).$$

Note that assumptions (A1)-(A3) are minimal in order to solve the problem, as for [A4], it is satisfied for e.g. for any $i \in \mathcal{J}$ and $\tau \in \mathcal{T}_0$, P-a.s. on $(\tau < T)$: $X^i_{\tau} < Y^i_{\tau}$ or $Q^i_{\tau} = Y^i_{\tau}$.

Next for T_1, T_2, \dots, T_N elements of \mathcal{T}_0 and for $i \in \mathcal{J}$, let us define $J_i(T_1, T_2, \dots, T_N)$, the payoff associated with the player i, as follows:

$$J_i(T_1, T_2, \cdots, T_N) := \mathbf{E} \Big\{ X_{T_i}^i \mathbf{1}_{\{T_i < R_i\}} + Q_{T_i}^i \mathbf{1}_{\{T_i = R_i\}} + Y_{R_i}^i \mathbf{1}_{\{T_i > R_i\}} \Big\}.$$
 (2.1)

where $R_i := \min\{T_j, j \neq i\} = \wedge_{j=1,N; j \neq i} T_j$.

The meaning of those payoffs in this nonzero-sum Dynkin game framework is the following: Assume that for any i = 1, ..., N, the player π_i makes the decision to use the stopping time T_i as a strategy of stopping. Let $i_0 \in \mathcal{J}$, then:

- π_{i_0} will receive an amount equal to $X_{T_{i_0}}^{i_0}$ if it decides unilaterally to stop controlling first. As for the other players π_j , $j \neq i_0$, each one will receive an amount which equals to $Y_{T_{i_0}}^j$;

- π_{i_0} will receive an amount equal to $Q_{T_{i_0}}^{i_0}$ if there is a commitment with one or more other players to stop first the game. Each one of the players j which do not get involved in the commitment will receive $Y_{T_{i_0}}^j$.

We next precise the notion of equilibrium that we are looking for.

Definition 2.2 We say that $(T_1^*, T_2^*, \dots, T_N^*) \in \mathcal{T}_0^N$ is a Nash equilibrium point of the Nonzero-sum Dynkin game associated with $(J_i)_{i \in \mathcal{J}}$ if for all $i = 1, \dots, N$ and all $T_1, \dots, T_N \in \mathcal{T}_0$ we have:

$$J_i(T_1^*, \cdots, T_{i-1}^*, T_i, T_{i+1}^*, \cdots, T_N^*) \le J_i(T_1^*, \cdots, T_{i-1}^*, T_i^*, T_{i+1}^*, \cdots, T_N^*).$$
(2.2)

The definition means that when the equilibrium is reached, is penalized each one of the players which makes the decision to change unilaterally its strategy of stopping. \Box

Next to begin with we give a result which in way is a streamline in order to construct a NEP for the nonzero-sum Dynkin game. Actually we have:

Proposition 2.1 Assume there exist N stopping times $(\tau_i^*)_{i=1,N}$ and N **F**-progressively measurable RCLL processes $(W^i)_{i=1,N}$, such that for any i = 1, ..., N, if we set $R_i = \tau_1^* \wedge ... \wedge \tau_{i-1}^* \wedge \tau_{i+1}^* \wedge ... \wedge \tau_N^*$ and $R = \tau_1^* \wedge ... \wedge \tau_N^* = \tau_i^* \wedge R_i$ then:

(i) $(W_{t\wedge R}^i)_{t\leq T}$ is an **F**-martingale and $(W_{t\wedge R_i}^i)_{t\leq T}$ is an **F**-supermatingale,

(*ii*)
$$\forall t \leq T, W_t^i \mathbb{1}_{\{t < R_i\}} \geq X_t^i \mathbb{1}_{\{t < R_i\}} \text{ and } W_{\tau_i^*}^i \mathbb{1}_{\{\tau_i^* < R_i\}} = X_{\tau_i^*}^i \mathbb{1}_{\{\tau_i^* < R_i\}},$$

(*iii*) $W_{R_i}^i = Y_{R_i}^i \mathbb{1}_{\{R_i < T\}} + Q_T^i \mathbb{1}_{\{R_i = T\}} \text{ and } (Y_{R_i}^i - Q_{R_i}^i) \mathbb{1}_{\{R_i = \tau_i^* < T\}} = 0.$

Then the N-uplet of stopping times $(\tau_i^*)_{i=1,N}$ is a Nash equilibrium point for the nonzerosum Dynkin game associated with $(J_i)_{i\in\mathcal{J}}$. Moreover for any $i\in\mathcal{J}$,

$$J_i((\tau_i^*)_{i=1,N}) = \mathbf{E}[W_0^i].$$

Proof: Actually for any $i \in \mathcal{J}$, since $(W_{t \wedge R}^i)_{t \leq T}$ is a martingale then

$$\mathbf{E}[W_0^i] = \mathbf{E}[W_R^i]
= \mathbf{E}[W_{\tau_i^*}^i \mathbb{1}_{\{\tau_i^* < R_i\}} + W_{R_i}^i \mathbb{1}_{\{\tau_i^* = R_i\}} + W_{R_i}^i \mathbb{1}_{\{\tau_i^* > R_i\}}].$$
(2.3)

But taking into account the equalities of (iii) we get:

$$\begin{split} W_{R_{i}}^{i} 1\!\!1_{\{\tau_{i}^{*}=R_{i}\}} + W_{R_{i}}^{i} 1\!\!1_{\{\tau_{i}^{*}>R_{i}\}} \\ &= (Y_{R_{i}}^{i} 1\!\!1_{\{R_{i}R_{i}\}} \\ &= Q_{R_{i}}^{i} 1\!\!1_{\{R_{i}=\tau_{i}^{*}$$

Making now the substitution in (2.3) we deduce that:

$$\begin{split} \mathbf{E}[W_0^i] &= \mathbf{E}[X_{\tau_i^*}^i \mathbbm{1}_{\{\tau_i^* < R_i\}} + Q_{\tau_i^*}^i \mathbbm{1}_{\{\tau_i^* = R_i\}} + Y_{R_i}^i \mathbbm{1}_{\{\tau_i^* > R_i\}}] \\ &= J_i(\tau_1^*, ..., \tau_N^*). \end{split}$$

On the other hand since $(W_{t \wedge R_i}^i)_{t \leq T}$ is an **F**-supermatingale then for any stopping time $\gamma \in \mathcal{T}_0$ we have,

$$\mathbf{E}[W_0^i] \geq \mathbf{E}[W_{\gamma \wedge R_i}^i] \\
= \mathbf{E}[W_{\gamma}^i \mathbb{1}_{\{\gamma < R_i\}} + W_{R_i}^i \mathbb{1}_{\{\gamma = R_i\}} + W_{R_i}^i \mathbb{1}_{\{\gamma > R_i\}}].$$
(2.4)

