TIME-DISCRETE HIGHER ORDER ALE FORMULATIONS: STABILITY

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ABSTRACT. Arbitrary Lagrangian Eulerian (ALE) formulations deal with PDEs on deformable domains upon extending the domain velocity from the boundary into the bulk with the purpose of keeping mesh regularity. This arbitrary extension has no effect on the stability of the PDE but may influence that of a discrete scheme. We propose time-discrete discontinuous Galerkin (dG) numerical schemes of any order for a time-dependent advection-diffusion model problem in moving domains, and study their stability properties. The analysis hinges on the validity of the Reynolds' identity for dG. Exploiting the variational structure and assuming exact integration, we prove that our conservative and non-conservative dG schemes are equivalent and unconditionally stable. The same results remain true for piecewise polynomial ALE maps of any degree and suitable quadrature that guarantees the validity of the Reynolds' identity. This approach generalizes the so-called geometric conservation law (GCL) to higher order methods. We also prove that simpler Runge-Kutta-Radau (RKR) methods of any order are conditionally stable, that is subject to a mild ALE constraint on the time steps. Numerical experiments corroborate and complement our theoretical results.

1. INTRODUCTION

Problems governed by partial differential equations (PDEs) on deformable domains $\Omega_t \subset \mathbb{R}^d$, which change in time $0 \leq t \leq T < \infty$, are of fundamental importance in science and engineering, especially for space dimensions $d \geq 2$. The boundary $\partial\Omega_t$ of Ω_t may move according to a law given a priori (moving boundary) or a law we need to solve for (free boundary). The latter are of course more common and much more challenging to study theoretically and solve numerically. This is, for instance, the case of fluid-structure interactions.

Two main classes of algorithms are available, which differ on their treatment of $\partial\Omega_t$. In the first class, a discrete version of $\partial\Omega_t$ moves across a fixed mesh in space (*Eulerian approach*). This requires an additional quantity to track the interface such as a level-set function, a phase-field indicator, an immersed structure (immersed boundary), or a Lagrange multiplier (fictitious domains). In the second class, both the interface and mesh in space move together keeping conformity (*Lagrangian approach*). The latter is advantageous whenever the flow involves higher order geometric quantities, such as curvature or Willmore forces [3, 7, 8], or to design higher order accurate schemes, provided no topological changes are expected. However, pure Lagrangian schemes deform the mesh according to the fluid velocity, which is proned to excessive mesh distorsions and thus require mesh smooting and frequent (and expensive) mesh regeneration. But having direct access to the geometry of $\partial\Omega_t$ and the design of higher order schemes make them quite competitive and of great interest to practitioners.

The *Arbitrary Lagrangian Eulerian (ALE) approach* was introduced in [12, 21, 22] to prevent excessive mesh distortion within the Lagrangian approach. The mesh boundary is deformed according to the prescribed boundary velocity $\mathbf{w}$, but an *arbitrary*, yet adequate, extension is used to

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perform the bulk deformation. This extension of $\mathbf{w}$ from $\partial\Omega_t$ to Ω_t can be performed using various techniques such as solving for a suitable boundary value problem with Dirichlet boundary condition $\mathbf{w}$; see [15, 26, 18, 24], and the references therein. This extension induces a map $\mathcal{A}_t : \Omega_0 \to \Omega_t$, the so-called *ALE map*, with the key property that

$$\mathbf{w}(\mathbf{x}, t) = \frac{d}{dt}\mathcal{A}_t(\mathbf{y}), \qquad \mathbf{x} = \mathcal{A}_t(\mathbf{y}).$$

The ALE velocity $\mathbf{w}$ is unrelated to the fluid velocity $\mathbf{b}$ and dictated mostly by the geometric principle of preserving mesh regularity. The pure Lagrangian approach corresponds to the choice $\mathbf{w} = \mathbf{b}$, whereas in the ALE approach $\mathbf{w} \neq \mathbf{b}$ generically. In the extreme case that the domain does not deform, then $\mathbf{w} = 0$ in $\Omega_t = \Omega_0$ irrespective of the value of $\mathbf{b}$. We refer to [10] for an analysis of the pure Lagrangian approach with emphasis on the convection-dominated diffusion regime.

For the study of the ALE approach we consider, as in [1, 4, 17, 16, 18, 23], a model problem consisting of a prescribed domain deformation Ω_t given by an ALE map $\mathcal{A}_t$ and the scalar advection-diffusion equation on Ω_t with vanishing Dirichlet boundary condition:

$$\begin{cases} \partial_t u + \nabla_{\mathbf{x}} \cdot (\mathbf{b}u) - \mu \Delta_{\mathbf{x}} u = f & \mathbf{x} \in \Omega_t, \ t \in [0, T] \\ u(\mathbf{x}, t) = 0 & \mathbf{x} \in \partial\Omega_t, \ t \in [0, T] \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \mathbf{x} \in \Omega_0. \end{cases} \tag{1.1}$$

Hereafter, $\mu > 0$ is a constant diffusion parameter, $\mathbf{b}$ is a convective velocity, f is a forcing term, and u_0 is an initial condition. Of course this is a prototype PDE for the more interesting, practically relevant, and technically demanding, Navier-Stokes equation for incompressible fluids typical of fluid-structure interactions; notice that in this context it does make sense to consider divergence free velocities: $\nabla_{\mathbf{x}} \cdot \mathbf{b} = 0$. We are not interested in the convection-dominated diffusion regime, in which $\mathbf{b}$ dominates μ, but rather on the design of higher order methods and the effect of the ALE map $\mathcal{A}_t$ on their stability. Multiplying the PDE in (1.1) by u and integrating by parts yields the usual *energy estimate*, provided $\nabla_{\mathbf{x}} \cdot \mathbf{b} = 0$,

$$\|u(t)\|^2_{L^2(\Omega_t)} + \mu \int_\tau^t \|\nabla_{\mathbf{x}} u(s)\|^2_{L^2(\Omega_s)}\, ds \le \|u(\tau)\|^2_{L^2(\Omega_\tau)} + \frac{1}{\mu}\int_\tau^t \|f(s)\|^2_{H^{-1}(\Omega_s)}\, ds, \tag{1.2}$$

for $0 \le \tau \le t \le T$. This estimate is insensitive of the geometry of the deformation built into the ALE map $\mathcal{A}_t$ and exhibits monotone behavior of the norm $\|u(t)\|_{L^2(\Omega_t)}$ provided $f = 0$. We say that a numerical method is *ALE-free stable* with respect to the energy norm if it reproduces (1.2); otherwise, if (1.2) is valid with a stability constant depending on $\mathcal{A}_t$ we say that the method is *ALE stable*. ALE-free stable schemes are desirable because they are qualitatively correct. The only provable ALE-free stable scheme based on finite element discretizations in space and without time-step constraints (*unconditional stability*) is the backward Euler-method [4, 17, 16, 18, 23]. This raises a couple of fundamental questions discussed later in this paper in the context of discontinuous Galerkin (dG) methods for (1.1):

- *Can higher order methods be ALE-free stable?*
- *Can higher order methods be unconditionally stable? If not, then how does the time-step restriction relate to the domain velocity* $\mathbf{w}$ *and the diffusion* μ*?*

The ALE framework is based on replacing the (Eulerian) partial time derivative $\partial_t u$ in (1.1) by the ALE-time derivative (or material derivative), which is the partial derivative along the trajectories induced by the ALE map while keeping the ALE-coordinate $\mathbf{y} \in \Omega_0$ fixed:

$$D_t u(\mathbf{x}, t) := \frac{d}{dt} u\big(\mathcal{A}_t(\mathbf{y}), t\big) = \partial_t u(\mathbf{x}, t) + \mathbf{w}(\mathbf{x}, t) \cdot \nabla_{\mathbf{x}} u(\mathbf{x}, t). \tag{1.3}$$

Inserting (1.3) into the PDE in (1.1) we end up with the *non-conservative* formulation

$$D_t u - \mathbf{w} \cdot \nabla_{\mathbf{x}} u + \nabla_{\mathbf{x}} \cdot (\mathbf{b}u) - \mu \Delta_{\mathbf{x}} u = f, \tag{1.4}$$

or its equivalent *conservative* counterpart

$$D_t u + (\nabla_{\mathbf{x}} \cdot \mathbf{w})u + \nabla_{\mathbf{x}} \cdot [(\mathbf{b} - \mathbf{w})u] - \mu \Delta_{\mathbf{x}} u = f. \tag{1.5}$$

It is worth noticing that (1.1) is equivalent to either (1.4) or (1.5), the latter two being more convenient numerically because the geometry of the domain deformation is built in explicitly. In this vein, it is also clear that $u = 1$ is a solution provided $f = \nabla_{\mathbf{x}} \cdot \mathbf{b} = 0$ and the Dirichlet condition in (1.1) changed to $u = 1$. A numerical scheme is said to satisfy the *geometric conservation law (GCL)* if it admits 1 as a discrete solution. The GCL was originally introduced for finite volume schemes as a minimum criterion for unconditional stability [19, 14], and it turns out to be closely related to quadrature for approximating integrals in time [17, 16, 23]. However, its role for ALE-free and unconditional stability, as well as its impact in accuracy, are not well understood.

There are two type of algorithms with supporting stability theory depending on their order of accuracy. The first class of first-order schemes hinges on the backward Euler method. Formaggia and Nobile [16] study a conservative finite element scheme for (1.1) with $\nabla_{\mathbf{x}} \cdot \mathbf{b} = 0$, which satisfies the GCL, and prove that it is ALE-free stable. Gastaldi [18], Boffi and Gastaldi [4], and Nobile [23] give an a priori error analysis. Moreover, Formaggia and Nobile [16], Boffi and Gastaldi [4], and Badia and Codina [1], propose ALE stable schemes which fail to satisfy the GCL.

The second class of second order schemes hinges on the Crank-Nicolson and backward differentiation formula (BDF) schemes; see Formaggia and Nobile [17], Boffi and Gastaldi [4], and Badia and Codina [1]. Even when the GCL condition is valid, the ensuing schemes are shown to be ALE stable and conditionally stable only. In fact, simulations show that the monotonicity of $\|u(t)\|_{L^2(\Omega_t)}$ does not hold at the discrete level.

The analysis of both first and second order schemes indicates that the ALE velocity $\mathbf{w}$ plays the role of an extra advection for the method, despite the fact that (1.2) is insensitive to $\mathbf{w}$. This leads to Gronwall-type arguments, time-step constraints and stability constants depending on the ALE map. The critical issue is to devise a time-discrete form of the so-called *Reynolds' identity*

$$\frac{d}{dt} \int_{\Omega_t} v \, d\mathbf{x} = \int_{\Omega_t} \partial_t v + \nabla_{\mathbf{x}} \cdot (v\mathbf{w}) \, d\mathbf{x} = \int_{\Omega_t} D_t v + v \nabla_{\mathbf{x}} \cdot \mathbf{w} \, d\mathbf{x} \tag{1.6}$$

that allows for the cancellation that happens at the continuous level. This basic property is not even clear for first-order schemes, which explains the lack of equivalence between conservative and non-conservative schemes as well as their stability properties [1, 4, 17, 16, 18, 23]. Moreover, it turns out that the GCL is equivalent to satisfying a discrete version of (1.6) for $v = 1$ [16].

We propose a family of discontinuous Galerkin (dG) methods of arbitrary order $q \geq 0$ and study their stability properties, for *time discretization* of (1.1). We believe that such a discretization is the key obstruction for the design of ALE-free stable schemes, and so refer to [1, 4, 17, 16, 18, 23] where C^0-finite elements are used for space discretization. The variational structure of dG allows for a direct implementation of (1.6) with exact integration and a separate analysis of the effect of quadrature. Our main contributions, valid for all $q \geq 0$, are as follows:

- *dG with exact integration*: ALE-free stability at the nodes $t = t_n$ (*nodal stability*) and ALE stability for all $t \in [0, T]$ (*global stability*) both without any time constraints (*unconditional stability*);
- *dG with Reynolds quadrature*: ALE-free nodal stability and ALE global stability both without any time constraints but assuming that the ALE map is piecewise polynomial in time;
- *dG with Radau quadrature*: ALE-free nodal stability and ALE global stability both with an ALE time constraint (conditional stability) but for any ALE map $W^2_\infty(W^1_\infty)$ piecewise in time.

We corroborate these findings with numerical experiments for several orders $0 \leq q \leq 3$, which show that our theory is sharp. The dG methods with quadrature are practical, with Radau quadrature being the minimal one that preserves the accuracy of dG and leads to the so-called *Runge-Kutta-Radau methods (RKR)* of order q for fixed domains. It turns out that all our unconditionally stable methods satisfy the classical GCL but this is not the mechanism that ensures stability; it is rather their ability to exactly reproduce the *Reynolds' identity*. This paper is the first in a series devoted to the analysis of dG for ALE formulations. We perform an a priori error analysis in [6] and an a posteriori error analysis in [5], both based on the stability notions developed here.

This paper extends the analysis of dG methods of any order for non-moving domains [28, Chapter 12] to time-dependent domains within the ALE framework. We also refer to [20] for the implementation of first-order dG methods in the context of fluid-structure interactions. It is worth comparing our results for dG methods with the pure Lagrangian framework for the advection-dominated diffusion equation on time-independent domains proposed by Chrysafinos and Walkington [10]. Both analyses have some conceptual similarities but the overall purposes are distinct. We are concerned with the design of arbitrary order dG methods for domains undergoing time-dependent Lipschitz deformations, the influence of the ALE map in their stability properties for moderate fluid velocities $\mathbf{b}$, and the analysis of the effect of quadrature in time. We emphasize that the latter plays a significant role in the design of implementable dG schemes and is an important aspect of our present contribution. In our approach, the ALE velocity $\mathbf{w}$ does not play the role of an advective velocity. In fact, dG schemes able to reproduce (1.6) are unconditionally stable schemes irrespective of the ALE map. In contrast, Chrysafinos and Walkington [10] consider a fixed domain but tackle the notoriously difficult hyperbolic regime $\mu \ll \|\mathbf{b}\|_{L^\infty}$. In their framework, the ALE velocity $\mathbf{w}$ is designed to compensate for large $\mathbf{b}$ and is thus chosen to satisfy $\mathbf{w} \approx \mathbf{b}$. They assume exact integration in space-time and advocate discontinuous maps in time to account for frequent remeshing.

We organize the paper as follows. In Section 2 we introduce some notation, and provide regularity assumptions on the ALE map so that the chain rule (1.3) and Reynolds' identity (1.6) are valid weakly. This allows us to prove existence and uniqueness of a function u solving the boundary value problem (1.1) weakly, and also being continuous with values in $L^2(\Omega_t)$ so that the initial condition in (1.1) makes sense. In Sections 3–5 we study the stability of our time-discrete dG schemes of any order $q \geq 0$ for divergence free advections $\nabla_{\mathbf{x}} \cdot \mathbf{b} = 0$. In particular, we devote Section 3 to dG methods with exact integration, Section 4 to Reynolds' quadrature and discussions of the GCL, and Section 5 to RKR methods. We conclude in Section 6 with extensions of the previous results to problems (1.1) with $\nabla_{\mathbf{x}} \cdot \mathbf{b} \neq 0$. It is only then that we get exponentials of $\|(\nabla_{\mathbf{x}} \cdot \mathbf{b})^-\|_{L^\infty}$ but never of geometric quantities. This is a distinctive feature of our analysis.

