Long-time behavior for a hydrodynamic model on nematic liquid crystal flows with asymptotic stabilizing boundary condition and external force

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Abstract

In this paper, we consider a simplified Ericksen–Leslie model for the nematic liquid crystal flow. The evolution system consists of the Navier–Stokes equations coupled with a convective Ginzburg–Landau type equation for the averaged molecular orientation. We suppose that the Navier–Stokes equations are characterized by a no-slip boundary condition and a timedependent external force $\mathbf{g}(t)$, while the equation for the molecular director is subject to a time-dependent Dirichlet boundary condition $\mathbf{h}(t)$. We show that, in 2D, each global weak solution converges to a single stationary state when $\mathbf{h}(t)$ and $\mathbf{g}(t)$ converge to a timeindependent boundary datum \mathbf{h}_{∞} and $\mathbf{0}$, respectively. Estimates on the convergence rate are also obtained. In the 3D case, we prove that global weak solutions are eventually strong so that results similar to the 2D case can be proven. We also show the existence of global strong solutions, provided that either the viscosity is large enough or the initial datum is close to a given equilibrium.

Keywords: Nematic liquid crystal flow, non-autonomous Navier–Stokes equations, timedependent Dirichlet boundary condition, long-time behavior, Lojasiewicz–Simon inequality.

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1 Introduction

We consider the following hydrodynamical model for the flow of nematic liquid crystals

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} + \nabla \pi = -\lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) + \mathbf{g}(t), \tag{1.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.2}$$

$$\mathbf{d}_t + \mathbf{v} \cdot \nabla \mathbf{d} = \eta (\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})), \tag{1.3}$$

in $\Omega \times \mathbb{R}^+$, where $\Omega \subset \mathbb{R}^n$ (n = 2, 3) is a bounded domain with sufficiently smooth boundary Γ , $\mathbf{v} = (v_1, ..., v_n)^{tr}$ is the velocity field of the flow and $\mathbf{d} = (d_1, ..., d_n)^{tr}$ represents the averaged macroscopic/continuum molecular orientations in \mathbb{R}^n (n = 2, 3). π is a scalar function representing the pressure (including both the hydrostatic and the induced elastic part from the orientation field). The external volume force is represented by \mathbf{g} . The positive constants ν , λ and η stand for

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viscosity, the competition between kinetic energy and potential energy, and macroscopic elastic relaxation time (Deborah number) for the molecular orientation field. $\nabla \mathbf{d} \odot \nabla \mathbf{d}$ denotes the $n \times n$ matrix whose (i, j)-th entry is given by $\nabla_i \mathbf{d} \cdot \nabla_j \mathbf{d}$, for $1 \leq i, j \leq n$. We assume that $\mathbf{f}(\mathbf{d}) = \nabla_{\mathbf{d}} F(\mathbf{d})$ for some smooth bounded function $F : \mathbb{R}^n \to \mathbb{R}$. In particular, one uses the Ginzburg–Landau approximation $\mathbf{f}(\mathbf{d}) = \frac{1}{\epsilon^2} (|\mathbf{d}|^2 - 1)\mathbf{d}$ to relax the nonlinear constraint $|\mathbf{d}| = 1$ on molecule length (cf. [19,20]).

System (1.1)–(1.3) was firstly proposed in [18] as a simplified approximate system of the original Ericksen–Leslie model for the nematic liquid crystal flows (cf. [7,17]). Well-posedness of the autonomous version of system (1.1)–(1.3) (namely, with $\mathbf{g} = \mathbf{0}$, no-slip boundary condition for \mathbf{v} , and time-independent Dirichlet boundary condition for \mathbf{d}) has been analyzed in [20] (see also [8,21] and, for different boundary conditions, [25]). For numerical approximation we refer to [23,26,27]. Problem (1.1)–(1.3) has also been investigated on a Riemannian manifold in [29], where the existence of a global attractor in the 2D case was proven. As far as the long-time behavior of the single trajectory is concerned, in [20], a natural question on the uniqueness of asymptotic limit for global solutions (to the autonomous system) was raised. This question was answered in [35], where it is proven that each trajectory converges to a single steady state (cf. [28,36] for some generalization). The proof is based on a suitable Lojasiewicz–Simon type inequality (see [30], cf. also [13] and references cited therein).

The technically more challenging case of time-dependent Dirichlet boundary conditions for **d** has been recently analyzed in [1,5,6,11]. For instance, under proper assumptions on the time-dependent boundary condition and assuming that $\mathbf{g} = \mathbf{0}$, the existence of global weak solution, the existence of global regular solution for viscosity coefficient big enough, and the weak/strong uniqueness were obtained in [6]. Regularity criteria for solutions in the 3D case can be found in [11]. Besides, the presence of a time-dependent external force is allowed in [1] and existence of global and exponential attractors is proven in the 2D case. In this paper, we want to extend the results of [35] to the non-autonomous case treated in [1]. Thus we consider system (1.1)–(1.3) subject to the boundary conditions

$$\mathbf{v}(x,t) = \mathbf{0}, \quad \mathbf{d}(x,t) = \mathbf{h}(x,t), \qquad (x,t) \in \Gamma \times \mathbb{R}^+, \tag{1.4}$$

and the initial conditions

$$\mathbf{v}|_{t=0} = \mathbf{v}_0(x) \quad \text{with } \nabla \cdot \mathbf{v}_0 = 0, \quad \mathbf{d}|_{t=0} = \mathbf{d}_0(x), \qquad x \in \Omega.$$
(1.5)

In the 2D case, we prove that each weak/strong solution converges to a single stationary state when $\mathbf{h}(t)$ and $\mathbf{g}(t)$ converge to a time-independent boundary datum \mathbf{h}_{∞} and $\mathbf{0}$, respectively. In the 3D case, we first show the eventual regularity of global weak solutions, and the existence of global strong solutions provided that either the viscosity is large enough or the initial datum is close to a given equilibrium. Then an analogous result on the long-time behavior as in 2D is also obtained. In both cases, we provide an estimate on the convergence rate.

Before ending this section, we state some key ingredients of the present paper. System (1.1)-(1.5) is non-autonomous due to the time-dependent boundary data **h** and external force **g**. This brings some additional difficulties into our subsequent proofs. First, in order to obtain the energy inequalities that play crucial roles in the proof of well-posedness as well as in the long-time behavior of global solutions (cf. Lemmas 2.2, 2.5, 2.6, 5.1), we have to introduce proper lifting functions (cf. (2.7) and (2.21) below). The idea was first used in [5,6], but the lifting functions introduced in this paper are different from those in [6]. This is due to the

fact that we need some specific energy inequalities which not only yield uniform estimates of the solutions, but also provide estimates of the convergence rate (cf. Section 4). The second issue regards the application of the Lojasiewicz–Simon approach (cf. [30]) which has been shown to be very useful in the study of long-time behavior of global solutions to nonlinear evolution equations (see, for instance, [12,13,15,34,35,38] and references therein). In particular, convergent results related to various evolution equations with asymptotically autonomous source terms were established, e.g., in [4,9,13,14]. However, our current case is much more complicated than the previous cases, because the Lojasiewicz–Simon inequality involves the vector \mathbf{d} that is subject to a time-dependent boundary datum. To overcome this difficulty, we derive an extended Lojasiewicz–Simon type inequality for vector functions with arbitrary nonhomogeneous Dirichlet boundary data, which is associated with the lifted energy (cf. Corollary 3.1). This generalizes the results in [13,35] and should have its own interest. Third, in the 3D case, we also apply the Lojasiewicz–Simon approach to prove the existence of global strong solutions provided that the initial datum is close to a local minimizer of the elastic energy and the non-autonomous terms are properly small perturbations of their asymptotic limits (cf. Section 5). Then we further discuss the stability of these energy minimizers. This extends the previous results in [20, 35] for the autonomous system, where the initial datum was required to be sufficiently close to a global energy minimizer. For the stability of the general Ericksen–Leslie system [22], we refer to the recent work [37].

The remaining part of the paper is organized as follows. The next section is devoted to report some existence and uniqueness results and basic *a priori* estimates for the solutions. The extended Lojasiewicz–Simon inequality we need is derived in Section 3. In Section 4 we show the convergence of each global weak/strong solution to a single steady state and provide uniform estimates on the convergence rate in 2D. Results in 3D are presented in Section 5. In particular, we study the eventual regularity of global weak solutions as well as the well-posedness when the initial data are close to local minimizers of the elastic energy. Long-time convergence of global solutions and stability of such minimizers are also proved. In the final Section 6, some useful properties of the lifting functions are reported.

2 Preliminaries: well-posedness and *a priori* estimates

Without loss of generality, from now on we set $\lambda = \eta = 1$. Let us introduce the function spaces we shall work with. As usual, $L^p(\Omega)$ and $W^{k,p}(\Omega)$ stand for the Lebesgue and the Sobolev spaces of real valued functions, with the convention that $H^k(\Omega) = W^{k,2}(\Omega)$. The spaces of vector-valued functions are denoted by bold letters, correspondingly. Without any further specification, $\|\cdot\|$ stands for the norm in $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$. This norm is induced by the scalar inner product $(u, v) = \int_{\Omega} uv dx$, where for vector valued functions the product uv is replaced by the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$. We set, as usual,

$$\mathbf{H} = \overline{\mathcal{V}}^{\mathbf{L}^2(\Omega)}, \quad \mathbf{V} = \overline{\mathcal{V}}^{\mathbf{H}_0^1(\Omega)}, \quad \text{where } \mathcal{V} = \left\{ \mathbf{v} \in C_0^\infty(\Omega, \mathbb{R}^n) : \, \nabla \cdot \mathbf{v} = 0 \right\}.$$

For any Banach space B, we denote its dual space by B^* . In particular, we denote the dual space of $\mathbf{H}_0^1(\Omega)$ by $\mathbf{H}^{-1}(\Omega)$.

