A NON-LOCAL MEAN CURVATURE FLOW AND ITS SEMI-IMPLICIT TIME-DISCRETE APPROXIMATION

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Abstract. We address in this paper the study of a geometric evolution, corresponding to a curvature which is non-local and singular at the origin. The curvature represents the first variation of the energy $\mathcal{M}_\rho(E)$ defined in (1.1), proposed in a recent work [5] as a variant of the standard perimeter penalization for the denoising of nonsmooth curves.

To deal with such degeneracies, we first give an abstract existence and uniqueness result for viscosity solutions of non-local degenerate Hamiltonians, satisfying suitable continuity assumption with respect to Kuratowsky convergence of the level sets. This abstract setting applies to an approximated flow. Then, by the method of minimizing movements, we also build an "exact" curvature flow, and we illustrate some examples, comparing the results with the standard mean curvature flow.

1. Introduction

In a recent paper [5], the last two authors, together with M. Barchiesi, S. H. Kang and T. Le, proposed a variational model for (binary) image denoising, which was supposed to preserve small scale details (or small oscillations of the boundary) while regularizing the large scales. This model is a variant of the celebrated Mumford-Shah functional, where the perimeter term is replaced by the following one

$$\mathcal{M}_\rho(E) \;=\; \frac{1}{2\rho}\left|\left\{x \in \mathbb{R}^d \,:\, \mathrm{dist}(x,E) \le \rho\,, \mathrm{dist}(x,\mathbb{R}^d \setminus E) \le \rho\right\}\right|, \tag{1.1}$$

defined for any $E \subseteq \mathbb{R}^d$, where $\rho > 0$ acts as a scale selection parameter. Here and throughout the paper, given $A \subset \mathbb{R}^d$ measurable, we denote by $|A|$ its Lebesgue measure. Notice that the energy is finite if and only if ∂E is compact. The idea behind such a variant is that fluctuations of ∂E at lengths much smaller than $\rho > 0$ will have very little influence on the energy; on the other hand, it behaves as the standard perimeter on much larger smooth boundaries, and in a more complicated non-local way on sets with fine microstructures on scales of order ρ.

The sort of denoising which is obtained in [5] is shown in Fig. 1, where small oscillations (here the stripes of the fingerprints) are almost untouched, while the noise has mostly been removed.

In this paper, we try to investigate some mathematical analysis aspects of this model. More precisely, we want to study the geometric evolution of curves and shapes by the gradient flow of the functional proposed by these authors.

To this purpose, we first extend our energy to L^1 functions, and express it in terms of a function depending on the oscillation of u on balls of radius ρ, following

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FIGURE 1. An example from [5, Fig. 4.1]: the fingerprint (left: noisy, right: denoised).

the approach in [16]. With this point of view, it turns out that (1.1) is the restriction to characteristics functions of a convex, l.s.c. functional, satisfying a suitable "coarea formula". Then, we introduce the "curvature" as the first variation of this functional with respect to inner variations of the sets. This curvature is not continuous and it is not well defined for all smooth sets. Therefore, in (2.6) we introduce a smoother version $\mathcal{M}^f$ of (1.1), which roughly speaking consists in averaging $\mathcal{M}_r$ over r, for r varying in a neighborhood of ρ. The corresponding curvature is now well defined on smooth sets.

After this preliminary analysis to define a proper notion of curvature, we study the corresponding geometric flow. Using a level set approach and working in the framework of viscosity solutions, we define a mean curvature flow equation, which is both non-local and singular. Indeed, our Hamiltonian $F(x, Du, D^2u, K)$ depends in a non-local way on the level set K, and behaves like a power $(d-1)$ of the curvature tensor of ∂K for vanishing sets, being thus singular in dimension $d \geq 3$ (see (3.27)). To deal with such degeneracy we combine the approach by Slepčev [26] to non-local Hamiltonians with the approach by Ishii and Souganidis [23] and Goto [20] to degenerate Hamiltonians. However, the approach in [26] is based on the assumption that the Hamiltonian is continuous with respect to all its variables, in particular with respect to L^1 convergence of the sets. This is not the case of our Hamiltonian (and of any reasonable regularization of it). Therefore, we build up a variant of the approach in [26] that works for a general class of Hamiltonians satisfying suitable continuity properties with respect to the Kuratowski convergence instead of L^1 convergence of sets. We adapt the notion in [23] of viscosity solutions for singular Hamiltonians to our non local setting, and we show a corresponding result of existence and uniqueness. This result will apply to a general class of Hamiltonians, which does not include the Hamiltonian corresponding to our non-local curvature flow, but only a suitable continuous approximation of it.

Finally, we study the minimizing movements corresponding to the energy $\mathcal{M}^f$. We introduce an implicit time-discretization of the motion, and we show that it converges, up to a subsequence, to a solution of the level set equation in the viscosity sense. In this way we recover an existence result for viscosity solutions also for the exact Hamiltonian corresponding to the first variation of $\mathcal{M}^f$, yet without uniqueness. We mention that in a recent paper of Caffarelli and Souganidis [9], a similar strategy (based, this time, on a diffusion/thresholding time-discrete scheme) has been successfully implemented to build up a non-local curvature flows associated to fractional diffusions. Our time-discretization approach can be numerically implemented, following the approach in [16]: we show eventually in Section 5 a few examples which are compared with the classical mean curvature flow, and seem to confirm a slower smoothing of oscillatory boundaries.

We mention the existence of a few interesting alternative approaches to non-local evolutions. The recent papers of [21, 8] provide a point of view slightly different from ours, and address different kinds of evolutions. In particular, [8] also deals with non-monotone evolutions, such as the one describing the motion of dislocation lines in crystals (see also [2, 3, 6, 7, 12, 22]). Another approach is described in the papers of Cardaliaguet [10, 11], Cardaliaguet and Rouy [15], Cardaliaguet and Ley [13, 14]. There, appropriate definition for evolving tubes are proposed and the convergence of a time-discrete scheme (of the same kind as ours) is addressed in [14]. Moreover, except in the preliminary work [10], the authors of this series of papers have taken care to never need to evaluate the velocity on arguments which are not "natural" (such as smooth level sets and their normal or second fundamental form), contrarily to what is needed in our proof of uniqueness (as in [26]). Unfortunately, extending their work to our approach raises complicated technical issues, since in particular our velocity does not have the required continuity properties, and our minimizers have unknown regularity. This is an interesting direction for future research, but we also believe it is useful to develop the level-set approach in the non-local geometric setting.

To summarize, the first goal of this paper is to investigate the geometric flow corresponding to a non-local variant of the perimeter introduced in [5], in connection with image denoising. We have developed a viscosity approach to non-local singular Hamiltonians, combining many ideas from [26], [23] and [20]. Through the viscosity approach we have obtained existence and uniqueness for a suitable regularization of our Hamiltonian, while a minimizing movements approach yields a solution for the original Hamiltonian. The abstract approach is itself interesting and stimulating: a complete picture at the moment is still missing, and this paper represents a first attempt to study singular non-local Hamiltonians, not even continuous with respect to L^1 convergence, but only with respect to Kuratowski convergence of level sets. The combination of the minimizing movements variational method with viscosity techniques seems to be a promising approach to complete the picture. To our knowledge, up to now this kind of study has been carried out only in [14], [18], and a few papers by the first author and co-workers. We hope that (borrowing in particular from [14]) we will be able to extend these ideas to other motions, and also, understand how to make the proof of the comparison result less dependent on the extension of the Hamiltonian out of its natural domain of definition.

2. The energy functionals

2.1. The ρ-neighborhood. As mentioned, we focus on the study of (1.1). It is well-known that, under mild regularity assumption on E (see for instance [4]) we have

$$\lim_{\rho\to 0}\mathcal{M}_\rho(E) \;=\; \mathcal{H}^{d-1}(\partial E) \;=\; Per(E),$$

where $Per(E)$ is the standard perimeter of E. It is also very easy to show that $\mathcal{M}_\rho$ Γ-converges to the standard perimeter [17]. An issue with definition (1.1) is that it depends on the choice of the representative within the Lebesgue equivalence class of the set E. For this reason, one may introduce the following variant:

$$\mathcal{E}_\rho(E) \;=\; \frac{1}{2\rho}\int_{\mathbb{R}^d} \operatorname{osc}_{B(x,\rho)}(\chi_E)\,dx \tag{2.1}$$

where $\operatorname{osc}_A(u)$ denotes the *essential oscillation* of the measurable function u over a measurable set A, defined by

$$\operatorname{osc}_A(u) \;=\; \operatorname*{ess\,sup}_A u \;-\; \operatorname*{ess\,inf}_A u.$$

One checks that $\mathcal{E}_\rho(E)$ coincides with the measure of the ρ-neighborhood of the essential boundary of E. Moreover,

$$\mathcal{E}_\rho(E) \;=\; \inf\{\mathcal{M}_\rho(E') : \, |E\triangle E'| = 0\},$$

where $E\triangle E'$ denotes the symmetric difference $(E\setminus E')\cup(E'\setminus E)$. Finally, $\mathcal{E}_\rho(E) = \mathcal{E}_\rho(E^c)$ and it is finite if and only if either E or E^c is (essentially) bounded (where $E^c := \mathbb{R}^d\setminus E$). An advantage of Definition (2.1) is that can be easily generalized to a measurable function $u\in L^1_{loc}(\mathbb{R}^d)$. By a slight abuse of notation, we still denote $\mathcal{E}_\rho(u)$ the functional:

$$\mathcal{E}_\rho(u) \;=\; \frac{1}{2\rho}\int_{\mathbb{R}^d} \operatorname{osc}_{B(x,\rho)}(u)\,dx. \tag{2.2}$$

One can check that this energy is one-homogeneous, convex and therefore subadditive. Moreover, it is lower semicontinuous with respect to weak* convergence in L^∞_{loc}, and satisfies the following generalized coarea formula

$$\mathcal{E}_\rho(u) \;=\; \int_{-\infty}^{\infty}\mathcal{E}_\rho(\{u>s\})\,ds. \tag{2.3}$$

This follows from the fact that for any A and u we have

$$\operatorname{osc}_A(u) \;=\; \int_{-\infty}^{\infty}\operatorname{osc}_A(\chi_{\{u>s\}})\,ds.$$

Moreover, one easily deduces that, given two measurable sets $A, B\subset\mathbb{R}^d$,

$$\mathcal{E}_\rho(A\cup B) \;+\; \mathcal{E}_\rho(A\cap B) \;\le\; \mathcal{E}_\rho(A) \;+\; \mathcal{E}_\rho(B)\,. \tag{2.4}$$

Indeed, observing that $\chi_{A\cup B}+\chi_{A\cap B} = \chi_A+\chi_B$, by coarea formula and in view of the subadditivity of $\mathcal{E}_\rho$, we have

$$\begin{aligned}\mathcal{E}_\rho(A\cup B)+\mathcal{E}_\rho(A\cap B) &= \int_{-\infty}^{\infty}\mathcal{E}_\rho(\{\chi_{A\cup B}+\chi_{A\cap B}>s\})\,ds\\ &= \mathcal{E}_\rho(\chi_A+\chi_B)\le\mathcal{E}_\rho(\chi_A)+\mathcal{E}_\rho(\chi_B) = \mathcal{E}_\rho(A)+\mathcal{E}_\rho(B).\end{aligned} \tag{2.5}$$

2.2. **The continuous energy functional.** Fix $\rho_0 > 0$ and $\underline{\delta} \in (0, \rho_0)$. We consider now a Lipschitz function $f : \mathbb{R} \to \mathbb{R}_+$ which is even, with $\mathrm{supp} f = [-\rho_0, \rho_0]$, constant in $[-\underline{\delta}, \underline{\delta}]$ and nonincreasing in $\mathbb{R}_+$.

We then introduce the following variant of (1.1):

$$\mathcal{M}^f(E) \; = \; \int_{\mathbb{R}^d} f(d_E(x))\, dx, \tag{2.6}$$

where d_E is the signed distance to ∂E (negative inside E and positive outside). Notice that $\mathcal{M}^f(E)$ is finite if and only if ∂E is compact, i.e., E or its complement is bounded. We now show that

$$\mathcal{M}^f(E) = \int_0^{\rho_0} (-2sf'(s))\mathcal{M}_s(E)\, ds.$$

Since $\mathcal{M}^f(E) = \mathcal{M}^f(E^c)$ we can assume that E^c is bounded. Then, thanks to the co-area formula we have

$$\begin{aligned}
\mathcal{M}^f(E) \; &= \; \int_{\mathbb{R}^d} f(d_E(x))|Dd_E(x)|\, dx = \int_{-\rho_0}^{\rho_0} f(s)\mathcal{H}^{d-1}(\partial\{d_E > s\})\, ds \\
&= - \int_{-\rho_0}^{\rho_0} -f'(s)|\{s < d_E < \rho_0\}|\, ds \\
&= \int_0^{\rho_0} -f'(s)(|\{-s < d_E < \rho_0\}| - |\{s < d_E < \rho_0\}|)\, ds \\
&= \int_0^{\rho_0} (-2sf'(s))\mathcal{M}_s(E)\, ds.
\end{aligned}$$

Thanks to this, we can introduce the following variant of $\mathcal{M}^f$, defined on Borel sets and which depends only on the Lebesgue equivalence class

$$\mathcal{E}^f(E) \; = \; \int_0^{\rho_0} (-2sf'(s))\mathcal{E}_s(E)\, ds. \tag{2.7}$$

As before, we consider the convex extension of $\mathcal{E}^f$, defined for all functions $u \in L^1_{loc}(\mathbb{R}^d)$ by

$$\mathcal{E}^f(u) \; = \; \int_0^{\rho_0} (-2sf'(s))\mathcal{E}_s(u)\, ds = \int_\Omega \int_0^{\rho_0} (-2sf'(s))\mathrm{osc}_{B(x,s)}(u)\, ds\, dx. \tag{2.8}$$

By construction, $\mathcal{E}^f$ is a convex, lower semicontinuous energy which satisfies the generalized coarea formula

$$\mathcal{E}^f(u) \; = \; \int_{-\infty}^{\infty} \mathcal{E}^f(\{u > s\})\, ds. \tag{2.9}$$

Clearly, (2.4) is still true also for $\mathcal{E}^f$.