But once more

$$\begin{split} & W_{R_{i}}^{i} \mathbb{1}_{\{\gamma = R_{i}\}} + W_{R_{i}}^{i} \mathbb{1}_{\{\gamma > R_{i}\}} \\ &= (Y_{R_{i}}^{i} \mathbb{1}_{\{R_{i} < T\}} + Q_{T}^{i} \mathbb{1}_{\{R_{i} = T\}}) \mathbb{1}_{\{\gamma = R_{i}\}} + (Y_{R_{i}}^{i} \mathbb{1}_{\{R_{i} < T\}} + Q_{T}^{i} \mathbb{1}_{\{R_{i} = T\}}) \mathbb{1}_{\{\gamma > R_{i}\}} \\ &= (Y_{R_{i}}^{i} \mathbb{1}_{\{R_{i} < T\}} + Q_{T}^{i} \mathbb{1}_{\{R_{i} = T\}}) \mathbb{1}_{\{\gamma = R_{i}\}} + Y_{R_{i}}^{i} \mathbb{1}_{\{\gamma > R_{i}\}} \\ &\geq Q_{R_{i}}^{i} \mathbb{1}_{\{\gamma = R_{i}\}} + Y_{R_{i}}^{i} \mathbb{1}_{\{\gamma > R_{i}\}} \end{split}$$

since $Y^i \ge Q^i$. Plugging now this last term in (2.4) and since $W^i_{\gamma} \mathbb{1}_{\{\gamma < R_i\}} \ge X^i_{\gamma} \mathbb{1}_{\{\gamma < R_i\}}$ we obtain:

$$J_{i}(\tau_{1}^{*},...,\tau_{N}^{*}) = \mathbf{E}[W_{0}^{i}]$$

$$\geq \mathbf{E}[X_{\gamma}^{i}1_{\{\gamma < R_{i}\}} + Q_{R_{i}}^{i}1_{\{\gamma = R_{i}\}} + Y_{R_{i}}^{i}1_{\{\gamma > R_{i}\}}] = J_{i}(\tau_{1}^{*},...,\tau_{i-1}^{*},\gamma,\tau_{i+1}^{*},...,\tau_{N}^{*}).$$

Thus the N-uplet of stopping times $(\tau_i^*)_{i=1,N}$ is a Nash equilibrium for the nonzero-sum Dynkin game.

To tackle the game problem, we mainly use the notion of Snell envelope of processes which we introduce briefly now. For more details on this subject one can refer e.g. to El-Karoui [9] or Dellacherie and Meyer [7].

Theorem 1 ([7], pp. 431 or [9], pp. 140) : Let $U = (U_t)_{0 \le t \le T}$ be an **F**-adapted *R*-valued RCLL process that belongs to class [D]. Then there exists $Z := (Z_t)_{0 \le t \le T}$ an **F**-adapted *R*-valued RCLL process of class [D], such that Z is the smallest super-martingale which dominates U, i.e, if $(\bar{Z}_t)_{0 \le t \le T}$ is another RCLL supermartingale of class [D] such that $\bar{Z} \ge U$ then $\bar{Z} \ge Z$, P-a.s.. The process Z is called the Snell envelope of U. It satisfies the following properties:

(i) For any **F**-stopping time θ we have:

$$Z_{\theta} = \operatorname{essup}_{\tau \in \mathcal{T}_{\theta}} \mathbf{E}[U_{\tau} | \mathcal{F}_{\theta}] \quad (and \ then \ Z_T = U_T).$$
(2.5)

(ii) If the predictable jumps of U are only positive, then the stopping time

$$\tau^* = \inf\{s \ge 0, \quad Z_s = U_s\}$$

is optimal, i.e.,

$$\mathbf{E}[Z_0] = \mathbf{E}[Z_{\tau^*}] = \mathbf{E}[U_{\tau^*}] = \sup_{\tau \ge 0} \mathbf{E}[U_{\tau}].$$
(2.6)

Remark 2.3 As a by-product of (2.6) we have $Z_{\tau^*} = U_{\tau^*}$ and the process $(Z_{t \wedge \tau^*})_{t \leq T}$ is a martingale.

3 The approximating scheme and its properties

We are now going to introduce sequences of stopping times which, as we will show it later, converge to a NEP of the game. So let us consider the sequence of **F**-stopping times $(\tau_n)_{n\geq 1}$ defined, by induction, as follows:

(i) $\tau_1 = \cdots \tau_N = T$. (ii) For $n \ge N+1$, let $(i,q) = (i_n,q_n) \in \mathbb{N}^2$ be such that n = Nq + i with $1 \le i \le N$. Then let us set:

$$\begin{aligned} &- \theta_{n} = \min\{\tau_{n-1}, \tau_{n-2}, \cdots, \tau_{n-N+1}\}; \\ &- U_{t}^{n} = X_{t}^{i} \mathbf{1}_{\{t < \theta_{n}\}} + \tilde{Y}_{\theta_{n}}^{i} \mathbf{1}_{\{t \ge \theta_{n}\}} \quad \text{with } \tilde{Y}_{t}^{i} = Q_{T}^{i} \mathbf{1}_{\{t=T\}} + Y_{t}^{i} \mathbf{1}_{\{t$$

Remark 3.1 As a direct consequence of the above definitions and the properties of the Snell envelope, the following relations or properties hold true: for any $n \ge N + 1$, (i) W^n is RCLL and for any $t \ge \theta_n$,

$$W_t^n = U_t^n = \widetilde{Y}_{\theta_n}^i;$$

(ii)

$$\mu_n \leq \theta_n, \ \tau_n \leq \tau_{n-N} \ and \ \theta_n \leq \theta_{n-N};$$

(iii) since the predictable jumps of U^n are only positive and taking into account Assumption (A3), we deduce from Theorem 1-(ii) that μ_n is optimal, i.e.,

$$\mathbf{E}[W_0^n] = \mathbf{E}[W_{\mu_n}^n] = \mathbf{E}[U_{\mu_n}^n] = \sup_{\tau \ge 0} \mathbf{E}[U_{\tau}^n].$$

(iv) Let n = Nq + i where the pair (i, q) is as above, therefore even if this is not explicitly mentioned in the definition, the stopping time $\theta_n = \theta_{Nq+i}$ depends on *i*. The same happens for U^n , τ_n , μ_n and W^n . \Box

Additionally we have:

Proposition 3.1 For any $n \ge 1$, $\mu_{n+N} \le \tau_n$, *P*-a.s..

Proof: Actually suppose there exists $m \ge 1$ such that $P[\tau_m < \mu_{m+N}] > 0$ and let us set $n = \min\{m \ge 1 \text{ s.t. } P[\tau_m < \mu_{m+N}] > 0\}$. Then we obviously have $n \ge N + 1$. Next on the set $\{\tau_n < \mu_{n+N}\}$ we have:

$$\tau_n < \theta_{n+N} := \tau_{n+N-1} \wedge \tau_{n+N-2} \wedge \cdots \tau_{n+1} \tag{3.1}$$

since $\mu_{n+N} \leq \theta_{n+N}$ (Rem.3.1-(ii)). But the definition of *n* implies that $\mu_{n+N-1} \leq \tau_{n-1}$ and then

$$\tau_{n+N-1} = \mu_{n+N-1} \mathbf{1}_{\{\mu_{n+N-1} < \theta_{n+N-1}\}} + \tau_{n-1} \mathbf{1}_{\{\mu_{n+N-1} = \theta_{n+N-1}\}}$$

and from (3.1) and the definition of θ_{n+N-1} we deduce that $\theta_{n+N-1} = \tau_n$. It follows that:

$$\tau_{n+N-1} = \mu_{n+N-1} \mathbf{1}_{\{\mu_{n+N-1} < \tau_n\}} + \tau_{n-1} \mathbf{1}_{\{\mu_{n+N-1} = \tau_n\}}$$

Therefore

$$\tau_n < \tau_{n+N-1} = \tau_{n-1}. \tag{3.2}$$

The strict inequality stems from (3.1) as for the equality it holds true since $\mu_{n+N-1} \leq \theta_{n+N-1} = \tau_n$ and $\tau_n < \tau_{n+N-1}$.

Next

$$\theta_{n+N-2} := \tau_{n+N-3} \wedge \tau_{n+N-4} \wedge \cdots \tau_{n+1} \wedge \tau_n \wedge \tau_{n-1} = \tau_n$$

since from (3.1) for any k = 1, ..., N - 1 we have $\tau_n < \tau_{n+k}$ and from (3.2) $\tau_n < \tau_{n-1}$. But once more the definitions of n and τ_{n+N-2} imply that:

$$\tau_{n+N-2} = \mu_{n+N-2} \mathbf{1}_{\{\mu_{n+N-2} < \tau_n\}} + \tau_{n-2} \mathbf{1}_{\{\mu_{n+N-2} = \tau_n\}}$$

As we know that $\tau_{n+N-2} > \tau_n$ then $\tau_{n+N-2} = \tau_{n-2} > \tau_n$.