In the following text, we will use the regularity result for Stokes problem (see, e.g., [33])

Lemma 2.1. For the Stokes operator $S: D(S) = \mathbf{V} \cap \mathbf{H}^2(\Omega) \to \mathbf{H}$ defined by

$$S\mathbf{u} = -\Delta \mathbf{u} + \nabla \pi \in \mathbf{H}, \quad \forall \mathbf{u} \in D(S),$$

it holds

$$\|\mathbf{u}\|_{\mathbf{H}^2} + \|\pi\|_{H^1 \setminus \mathbb{R}} \le C \|S\mathbf{u}\|, \quad \forall \mathbf{u} \in D(S),$$

for some positive constant C only depending on Ω and the spatial dimension.

We begin to report the existence of a weak solution (see [1, Corollary 1.1, Theorem 1.4]).

Proposition 2.1. Suppose n = 2, 3. For any given T > 0, assume

$$\mathbf{g} \in L^2(0,T;\mathbf{V}^*),\tag{2.1}$$

$$\mathbf{h} \in L^2(0, T; \mathbf{H}^{\frac{3}{2}}(\Gamma)), \tag{2.2}$$

$$\mathbf{h}_t \in L^2(0, T; \mathbf{H}^{-\frac{1}{2}}(\Gamma)) \tag{2.3}$$

$$|\mathbf{h}|_{\mathbb{R}^n} \le 1, \quad a.e. \text{ on } \Gamma \times [0,T],$$

$$(2.4)$$

$$\mathbf{d}_0|_{\Gamma} = \mathbf{h}|_{t=0}.\tag{2.5}$$

Then for any $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{H} \times \mathbf{H}^1(\Omega)$ with $|\mathbf{d}_0|_{\mathbb{R}^n} \leq 1$ almost everywhere in Ω , problem (1.1)–(1.5) admits a weak solution (\mathbf{v}, \mathbf{d}) such that

$$\mathbf{v} \in L^{\infty}(0, T; \mathbf{H}) \cap L^{2}(0, T; \mathbf{V}),$$

$$\mathbf{d} \in L^{\infty}(0, T; \mathbf{H}^{1}(\Omega)) \cap L^{2}(0, T; \mathbf{H}^{2}(\Omega)),$$

$$|\mathbf{d}(x, t)|_{\mathbb{R}^{n}} \leq 1, \quad \text{a.e. on } \Omega \times [0, T].$$
(2.6)

If n = 2, then the weak solution (\mathbf{v}, \mathbf{d}) to problem (1.1)–(1.5) is unique. Moreover, we have $(\mathbf{v}, \mathbf{d}) \in C([0, T]; \mathbf{H} \times \mathbf{H}^1(\Omega)).$

Remark 2.1. The weak maximum principle (2.6) plays an important role in the analysis of system (1.1)–(1.5) (cf. [20] for the autonomous case). We recall that system (1.1)–(1.5) is a simplified version of the Ericksen–Leslie system for the liquid crystal flow of nematic type, in which the molecule is assumed to be "small" such that the stretching and rotating effects in the fluid are neglected. In particular, when the stretching effect is taken into account (cf. [31]), the weak maximum principle (2.6) fails. The lack of control of the $L_t^{\infty} L_x^{\infty}$ -norm of **d** brings extra difficulties in the analysis. For instance, in this case, it is not clear how to define weak solutions (compare with [20, 22]). We refer to [2, 10, 28, 31, 36] for extensive studies (well-posedness, long-time behavior, and so on) on more general liquid crystal systems with stretching terms (see also [3, 37] for the full Ericksen–Leslie system). Whether the results obtained in this paper can be extended to those nonautonomous general liquid crystal systems involving stretching effect remains a challenging open problem.

In order to obtain proper energy inequalities for the system (1.1)–(1.5), we recall that suitable lifting functions were introduced in [5,6] to overcome the technical difficulties related to the timedependent boundary datum for **d**. The first lifting function $\mathbf{d}_E = \mathbf{d}_E(x,t)$ is of elliptic type (cf. [6]):

$$\begin{cases} -\Delta \mathbf{d}_E = \mathbf{0}, & \text{in } \Omega \times \mathbb{R}^+, \\ \mathbf{d}_E = \mathbf{h}, & \text{on } \Gamma \times \mathbb{R}^+. \end{cases}$$
(2.7)

In particular, we define the lifting function \mathbf{d}_{E0} for the initial datum:

$$\begin{cases} -\Delta \mathbf{d}_{E0} = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{d}_{E0} = \mathbf{d}_0, & \text{on } \Gamma. \end{cases}$$
(2.8)

Set now

$$\widehat{\mathbf{d}} = \mathbf{d} - \mathbf{d}_E. \tag{2.9}$$

Then system (1.1)-(1.5) can be rewritten into the following form:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} + \nabla \pi = -\Delta \widehat{\mathbf{d}} \cdot \nabla \mathbf{d} + \mathbf{g}(t), \qquad (2.10)$$

$$\nabla \cdot \mathbf{v} = 0, \qquad (2.11)$$

$$\widehat{\mathbf{d}}_t + \mathbf{v} \cdot \nabla \mathbf{d} = \Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d}) - \partial_t \mathbf{d}_E(t)$$
(2.12)

with homogeneous Dirichlet boundary conditions and initial conditions

$$\mathbf{v} = \mathbf{0}, \quad \widehat{\mathbf{d}} = \mathbf{0}, \qquad \text{on } \Gamma \times \mathbb{R}^+,$$

$$(2.13)$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \widehat{\mathbf{d}}|_{t=0} = \mathbf{d}_0 - \mathbf{d}_{E0}, \qquad \text{in } \Omega.$$
(2.14)

Note that we have used the well-known identity $\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) = \frac{1}{2} \nabla (|\nabla \mathbf{d}|^2) + \Delta \mathbf{d} \cdot \nabla \mathbf{d}$ to absorb the gradient term into pressure (cf. [20]).

Let us introduce the lifted energy

$$\widehat{\mathcal{E}}(t) = \frac{1}{2} \|\mathbf{v}(t)\|^2 + \frac{1}{2} \|\nabla\widehat{\mathbf{d}}(t)\|^2 + \int_{\Omega} F(\mathbf{d}(t)) dx, \quad t \ge 0.$$
(2.15)

Then we can derive the *basic energy inequality* for system (1.1)-(1.5).

Lemma 2.2. Let the assumptions of Proposition 2.1 be satisfied for all T > 0. Then, any weak solution which is smooth enough satisfies the following inequality for $t \ge 0$

$$\frac{d}{dt}\widehat{\mathcal{E}}(t) + \frac{\nu}{2} \|\nabla \mathbf{v}\|^2 + \frac{1}{2} \|\Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^2 \le \frac{1}{2} \|\partial_t \mathbf{d}_E\|^2 + C \|\partial_t \mathbf{d}_E\| + C \|\mathbf{g}\|_{\mathbf{V}^*}^2 := r(t), \quad (2.16)$$

where C is a positive constant independent of \mathbf{v} and \mathbf{d} .

Proof. Multiplying (2.10) and (2.12) by \mathbf{v} and $-\Delta \hat{\mathbf{d}} + \mathbf{f}(\mathbf{d})$, respectively, integrating over Ω and adding the results together, we get

$$\frac{d}{dt} \left(\frac{1}{2} \| \mathbf{v} \|^2 + \frac{1}{2} \| \nabla \widehat{\mathbf{d}} \|^2 + \int_{\Omega} F(\mathbf{d}) dx \right) + \nu \| \nabla \mathbf{v} \|^2 + \| \Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d}) \|^2$$

$$= (\partial_t \mathbf{d}_E, \Delta \widehat{\mathbf{d}}) + (\mathbf{g}, \mathbf{v}).$$
(2.17)

In above, we have used the facts $(\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{v}) = (\nabla P, \mathbf{v}) = (\mathbf{v} \cdot \nabla \mathbf{d}, \mathbf{f}(\mathbf{d})) = 0$ due to the impressibility condition $\nabla \cdot \mathbf{v} = 0$. By the Poincaré inequality $\|\mathbf{v}\| \leq C_P \|\nabla \mathbf{v}\|$ and (2.6), the right-hand side of (2.17) can be estimated as follows

$$\begin{aligned} &|(\partial_t \mathbf{d}_E, \Delta \widehat{\mathbf{d}}) + (\mathbf{g}, \mathbf{v})| \\ &\leq |(\partial_t \mathbf{d}_E, \Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d}))| + |(\partial_t \mathbf{d}_E, \mathbf{f}(\mathbf{d}))| + |(\mathbf{g}, \mathbf{v})| \\ &\leq \|\Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| \|\partial_t \mathbf{d}_E\| + \|\mathbf{f}(\mathbf{d})\| \|\partial_t \mathbf{d}_E\| + \|\mathbf{v}\|_{\mathbf{V}} \|\mathbf{g}\|_{\mathbf{V}^*} \\ &\leq \frac{\nu}{2} \|\nabla \mathbf{v}\|^2 + \frac{1}{2} \|\Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^2 + \frac{1}{2} \|\partial_t \mathbf{d}_E\|^2 + C \|\partial_t \mathbf{d}_E\| + C \|\mathbf{g}\|_{\mathbf{V}^*}^2. \end{aligned}$$

The proof is complete.