2. Preliminaries

2.1. **Notation and Regularity Assumptions.** For any Lipschitz domain D of $\mathbb{R}^m$, $m = d$ or $m = d+1$, we let $L^r(D)$, $1 \leq r \leq \infty$, be the usual Lebesgue space and $W^1_r(D)$ be the corresponding Sobolev space with differentiability 1. We let $H^1(D) = W^1_2(D)$ and $H^1_0(D)$ be the closure in $H^1(D)$ of smooth functions with compact support. We equip the space $H^1_0(D)$ with the norm $\|\nabla_{\mathbf{x}} v\|_{L^2(D)} = \left(\int_D |\nabla_{\mathbf{x}} v|^2 d\mathbf{x}\right)^{1/2}$ and denote by $H^{-1}(D)$ its dual space. Spaces of vector-valued functions are written in boldface.

Let $\Omega_0 \subseteq \mathbb{R}^d$ be the reference domain with Lipschitz boundary $\partial\Omega_0$ and $\Omega_t \subseteq \mathbb{R}^d$ be a deformable domain at time $t \in [0,T]$, with $T < \infty$ fixed. For every $t \in [0,T]$, we associate points $\mathbf{y} \in \Omega_0$ and $\mathbf{x} \in \Omega_t$ via a family of mappings $\{\mathcal{A}_t\}_{t\in(0,T]}$ with $\mathcal{A}_0 := \mathrm{I}_d$, the identity mapping, as follows:

$$\mathcal{A}_t : \Omega_0 \subseteq \mathbb{R}^d \to \Omega_t \subseteq \mathbb{R}^d, \quad \mathbf{x}(\mathbf{y}, t) = \mathcal{A}_t(\mathbf{y}).$$

We frequently regard $\mathcal{A}_t$ as a space-time function $\mathcal{A}(\mathbf{y}, t) := \mathcal{A}_t(\mathbf{y})$, and we refer to $\mathbf{y} \in \Omega_0$ as the ALE coordinate and $\mathbf{x} = \mathbf{x}(\mathbf{y}, t)$ as the spatial or Eulerian coordinate. Using $\{\mathcal{A}_t\}_{t\in[0,T]}$, we set

$$\mathcal{Q}_T := \{(\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R} : t \in [0,T],\, \mathbf{x} = \mathcal{A}_t(\mathbf{y}),\, \mathbf{y} \in \Omega_0\}.$$

Definition 2.1 (ALE maps). *We say that $\{\mathcal{A}_t\}_{t\in[0,T]}$ is a family of ALE maps if the following conditions are satisfied:*

- *Regularity:* $\mathcal{A}(\cdot,\cdot) \in \mathbf{W}^1_\infty\big((0,T);\mathbf{W}^1_\infty(\Omega_0)\big)$;
- *One to one: there exists a constant* $\gamma > 0$ *such that for all* $t \in [0,T]$,

$$\|\mathcal{A}_t(\mathbf{y}_1) - \mathcal{A}_t(\mathbf{y}_2)\|_{\ell_\infty} \ge \gamma\|\mathbf{y}_1 - \mathbf{y}_2\|_{\ell_\infty}, \qquad \forall \mathbf{y}_1,\ \mathbf{y}_2 \in \Omega_0.$$

The one-to-one requirement implies that $\mathcal{A}_t : \Omega_0 \to \Omega_t$ is invertible with Lipschitz inverse, i.e., $\mathcal{A}_t$ is bi-Lipschitz and so a homeomorphism. This implies that $v = \widehat{v} \circ \mathcal{A}_t^{-1} \in H^1_0(\Omega_t)$ if and only if $\widehat{v} \in H^1_0(\Omega_0)$; cf. [17, Proposition 1]. Moreover, since Ω_0 is Lipschitz so is Ω_t and $\Omega_t \subset \Omega$ for some bounded domain Ω, for all $t \in [0,T]$. Hence, the Poincaré inequality in Ω implies the existence of an absolute constant C_Ω, independent of t, so that

$$\|v\|_{L^2(\Omega_t)} \le C_\Omega \|\nabla_{\mathbf{x}} v\|_{L^2(\Omega_t)}, \quad \forall v \in H^1(\Omega_t). \tag{2.1}$$

In addition, since the Jacobian matrix of $\mathcal{A}_t$, $\mathbf{J}_{\mathcal{A}_t} := \frac{\partial \mathbf{x}}{\partial \mathbf{y}}$, is Lipschitz in time we deduce that

$$\frac{d}{dt}\det \mathbf{J}_{\mathcal{A}_t}(\mathbf{y},t) = \nabla_{\mathbf{x}} \cdot \mathbf{w}(\mathcal{A}_t(\mathbf{y}),t)\det \mathbf{J}_{\mathcal{A}_t}(\mathbf{y},t) \quad \Rightarrow \quad \det \mathbf{J}_{\mathcal{A}_t}(\mathbf{y},t) = \mathrm{e}^{\int_0^t \nabla_{\mathbf{x}}\cdot\mathbf{w}(\mathcal{A}_s(\mathbf{y}),s)\,ds}. \tag{2.2}$$

As a consequence, $\det \mathbf{J}_{\mathcal{A}_t}$ is positive and bounded away from 0 and ∞ uniformly for $t \in [0,T]$. Only in Section 5 we will need additional regularity assumptions on $\mathcal{A}_t$ beyond Definition 2.1.

Sometimes later it will be more convenient to use Ω_τ, $\tau \in (0,T]$ as reference domain rather than Ω_0. In such a case, the letter $\mathbf{y} \in \Omega_\tau$ will still indicate points in the reference domain and the letter $\mathbf{x} \in \Omega_t$ indicate points in any other domain Ω_t, $t \in [0,T]\setminus\{\tau\}$. Moreover, for $\tau, s \in [0,T]$, we denote by $\mathcal{A}_{\tau\to s} : \Omega_\tau \to \Omega_s$ the map

$$\mathcal{A}_{\tau\to s} := \mathcal{A}_s \circ \mathcal{A}_\tau^{-1},$$

whence $\mathcal{A}_s = \mathcal{A}_{0\to s}$. Taking Ω_τ, $\tau \in [0,T]$ as the reference domain, to every function $g : \mathcal{Q}_T \to \mathbb{R}$ we associate the function $\widehat{g} : \Omega_\tau \times [0,T] \to \mathbb{R}$ defined by

$$\widehat{g}(\mathbf{y},t) := g\big(\mathcal{A}_{\tau\to t}(\mathbf{y}),t\big).$$

We use the notation $\langle\cdot,\cdot\rangle_D$ for both the duality pairing and the L^2-inner product in D, depending on the context. For $Y = L^r$ or W^1_r, $1 \le r < \infty$, $Y = H^1_0$ or $Y = H^{-1}$, we define the spaces

$$L^2(Y;\mathcal{Q}_T) := \{v : \mathcal{Q}_T \to \mathbb{R} : \int_0^T \|v(t)\|^2_{Y(\Omega_t)}\,dt < \infty\}.$$

We similarly define the space $C(Y;\mathcal{Q}_T)$ of continuous functions with values in Y, as well as

$$L^\infty(\mathrm{div};\mathcal{Q}_T) := \{\mathbf{c} : \mathcal{Q}_T \to \mathbb{R}^d : \operatorname{ess\,sup}_{t\in(0,T)}\big(\|\mathbf{c}(t)\|_{L^\infty(\Omega_t)} + \|\nabla_{\mathbf{x}}\cdot\mathbf{c}(t)\|_{L^\infty(\Omega_t)}\big) < \infty\}.$$

To simplify the notation, we omit writing the dependency in $\mathcal{Q}_T$ when there is no confusion.

2.2. **Material Derivative and Reynolds' Identities.** We denote by ∂_t the usual partial time derivative holding the space variable constant. Given $g : \mathcal{Q}_T \to \mathbb{R}$, we indicate with $D_t g$ the *material (or ALE) derivative*, namely the partial time derivative keeping the ALE coordinate $\mathbf{y}$ fixed

$$(D_t g)(\mathbf{x},t) := (\partial_t \widehat{g})(\mathbf{y},t).$$

The *domain velocity* $\widehat{\mathbf{w}} : \Omega_0 \times [0,T] \to \mathbb{R}^d$ on the ALE frame is defined as

$$\widehat{\mathbf{w}}(\mathbf{y},t) := \partial_t \mathbf{x}(\mathbf{y},t),$$

whereas $\mathbf{w} : \mathcal{Q}_T \to \mathbb{R}^d$ indicates the corresponding function on the Eulerian frame, i.e.,

$$\mathbf{w}(\mathbf{x},t) := \widehat{\mathbf{w}}\big(\mathcal{A}_t^{-1}(\mathbf{x}),t\big). \tag{2.3}$$

The following lemma justifies the chain rule for weak material derivatives.

Lemma 2.1 (Leibnitz formula in $W^1_1(\mathcal{Q}_T)$)**.** *Let* $g \in W^1_1(\mathcal{Q}_T)$ *and* $\{\mathcal{A}_t\}_{t\in[0,T]}$ *be a family of ALE maps. Then,* $D_t g \in L^1(\mathcal{Q}_T)$ *and*

$$D_t g = \partial_t g + \mathbf{w}\cdot\nabla_{\mathbf{x}} g. \tag{2.4}$$

Proof. The Lipschitz regularity of the ALE map $\mathbf{x}(\mathbf{y},t)$ implies that the standard Leibniz formula is valid for weak derivatives of the composite map $\widehat{g}(\mathbf{y},t) = g(\mathbf{x}(\mathbf{y},t),t)$ [29]. This yields (2.4). □

The Reynolds' identities reported below are weak versions of the *Reynolds' Transport Theorem.*

Lemma 2.2 (Reynolds' identities). *Let $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a family of ALE maps. For any $\upsilon \in W_1^1(\mathcal{Q}_T)$ there holds*

$$\frac{d}{dt}\int_{\Omega_t} \upsilon \, d\mathbf{x} = \int_{\Omega_t} (D_t\upsilon + \upsilon\nabla_{\mathbf{x}}\cdot\mathbf{w})\, d\mathbf{x}. \tag{2.5}$$

In particular, for $w,\ \upsilon \in H^1(\mathcal{Q}_T)$ we have

$$\frac{d}{dt}\int_{\Omega_t} \upsilon w \, d\mathbf{x} = \int_{\Omega_t} w(D_t\upsilon + \upsilon\nabla_{\mathbf{x}}\cdot\mathbf{w})\, d\mathbf{x} + \int_{\Omega_t} \upsilon D_t w\, d\mathbf{x}. \tag{2.6}$$

Proof. The Reynolds' Transport Theorem gives (2.5) for smooth functions υ and ALE maps $\mathcal{A}_t$; see for instance [17]. We invoke a density argument to extend (2.5) to $\upsilon \in W_1^1(\mathcal{Q}_T)$ and $\mathcal{A}_t \in \mathbf{W}^1_\infty\big((0,T);\mathbf{W}^1_\infty(\Omega_0)\big)$. Expression (2.6) follows from (2.5) and Sobolev embeddings. □

2.3. **The Continuous Problem in the ALE Framework.** We assume that $u_0 \in H_0^1(\Omega_0)$, $f \in L^2(\mathcal{Q}_T)$ and $\mathbf{b} \in L^\infty(\mathrm{div};\mathcal{Q}_T)$. In view of the chain rule (2.4), the PDE in (1.1) can be rewritten as

$$D_t u + (\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}} u + (\nabla_{\mathbf{x}}\cdot\mathbf{b})u - \mu\Delta_{\mathbf{x}} u = f \quad \text{in } \mathcal{Q}_T. \tag{2.7}$$

A variational formulation of problem (2.7) reads as follows: seek $u \in L^2(H_0^1;\mathcal{Q}_T)\cap H^1(L^2;\mathcal{Q}_T)$ satisfying $u(\cdot,0)=u_0$ and such that for all $\upsilon \in L^2(H_0^1)$ and $\tau, t\in[0,T]$ with $\tau<t$,

$$\begin{aligned}\int_\tau^t \langle D_t u,\upsilon\rangle_{\Omega_s}\, ds &+ \int_\tau^t \langle (\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}} u,\upsilon\rangle_{\Omega_s}\, ds \\ &+ \int_\tau^t \langle(\nabla_{\mathbf{x}}\cdot\mathbf{b})u,\upsilon\rangle_{\Omega_s} + \mu\int_\tau^t \langle\nabla_{\mathbf{x}} u,\nabla_{\mathbf{x}}\upsilon\rangle_{\Omega_s}\, ds = \int_\tau^t\langle f,\upsilon\rangle_{\Omega_s}\, ds.\end{aligned} \tag{2.8}$$

Equation (2.8) is a *non-conservative* weak ALE formulation for problem (1.1). With formula (2.4) at hand, we can reformulate problem (1.1) as a time-dependent advection-diffusion system with variable coefficients on the reference domain Ω_0. The regularity of the ALE maps guarantees the parabolic nature of the ensuing equation and the existence of a unique solution u satisfying

$$u \in H^1(\mathcal{Q}_T) \subset C(L^2;\mathcal{Q}_T),$$

via energy techniques [11, 13, 27]. This thereby justifies the meaning of $u(\cdot,0)=u_0$, as well as the further regularity $\Delta_{\mathbf{x}} u, D_t u \in L^2(\mathcal{Q}_T)$.

Using the Reynolds' identity (2.6), the variational formulation (2.8) can be rewritten as follows:

$$\begin{aligned}\langle u(t),\upsilon(t)\rangle_{\Omega_t} &+ \int_\tau^t \langle \nabla_{\mathbf{x}}\cdot\big[(\mathbf{b}-\mathbf{w})u\big],\upsilon\rangle_{\Omega_s}\, ds + \mu\int_\tau^t\langle\nabla_{\mathbf{x}} u,\nabla_{\mathbf{x}}\upsilon\rangle_{\Omega_s}\, ds \\ &- \int_\tau^t \langle u, D_t\upsilon\rangle_{\Omega_s}\, ds = \langle u(\tau),\upsilon(\tau)\rangle_{\Omega_\tau}\, d\tau + \int_\tau^t \langle f,\upsilon\rangle_{\Omega_s}\, ds, \quad \forall \upsilon\in H_0^1(\mathcal{Q}_T).\end{aligned} \tag{2.9}$$

Equation (2.9) is the *conservative* weak ALE formulation for problem (1.1). We emphasize that non-conservative and conservative formulations (2.8) and (2.9) are equivalent.