Repeating now this procedure as many times as necessary we deduce that for any $j = 1, \ldots, N-1$,

$$\tau_n < \tau_{n+N-j} = \tau_{n-j}$$

and then on the set $\{\tau_n < \mu_{n+N}\}$ we have

$$\tau_n < \theta_{n+N} = \theta_n$$

and thanks to the definitions of n and τ_n we also have on $\Gamma := \{\tau_n < \mu_{n+N}\}$

$$\tau_n = \mu_n \mathbf{1}_{\{\mu_n < \theta_n\}} + \tau_{n-N} \mathbf{1}_{\{\mu_n = \theta_n\}} = \mu_n$$

since $\mu_n \leq \tau_{n-N}$. Therefore on the set $\Gamma \in \mathcal{F}_{\tau_n}$ we have $U^n = U^{n+N}$ since $\theta_{n+N} = \theta_n$ and

$$\begin{split} \mathbb{1}_{\Gamma} W_{\mu_n}^{n+N} &= \mathbb{1}_{\Gamma} W_{\tau_n}^{n+N} = \mathbb{1}_{\Gamma} \operatorname{esssup}_{\nu \in \mathcal{T}_{\tau_n}} \mathbf{E}[U_{\nu}^{n+N} | \mathcal{F}_{\tau_n}] = \operatorname{esssup}_{\nu \in \mathcal{T}_{\tau_n}} \mathbf{E}[\mathbb{1}_{\Gamma} U_{\nu}^{n} | \mathcal{F}_{\tau_n}] \\ &= \operatorname{esssup}_{\nu \in \mathcal{T}_{\tau_n}} \mathbf{E}[\mathbb{1}_{\Gamma} U_{\nu}^{n} | \mathcal{F}_{\tau_n}] \\ &= \mathbb{1}_{\Gamma} W_{\tau_n}^n = \mathbb{1}_{\Gamma} W_{\mu_n}^n \\ &= \mathbb{1}_{\Gamma} U_{\mu_n}^n \\ &= \mathbb{1}_{\Gamma} U_{\mu_n}^{n+N}, \end{split}$$

i.e., $\mathbb{1}_{\Gamma}W_{\mu_n}^{n+N} = \mathbb{1}_{\Gamma}U_{\mu_n}^{n+N}$ and then $\mu_{n+N} \leq \mu_n$ on Γ since μ_{n+N} is the first time that W^{n+N} reaches U^{n+N} . As on Γ we have $\mu_n = \tau_n < \mu_{n+N}$ then this is contradictory with the previous inequality. It follows that $P[\Gamma] = 0$ and for any $m \geq 1$ we have $\mu_{m+N} \leq \tau_m$, P-a.s.. The proof is complete.

As a by-product we obtain:

Corollary 3.2 For any $n \ge N+1$,

(*i*)
$$\tau_n = \mu_n \mathbf{1}_{\{\mu_n < \theta_n\}} + \tau_{n-N} \mathbf{1}_{\{\mu_n = \theta_n\}}$$
;
(*ii*) $\mu_n = \tau_n \land \theta_n = \tau_n \land \tau_{n-1} \land \cdots \tau_{n-N+1}$

Proof: Indeed, (i) is a direct consequence of the previous proposition and the definition of τ_n . As for (ii), we have

$$\tau_n \wedge \theta_n = \tau_n \mathbb{1}_{\{\tau_n < \theta_n\}} + \theta_n \mathbb{1}_{\{\tau_n \ge \theta_n\}}.$$

But $\tau_n \mathbb{1}_{\{\tau_n < \theta_n\}} = \mu_n \mathbb{1}_{\{\mu_n < \theta_n\}}$ and on $[\tau_n \ge \theta_n]$ we have $\theta_n = \mu_n$. Therefore $\theta_n \mathbb{1}_{\{\tau_n \ge \theta_n\}} = \mu_n \mathbb{1}_{\{\tau_n \ge \theta_n\}}$. Gathering now those relations yields $\mu_n = \tau_n \land \theta_n$. Finally the second equality is just the definition of θ_n .

4 Existence of a Nash equilibrium point

For any $i \in \{1, \dots, N\}$, let us define:

$$T_i^* = \lim_{n \to \infty} \tau_{Nn+i}$$
 and $R_i^* = \lim_{n \to \infty} \theta_{Nn+i} = \min\{T_j^*; j \neq i\}$

Those limits exist since for any $n \ge N+1$, we know that $\tau_n \le \tau_{n-N}$ therefore the sequences $(\tau_{Nn+i})_{n\ge 0}$ are non-increasing for fixed *i*.

We have also, for all $i \in \{1, \cdots, N\}$

$$R^* := T_1^* \wedge \dots \wedge T_N^* = \lim_{n \to \infty} \mu_{Nn+i} = \lim_{n \to \infty} \mu_n.\square$$

We are going now to show that the N-uplet of stopping times $(T_i^*)_{i=1,...,N}$ is a Nash equilibrium point for the N-players nonzero-sum Dynkin game associated with $(J_i)_{i \in \mathcal{J}}$. The proof will be obtained after several intermediary results given below.

Lemma 4.1 For any $n \ge 1$, $i \in \{1, \dots, N\}$ and any $\theta \in \mathcal{T}_0$ we have:

$$J_{i}(\tau_{Nn+N+1},\tau_{Nn+N+2},\cdots,\tau_{Nn+N+i-1},\theta,\tau_{Nn+i+1},\cdots,\tau_{Nn+N})$$

$$\leq J_{i}(\tau_{Nn+N+1},\tau_{Nn+N+2},\cdots,\tau_{Nn+N+i-1},\tau_{Nn+N+i},\tau_{Nn+i+1},\cdots,\tau_{Nn+N})$$

$$+\mathbf{E}[(Y_{\tau_{Nn+N+i}}^{i}-Q_{\tau_{Nn+N+i}}^{i})1_{\{\tau_{Nn+N+i}=\theta_{Nn+N+i}< T\}}].$$

Proof: For any $n \ge 1, i \in \{1, \cdots, N\}$ and $\theta \in \mathcal{T}_0$,

$$J_i(\tau_{Nn+N+1},\tau_{Nn+N+2},\cdots,\tau_{Nn+N+i-1},\theta,\tau_{Nn+i+1},\cdots,\tau_{Nn+N}) = E\Big\{X_{\theta}^i \mathbb{1}_{\{\theta < \theta_{Nn+N+i}\}} + Q_{\theta}^i \mathbb{1}_{\{\theta = \theta_{Nn+N+i}\}} + Y_{\theta_{Nn+N+i}}^i \mathbb{1}_{\{\theta > \theta_{Nn+N+i}\}}\Big\}.$$

But

$$X^{i}_{\theta} \mathbb{1}_{\{\theta < \theta_{Nn+N+i}\}} \le W^{Nn+N+i}_{\theta} \mathbb{1}_{\{\theta < \theta_{Nn+N+i}\}}$$

and since $Q^i \leq Y^i$ we have

$$Q_{\theta}^{i} \mathbb{1}_{\{\theta=\theta_{Nn+N+i}\}} + Y_{\theta_{Nn+N+i}}^{i} \mathbb{1}_{\{\theta>\theta_{Nn+N+i}\}}$$

$$\leq (Q_{T}^{i} \mathbb{1}_{\{\theta_{Nn+N+i}=T\}} + Y_{\theta_{Nn+N+i}}^{i} \mathbb{1}_{\{T>\theta_{Nn+N+i}\}}) \mathbb{1}_{\{\theta\geq\theta_{Nn+N+i}\}}$$

$$\leq W_{\theta_{Nn+N+i}}^{Nn+N+i} \mathbb{1}_{\{\theta\geq\theta_{Nn+N+i}\}}.$$

Therefore

$$J_i(\tau_{Nn+N+1}, \tau_{Nn+N+2}, \cdots, \tau_{Nn+N+i-1}, \theta, \tau_{Nn+i+1}, \cdots, \tau_{Nn+N}) \leq \mathbf{E}[W_{\theta \land \theta_{Nn+N+i}}^{Nn+N+i}] \leq \mathbf{E}[W_0^{Nn+N+i}]$$

$$(4.3)$$

since W^{Nn+N+i} is a supermartingale.