Remark 2.2. We fix the calculations in [6, Lemma 2] where the term $(\partial_t \mathbf{d}_E, \Delta \widehat{\mathbf{d}})$ is missing. Though it does not affect the proof of existence result, it does have influence on the long-time behavior of global solutions (especially on the convergence rate).

Let us now introduce the following (Banach) spaces of translation bounded functions

$$\begin{split} L^q_{tb}(0,+\infty;X) &:= \Bigl\{ \mathbf{h} \in L^q_{loc}([0,+\infty);X) : \\ \|\mathbf{h}\|^q_{L^q_{tb}(0,+\infty;X)} &:= \sup_{t \ge 0} \int_t^{t+1} \|\mathbf{h}(\tau)\|^q_X d\tau < +\infty \Bigr\} \end{split}$$

where X is a (real) Banach space and $q \in [1, +\infty)$ is given.

From the basic energy inequality (2.16), through a suitable Galerkin approximation scheme, one can derive uniform-in-time estimates for any weak solution (the proof is a minor modification of [1, Lemma 1.2, Remark 1.1]).

Lemma 2.3. Let the assumptions of Proposition 2.1 hold for all T > 0. In addition, suppose that

$$\mathbf{g} \in L^2(0, +\infty; \mathbf{V}^*), \tag{2.18}$$

$$\mathbf{h} \in L^2_{tb}(0, +\infty; \mathbf{H}^{\frac{3}{2}}(\Gamma)), \tag{2.19}$$

$$\mathbf{h}_t \in L^2(0, +\infty; \mathbf{H}^{-\frac{1}{2}}(\Gamma)) \cap L^1(0, +\infty; \mathbf{H}^{-\frac{1}{2}}(\Gamma)).$$
(2.20)

Then a weak solution (\mathbf{v}, \mathbf{d}) to problem (1.1)–(1.5) given by Proposition 2.1 is a global solution on $[0, +\infty)$ and fulfills the following uniform bounds

$$\begin{aligned} \|\mathbf{v}(t)\| &\leq C, \quad \|\mathbf{d}(t)\|_{\mathbf{H}^{1}} \leq C, \quad \forall \ t \geq 0, \\ \int_{0}^{t} (\nu \|\nabla \mathbf{v}(\tau)\|^{2} + \|(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}))(\tau)\|^{2}) d\tau \leq C, \quad \forall \ t \geq 0. \end{aligned}$$

Here C is a positive constant depending on $\|\mathbf{v}_0\|$, $\|\mathbf{d}_0\|_{\mathbf{H}^1}$, $\|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{V}^*)}$, $\|\mathbf{h}\|_{L^2_{tb}(0,+\infty;\mathbf{H}^{\frac{3}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^2(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$ and $\|\mathbf{h}_t\|_{L^1(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$.

Next, we introduce the lifting function $\mathbf{d}_P = \mathbf{d}_P(x, t)$ of parabolic type, which satisfies

$$\begin{cases} \partial_t \mathbf{d}_P - \Delta \mathbf{d}_P = \mathbf{0}, & \text{in } \Omega \times \mathbb{R}^+, \\ \mathbf{d}_P = \mathbf{h}, & \text{on } \Gamma \times \mathbb{R}^+, \\ \mathbf{d}_P(0) = \mathbf{d}_{E0}, & \text{in } \Omega. \end{cases}$$
(2.21)

The motivation of introducing the parabolic lifting function \mathbf{d}_P is that we now have, by definition, $\Delta(\mathbf{d} - \mathbf{d}_P) - \mathbf{f}(\mathbf{d})|_{\Gamma} = \mathbf{0}$. This fact is crucial when we use integration by parts to derive some higher-order differential inequalities of system (1.1)–(1.5) (cf. [6,11]). We note that \mathbf{d}_P in (2.21) is different from the one introduced in [6] as they have different initial values. Both choices are valid for the proof of existence result, but the current definition of \mathbf{d}_P is necessary for the study of long-time behavior. Denote

$$\mathbf{d} = \mathbf{d} - \mathbf{d}_P$$

System (1.1)–(1.5) can now be rewritten into the following form:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} + \nabla P = -\Delta \mathbf{d} \cdot \nabla \mathbf{d} + \mathbf{g}(t), \qquad (2.22)$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2.23}$$

$$\widetilde{\mathbf{d}}_t + \mathbf{v} \cdot \nabla \mathbf{d} = \Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})$$
(2.24)

with homogeneous Dirichlet boundary conditions and initial conditions

$$\mathbf{v} = \mathbf{0}, \quad \widetilde{\mathbf{d}} = \mathbf{0}, \qquad \text{on } \Gamma \times \mathbb{R}^+,$$

$$(2.25)$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \widetilde{\mathbf{d}}|_{t=0} = \mathbf{d}_0 - \mathbf{d}_{E0}, \qquad \text{in } \Omega.$$
(2.26)

In the sequel, we shall frequently use the following lemma (cf. [6])

Lemma 2.4. The following equivalence between norms hold

$$\begin{split} \|\mathbf{v}\|_{\mathbf{H}^{1}} &\approx \|\nabla \mathbf{v}\|, \quad \|\widetilde{\mathbf{d}}\|_{\mathbf{H}^{1}} \approx \|\nabla \widetilde{\mathbf{d}}\|, \quad \text{in } \mathbf{H}_{0}^{1}(\Omega), \\ \|\mathbf{v}\|_{\mathbf{H}^{2}} &\approx \|\Delta \mathbf{v}\|, \quad \|\widetilde{\mathbf{d}}\|_{\mathbf{H}^{2}} \approx \|\Delta \widetilde{\mathbf{d}}\|, \quad \text{in } \mathbf{H}_{0}^{1}(\Omega) \cap \mathbf{H}^{2}(\Omega), \\ \|\widetilde{\mathbf{d}}\|_{\mathbf{H}^{3}} &\approx \|\nabla(\Delta \widetilde{\mathbf{d}})\| + \|\Delta \widetilde{\mathbf{d}}\|, \quad \text{in } \mathbf{H}_{0}^{1}(\Omega) \cap \mathbf{H}^{3}(\Omega). \end{split}$$

If **d** and **d**_P are functions that are smooth enough and $|\mathbf{d}|_{\mathbb{R}^n} \leq 1$, $|\mathbf{d}_P|_{\mathbb{R}^n} \leq 1$, then we have

$$\begin{aligned} \|\Delta \mathbf{d}\| &\leq \|\Delta \mathbf{d}_P\| + \|\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})\| + C, \\ \|\nabla \Delta \mathbf{d}\| &\leq \|\nabla \Delta \mathbf{d}_P\| + \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\| + C \|\nabla \mathbf{d}\|, \end{aligned}$$

where C is a positive constant independent of \mathbf{d} and \mathbf{d}_P .

Let us introduce the quantity

$$\mathcal{A}_P(t) = \|\nabla \mathbf{v}(t)\|^2 + \|\Delta \mathbf{d}(t) - \mathbf{f}(\mathbf{d}(t))\|^2, \quad t \ge 0.$$

Lemma 2.5. Let n = 2 and let the assumptions of Lemma 2.3 hold. If the weak solution (\mathbf{v}, \mathbf{d}) is smooth enough then it satisfies the following inequality

$$\frac{d}{dt}\mathcal{A}_P(t) \le C(\mathcal{A}_P^2(t) + \mathcal{A}_P(t) + R_1(t)), \qquad (2.27)$$

where

$$R_1(t) = \|\partial_t \mathbf{d}_P(t)\|^4 + \|\partial_t \mathbf{d}_P(t)\|^2 + \|\nabla \Delta \mathbf{d}_P(t)\|^2 + \|\mathbf{g}(t)\|^2.$$
(2.28)

Here C is a positive constant depending on ν , $\|\mathbf{v}_0\|$, $\|\mathbf{d}_0\|_{\mathbf{H}^1}$, $\|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{V}^*)}$, $\|\mathbf{h}_t\|_{L^2(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^1(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$ and $\|\mathbf{h}\|_{L^2_{tb}(0,+\infty;\mathbf{H}^{\frac{3}{2}}(\Gamma))}$.