Remark 2.1 (test functions). In contrast to the existing literature [1, 4, 16, 17, 18, 20, 23], in both (2.8) and (2.9) the test-functions υ do not have vanishing material derivative. This mimics the usual approach for time-independent domains and is consistent with the definition of discrete spaces and the dG methods in time in the ALE frame in Section 3. This approach is crucial for stability.

Remark 2.2 (H^{-1}-functional setting)**.** One might wonder about the formulation of (1.1) in the weaker setting $u_0 \in L^2(\Omega_0)$, $f \in L^2(H^{-1}; \mathcal{Q}_T)$ whence $u \in L^2(H^1_0; \mathcal{Q}_T)$ with $\Delta_{\mathbf{x}} u, D_t u \in L^2(H^{-1}; \mathcal{Q}_T)$, typical of parabolic problems. However, the energy argument on the reference domain Ω_0 would require the additional *space* regularity $\mathcal{A}_t, \mathcal{A}_t^{-1} \in \mathbf{W}^2_\infty$ to ensure that a functional in $L^2\big((0,T); H^{-1}(\Omega_0)\big)$ defines a functional in $L^2(H^{-1}; \mathcal{Q}_T)$ via the ALE map (and vice versa). This would imply that $\mathcal{A}_t, \mathcal{A}_t^{-1} \in \mathbf{C}^1$ in space, which is too strong as an assumption on the ALE maps because they are usually made of continuous finite element approximations.

3. Discontinuous Galerkin Method in Time: Exact Integration

In this section, we employ both (2.8) and(2.9) to construct the discontinuous Galerkin (dG) method within the ALE framework for moving domains. We assume exact integration with the purpose of emphasizing the essential arguments, but we discuss numerical integration in Sections 4 and 5. We also assume that $\nabla_{\mathbf{x}} \cdot \mathbf{b} = 0$ to simplify the arguments and postpone to Section 6 the extensions to the general case $\nabla_{\mathbf{x}} \cdot \mathbf{b} \neq 0$.

3.1. **The dG Methods and Nodal Stability.** Let $0 =: t_0 < t_1 < \cdots < t_N := T$ be a partition of $[0,T]$, and for $n = 0, 1, \ldots, N-1$, let $I_n := (t_n, t_{n+1}]$, $k_n := t_{n+1} - t_n$ be the variable time steps, and

$$\mathcal{Q}_n := \{(\mathbf{x}, t) \in \mathcal{Q}_T : \, t \in I_n\}.$$

In the forthcoming analysis there will be constants depending explicitly on $D\mathcal{A}_t$, the space differential of the ALE map $\mathcal{A}_t$, and its first time derivative. They may change at each appearance and be multiplied by other constants depending on the polynomial degree in time (later denoted by q) and the space dimension d. To simplify the notation, and make it clear that the constants are explicit we now introduce two characteristic constants:

$$\begin{aligned} A_n &:= \|D\mathcal{A}_{t_n \to t}\|^r_{\mathbf{L}^\infty(I_n; \mathbf{L}^\infty(\Omega_{t_n}))} \|(D\mathcal{A}_{t_n \to t})^{-1}\|^\ell_{\mathbf{L}^\infty(I_n; \mathbf{L}^\infty(\Omega_{t_n}))}, \\ B_n &:= \|D\mathcal{A}_{t_n \to t}\|_{\mathbf{W}^1_\infty(I_n; \mathbf{L}^\infty(\Omega_{t_n}))}, \end{aligned} \tag{3.1}$$

where the powers $r, \ell \geq 0$ with $r + \ell \geq 1$ will not be specified, but they can be equal to $0, 1, d, d+2$ depending on the context . We do not specify the norm used in (3.1) for the finite dimensional space $\mathbb{R}^{d \times d}$ due to the equivalence of norms. It is important to realize that $\mathcal{A}_{t_n \to t} = \mathcal{A}_t \circ A_{t_n}^{-1}$ implies

$$\lim_{t \to t_n} \|D\mathcal{A}_{t_n \to t}\|_{\mathbf{L}^\infty(\Omega_{t_n})} = \|\mathrm{I}_d\|_{\mathbf{L}^\infty(\Omega_{t_n})} = 1, \tag{3.2}$$

because $\|D\mathcal{A}_{t_n \to t}\|_{L^\infty(\Omega_{t_n})}$ is Lipschitz, whence $A_n, B_n = O(1)$ are local constants in I_n which do not involve exponentials of either geometric quantities or T. In contrast, from (2.2) we deduce that

$$\|\det \mathbf{J}_{\mathcal{A}_t}\|_{L^\infty((0,T); L^\infty(\Omega_0))} \leq e^{\int_0^T \|\nabla_{\mathbf{x}} \cdot \mathbf{w}(t)\|_{L^\infty(\Omega_t)} dt};$$

similar estimates are valid for $D\mathcal{A}_t, D\mathcal{A}_t^{-1}$ and $\frac{d}{dt} D\mathcal{A}_t = D\mathbf{w}$. We avoid constants depending on these global geometric quantities, which are typical of the pure Lagrangian approach $\mathbf{w} = \mathbf{b}$ [10].

To indicate absolute constants depending only on the polynomial degree q, the space dimension d and the constant C_Ω in (2.1) we frequently use the notation $\lesssim$ in the subsequent analysis.

For $q \geq 0$, the discrete space $\mathcal{V}_q$ associated with the dG method in time of order $q+1$, for problems defined on moving domains, is defined as follows:

$$\mathcal{V}_q := \{V : \mathcal{Q}_T \to \mathbb{R} : \, V|_{I_n} = \sum_{j=0}^q \varphi_j t^j \text{ where } \varphi_j \in L^2(H^1_0) \text{ with } D_t \varphi_j = 0, \, j = 0, \ldots, q\}.$$

Therefore, the dG space $\mathcal{V}_q$ consists of functions which are piecewise polynomials in time of degree at most q along the trajectories defined by the ALE map, and with coefficients in H^1_0; this space was considered in [20] for $q = 0$. Such a $\mathcal{V}_q$ extends the corresponding discrete space associated with the dG method in time for problems defined on non-moving domains [28]. Moreover,

$$\mathcal{V}_q(I_n) := \{V : \mathcal{Q}_n \to \mathbb{R} : V = W|_{\mathcal{Q}_n}, W \in \mathcal{V}_q\}, \quad n = 0, 1, \dots, N-1,$$

is the space of restrictions to $\mathcal{Q}_n$ of functions in $\mathcal{V}_q$.

In this paper, we consider semidiscrete schemes with discretization only in time. Thus, for $n = 0, 1, \dots, N-1$, $\mathcal{V}_q(I_n)$ is not a finite-dimensional space. This follows a similar approach to semidiscrete dG for time-independent domains [28, Chapter 12].

Remark 3.1 (finite-dimensional arguments)**.** Any function $V \in \mathcal{V}_q(I_n)$ is a polynomial in time of degree at most q when viewed on a reference domain Ω_τ. Specifically, the quantity

$$I_n \ni t \mapsto \|\widehat{V}(t)\|^2_{\mathcal{H}(\Omega_\tau)} \tag{3.3}$$

is a polynomial in time of degree at most $2q$, where $\|\cdot\|_{\mathcal{H}(\Omega_\tau)}$ denotes the norm in the Hilbert space $\mathcal{H}(\Omega_\tau)$. Therefore, finite-dimensional arguments such as inverse inequalities ([9, Chapter 4, Lemma 4.5.3]) and the equivalence of norms in finite-dimensional spaces of polynomials, can be applied to (3.3). Quantities of the form (3.3) appear often in our subsequent analysis.

The discontinuous Galerkin approximation U to u for the *non-conservative* ALE formulation (2.8) with $\nabla_{\mathbf{x}} \cdot \mathbf{b} = 0$ is defined as follows: we seek a $U \in \mathcal{V}_q$ such that

$$U(\cdot, 0) = u_0 \quad \text{in} \quad \varOmega_0, \tag{3.4}$$

and for $n = 0, 1, \dots, N-1$,

$$\begin{aligned} &\int_{I_n} \langle D_t U, V\rangle_{\varOmega_t}\,dt + \langle U(t_n^+) - U(t_n), V(t_n^+)\rangle_{\varOmega_{t_n}} + \int_{I_n} \langle (\mathbf{b} - \mathbf{w}) \cdot \nabla_{\mathbf{x}} U, V\rangle_{\varOmega_t}\,dt \\ &\qquad + \mu \int_{I_n} \langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\varOmega_t}\,dt = \int_{I_n} \langle f, V\rangle_{\varOmega_t}\,dt, \quad \forall V \in \mathcal{V}_q(I_n). \end{aligned} \tag{3.5}$$

The *conservative* dG formulation is based on (2.9) and reads: seek $U \in \mathcal{V}_q$ satisfying (3.4) and

$$\begin{aligned} &\langle U(t_{n+1}), V(t_{n+1})\rangle_{\varOmega_{t_{n+1}}} - \langle U(t_n), V(t_n^+)\rangle_{\varOmega_{t_n}} + \int_{I_n} \langle \nabla_{\mathbf{x}} \cdot \big((\mathbf{b} - \mathbf{w})U\big), V\rangle_{\varOmega_t}\,dt \\ &\qquad + \mu \int_{I_n} \langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\varOmega_t}\,dt - \int_{I_n} \langle U, D_t V\rangle_{\varOmega_t}\,dt = \int_{I_n} \langle f, V\rangle_{\varOmega_t}\,dt, \quad \forall V \in V_q(I_n). \end{aligned} \tag{3.6}$$

We again have that both (3.5) and (3.6) are equivalent, a property we will exploit in our analysis. We stress that dG produces approximations defined for all times t with a consistent domain for the approximation to lie in. The importance of the latter was first observed by Pironneau, Liou, and Tezduyar [25], who studied a time-dependent advection-diffusion model problem defined on moving domains and used Characteristic-Galerkin type formulations. However, they assumed that the time-dependent domains had to be "close to each other" between two consecutive time steps in order to derive stability and optimal order error bounds.

We also point out that for non-moving domains we have $\varOmega_t = \varOmega_0$ for all $t \in [0, T]$, we can choose the ALE map to be the identity, and $\mathbf{w} \equiv 0$. This implies that the material derivative becomes the usual partial derivative in time, whence both (3.5) and (3.6) generalize dG in time for problems (1.1) defined on non-moving domains [28].

Remark 3.2 (continuity of ALE map)**.** Since the ALE map is time-continuous, we have that $\mathcal{A}_{t_n^+} = \mathcal{A}_{t_n}$, i.e., $\varOmega_{t_n^+} = \varOmega_{t_n}$. This fact has been used for the definition of both (3.5) and (3.6) and it will also be used in the analysis below. Discontinuous maps are proposed in [10] within a Lagrangian approach to reduce the effect of large $\mathbf{b}$ for a hyperbolic-type problem on time-independent domains.

We are after stability estimates for dG insensitive to the ALE velocity $\mathbf{w}$ and the polynomial degree q. The following identity, based on (2.6), plays a significant role in this respect.

Lemma 3.1 (discrete Reynolds' identity)**.** *For every* $V \in \mathcal{V}_q(I_n)$*,* $n = 0, 1, \ldots, N-1$*, we have*

$$\int_{I_n} \Big(\langle D_t V, V \rangle_{\Omega_t} - \langle \mathbf{w} \cdot \nabla_{\mathbf{x}} V, V \rangle_{\Omega_t} \Big) dt = \frac{1}{2} \| V(t_{n+1}) \|^2_{L^2(\Omega_{t_{n+1}})} - \frac{1}{2} \| V(t_n^+) \|^2_{L^2(\Omega_{t_n})}. \tag{3.7}$$

Proof. Take $v = w = V \in \mathcal{V}_q(I_n)$ in (2.6) and integrate in time over I_n. Integrating by parts the term involving $\mathbf{w}$, and using that V has a vanishing trace, we get

$$\int_{I_n} \langle V^2, \nabla_{\mathbf{x}} \cdot \mathbf{w} \rangle_{\Omega_t} dt = -2 \int_{I_n} \langle \mathbf{w} \cdot \nabla_{\mathbf{x}} V, V \rangle_{\Omega_t} dt.$$

This leads to (3.7) and completes the proof. □

We next apply Lemma 3.1 to prove that dG admits a unique solution and it is stable. The difficulty is that dG is semidiscrete and thus we must cope with a continuous space.

Proposition 3.1 (existence and uniqueness)**.** *There exists a unique solution* $U \in \mathcal{V}_q$ *of* (3.5) *and* (3.6) *satisfying* (3.4)*.*

Proof. Since (3.5) and (3.6) are equivalent, we focus on (3.5). For $t = 0$, $U(\cdot, 0) = u_0$ in Ω_0 is well defined. We assume that for $0 \le n \le N-2$, the terminal value $U(\cdot, t_n)$ is well defined in Ω_{t_n}, and proceed by induction to prove that there exists a unique solution of (3.5) over I_n. Changing variables from Ω_t to Ω_{t_n} and using that $\det \mathbf{J}_{\mathcal{A}_{t_n \to t}}$ is uniformly positive and bounded, we see that $\mathcal{V}_q(I_n)$ is a Hilbert space with respect to the $L^2(H_0^1)-$inner product

$$(V, W)_{L^2(H_0^1)} := \int_{I_n} \langle V, W \rangle_{\Omega_t} \, dt + \int_{I_n} \langle \nabla_{\mathbf{x}} V, \nabla_{\mathbf{x}} W \rangle_{\Omega_t} \, dt. \tag{3.8}$$

Moreover, we consider the following bilinear form in $\mathcal{V}_q(I_n)$ which appears in (3.5)

$$\begin{aligned} b(V, W) := & \int_{I_n} \langle D_t V, W \rangle_{\Omega_t} \, dt + \langle V(t_n^+), W(t_n^+) \rangle_{\Omega_{t_n}} \\ & + \int_{I_n} \langle (\mathbf{b} - \mathbf{w}) \cdot \nabla_{\mathbf{x}} V, W \rangle_{\Omega_t} \, dt + \mu \int_{I_n} \langle \nabla_{\mathbf{x}} V, \nabla_{\mathbf{x}} W \rangle_{\Omega_t} \, dt. \end{aligned}$$