Next

$$J_{i}(\tau_{Nn+N+1},\tau_{Nn+N+2},\cdots,\tau_{Nn+N+i-1},\tau_{Nn+N+i},\tau_{Nn+i+1},\cdots,\tau_{Nn+N})$$

$$= \mathbf{E}[U_{\tau_{Nn+N+i}\wedge\theta_{Nn+N+i}}^{Nn+N+i}] + \mathbf{E}[(Q_{\theta_{Nn+N+i}}^{i} - Y_{\theta_{Nn+N+i}}^{i}) \mathbb{1}_{(\tau_{Nn+N+i}=\theta_{Nn+N+i}

$$= E[U_{\mu_{Nn+i+1}}^{Nn+N+i}] + \mathbf{E}[(Q_{\theta_{Nn+N+i}}^{i} - Y_{\theta_{Nn+N+i}}^{i}) \mathbb{1}_{\{\tau_{Nn+N+i}=\theta_{Nn+N+i}

$$= E[W_{\mu_{Nn+i+1}}^{Nn+N+i}] + \mathbf{E}[(Q_{\theta_{Nn+N+i}}^{i} - Y_{\theta_{Nn+N+i}}^{i}) \mathbb{1}_{\{\tau_{Nn+N+i}=\theta_{Nn+N+i}

$$= \mathbf{E}[W_{0}^{Nn+N+i}] + \mathbf{E}[(Q_{\theta_{Nn+N+i}}^{i} - Y_{\theta_{Nn+N+i}}^{i}) \mathbb{1}_{\{\tau_{Nn+N+i}=\theta_{Nn+N+i}

$$= \mathbf{E}[W_{0}^{Nn+N+i}] + \mathbf{E}[(Q_{\theta_{Nn+N+i}}^{i} - Y_{\theta_{Nn+N+i}}^{i}) \mathbb{1}_{\{\tau_{Nn+N+i}=\theta_{Nn+N+i}$$$$$$$$$$

since $\tau_{Nn+N+i} \wedge \theta_{Nn+N+i} = \mu_{Nn+i+1}$ (Cor. 4.1-(ii)), $(W_{t \wedge \mu_{Nn+i+1}}^{Nn+N+i})_{t \leq T}$ is a martingale (Remark 2.3) and finally by (i) of Remark 3.1. Comparing now (4.3) and (4.4) to obtain the desired result.

We now focus on the limits of the terms that appear in the inequality of the previous lemma.

Lemma 4.2 : The following asymptotic inequalities hold true: (i) For all $\theta \in \mathcal{T}_0$ and $i \in \mathcal{J}$, we have:

$$\lim_{n \to \infty} J_i(\tau_{Nn+N+1}, \tau_{Nn+N+2}, \cdots, \tau_{Nn+N+i-1}, \theta, \tau_{Nn+i+1}, \cdots, \tau_{Nn+N}) = J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, \theta, T_{i+1}^*, \cdots, T_N^*) - \mathbf{E}[(Q_{\theta}^i - X_{\theta}^i) \ 1 \bigcap_{n \ge 0} \{\theta = R_i^* < \theta_{Nn+i}\}].$$

$$\underline{\lim}_{n \to \infty} \left(J_i(\tau_{Nn+N+1}, \tau_{Nn+N+2}, \cdots, \tau_{Nn+N+i-1}, \tau_{Nn+N+i}, \tau_{Nn+i+1}, \cdots, \tau_{Nn+N}) + \mathbf{E}[(Y_{\tau_{Nn+N+i}}^i - Q_{\tau_{Nn+N+i}}^i) 1_{\{\tau_{Nn+N+i} = \theta_{Nn+N+i} < T\}}] \right) = J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, T_i^*, T_{i+1}^*, \cdots, T_N^*) + \mathbf{E}[(Y_{T_i^*}^i - Q_{T_i^*}^i) 1_{\{T_i^* = R_i^* < T\}}] + \underline{\lim}_{n \to \infty} \mathbf{E}[(X_{T_i^*}^i - Y_{T_i^*}^i) 1_{\{\tau_{Nn+N+i} < \theta_{Nn+N+i}, T_i^* = R_i^* < T\}}].$$

Proof: (i) Actually

$$J_{i}(\tau_{Nn+N+1},\tau_{Nn+N+2},\cdots,\tau_{Nn+N+i-1},\theta,\tau_{Nn+i+1},\cdots,\tau_{Nn+N})$$

$$= \mathbf{E}[X_{\theta}^{i}\mathbb{1}_{\{\theta \leq \theta_{Nn+N+i}\}} + Y_{\theta_{Nn+N+i}}^{i}\mathbb{1}_{\{\theta > \theta_{Nn+N+i}\}} + (Q_{\theta}^{i} - X_{\theta}^{i})\mathbb{1}_{\{\theta = \theta_{Nn+N+i}\}}].$$

$$(4.5)$$

As the process Y^i is RCLL and of class [D] and the sequence $(\theta_{Nn+N+i})_n$ is decreasing then, when $n \to \infty$,

$$\mathbf{E}[X_{\theta}^{i}1\!\!1_{\{\theta \le \theta_{Nn+N+i}\}} + Y_{\theta_{Nn+N+i}}^{i}1\!\!1_{\{\theta > \theta_{Nn+N+i}\}}] \to \mathbf{E}[X_{\theta}^{i}1\!\!1_{\{\theta \le R_{i}^{*}\}} + Y_{R_{i}^{*}}^{i}1\!\!1_{\{\theta > R_{i}^{*}\}}].$$
(4.6)

On the other hand, when $n \to \infty$,

$$\mathbf{E}[(Q_{\theta}^{i} - X_{\theta}^{i})\mathbb{1}_{\{\theta = \theta_{Nn+N+i}\}}] \to \mathbf{E}[(Q_{\theta}^{i} - X_{\theta}^{i})\mathbb{1}_{\{\theta = R_{i}^{*}\}}] - \mathbf{E}[(Q_{\theta}^{i} - X_{\theta}^{i})\mathbb{1}_{\{\theta = \theta_{Nn+N+i}\}}] \cap \mathbf{E}[(Q_{\theta}^{i} - X_{\theta}^{i})\mathbb{1}_{\{\theta = \theta_{Nn+N+i}\}}].$$

$$(4.7)$$

Actually (4.7) is obtained in paying attention whether the sequence $(\theta_{Nn+N+i})_n$ is of stationary type or not. Going back now to (4.5), take the limit and make use of (4.6) and (4.7) to obtain the desired result. \Box