Proof. Taking the time derivative of $\mathcal{A}_P(t)$, we obtain by a direct calculation that

$$\frac{1}{2} \frac{d}{dt} \mathcal{A}_{P}(t) + (\nu \| S \mathbf{v} \|^{2} + \| \nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}) \|^{2}) \\
= -(S \mathbf{v}, \mathbf{v} \cdot \nabla \mathbf{v}) + (S \mathbf{v}, \mathbf{g}) - (S \mathbf{v}, \Delta \mathbf{d} \cdot \nabla \mathbf{d}) - (\nabla (\mathbf{v} \cdot \nabla \mathbf{d}), \nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))) \\
-(\mathbf{f}'(\mathbf{d}) \mathbf{d}_{t}, \Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})) \\
\coloneqq \sum_{j=1}^{5} I_{j}.$$
(2.29)

To get this identity we have used the fact that $\Delta \tilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})|_{\Gamma} = \mathbf{0}$ as well as $(S\mathbf{v}, \mathbf{v}_t) = (-\Delta \mathbf{v}, \mathbf{v}_t)$. It is not difficult to see that

$$|I_1| \leq \|S\mathbf{v}\| \|\mathbf{v}\|_{\mathbf{L}^4} \|\nabla \mathbf{v}\|_{\mathbf{L}^4}$$

$$\leq C \|S\mathbf{v}\| (\|\nabla \mathbf{v}\|^{\frac{1}{2}} \|\mathbf{v}\|^{\frac{1}{2}}) (\|\Delta \mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\|^{\frac{1}{2}})$$

$$\leq C \|S\mathbf{v}\|^{\frac{3}{2}} \|\nabla \mathbf{v}\| \leq \varepsilon \|S\mathbf{v}\|^{2} + C \|\nabla \mathbf{v}\|^{2},$$
$$|I_{2}| \leq \varepsilon \|S\mathbf{v}\|^{2} + C \|\mathbf{g}\|^{2}.$$

For I_3 , we have

$$\begin{aligned} |I_3| &= |(S\mathbf{v}, (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})) \cdot \nabla \mathbf{d}) + (S\mathbf{v}, \mathbf{f}(\mathbf{d}) \cdot \nabla \mathbf{d}) + (S\mathbf{v}, \partial_t \mathbf{d}_P \cdot \nabla \mathbf{d})| \\ &\leq ||S\mathbf{v}|| ||\nabla \mathbf{d}||_{\mathbf{L}^4} ||\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})||_{\mathbf{L}^4} + ||S\mathbf{v}|| ||\nabla \mathbf{d}||_{\mathbf{L}^{\infty}} ||\partial_t \mathbf{d}_P|| \\ &\leq \varepsilon ||S\mathbf{v}||^2 + C ||\nabla \mathbf{d}||_{\mathbf{L}^4}^2 ||\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})||_{\mathbf{L}^4}^2 + C ||\nabla \mathbf{d}||_{\mathbf{L}^{\infty}}^2 ||\partial_t \mathbf{d}_P||^2. \end{aligned}$$

On account of Lemma 2.3, we infer from the Sobolev embedding theorems that

$$\begin{aligned} \|\nabla \mathbf{d}\|_{\mathbf{L}^{4}}^{2} &\leq C \|\Delta \mathbf{d}\| \|\nabla \mathbf{d}\| + C \|\nabla \mathbf{d}\|^{2} \leq C \|\Delta \widetilde{\mathbf{d}}\| + C \|\partial_{t} \mathbf{d}_{P}\| + C \\ &\leq C \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + C \|\partial_{t} \mathbf{d}_{P}\| + C, \end{aligned}$$

$$\begin{aligned} \|\nabla \mathbf{d}\|_{\mathbf{L}^{\infty}}^2 &\leq C \|\nabla \Delta \mathbf{d}\| \|\nabla \mathbf{d}\| + C \|\nabla \mathbf{d}\|^2 \\ &\leq C \|\nabla \Delta \mathbf{d}_P\| + C(1 + \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|), \end{aligned}$$

$$\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|_{\mathbf{L}^4}^2 \leq C \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\| \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|_{\mathbf{d}}$$

Using the above estimates, we obtain the estimates for I_3 and I_4 :

$$\begin{aligned} |I_{3}| &\leq \varepsilon \|S\mathbf{v}\|^{2} + C\|\nabla(\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|\|\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|(\|\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\partial_{t}\mathbf{d}_{P}\| + 1) \\ &+ C\|\partial_{t}\mathbf{d}_{P}\|^{2}(\|\nabla\Delta\mathbf{d}_{P}\| + \|\nabla(\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\| + 1) \\ &\leq \varepsilon \|S\mathbf{v}\|^{2} + \varepsilon \|\nabla(\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{2} + C\|\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{4} + C\|\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2} \\ &+ C\|\partial_{t}\mathbf{d}_{P}\|^{2}(\|\partial_{t}\mathbf{d}_{P}\|^{2} + \|\nabla\Delta\mathbf{d}_{P}\| + 1), \end{aligned}$$

$$\begin{aligned} |I_4| &\leq \|\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|(\|\nabla \mathbf{v}\|_{\mathbf{L}^4}\|\nabla \mathbf{d}\|_{\mathbf{L}^4} + \|\mathbf{v}\|_{\mathbf{L}^{\infty}}\|\mathbf{d}\|_{\mathbf{H}^2}) \\ &\leq \varepsilon \|\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^2 + C\|\nabla \mathbf{v}\|_{\mathbf{L}^4}^2 \|\nabla \mathbf{d}\|_{\mathbf{L}^4}^2 + C\|\mathbf{v}\|_{\mathbf{L}^{\infty}}^2 (\|\Delta \mathbf{d}\|^2 + 1) \\ &\leq \varepsilon \|\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^2 + C\|\Delta \mathbf{v}\|\|\nabla \mathbf{v}\|(\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\partial_t \mathbf{d}_P\| + 1) \\ &+ C\|\Delta \mathbf{v}\|\|\mathbf{v}\|(\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^2 + \|\partial_t \mathbf{d}_P\|^2 + 1) \\ &\leq \varepsilon \|S \mathbf{v}\|^2 + \varepsilon \|\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^2 + C\|\nabla \mathbf{v}\|^4 + C\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^4 \\ &+ C\|\nabla \mathbf{v}\|^2 + C\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^2 + \|\partial_t \mathbf{d}_P\|^4. \end{aligned}$$

We now observe that

$$I_{5} = -(\mathbf{f}'(\mathbf{d})\widetilde{\mathbf{d}}_{t}, \Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})) - (\mathbf{f}'(\mathbf{d})\partial_{t}\mathbf{d}_{P}, \Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})) := I_{5a} + I_{5b}.$$
 (2.30)

Recalling (2.24), we have

$$\begin{aligned} |I_{5a}| &= |(\mathbf{f}'(\mathbf{d})(\mathbf{v}\cdot\nabla)\mathbf{d},\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})) - (\mathbf{f}'(\mathbf{d})(\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})),\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))| \\ &\leq \|\mathbf{f}'(\mathbf{d})\|_{\mathbf{L}^{\infty}}(\|\mathbf{v}\|_{\mathbf{L}^{4}}\|\nabla\mathbf{d}\|_{\mathbf{L}^{4}}\|\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2}) \\ &\leq C\|\nabla\mathbf{v}\|\|\|\mathbf{v}\|\|\nabla\mathbf{d}\|_{\mathbf{L}^{4}}^{2} + C\|\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2} \\ &\leq C\|\nabla\mathbf{v}\|^{2} + C\|\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2} + C\|\partial_{t}\mathbf{d}_{P}\|^{2}, \end{aligned}$$

$$|I_{5b}| \leq \|\mathbf{f}'(\mathbf{d})\|_{\mathbf{L}^{\infty}} \|\partial_t \mathbf{d}_P\| \|\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})\| \leq C \|\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})\|^2 + C \|\partial_t \mathbf{d}_P\|^2.$$
(2.31)

Finally, collecting the above estimates and taking ε sufficiently small, we deduce that

$$\frac{d}{dt}\mathcal{A}_{P}(t) + (\nu \|S\mathbf{v}\|^{2} + \|\nabla(\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{2})$$

$$\leq C(\mathcal{A}_{P}^{2}(t) + \mathcal{A}_{P}(t)) + C\|\partial_{t}\mathbf{d}_{P}\|^{2}(\|\partial_{t}\mathbf{d}_{P}\|^{2} + \|\nabla\Delta\mathbf{d}_{P}\| + 1) + \|\mathbf{g}\|^{2},$$

which easily implies the inequality (2.27).

Taking advantage of Lemmas 2.5, 6.1 and 6.2, one can deduce the following results on the regularity of weak solutions as well as the existence of strong solutions to system (1.1)-(1.5) in 2D.

Theorem 2.1. Let n = 2 and let the assumptions of Proposition 2.1 hold for all T > 0. In addition, suppose that

$$\mathbf{g} \in L^2(0, +\infty; \mathbf{H}),\tag{2.32}$$

$$\mathbf{h} \in L^2_{tb}(0, +\infty; \mathbf{H}^{\frac{5}{2}}(\Gamma)), \tag{2.33}$$

$$\mathbf{h}_t \in L^2(0, +\infty; \mathbf{H}^{\frac{1}{2}}(\Gamma)) \cap L^1(0, +\infty; \mathbf{H}^{-\frac{1}{2}}(\Gamma)).$$
(2.34)

(i) System (1.1)–(1.5) admits a unique global weak solution (\mathbf{v}, \mathbf{d}) satisfying

$$\begin{aligned} \|\mathbf{v}(t)\|_{\mathbf{V}} &\leq C(1+t^{-1}), \quad \|\mathbf{d}(t)\|_{\mathbf{H}^2} \leq C(1+t^{-1}), \quad \forall t > 0, \\ \int_{\delta}^{t} (\|\mathbf{v}(\tau)\|_{\mathbf{H}^2}^2 + \|\mathbf{d}(\tau)\|_{\mathbf{H}^3}^2) d\tau \leq C(1+\delta^{-1})T, \quad t \in [\delta, T], \end{aligned}$$

where C is a positive constant depending on ν , $\|\mathbf{v}_0\|$, $\|\mathbf{d}_0\|_{\mathbf{H}^1}$, $\|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{H})}$, $\|\mathbf{h}\|_{L^2_{tb}(0,+\infty;\mathbf{H}^{\frac{5}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^2(0,+\infty;\mathbf{H}^{\frac{1}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^1(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$.