We observe that b is bounded in $\mathcal{V}_q(I_n)$ because the space of polynomials of degree $\le q$ is finite dimensional and all norms are equivalent; see Remark 3.1. In addition, b is coercive: take $W = V$ and notice that

$$\int_{I_n} \langle \mathbf{b} \cdot \nabla_{\mathbf{x}} V, V \rangle_{\Omega_t} dt = \frac{1}{2} \int_{I_n} \langle \mathbf{b}, \nabla_{\mathbf{x}} V^2 \rangle_{\Omega_t} dt = -\frac{1}{2} \int_{I_n} \langle \nabla_{\mathbf{x}} \cdot \mathbf{b}, V^2 \rangle_{\Omega_t} dt = 0, \tag{3.9}$$

because V has a vanishing trace and $\mathbf{b}$ is divergence free. Moreover, Lemma 3.1 yields

$$\int_{I_n} \Big(\langle D_t V, V \rangle_{\Omega_t} - \langle \mathbf{w} \cdot \nabla_{\mathbf{x}} V, V \rangle_{\Omega_t} \Big) \, dt + \langle V(t_n^+), V(t_n^+) \rangle_{\Omega_{t_n}} = \frac{1}{2} \| V(t_{n+1}) \|^2_{L^2(\Omega_{t_{n+1}})} + \frac{1}{2} \| V(t_n^+) \|^2_{L^2(\Omega_{t_n})},$$

whence together with a Poincaré inequality (2.1), which holds uniformly in Ω_t, we derive

$$\mu (V, V)_{L^2(H_0^1)} \lesssim \mu \int_{I_n} \| \nabla_{\mathbf{x}} V(t) \|^2_{L^2(\Omega_t)} dt \le b(V, V).$$

This coercivity of b, in conjunction with the continuity of $F(V) := \int_{I_n} \langle f, V \rangle$, yields the existence of a unique $U \in \mathcal{V}_q(I_n)$ satisfying (3.5) via the Lax-Milgram Theorem. This implies that $U(\cdot, t_{n+1})$ is well defined, and concludes the induction argument and the proof. □

Theorem 3.1 (stability with exact integration)**.** *The solution $U \in \mathcal{V}_q$ of* (3.5) *or* (3.6)*, both supplemented by the initial condition* (3.4)*, satisfies for* $0 \le m < n \le N$:

$$\|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \sum_{j=m}^{n-1} \|U(t_j^+) - U(t_j)\|^2_{L^2(\Omega_{t_j})} + \mu \int_{t_m}^{t_n} \|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}\, dt \tag{3.10}$$
$$\le \|U(t_m)\|^2_{L^2(\Omega_{t_m})} + \frac{1}{\mu}\int_{t_m}^{t_n} \|f(t)\|^2_{H^{-1}(\Omega_t)}\, dt.$$

Proof. Take $V = U$ in (3.5). The coercivity argument in Proposition 3.1 gives

$$\int_{I_j} \Big[\langle D_t U, U\rangle_{\Omega_t} + \langle (\mathbf{b} - \mathbf{w})\cdot \nabla_{\mathbf{x}} U, U\rangle_{\Omega_t}\Big]\, dt = \frac{1}{2}\|U(t_{j+1})\|^2_{L^2(\Omega_{t_{j+1}})} - \frac{1}{2}\|U(t_j^+)\|^2_{L^2(\Omega_{t_j})},$$

whereas a simple calculation reveals

$$2\langle U(t_j^+) - U(t_j), U(t_j^+)\rangle_{\Omega_{t_j}} = \|U(t_j^+)\|^2_{L^2(\Omega_{t_j})} - \|U(t_j)\|^2_{L^2(\Omega_{t_j})} + \|U(t_j^+) - U(t_j)\|^2_{L^2(\Omega_{t_j})}.$$

Finally, the Cauchy-Schwarz and Young inequalities yield

$$\int_{I_j} \langle f, U\rangle_{\Omega_t}\, dt \le \frac{\mu}{2}\int_{I_j} \|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}\, dt + \frac{1}{2\mu}\int_{I_j} \|f(t)\|^2_{H^{-1}(\Omega_t)}\, dt.$$

Inserting these expressions in (3.5) and adding from $j = m$ to $j = n-1$, we obtain (3.10). □

Remark 3.3 (monotonicity property)**.** If $f \equiv 0$ and $m = n-1$, (3.10) implies the relation

$$\|U(t_n)\|_{L^2(\Omega_{t_n})} \le \|U(t_{n-1})\|_{L^2(\Omega_{t_{n-1}})}, \qquad \forall\, 1 \le n \le N. \tag{3.11}$$

This important relation, valid for any time step k_n, polynomial degree $q \ge 0$ and diffusion coefficient μ, is not observed in [1, 4, 17] for second order schemes. Relation (3.11) is a discrete version of the monotonicity property (1.2), which holds for the continuous problem.

3.2. **Global Stability.** The purpose of this section is to derive a stability result for the continuous $L^\infty(L^2)-$norm, i.e., on the whole time interval, without any constraint on the time steps. The arguments below extend techniques for non-moving domains [28, Chapter 12].

Lemma 3.2 (relation between U and D_tU)**.** *If A_n is defined in* (3.1)*, then there holds for all $t \in I_n$*

$$\|U(t)\|^2_{L^2(\Omega_t)} \lesssim A_n \|U(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} + A_n k_n \int_{I_n} \|D_t U(t)\|^2_{L^2(\Omega_t)}\, dt. \tag{3.12}$$

Proof. We consider $\Omega_{t_{n+1}}$ as the reference domain. For all $t \in I_n$, we have that $\widehat{U}(t) = \widehat{U}(t_{n+1}) - \int_t^{t_{n+1}} \partial_s \widehat{U}(s)\, ds$. Consequently, upon squaring, applying the Cauchy-Schwarz inequality, and integrating over $\Omega_{t_{n+1}}$, we obtain

$$\|\widehat{U}(t)\|^2_{L^2(\Omega_{t_{n+1}})} \le 2\|\widehat{U}(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} + 2k_n \int_{I_n} \|\partial_t \widehat{U}(t)\|^2_{L^2(\Omega_{t_{n+1}})}\, dt.$$

We easily deduce (3.12) upon changing variables from $\Omega_{t_{n+1}}$ to Ω_t and using (3.1). □

The above lemma is instrumental to obtain the stability result on the whole interval.

Theorem 3.2 (global stability with exact integration)**.** *Let $f \in L^2(\mathcal{Q}_T)$ and $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a family of ALE maps. Then, the solution $U \in \mathcal{V}_q$ of problems* (3.5) *or* (3.6) *both supplemented by the initial* *condition* (3.4) *satisfies for* $n = 0, 1, \ldots, N$:

$$\begin{aligned}\sup_{t\in[0,t_n]} \|U(t)\|^2_{L^2(\Omega_t)} \lesssim \max_{0\le j\le n-1} \big\{A_j(1+F_jk_j)\big\}\Big(\|U(0)\|^2_{L^2(\Omega_0)} + \frac{1}{\mu}\int_0^{t_n} \|f(t)\|^2_{H^{-1}(\Omega_t)}\,dt\Big) \\ + \max_{0\le j\le n-1} A_jk_j \int_{I_j} \|f(t)\|^2_{L^2(\Omega_t)}\,dt,\end{aligned} \tag{3.13}$$

where the constants A_j, B_j *are defined in* (3.1) *and* F_j *is given by*

$$F_j := B_j + \frac{\|\mathbf{b}-\mathbf{w}\|^2_{L^\infty(\mathcal{Q}_j)}}{\mu} \qquad j = 0, 1, \ldots, n. \tag{3.14}$$

Proof. The idea is to derive a bound for D_tU and apply Lemma 3.2. We proceed in several steps.
$\boxed{1}$ We set $V = (t-t_n)D_tU \in \mathcal{V}_q(I_n)$ in (3.5). Taking advantage of the fact that $V(t_n^+) = 0$, we have

$$\begin{aligned}\int_{I_n} (t-t_n)\|D_tU(t)\|^2_{L^2(\Omega_t)}\,dt + \int_{I_n} (t-t_n)\langle(\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}}U, D_tU\rangle_{\Omega_t}\,dt \\ + \mu\int_{I_n} (t-t_n)\langle\nabla_{\mathbf{x}}U, \nabla_{\mathbf{x}}D_tU\rangle_{\Omega_t}\,dt = \int_{I_n}(t-t_n)\langle f, D_tU\rangle_{\Omega_t}\,dt,\end{aligned} \tag{3.15}$$

and estimate each term in (3.15) separately.
$\boxed{2}$ Applying the Cauchy-Schwarz and Young inequalities yields

$$\begin{aligned}&\int_{I_n} (t-t_n)\langle(\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}}U, D_tU\rangle_{\Omega_t}\,dt \\ &\quad\le k_n\|\mathbf{b}-\mathbf{w}\|^2_{L^\infty(\mathcal{Q}_n)}\int_{I_n}\|\nabla_{\mathbf{x}}U(t)\|^2_{L^2(\Omega_t)}\,dt + \frac{1}{4}\int_{I_n}(t-t_n)\|D_tU(t)\|^2_{L^2(\Omega_t)}\,dt.\end{aligned} \tag{3.16}$$

Proceeding similarly gives

$$\int_{I_n}(t-t_n)\langle f, D_tU\rangle_{\Omega_t}\,dt \le \int_{I_n}(t-t_n)\|f(t)\|^2_{L^2(\Omega_t)}\,dt + \frac{1}{4}\int_{I_n}(t-t_n)\|D_tU(t)\|^2_{L^2(\Omega_t)}\,dt. \tag{3.17}$$

$\boxed{3}$ We use Ω_{t_n} as reference domain and change variables to obtain

$$\int_{I_n}(t-t_n)\langle\nabla_{\mathbf{x}}U, \nabla_{\mathbf{x}}D_tU\rangle_{\Omega_t}\,dt = \int_{I_n}(t-t_n)\langle\mathbf{K}_{\mathcal{A}_{t_n\to t}}\nabla_{\mathbf{y}}\widehat{U}, \nabla_{\mathbf{y}}\partial_t\widehat{U}\rangle_{\Omega_{t_n}}\,dt \tag{3.18}$$

with $\mathbf{K}_{\mathcal{A}_{t_n\to t}} := \det\mathbf{J}_{\mathcal{A}_{t_n\to t}}\mathbf{J}^{-1}_{\mathcal{A}_{t_n\to t}}\mathbf{J}^{-\mathrm{T}}_{\mathcal{A}_{t_n\to t}}$. Integration by parts in time yields

$$\begin{aligned}\int_{I_n}(t-t_n)\langle\nabla_{\mathbf{x}}U, \nabla_{\mathbf{x}}D_tU\rangle_{\Omega_t}\,dt = \frac{1}{2}k_n\|\nabla_{\mathbf{x}}U(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} \\ -\frac{1}{2}\int_{I_n}(t-t_n)\langle\partial_t\mathbf{K}_{\mathcal{A}_{t_n\to t}}\nabla_{\mathbf{y}}\widehat{U}, \nabla_{\mathbf{y}}\widehat{U}\rangle_{\Omega_t}\,dt - \frac{1}{2}\int_{I_n}\|\nabla_{\mathbf{x}}U(t)\|^2_{L^2(\Omega_t)}\,dt.\end{aligned} \tag{3.19}$$

Moreover, exploiting the regularity of the ALE maps $\mathcal{A}_t$ we get, with B_n defined in (3.1),

$$\int_{I_n}(t-t_n)\langle\partial_t\mathbf{K}_{\mathcal{A}_{t_n\to t}}\nabla_{\mathbf{y}}\widehat{U}, \nabla_{\mathbf{y}}\widehat{U}\rangle_{\Omega_{t_n}}\,dt \lesssim B_nk_n\int_{I_n}\|\nabla_{\mathbf{x}}U(t)\|^2_{L^2(\Omega_t)}\,dt. \tag{3.20}$$

$\boxed{4}$ Inserting (3.16)-(3.17) and (3.19)-(3.20) into (3.15), we easily arrive at

$$\begin{aligned}&\int_{I_n}(t-t_n)\|D_tU(t)\|^2_{L^2(\Omega_t)}\,dt + \mu k_n\|\nabla_{\mathbf{x}}U(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} \\ &\lesssim \mu\Big(1 + B_nk_n + k_n\frac{\|\mathbf{b}-\mathbf{w}\|^2_{L^\infty(\mathcal{Q}_n)}}{\mu}\Big)\int_{I_n}\|\nabla_{\mathbf{x}}U(t)\|^2_{L^2(\Omega_t)}\,dt + \int_{I_n}(t-t_n)\|f(t)\|^2_{L^2(\Omega_t)}\,dt.\end{aligned} \tag{3.21}$$

$\boxed{5}$ It remains to estimate $\int_{I_n} \|D_t U(t)\|^2_{L^2(\Omega_t)}\,dt$. We make use of the equivalence of norms in the finite-dimensional space of polynomials of degree q, stated in Remark 3.1, to write

$$k_n \int_{I_n} \|D_t U(t)\|^2_{L^2(\Omega_t)}\,dt = k_n \int_{I_n} \left(\int_{\Omega_{t_n}} |\partial_t \widehat{U}(t)|^2 \det \mathbf{J}_{\mathcal{A}_{t_n\to t}} \right) dt \lesssim A_n \int_{I_n} (t-t_n)\|D_t U(t)\|^2_{L^2(\Omega_t)}\,dt,$$

where A_n is defined in (3.1). Thus, estimate (3.12) together with (3.21) yield

$$\begin{aligned}\|U(t)\|^2_{L^2(\Omega_t)} &\lesssim A_n \|U(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} \\ &\quad + \mu A_n(1+F_n k_n)\int_{I_n} \|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}\,dt + A_n \int_{I_n} (t-t_n)\|f(t)\|^2_{L^2(\Omega_t)}\,dt,\end{aligned} \tag{3.22}$$

where F_n is given by (3.14). Combining (3.22) with (3.10) leads to the asserted estimate. □

Remark 3.4 (global monotonicity)**.** In contrast to Remark 3.3, the upper bound in (3.13) with $f \equiv 0$ involves constants depending on the ALE map. One might thus wonder whether monotonicity of $\|U(t)\|_{L^2(\Omega_t)}$ holds for all $t \in (0,T)$ and not just for the breakpoints t_n. Figure 1 in Section 4 documents that this cannot be expected in general.

4. Reynolds' Quadrature and Geometric Conservation Law

In Section 3, we proved the unconditional stability of the dG methods (3.5) and (3.6) with exact integration in time. In this section, we let the *ALE map be a continuous piecewise polynomial of degree* $\le q'$ *in time*, which is not a restrictive assumption in the context of finite element applications, and explore the use of quadrature; $q' = 0$ corresponds to a constant ALE map in time (no domain motion). Using quadrature in either (3.5) or (3.6) leads to a practical scheme, but not yet discretized in space. Full discretization will be the subject of a forthcoming paper.

For an arbitrary ALE map $\mathcal{A}_t$, we can use its L^2-projection in time onto the space of continuous piecewise polynomials of degree $\le q'$ and consider the discrete scheme (3.5) or (3.6) with respect to the L^2 projection instead of the original ALE map. Invoking a perturbation argument, we prove in [6] that there is no loss of accuracy by replacing the exact map with its L^2-projection.