Next let us focus on (ii). Let $i \in \mathcal{J}$ be fixed, then:

$$\underline{\lim}_{n \to \infty} \left(J_i(\tau_{Nn+N+1}, \tau_{Nn+N+2}, \cdots, \tau_{Nn+N+i-1}, \tau_{Nn+N+i}, \tau_{Nn+i+1}, \cdots, \tau_{Nn+N}) + \mathbf{E}[(Y^i_{\tau_{Nn+N+i}} - Q^i_{\tau_{Nn+N+i}}) \mathbb{1}_{\{\tau_{Nn+N+i} = \theta_{Nn+N+i} < T\}}] \right)$$

$$= \underline{\lim}_{n \to \infty} E[X^{i}_{\tau_{Nn+N+i}} 1\!\!1_{\{\tau_{Nn+N+i} < \theta_{Nn+N+i}\}} + Y^{i}_{\theta_{Nn+N+i}} 1\!\!1_{\{\tau_{Nn+N+i} > \theta_{Nn+N+i}\}} + Y^{i}_{\theta_{Nn+N+i}} 1\!\!1_{\{\tau_{Nn+N+i} = \theta_{Nn+N+i} < T\}} + Q^{i}_{T} 1\!\!1_{\{\tau_{Nn+N+i} = \theta_{Nn+N+i} = T\}}]$$

$$= \underline{\lim}_{n \to \infty} E[X^{i}_{\tau_{Nn+N+i}} \ \mathbb{1}_{\{\tau_{Nn+N+i} < \theta_{Nn+N+i}; \ T^{*}_{i} \le R^{*}_{i}\}} + Y^{i}_{\theta_{Nn+N+i}} \ \mathbb{1}_{\{\tau_{Nn+N+i} > \theta_{Nn+N+i}; \ T^{*}_{i} \ge R^{*}_{i}\}} + Y^{i}_{\theta_{Nn+N+i}} \ \mathbb{1}_{\{\tau_{Nn+N+i} = \theta_{Nn+N+i}; \ T^{*}_{i} = R^{*}_{i} < T\}} + Q^{i}_{T} \ \mathbb{1}_{\{T^{*}_{i} = R^{*}_{i} = T\}}]$$

$$= E[X_{T_i^*}^i \ \mathbb{1}_{\{T_i^* < R_i^*\}} + Q_{T_i^*}^i \ \mathbb{1}_{\{T_i^* = R_i^*\}} + Y_{R_i^*}^i \ \mathbb{1}_{\{T_i^* > R_i^*\}}] - \mathbf{E}[Q_{T_i^*}^i \ \mathbb{1}_{\{T_i^* = R_i^* < T\}}] + \underbrace{\lim_{n \to \infty} E[X_{T_i^*}^i \ \mathbb{1}_{\{\tau_{Nn+N+i} < \theta_{Nn+N+i}; \ T_i^* = R_i^*\}} + Y_{R_i^*}^i \ \mathbb{1}_{\{\tau_{Nn+N+i} \ge \theta_{Nn+N+i}; \ T_i^* = R_i^* < T\}}]$$

$$= J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, T_i^*, T_{i+1}^*, \cdots, T_N^*) + E[(Y_{T_i^*}^i - Q_{T_i^*}^i) \ \mathbf{1}_{\{T_i^* = R_i^* < T\}}] + \underbrace{\lim_{n \to \infty} \mathbf{E}[(X_{T_i^*}^i - Y_{T_i^*}^i) \ \mathbf{1}_{\{\tau_{Nn+N+i} < \theta_{Nn+N+i}, T_i^* = R_i^* < T\}}]$$

which is the desired result. Note that in the fourth inequality we have taken into account the fact that the processes X^i and Y^i are of RCLL and of class [D].

An obvious consequence of Lemmata 4.1 and 4.2 is:

Corollary 4.1 For all $\theta \in \mathcal{T}_0$ and all $i \in \mathcal{J}$

$$J_{i}(T_{1}^{*}, T_{2}^{*}, \cdots, T_{i-1}^{*}, \theta, T_{i+1}^{*}, \cdots, T_{N}^{*}) - \mathbf{E}[(Q_{\theta}^{i} - X_{\theta}^{i}) \ 1\!\!1_{\bigcap_{n \ge 0}} \{\theta = R_{i}^{*} < \theta_{Nn+i}\}]$$

$$\leq J_{i}(T_{1}^{*}, T_{2}^{*}, \cdots, T_{i-1}^{*}, T_{i}^{*}, T_{i+1}^{*}, \cdots, T_{N}^{*}) + E[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) \ 1\!\!1_{\{T_{i}^{*} = R_{i}^{*} < T\}}]$$

$$+ \underline{\lim}_{n \to \infty} \mathbf{E}[(X_{T_{i}^{*}}^{i} - Y_{T_{i}^{*}}^{i}) \ 1\!\!1_{\{\tau_{Nn+N+i} < \theta_{Nn+N+i}, T_{i}^{*} = R_{i}^{*} < T\}}]. \qquad \blacksquare$$

$$(4.8)$$

Lemma 4.3 (i) We have:

$$\lim_{n \to \infty} \mathbf{E}[(X_{T_i^*}^i - Y_{T_i^*}^i) \, \mathbb{1}_{\{\tau_{Nn+N+i} < \theta_{Nn+N+i}, \, T_i^* = R_i^* < T\}}] = 0$$

and then for all $\varepsilon > 0$

$$\lim_{n \to \infty} P[Y_{T_i^*}^i - X_{T_i^*}^i > \varepsilon, \ \tau_{Nn+N+i} < \theta_{Nn+N+i}, \ T_i^* = R_i^* < T] = 0.$$

(ii) For all $\theta \in \mathcal{T}_0$ and all $i \in \mathcal{J}$,

$$\begin{aligned} J_{i}(T_{1}^{*}, T_{2}^{*}, \cdots, T_{i-1}^{*}, \theta, T_{i+1}^{*}, \cdots, T_{N}^{*}) + E[(Y_{R_{i}^{*}}^{i} - Q_{R_{i}^{*}}^{i}) 1_{\{\theta = R_{i}^{*} < T\}}] \\ & \leq J_{i}(T_{1}^{*}, T_{2}^{*}, \cdots, T_{i-1}^{*}, T_{i}^{*}, T_{i+1}^{*}, \cdots, T_{N}^{*}) + \mathbf{E}[(Y_{R_{i}^{*}}^{i} - Q_{R_{i}^{*}}^{i}) 1_{\{T_{i}^{*} = R_{i}^{*} < T\}}]. \end{aligned}$$

Proof: (i) Actually let θ be the following **F**-stopping time:

$$\theta = T_i^* 1_{\{T_i^* < R_i^*\}} + T 1_{\{T_i^* \ge R_i^*\}}$$

Then using inequality (4.8) yields:

$$\begin{split} J_{i}(T_{1}^{*}, T_{2}^{*}, \cdots, T_{i-1}^{*}, \theta, T_{i+1}^{*}, \cdots, T_{N}^{*}) &- \mathbf{E}[(Q_{\theta}^{i} - X_{\theta}^{i}) \ \mathbb{1} \bigcap_{n \geq 0} \{\theta = R_{i}^{*} < \theta_{Nn+i}\}] \\ &= J_{i}(T_{1}^{*}, T_{2}^{*}, \cdots, T_{i-1}^{*}, T_{i}^{*}, T_{i+1}^{*}, \cdots, T_{N}^{*}) + E[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) \ \mathbb{1}_{\{T_{i}^{*} = R_{i}^{*} < T\}}] \\ &\leq J_{i}(T_{1}^{*}, T_{2}^{*}, \cdots, T_{i-1}^{*}, T_{i}^{*}, T_{i+1}^{*}, \cdots, T_{N}^{*}) + \mathbf{E}[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) \ \mathbb{1}_{\{T_{i}^{*} = R_{i}^{*} < T\}}] \\ &+ \underline{\lim}_{n \to \infty} \mathbf{E}[(X_{T_{i}^{*}}^{i} - Y_{T_{i}^{*}}^{i}) \ \mathbb{1}_{\{\tau_{Nn+N+i} < \theta_{Nn+N+i}, T_{i}^{*} = R_{i}^{*} < T\}}]. \end{split}$$

Hence

$$\lim_{n \to \infty} \mathbf{E}[(X_{T_i^*}^i - Y_{T_i^*}^i) 1_{\{\tau_{Nn+N+i} < \theta_{Nn+N+i}, T_i^* = R_i^* < T\}}] \ge 0$$

which completes the proof since $X_{T_i^*}^i - Y_{T_i^*}^i \leq 0$ thanks to Assumption (A3).