(ii) If $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$, then problem (1.1)–(1.5) admits a unique global strong solution (\mathbf{v}, \mathbf{d}) satisfying

$$\|\mathbf{v}(t)\|_{\mathbf{V}} \le C, \quad \|\mathbf{d}(t)\|_{\mathbf{H}^2} \le C, \quad \forall t \ge 0,$$
(2.35)

$$\int_{0}^{t} (\|\mathbf{v}(\tau)\|_{\mathbf{H}^{2}}^{2} + \|\mathbf{d}(\tau)\|_{\mathbf{H}^{3}}^{2}) d\tau \le CT, \quad t \in [0, T],$$
(2.36)

where C is a positive constant depending on ν , $\|\mathbf{v}_0\|_{\mathbf{V}}$, $\|\mathbf{d}_0\|_{\mathbf{H}^2}$, $\|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{H})}$, $\|\mathbf{h}\|_{L^2_{tb}(0,+\infty;\mathbf{H}^{\frac{5}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^2(0,+\infty;\mathbf{H}^{\frac{1}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^1(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$.

Remark 2.3. Lemma 2.3 and Theorem 2.1 still hold when \mathbf{g} and \mathbf{h}_t are translation bounded with respect to time (see [1]).

Next, we consider the 3D case. Instead of Lemma 2.5, we have the following higher-order energy inequality

Lemma 2.6. Let n = 3 and let the assumptions of Lemma 2.3 hold. If a weak solution (\mathbf{v}, \mathbf{d}) is smooth enough then it satisfies the following inequality

$$\frac{d}{dt}\widetilde{\mathcal{A}}_{P}(t) + \left(\nu - c_{1}\widetilde{\mathcal{A}}_{P}(t)\right) \|S\mathbf{v}\|^{2} + \left(1 - \frac{c_{2}}{\nu^{\frac{1}{2}}}\widetilde{\mathcal{A}}_{P}(t)\right) \|\nabla(\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{2}$$

$$\leq C(1 + \nu^{-2})(\mathcal{A}_{P}(t) + R_{2}(t)), \quad t \geq 0,$$
(2.37)

where $\widetilde{\mathcal{A}}_P(t) = \mathcal{A}_P(t) + 1$ and

$$R_2(t) = \|\partial_t \mathbf{d}_P(t)\|^2 + \|\partial_t \mathbf{d}_P(t)\|^6 + \|\nabla \partial_t \mathbf{d}_P(t)\|^2 + \|\mathbf{g}(t)\|^2.$$
(2.38)

Here c_1, c_2, C are positive constants that may depend on $\|\mathbf{v}_0\|$, $\|\mathbf{d}_0\|_{\mathbf{H}^1}$ and on $\|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{V}^*)}$, $\|\mathbf{h}\|_{L^2_{tb}(0,+\infty;\mathbf{H}^{\frac{3}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^2(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^1(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$, but they are independent of ν .

Proof. We estimate the right-hand side of (2.29) by using the 3D version of Sobolev embedding theorems. We have

$$|I_1| \leq \|S\mathbf{v}\| \|\mathbf{v}\|_{\mathbf{L}^6} \|\nabla \mathbf{v}\|_{\mathbf{L}^3}$$

$$\leq C \|S\mathbf{v}\| \|\nabla \mathbf{v}\| (\|\Delta \mathbf{v}\|^{\frac{1}{2}} \|\nabla \mathbf{v}\|^{\frac{1}{2}})$$

$$\leq C \|S\mathbf{v}\|^{\frac{3}{2}} \|\nabla \mathbf{v}\|^{\frac{3}{2}} \leq \frac{1}{2} \|\nabla \mathbf{v}\|^{\frac{4}{3}} \|S\mathbf{v}\|^2 + C \|\nabla \mathbf{v}\|^2,$$

$$|I_2| \le \frac{\nu}{8} ||S\mathbf{v}||^2 + \frac{2}{\nu} ||\mathbf{g}||^2.$$

Recalling that $\|\mathbf{d}\|_{\mathbf{H}^1} \leq C$ (cf. Lemma 2.3), from the Sobolev embedding theorems as well as Agmon's inequality in dimension three, we infer

$$\begin{aligned} \|\nabla \mathbf{d}\|_{\mathbf{L}^{3}} &\leq C \|\Delta \mathbf{d}\|^{\frac{1}{2}} \|\nabla \mathbf{d}\|^{\frac{1}{2}} + C \|\nabla \mathbf{d}\| \leq C (\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\partial_{t} \mathbf{d}_{P}\|)^{\frac{1}{2}} + C, \\ \|\nabla \mathbf{d}\|_{\mathbf{L}^{6}} &\leq C \|\Delta \mathbf{d}\| + C \|\nabla \mathbf{d}\| \leq C \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + C \|\partial_{t} \mathbf{d}_{P}\| + C, \end{aligned}$$

$$\begin{split} \|\nabla \mathbf{d}\|_{\mathbf{L}^{\infty}} &\leq C \|\nabla \mathbf{d}\|_{\mathbf{H}^{1}}^{\frac{1}{2}} \|\nabla \mathbf{d}\|_{\mathbf{H}^{2}}^{\frac{1}{2}} \leq C(\|\nabla \Delta \mathbf{d}\|^{\frac{1}{2}} \|\Delta \mathbf{d}\|^{\frac{1}{2}} + \|\Delta \mathbf{d}\| + 1) \\ &\leq C(\|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \|\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{\frac{1}{2}} + \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{\frac{1}{2}} \|\nabla \Delta \mathbf{d}_{P}\|^{\frac{1}{2}} \\ &+ \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \|\Delta \mathbf{d}_{P}\|^{\frac{1}{2}} + \|\nabla \Delta \mathbf{d}_{P}\|^{\frac{1}{2}} \|\Delta \mathbf{d}_{P}\|^{\frac{1}{2}} + \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \\ &+ \|\nabla \Delta \mathbf{d}_{P}\|^{\frac{1}{2}} + \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\Delta \mathbf{d}_{P}\| + 1), \\ &\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{3}} \leq C \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{\frac{1}{2}}. \end{split}$$

Thus we have

$$\begin{split} |I_{3}| &\leq \|S\mathbf{v}\| \|\nabla \mathbf{d}\|_{\mathbf{L}^{6}} \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{3}} + \|S\mathbf{v}\| \|\nabla \mathbf{d}\|_{\mathbf{L}^{\infty}} \|\partial_{t} \mathbf{d}_{P}\| \\ &\leq C\|S\mathbf{v}\| (\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\partial_{t} \mathbf{d}_{P}\| + 1) \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{\frac{1}{2}} \\ &+ C\|S\mathbf{v}\| \|\partial_{t} \mathbf{d}_{P}\| \left(\|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{\frac{1}{2}} + \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{\frac{1}{2}} \|\nabla \partial_{t} \mathbf{d}_{P}\|^{\frac{1}{2}} \\ &+ \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \|\partial_{t} \mathbf{d}_{P}\|^{\frac{1}{2}} + \|\nabla \partial_{t} \mathbf{d}_{P}\|^{\frac{1}{2}} \|\partial_{t} \mathbf{d}_{P}\|^{\frac{1}{2}} + \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \\ &+ \|\nabla \partial_{t} \mathbf{d}_{P}\|^{\frac{1}{2}} + \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\partial_{t} \mathbf{d}_{P}\| + 1 \Big) \\ &\leq \left(\frac{\nu}{8} + \frac{1}{2} \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2}\right) \|S\mathbf{v}\|^{2} + \frac{1}{8} \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{2} \\ &+ C(1 + \nu^{-2})(\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2} + \|\partial_{t} \mathbf{d}_{P}\|^{6} + \|\partial_{t} \mathbf{d}_{P}\|^{2} + \|\nabla \Delta \mathbf{d}_{P}\|^{2}), \end{split}$$

$$\begin{aligned} |I_4| &\leq \|\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\| (\|\nabla \mathbf{v}\|_{\mathbf{L}^3} \|\nabla \mathbf{d}\|_{\mathbf{L}^6} + \|\mathbf{v}\|_{\mathbf{L}^{\infty}} \|\mathbf{d}\|_{\mathbf{H}^2}) \\ &\leq C \|\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\| \|\nabla \mathbf{v}\|^{\frac{1}{2}} \|\Delta \mathbf{v}\|^{\frac{1}{2}} (\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\partial_t \mathbf{d}_P\| + 1) \\ &\leq \left(\frac{\nu}{8} + \frac{1}{2} \|\nabla \mathbf{v}\|^2\right) \|S \mathbf{v}\|^2 + \left(\frac{1}{8} + \frac{1}{2\nu^{\frac{1}{2}}} \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^2\right) \|\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^2 \end{aligned}$$

$$+C\|\Delta\widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^2+C(1+\nu^{-1})\|\nabla\mathbf{v}\|^2+C\|\partial_t\mathbf{d}_P\|^4,$$

$$\begin{aligned} |I_{5a}| &\leq \|\mathbf{f}'(\mathbf{d})\|_{\mathbf{L}^{\infty}}(\|\mathbf{v}\|_{\mathbf{L}^{6}}\|\nabla \mathbf{d}\|_{\mathbf{L}^{3}}\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2}) \\ &\leq C\|\nabla \mathbf{v}\|^{2}\|\nabla \mathbf{d}\|_{\mathbf{L}^{3}}^{2} + C\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2} \\ &\leq C\|\Delta \mathbf{v}\|\|\mathbf{v}\|(\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\| + \|\partial_{t}\mathbf{d}_{P}\| + 1) + C\|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2} \\ &\leq \frac{\nu}{8}\|S\mathbf{v}\|^{2} + C(1 + \nu^{-1})(\|\nabla \mathbf{v}\|^{2} + \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2} + \|\partial_{t}\mathbf{d}_{P}\|^{2}). \end{aligned}$$