4.1. **Reynolds' Quadrature and the dG Methods.**

The key relation for the stability estimate of Theorem 3.1 is the Reynolds' identity (3.7). In this section, we use quadrature in time of sufficiently high order so as to make (3.7) valid, and refer to such an integration rule as *Reynolds' quadrature*. The effect of Radau quadrature with $q+1$ Radau points is discussed in Section 5. The latter is computational less intensive that Reynolds' quadrature and leads to dG schemes with the same accuracy; cf. [6]. However, Radau quadrature requires a restriction on the time steps for stability.

Lemma 4.1 (polynomial degree)**.** *Let the ALE map $\mathcal{A}_t$ be a continuous piecewise polynomial in time of degree $q' \ge 0$. Let $t \in I_n$ and $V, W \in \mathcal{V}_q(I_n)$, $n = 0, 1, \dots, N-1$. If $q' \ge 1$, then the terms*

$$\int_{\Omega_t} D_t V\, W\,d\mathbf{x}, \qquad \int_{\Omega_t} \nabla_{\mathbf{x}} \cdot \big(\mathbf{w}\ V\big) W\,d\mathbf{x}, \qquad \int_{\Omega_t} (\nabla_{\mathbf{x}} \cdot \mathbf{w})\, V\ W\,d\mathbf{x}$$

are polynomials in time of degree

$$p := 2q + \max\{dq' - 1, 0\}. \tag{4.1}$$

If $q' = 0$, then the second and the third terms vanish, whereas the first term has polynomial degree $p-1$ provided $q \ge 1$ and vanishes otherwise.

Proof. For $n = 0, 1, \dots, N-1$, we consider Ω_{t_n} as the reference domain. Then

$$\int_{\Omega_t} D_t V\, W\,d\mathbf{x} = \int_{\Omega_{t_n}} \partial_t \widehat{V}\, \widehat{W} \det \mathbf{J}_{\mathcal{A}_{t_n\to t}}\,d\mathbf{y}.$$

Since $\mathcal{A}_{t_n\to t}$ is a polynomial of degree q' in time, so are the entries of $\mathbf{J}_{\mathcal{A}_{t_n\to t}}$, whence $\det \mathbf{J}_{\mathcal{A}_{t_n\to t}}$ is a polynomial of degree dq'. If $q' \geq 1$, then clearly $\int_{\Omega_{t_n}} \partial_t \widehat{V}\, \widehat{W} \det \mathbf{J}_{\mathcal{A}_{t_n\to t}}\, d\mathbf{y}$ is a polynomial of degree $(q-1)+q+dq' = p$ in time, whereas if $q'=0$ then the polynomial degree is $2q-1 = p-1$.

For the second term we argue as in [17, Proposition 4]. For $q' \geq 1$, we have

$$\int_{\Omega_t} (\nabla_{\mathbf{x}} \cdot \mathbf{w})\, V\; W d\mathbf{x} = \int_{\Omega_{t_n}} \left(\left[\mathbf{J}^{-\mathrm{T}}_{\mathcal{A}_{t_n\to t}} \nabla_{\mathbf{y}} \right] \cdot \widehat{\mathbf{w}} \right) \widehat{V}\; \widehat{W} \det \mathbf{J}_{\mathcal{A}_{t_n\to t}}\, d\mathbf{y}.$$

Since $\left[\mathbf{J}^{-\mathrm{T}}_{\mathcal{A}_{t_n\to t}} \nabla_{\mathbf{y}}\right] \cdot \widehat{\mathbf{w}} \det \mathbf{J}_{\mathcal{A}_{t_n\to t}} = [\mathrm{cof}\; (\mathbf{J}_{\mathcal{A}_{t_n\to t}})^{\mathrm{T}} \nabla_{\mathbf{y}}] \cdot \widehat{\mathbf{w}}$ is a polynomial of degree $(d-1)q' + q' - 1 = dq'-1$, we infer that $\int_{\Omega_{t_n}} \left(\left[\mathbf{J}^{-\mathrm{T}}_{\mathcal{A}_{t_n\to t}} \nabla_{\mathbf{y}}\right] \cdot \widehat{\mathbf{w}} \right) \widehat{V}\; \widehat{W} \det \mathbf{J}_{\mathcal{A}_{t_n\to t}}\, d\mathbf{y}$ is a polynomial of degree p in time. If $q'=0$, then $\mathcal{A}_{t_n\to t}$ is the identity and $\mathbf{w}=0$, which implies that the term above vanishes.

We finally handle the term $\int_{\Omega_t} \nabla_{\mathbf{x}} \cdot \big(\mathbf{w}V\big) W\, d\mathbf{x}$ similarly to the second one for all $q' \geq 0$. $\square$

The above lemma motivates the following definition of Reynolds' quadrature, which integrates exactly all the terms in the Reynolds' identity (3.7).

Definition 4.1 (Reynolds' quadrature)**.** *We say that a quadrature Q on $(0,1]$ with positive weights ω_j and nodes τ_j, $j = 0,1,\ldots,r$, is a* Reynolds' quadrature *if it is exact for polynomials of degree p defined in (4.1). The corresponding weights $\{\omega_{n,j}\}_{j=0}^r$ and quadrature points $\{t_{n,j}\}_{j=0}^r$ in $I_n = (t_n, t_{n+1}]$ are*

$$\omega_{n,j} = k_n \omega_j, \quad t_{n,j} = t_n + k_n \tau_j, \quad j = 0,1,\ldots,r,$$

for all $n = 0,1,\ldots,N-1$, and the quadrature Q_n reads as follows for all $g \in C(I_n)$:

$$Q_n(g) := \sum_{j=0}^{r} \omega_{n,j} g(t_{n,j}) \approx \int_{I_n} g(t)\, dt. \tag{4.2}$$

We prove below that the computationally practical schemes obtained when applying (4.2) to either (3.5) or (3.6) enjoy the same stability properties as the schemes with exact integration and so require no restriction on the time-steps for stability. Whether or not the quadrature order p is necessary for unconditional stability is yet to be established.

We next discuss the discrete Reynolds' identities that are crucial for stability.

Lemma 4.2 (discrete Reynolds' identities)**.** *Let the ALE map $\mathcal{A}_t$ be a continuous piecewise polynomial in time of degree q' and let Q_n be a Reynolds' quadrature over I_n, $n = 0,1,\ldots,N-1$. Then the following discrete Reynolds' identity holds true*

$$\begin{aligned} \langle V(t_{n+1}), W(t_{n+1})\rangle_{\Omega_{t_{n+1}}} &- \langle V(t_n^+), W(t_n^+)\rangle_{\Omega_{t_n}} \\ &= Q_n\big(\langle D_t V + V\nabla_{\mathbf{x}} \cdot \mathbf{w}, W\rangle_{\Omega_t}\big) + Q_n\big(\langle V, D_t W\rangle_{\Omega_t}\big), \end{aligned} \tag{4.3}$$

for all $V,\ W \in \mathcal{V}_q(I_n)$. In particular, for every $V \in \mathcal{V}_q(I_n)$

$$\frac{1}{2}\|V(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} - \frac{1}{2}\|V(t_n^+)\|^2_{L^2(\Omega_{t_n})} = Q_n\big(\langle D_t V + \mathbf{w}\cdot\nabla_{\mathbf{x}} V, V\rangle_{\Omega_t}\big). \tag{4.4}$$

Proof. In view of Lemma 4.1 and (4.2), we realize that

$$Q_n\big(\langle D_t V, W\rangle_{\Omega_t}\big) = \int_{I_n} \langle D_t V, W\rangle_{\Omega_t}\, dt, \quad Q_n(\langle V\nabla_{\mathbf{x}}\cdot\mathbf{w}, W\rangle_{\Omega_t}) = \int_{I_n} \langle V\nabla_{\mathbf{x}}\cdot\mathbf{w}, W\rangle_{\Omega_t}\, dt, \tag{4.5}$$

and the proof continues as that of Lemma 3.1. $\square$

Remark 4.1 (Reynolds' quadrature and GCL)**.** For $q=0$, (4.4) is the geometric conservation law (GCL) appearing in the papers by Formaggia and Nobile, [17, 16] (see also [23]), Gastaldi, [18], and Boffi and Gastaldi, [4]. Therefore, (4.4) may be regarded as a generalization of the GCL to higher polynomial degree $q > 0$ and test functions with non-zero material derivative.

We can now write the *non-conservative* dG scheme (3.5) in terms of Reynolds' quadrature for $n = 0, 1, \dots, N-1$, as follows:

$$\begin{aligned} Q_n\big(\langle D_t U, V\rangle_{\Omega_t}\big) + \langle U(t_n^+) - U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} + Q_n\big(\langle(\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}} U, V\rangle_{\Omega_t}\big) \\ + \mu Q_n\big(\langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\Omega_t}\big) = Q_n\big(\langle f, V\rangle_{\Omega_t}\big), \quad \forall\, V \in \mathcal{V}_q(I_n). \end{aligned} \tag{4.6}$$

Likewise, in view of (4.3), we see that the *conservative* formulation (3.6) reads

$$\begin{aligned} \langle U(t_{n+1}), V(t_{n+1})\rangle_{\Omega_{t_{n+1}}} - \langle U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} + Q_n\big(\langle \nabla_{\mathbf{x}}\cdot\big((\mathbf{b}-\mathbf{w})U\big), V\rangle_{\Omega_t}\big) \\ + \mu Q_n(\langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\Omega_t}) - Q_n(\langle U, D_t V\rangle_{\Omega_t}) = Q_n(\langle f, V\rangle_{\Omega_t}), \quad \forall\, V \in \mathcal{V}_q(I_n). \end{aligned} \tag{4.7}$$

Lemma 4.2, in conjunction with the argument in Proposition 3.1, leads to the existence and uniqueness of $U \in \mathcal{V}_q$ solving either (4.6) or (4.7). We again insist that (4.3) implies that (4.6) and (4.7) are equivalent.

Remark 4.2 (non-conservative backward Euler method)**.** The non-conservative backward Euler formulation proposed by Formaggia and Nobile is not equivalent to the conservative one [17]. The difference with our non-conservative backward Euler formulation (4.6) (q=0) lies on the term $\langle U(t_n^+) - U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}}$, which is computed in $\Omega_{t_{n+1}}$ instead of Ω_{t_n}. Our choice is natural when the backward Euler method is viewed as a dG method, and thus variationally. We will show below that (4.6) is unconditionally stable for all $q \geq 0$.

Remark 4.3 (conservative backward Euler method)**.** For $q = 0$ and the mid-point integration rule, (4.7) reduces to the unconditional stable backward Euler method proposed by Formaggia and Nobile in [17]. We will show below that (4.7) is unconditionally stable for all $q \geq 0$.

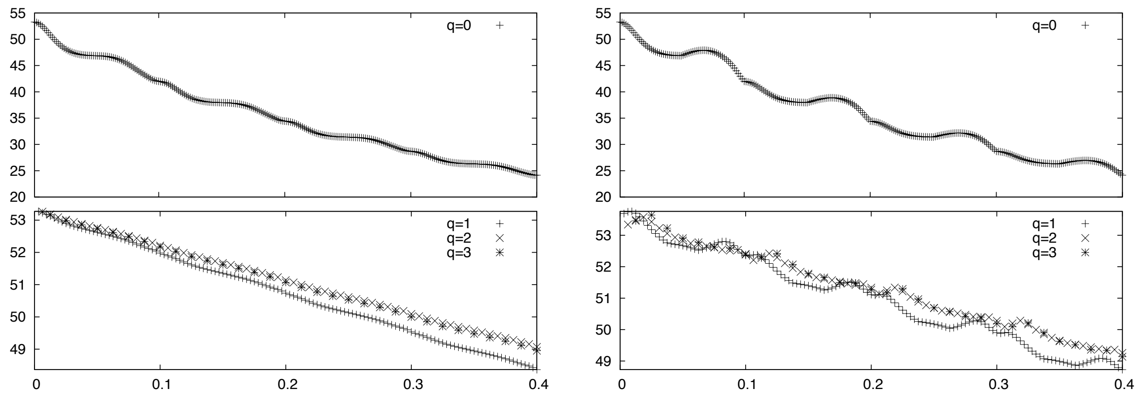

FIGURE 1. Evolution of $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ (left) and $\max_{t\in I_n}\|U(t)\|_{L^2(\Omega_t)}$ (right) for $q = 0$ with 2^8 uniform time steps (top) and $q = 1, 2, 3$ with respectively 2^7, 2^6, 2^5 uniform time steps (bottom). The space discretization is fine enough not to influence the time discretization. The reference domain is $\Omega_0 := [0,1]\times[0,1]$, the time interval is $[0, 0.4]$, the diffusivity is $\mu = 0.01$, the domain velocity $\mathbf{w}$ is the L^2-projection over piecewise polynomials of degree q of the time derivative of $\mathcal{A}_t(\mathbf{y}) := \mathbf{y}(2-\cos(20\pi\, t))$, with $\mathbf{y} \in \Omega_0$, $t \in (0, 0.4)$, and the forcing is $f = 0$ [17]. The ALE map is obtained by integration in each time interval I_n, enforcing continuity at the nodes. All schemes display monotone $\|U(t)\|_{L^2(\Omega_t)}$ when restricted to the breakpoints $t = t_n$, as predicted by Theorems 3.1 and 4.1, the backward Euler scheme ($q = 0$) being much more dissipative than the others. Oscillations of the ALE map destroy this monotonicity property over the whole time interval, thereby corroborating Theorem 4.3.

4.2. **Nodal Stability.** We now discuss stability in the $\ell^\infty(L^2)$ and the $L^2(H^1)$-norms and prove a bound similar to (3.10), which requires *no constraint on the time-step nor it involves any constants depending on the ALE map.* Several plots of $\|U(t)\|_{L^2(\Omega_t)}$ are depicted in Figure 1.

Theorem 4.1 (nodal stability with Reynolds' quadrature)**.** *Let $f \in C(H^{-1}; \mathcal{Q}_T) \cap L^2(\mathcal{Q}_T)$ and the ALE map $\mathcal{A}_t$ be a continuous piecewise polynomial in time of degree q'. Let $U \in \mathcal{V}_q$ be the solution of problem* (4.6) *or* (4.7)*, together with* (3.4)*, using a Reynolds' quadrature Q_n over I_n. Then*

$$
\begin{aligned}
&\|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \sum_{j=m}^{n-1} \|U(t_j^+) - U(t_j)\|^2_{L^2(\Omega_{t_j})} + \mu \sum_{j=m}^{n-1} Q_j(\|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}) \\
&\qquad \le \|U(t_m)\|^2_{L^2(\Omega_{t_m})} + \frac{1}{\mu}\sum_{j=m}^{n-1} Q_j(\|f(t)\|^2_{H^{-1}(\Omega_t)}), \qquad 0 \le m < n \le N.
\end{aligned}
\tag{4.8}
$$

Proof. We proceed as in Theorem 3.1, taking $V = U$ in (4.6) but now using (4.4). For the forcing term, we employ the Cauchy-Schwarz and Young inequalities for Q_n

$$
Q_n\big(\langle f, U\rangle_{\Omega_t}\big) \le \frac{\mu}{2} Q_n\big(\|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}\big) + \frac{1}{2\mu} Q_n\big(\|f(t)\|^2_{H^{-1}(\Omega_t)}\big),
$$

the former being a consequence of the positivity of the weights. This completes the proof. □

The monotonicity of $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ given by (4.8) for all polynomial degree $q \ge 0$ and $f = 0$ is consistent with Remark 3.3 and the experiments depicted in Figure 1 for the oscillatory ALE map of [17]. However, in contrast to [17], all our higher order schemes are monotone at the breakpoints t_n, a property that fails in the whole time interval irrespective of quadrature (see Remark 3.4). We will see in Section 4.3 that a stability result holds for the full $L^\infty(L^2)-$norm for the methods (4.6) and (4.7) without any constraint on the time steps, as in the case of exact integration; nevertheless, the constants involved in the upper bound depend on the ALE map. We also point out that the higher order dG schemes ($q > 0$) behave quite similarly and are much less dissipative than the backward Euler method ($q = 0$); see botton row of Figure 1.