(ii) Let $\tilde{\theta}$ be the following **F**-stopping time:

$$\widetilde{\theta} = \theta \ \mathbbm{1}_{\{\theta = R_i^* < T\}^c} + T \ \mathbbm{1}_{\{\theta = R_i^* < T\}}$$

where the superscript (^c) stands for the complement. Since $P[\tilde{\theta} = R_i^* < T] = 0$ we obtain from (4.8) and (i),

$$\begin{split} J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, \widetilde{\theta}, T_{i+1}^*, \cdots, T_N^*) &- \mathbf{E}[(Q_{\widetilde{\theta}}^i - X_{\widetilde{\theta}}^i) \ 1\!\!1 \bigcap_{n \ge 0} \{\widetilde{\theta} = R_i^* < \theta_{Nn+i} \}^{]} \\ &= J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, \theta, T_{i+1}^*, \cdots, T_N^*) + E[(Y_{R_i^*}^i - Q_{R_i^*}^i) \ 1\!\!1_{\{\theta = R_i^* < T\}}] \\ &\le J_i(T_1^*, T_2^*, \cdots, T_{i-1}^*, T_i^*, T_{i+1}^*, \cdots, T_N^*) + E[(Y_{R_i^*}^i - Q_{R_i^*}^i) \ 1\!\!1_{\{T_i^* = R_i^* < T\}}], \end{split}$$

whence the desired result.

We now give a key-result which allows us to conclude.

Proposition 4.1 Under Assumption (A4), for all $i \in \mathcal{J}$ we have:

$$E[(Y_{T_i^*}^i - Q_{T_i^*}^i) 1_{\{T_i^* = R_i^* < T\}}] = 0.$$
(4.9)

Proof: First note that for any $i \in \mathcal{J}$ we have:

$$\begin{split} \mathbf{E}[(Y_{T_i^*}^i - Q_{T_i^*}^i) 1\!\!1_{\{T_i^* = R_i^* < T\}}] \leq \\ \sum_{I = \{i_1, \dots, i_k\} \subset \mathcal{J}, \ i \in I \text{ and } k \geq 2} E[(Y_{T_i^*}^i - Q_{T_i^*}^i) 1\!\!1_{\{T_{i_1}^* = T_{i_2}^* = \dots = T_{i_k}^* < R_I^*\}}\}, \end{split}$$

where $R_I^* = \min\{T_j^*; j \notin I\}$ with $\min \emptyset = T$. Therefore it is enough to show that for any $i_1, ..., i_k \in \mathcal{J}$, which we assume w.l.o.g satisfying $i_1 < i_2 < \cdots < i_k$, we have:

$$E[(Y_{T_i^*}^i - Q_{T_i^*}^i) \mathbb{1}_{\{T_{i_1}^* = T_{i_2}^* = \dots = T_{i_k}^* < R_I^*\}}] = 0$$

for any $i \in I = \{i_1, \dots, i_k\}.$

Step 1: For any $n \ge 0$,

$$P[A_n := \bigcap_{j=1}^k \{R_I^* > \tau_{Nn+i_j} \ge \theta_{Nn+i_j}\}] = 0.$$
(4.10)

Actually, first note that $P[A_0] = P[A_1] = 0$. Next let us show that $A_n \subset A_{n-1}$ for any $n \ge 2$.

On A_n :

By the definitions of τ_n and R_I^* , we have: $\forall j \in \{1, \dots, k\}, \forall \alpha \notin I$,

$$\tau_{Nn+i_j} = \tau_{N(n-1)+i_j} \quad \text{and} \quad \tau_{Nn+i_j} < R_I^* \le \tau_{Nn+\alpha} \le \tau_{N(n-\ell)+\alpha}, \quad \ell \ge 0.$$
(4.11)

Therefore in using those properties we deduce that:

$$\theta_{Nn+i_j} := \tau_{Nn+i_j-1} \wedge \tau_{Nn+i_j-2} \wedge \dots \wedge \tau_{Nn+i_j-N+1}$$

$$= \tau_{Nn+i_{j-1}} \wedge \tau_{Nn+i_{j-2}} \wedge \dots \wedge \tau_{Nn+i_1} \wedge \tau_{Nn+i_k-N} \wedge \tau_{Nn+i_{k-1}-N} \wedge \dots \wedge \tau_{Nn+i_{j+1}-N}$$

$$(4.12)$$

Let us give briefly the justification of the second equality. Indeed for some $\ell \in \{1, ..., N-1\}$ either $i_j - \ell > 0$ or $i_j - \ell \leq 0$. Case (i): $i_j - \ell > 0$. Then if $i_j - \ell \in \{i_1, ..., i_{j-1}\}$ then we keep it in the expression of θ_{Nn+i_j} and if $i_j - \ell \notin \{i_1, ..., i_{j-1}\}$ then we know from (4.11) that, e.g., $\tau_{Nn+i_j-\ell} \geq \tau_{Nn+i_{j-1}}$ and then $\tau_{Nn+i_j-\ell}$ is deleted from the expression of θ_{Nn+i_j} . Case (ii): $i_j - \ell \leq 0$. Then $\tau_{Nn+i_j-\ell} = \tau_{N(n-1)+i_j-\ell+N}$ with $i_j - \ell + N \geq i_j + 1$. Once more if $i_j - \ell + N \in \{i_{j+1}, ..., i_k\}$ then we keep $\tau_{Nn+i_j-\ell}$ it in the expression of θ_{Nn+i_j} . Otherwise, i.e., if $i_j - \ell + N \notin \{i_{j+1}, ..., i_k\}$, then $\tau_{Nn+i_j-\ell} = \tau_{N(n-1)+i_j-\ell+N} \geq \tau_{Nn+i_j-\ell+N} \geq \tau_{Nn+i_j-1}$ and $\tau_{Nn+i_j-\ell}$ is deleted from the expression of θ_{Nn+i_j} .

Now the first equality of (4.11) yields:

$$\theta_{Nn+i_j} = \tau_{Nn+i_{j-1}} \wedge \tau_{Nn+i_{j-2}} \wedge \dots \wedge \tau_{Nn+i_1} \wedge \tau_{Nn+i_k} \wedge \tau_{Nn+i_{k-1}} \wedge \dots \wedge \tau_{Nn+i_{j+1}}.$$
(4.13)

Next by a backward induction argument we have that for any $j \in \{1, \dots, k\}$,

$$\tau_{Nn+i_j} = \tau_{N(n-1)+i_j} = \tau_{N(n-2)+i_j}$$
 and $\theta_{Nn+i_j} = \theta_{N(n-1)+i_j}$.

Actually for j = k, by (4.11) and (4.13) we have:

$$\tau_{Nn+i_k} = \tau_{N(n-1)+i_k} \ge \theta_{Nn+i_k} = \tau_{Nn+i_{k-1}} \land \tau_{Nn+i_{k-2}} \land \dots \land \tau_{Nn+i_1}$$
$$= \tau_{N(n-1)+i_{k-1}} \land \tau_{N(n-1)+i_{k-2}} \land \dots \land \tau_{N(n-1)+i_1}$$
$$= \theta_{N(n-1)+i_k}.$$

The last equality holds true since by monotonicity we have $\theta_{N(n-1)+i_k} \ge \theta_{Nn+i_k}$ and by definition

$$\theta_{N(n-1)+i_k} \leq \tau_{N(n-1)+i_{k-1}} \wedge \tau_{N(n-1)+i_{k-2}} \wedge \dots \wedge \tau_{N(n-1)+i_1} = \theta_{Nn+i_k}.$$

Therefore by definition of $\tau_{N(n-1)+i_k}$ we have $\tau_{Nn+i_k} = \tau_{N(n-1)+i_k} = \tau_{N(n-2)+i_k}$. Thus the property is satisfied for j = k.