We observe that I_{5b} can be estimated as in (2.31). Then, collecting all the estimates of I_j , we have

$$\begin{aligned} \frac{d}{dt}\mathcal{A}_{P}(t) + \left(\nu - \|\nabla \mathbf{v}\|^{\frac{4}{3}} - \|\nabla \mathbf{v}\|^{2}\right) \|S\mathbf{v}\|^{2} \\ + \left(1 - \frac{1}{\nu^{\frac{1}{2}}} \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2}\right) \|\nabla (\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{2} \\ \leq C(1 + \nu^{-2})(\mathcal{A}_{P}(t) + \|\partial_{t}\mathbf{d}_{P}\|^{2} + \|\partial_{t}\mathbf{d}_{P}\|^{6} + \|\nabla \Delta \mathbf{d}_{P}\|^{2} + \|\mathbf{g}\|^{2}). \end{aligned}$$

As a result, there exist constants $c_1, c_2 > 0$ independent of ν such that the following inequality holds

$$\frac{d}{dt}\mathcal{A}_{P}(t) + (\nu - c_{1}\widetilde{\mathcal{A}}_{P}(t))\|S\mathbf{v}\|^{2} + \left(1 - \frac{c_{2}}{\nu^{\frac{1}{2}}}\widetilde{\mathcal{A}}_{P}(t)\right)\|\nabla(\Delta\widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^{2}$$

$$\leq C(1 + \nu^{-2})(\mathcal{A}_{P}(t) + \|\partial_{t}\mathbf{d}_{P}\|^{2} + \|\partial_{t}\mathbf{d}_{P}\|^{6} + \|\nabla\Delta\mathbf{d}_{P}\|^{2} + \|\mathbf{g}\|^{2}),$$

which implies (2.37).

On account of Lemma 2.6, one can deduce that system (1.1)-(1.5) admits at least one global strong solution, provided that the viscosity is large enough (see [6, Theorem 7] for the case $\mathbf{g} = \mathbf{0}$, cf. also [20, 35] for the autonomous case). We just report a result under weaker assumptions than that in [6] and omit the detailed proof.

Theorem 2.2. Let n = 3 and assume that (2.32)-(2.34) and (2.4) are satisfied. For any $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$ satisfying (2.5) and $|\mathbf{d}_0|_{\mathbb{R}^3} \leq 1$, there exists a $\nu_0 > 0$, depending on $\|(\mathbf{v}_0, \mathbf{d}_0)\|_{\mathbf{V}\times\mathbf{H}^2}$ and $\|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{H})}$, $\|\mathbf{h}\|_{L^2_{tb}(0,+\infty;\mathbf{H}^{\frac{5}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^2(0,+\infty;\mathbf{H}^{\frac{1}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^1(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$, such that, for any $\nu \geq \nu_0$, problem (1.1)–(1.5) admits a global strong solution (\mathbf{v}, \mathbf{d}) which satisfies the same uniform estimates as in the 2D case (cf. (2.35) and (2.36)).

Remark 2.4. When n = 3, the weak-strong uniqueness result obtained in [6, Theorem 7] still holds in our case. Thus, the global strong solution (\mathbf{v}, \mathbf{d}) obtained in Theorem 2.2 is unique.

3 Extended Lojasiewicz–Simon type inequality

For all $\mathbf{d} \in \mathcal{N} := \{ \phi \in \mathbf{H}^1(\Omega) : \phi|_{\Gamma} = \mathbf{h}_{\infty} \}$, where $\mathbf{h}_{\infty} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ is given, we consider the functional

$$E(\mathbf{d}) = \frac{1}{2} \|\nabla \mathbf{d}\|^2 + \int_{\Omega} F(\mathbf{d}) dx.$$
(3.1)

It is straightforward to verify that

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Lemma 3.1. If $\psi \in \mathbf{H}^1(\Omega)$ is a weak solution to the elliptic problem

$$\begin{cases} -\Delta \phi + \mathbf{f}(\phi) = \mathbf{0}, \\ \phi|_{\Gamma} = \mathbf{h}_{\infty}, \end{cases}$$
(3.2)

then ψ is a critical point of the functional $E(\mathbf{d})$ in \mathcal{N} . Conversely, if ψ is a critical point of the functional $E(\mathbf{d})$ in \mathcal{N} , then ψ is a weak solution to problem (3.2).

Remark 3.1. Due to the elliptic regularity theory, if \mathbf{h}_{∞} is more regular, then ψ is more regular. For instance, if $\mathbf{h}_{\infty} \in \mathbf{H}^{\frac{3}{2}}(\Gamma)$, then $\psi \in \mathbf{H}^{2}(\Omega)$.

Then we have

Lemma 3.2. Suppose that ψ is a critical point of $E(\mathbf{d})$ in \mathcal{N} . Then there exist constants $\beta_1 > 0$, $\theta \in (0, \frac{1}{2})$ depending on ψ such that, for any $\mathbf{w} \in \mathcal{N}$ that satisfies $\|\mathbf{w} - \psi\|_{\mathbf{H}^1} < \beta_1$, there holds

$$\| -\Delta \mathbf{w} + \mathbf{f}(\mathbf{w}) \|_{\mathbf{H}^{-1}} \ge |E(\mathbf{w}) - E(\psi)|^{1-\theta}.$$
(3.3)

Remark 3.2. The above lemma can be viewed as an extended version of Simon's result [30] for scalar function under L^2 -norm. We can refer to [13, Chapter 2, Theorem 5.2], in which the vector case subject to homogeneous Dirichlet boundary condition was considered. We observe that the result can be easily proved by modifying the argument in [13] using a simple transformation (cf. also [35, Remark 2.1]).

The Lojasiewicz–Simon type inequality (3.3) only applies to proper perturbations of the critical point of energy E in the set \mathcal{N} and it is not enough for our evolutionary problem (1.1)–(1.5), whose boundary datum is time-dependent (not necessary in \mathcal{N}). In order to overcome this difficulty, we prove the following extended result that also involves the perturbation of boundary:

Theorem 3.1. Suppose that ψ is a critical point of $E(\mathbf{d})$ in \mathcal{N} . Then there exists a constant $\beta \in (0,1)$ depending on ψ such that, for any $\mathbf{d} \in \mathbf{H}^1(\Omega)$ satisfying $\|\mathbf{d} - \psi\|_{\mathbf{H}^1} < \beta$, there holds

$$C\left(\|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} + \|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}\right) + \| - \Delta \mathbf{d} + \mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}} \ge |E(\mathbf{d}) - E(\psi)|^{1-\theta}, \quad (3.4)$$

where $\theta \in (0, \frac{1}{2})$ is the same constant as in Lemma 3.2, while C is a positive constant depending on ψ .

Proof. For any $\mathbf{d} \in \mathbf{H}^1(\Omega)$, we have that $\Delta \mathbf{d} \in \mathbf{H}^{-1}(\Omega)$. Then we consider the elliptic boundary value problem

$$\begin{cases} \Delta \mathbf{w} = \Delta \mathbf{d}, \\ \mathbf{w}|_{\Gamma} = \mathbf{h}_{\infty}. \end{cases}$$
(3.5)

It easily follows from the elliptic regularity theory (cf. e.g., [32, Proposition 5.1.7]) that

$$\|\mathbf{w} - \mathbf{d}\|_{\mathbf{H}^1} \le C \|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)},\tag{3.6}$$

which implies

$$\begin{aligned} \|\mathbf{w} - \psi\|_{\mathbf{H}^{1}} &\leq \|\mathbf{w} - \mathbf{d}\|_{\mathbf{H}^{1}} + \|\mathbf{d} - \psi\|_{\mathbf{H}^{1}} \\ &\leq C \|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} + \|\mathbf{d} - \psi\|_{\mathbf{H}^{1}} \end{aligned}$$

$$\leq C \|\mathbf{d} - \psi\|_{\mathbf{H}^1}. \tag{3.7}$$

Let β_1 be the constant in Lemma 3.2. We infer from the above inequality that if $\beta \in (0,1)$ is chosen sufficiently small, then we have $\|\mathbf{w} - \psi\|_{\mathbf{H}^1} < \beta_1$. As a consequence of Lemma 3.2, we have

$$\| -\Delta \mathbf{w} + \mathbf{f}(\mathbf{w}) \|_{\mathbf{H}^{-1}} \ge |E(\mathbf{w}) - E(\psi)|^{1-\theta}.$$
(3.8)

On the other hand, by the definition of \mathbf{w} , we can see that

$$|E(\mathbf{w}) - E(\psi)|^{1-\theta} \leq || - \Delta \mathbf{w} + \mathbf{f}(\mathbf{w})||_{\mathbf{H}^{-1}}$$

$$\leq || - \Delta \mathbf{d} + \mathbf{f}(\mathbf{d})||_{\mathbf{H}^{-1}} + C||\mathbf{f}(\mathbf{d}) - \mathbf{f}(\mathbf{w})||_{\mathbf{L}^{\frac{6}{5}}(\Omega)}$$

$$\leq || - \Delta \mathbf{d} + \mathbf{f}(\mathbf{d})||_{\mathbf{H}^{-1}} + C||\mathbf{d} - \mathbf{w}||_{\mathbf{H}^{1}}$$

$$\leq || - \Delta \mathbf{d} + \mathbf{f}(\mathbf{d})||_{\mathbf{H}^{-1}} + C||\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}||_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}.$$
(3.9)