2.3. **The non-local curvature.** Let $E \subset \mathbb{R}^d$ be a smooth set with compact boundary. We denote by $\nu_E(x)$ the outer normal unit vector to ∂E at x. The non-local curvature κ_ρ is formally defined as the first variation of the energy $\mathcal{E}_\rho$ in (2.1). Set

$$\kappa_\rho(E, x) \; = \; \kappa^+_\rho(E, x) \, + \, \kappa^-_\rho(E, x), \tag{2.10}$$

where

$$\kappa_\rho^+(E,x) \;=\; \begin{cases} \dfrac{1}{2\rho}\det(I+\rho\nabla\nu_E(x)) & \text{if } \operatorname{dist}(x+\rho\nu_E(x),E)=\rho\,,\\ \\ 0 & \text{otherwise,}\end{cases} \tag{2.11}$$

$$\kappa_\rho^-(E,x) \;=\; \begin{cases} -\dfrac{1}{2\rho}\det(I-\rho\nabla\nu_E(x)) & \text{if } \operatorname{dist}(x-\rho\nu_E(x),E^c)=\rho\,,\\ \\ 0 & \text{otherwise.}\end{cases} \tag{2.12}$$

Let us decompose ∂E into three sets: $\partial E = A_\rho^+ \cup B_\rho^+ \cup \mathcal{N}_\rho^+$, where

$A_\rho^+ := \{x\in\partial E:\ \text{there exists } t>\rho:\ \operatorname{dist}(x+t\nu_E(x),E)=t\}$,
$B_\rho^+ := \{x\in\partial E : \operatorname{dist}(x+\rho\nu_E(x),E)<\rho\}$.
$\mathcal{N}_\rho^+ = \partial E\setminus(A_\rho^+\cup B_\rho^+)$.

Analogously, we defineA_ρ^-, B_ρ^-, $\mathcal{N}_\rho^-$ with $\nu_E(x)$ replaced by $-\nu_E(x)$, and we set $\mathcal{N}_\rho := \mathcal{N}_\rho^+\cup\mathcal{N}_\rho^-$.

Lemma 2.1. *Let $E\subset\mathbb{R}^n$ be a set of class C^2 with compact boundary, such that $\mathcal{H}^{d-1}(\mathcal{N}_\rho)=0$. Then, for every $\varphi\in C^2(\partial E;\mathbb{R})$ we have*

$$\frac{d}{d\varepsilon}\mathcal{E}_\rho\big(\Phi_\varepsilon(E)\big)_{|\varepsilon=0} = \int_{\partial E}\kappa_\rho\varphi\,d\mathcal{H}^{d-1}, \tag{2.13}$$

where Φ_ε is a diffeomorphism such that $\Phi(x)=x+\varepsilon\varphi(x)\nu_E(x)$ for $x\in\partial E$.

Remark 2.2. Notice that the assumption of Lemma 2.1 holds true for generic smooth sets. More precisely, given a smooth set E, then for almost all positive ρ one has $\mathcal{H}^{d-1}(\mathcal{N}_\rho)=0$. Moreover, such assumption is crucial. Indeed, let E be a rectangle of sides 2 and 4, respectively, and set $\rho=1$. In this case, a curvature κ_ρ satisfying (2.13) is not well defined. Indeed, one readily sees that such curvature κ_ρ should depend in a non-local way on φ. More precisely, let $\varphi=\varphi_1+\varphi_2$, where φ_i are defined in a small neighborhood of the middle points p_i of the large sides L_i of E, and assume that φ_i have constant sign. Then, if $\varphi_1(p_1)+\varphi_2(p_2)>0$, then (2.13) holds true, while if $\varphi_1(p_1)+\varphi_2(p_2)<0$, then (2.13) holds true with κ_ρ replaced by κ_ρ^+. In particular, the "curvature" at p_1 depends on the value of φ at p_2.

Proof of Lemma 2.1. We will show that

$$\frac{d}{d\varepsilon}\mathcal{E}_\rho^\pm\big(\Phi_\varepsilon(E)|E\big)_{|\varepsilon=0} = \int_{\partial E}\kappa_\rho^\pm\varphi\,d\mathcal{H}^{d-1}, \tag{2.14}$$

where, for every set F

$$\mathcal{E}_\rho^+(F|E) := \frac{1}{2\rho}\int_{E^c}\operatorname{osc}_{B(x,\rho)}(\chi_F)\,dx \qquad\text{and}\qquad \mathcal{E}_\rho^-(F|E) = \frac{1}{2\rho}\int_{E}\operatorname{osc}_{B(x,\rho)}(\chi_F)\,dx.$$

In order to prove (2.13), we will focus on the identity

$$\frac{d}{d\varepsilon}\mathcal{E}_\rho^+\big(\Phi_\varepsilon(E)\big)_{|\varepsilon=0} = \int_{\partial E}\kappa_\rho^+\varphi\,d\mathcal{H}^{d-1}, \tag{2.15}$$

the variation of $\mathcal{E}_\rho^-$ being analogous. The proof is divided in three steps: first, we prove that (2.15) holds if the support of φ is contained in A_ρ^+. Then, we show that the variation vanishes on B^+_ρ. Finally, by a localization argument, recalling also $\mathcal{H}^{d-1}(N^+_\rho)=0$, we deduce the validity of (2.15).

Step 1. In this Step we assume that $\mathrm{supp}(\varphi)\subseteq A^+_\rho$, and then prove (2.15). For every $x\in A^+_\rho$ let $y_\varepsilon(x):=\Phi_\varepsilon(x)=x+\varepsilon\varphi(x)\nu_E(x)$, and set

$$E^+_{\varphi,\varepsilon}:=\big\{y_\varepsilon(x)+t\nu_{\Phi_\varepsilon(E)}(y_\varepsilon(x)),\,x\in\partial E,\,t\in(-\rho,\rho)\big\}\setminus E, \tag{2.16}$$

so that, for ε small enough

$$\mathcal{E}^+_\rho\big(\Phi_\varepsilon(E)|E\big)=|E^+_{\varphi,\varepsilon}|.$$

Set now

$$\tilde{E}^+_{\varphi,\varepsilon}:=\{x+t\nu_E(x),x\in\partial E,\,t\in(0,\rho+\varepsilon\varphi(x))\}\setminus E.$$

By construction, since ∂E is of class C^2, one can see that $|E^+_{\varphi,\varepsilon}\triangle\tilde{E}^+_{\varphi,\varepsilon}|=o(\varepsilon)$. Finally, we have

$$\begin{aligned}\frac{\mathcal{E}^+_\rho\big(\Phi_\varepsilon(E)|E\big)-\mathcal{E}^+_\rho\big(E|E\big)}{\varepsilon}&=\frac{|E^+_{\varphi,\varepsilon}|-|E^+_{\varphi,0}|}{\varepsilon}=\frac{|\tilde{E}^+_{\varphi,\varepsilon}|-|E^+_{\varphi,0}|}{\varepsilon}+o(1)\\&=\frac{1}{\varepsilon}\int_{\partial E}dx\int_0^{\varepsilon\varphi(x)}|\det\big(I+(\rho+t)\nabla\nu_E(x)\big)|dt+o(1).\end{aligned}$$

For $\varepsilon\to0$ we recover (2.15).

Step 2. In this step we show that the curvature κ^+_ρ vanishes on B^+_ρ. This amounts to show that, if φ has support in B^+_ρ, then

$$\frac{d}{d\varepsilon}\mathcal{E}^+_\rho\big(\Phi_\varepsilon(E)\big)_{|\varepsilon=0}=0.$$

This is readily seen, since by definition of B^+_ρ we have that, for ε small enough, $E^+_{\varphi,\varepsilon}=E^+_{\varphi,0}$, so that

$$\mathcal{E}^+_\rho\big(\Phi_\varepsilon(E)|E\big)=|E^+_{\varphi,\varepsilon}|=|E^+_{\varphi,0}|=\mathcal{E}^+_\rho\big(E|E\big).$$

Step 3. In this step we conclude the proof by standard localization arguments. Given $\delta>0$, we can always write $\varphi=\varphi_1+\varphi_2+\varphi_3$ where $\mathrm{supp}(\varphi_1)\subset A^+_\rho$, $\mathrm{supp}(\varphi_2)\subset B^+_\rho$, $\mathcal{H}^{d-1}(\mathrm{supp}(\varphi_3))\le\delta$ and $|\varphi_i|\le|\varphi|$. Notice that

$$\mathcal{E}^+_\rho\big(\Phi_\varepsilon(E)|E\big)=|E^+_{\varphi_1+\varphi_2+\varphi_3,\varepsilon}|=|E^+_{\varphi_1+\varphi_2,\varepsilon}|+r,$$

where $|r|\le C\varepsilon\delta$. Moreover, for ε small enough

$$E^+_{\varphi_1+\varphi_2,\varepsilon}=E^+_{\varphi_1,\varepsilon}.$$

Therefore, using Step 1, and since $\kappa^+_\rho\varphi_2\equiv0$ by Step 2, we conclude

$$\begin{aligned}&\limsup_{\varepsilon\to0}\left|\frac{\mathcal{E}^+_\rho\big(\Phi_\varepsilon(E)|E\big)-\mathcal{E}^+_\rho\big(E|E\big)}{\varepsilon}-\int_{\partial E}\kappa^+_\rho\varphi\,d\mathcal{H}^{d-1}\right|\le\\&\lim_{\varepsilon\to0}\left|\frac{|E^+_{\varphi_1,\varepsilon}|-|E^+_{\varphi_1,0}|}{\varepsilon}-\int_{\partial E}\kappa^+_\rho\varphi_1\,d\mathcal{H}^{d-1}\right|+C'\delta=C'\delta.\end{aligned}\tag{2.17}$$

We conclude by the arbitrariness of δ.

□

Now we introduce the non-local curvature κ_f associated to the energy (2.7):

$$\kappa_f(E,x) = \kappa_f^+(E,x) + \kappa_f^-(E,x), \tag{2.18}$$

where

$$\kappa_f^\pm(E,x) = \int_0^\rho (-2sf'(s))\kappa_s^\pm(E,x)\,ds. \tag{2.19}$$

The following result is a direct consequence of (2.7), Lemma 2.1 and Remark 2.2:

Theorem 2.3. *Let $E \subset \mathbb{R}^n$ be an open bounded set of class C^2. Then, for every $\varphi \in C^2(\partial E;\mathbb{R})$ we have*

$$\frac{d}{d\varepsilon}\mathcal{E}\big(\Phi_\varepsilon(E)\big)_{|\varepsilon=0} = \int_{\partial E} \kappa_f(E,x)\varphi(x)\,d\mathcal{H}^{d-1}(x), \tag{2.20}$$

where Φ_ε is a diffeomorphism such that $\Phi(x) = x + \varepsilon\varphi(x)\nu_E(x)$ for $x \in \partial E$.

3. Viscosity solutions of the non-local level-set equation

In this section we introduce the level set formulation of the geometric evolution problem

$$V = \kappa_f, \tag{3.1}$$

where V represents the normal velocity of the boundary of the evolving sets $t \mapsto E_t$, and we give a proper notion of viscosity solution. Then, we develop an abstract setting where we provide existence and uniqueness results, for a suitable class of Hamiltonians. Unfortunately, as already said in the introduction, this theory applies only to a suitable regularization of the curvature κ_f. However, we will also provide later on (Section 4) an existence result, without uniqueness, for equation (3.1).

3.1. The non-local evolution.

Here we introduce the level set formulation of the geometric evolution problem (3.1). To this aim, following the level set approach, we identify E_t with the superlevel set $\{u \geq 0\}$ of a function $u : \mathbb{R}\times\mathbb{R}^d \mapsto \mathbb{R}$, and study the corresponding degenerate parabolic equation in u. Let $\mathbb{M}^{d\times d}_{\rm sym}$ denote the class of $d\times d$ symmetric matrices, and $\mathcal{C}(\mathbb{R}^d)$ the class of closed subsets of $\mathbb{R}^d$. We introduce the Hamiltonian $F_f : \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{M}^{d\times d}_{\rm sym}\times\mathcal{C}(\mathbb{R}^d) \mapsto \mathbb{R}$ defined by

$$F_f(x,p,X,K) := \int_0^{\rho_0} (-2sf'(s))F_s(x,p,X,K)\,ds\,, \tag{3.2}$$

where

$$F_s(x,p,X,K) \;=\; F_s^+(x,p,X,K) \,+\, F_s^-(x,p,X,K) \tag{3.3}$$

and the functions F_s^+ and F_s^- are defined as follows:

$$F_s^+(x,p,X,K) \;=\; \begin{cases} \dfrac{|p|}{2s}\det\left[I - \dfrac{s}{|p|}\mathcal{P}_{\hat p}X\mathcal{P}_{\hat p}\right]^+ & \text{if } \begin{cases} p\neq 0,\\ \operatorname{dist}(x - s\hat p, K) \geq s,\end{cases}\\[2ex] 0 & \text{otherwise,}\end{cases}$$

$$F_s^-(x,p,X,K) \; = \begin{cases} -\dfrac{|p|}{2s}\det\left[I+\dfrac{s}{|p|}\mathcal{P}_{\hat p}X\mathcal{P}_{\hat p}\right]^+ & \text{if } \begin{cases} p\neq 0, \\ \operatorname{dist}(x+s\hat p, K^c)\geq s, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathcal{P}_{\hat p} := (I-\hat p\otimes\hat p)$, where, for $p\neq 0$, $\hat p = p/|p|$, and for X a symmetric matrix, $[X]^+$ is the matrix with all eigenvalues replaced with their positive part (in particular, $\det[X]^+ = 0$ for any X which is not positive definite).