Estimate (4.8) provides unconditional stability of the discrete $L^2(H^1)-$norm. However, to obtain stability in the continuous $L^2(H^1)-$norm, we need to relate the discrete norm $Q_n(\|\nabla_{\mathbf{x}} V(t)\|^2_{L^2(\Omega_t)})$ with $\int_{I_n} \|\nabla_{\mathbf{x}} V(t)\|^2_{L^2(\Omega_t)}\, dt$. We discuss this next.

Lemma 4.3 (discrete and continuous norms)**.** *Let $\mathcal{A}_t$ be an arbitrary ALE map and let Q_n be a quadrature in time over I_n which is exact for polynomials of degree at least $2q$. Then, for $n = 0, 1, \dots, N-1$, $V \in \mathcal{V}_q(I_n)$, we have*

$$
\frac{1}{A_n} Q_n\big(\|W(t)\|^2_{L^2(\Omega_t)}\big) \lesssim \int_{I_n} \|W(t)\|^2_{L^2(\Omega_t)}\, dt \lesssim A_n Q_n\big(\|W(t)\|^2_{L^2(\Omega_t)}\big),
\tag{4.9}
$$

where $W = V$ or $W = \nabla_{\mathbf{x}} V$ and the constants A_n are defined in (3.1)*.*

Proof. We let $W = \nabla_{\mathbf{x}} V$ and write $\int_{I_n} \|\nabla_{\mathbf{x}} V(t)\|^2_{L^2(\Omega_t)}\, dt$ in the reference domain Ω_{t_n} to obtain

$$
\int_{I_n} \|\nabla_{\mathbf{x}} V(t)\|^2_{L^2(\Omega_t)}\, dt = \int_{I_n}\int_{\Omega_{t_n}} |\mathbf{J}^{-\mathrm{T}}_{\mathcal{A}_{t_n\to t}} \nabla_{\mathbf{y}} \widehat{V}(t)|^2 \det \mathbf{J}_{\mathcal{A}_{t_n\to t}}\, d\mathbf{y}\, dt \lesssim A_n \int_{I_n}\int_{\Omega_{t_n}} |\nabla_{\mathbf{y}} \widehat{V}|^2\, d\mathbf{y}\, dt.
$$

Since $\int_{\Omega_{t_n}} |\nabla_{\mathbf{y}} \widehat{V}|^2\, d\mathbf{y}$ is a polynomial in time of degree $2q$ which is exactly integrated by the quadrature Q_n, we find that

$$
\int_{I_n}\int_{\Omega_{t_n}} |\nabla_{\mathbf{y}} \widehat{V}|^2\, d\mathbf{y}\, dt = \sum_{j=0}^{r} \omega_{n,j} \int_{\Omega_{t_n}} |\nabla_{\mathbf{y}} \widehat{V}(\mathbf{y}, t_{n,j})|^2 d\mathbf{y} dt \lesssim A_n Q_n(\|\nabla_{\mathbf{x}} V(t)\|^2_{L^2(\Omega_t)}).
\tag{4.10}
$$

Combining the previous two estimates gives the right inequality in (4.9). Similar arguments lead to the left inequality in (4.9) as well as the case $W = V$. $\square$

We can now state the stability result for the continuous $L^2(H^1)-$ norm.

Theorem 4.2 (stability in the continuous energy norm). *Let $f \in C(H^{-1}; \mathcal{Q}_T) \cap L^2(\mathcal{Q}_T)$ and the ALE map $\mathcal{A}_t$ be a continuous piecewise polynomial of degree q'. Let $U \in \mathcal{V}_q$ be the solution of* (4.6) *or* (4.7)*, together with* (3.4)*, using a Reynolds' quadrature Q_n over I_n. Then, for $0 \le m < n \le N$, we have*

$$\|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \sum_{j=m}^{n-1} \|U(t_j^+) - U(t_j)\|^2_{L^2(\Omega_{t_j})} + \mu \sum_{j=m}^{n-1} \frac{1}{A_j} \int_{I_j} \|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}\, dt$$
$$\le \|U(t_m)\|^2_{L^2(\Omega_{t_m})} + \frac{1}{\mu} \sum_{j=m}^{n-1} Q_j(\|f(t)\|^2_{H^{-1}(\Omega_t)}).$$

Proof. The proof is a direct consequence of (4.8) and (4.9). $\square$

4.3. **Global Stability.** We now discuss the stability in the $L^\infty(L^2)$-norm when using Reynolds' quadrature for the nontrivial case $q > 0$. We start with an observation about quadrature error.

Lemma 4.4 (Reynolds' quadrature error). *If Q is a Reynolds' quadrature on the interval $I := (0, 1]$ and $\Psi \in \mathbb{P}_{2q}$ is a polynomial of degree $\le 2q$ over I, the quadrature error*

$$E(\varphi) := \int_I \Psi\varphi - Q(\Psi\varphi), \tag{4.11}$$

satisfies for $2q - 1 \le \ell \le p$, where p is as in (4.1)*,*

$$|E(\varphi)| \lesssim \|\Psi\|_{L^1(I)} |\varphi|_{W_\infty^{\ell-2q+1}(I)}, \qquad \forall\, \varphi \in W_\infty^{\ell-2q+1}(I). \tag{4.12}$$

Proof. Since Q is exact for polynomials of degree $\le p$ and $p \ge 2q$, we directly obtain that

$$E(\varphi) = 0, \quad \forall \varphi \in \mathbb{P}_{p-2q}.$$

The fact that the weights of a Reynolds' quadrature are positive and sum-up to one implies

$$|E(\varphi)| \le 2\|\Psi\|_{L^\infty(0,1)} \|\varphi\|_{L^\infty(0,1)} \lesssim \|\Psi\|_{L^1(0,1)} \|\varphi\|_{L^\infty(0,1)},$$

because of the equivalence of norms in $\mathbb{P}_{2q}$. Applying the Bramble-Hilbert Lemma yields (4.12). $\square$

To derive a global bound in $L^\infty(L^2)$ we now review the proof of Theorem 3.2 and account for the effect of quadrature. We state the result as follows.

Theorem 4.3 (global stability with Reynolds' quadrature). *Let $f \in C(L^2)$ and the ALE map $\mathcal{A}_t$ be a continuous piecewise polynomial of degree q'. Then the solution $U \in \mathcal{V}_q$ of either* (4.6) *or* (4.7)*, together with* (3.4)*, satisfies for $n = 1, \dots, N$, the following stability result*

$$\sup_{t \in [0,t_n]} \|U(t)\|^2_{L^2(\Omega_t)} \lesssim G_n \Big(\|U(0)\|^2_{L^2(\Omega_0)} + \frac{1}{\mu} \sum_{j=0}^{n-1} Q_j(||f(t)||^2_{H^{-1}(\Omega_t)}) \Big) + \max_{0 \le j \le n-1} k_j A_j Q_j \Big(\|f(t)\|^2_{L^2(\Omega_t)} \Big),$$

where $G_n := \max_{0 \le j \le n-1} \{A_j(1 + k_j F_j)\}$, F_j is defined in (3.14) *and A_j is given in* (3.1)*.*

Proof. We start with the equality (3.15), which for the dG method (4.6) becomes

$$\begin{aligned} &\int_{I_n} (t - t_n) \|D_t U(t)\|^2_{L^2(\Omega_t)}\, dt + Q_n\Big((t - t_n) \langle (\mathbf{b} - \mathbf{w}) \cdot \nabla_{\mathbf{x}} U, D_t U \rangle_{\Omega_t} \Big). \\ &\qquad + \mu Q_n\Big((t - t_n) \langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} D_t U \rangle_{\Omega_t} \Big) = Q_n\Big((t - t_n) \langle f, D_t U \rangle_{\Omega_t} \Big). \end{aligned} \tag{4.13}$$

We estimate again each term separately, and thereby split the proof into several steps.

We first employ the Cauchy-Schwarz and Young inequalities to get

$$\Big|Q_n\Big((t-t_n)\langle(\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}}U, D_tU\rangle_{\Omega_t}\Big)\Big| = \Big|\sum_{j=0}^{r}\omega_{n,j}\Big((t-t_n)\int_{\Omega_t}(\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}}U\, D_tU\Big)(t_{n,j})\Big|$$
$$\le k_n\|\mathbf{b}-\mathbf{w}\|^2_{L^\infty(\mathcal{Q}_n)}Q_n(\|\nabla_{\mathbf{x}}U(t)\|^2_{L^2(\Omega_t)}) + \frac{1}{4}\int_{I_n}(t-t_n)\|D_tU(t)\|^2_{L^2(\Omega_t)}\,dt,$$

because Reynolds' quadrature integrates exactly the term $(t-t_n)\|D_tU(t)\|^2_{L^2(\Omega_t)}$ according to Lemma 4.1. Likewise, we obtain

$$|Q_n\Big((t-t_n)\langle f, D_tU\rangle_{\Omega_t}\Big)| \le Q_n\Big((t-t_n)\|f(t)\|^2_{L^2(\Omega_t)}\Big) + \frac{1}{4}\int_{I_n}(t-t_n)\|D_tU(t)\|^2_{L^2(\Omega_t)}.$$

Next, if E_n stands for the Reynolds' quadrature error in the interval I_n, we can write

$$Q_n\Big((t-t_n)\langle\nabla_{\mathbf{x}}U, \nabla_{\mathbf{x}}D_tU\rangle_{\Omega_t}\Big) = \int_{I_n}(t-t_n)\langle\nabla_{\mathbf{x}}U, \nabla_{\mathbf{x}}D_tU\rangle_{\Omega_t}\,dt - E_n\Big((t-t_n)\langle\nabla_{\mathbf{x}}U, \nabla_{\mathbf{x}}D_tU\rangle_{\Omega_t}\Big).$$

The first term appears in Theorem 3.2, and we thus make use again of (3.18)-(3.20). Once we write the second term over the reference domain Ω_{t_n} we observe the appearance of the weight function $\mathbf{K}_{\mathcal{A}_{t_n\to t}} := \det \mathbf{J}_{\mathcal{A}_{t_n\to t}}\mathbf{J}^{-1}_{\mathcal{A}_{t_n\to t}}\mathbf{J}^{-\mathrm{T}}_{\mathcal{A}_{t_n\to t}}$ which, in view of the regularity of the ALE maps, verifies $\mathbf{K}_{\mathcal{A}_{t_n\to t}} \in L^\infty(\mathcal{Q}_n)$. We next resort to Lemma 4.4 with $\ell = 2q-1$ to derive

$$\Big|E_n\Big((t-t_n)\langle\nabla_{\mathbf{x}}U, \nabla_{\mathbf{x}}D_tU\rangle_{\Omega_t}\Big)\Big| \lesssim \int_{\Omega_{t_n}}\int_{I_n}|\mathbf{K}_{\mathcal{A}_{t_n\to t}}\nabla_{\mathbf{y}}\widehat{U}|\,|(t-t_n)\nabla_{\mathbf{y}}\partial_t\widehat{U}|\,dt\,d\mathbf{y}$$
$$\lesssim A_n\Big(\int_{I_n}\|\nabla_{\mathbf{y}}\widehat{U}(t)\|^2_{L^2(\Omega_{t_n})}\,dt\Big)^{1/2}k_n\Big(\int_{I_n}\|\partial_t\nabla_{\mathbf{y}}\widehat{U}(t)\|^2_{L^2(\Omega_{t_n})}\,dt\Big)^{1/2}.$$

We now invoke the inverse inequality alluded to in Remark 3.1 to infer that

$$k_n\Big(\int_{I_n}\|\partial_t\nabla_{\mathbf{y}}\widehat{U}(t)\|^2_{L^2(\Omega_{t_n})}\,dt\Big)^{1/2} \lesssim \Big(\int_{I_n}\|\nabla_{\mathbf{y}}\widehat{U}(t)\|^2_{L^2(\Omega_{t_n})}\,dt\Big)^{1/2}.$$

whence

$$\Big|E_n\Big((t-t_n)\langle\nabla_{\mathbf{x}}U, \nabla_{\mathbf{x}}D_tU\rangle_{\Omega_t}\Big)\Big| \lesssim A_n\int_{I_n}\|\nabla_{\mathbf{x}}U(t)\|^2_{L^2(\Omega_t)}\,dt.$$

We finally proceed as in Theorem 3.2, but making use of (4.8), to deduce the asserted estimate. □

5. Runge-Kutta-Radau Methods in the ALE Framework: Conditional Stability

We have discussed so far dG methods of any order, with exact integration and Reynolds' quadrature, for approximating (1.1) and proved stability without any restriction on the time steps (unconditional stability). However, Reynolds' quadrature is dimension dependent and becomes computationally more intensive for higher dimensions: it integrates exactly polynomials of degree $p = 2q + \max\{dq'-1, 0\}$, where q' is the degree of the polynomial ALE map (see (4.1)).

Radau quadrature with $q+1$ nodes is enough for dG methods to be unconditionally stable, for time independent domains, and give optimal order a priori error estimates [28, Chapter 12]. Our aim now is to study the effect of such quadrature in the present context, whence we examine *Runge-Kutta-Radau (RKR) methods in the ALE framework for moving domains.*

We will see that RKR methods make use of the *exact* ALE map at the intermediate stages and are stable, but *under a mild constraint on the time steps depending on the ALE map (conditional stability).* Figure 2 documents the behavior of $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ for RKR methods of order $q = 0, 1, 2, 3$ and the same oscillatory test case of [17], already discussed in Figure 1 for Reynolds' quadrature.