We deduce from $\theta \in (0, \frac{1}{2})$ that

$$|E(\mathbf{d}) - E(\psi)|^{1-\theta} \le |E(\mathbf{w}) - E(\psi)|^{1-\theta} + |E(\mathbf{d}) - E(\mathbf{w})|^{1-\theta},$$
(3.10)

and

$$|E(\mathbf{d}) - E(\mathbf{w})|^{1-\theta}$$

$$\leq \left(\frac{1}{2}\right)^{1-\theta} \left| \|\nabla \mathbf{d}\|^2 - \|\nabla \mathbf{w}\|^2 \right|^{1-\theta} + \left| \int_{\Omega} (F(\mathbf{d}) - F(\mathbf{w})) dx \right|^{1-\theta}$$

$$\leq C(\|\mathbf{d}\|_{\mathbf{H}^1}, \|\mathbf{w}\|_{\mathbf{H}^1}) \|\mathbf{d} - \mathbf{w}\|_{\mathbf{H}^1}^{1-\theta} \leq C \|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}, \qquad (3.11)$$

where in (3.11) we use the facts that $\|\mathbf{d}\|_{\mathbf{H}^1} \le \|\psi\|_{\mathbf{H}^1} + \beta$ and $\|\mathbf{w}\|_{\mathbf{H}^1} \le \|\psi\|_{\mathbf{H}^1} + \beta_1$. Combining (3.9)–(3.11), we deduce (3.4).

Since the basic energy inequality (2.16) (cf. Lemma 2.2) is only valid for the lifted energy $\hat{\mathcal{E}}$ (2.15), in order to apply the Łojasiewicz–Simon approach to our problem, we need to consider the following auxiliary functional corresponding to energy E (cf. (3.1)):

$$\widehat{E}(\mathbf{d}) = \frac{1}{2} \|\nabla \widehat{\mathbf{d}}\|^2 + \int_{\Omega} F(\mathbf{d}) dx, \quad \forall \ \mathbf{d} \in \mathbf{H}^1(\Omega),$$
(3.12)

where

 $\widehat{\mathbf{d}} = \mathbf{d} - \mathbf{d}_E,$

and \mathbf{d}_E is the elliptic lifting function satisfying the following elliptic problem (cf. (2.7))

$$\begin{cases} -\Delta \mathbf{d}_E = \mathbf{0}, & x \in \Omega, \\ \mathbf{d}_E = \mathbf{d}|_{\Gamma}, & x \in \Gamma. \end{cases}$$
(3.13)

Then we have

Corollary 3.1. Suppose that ψ is a critical point of $E(\mathbf{d})$ in \mathcal{N} . Then there exist constants $\beta \in (0,1)$ and $\theta \in (0,\frac{1}{2})$ depending on ψ such that, for any $\mathbf{d} \in \mathbf{H}^1(\Omega)$ satisfying $\|\mathbf{d} - \psi\|_{\mathbf{H}^1} < \beta$, there holds

$$C\|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} + \| - \Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}} \ge |\widehat{E}(\mathbf{d}) - \widehat{E}(\psi)|^{1-\theta},$$
(3.14)

where C is a positive constant depending on ψ and \mathbf{h}_{∞} .

Proof. From the definition of $\widehat{E}(\mathbf{d})$, we set, for $\psi \in \mathcal{N}$,

$$\widehat{E}(\psi) = \frac{1}{2} \|\nabla\widehat{\psi}\|^2 + \int_{\Omega} F(\psi) dx, \qquad (3.15)$$

where $\widehat{\psi} = \psi - \psi_E$ and ψ_E satisfies

$$\begin{cases} -\Delta \psi_E = \mathbf{0}, & x \in \Omega, \\ \psi_E = \mathbf{h}_{\infty}, & x \in \Gamma. \end{cases}$$
(3.16)

A direct calculation yields that

$$\widehat{E}(\mathbf{d}) = E(\mathbf{d}) + \frac{1}{2} \|\nabla \mathbf{d}_E\|^2 - \int_{\Omega} \nabla \mathbf{d} : \nabla \mathbf{d}_E dx,$$

$$\widehat{E}(\psi) = E(\psi) + \frac{1}{2} \|\nabla \psi_E\|^2 - \int_{\Omega} \nabla \psi : \nabla \psi_E dx,$$

where we used the notation $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{n} A_{ij} B_{ij}$. Theorem 3.1 implies that there exist constants $\beta \in (0, 1)$ and $\theta \in (0, \frac{1}{2})$, such that for any $\mathbf{d} \in \mathbf{H}^{1}(\Omega)$ satisfying $\|\mathbf{d} - \psi\|_{\mathbf{H}^{1}} < \beta$, (3.4) holds. Next, we proceed to estimate the quantity $|\widehat{E}(\mathbf{d}) - \widehat{E}(\psi)|^{1-\theta}$

$$\begin{aligned} |\vec{E}(\mathbf{d}) - \vec{E}(\psi)|^{1-\theta} \\ &\leq |E(\mathbf{d}) - E(\psi)|^{1-\theta} + \left(\frac{1}{2}\right)^{1-\theta} \left| \int_{\Omega} \nabla(\mathbf{d}_E - \psi_E) : \nabla(\mathbf{d}_E + \psi_E) dx \right|^{1-\theta} \\ &+ \left| \int_{\Omega} (\nabla \mathbf{d} : \nabla \mathbf{d}_E - \nabla \psi : \nabla \psi_E) dx \right|^{1-\theta} \\ &\coloneqq J_1 + J_2 + J_3. \end{aligned}$$
(3.17)

The estimate for J_1 follows from (3.4). Since $\|\mathbf{d} - \psi\|_{\mathbf{H}^1} < \beta < 1$, then $\|\mathbf{d}\|_{\mathbf{H}^1} \le \|\psi\|_{\mathbf{H}^1} + 1$. For J_2 , we infer from the elliptic estimate (cf. [32, Proposition 5.1.7]) that

$$J_{2} \leq C \|\nabla(\mathbf{d}_{E} - \psi_{E})\|^{1-\theta} \|\nabla(\mathbf{d}_{E} + \psi_{E})\|^{1-\theta}$$

$$\leq C \|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} \left(\|\mathbf{d}|_{\Gamma}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} + \|\mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\right)^{1-\theta}$$

$$\leq C \|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} \left(\|\mathbf{d}\|_{\mathbf{H}^{1}(\Omega)} + \|\mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\right)^{1-\theta}$$

$$\leq C \|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}.$$
(3.18)

Recalling the function **w** introduced in (3.5), we estimate J_3 as follows

$$J_{3} = \left| \int_{\Omega} \left[\nabla (\mathbf{d} - \mathbf{w}) : \nabla \mathbf{d}_{E} + \nabla \mathbf{w} : \nabla (\mathbf{d}_{E} - \psi_{E}) + \nabla (\mathbf{w} - \psi) : \nabla \psi_{E} \right] dx \right|^{1-\theta}$$

$$\leq \left| \int_{\Omega} \nabla (\mathbf{d} - \mathbf{w}) : \nabla \mathbf{d}_{E} dx \right|^{1-\theta} + \left| \int_{\Omega} \nabla \mathbf{w} : \nabla (\mathbf{d}_{E} - \psi_{E}) \right|^{1-\theta}$$

$$+ \left| \int_{\Omega} \nabla (\mathbf{w} - \psi) : \nabla \psi_{E} dx \right|^{1-\theta}$$

$$:= J_{3a} + J_{3b} + J_{3c}. \tag{3.19}$$

Using (3.6) and (3.7) and the fact $\|\mathbf{d} - \psi\|_{\mathbf{H}^1} < \beta$, we observe that

$$J_{3a} \leq \|\nabla (\mathbf{d} - \mathbf{w})\|^{1-\theta} \|\nabla \mathbf{d}_E\|^{1-\theta}$$

$$\leq C \|\mathbf{d}\|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} \|\mathbf{d}\|_{\Gamma}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}$$

$$\leq C \|\mathbf{d}\|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} \|\mathbf{d}\|_{\mathbf{H}^{1}(\Omega)}^{1-\theta}$$

$$\leq C \|\mathbf{d}\|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta},$$

$$(3.20)$$

$$J_{3b} \leq \|\nabla \mathbf{w}\|^{1-\theta} \|\nabla (\mathbf{d}_E - \psi_E)\|^{1-\theta}$$

$$\leq C (\|\psi\|_{\mathbf{H}^1} + C\beta)^{1-\theta} \|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}$$

$$\leq C \|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}.$$
(3.21)

For J_{3c} , using integration by parts and noticing that $\Delta \psi_E = \mathbf{0}$, $(\mathbf{w} - \psi)|_{\Gamma} = \mathbf{0}$, we obtain

$$\int_{\Omega} \nabla(\mathbf{w} - \psi) : \nabla \psi_E dx = -\int_{\Omega} (\mathbf{w} - \psi) \cdot \Delta \psi_E dx + \int_{\Gamma} (\mathbf{w} - \psi)|_{\Gamma} \cdot \partial_{\mathbf{n}} \psi_E dS = 0, \quad (3.22)$$

where **n** is the unit outer normal to the boundary Γ . Thus (3.22) implies that

$$J_{3c} = 0. (3.23)$$

Finally, since $1-\theta \in (0,1)$, we have $\|\mathbf{d}\|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq C \|\mathbf{d}\|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}$. In summary, we can conclude from (3.4), (3.17)–(3.23), and $\Delta \widehat{\mathbf{d}} = \Delta \mathbf{d}$ that (3.14) holds. The proof is complete.