Remark 3.1. If u is a smooth function and $u(x)$ is not a critical level, by Theorem 2.3 we easily deduce that

$$F_f(x, Du(x), D^2u, \{y\in\mathbb{R}^d : u(y)\geq u(x)\}) = |Du(x)|\,\kappa_f(\{u\geq u(x)\}, x). \tag{3.4}$$

In this identity we use in particular that, if $p=\nabla u$, $X=D^2u$, $K=\{u\geq u(x)\}$, then $\det\left[I-\frac{s}{|p|}\mathcal{P}_{\hat p}X\mathcal{P}_{\hat p}\right]^+ = 0$ means that there is a direction along which the curvature is larger than $1/s$, so that $\kappa_s^+(K,x)=0$.

The level set approach consists in solving the following parabolic Cauchy problem

$$\begin{cases} u_t(x,t) + F_f(x, Du(x,t), D^2u(x,t), \{y:\ u(y,t)\geq u(x,t)\}) \;=\; 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t>0\,, x\in\mathbb{R}^d, \\ u(0,\cdot) = u_0, \end{cases} \tag{3.5}$$

in the viscosity sense. The definition of a viscosity solution for such a non-local Hamiltonian will be introduced in the next subsection. We will prove an existence and uniqueness result in this setting, which will be applied to a smoothed variant of F_f.

3.2. **The abstract setting.** We introduce here a notion of viscosity solutions for problems such as (3.5). The issues are of course that the Hamiltonian is non-local, but also that it is singular in $p=0$ (at least in dimension $d\geq 3$), in the sense that it grows as the set vanishes as a power ($d-1$, in dimension d) of the curvature tensor. For this reason, we have to adapt both the setting of Slepčev [26] for non-local evolutions (notice however that we will consider weaker continuity assumptions with respect to the set variable), and the one of Ishii and Souganidis [23] (see also Goto [20]) for singular Hamiltonians.

We will first list the properties which our Hamiltonians need to satisfy in order to show an existence and uniqueness result, and then introduce the appropriate definition of a viscosity solution (which is almost standard). Let $\mathcal{A}(\mathbb{R}^d)$ denote the family of open sets in $\mathbb{R}^d$, and $\mathcal{C}(\mathbb{R}^d)$ the family of closed sets. We consider Hamiltonians $F:\mathbb{R}^d\times\mathbb{R}^d\setminus\{0\}\times\mathbb{M}^{d\times d}_{\mathrm{sym}}\times\{\mathcal{C}(\mathbb{R}^d)\cup\mathcal{A}(\mathbb{R}^d)\}\mapsto\mathbb{R}$ satisfying the following properties:

i) Translational invariance: $F(x+r,p,X,E+r) = F(x,p,X,E)$ for every $r\in\mathbb{R}^d$;
ii) Degenerate ellipticity: $F(x,p,X,E)\geq F(x,p,Y,E)$ if $X\leq Y$;
iii) Monotonicity in the set variable: $F(x,p,X,E)\geq F(x,p,X,G)$ if $E\subseteq G$;
iv) Geometric property: $F(x,\lambda p,\lambda X+\mu p\otimes p,E) = \lambda F(x,p,X,E)$ for all $\lambda\geq 0$, $\mu\in\mathbb{R}$.

v) Continuity: F is continuous with respect to its first variable, moreover, the following properties hold:

v.1) If $x_n \to x$, $p_n \to p \neq 0$, $X_n \to X$ and $\{K_n\} \subset \mathcal{C}(\mathbb{R}^d)$ is a sequence converging to K in the Kuratowski sense, then

$$F(x,p,X,K) \leq \liminf_n F(x_n,p_n,X_n,K_n).$$

v.2) If $x_n \to x$, $p_n \to p \neq 0$, $X_n \to X$ and $\{A_n\} \subset \mathcal{A}(\mathbb{R}^d)$ is a sequence such that A_n^c converges to A^c in the Kuratowski sense, then

$$F(x,p,X,A) \geq \limsup_n F(x_n,p_n,X_n,A_n).$$

vi) There exists a continuous function $c : (0,+\infty) \mapsto (0,+\infty)$ such that, for all $x \in \mathbb{R}^d$, $p \in \mathbb{R}^d \setminus \{0\}$, $E \in \mathcal{C}(\mathbb{R}^d) \cup \mathcal{A}(\mathbb{R}^d)$ we have

$$-c(|p|) \leq F(x,p,\pm I,E) \leq c(|p|). \tag{3.6}$$

Following [23], we introduce the family $\mathcal{F}$ of functions $f \in C^2([0,\infty))$ such that $f(0) = f'(0) = f''(0) = 0$, and such that $f''(r) > 0$ for all $r > 0$ which satisfy

$$\lim_{p\to 0} \frac{f'(|p|)}{|p|} c(|p|) = 0. \tag{3.7}$$

We refer to [23, p. 229] for the proof that the family $\mathcal{F}$ is not empty.

Let $T > 0$ be fixed. As a slight variant to [23], we introduce the following definition.

Definition 3.2. *We will say that $\varphi \in C^0(\mathbb{R}^d \times (0,T))$ is* admissible at the point *$\hat{z} = (\hat{x},\hat{t})$ if it is of class C^2 in a neighborhood of $\hat{z}$ and, in case $D\varphi(\hat{z}) = 0$, the following holds: there exists $f \in \mathcal{F}$ and $\omega \in C^0([0,\infty))$ satisfying $\lim_{r\to 0} \omega(r)/r = 0$, such that*

$$|\varphi(x,t) - \varphi(\hat{z}) - \varphi_t(\hat{z})(t-\hat{t})| \leq f(|x-\hat{x}|) + \omega(|t-\hat{t}|)$$

for all (x,t) in a neighborhood of $\hat{z}$.

Given a function u_0, which is uniformly continuous in $\mathbb{R}^d$, we want to solve

$$u_t(x,t) + F(x, Du(x,t), D^2u(x,t), \{y \,:\, u(y,t) \geq u(x,t)\}) = 0 \qquad \text{for } (x,t) \in \mathbb{R}^d \times (0,T), \tag{3.8}$$

subject to the initial condition $u(0,\cdot) = u_0$. We introduce the following definition of a viscosity sub/supersolution, inspired from both frameworks of [23] and [26].

Definition 3.3. *An upper semicontinuous function $u : \mathbb{R}^d \times [0,T) \to \mathbb{R}$ is a viscosity subsolution of (3.8) if for all $z := (x,t) \in \mathbb{R}^d \times (0,T)$ and all $\varphi \in C^0(\mathbb{R}^d \times (0,T))$ such that $u - \varphi$ has a maximum at z and φ is admissible at z we have*

$$\begin{cases} \varphi_t(z) + F\left(x, D\varphi(z), D^2\varphi(z), \{y : \varphi(y,t) \geq \varphi(z)\}\right) \leq 0 & \text{if } D\varphi(z) \neq 0, \\ \varphi_t(z) \leq 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

A lower semicontinuous function is a viscosity supersolution of (3.8) if for all $z \in \mathbb{R}^d \times (0,T)$ and all $\varphi \in C^0(\mathbb{R}^d \times (0,T))$ such that $u - \varphi$ has a minimum at z and

φ is admissible at z we have

$$\begin{cases} \varphi_t(z) + F\left(x, D\varphi(z), D^2\varphi(z), \{y : \varphi(y,t) > \varphi(z)\}\right) \geq 0 & \text{if } D\varphi(z) \neq 0, \\ \varphi_t(z) \geq 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Finally, a function u is a viscosity solution of (3.8) *if its upper semicontinuous envelope is a subsolution and its lower semicontinuous envelope is a supersolution of* (3.8).

As it is standard in the theory of viscosity solutions, the maximum in the definition of subsolutions can be assumed to be strict, while the test functions φ can be assumed to be coercive (and similarly for supersolutions). For the reader's convenience, we show that this is the case also in our non-local setting. Assume for instance that u is a subsolution, $u - \varphi$ has a maximum at some (x,t), with φ admissible at (x,t). If $D\varphi(x,t) \neq 0$ we replace φ with

$$\varphi_\varepsilon(y,s) := \varphi(y,s) + \varepsilon|y-x|^2 + |t-s|^2.$$

Then the maximum of $u - \varphi_\varepsilon$ at (x,t) is strict, and we recover the inequality (3.9) for φ by letting $\varepsilon \to 0$ and using the semicontinuity of F, observing that the sets $\{\varphi(y,t) + \varepsilon|y-x|^2 \geq \varphi(x,t)\}$ converge to $\{\varphi(y,t) \geq \varphi(x,t)\}$ in the Kuratowski sense. We use then the semicontinuity property v.1) to conclude.

If $D\varphi(x,t) = 0$, we choose $f \in \mathcal{F}$ as in Definition 3.2, and replace φ by

$$\tilde{\varphi}(y,s) := \varphi(y,s) + f(y-x) + |t-s|^2.$$

We still have $D\tilde{\varphi}(x,t) = 0$, $\tilde{\varphi}$ is admissible at (x,t), $\tilde{\varphi}_t(x,t) = \varphi_t(x,t)$ and now the maximum of $u - \tilde{\varphi}$ is strict.

Notice that our definition of supersolutions and subsolutions is formally different than the one given in [26], that involves the superlevel sets of u instead of φ. Indeed, in the case of a subsolution we can assume that the test function φ is such that $u \leq \varphi$, and $u(x,t) = \varphi(x,t)$. Then, $\{y : u(y,t) \geq u(x,t)\} \subset \{y : \varphi(y,t) \geq \varphi(x,t)\}$ so that

$$\begin{aligned} &F\left(x, D\varphi(x,t), D^2\varphi(x,t), \{y : u(y,t) \geq u(x,t)\}\right) \\ &\qquad\qquad \geq\ F\left(x, D\varphi(x,t), D^2\varphi(x,t), \{y : \varphi(y,t) \geq \varphi(x,t)\}\right). \end{aligned}$$

Therefore, our definition seems actually weaker. The following Lemma shows that, in fact, it is equivalent.

Lemma 3.4. *Let u be a viscosity subsolution of* (3.8). *Then, for all (x,t) in $\mathbb{R}^d \times (0,T)$ and all $\varphi \in C^0(\mathbb{R}^d \times (0,T))$ admissible at (x,t), with $D\varphi(x,t) \neq 0$, and such that $u - \varphi$ has a maximum at (x,t) we have*

$$\varphi_t(x,t) + F\left(x, D\varphi(x,t), D^2\varphi(x,t), \{y : u(y,t) \geq u(x,t)\}\right) \leq 0. \tag{3.11}$$

A similar statement holds for supersolutions.

Proof. We can assume that the test function φ is such that $u \leq \varphi$ and $u(x,t) = \varphi(x,t)$. Consider a decreasing sequence ψ^n of functions which are smooth and such that $\inf_n \psi^n = u$, $\psi^n \geq u + 1/n$. Such a sequence exists because u is upper-semicontinuous. We consider now the test function $\varphi^n = \min\{\varphi, \psi^n\}$, and notice that $\varphi^n = \varphi$ in a neighborhood of (x,t), and hence $u - \varphi^n$ still has a maximum at (x,t). By the very definition of subsolutions we have

$$\varphi_t(x,t) \;+\; F\left(x, D\varphi(x,t), D^2\varphi(x,t), \{y : \varphi^n(y,t) \geq u(x,t)\}\right) \;\leq\; 0.$$

Consider $K_n = \{\varphi^n(\cdot,t) \geq u(x,t)\} \supseteq K = \{u(\cdot,t) \geq u(x,t)\}$. Since the sequence of the sets K_n is nonincreasing, $K_n \to \bigcap_k K_k$ in the Kuratowski sense, and by construction $K = \bigcap_k K_k$. It follows that

$$\begin{aligned}\liminf_{n\to\infty} F\left(x, D\varphi(x,t), D^2\varphi(x,t), \{y : \varphi^n(y,t) \geq u(x,t)\}\right)&\\ \geq\; F\left(x, D\varphi(x,t), D^2\varphi(x,t), \{y : u(y,t) \geq u(x,t)\}\right)&\end{aligned}$$

and (3.11) follows. □

Remark 3.5. By the assumption iv) on F, a standard argument shows that if u is a subsolution (supersolution) and $\theta : \mathbb{R} \to R$ is increasing, then $\theta \circ u$ is still a subsolution (supersolution).

3.3. **A comparison result.** Here we provide a comparison result, that is the main ingredient to get existence and uniqueness in the viscosity setting. Let us set

$$Q_T := \mathbb{R}^d \times (0,T), \qquad \partial_p Q_T = \mathbb{R}^d \times \{0\}, \qquad \mathcal{R}_T = Q_T \cup \partial_p Q_T.$$

Moreover, we denote by $USC(\mathcal{R}_T)$ and $LSC(\mathcal{R}_T)$ the space of upper and lower semicontinuous functions on $\mathcal{R}_T$, respectively. The following comparison principle is an extension of [23, Theorem 1.7] for non local evolutions.

Theorem 3.6. *Let $u \in USC(\mathcal{R}_T)$ and $v \in LSC(\mathcal{R}_T)$ be a subsolution and a supersolution of* (3.8)*, respectively. Assume that*

$$\lim_{r\downarrow 0} \sup\{u(z) - v(\zeta) : (z,\zeta) \in \partial_p Q_T \times \mathcal{R}_T) \cup (\mathcal{R}_T \times \partial_p Q_T),\ |z-\zeta| \leq r\} \leq 0. \tag{3.12}$$

Then $u \leq v$ in $\mathcal{R}_T$, and moreover,

$$\lim_{r\downarrow 0} \sup\{u(z) - v(\zeta) : z,\zeta \in \mathcal{R}_T,\ |z-\zeta| \leq r\} \leq 0.$$

Proof. The proof follows the line of the proof of [23, Theorem 1.7]. We do not provide a self-contained proof; we only indicate the changes needed to adapt that proof to the context of our non-local setting. For the reader's convenience, we will use the same notation as in [23], up to the fact that in our case, $\Omega = \mathbb{R}^d$ (and the space dimension is denoted by d instead of N).