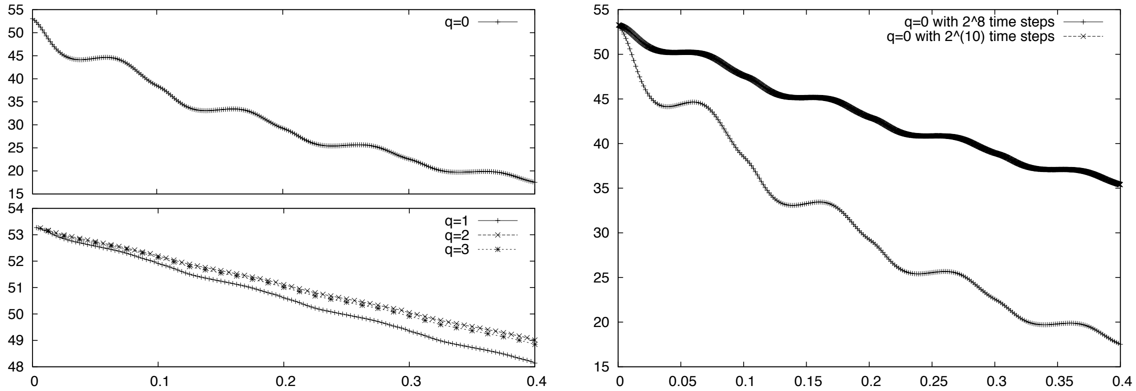

FIGURE 2. Evolution of $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ for $q = 0$ with 2^8 uniform time steps (top-left), for $q = 1, 2, 3$ with 2^7, 2^6, 2^5 uniform time steps respectively (bottom-left), and for $q = 0$ with 2^8 and 2^{10} uniform time steps (right). The space discretization is fine enough not to influence the time discretization. The test is the same as in Figure 1 and is taken from [17]. Monotonicity of $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ for $q = 0$ is sensitive to the time step size (conditional stability), a property of RKR methods proved in Theorem 5.1 for all $q \geq 0$ (see right). Stability of higher order RKR methods ($q > 0$) is less sensitive to the time steps (bottom-left).

We point out that *the time step restriction is not a CFL condition* because the space is continuous, and that RKR methods are much cheaper than Reynolds' methods for the same accuracy since the number of nodes $q+1$ compare favorably with p. A detailed discussion of the convergence rates for both methods is given in [6].

Our results are somewhat related to those of Badia and Codina [1], who proposed first and second order BDF schemes in the ALE framework for moving domains. These methods do not satisfy the GCL, and are stable and optimally accurate under a constraint on the time steps similar to ours.

Let ω_j and τ_j, $j = 0, 1, \ldots, q$, be the weights and nodes, respectively, for the Radau quadrature rule in $(0, 1]$: $Q^q(v) := \sum_{j=0}^{q} \omega_j v(\tau_j)$ for $v \in C([0, 1])$. Then $\{\omega_{n,j}\}_{j=0}^{q}$ and $\{\tau_{n,j}\}_{j=0}^{q}$ given by

$$\omega_{n,j} = k_n \omega_j \quad \text{and} \quad t_{n,j} = t_n + k_n \tau_j, \qquad j = 0, 1, \ldots, q$$

are the Radau weights and quadrature points for I_n, $n = 0, 1, \ldots, N - 1$. Employing such a quadrature in (3.5), say Q_n^q, the RKR method in the non-conservative ALE frame reads:

$$\begin{aligned} Q_n^q(\langle D_t U, V\rangle_{\Omega_t}) + \langle U(t_n^+) - U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} + Q_n^q(\langle(\mathbf{b} - \mathbf{w}) \cdot \nabla_{\mathbf{x}} U, V\rangle_{\Omega_t}) \\ + \mu Q_n^q(\langle \nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\Omega_t}) = Q_n^q(\langle f, V\rangle_{\Omega_t}), \quad \forall\, V \in \mathcal{V}_q(I_n), \end{aligned} \tag{5.1}$$

with $U(\cdot, 0) = u_0$ in Ω_0. In contrast to Reynolds' quadrature Q_n, Radau quadrature Q_n^q does not integrate exactly the terms appearing in Reynolds' identity (3.7). To compensate for this variational crime we impose an extra local time-regularity condition on the family of ALE maps $\{\mathcal{A}_t\}_{t\in[0,T]}$:

$$B_{n,2} := \|D\mathcal{A}_{t_n\to t}\|_{\mathbf{W}^2_\infty(I_n;\mathbf{L}^\infty(\Omega_{t_n}))} < \infty; \tag{5.2}$$

compare with constant B_n in (3.1).

5.1. **Conditional Nodal Stability.** We set $V = U$ in (5.1) and rely on (3.7) to obtain

$$\begin{aligned} \frac{1}{2}\|U(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} - \frac{1}{2}\|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \frac{1}{2}\|U(t_n^+) - U(t_n)\|^2_{L^2(\Omega_{t_n})} + \mu Q_n^q(\|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}) \\ = Q_n^q(\langle f, U\rangle_{\Omega_t}) + E_n^q(\langle D_t U, U\rangle_{\Omega_t}) + E_n^q\big(\langle(\mathbf{b} - \mathbf{w}) \cdot \nabla_{\mathbf{x}} U, U\rangle_{\Omega_t}\big), \end{aligned} \tag{5.3}$$

at the expense of two quadrature error terms $E_n^q(\cdot)$ over I_n written on the right-hand side. To estimate such terms, we resort to a variant of Lemma 4.4 that reads over $(0,1)$ as follows.

Lemma 5.1 (Radau quadrature error)**.** *If Q^q is the Radau quadrature rule over $I := (0,1]$ with $q+1$ nodes, then the quadrature error E^q satisfies*

$$E^q(\Psi\varphi) \lesssim \|\Psi\|_{L^1(I)}|\varphi|_{W_\infty^{m+1}(I)}, \quad \forall\,\Psi \in \mathbb{P}_{2q-m},\ \forall\,\varphi \in W_\infty^{m+1}(I),\ m=0,1. \tag{5.4}$$

Proof. We argue as in Lemma 4.4 using that Q^q is exact for polynomials of degree $\le 2q$. ☐

Theorem 5.1 (conditional nodal stability for RKR)**.** *Let $f \in C(H^{-1};\mathcal{Q}_T)\cap L^2(\mathcal{Q}_T)$ and (5.2) be valid. If*

$$A_n(1+B_{n,2})k_n \lesssim \mu, \quad \forall\, 0 \le n < N, \tag{5.5}$$

then the solution $U \in \mathcal{V}_q$ of problem (3.4) and (5.1) satisfies, for $0 \le m < n \le N$,

$$\begin{aligned} \|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \sum_{j=m}^{n-1}\|U(t_j^+)-U(t_j)\|^2_{L^2(\Omega_{t_j})} + \mu\sum_{j=m}^{n-1}Q_j(\|\nabla_{\mathbf{x}}U(t)\|^2_{\Omega_{t_j}}) \\ \le \|U(t_m)\|^2_{L^2(\Omega_{t_m})} + \frac{2}{\mu}\sum_{j=m}^{n-1}Q_j(\|f(t)\|^2_{H^{-1}(\Omega_t)}). \end{aligned} \tag{5.6}$$

Proof. We first write $E_n^q(\langle D_tU,U\rangle_{\Omega_t})$ over the reference domain Ω_{t_n}, and use (5.4) with $m=1$, to get

$$\begin{aligned} |E_n^q(\langle D_tU,U\rangle_{\Omega_t})| &\lesssim k_n^2\int_{\Omega_{t_n}}\Big(\int_{I_n}|\partial_t\widehat{U}|\,|\widehat{U}|\,dt\Big)|\det\mathbf{J}_{\mathcal{A}_{t_n\to t}}|_{W^2_\infty(I_n)}\,d\mathbf{y} \\ &\lesssim k_n^2B_{n,2}\Big(\int_{I_n}\|\partial_t\widehat{U}(t)\|^2_{L^2(\Omega_{t_n})}\,dt\Big)^{1/2}\Big(\int_{I_n}\|\widehat{U}(t)\|^2_{L^2(\Omega_{t_n})}\,dt\Big)^{1/2}. \end{aligned}$$

Using the inverse inequality mentioned in Remark 3.1), we can compensate the time derivative of $\|\partial_t\widehat{U}(t)\|_{L^2(\Omega_{t_n})}$ with k_n, and next appeal to Poincaré inequality (2.1) and (4.10) to get

$$|E_n^q(\langle D_tU,U\rangle_{\Omega_t})| \lesssim k_nA_nB_{n,2}Q_n^q\big(\|\nabla_{\mathbf{x}}U(t)\|^2_{L^2(\Omega_t)}\big).$$

Similar arguments, with $m=0$, lead to

$$\big|E_n^q\big(\langle(\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}}U,U\rangle_{\Omega_t}\big)\big| \lesssim k_nA_nQ_n^q\big(\|\nabla_{\mathbf{x}}U(t)\|^2_{L^2(\Omega_t)}\big);$$

note that we can eliminate $\mathbf{b}$ in (5.3) because it is divergence free and the argument in (3.9) applies.

For the forcing term in (5.3) we proceed exactly as in Theorem 4.1. Substituting these expressions into (5.3), choosing k_n according to (5.5), and adding over n, we easily obtain (5.6). ☐

Remark 5.1 (existence and uniqueness)**.** The argument in Proposition 3.1 applies, with the integrals defining the bilinear form b replaced by Radau quadrature, provided the time constraint (5.5) is valid and the error terms in (5.3) are handled as in Theorem 5.1 to prove coercivity of b.

Remark 5.2 (monotonicity of nodal values)**.** We retain monotonicity of $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ for $f=0$ provided the time step constraint (5.5) is enforced. This is consistent with the experiments in Figure 2 for $q=1,2,3,4$, but falls short of explaining the monotone behavior for $q>0$ regardless of (5.5). We were not able to produce a test case where the monotonicity property is lost for $q>0$.

Remark 5.3 (vanishing diffusion)**.** In view of (5.5), one might expect oscillations of $\|U(t_n)\|_{L^2(\Omega_{t_n})}$ for small diffusion μ. We were unable so far to observe this for $q>0$ as is reported in Figure 2. Whether or not (5.5) is necessary for stability for $q>0$ needs further investigation.

Remark 5.4 (conservative and non-conservative RKR methods)**.** For moving domains, the non-conservative RKR method (5.1) is no longer equivalent to the conservative one, which reads as (4.7). This is due to the violation of the Reynolds' identity by Radau quadrature. However, stability results (as well as error bounds; cf. [6]) for the conservative RKR method can be derived similarly.

Remark 5.5 (global stability for RKR methods)**.** Proceeding as in Subsection 4.3 we can prove a stability estimate for U in the $L^\infty(L^2)-$norm as well, but again, under the time constraint (5.5).

5.2. **An Implicit-Explicit Runge-Kutta Method.** We present now an interesting first-order method, natural for free boundary problems; cf. [7, 8]. This method is not a RKR method, but falls in the family of implicit-explicit Runge-Kutta (IERK) methods.

To this end, for $q=0$, we set $U_{n+1} := U(\mathcal{A}_{t_n\to t_{n+1}}(\cdot), t_{n+1})$ for $U \in \mathcal{V}_0(I_n)$. We approximate the integrals in (3.5) by the left-side rectangle quadrature rule and we end up with

$$\begin{aligned}\langle (U_{n+1}-U_n) + k_n\big((\mathbf{b}-\mathbf{w})(t_n)\cdot\nabla_{\mathbf{x}} U_{n+1}\big), V\rangle_{\Omega_{t_n}} \\ + \mu k_n \langle \nabla_{\mathbf{x}} U_{n+1}, \nabla_{\mathbf{x}} V\rangle_{\Omega_{t_n}} = k_n\langle f(t_n), V\rangle_{\Omega_{t_n}}, \quad \forall V \in \mathcal{V}_0(I_n).\end{aligned} \tag{5.7}$$

We note that the first order IERK method (5.7) can be obtained from (3.5) by aproximation of the integrals with the right-side rectangle quadrature. It is easily seen that the theory of the present section can be adapted to (5.7) as well. In particular, the stability result of Theorem 5.1 remains valid. The main advantage of method (5.7), which makes it appropriate for free boundary problems is that it is implicit with respect to the approximation U, but explicit with respect to the moving domain. The latter is beneficial whenever we do not know in advance $\Omega_{t_{n+1}}$ at step n, while the implicit nature of the method in U helps avoiding any CFL condition.

6. Advections with Non-Vanishing Divergence

In this last section we discuss well-posedness and stability of dG methods for (1.1) when $\nabla_{\mathbf{x}}\cdot\mathbf{b}\neq 0$ and present the limitations of theory due to such condition. We will see that the results of this section generalize in a natural way not only those for $\nabla_{\mathbf{x}}\cdot\mathbf{b}=0$, but also the corresponding results for $\nabla_{\mathbf{x}}\cdot\mathbf{b}\neq 0$ in non-moving domains. In fact, we will show nodal stability and in the $L^2(H^1)-$norm as well as in the $L^\infty(L^2)-$norm. The analysis of the section is slightly more technical than the case $\nabla_{\mathbf{x}}\cdot\mathbf{b}=0$; however we will rely on the main arguments and relations from the divergence-free case in order to carry out the analysis.