Assume now that the property is satisfied for $j = k, .., \ell + 1$ $(2 \le \ell + 1 \le k)$ and let us show it is also valid for $j = \ell$. From (4.11) and (4.13) we have

$$\tau_{Nn+i_{\ell}} = \tau_{N(n-1)+i_{\ell}} \ge \theta_{Nn+i_{\ell}}$$

and

$$\theta_{Nn+i_{\ell}} = \tau_{Nn+i_{\ell-1}} \wedge \tau_{Nn+i_{\ell-2}} \wedge \cdots \wedge \tau_{Nn+i_1} \wedge \tau_{Nn+i_k} \wedge \tau_{Nn+i_{k-1}} \wedge \cdots \wedge \tau_{Nn+i_{\ell+1}}$$

On the other hand

$$\theta_{N(n-1)+i_{\ell}} = \tau_{N(n-1)+i_{\ell}-1} \wedge \tau_{N(n-1)+i_{\ell}-2} \wedge \dots \wedge \tau_{N(n-1)+i_{\ell}-N+1}$$

= $\tau_{Nn+i_{\ell-1}} \wedge \dots \wedge \tau_{Nn+i_{1}} \wedge \tau_{N(n-1)+i_{k}-N} \dots \wedge \tau_{N(n-1)+i_{\ell+1}-N}$

This second equality is obtained in the same way as in (4.12) in using (4.11) and the induction hypothesis. Therefore, once more by the induction hypothesis, we have:

$$\theta_{N(n-1)+i_{\ell}} = \tau_{Nn+i_{\ell-1}} \wedge \dots \wedge \tau_{Nn+i_1} \wedge \tau_{Nn+i_k} \cdots \wedge \tau_{Nn+i_{\ell+1}}$$
$$= \theta_{Nn+i_{\ell}}.$$

It follows that

$$\tau_{Nn+i_{\ell}} = \tau_{N(n-1)+i_{\ell}} \ge \theta_{Nn+i_{\ell}} = \theta_{N(n-1)+i_{\ell}}$$

and then $\tau_{N(n-1)+i_{\ell}} = \tau_{N(n-2)+i_{\ell}}$. Thus the property is satisfied for ℓ .

Therefore for any $j \in \{1, ..., k\}$ we have

$$\tau_{Nn+i_j} = \tau_{N(n-1)+i_j} \ge \theta_{Nn+i_j} = \theta_{N(n-1)+i_j}$$

which implies that $A_n \subset A_{n-1}$, for any $n \ge 1$ and then $P(A_n) = 0$ for any $n \ge 0$ since $P(A_0) = 0$. \Box

Step 2: To proceed let $n \ge 0$ and $\varepsilon > 0$, then we have:

$$\begin{split} E[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) 1\!\!1_{\{T_{i_{1}}^{*} = T_{i_{2}}^{*} = \cdots = T_{i_{k}}^{*} < R_{I}^{*}\}}] \leq \\ \mathbf{E}[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) 1\!\!1_{\bigcap_{j=1}^{k} \{T_{i_{1}}^{*} = T_{i_{2}}^{*} = \cdots = T_{i_{k}}^{*} < R_{I}^{*}; R_{I}^{*} > \tau_{Nn+i_{j}} \ge \theta_{Nn+i_{j}}\}}] + \\ \sum_{j=1}^{k} \mathbf{E}[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) 1\!\!1_{\{T_{i_{1}}^{*} = T_{i_{2}}^{*} = \cdots = T_{i_{k}}^{*} < R_{I}^{*}; R_{I}^{*} \le \tau_{Nn+i_{j}}\}}] + \\ \sum_{j=1}^{k} \mathbf{E}[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) 1\!\!1_{\{Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i} \le \frac{1}{\varepsilon}; Y_{T_{i_{j}^{*}}^{i}}^{ij} - X_{T_{i_{j}^{*}}^{ij}}^{ij} > \varepsilon; T_{i_{1}}^{*} = T_{i_{2}}^{*} = \cdots = T_{i_{k}}^{*} < R_{I}^{*}; \tau_{Nn+i_{j}} < \theta_{Nn+i_{j}}\}}] + \\ \sum_{j=1}^{k} \mathbf{E}[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) 1\!\!1_{\{Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i} > \frac{1}{\varepsilon}; Y_{T_{i_{j}^{*}}^{ij}}^{ij} - X_{T_{i_{j}^{*}}^{ij}}^{ij} > \varepsilon; T_{i_{1}}^{*} = T_{i_{2}}^{*} = \cdots = T_{i_{k}}^{*} < R_{I}^{*}; \tau_{Nn+i_{j}} < \theta_{Nn+i_{j}}\}}] + \\ \sum_{j=1}^{k} \mathbf{E}[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) 1\!\!1_{\{Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{ij} > \frac{1}{\varepsilon}; Y_{T_{i_{j}^{*}}^{ij}}^{ij} - X_{T_{i_{j}^{*}}^{ij}}^{ij} > \varepsilon; T_{i_{1}^{*}}^{i} = T_{i_{2}}^{*} = \cdots = T_{i_{k}}^{*} < R_{I}^{*}; \tau_{Nn+i_{j}} < \theta_{Nn+i_{j}}\}}]. \end{split}$$

Therefore, in taking into account (4.10), we have:

$$\begin{split} E[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) \ 1\!\!1_{\{T_{i_{1}}^{*} = T_{i_{2}}^{*} = \cdots = T_{i_{k}}^{*} < R_{I}^{*}\}}] \leq \\ \sum_{j=1}^{k} \mathbf{E}[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) \ 1\!\!1_{\{T_{i_{j}}^{*} < R_{I}^{*} \in \tau_{Nn+i_{j}}\}}] + \\ \frac{1}{\varepsilon} \sum_{j=1}^{k} P(Y_{T_{i_{j}}^{i}}^{ij} - X_{T_{i_{j}}^{i}}^{ij}) \geq \varepsilon; \ T_{i_{j}}^{*} = R_{i_{j}}^{*} < T; \ \tau_{Nn+i_{j}} < \theta_{Nn+i_{j}}) + \\ kE[(Y_{T_{i}^{*}}^{i} - X_{T_{i}^{*}}^{i}) \ 1\!\!1_{\{Y_{T_{i}^{*}}^{i} - X_{T_{i}^{*}}^{i} > \frac{1}{\varepsilon}}\}] + \sum_{j=1}^{k} \mathbf{E}[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) \ 1\!\!1_{\{Y_{T_{i}^{*}}^{i} - X_{T_{i}^{*}}^{i} > \frac{1}{\varepsilon}}]] + \sum_{j=1}^{k} \mathbf{E}[(Y_{T_{i}^{*}}^{i} - Q_{T_{i}^{*}}^{i}) \ 1\!\!1_{\{Y_{T_{i}^{*}}^{ij} - X_{T_{i}^{*}}^{ij} \le \varepsilon, T_{i}^{*} < T\}}]. \end{split}$$

Now taking first the limit as $n \to \infty$ and using the second property of Lemma 4.3-(ii) for the third term of the right-hand side, then taking the limit as $\varepsilon \to 0$ to obtain:

$$E[(Y_{T_i^*}^i - Q_{T_i^*}^i) \ \mathbb{1}_{\{T_{i_1}^* = T_{i_2}^* = \dots = T_{i_k}^* < R_I^*\}}] \le \sum_{j=1}^k \mathbf{E}[(Y_{T_i^*}^i - Q_{T_i^*}^i) \ \mathbb{1}_{\{Y_{i_j}^{i_j} - X_{T_i^*}^{i_j} = 0, T_i^* < T\}}]$$

But by Assumption (A4) we have $\mathbf{E}[(Y_{T_i^*}^i - Q_{T_i^*}^i) \mathbbm{1}_{\{Y_{i_j}^{i_j} - X_{T_i^*}^{i_j} = 0, T_i^* < T\}}] = 0$ therefore

$$\mathbf{E}[(Y_{T_i^*}^i - Q_{T_i^*}^i) \, \mathbb{1}_{\{T_{i_1}^* = T_{i_2}^* = \dots = T_{i_k}^* < R_I^*\}}] = 0 \text{ and } E[(Y_{T_i^*}^i - Q_{T_i^*}^i) \, \mathbb{1}_{\{T_i^* = R_i^* < T\}}] = 0$$

which completes the proof of (4.9).