As in [23], by Remark 3.5 we may assume without loss of generality that u and v are bounded, and we extend their domain of definition on $\overline{Q}_T$ by setting

$$\begin{aligned} u(x,T) &= \lim_{r\downarrow 0} \sup\{u(y,s) : (y,s) \in \mathcal{R}_T\,, |y-x| + |s-T| \leq r\}\,,\\ v(x,T) &= \lim_{r\downarrow 0} \inf\{v(y,s) : (y,s) \in \mathcal{R}_T\,, |y-x| + |s-T| \leq r\}\,. \end{aligned}$$

The functions u and v are still upper and lower semicontinuous in $\overline{Q}_T$, respectively.

We first show that u is still a subsolution (and v a supersolution) in $\mathbb{R}^d \times (0,T]$ in the obvious sense. Assume indeed that $u - \varphi$ has a strict maximum at $z = (y,T)$, where φ is admissible and coercive.

Assume first that $D\varphi(z) \neq 0$. For any $n \in \mathbb{N}$ large enough, the function $(x,t) \mapsto u(x,t) - \varphi(x,t) - 1/[n(T-t)]$ attains a maximum at a point $z_n = (y_n, t_n) \in Q_T$, where $z_n \to z$ as $n \to \infty$, moreover we have $D\varphi(z_n) \neq 0$ for n large. Hence,

$$\varphi_t(z_n) + \frac{1}{n(T-t_n)^2} + F(x_n, D\varphi(z_n), D^2\varphi(z_n), \{\varphi(\cdot, t_n) \geq \varphi(z_n)\}) \leq 0.$$

Since any Kuratowski limit of $\{\varphi(\cdot, t_n) \geq \varphi(z_n)\}$ is contained in $\{\varphi(\cdot, t) \geq \varphi(z)\}$, using properties iii) and v.1) of F we deduce

$$\varphi_t(z) + F(x, D\varphi(z), D^2\varphi(z), \{\varphi(\cdot, t) \geq \varphi(z)\}) \leq 0.$$

If now $D\varphi(z) = 0$, we follow the lines of [23, Proposition 1.3]. Since φ is admissible at $z = (y, T)$, there are $\delta > 0$, $f \in \mathcal{F}$ and $\omega \in C^0(\mathbb{R})$ with $\omega(r)/r \to 0$ as $r \to 0$ such that

$$|\varphi(x,t) - \varphi(z) - \varphi_t(z)(t-T)| \leq f(|x-y|) + \omega(t-T)$$

for all $(x,t) \in B(z, \delta)$. Without loss of generality we assume that $\omega \in C^1(\mathbb{R})$ and $\omega(0) = \omega'(0) = 0$ and also that $\omega(r) > 0$ for $r \neq 0$. Next choose a sequence $\omega_n \in C^2(\mathbb{R})$ such that $\omega_n(r) \to \omega(r)$ and $\omega_n'(r) \to \omega_n'(r)$ locally uniformly in $\mathbb{R}$ and set

$$\begin{aligned} &\psi(x,t) = \varphi_t(z)(t-T) + 2f(|x-y|) + 2\omega(t-T)\,, \\ &\psi_n(x,t) = \varphi_t(z)(t-T) + 2f(|x-y|) + 2\omega_n(t-T) - \frac{1}{n(T-t)}\,. \end{aligned} \tag{3.13}$$

We have $u - \psi$ has a strict maximum at z. Hence for n large enough $u - \psi_n$ has a strict maximum at $z_n = (y_n, t_n) \in Q_T$, with $z_n \to z$, and ψ_n is admissible at z_n. As u is a subsolution, we have, using also property iv) of F,

$$\begin{aligned} \varphi_t(z) + 2\omega_n'(t_n - T) + \frac{1}{n(T-t_n)^2} \\ + \frac{2f'(|y_n - y|)}{|y_n - y|} F(y_n, y_n - y, I, \{\psi_n(\cdot, t_n) \geq \psi_n(z_n)\}) \leq 0 \end{aligned} \tag{3.14}$$

if $y_n \neq y$, while $\varphi_t(z) + 2\omega_n'(T - t_n) + 1/[n(T-t_n)^2] \leq 0$ if $y_n = y$. Letting $n \to \infty$, we get $\varphi_t(z) \leq 0$ thanks to (3.7). Hence, as claimed, u is a subsolution in $\mathbb{R}^d \times (0, T]$.

Now, as in [23], we assume that

$$\theta_0 := \limsup_{r \downarrow 0} \Big\{ u(z) - v(\zeta) \,:\, (z, \zeta) \in \overline{Q}_T^2, |z - \zeta| \leq r \Big\} > 0 \tag{3.15}$$

and try to get a contradiction. The proof then follows identically the proof in [23] from page 238 until the middle of page 241. In particular (using exactly the same notation), the case "$\hat{\theta} = \theta$" is identical (since the non-locality does not play any role in that case), and we may jump to the case $\hat{\theta} < \theta$. As in [23], we then let $(\hat{x}, \hat{t}, \hat{y}, \hat{s}) \in \overline{Q}_T \times \overline{Q}_T$ be the maximum point of

$$u(x,t) - v(y,s) - \alpha f(|x-y|) - \alpha(t-s)^2 - \varepsilon t - \varepsilon s - \delta|x|^2 - \delta|y|^2, \tag{3.16}$$

where $f \in \mathcal{F}$, and ε, $\alpha > 0$ are suitable positive constants, and $\delta > 0$ is chosen in such a way that the value of this maximum point is strictly positive. We then can jump to the middle of page 241 (more precisely, up to "Now the definition of viscosity solution yields"). Here, the situation is a bit changed. By Lemma 3.4 we get

$$2\alpha(\hat{t}-\hat{s})+\varepsilon+F\left(\hat{x},\alpha f'(|\hat{p}|)\frac{\hat{p}}{|\hat{p}|}+2\delta\hat{x},X+2\delta I,\{u(\cdot,\hat{t})\geq u(\hat{x},\hat{t})\}\right)\leq 0,$$

and

$$2\alpha(\hat{t}-\hat{s})-\varepsilon+F\left(\hat{y},\alpha f'(|\hat{p}|)\frac{\hat{p}}{|\hat{p}|}-2\delta\hat{y},X-2\delta I,\{v(\cdot,\hat{s})>v(\hat{y},\hat{s})\}\right)\geq 0.$$

Here X is a suitable symmetric matrix which also depends on δ, and $\hat{p}:=\hat{x}-\hat{y}$. Moreover, X, $\hat{p}$, $\hat{t}$ and $\hat{s}$ are uniformly bounded, while $|\hat{p}|$ is bounded from below. Therefore, we may assume that they converge, as $\delta\to 0$, to some limit denoted in [23] by Y, $\bar{p}\neq 0$, $\bar{t}$, $\bar{s}$, respectively. Denote

$$K_\delta:=\{u(\cdot,\hat{t})\geq u(\hat{x},\hat{t})\}-\hat{x},\qquad L_\delta:=\{v(\cdot,\hat{s})>v(\hat{y},\hat{s})\}-\hat{y}.$$

We may also assume that $K_\delta\to K$, $L_\delta^c\to L^c$ in the Kuratowski sense, for some $K\in\mathcal{C}(\mathbb{R}^d)$, $L\in\mathcal{A}(\mathbb{R}^d)$. We deduce (using the semicontinuity properties of F and the translational invariance) that

$$\begin{aligned}&2\alpha(\bar{t}-\bar{s})+\varepsilon+F\left(0,\alpha f'(|\bar{p}|)\frac{\bar{p}}{|\bar{p}|},Y,K\right)\leq 0,\\&2\alpha(\bar{t}-\bar{s})-\varepsilon+F\left(0,\alpha f'(|\bar{p}|)\frac{\bar{p}}{|\bar{p}|},Y,L\right)\geq 0.\end{aligned}\tag{3.17}$$

By (3.16) we have

$$\begin{aligned}u(x,\hat{t})-u(\hat{x},\hat{t})\leq v(y,\hat{s})-v(\hat{y},\hat{s})-&\\ \left(\alpha f(|\hat{x}-\hat{y}|)-\alpha f(|x-y|)+\delta|\hat{x}|^2-\delta|x|^2+\delta|\hat{y}|^2-\delta|y|^2\right).&\end{aligned}\tag{3.18}$$

Let $R>0$ and choose $\xi\in K_\delta\cap B_R$. Let also $\eta\in(0,1/2)$ and $q=2\eta\hat{p}$ (recall $\hat{p}=\hat{x}-\hat{y}$). Choose z with $|z|\leq\eta|\hat{p}|$. Choosing $x=\hat{x}+\xi$ and $y=\hat{y}+\xi+q+z$ in (3.18), and observing that $x-y=(1-2\eta)\hat{p}-z$ so that $|x-y|\leq(1-\eta)|\hat{p}|$, we obtain since $\xi\in K_\delta$

$$\begin{aligned}0\leq u(\hat{x}+\xi,\hat{t})-u(\hat{x},\hat{t})\leq v(\hat{y}+\xi+q+z,\hat{s})-v(\hat{y},\hat{s})-&\\ \Big(\alpha f(|\hat{p}|)-\alpha f((1-\eta)|\hat{p}|)-\delta\xi\cdot(2\hat{x}+\xi)-\delta(\xi+q+z)\cdot(2\hat{y}+\xi+q+z)\Big).&\end{aligned}$$

Since $\delta(|\hat{x}|+|\hat{y}|)\to 0$ (see [23]), $|\xi|\leq R$, and

$$\alpha f(|\hat{p}|)-\alpha f((1-\eta)|\hat{p}|)\ \geq c>\ 0$$

for some c independent of δ (as $|\hat{p}|$ is bounded away from zero), we have

$$\alpha f(|\hat{p}|)-\alpha f((1-\eta)|\hat{p}|)-\delta\xi\cdot(2\hat{x}+\xi)-\delta(\xi+q+z)\cdot(2\hat{y}+\xi+q+z)>0$$

for δ small. Thus, $\xi+q+z\in L_\delta$. As this is true for all $|z|\leq\eta|\hat{p}|$, we find that for δ small enough,

$$q+(K_\delta\cap B_R(0))+B_{\eta|\hat{p}|}(0)\ \subseteq\ L_\delta.$$

In other words, the sets $q+(K_\delta\cap B_R(0))$ are at distance at least $\eta|\hat{p}|$ from L_δ^c. Taking the (Kuratowski) limits as $\delta\to 0$ we deduce that $\operatorname{dist}(2\eta\bar{p}+K,L^c)\geq\eta|\bar{p}|$, and in particular that $2\eta\bar{p} + K \subset L$. Using property iii) of F and the translational invariance again, we deduce that

$$F\left(-2\eta\bar{p}, \alpha f'(|\bar{p}|)\frac{\bar{p}}{|\bar{p}|}, Y, K\right) \geq F\left(0, \alpha f'(|\bar{p}|)\frac{\bar{p}}{|\bar{p}|}, Y, L\right)$$

for any $\eta \in (0, 1/2)$. Taking the limit $\eta \to 0$, and using the continuity of F with respect its first variable together with (3.17), we obtain that $2\varepsilon \leq 0$, a contradiction. □

3.4. **Existence and uniqueness of viscosity solutions.** To show the existence of viscosity solutions, we need the following stability result.

Proposition 3.7. *Let $(u_n)_{n\geq 1}$ be a sequence of upper semicontinuous subsolutions of* (3.8) *and let, for any $z = (x, t)$,*

$$u^*(z) \;=\; \lim_{r\downarrow 0} \sup \left\{ u_n(\zeta) \,:\, |z - \zeta| \leq r \,, n \geq \frac{1}{r} \right\}.$$

Then u^* is also a subsolution of (3.8)*.*

Of course, a symmetric result holds for supersolutions.

Proof. The proof of this result is a variant of the proof of [23, Prop. 1.3] (see also the proof of property **(P2)** in [26]), observing that if $z_n = (x_n, t_n) \to z = (x, t)$ and φ is a test function, then the sets $K_n := \{\varphi(\cdot, t_n) \geq \varphi(z_n)\}$ converge (up to a subsequence) in the Kuratowski sense to a set $K \subseteq \{f(\cdot, t) \geq \varphi(z)\}$. We conclude using the monotonicity and the semicontinuity properties of F. □

Given $A \subset \mathbb{R}^n$, we denote by $BUC(A)$ the space of bounded, uniformly continuous functions from A to $\mathbb{R}$. We now can state a general existence and uniqueness result:

Theorem 3.8. *Let $u_0 \in BUC(\mathbb{R}^d)$. Then, there exists a unique viscosity solution $u \in BUC(\mathbb{R}^d \times [0, \infty))$ of (3.8) with initial condition u_0.*

Proof. The proof of this result is very classical, see [19, 23] and based on Perron's method. We introduce

$$\bar{u}(x, t) = \sup \Big\{ u(x, t) \,:\, u \text{ subsolution of (3.8)}, \min u_0 \leq u \leq \max u_0 \,, u(\cdot, 0) \leq u_0 \Big\},$$

and u^*, u_*, its upper and lower semicontinuous envelopes. The fact that u^* is a subsolution follows from Proposition 3.7, observing that at each point (x, t) we can find a suitable sequence of subsolutions $(u_n)_{n\geq 1}$ whose relaxed upper limit is $u^*(x, t)$.