We assume exact integration. The non-conservative dG method (3.5) reads

$$\begin{aligned}&\int_{I_n}\langle D_t U, V\rangle_{\Omega_t}\,dt + \langle U(t_n^+) - U(t_n), V(t_n^+)\rangle_{\Omega_{t_n}} + \int_{I_n}\langle (\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}} U, V\rangle_{\Omega_t}\,dt \\ &+ \int_{I_n}\langle U\nabla_{\mathbf{x}}\cdot\mathbf{b}, V\rangle_{\Omega_t}\,dt + \mu\int_{I_n}\langle\nabla_{\mathbf{x}} U, \nabla_{\mathbf{x}} V\rangle_{\Omega_t}\,dt = \int_{I_n}\langle f, V\rangle_{\Omega_t}\,dt, \quad \forall V\in\mathcal{V}_q(I_n).\end{aligned} \tag{6.1}$$

The conservative dG method reads exactly as (3.6). Since both methods coincide, as in Section 3, we only deal with (6.1) here. Setting $V=U$ in (6.1) yields

$$\begin{aligned}&\frac{1}{2}\|U(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} - \frac{1}{2}\|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \frac{1}{2}\|U(t_n^+) - U(t_n)\|^2_{L^2(\Omega_{t_n})} \\ &\qquad + \frac{1}{2}\int_{I_n}\langle\nabla_{\mathbf{x}}\cdot\mathbf{b}, U^2\rangle_{\Omega_t}\,dt + \mu\int_{I_n}\|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}\,dt = \int_{I_n}\langle f, U\rangle_{\Omega_t}\,dt.\end{aligned} \tag{6.2}$$

in view of (3.7) and (3.9). If $\nabla_{\mathbf{x}}\cdot\mathbf{b}\geq 0$, then we have $\int_{I_n}\langle\nabla_{\mathbf{x}}\cdot\mathbf{b}, U^2\rangle_{\Omega_t}\,dt \geq 0$ and the analysis continues as for $\nabla_{\mathbf{x}}\cdot\mathbf{b}=0$. This suggests decomposing $\nabla_{\mathbf{x}}\cdot\mathbf{b}$ into its positive and negative parts:

$$\nabla_{\mathbf{x}}\cdot\mathbf{b} = (\nabla_{\mathbf{x}}\cdot\mathbf{b})^+ - (\nabla_{\mathbf{x}}\cdot\mathbf{b})^-.$$

Proposition 6.1 (existence and uniqueness)**.** *There exists an absolute constant Λ independent of the ALE map and the time steps k_n so that if we choose the time steps k_n such that*

$$2k_n A_n \Lambda \max\{1,\mu^{-1}\}\|(\nabla_{\mathbf{x}}\cdot\mathbf{b})^-\|_{L^\infty(\mathcal{Q}_n)} \le 1, \tag{6.3}$$

then problem (3.4)-(6.1) *admits a unique solution $U \in \mathcal{V}_q(I_n)$.*

Proof. We follow the steps of the proof of Proposition 3.1, i.e., we proceed by induction on n considering Ω_{t_n} as reference domain and the Hilbert space $\big(\mathcal{V}_q(I_n),(\cdot,\cdot)_{L^2(H_0^1)}\big)$; cf. (3.8). In particular, the bilinear form b associated with (6.1)

$$\begin{aligned} b(V,W) := &\int_{I_n}\langle D_t V, W\rangle_{\Omega_t}\,dt + \langle V(t_n^+),W(t_n^+)\rangle_{\Omega_t}\\ &+\int_{I_n}\langle(\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}}V,W\rangle_{\Omega_t}\,dt + \int_{I_n}\langle V\nabla_{\mathbf{x}}\cdot\mathbf{b},W\rangle_{\Omega_t}\,dt + \mu\int_{I_n}\langle\nabla_{\mathbf{x}}V,\nabla_{\mathbf{x}}W\rangle_{\Omega_t}\,dt \end{aligned}$$

is bounded. Furthermore, integrating (3.12) over I_n and using the inverse inequality with respect to time (cf. Remark 3.1) and Poincaré's inequality (2.1) we obtain

$$\int_{I_n}\|V(t)\|^2_{L^2(\Omega_t)}\,dt \le \Lambda A_n k_n\|V(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} + \Lambda A_n k_n\int_{I_n}\|\nabla_{\mathbf{x}}V(t)\|^2_{L^2(\Omega_t)}\,dt. \tag{6.4}$$

Taking $V=W$ in the bilinear form and using (6.4) in conjunction with

$$\int_{I_n}\langle(\nabla_{\mathbf{x}}\cdot\mathbf{b})^-,V^2\rangle_{\Omega_t}\,dt \le \|(\nabla_{\mathbf{x}}\cdot\mathbf{b})^-\|_{L^\infty(\mathcal{Q}_n)}\int_{I_n}\|V(t)\|^2_{L^2(\Omega_t)}\,dt, \tag{6.5}$$

we arrive at

$$\begin{aligned} b(V,V) \ge &\frac{1}{2}\Big(1-k_n\Lambda A_n\|(\nabla_{\mathbf{x}}\cdot\mathbf{b})^-\|_{L^\infty(\mathcal{Q}_n)}\Big)\|V(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} + \frac{1}{2}\|V(t_n^+)\|^2_{L^2(\Omega_{t_n})}\\ &+\frac{1}{2}\int_{I_n}\langle(\nabla_{\mathbf{x}}\cdot\mathbf{b})^+,U^2\rangle_{\Omega_t}\,dt + \mu\Big(1-k_n\frac{\Lambda A_n}{2\mu}\|(\nabla_{\mathbf{x}}\cdot\mathbf{b})^-\|_{L^\infty(\mathcal{Q}_n)}\Big)\int_{I_n}\|\nabla_{\mathbf{x}}V(t)\|^2_{L^2(\Omega_t)}\,dt. \end{aligned}$$

Thus, in view of (6.3), we get $b(V,V) \ge \frac{\mu}{2}\int_{I_n}\|\nabla_{\mathbf{x}}V(t)\|^2_{L^2(\Omega_t)}\,dt$, and the proof concludes as that of Proposition 3.1. □

Before showing stability for (6.1), we establish a discrete Gronwall lemma. Even though this result is well known, we give a brief proof for completeness.

Lemma 6.1 (discrete Gronwall lemma)**.** *Let $\{\varphi_j\}_{j=0}^N, \{\psi_j\}_{j=0}^N, \{f_j\}_{j=0}^N$ be non-negative and satisfy*

$$(1-\chi_{j+1})\varphi_{j+1} + \psi_j \le \varphi_j + f_j \qquad 0 \le j < N, \tag{6.6}$$

with $0 < \chi_j \le 1/2$. If $G_n = \mathrm{e}^{2\sum_{j=0}^n \chi_j}$, then for $0 \le m < n \le N$

$$\varphi_n + \sum_{j=m}^{n-1}\psi_j \le \mathrm{e}^{G_n-G_m}\varphi_m + \sum_{j=m}^{n-1}\mathrm{e}^{G_n-G_j}f_j. \tag{6.7}$$

Proof. We multiply (6.6) by the 'integrating factor' $\prod_{i=0}^j(1-\chi_i)$ and sum from $j=m$ to $j=n-1$. After exploiting a telescopic cancellation, the asserted estimate (6.7) follows from the elementary relation $\frac{1}{1-x} \le \mathrm{e}^{2x}$, valid for all $0 < x \le 1/2$. □

We now prove nodal stability for advections with non-zero divergence. We stress that this is the sole instance in our analysis where an exponential involving $\nabla_{\mathbf{x}}\cdot\mathbf{b}$, but not the ALE map, appears.

Theorem 6.1 (nodal stability for $\nabla_{\mathbf{x}} \cdot \mathbf{b} \neq 0$)**.** *Assume that $\{\mathcal{A}_t\}_{t\in[0,T]}$ be a family of ALE maps, $\mathbf{b} \in L^\infty(\mathrm{div}; \mathcal{Q}_T)$, and that the time-steps k_n satisfy* (6.3) *for $n = 0, 1, \ldots, N-1$. If $G_n := 2\Lambda \sum_{j=0}^{n} A_n \|(\nabla_{\mathbf{x}} \cdot \mathbf{b})^-\|_{L^\infty(\mathcal{Q}_j)} k_j$, with Λ the constant in* (6.3)*, then the following stability bound holds true for $0 \leq m < n \leq N$*

$$
\begin{aligned}
&\|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \sum_{j=m}^{n-1} \|U(t_j^+) - U(t_j)\|^2_{L^2(\Omega_{t_j})} + \mu \sum_{j=m}^{n-1} \int_{t_m}^{t_n} \|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}\, dt \\
&\qquad \leq \mathrm{e}^{G_n - G_m} \|U(t_m)\|^2_{L^2(\Omega_{t_m})} + \frac{2}{\mu} \sum_{j=m}^{n-1} \mathrm{e}^{G_n - G_j} \int_{I_j} \|f(t)\|^2_{H^{-1}(\Omega_t)}\, dt.
\end{aligned} \tag{6.8}
$$

Proof. Let $\chi_{n+1} := \Lambda A_n \|(\nabla_{\mathbf{x}} \cdot \mathbf{b})^-\|_{L^\infty(\mathcal{Q}_n)} k_n$, which satisfies $\chi_{n+1} \leq 1/2$ according to (6.3). Arguing as in Proposition 6.1, we deduce

$$
\begin{aligned}
&\big(1 - \chi_{n+1}\big) \|U(t_{n+1})\|^2_{L^2(\Omega_{t_{n+1}})} + \|U(t_n^+) - U(t_n)\|^2_{L^2(\Omega_{t_n})} + \mu \int_{I_n} \|\nabla_{\mathbf{x}} U(t)\|^2_{L^2(\Omega_t)}\, dt \\
&\qquad \leq \|U(t_n)\|^2_{L^2(\Omega_{t_n})} + \frac{2}{\mu} \int_{I_n} \|f(t)\|^2_{H^{-1}(\Omega_t)}\, dt.
\end{aligned}
$$

The asserted estimate (6.8) follows directly from Lemma 6.1. □

Remark 6.1 (time constraint for $\nabla_{\mathbf{x}} \cdot \mathbf{b} \neq 0$)**.** Estimate (6.8) extends (3.10) to the case $(\nabla_{\mathbf{x}} \cdot \mathbf{b})^- \neq 0$ because when $(\nabla_{\mathbf{x}} \cdot \mathbf{b})^- = 0$ both estimates coincide. The time-step constraint (6.3) depends explicitly on $\|(\nabla_{\mathbf{x}} \cdot \mathbf{b})^-\|_{L^\infty(\mathcal{Q}_n)}$ and the the ALE constant $A_n = O(1)$ (see (3.2)). Since they interact in a multiplicative fashion, for moderate convection this constraint may be unnoticeable. This agrees with the corresponding theory on non-moving domains, and it is important when studying numerically the incompressible Navier-Stokes equation defined on moving domains [8, 12, 22, 23, 26].

Remark 6.2 (Lagrangian approach)**.** Advection-dominated diffusion problems on fixed domains have been examined by Chrysafinos and Walkington in [10] using the dG method within a Lagrangian framework with exact integration. This corresponds to choosing $\mathbf{w} \approx \mathbf{b}$, to compensate for large advection $\mathbf{b}$, and assuming that $\mathcal{A}_t$ reduces to the identity on the boundary of Ω_0. The dependence on $\mathbf{b}$ of the stability constants is similar in both works. Compared to the present work, the time step restriction in [10] is weaker in the regime μ small but at the expense of stability constants less robust in terms of the ALE map. Notice that the results proposed in [10] are subject to an additional CFL condition due to the use of inverse inequalities in space.

We conclude with a global $L^\infty(L^2)$ stability bound for advections with non-zero divergence.

Theorem 6.2 (global stability for $\nabla_{\mathbf{x}} \cdot \mathbf{b} \neq 0$)**.** *If the conditions of Theorem 6.1 are valid and $f \in L^2(\mathcal{Q}_T)$, then the estimate below holds for the solution $U \in \mathcal{V}_q$ of problem* (3.4)-(6.1)

$$
\begin{aligned}
\sup_{t\in[0,t_n]} \|U(t)\|^2_{L^2(\Omega_t)} &\lesssim \max_{0\leq j\leq n-1} \big\{A_j(1 + F_j k_j)\big\} \Big(\mathrm{e}^{G_n} \|U(0)\|^2_{L^2(\Omega_0)} + \sum_{j=0}^{n-1} \mathrm{e}^{G_n - G_j} \int_{I_j} \|f(t)\|^2_{H^{-1}(\Omega_t)}\, dt\Big) \\
&\quad + \max_{0\leq j\leq n-1} A_j k_j \int_{I_j} \|f(t)\|^2_{L^2(\Omega_t)} \qquad n = 1, \ldots, N,
\end{aligned}
$$

where the constant F_j is given by

$$
F_j := B_j + \mu^{-1}\big(\|\mathbf{b} - \mathbf{w}\|_{L^\infty(\mathcal{Q}_n)} + A_n \|\nabla_{\mathbf{x}} \cdot \mathbf{b}\|_{L^\infty(\mathcal{Q}_n)}\big).
$$

Proof. We go back to the proof of Theorem 3.2 and modify the key relation (3.15) as follows:

$$\begin{aligned}\int_{I_n}(t-t_n)\|D_tU(t)\|^2_{L^2(\Omega_t)}\,dt+\int_{I_n}(t-t_n)\langle(\mathbf{b}-\mathbf{w})\cdot\nabla_{\mathbf{x}}U,D_tU\rangle_{\Omega_t}\,dt&\\+\int_{I_n}(t-t_n)\langle U\nabla_{\mathbf{x}}\cdot\mathbf{b},D_tU\rangle_{\Omega_t}\,dt&\\+\mu\int_{I_n}(t-t_n)\langle\nabla_{\mathbf{x}}U,\nabla_{\mathbf{x}}D_tU\rangle_{\Omega_t}\,dt&=\int_{I_n}(t-t_n)\langle f,D_tU\rangle_{\Omega_t}\,dt.\end{aligned}$$

All terms are exactly the same as in Theorem 3.2, except for that involving $\nabla_{\mathbf{x}}\cdot\mathbf{b}$. Using Cauchy-Schwarz and Poincaré inequalities, such a term becomes

$$\begin{aligned}\int_{I_n}(t-t_n)\langle U\nabla_{\mathbf{x}}\cdot\mathbf{b},D_tU\rangle_{\Omega_t}\,dt\le\frac14\int_{I_n}(t-t_n)\|D_tU(t)\|^2_{L^2(\Omega_t)}\,dt,&\\+cA_nk_n\|\nabla_{\mathbf{x}}\cdot\mathbf{b}\|^2_{L^2(\mathcal{Q}_n)}\int_{I_n}\|\nabla_{\mathbf{x}}U(t)\|^2_{L^2(\Omega_t)}\,dt,&\end{aligned}$$

with c an absolute constant (independent of the ALE map and the time-steps). Inserting this back into the first relation, and invoking the remaining terms (3.16)-(3.17) and (3.19)-(3.20) from Theorem 3.2, we readily end up with the expression (3.21), except that F_n now contains the additional term $A_n\|\nabla_{\mathbf{x}}\cdot\mathbf{b}\|_{L^\infty(\mathcal{Q}_n)}$. We next conclude as in Theorem 3.2. □

Remark 6.3 (nodal stability for dG with quadrature and $\nabla_{\mathbf{x}}\cdot\mathbf{b}\neq 0$)**.** To prove nodal stability for the non-conservative dG scheme with Reynolds' quadrature corresponding to (4.6), we need to handle the term $Q_n\big(\langle\nabla_{\mathbf{x}}\cdot\mathbf{b},U^2\rangle_{\Omega_t}\big)$ in (6.2) instead of $\int_{I_n}\langle\nabla_{\mathbf{x}}\cdot\mathbf{b},U^2\rangle_{\Omega_t}\,dt$. We observe that

$$\begin{aligned}Q_n\big(\langle\nabla_{\mathbf{x}}\cdot\mathbf{b},U^2\rangle_{\Omega_t}\big)&=Q_n\big(\langle(\nabla_{\mathbf{x}}\cdot\mathbf{b})^+U^2\rangle_{\Omega_t}\big)-Q_n(\langle(\nabla_{\mathbf{x}}\cdot\mathbf{b})^-,U^2\rangle_{\Omega_t}\big)\\&\ge-\|(\nabla_{\mathbf{x}}\cdot\mathbf{b})^-\|_{L^2(\mathcal{Q}_n)}\int_{I_n}\|U(t)\|^2_{L^2(\Omega_t)}\,dt,\end{aligned}$$

and next proceed as in Proposition 6.1 and Theorem 6.1. The same result is valid for the conservative version (4.7), because both (4.6) and (4.7) are equivalent, as well as the dG method with Radau quadrature of Section 5.

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