We are now ready to give the main result of this paper which is a direct consequence of Lemma 4.3-(ii), Proposition 4.1 and the fact that $Y^i \ge Q^i$ for any $i \in \mathcal{J}$.

Theorem 4.2 Under assumptions (A1)-(A4), the N-uplet of stopping times $(T_i^*)_{i=1,...,N}$ is a Nash equilibrium point for the N-player nonzero-sum Dynkin game associated with $(J_i)_{i \in \mathcal{J}}$.

Remark 4.3 Note that $(T_i^*)_{i \in \mathcal{J}}$ and $(J_i(T_1^*, T_2^*, \dots, T_N^*))_{i \in \mathcal{J}}$ do not depend on the processes $(Q_t^i)_{t < T}$ for any $i \in \mathcal{J}$. On the other hand the NEP of a Dynkin game is not unique. Actually assume that for any $i \in \mathcal{J}$ and $t \leq T$, $X_t^i = \frac{1}{2}$ and $Q_t^i = Y_t^i = 1$. Therefore one can easily show, directly or in using Proposition 2.1, that for any $t_0 \in [0, T]$, $(t_0, ..., t_0)$ is a NEP for this game and $J_i(t_0, ..., t_0) = 1$, $\forall i \in \mathcal{J}$.

Finally we have the following result related to Proposition 2.1 which is a direct consequence of the fact that $(T_i^*)_{i \in \mathcal{J}}$ is a NEP for the nonzero-sum Dynkin game and, Proposition 1 and Remark 2.3 and Proposition 4.1.

Proposition 4.4 For $i \in \mathcal{J}$, let us set:

$$W^{i} := (W^{i}_{t})_{t \leq T} := \mathbf{R}(X^{i}_{t} \mathbb{1}_{\{t < R^{*}_{i}\}} + \tilde{Y}^{i}_{R^{*}_{i}} \mathbb{1}_{\{t \geq R^{*}_{i}\}})$$

where **R** is the Snell envelope operator. Then $(T_i^*)_{i \in \mathcal{J}}$ and $(W^1, ..., W^N)$ satisfy (i)-(iii) of Proposition 2.1.

References

- [1] Bensoussan, A. Friedman, A.: Nonzero-sum stochastic differential games with stopping times and free boundary value problem. *Trans. A.M.S. vol.213, pp.275-327, no 2 (1977).*
- [2] Bensoussan, A. and Friedman, A.: Non-linear variational inequalities and differential games with stopping times. J. Funct. Anal. 16, pp. 305-352, (1974).
- [3] Bismut, J.M.: Sur un problème de Dynkin. Z. Wahrsch. Verw. Geb., 39 (1977), pp. 31-53.
- [4] Cattiaux, P., Lepeltier, J.P.: Existence of an quasi-Markov Nash equilibrium for non-zero sun Markov stopping games. Stochastics and stochastics reports, vol. 30, no2, pp. 85-103, (1990).
- [5] Bielecki, T.R., Crépey, S., Jeanblanc, M., Rutkowski, M.: Convertible Bonds in a Defaultable Diffusion Model. In Stochastic Analysis with Financial Applications, Progress in Probability, (2011), Vol. 65, Part 2, pp. 255-298, (2011).
- [6] Cvitanic, J. and Karatzas, I. : Backward SDEs with reflection and Dynkin games. Annals of Probability 24 (4), pp. 2024-2056, (1996).

- [7] Dellacherie, C. and Meyer, P. A. : Probabilités et Potentiel, Chapitres 1-8. Hermann, Paris, (1980).
- [8] Dynkin, E.B.: The game variant of a problem on optimal stopping. Soviet Math. Dokl. 10, pp.270-274, (1969).
- [9] El Karoui, N.: Les aspects probabilistes du contrôle stochastique. Ecole d'été de probabilités de Saint-Flour, Lect. Notes in Math. No 876 (1980), Springer Verlag.
- [10] Etourneau, E.: Résolution d'un problème de jeu de somme non nulle sur les temps d'arrêt. Thèse de 3-ième cycle, Université Pierre et Marie Curie, Paris 6 (Fr.), (1986).
- [11] Friedman, A.: Stochastic games and variational inequalities. Arch. Mech. Anal. 51, pp. 321-346 (1973).
- [12] Hamadène, S: Mixed Zero-sum differential game and American game options. SIAM J. Control Optim. 45 (2006), pp. 496-518.
- [13] Hamadène, S., Zhang, J.: The Continuous Time Nonzero-sum Dynkin Game Problem and Application in Game Options. SIAM J. Control Optim. Volume 48, Issue 5, pp. 3659-3669 (2009).
- [14] Kallsen, J., Kuhn, C.: Pricing Derivatives of American and Game Type in Incomplete Markets. Finance & Stochastics, 8(2), pp. 261-284 (2004).
- [15] Kifer, Y.: Game options. Finance and Stochastics 4, pp. 443-463 (2000).
- [16] Krylov, N.V.: The problem with two free boundaries and optimal stopping of Markov processes. Dokl. Akad. Nauk. USSR 194, 1263-1265 (1970).
- [17] Krylov, N.V.: Control of Markov processes and W spaces. Izv. Math. Nauk. USSR 5, 233-266 (1971).
- [18] Kuhn, C.: Game contingent claims in complete and incomplete markets. Journal of Mathematical Economics 40 (2004), pp. 889-902.
- [19] Laraki, R., Solan, E.: The value function of zero-sum stopping games in continuous time, SIAM Journal on Control and Optimization, vol 43, pp. 1913-1922 (2005).
- [20] Lepeltier, J. P., Maingueneau, M.A.: Le Jeu de Dynkin en Théorie Générale sans l'Hypothèse de Mokobodzki. Stochastics, 13 (1984), pp. 25-44.
- [21] Morimoto, H.: Dynkin games and martingale methods. Stochastics, 13 (1984), pp. 213-228.
- [22] Morimoto, H.: Non-Zero-Sum Discrete Parameter Stochastic Games with Stopping Times. Probab. Th. Rel. Fields 72, pp. 155-160 (1986)

- [23] Nagai, H.: Nonzero-sum stopping games of symmetric Markov processes. Probability Theory and Related Fields 75, Number 4, pp.487-497, (1987).
- [24] Sirbu, M., Shreve, S.E.: A two-person game for pricing convertible bonds. SIAM J. Control Optim. Vol. 45, No. 4, pp. 1508-1539, (2006).
- [25] Stettner, L.: Zero-sum Markov games with stopping and impulsive strategies. Appl. Math. Optim., 9 (1982), pp. 1-24.
- [26] Touzi, N., Vieille, N.: Continuous-time Dynkin games with mixed strategies, SIAM Journal on Control and Optimization 41, pp. 1073-1088 (2002).
- [27] Zabczyk, J.: Stopping problems in stochastic control. Proceedings of the International Congress of Mathematicians pp. 1425-1437 (1983).