The fact that u_* is also a supersolution is classical and obtained by contradiction, assuming that at some point $\bar{z} = (\bar{x}, \bar{t})$ of (strict) contact with a test function $\varphi \leq u_*$, φ does not satisfy (3.10). If $D\varphi(\bar{z}) \neq 0$, one can use the test function φ to construct a new subsolution $\bar{u} > u_*$ in a neighborhood of $\bar{z}$, thus contradicting the maximality of u_*. To treat the case $D\varphi(\bar{z}) = 0$ one repeats the same construction, but (as in the proof of [23, Prop. 1.3]) with φ replaced by

$$\psi(x, t) = \varphi(\bar{z}) + \varphi_t(\bar{z})(t - \bar{t}) - 2f(|x - \bar{x}|) - 2\omega(t - \bar{t}).$$

The regularity properties of u and the fact that the initial condition is attained, can be shown as in the last part of the proof of [23, Theorem 1.8]. $\square$

3.5. **Application to our evolution problem.** Here we show how to apply the viscosity approach developed above to our specific problem. First, we extend the Hamiltonian F_f in (3.2) to open sets, enforcing that the evolution of a closed set K agrees with the evolution of its complement, i.e., setting for every $A \in \mathcal{A}(\mathbb{R}^d)$:

$$F_f(x,p,X,A) := -F_f(x,-p,-X,A^c) = F_f(x,p,X,A^c), \tag{3.19}$$

where the last identity follows by the very definition (3.2), (3.3) of F_f. It turns out, however, that the Hamiltonian F_f in (3.2) does *not* satisfy all the assumptions which are required in Theorem 3.8. In fact, it lacks assuptions v), v.1), v.2). A basic counterexample is as follows: Let K be a ball, and $x_n \to x \in \partial K$ with $x_n \notin K$ for all n. Then, $F^-_\rho(x_n,p,X,K) = 0$ for all n and any p, X, while if p is the inner normal to ∂K, X is small enough and the radius of the ball K is large enough, then $F^-_\rho(x,p,X,K) < 0$. On the other hand, $F^+_\rho(x_n,p,X,K)$ will be constant (and positive). Hence F_f, in that case, will be l.s.c., but not continuous, and in particular v.2) does not hold, neither the continuity with respect to x.

In fact, we observe now that a continuous Hamiltonian extending the non-local curvature (2.18) does not exist. Indeed, let $K = \overline{B} := \overline{B_1(0)} \subset \mathbb{R}^2$ and $x \in \partial B$. Let moreover A_n be open smooth subsets of $\bar{B}$ and $x_n \in \partial A_n$ be satisfying the following properties: 1) A_n have vanishing diameter; 2) $x_n \to x$; 3) the outer normal and the (euclidean) curvature of A_n at x_n agree with the outer normal and curvature of B at x, respectively. These conditions are clearly compatible. Set $K_n := B \setminus A_n$. Then $\kappa^{\pm}_f(K_n,x_n) = 0$ for n large enough (remember that $f' = 0$ near 0). The idea now is that, if we could extend κ_f into a semi-continuous Hamiltonian in the sense of v), it would follow that $\kappa_f(K,x) \le 0$, which is not true. More precisely, let $u_n = -d_{K_n}$. By (3.4) and since $Du_n(x_n) = Du(x)$, $D^2u_n(x_n) = D^2u(x)$ we have

$$\begin{aligned} F_f(x, Du(x), D^2u(x), K) &= \kappa_f(K,x) > 0 = \liminf_{n\to\infty} \kappa_f(K_n,x_n) \\ = \liminf_{n\to\infty} F_f(x_n, Du_n(x_n), D^2u_n(x_n), K_n) &= \liminf_{n\to\infty} F_f(x_n, Du(x), D^2u(x), K_n). \end{aligned} \tag{3.20}$$

Since $x_n \to x$ and $K_n \to K$, we conclude that property v.1) does not hold. This means that Theorem 3.8 does not apply for our particular problem, without further smoothing (see Proposition 3.10 below).

We can show a continuity slightly weaker than assumptions v), v.1), v.2), which however will have some utility in the sequel. The following result shows that these properties are essentially true if (x,p,X,K) are of the form $(x, D\varphi(x), D^2\varphi(x), \{\varphi \ge \varphi(x)\})$, when $D\varphi(x) \neq 0$. Finding a result similar to Theorem 3.8 but under this weaker assumption would be very interesting, and is a subject for future study.

Lemma 3.9. *Let $\varphi_n, \varphi : \mathbb{R}^d \to \mathbb{R}$ be C^2_{loc} functions, and assume that $\varphi_n \to \varphi$ in C^2_{loc} as $n \to \infty$. Let $x \in \mathbb{R}^d$ with $D\varphi(x) \neq 0$ and consider a sequence (x_n) with*

$x_n \to x$. Then,

$$F_f(x, D\varphi(x), D^2\varphi(x), \{\varphi \geq \varphi(x)\}) \leq \liminf_{n\to\infty} F_f(x_n, D\varphi_n(x_n), D^2\varphi_n(x_n), \{\varphi_n \geq \varphi_n(x_n)\}) \tag{3.21}$$

and

$$F_f(x, D\varphi(x), D^2\varphi(x), \{\varphi > \varphi(x)\}) \geq \limsup_{n\to\infty} F_f(x_n, D\varphi_n(x_n), D^2\varphi_n(x_n), \{\varphi_n > \varphi_n(x_n)\}) . \tag{3.22}$$

Proof. We prove only the first inequality, the second one being a consequence of the first one and the identity

$$F_f(x, D\varphi(x), D^2\varphi(x), \{\varphi > \varphi(x)\}) = -F_f(x, -D\varphi(x), -D^2\varphi(x), \{-\varphi \geq \varphi(x)\}).$$

First of all, replacing φ_n with $y \mapsto \varphi_n(y - x + x_n)$ we may assume (by translation invariance of the Hamiltonian) that $x_n = x$ for all $n \geq 1$. Denote $p = D\varphi(x)$, $p_n = D\varphi_n(x)$, $\hat{p} = D\varphi(x)/|D\varphi(x)|$, and $\hat{p}_n = D\varphi_n(x)/|D\varphi_n(x)|$, $X = D^2\varphi(x)$, $X_n = D^2\varphi_n(x)$, $K = \{\varphi \geq \varphi(x)\}$, $K_n = \{\varphi_n \geq \varphi_n(x)\}$. One has that $p_n \to p$, etc, except for one detail: K_n may not converge to K: more precisely any Kuratowski limit of a subsequence of K_n is a set in between $\{\varphi > \varphi(x)\}$ and K.

Consider now $s \in [\underline{\delta}, \delta]$ (recall that f is constant on $[0, \underline{\delta}]$). Since K is C^2 near x (by the implicit function theorem), there exists a positive $s^* \in (0, \delta]$, such that $\mathrm{dist}(x - s\hat{p}, K) = s$ if $s \leq s^*$, and $\mathrm{dist}(x - s\hat{p}, K) < s$ if $s \in (s^*, \delta]$ (possibly empty).

We want to prove that for a.e. s (in fact, for all $s \neq s^*$),

$$\kappa_s^+(K, x) \leq \liminf_{n\to\infty} \kappa_s^+(K_n, x). \tag{3.23}$$

Since, clearly,

$$\frac{|p_n|}{2s} \det\left[I - \frac{s}{|p_n|}\mathcal{P}_{p_n} X_n \mathcal{P}_{p_n}\right]^+ \overset{n\to\infty}{\longrightarrow} \frac{|p|}{2s} \det\left[I - \frac{s}{|p|}\mathcal{P}_{\hat{p}} X \mathcal{P}_{\hat{p}}\right]^+ ,$$

we need to show (3.23) only when the right-hand side is zero, or more precisely, when $\mathrm{dist}(x - s\hat{p}_n, K_n) < s$ for infinitely many $n \geq 1$. In this case, let n_k be a subsequence such that $K_{n_k} \to \tilde{K}$ (Kuratowski) and $\mathrm{dist}(x - s\hat{p}_{n_k}, K_{n_k}) < s$ for all k. Let $y_{n_k} \in K_{n_k}$ such that $|x - s\hat{p}_{n_k} - y_{n_k}| < s$, and we can also assume that $y_{n_k} \to y \in \tilde{K} \subset K$. There are two situations:

- either $y \neq x$, in which case, since $|x - s\hat{p} - y| \leq s$, we have $s \geq s^*$. Since for $s > s^*$ $\kappa_s^+(K, x) = 0$, we conclude that (3.23) holds for all $s \neq s^*$;
- or $y = x$, in which case there exists $z_k \in [x, y_{n_k}]$ such that

$$\varphi_{n_k}(x) \leq \varphi_{n_k}(y_{n_k}) = \varphi_{n_k}(x) + D\varphi_{n_k}(x)\cdot(y_{n_k} - x) + \frac{1}{2}(D^2\varphi_{n_k}(z_k)(y_{n_k} - x))\cdot(y_{n_k} - x),$$

hence

$$0 \leq p_{n_k} \cdot (y_{n_k} - x) + \frac{1}{2}(X_{n_k}(z_k)(y_{n_k} - x)) \cdot (y_{n_k} - x). \tag{3.24}$$

Now,

$$s^2 > |x - s\hat{p}_{n_k} - y_{n_k}|^2 = |y_{n_k} - x|^2 + s^2 + \frac{2s}{|p_{n_k}|}(p_{n_k} \cdot (y_{n_k} - x)),$$

and hence, dividing by $|y_{n_k}-x|^2$ we have

$$0 > 1 + \frac{2s}{|p_{n_k}|}\frac{(p_{n_k}\cdot(y_{n_k}-x))}{|y_{n_k}-x|^2}, \tag{3.25}$$

Set $\xi_k = (y_{n_k}-x)/|y_{n_k}-x|$. Up to a subsequence $\xi_k \to \xi \perp \hat p$. By (3.24) and (3.25) we conclude

$$\frac{s}{|p|}X\xi\cdot\xi \;\geq\; 1$$

and in particular, $I - s'/|p|\mathcal{P}_{\hat p}X\mathcal{P}_{\hat p}$ has a negative eigenvalue as soon as $s' > s$. It follows that $s^* \leq s$ and, again, we deduce (3.23) for all $s \neq s^*$.

Now, it remains to take the integral for $s \in [\underline{\delta}, \delta]$, and it follows, using Fatou's lemma, that:

$$\begin{aligned} &F^+_f(x, D\varphi(x), D^2\varphi(x), \{\varphi \geq \varphi(x)\}) \\ &\qquad\qquad \leq \liminf_{n\to\infty} F^+_f(x, D\varphi_n(x), D^2\varphi_n(x), \{\varphi_n \geq \varphi_n(x)\})\,. \end{aligned}$$

In order to show (3.21), it remains to show a similar inequality for F^-_f.

This time, we need to show that for almost any $s \in [\underline{\delta}, \delta]$, if, for a subsequence, $\kappa^-_s(K_{n_k}, x) < 0$, then $\kappa^-_s(K, x)$ must also take the value $-1/(2s)\det[I + (s/|p|)\mathcal{P}_{\hat p}X\mathcal{P}_{\hat p}]^+$. But it means precisely that $\mathrm{dist}(x_{n_k} + s\hat p_{n_k}, K^c_{n_k}) = s$ for all k, hence $\varphi \geq \varphi(x_{n_k})$ on the ball of center $x_{n_k} + s\hat p_{n_k}$ and radius s. Passing to the limit, we deduce that $\varphi \geq \varphi(x)$ on the ball of center $x + s\hat p$ and radius s, so that $\mathrm{dist}(x + s\hat p, K^c) \geq s$. The thesis follows. □

Finally, we build an approximation of the Hamiltonian F_f which will fulfill the assumptions (i–v) of Section 3. To this purpose it is clearly enough to approximate F_ρ for fixed ρ; a possibility is a follows,

$$\begin{aligned} F_\varepsilon(x,p,X,E) \;:=\; &\frac{|p|}{2\rho}\det\left[I - \frac{\rho}{|p|}\mathcal{P}_{\hat p}X\mathcal{P}_{\hat p}\right]^+ H_\varepsilon(\mathrm{dist}(x-\rho\hat p, E) - \rho) \\ -\; &\frac{|p|}{2\rho}\det\left[I + \frac{\rho}{|p|}\mathcal{P}_{\hat p}X\mathcal{P}_{\hat p}\right]^+ H_\varepsilon(\mathrm{dist}(x+\rho\hat p, E^c) - \rho), \end{aligned} \tag{3.26}$$

where $H_\varepsilon(t)$ is a continuous approximation of the Heavyside function, which is 1 for $t \geq 0$, 0 for $t \leq -\varepsilon$, and nondecreasing.

In that case,

$$|F_\varepsilon(x,p,\pm I,E)| \leq \frac{|p|}{2\rho}\left(\left(1+\frac{\rho}{|p|}\right)^{d-1} + \left[\left(1-\frac{\rho}{|p|}\right)^+\right]^{d-1}\right) \leq c(|p|) \sim |p|^{2-d} \tag{3.27}$$

as $|p| \to 0$, and in dimension $d \geq 3$ this Hamiltonian is indeed singular. The following result is straightforward:

Proposition 3.10. The Hamiltonian F_ε satisties all the properties required in Section 3.

In particular, by Theorem 3.8 we deduce existence and uniqueness, in the viscosity setting, of the geometric flow corresponding to the regularized non local curvature (3.26). In the next section we show an existence result (but with no proof of uniqueness) for the original non-local flow (3.5).

4. The geometric evolution associated to $\mathcal{E}^f$

We will now follow a different approach, in order to construct a (level-set) flow of our curvature which is actually a viscosity solution of the equation (3.5), with F_f the Hamiltonian defined in (3.2). Here we assume that u_0 is an initial datum with compact support, and bounded, uniformly continuous ($u_0 \in BUC_c(\mathbb{R}^d)$).

The construction follows the approach first suggested by Luckhaus and Sturzenhecker, and Almgren, Taylor and Wang [24, 1]. We follow here a simple strategy which has been elaborated in [18] for the classical Mean Curvature Flow, and which we adapt to our setting.

First, given a time-step $h > 0$ and a compact set E, we define $T_h^- E$ (resp, $T_h^+ E$) as the minimal (resp., maximal) solution to

$$
\begin{aligned}
(4.1) \qquad \min_{F \subset \mathbb{R}^d} & \left\{ \mathcal{E}^f(F) \; + \; \frac{1}{h} \int_{F \triangle E} \operatorname{dist}(x, \partial E)\, dx \right\} \\
&= \min_{F \subset \mathbb{R}^d} \left\{ \mathcal{E}^f(F) \; + \; \frac{1}{h} \int_F d_E(x)\, dx \right\} - \frac{1}{h} \int_E d_E(x)\, dx,
\end{aligned}
$$

where $d_E(x) = \operatorname{dist}(x, E) - \operatorname{dist}(x, \mathbb{R}^d \setminus E)$. The existence of a solution to (4.1) is not totally obvious, however, it can be established by considering the equivalent convex variational problem

$$
\min_{u \in L^1(\mathbb{R}^d;[0,1])} \left\{ \mathcal{E}^f(u) \; + \; \frac{1}{h} \int_{\mathbb{R}^d} u(x) d_E(x)\, dx \right\}
$$

with $\mathcal{E}^f$ defined in (2.9), and observing that, given a solution of that problem, for a.e. $s \in (0,1)$ the sets $\{u > s\}$ and $\{u \geq s\}$ are a solution to (4.1). The existence of a minimal (or maximal) solution follows from the fact that if E, E' are solutions, then also $E \cap E'$ and $E \cup E'$ are, thanks to (2.4). Moreover, it is not difficult to see that if F solves (4.1), then

$$
\mathcal{M}^f(F) = \mathcal{E}^f(F)\,,
$$

where $\mathcal{M}^f(F)$ is defined in (2.6). The following classical lemmas hold.

Lemma 4.1. *If $E \subset\subset E'$, then $T_h^+ E \subseteq T_h^- E'$. Moreover, if $E \subseteq E'$, then $T_h^\pm E \subseteq T_h^\pm E'$.*

Proof. The proof is classical and we just sketch it. We first assume that $E \subset\subset E'$, so that $d_E > d_{E'}$ a.e. We compare the energy (4.1) of $F = T_h^+ E$ with the one of $F \cap F'$, where $F' = T_h^- E'$, and the energy (4.1) (with E replaced by E') of F' with the one of $F \cup F'$. We sum both inequalities and use (2.4) to deduce that $F \subseteq F'$.

Now, if $d_E \geq d_{E'}$, we replace d_E with $d_E + \varepsilon$ and observe that the corresponding minimal solutions F_ε and F' satisfy $F_\varepsilon \subseteq F'$. Let F_0 be the Kuratowsky limit of F_ε (up to a subsequence). Then, it is easy to see that F_0 is a solution, and $T_h^- E \subseteq F_0 \subseteq F' = T_h^- E'$. The proof for T_h^+ is almost identical. $\square$

Lemma 4.2. *Let $E \subset\subset E'$ and let $\delta = \operatorname{dist}(\partial E, \partial E') > 0$. Then $T_h^+ E \subset\subset T_h^- E'$ and, more precisely, $\operatorname{dist}(\partial T_h^+ E, \partial T_h^- E') \geq \delta$.*

Proof. Let $z \in \mathbb{R}^d$ with $|z| < \delta$: $z + E \subset\subset E'$ so that $T_h^+(z+E) \subseteq T_h^-(E)$. By translation invariance of the scheme it follows that $z + T_h^+(E) \subseteq T_h^-(E)$, and we deduce the thesis. $\square$

If E is a non-compact set with compact boundary, we can define $T_h^{\pm} E$ in a similar way (or simply let $T_h^{\pm} E = \mathbb{R}^d \setminus (T_h^{\mp}(\mathbb{R}^d \setminus E)))$, and still the comparison holds. Thanks to the comparison lemma 4.1, starting from a function $u \in BUC_c(\mathbb{R}^d)$ (with compact support, or constant outside of a compact set), for $s > s'$ we have $T_h^+\{u \geq s\} \subseteq T_h^-\{u \geq s'\}$. It follows that we can define a function

$$T_h u(x) \ := \ \sup\{s \ : \ x \in T_h^+\{u \geq s\}\} \ = \ \sup\{s \ : \ x \in T_h^-\{u \geq s\}\}.$$

We easily see that for a.e. s, $\{T_h u \geq s\} = T_h^{\pm}\{u \geq s\}$. Using Lemma 4.2, we find that the distance between two such level sets of $T_h u$ is larger than the distance between the corresponding level sets of u: hence $T_h u \in BUC_c(\mathbb{R}^d)$, with the same modulus of continuity. Finally, we can deduce (by approximation) that for any level $s \in \mathbb{R}$,

$$T_h^-\{u \geq s\} \ = \ \{T_h u > s\}\ , \qquad T_h^+\{u \geq s\} \ = \ \{T_h u \geq s\}.$$

Now, starting from u_0, we build a function $u_h(x,t) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ by letting

$$u_h(x,t) \ := \ (T_h)^{[\frac{t}{h}]} u_0$$

where $[\cdot]$ is the integer part. By construction, u_h has a uniform spatial modulus of continuity. The next lemma deals with the non-local evolution of balls.

Lemma 4.3. *Let $x \in \mathbb{R}^d$, $r_0 > 0$ and let $E_0 = B(x, r_0)$. Then for every $h, t > 0$ we have*

$$(T_h^{\pm})^{[\frac{t}{h}]}(E_0) = B(x, r_h^{\pm}(t)),$$

for some $r_h^{\pm}(t) \geq 0$. Moreover, $r_h^{\pm}(t) \to r(t)$ uniformly in $[0, T^*(r_0)]$ as $h \to 0$, where r is the solution to

$$\left\{ \begin{array}{l} \dot r(t) = \displaystyle\int_{\bar\delta}^{\delta} f'(s) \left[(1 + \frac{s}{r})^{d-1} - \left((1 - \frac{s}{r})^+\right)^{d-1} \right] ds, \\ r(0) = r_0, \end{array} \right. \tag{4.2}$$

and $T^*(r_0)$ is extinction time of $r(t)$ (i.e, such that $r(T^*(r_0)) = 0$). Finally, there exists $c_0 > 0$ such that for every $r_0 \leq 1$ we have

$$T^*(r_0) \geq c_0\, r_0^d. \tag{4.3}$$

Proof. By translation invariance we may assume $x = 0$. Since the union of any family of minimizers of (4.1) is still a minimizer, we deduce that any rotation of $(T_h^+)^{[\frac{t}{h}]}(E_0)$ is contained in $(T_h^+)^{[\frac{t}{h}]}(E_0)$ i.e., the maximal solution is radially symmetric. Analogously, by the stability of the minimality property with respect to intersection we deduce that $(T_h^-)^{[\frac{t}{h}]}(E_0)$ is radially symmetric. By a rearrangement procedure it can be readily seen that the maximal and minimal solutions are in fact balls. Indeed, let $r \geq 0$ be determined by $|B(r)| = (T_h^+)^{[\frac{t}{h}]}(E_0)$. Then it is easy to see that

$$\begin{aligned} &\mathcal{E}^f(B(r)) \leq \mathcal{E}^f((T_h^{\pm})^{[\frac{t}{h}]}(E_0))\,, \\ &\int_{B(r)\triangle E_0} \mathrm{dist}(x, \partial E_0)\, dx \leq \int_{(T_h^{\pm})^{[\frac{t}{h}]}(E_0)\triangle E_0} \mathrm{dist}(x, \partial E_0)\, dx \end{aligned}$$

with strict inequality whenever the radially symmetric set $(T_h^{\pm})^{[\frac{t}{h}]}(E_0)$ is not a ball.

For $0 < r < R$ let $e(r,R)$ be the total energy in (4.1) for $E = B_R$ and $F = B(r)$, i.e.,

$$e(r,R) \;=\; -\int_{\underline{\delta}}^{\delta} f'(s)\omega_d[(r+s)^d - ((r-s)^+)^d]\,ds + \frac{d\omega_d}{h}\int_r^R (R-s)s^{d-1}\,ds, \tag{4.4}$$

where ω_d denotes, as usual, the volume of the unit ball in $\mathbb{R}^d$. A straightforward computation shows that $\frac{\partial}{\partial r}e(r,R) = 0$ is equivalent to

$$\frac{1}{h}(r-R) = \int_{\underline{\delta}}^{\delta} f'(s)[(1+\frac{s}{r})^{d-1} - ((1-\frac{s}{r})^+)^{d-1}]ds. \tag{4.5}$$

Now we construct the approximated evolution starting from $B(r_0)$. To this purpose, let us set $r_{h,0} = r_0$ and define $r_{h,i}$ recursively, as the minimum point of $e(r, r_{h,i-1})$ (and we stop if $r_{h,i} = 0$). Denote by $\hat{r}_h(t)$ the piecewise affine interpolation of $r_{i,h}$ given by

$$\hat{r}_h(t) = r_{h,[\frac{t}{h}]} + \left(t - \left[\tfrac{t}{h}\right]\right)\left(r_{h,[\frac{t+1}{h}]} - r_{h,[\frac{t}{h}]}\right).$$

Then, by (4.5), $\hat{r}_h(t)$ solves

$$\begin{cases} \dfrac{d}{dt}\hat{r}_h(t) = g\left(\hat{r}_h\left(\left[\tfrac{t+1}{h}\right]\right)\right); \\ \hat{r}_h(0) = r_0, \end{cases}$$

where

$$g(r) := \int_{\bar{\delta}}^{\delta} f'(s)\left[(1+\frac{s}{r})^{d-1} - \left((1-\frac{s}{r})^+\right)^{d-1}\right] ds.$$

Let $[0, T^*(r_0))$ be the maximal interval of definition for the solution to problem (4.2). Clearly, we have $r(T^*(r_0)) = 0$. Moreover, standards stability arguments in ODE yield that $\hat{r}_h$, and in turn r_h, converge uniformly to r in $[0,T]$ for every $T < T^*(r_0)$. The uniform convergence in $[0, T^*(r_0)]$ follows by monotonicity.

Noticing that, for $r \le 1$, $|g(r)| \le c\,r^{1-d}$ for some $c > 0$, the final bound on $T^*(r_0)$ follows by comparing with the solution to

$$\begin{cases} \dot{r}(t) = -cr^{1-d}, \\ r(0) = r_0. \end{cases}$$

□

Lemma 4.4. *There exists a time modulus of continuity $\hat{\omega}$, such that for any $\delta > 0$, there exists $h(\delta)$ such that if $x \in \mathbb{R}^d$, $0 < h \le h(\delta)$, and $t, s \ge 0$ with $|t-s| \le \delta$ then*

$$|u_h(x,t) - u_h(x,s)| \;\le\; \hat{\omega}(\delta)$$

Proof. Let ω be a spatial modulus of continuity for u_0, and therefore also for $u_h(\cdot, t)$ with $t \ge 0$. Fix $r_0 > 0$. Then, $u_h(y,t) \le u_h(x,t) + \omega(r_0)$ for all $y \in B_{r_0}(x)$. Lemma 4.3 shows that if h is small enough, then $u_h(x, t+s) \le u_h(x,t) + \omega(r_0)$ for $s \le c_0 r_0^d/2$. Analogously, by $u_h(y,t) \ge u_h(x,t) - \omega(r_0)$ for all $y \in B_{r_0}(x)$ we deduce $u_h(x,t+s) \ge u_h(x,t) - \omega(r_0)$ for $s \le c_0 r_0^d/2$. The thesis follows if we choose $r_0 = (2\delta/c_0)^{1/d}$, $\hat{\omega}(\delta) = \omega(r_0)$. □

Thanks to Lemma 4.4, we can extract a subsequence $(h_k)_{k\geq 1}$ such that u_{h_k} converges locally uniformly in $\mathbb{R}^d \times \mathbb{R}_+$ to a function $u(x,t)$ which is bounded and uniformly continuous in space and time.

Remark 4.5. Let $h_n \to 0$ be such that u_{h_n} admits a limit u. Then, as a straightforward consequence of Lemma 4.3, we deduce that if for some level $s \in \mathbb{R}$, $u(t,\cdot) \geq s$ (resp., $\leq s$) on a ball of radius r_0, then $u(\cdot,t') \geq s$ (resp., $\leq s$) on the concentric ball with radius $r(t'-t)$, for $t' \geq t$, provided that $r(t'-t) > 0$ (here $r(\cdot)$ solves (4.2)).

We can now show the main result of this section.

Theorem 4.6. *The limit u is a viscosity solution of* (3.5)*, in the sense of Definition 3.3.*

Proof. First it is clear, by construction, that $u(0,\cdot) = u_0$. Hence we need to show that the equation holds for $t > 0$. We only prove that it is a subsolution, the proof that it is a supersolution being identical. Let $\varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ and $(\bar{x},\bar{t}) \in \mathbb{R}^d \times \mathbb{R}_+$ be a maximum point of $u - \varphi$. We may assume that this is a strict maximum point and that φ is coercive : if not, we should first replace (as usual) φ with $\varphi(x,t) + \eta(|x-\bar{x}|^2 + |t-\bar{t}|^2)$, derive an inequality for this modified function, and send $\eta \to 0$, which will give the desired inequality thanks to (3.21).

By standard methods, we can then find $(x_k,t_k) \to (\bar{x},\bar{t})$ such that $t_k > 0$ and $u_{h_k} - \varphi$ has a maximum at (x_k,t_k).

Step 1. Let us first assume that $D\varphi(\bar{x},\bar{t}) \neq 0$ so that in particular, for k large enough, $D\varphi(x_k,t_k) \neq 0$. We have that for all (x,t),

$$u_{h_k}(x,t) \;\leq\; \varphi(x,t) + c_k \tag{4.6}$$

where $c_k := [u_{h_k}(x_k,t_k) - \varphi(x_k,t_k)]$, with equality if $(x,t) = (x_k,t_k)$. Let $\eta > 0$ and $\varphi^\eta_{h_k} : \mathbb{R}^d \to \mathbb{R}$ given by

$$\varphi^\eta_{h_k}(x) \;=\; \varphi(t_k,x) \,+\, c_k \,+\, \frac{\eta}{2}|x-x_k|^2\,,$$

then, for all $x \in \mathbb{R}^d$,

$$u_{h_k}(t_k,x) \;\leq\; \varphi^\eta_{h_k}(x)$$

with equality *if and only if* $x = x_k$. Let $\varepsilon > 0$ and consider the open, nonempty set $V_\varepsilon = \{x \,:\, u_{h_k}(t_k,x) > \varphi^\eta_{h_k}(x) - \varepsilon\}$, which has positive measure, contains x_k, and converges to $\{x_k\}$ in the Hausdorff sense as $\varepsilon \to 0$. In particular, setting $s_\varepsilon := u_{h_k}(x_k,t_k) - \varepsilon/2$, we have that for $\varepsilon > 0$ sufficiently small $|W_\varepsilon| > 0$, where

$$W_\varepsilon \;:=\; \{x \in \mathbb{R}^d \,:\, u_{h_k}(t_k,x) \geq s_\varepsilon\} \setminus \{x \in \mathbb{R}^d \,:\, \varphi^\eta_{h_k}(x) \geq \varepsilon + s_\varepsilon\} \;\subseteq V_\varepsilon\,,$$

Now, by minimality, we have

$$\begin{aligned}\mathcal{E}^f(\{u_{h_k}(\cdot,t_k) \geq s_\varepsilon\}) \,+\, \frac{1}{h_k}\int_{\{u_{h_k}(\cdot,t_k)\geq s_\varepsilon\}} d_{\{u_{h_k}(\cdot,t_k-h_k)\geq s_\varepsilon\}}(x)\,dx \\ \leq\, \mathcal{E}^f(\{u_{h_k}(\cdot,t_k) \geq s_\varepsilon\} \cap \{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\}) \\ +\, \frac{1}{h_k}\int_{\{u_{h_k}(\cdot,t_k)\geq s_\varepsilon\}\cap\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}} d_{\{u_{h_k}(\cdot,t_k-h_k)\geq s_\varepsilon\}}(x)\,dx\,.\end{aligned}$$

Adding to both sides the term $\mathcal{E}^f(\{u_{h_k}(\cdot,t_k)\geq s_\varepsilon\}\cup\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\})$ and using (2.4), we obtain

$$\mathcal{E}^f(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\cup W_\varepsilon)-\mathcal{E}^f(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\})+\frac{1}{h_k}\int_{W_\varepsilon}d_{\{u_{h_k}(\cdot,t_k-h_k)\geq s_\varepsilon\}}(x)\,dx\;\leq\;0\,.$$

Observing that by (4.6), $\{u_{h_k}(\cdot,t_k-h_k)\geq s_\varepsilon\}\subseteq\{\varphi(\cdot,t_k-h_k)\geq s_\varepsilon-c_k\}$, we also have

$$\mathcal{E}^f(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\cup W_\varepsilon)-\mathcal{E}^f(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\})+\frac{1}{h_k}\int_{W_\varepsilon}d_{\{\varphi(\cdot,t_k-h_k)\geq s_\varepsilon-c_k\}}(x)\,dx\;\leq\;0\,. \tag{4.7}$$

Now notice that for $z\in W_\varepsilon$ we have

$$s_\varepsilon\;\leq\;\varphi(z,t_k)+c_k+\frac{\eta}{2}|z-x_k|^2\;<\;\varepsilon+s_\varepsilon. \tag{4.8}$$

In particular,

$$W_\varepsilon\subseteq B_{C\sqrt{\varepsilon}}(x_k). \tag{4.9}$$

Moreover,

$$\varphi(z,t_k-h_k)=\varphi(z,t_k)-h_k\partial_t\varphi(z,t_k)+h_k^2\int_0^1(1-s)\partial^2_{tt}\varphi(z,t_k-sh_k)\,ds\,. \tag{4.10}$$

If y is the point closest to z with $\varphi(y,t_k-h_k)=s_\varepsilon-c_k$, so that $|y-z|=|d_{\{\varphi(\cdot,t_k-h_k)\geq s_\varepsilon-c_k\}}(z)|$, then

$$\begin{aligned}\varphi(z,t_k-h_k)\;=\;&\varphi(y,t_k-h_k)+(z-y)\cdot D\varphi(y,t_k-h_k)\\&+\int_0^1(1-s)(D^2\varphi(y+s(z-y),t_k-h_k)(z-y))\cdot(z-y)\,ds\\=\;&s_\varepsilon-c_k-d_{\{\varphi(\cdot,t_k-h_k)\geq s_\varepsilon-c_k\}}(z)|D\varphi(y,t_k-h_k)|\\&+\int_0^1(1-s)(D^2\varphi(y+s(z-y),t_k-h_k)(z-y))\cdot(z-y)\,ds\,.\end{aligned} \tag{4.11}$$

Combining (4.8), (4.10), and (4.11), we deduce

$$\begin{aligned}&d_{\{\varphi(\cdot,t_k-h_k)\geq s_\varepsilon-c_k\}}(z)|D\varphi(y,t_k-h_k)|\\&\qquad\geq\;-\varepsilon+h_k\partial_t\varphi(z,t_k)\;-\;h_k^2\int_0^1(1-s)\partial^2_{tt}\varphi(z,t_k-sh_k)\,ds\\&\qquad\quad+\int_0^1(1-s)(D^2\varphi(y+s(z-y),t_k-h_k)(z-y))\cdot(z-y)\,ds\,.\end{aligned}$$

Note that, in view of (4.8), $|\varphi(z,t_k)-\varphi(y,t_k)|\leq\varepsilon+Ch_k=O(h_k)$, provided that $\varepsilon<<h_k$ are small enough. In turn, as $|D\varphi(x_k,t_k)|\neq0$, we have $|z-y|=O(h_k)$ and, using also (4.9), we deduce

$$\begin{aligned}\frac{1}{h_k}d_{\{\varphi(\cdot,t_k-h_k)\geq s_\varepsilon-c_k\}}(z)\;\geq\;&\frac{\partial_t\varphi(z,t_k)-\frac{\varepsilon}{h_k}+O(h_k)}{|D\varphi(y,t_k-h_k)|}\\&=\;\frac{\partial_t\varphi(x_k,t_k)+O(\sqrt{\varepsilon})-\frac{\varepsilon}{h_k}+O(h_k)}{|D\varphi(x_k,t_k)|+O(h_k)}\,.\end{aligned} \tag{4.12}$$

We now focus on the term

$$\mathcal{E}^f(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\cup W_\varepsilon)-\mathcal{E}^f(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\})$$

of inequality (4.7). This is the sum of the two following expressions, which we will estimate separately:

$$\int_0^\delta -f'(s)\big(|(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\cup W_\varepsilon)+B_s|-|\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}+B_s|\big)\,ds\,, \tag{4.13}$$

$$\int_0^\delta -f'(s)\big(|\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\ominus B_s|-|(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\cup W_\varepsilon)\ominus B_s|\big)\,ds\,, \tag{4.14}$$

where $A\ominus B$ denotes the set $\{x\,:\,x+B\subseteq A\}$. We recall that by assumption, $f'(s)=0$ for $s\leq\underline{\delta}$, so that the integrals are in fact on $[\underline{\delta},\delta]$.

Let us first consider (4.13). For any x in a neighborhood of x_k, we have $x\in\partial\{\varphi^\eta_{h_k}\geq\varphi^\eta_{h_k}(x)\}$ and we can define $s^*(x)\in(0,\delta]$ such that for $s\in(0,s^*(x)]$, $\mathrm{dist}(x+s\nu(x),\{\varphi^\eta_{h_k}\geq\varphi^\eta_{h_k}(x)\})=s$ and for $s\in(s^*(x),\delta]$ (possibly empty), $\mathrm{dist}(x+s\nu(x),\{\varphi^\eta_{h_k}\geq\varphi^\eta_{h_k}(x)\})<s$. Here $\nu(x)=-D\varphi^\eta_{h_k}(x)/|D\varphi^\eta_{h_k}(x)|$, and it is important to observe that thanks to the regularity of $\varphi^\eta_{h_k}$, $s^*(x)$ is continuous near x_k.

If ε is small, for $x\in\partial\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}$, there exists a minimal $\overline{h}^\varepsilon(x)\geq 0$, with $\overline{h}^\varepsilon(x)\leq C\sqrt{\varepsilon}$, such that $W_\varepsilon\cap\{x+t\nu(x),t\in[0,\delta]\}\subseteq\{x+t\nu(x),t\in[0,\overline{h}^\varepsilon(x)]\}$. Clearly, for $s\geq\underline{\delta}$ and ε small enough,

$$\begin{aligned}\big(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\cup W_\varepsilon\,+\,B_s\big)\setminus\big(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\,+\,B_s\big)&\\ \supseteq\ \big\{x+t\nu(x)\,:\,x\in\partial\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\,,s\leq t\leq\min\{s^*(x),s+\overline{h}^\varepsilon(x)\}\big\}&\end{aligned}$$

The volume of this latter set is

$$\int_{\partial\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}}\int_{I(x)}\det(I+t\nabla\nu(x))\,dt\,d\mathcal{H}^{d-1}(x)$$

where $I(x)$ is the interval (possibly empty) $\{t\,:\,s\leq t\leq\min\{s^*(x),s+\overline{h}^\varepsilon(x)\}$. Fix $\sigma>0$. A simple continuity argument yields that, for ε sufficiently small, if $\underline{\delta}\leq s\leq s^*(x_k)-\sigma$, then $s+\overline{h}^\varepsilon(x)\leq s^*(x)$. We deduce that

$$\begin{aligned}&|(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\cup W_\varepsilon\,+\,B_s)\setminus(\{\varphi^\eta_{h_k}\geq\varepsilon+s_\varepsilon\}\,+\,B_s)|\\ &\qquad\geq\int_{\{\varphi^\eta_{h_k}=\varepsilon+s_\varepsilon,\ \overline{h}^\varepsilon>0\}}\int_s^{s+\overline{h}^\varepsilon(x)}\det(I+t\nabla\nu(x))\,dt\,d\mathcal{H}^{d-1}(x)\\ &=\int_{\{\varphi^\eta_{h_k}=\varepsilon+s_\varepsilon,\ \overline{h}^\varepsilon>0\}}\det(I+s\nabla\nu(x))\int_0^{\overline{h}^\varepsilon(x)}\frac{\det(I+(s+t)\nabla\nu(x))}{\det(I+s\nabla\nu(x))}\,dt\,d\mathcal{H}^{d-1}(x)\\ &\qquad\geq\ (\det(I+s\nabla\nu(x_k))+O(\sqrt{\varepsilon}))(1+O(\sqrt{\varepsilon}))|W_\varepsilon|\,,\end{aligned} \tag{4.15}$$

where we have used that

$$|W_\varepsilon|\ \leq\ \int_{\{\varphi^\eta_{h_k}=\varepsilon+s_\varepsilon,\ \overline{h}^\varepsilon>0\}}\int_0^{\overline{h}^\varepsilon(x)}\det(I+t\nabla\nu(x))\,dt\,d\mathcal{H}^{d-1}(x)\,.$$

Finally, integrating over $s\in[\underline{\delta},s^*(x^k)-\sigma]$ we deduce that (4.13) is larger than

$$\begin{aligned}&(1+O(\sqrt{\varepsilon}))|W_\varepsilon|\int_{\underline{\delta}}^{s^*(x^k)-\sigma}-(2sf'(s))(\kappa^+_s(K^\eta_k,x_k)+O(\sqrt{\varepsilon}))\,ds\\ &\qquad\qquad=\ |W_\varepsilon|(\kappa^+_f(K^\eta_k,x_k)+O(\sigma))\,,\end{aligned} \tag{4.16}$$

where the "curvature" (defined in (2.11) and (2.19)) is relative to the set $K_k^\eta = \{\varphi^\eta_{h_k} \geq \varphi^\eta_{h_k}(x_k)\}$. Here we have used the fact that $\sqrt{\varepsilon} < \sigma$, so that $O(\sqrt{\varepsilon}) = O(\sigma)$.

Now let us estimate the negative quantity (4.14). It is very similar, but not equivalent.

We now need to introduce the function $\underline{h}^\varepsilon$, defined for $x \in \partial\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\}$, which is the largest real number (which can be nonzero only in a neighborhood of x_k) such that $x + s\nu(x) \in W_\varepsilon$ for all $s \in (0, \underline{h}^\varepsilon(x))$. For x in a neighborhood of x_k, we also introduce $s_*(x) \in (0, \delta]$, such that $\mathrm{dist}(x - s\nu(x), \partial\{\varphi^\eta_{h_k} \geq \varphi^\eta_{h_k}(x)\}) = s$ for $s \leq s_*(x)$ and $\mathrm{dist}(x - s\nu(x), \partial\{\varphi^\eta_{h_k} \geq \varphi^\eta_{h_k}(x)\}) < s$ for $s_*(x) < s \leq \delta$. Just as s^*, this quantity is continuous with respect to x, thanks to the regularity of $\varphi^\eta_{h_k}$. If $x \in \partial\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\}$ and $s \leq s_*(x)$, one has that

$$\bigcup_{\{\varphi^\eta_{h_k} = \varepsilon + s_\varepsilon,\ \underline{h}^\varepsilon > 0\}} \left\{x + t\nu(x) \,:\, -s \leq t \leq -s + \underline{h}^\varepsilon(x)\right\} \supseteq \big((\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\} \cup W_\varepsilon) \ominus B_s\big) \setminus \big(\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\} \ominus B_s\big)\,,$$

at least when W_ε is small enough with respect to the scale s (which can be ensured, as we need to consider only $s \geq \underline{\delta}$). It follows that

$$\begin{aligned}
&\left|\big((\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\} \cup W_\varepsilon) \ominus B_s\big) \setminus \big(\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\} \ominus B_s\big)\right| \\
&\qquad \leq \int_{\{\varphi^\eta_{h_k} = \varepsilon + s_\varepsilon,\ \underline{h}^\varepsilon > 0\}} \int_{-s}^{-s + \underline{h}^\varepsilon(x)} \det(I + t\nabla\nu(x))\, dt\, d\mathcal{H}^{d-1}(x) \\
&= \int_{\{\varphi^\eta_{h_k} = \varepsilon + s_\varepsilon,\ \underline{h}^\varepsilon > 0\}} \det(I - s\nabla\nu(x)) \int_0^{\underline{h}^\varepsilon(x)} \frac{\det(I - (s-t)\nabla\nu(x))}{\det(I - s\nabla\nu(x))}\, dt\, d\mathcal{H}^{d-1}(x) \\
&\qquad \leq (\det(I - s\nabla\nu(x_k)) + O(\sqrt{\varepsilon}))(1 + O(\sqrt{\varepsilon}))|W_\varepsilon|\,,
\end{aligned} \tag{4.17}$$

where we have used, this time, that

$$|W_\varepsilon| \;\geq\; \int_{\{\varphi^\eta_{h_k} = \varepsilon + s_\varepsilon,\ \underline{h}^\varepsilon > 0\}} \int_0^{\underline{h}^\varepsilon(x)} \det(I + t\nabla\nu(x))\, dt\, d\mathcal{H}^{d-1}(x)\,.$$

Fix $\sigma > 0$ as before. If $s \geq s_*(x_k) + \sigma$, then $s \geq s_*(x) + \sigma/2$ near x_k and we see that

$$\left|\big((\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\} \cup W_\varepsilon) \ominus B_s\big) \setminus \big(\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\} \ominus B_s\big)\right| = 0\,, \tag{4.18}$$

provided that ε is small enough. Integrating (4.17)-(4.18) over $s \in [\underline{\delta}, \delta]$ we deduce that (4.14) is larger than

$$\begin{aligned}
&(1 + O(\sqrt{\varepsilon}))|W_\varepsilon|\Big(\int_{\underline{\delta}}^{\delta} -(2sf'(s))(\kappa_s^-(K_k^\eta, x_k) + O(\sqrt{\varepsilon}))\, ds + O(\sigma)\Big) \\
&\qquad\qquad = \; |W_\varepsilon|(\kappa_f^-(K_k^\eta, x_k) + O(\sigma))\,.
\end{aligned} \tag{4.19}$$

It therefore follows from (4.16) and (4.19) that

$$\mathcal{E}^f(\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\} \cup W_\varepsilon) - \mathcal{E}^f(\{\varphi^\eta_{h_k} \geq \varepsilon + s_\varepsilon\}) \;\geq\; |W_\varepsilon|(\kappa_f(K_k^\eta, x_k) + O(\sigma))\,.$$

Thanks to (4.12), we deduce, after dividing (4.7) by $|W_\varepsilon|$ and sending $\varepsilon \to 0$ and $\sigma \to 0$, that

$$\frac{\partial_t \varphi(x_k, t_k)}{|D\varphi(x_k, t_k)|} \; + \; \kappa_f(K_k^\eta, x_k) \; + \; O(h_k) \; \le \; 0 \tag{4.20}$$

where $O(h_k)$ depends only on the regularity of φ. We may therefore send η to zero to deduce that (4.20) also holds also with K_k^η replaced by $K_k = \{\varphi(\cdot, t_k) \ge \varphi(x_k, t_k)\}$, thanks to (3.21) and (3.4). Letting now $k \to \infty$, using (3.21) and the monotonicity property iii), we deduce

$$\partial_t \varphi(\bar{x}, \bar{t}) \; + \; F_f(\bar{x}, D\varphi(\bar{x}, \bar{t}), D^2\varphi(\bar{x}, \bar{t}), \{\varphi(\bar{t}, \cdot) \ge \varphi(\bar{x}, \bar{t})\}) \; \le \; 0\,, \tag{4.21}$$

that is, u is a viscosity subsolution at $(\bar{t}, \bar{x})$.

Step 2. Now we consider the case $D\varphi(\bar{z}) = 0$ and we show that $\varphi_t(\bar{z}) \le 0$. Let ψ_n be defined as in (3.13), with T replaced by $\bar{t}$, and let $z_n = (x_n, t_n)$ be a sequence of maximizers of $u - \psi_n$, such that $x_n \to \bar{x}$ and $t_n \to \bar{t}^-$. If $x_n \ne \bar{x}$ for a (not relabeled) subsequence, then $D\psi_n(x_n, t_n) \ne 0$ and (4.21) holds for ψ_n at z_n. Passing to the limit and using the properties of f, we deduce that $\varphi_t(\bar{z}) \le 0$ (see (3.14) for the details).

We now assume that $z_n = (\bar{x}, t_n)$ for all n sufficiently large. Set $h_n := \bar{t} - t_n$ and

$$r_n := \sqrt[d]{\frac{2h_n}{c_0}}\,,$$

where c_0 is the constant in (4.3). Note now that by (3.7) and (3.27), the function f appearing in the definition of ψ_n is of the form $f(r) = g(r) r^d$, for a suitable function g such that $g(r) \to 0^+$ as $r \to 0^+$. It easily follows that

$$\begin{aligned} B(\bar{x}, r_n) &\subset \{\psi_n(\cdot, t_n) \le \psi_n(\bar{x}, t_n) + f(r_n)\} \\ &= \left\{\psi_n(\cdot, t_n) \le \psi_n(\bar{x}, t_n) + g(r_n)\frac{2h_n}{c_0}\right\} \\ &\subset \left\{u(\cdot, t_n) \le u(\bar{x}, t_n) + g(r_n)\frac{2h_n}{c_0}\right\}\,, \end{aligned}$$

Note that the last inclusion follows from the maximality of $u - \psi_n$ at z_n and the fact that $u(z_n) = \psi_n(z_n)$. By (4.3), the extinction time $T^*(r_n)$ of the ball $B(\bar{x}, r_n)$ under the non-local evolution satisfies $T^*(r_n) \ge c_0 r^d = 2h_n$. Hence, by comparison (see Remark 4.5), we deduce that

$$\bar{x} \in \left\{u(\cdot, \bar{t}) \le u(\bar{x}, t_n) + g(r_n)\frac{2h_n}{c_0}\right\}\,.$$

Thus, using also the maximality of $u - \varphi$ at $\bar{z}$,

$$\frac{\varphi(\bar{x}, t_n) - \varphi(\bar{z})}{-h_n} \le \frac{u(\bar{x}, t_n) - u(\bar{x}, \bar{t})}{-h_n} \le g(r_n)\frac{2}{c_0}\,.$$

Passing to the limit, we conclude that $\varphi_t(\bar{z}) \le 0$.

$\square$

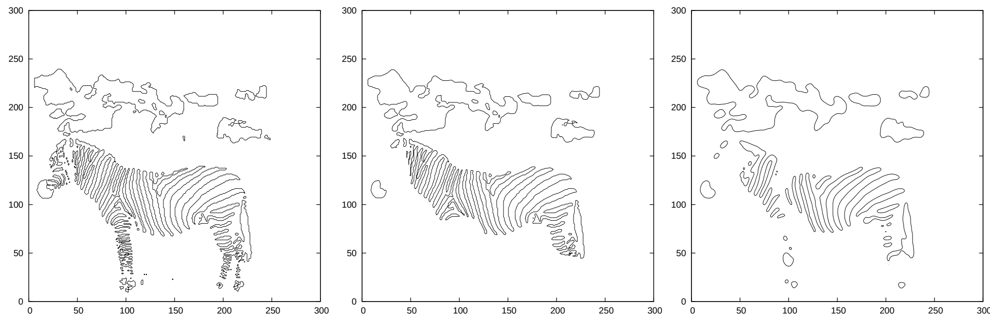

FIGURE 2. A zebra and its smoothing (left: starting image, center: the non-local motion, right: the standard curvature flow), at a small time.

5. ALGORITHM AND NUMERICAL EXAMPLES

5.1. **A numerical implementation of the time-discrete scheme.** We show in this section an example of evolution with the motion studied in this paper, in dimension two. Actually, the implementation is not straightforward and only an approximate motion is computed, on a discrete rectangular grid. The approach we follow is described in [16]. It consists in minimizing, given a discretization of the signed distance function $d^{E^{n-1}}$ to the boundary of the set E^{n-1} (negative inside, positive outside), the energy

$$\min_u J(u) \ + \ \frac{1}{2h}\|u - d^{E^{n-1}}\|^2 \tag{5.1}$$

and define d^{E^n} as the signed distance function to $\{u \leq 0\}$, computed as precisely as possible using a Fast-Marching algorithm [27, 25]. Here, u, $d^{E^{n-1}}$ are defined on the discrete points $\{(i,j) \, : \, 0 \leq i \leq N-1\,, 0 \leq j \leq M-1\}$, and the term

$$\|u - d^{E^{n-1}}\|^2 \ = \ \sum_{i,j} (u_{i,j} - d^{E^{n-1}}_{i,j})^2$$

is the Euclidean norm. A spatial discretization term can be introduced in an obvious way. It turns out that if J is a correct approximation of the functional (2.2), then the algorithm is an approximation of the time-discrete scheme (4.1) studied in Section 4. In this case, the iterations should be an approximation of the motion driven by the energy.

The discretization of the "total variation" $J(u)$ is more complicated. Actually, the simplest here is to approximate (2.2) rather than (2.8). We fix $\rho > 0$, Let B be the discrete ball $\{(i,j) \in \mathbb{Z}^2 \, : \, i^2 + j^2 \leq \rho\}$, and let

$$J(u) \ = \ \frac{1}{2\rho} \sum_{i,j} \mathrm{osc}_{(i,j)+B}(u)$$

where the oscillation (here simply the max minus the min) is computed on the finite sets $((i,j) + B) \cap [0, N-1] \times [0, M-1]$.

It turns out that for this particular energy, there is an approach, based on a graph representation and the maxflow/mincut duality, for minimizing binary problems

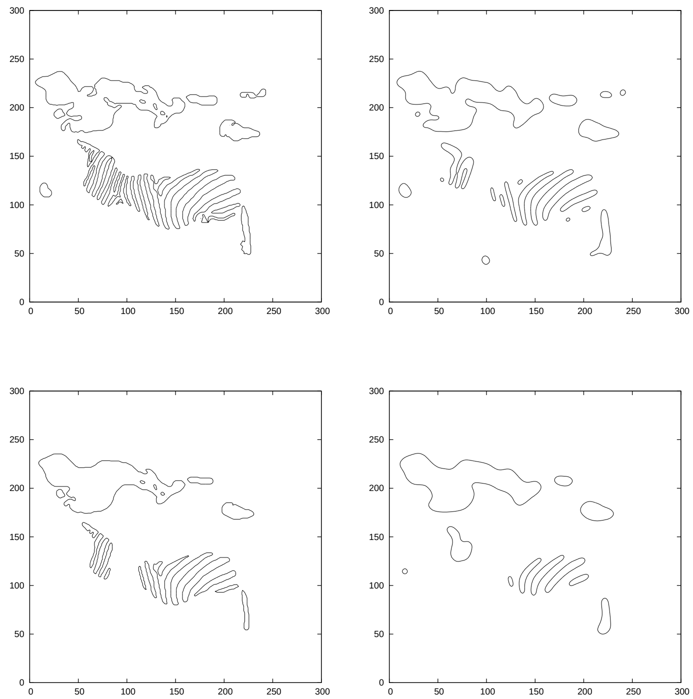

FIGURE 3. and at later times (left: the non-local motion, right: the standard curvature flow).

such as

$$\min_{u_{i,j}\in\{0,1\}} J(u) + \sum_{i,j} f_{i,j}u_{i,j}$$

(given any real-valued matrix $(f_{i,j})_{0\le i<N,\,0\le j<M}$), and an algorithm for minimizing (5.1) is easily derived. See [16] for details and in particular [16, Appendix B] for how this particular J can be implemented.

5.2. **Examples: two ways to shrink a Zebra.** Figures 2, 3 and 4 show the motion applied to an initial set of curves with a lot of oscillations. As expected, the standard curvature motion shrinks the small scale objects much faster than the one based on the oscillation, in particular the stripes are preserved longer by the non-local flow. Notice that it is very difficult to estimate the exact corresponding times for the two flows, moreover, the numerical imprecision may provoke sometimes the "fusion" of the stripes in the classical curvature flow (wich is computed also using (5.1), but now J is a discretization of the standard total variation).

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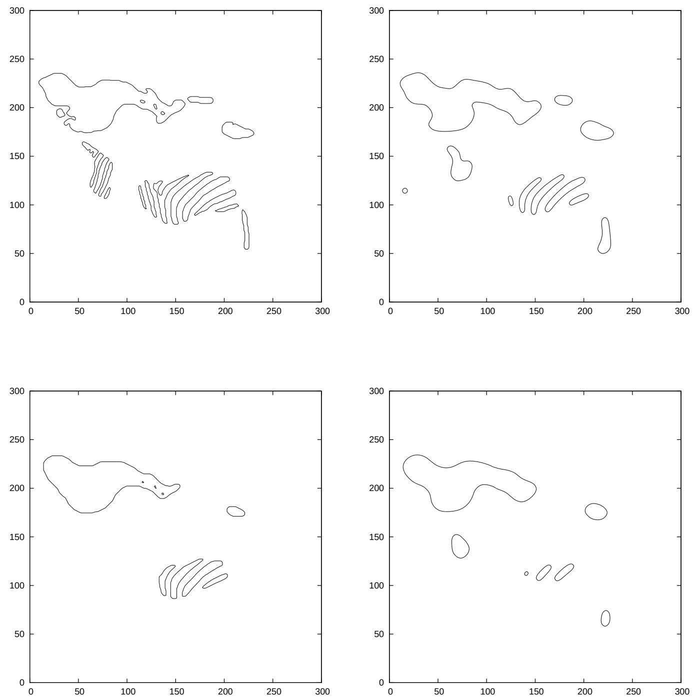

Figure 4. and later...

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