Remark 3.3. If $\theta \in (0, \frac{1}{2})$ is such that (3.14) holds, then, for all $\theta' \in (0, \theta)$ and any $\mathbf{d} \in \mathbf{H}^1(\Omega)$ satisfying $\|\mathbf{d} - \psi\|_{\mathbf{H}^1} < \beta$, we still have

$$C\left(\|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta'} + \| - \Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}}\right) \ge |\widehat{E}(\mathbf{d}) - \widehat{E}(\psi)|^{1-\theta'}, \quad (3.24)$$

where C is a (properly adjusted) positive constant depending on ψ and \mathbf{h}_{∞} . To see this, we first notice that, since $2 > \frac{1-\theta'}{1-\theta} > 1$, for any $a, b \ge 0$, it holds $(a+b)^{\frac{1-\theta'}{1-\theta}} \le 2(a^{\frac{1-\theta'}{1-\theta}} + b^{\frac{1-\theta'}{1-\theta}})$. Then it follows from (3.14) that

$$\begin{split} |\widehat{E}(\mathbf{d}) - \widehat{E}(\psi)|^{1-\theta'} &= \left(|\widehat{E}(\mathbf{d}) - \widehat{E}(\psi)|^{1-\theta} \right)^{\frac{1-\theta'}{1-\theta}} \\ \leq & C^{\frac{1-\theta'}{1-\theta}} \left(\|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} + \| - \Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}} \right)^{\frac{1-\theta'}{1-\theta}} \\ \leq & C \left(\|\mathbf{d}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta'} + \| - \Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}} \right). \end{split}$$

4 Long-time behavior in 2D

In this section, we focus on the case n = 2. In order to study the long-time behavior of global solutions to problem (1.1)–(1.5), we need some decay conditions on the time-dependent external force **g** and boundary data **h**, namely,

(H1)
$$\int_{t}^{+\infty} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d\tau \leq C(1+t)^{-1-\gamma};$$

(H2) $\int_{t}^{+\infty} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{2} d\tau \leq C(1+t)^{-1-\gamma};$

- (H3) $\int_{t}^{+\infty} \|\mathbf{g}(\tau)\|^2 d\tau \leq C(1+t)^{-1-\gamma};$
- (H4) $\|\mathbf{g}(t)\|^2 \leq C(1+t)^{-2-\gamma};$
- (H5) $\|\mathbf{h}_t(t)\|_{\mathbf{L}^2(\Gamma)} \le C(1+t)^{-1-\gamma};$

for all $t \ge 0$. Here C and γ are given positive constants. We also note that (H4) entails (H3).

Since in the 2D case weak solutions become strong for positive times (cf. Theorem 2.1), we can confine ourselves to consider strong solutions. We recall that, for any given global strong solution (\mathbf{v}, \mathbf{d}) , we have the uniform estimate (2.35). It follows that the ω -limit set of the corresponding initial datum $(\mathbf{v}_0, \mathbf{d}_0)$ is non-empty. Namely, for any unbounded increasing sequence $\{t_n\}_{n=1}^{\infty}$, there are functions $\mathbf{v}_{\infty} \in \mathbf{V}$ and $\mathbf{d}_{\infty} \in \mathbf{H}^2(\Omega)$ such that, up to a subsequence $\{t_j\}_{j=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$, we have

$$\lim_{j \to +\infty} \|\mathbf{v}(t_j) - \mathbf{v}_{\infty}\| = 0, \quad \lim_{j \to +\infty} \|\mathbf{d}(t_j) - \mathbf{d}_{\infty}\|_{\mathbf{H}^1} = 0.$$
(4.1)

Next, we characterize the structure of the ω -limit set. In order to do that, we first recall a technical lemma (see [38, Lemma 6.2.1])

Lemma 4.1. Let T be given with $0 < T \le +\infty$. Suppose that y and h are nonnegative continuous functions defined on [0,T] and satisfy the following conditions: $\frac{dy}{dt} \le c_1y^2 + c_2 + h$, with $\int_0^T y(t)dt \le c_3$, $\int_0^T h(t)dt \le c_4$, where c_i (i = 1, 2, 3, 4) are given nonnegative constants. Then for any $\rho \in (0,T)$, the following estimates holds: $y(t + \rho) \le \left(\frac{c_3}{\rho} + c_2\rho + c_4\right)e^{c_1c_3}$, for all $t \in [0, T - \rho]$. Furthermore, if $T = +\infty$, then $\lim_{t \to +\infty} y(t) = 0$.

Proposition 4.1. Let the assumptions of Theorem 2.1 hold. Then the ω -limit set $\omega(\mathbf{v}_0, \mathbf{d}_0)$ is a subset of

$$\mathcal{S} = \{ (\mathbf{0}, \mathbf{u}) : \mathbf{u} \in \mathcal{N} \cap \mathbf{H}^2(\Omega) \text{ such that } -\Delta \mathbf{u} + \mathbf{f}(\mathbf{u}) = 0 \text{ in } \Omega \}.$$

Moreover, we have

$$\lim_{t \to +\infty} \|\mathbf{v}(t)\|_{\mathbf{V}} = 0,\tag{4.2}$$

$$\lim_{t \to +\infty} \| -\Delta \mathbf{d}(t) + \mathbf{f}(\mathbf{d}(t)) \| = 0.$$
(4.3)

Proof. It follows from Lemma 2.2 that

$$\int_0^{+\infty} \|\nabla \mathbf{v}(t)\|^2 + \|\Delta \widehat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))\|^2 dt < +\infty,$$

which together with the definition of \mathcal{A}_P and (6.3) yields

$$\int_{0}^{+\infty} \mathcal{A}_{P}(t) dt \leq \int_{0}^{+\infty} (\|\nabla \mathbf{v}(t)\|^{2} + 2\|(\Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d}))(t)\|^{2} + 2\|\partial_{t} \mathbf{d}_{P}(t)\|^{2}) dt < +\infty.$$
(4.4)

Using Lemma 2.5 and Lemma 4.1, we can see that

$$\lim_{t \to +\infty} \mathcal{A}_P(t) = 0,$$

which implies $\lim_{t\to+\infty} \|\nabla \mathbf{v}(t)\| = 0$. Hence, for any $(\mathbf{v}_{\infty}, \mathbf{d}_{\infty}) \in \omega(\mathbf{v}_0, \mathbf{d}_0)$, we have $\mathbf{v}_{\infty} = \mathbf{0}$. On the other hand, by definition of \mathcal{A}_P , (4.4) also yields that

$$\lim_{t \to +\infty} \| -\Delta \widetilde{\mathbf{d}}(t) + \mathbf{f}(\mathbf{d}(t)) \| = 0.$$
(4.5)

From Lemma 6.2, we have $\lim_{t\to+\infty} \|\partial_t \mathbf{d}_P(t)\| = 0$ (cf. (6.5)). As a result, it follows from the inequality

$$0 \le \| -\Delta \mathbf{d}(t) + \mathbf{f}(\mathbf{d}(t)) \| \le \| -\Delta \mathbf{d}(t) + \mathbf{f}(\mathbf{d}(t)) \| + \| \partial_t \mathbf{d}_P(t) \|, \quad \forall t \ge 0$$

$$(4.6)$$

that (4.3) holds. Concerning the limit function \mathbf{d}_{∞} , we infer from (2.35) that $\mathbf{d}_{\infty} \in \mathbf{H}^{2}(\Omega)$ and (4.1) holds. We now check the boundary condition for \mathbf{d}_{∞} . Since $\mathbf{h}_{t} \in L^{1}(0, +\infty; \mathbf{H}^{-\frac{1}{2}}(\Gamma))$, $\mathbf{h}(t)$ strongly converges to a certain function $\mathbf{h}_{\infty} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ as time goes to infinity with a controlled rate, namely,

$$\|\mathbf{h}(t) - \mathbf{h}_{\infty}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \le \int_{t}^{+\infty} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} d\tau \to 0, \quad \text{as } t \to +\infty.$$

$$(4.7)$$

On the other hand, we infer from (2.33) and (2.34) that $\mathbf{h} \in L^{\infty}(0, +\infty; \mathbf{H}^{\frac{3}{2}}(\Gamma))$. Consequently, $\mathbf{h}_{\infty} \in \mathbf{H}^{\frac{3}{2}}(\Gamma)$ and \mathbf{h} weakly-star converges to \mathbf{h}_{∞} in $L^{\infty}(0, +\infty; \mathbf{H}^{\frac{3}{2}}(\Gamma))$. By interpolation, we have $\lim_{t \to +\infty} \|\mathbf{h}(t) - \mathbf{h}_{\infty}\|_{\mathbf{L}^{2}(\Gamma)} = 0$. Thus, from the asymptotic behavior of the boundary datum \mathbf{h} , we have for any $j \in \mathbb{N}$,

$$\begin{aligned} |\mathbf{d}_{\infty}|_{\Gamma} - \mathbf{h}_{\infty} \|_{\mathbf{L}^{2}(\Gamma)} &\leq \|\mathbf{d}_{\infty}|_{\Gamma} - \mathbf{h}(t_{j})\|_{\mathbf{L}^{2}(\Gamma)} + \|\mathbf{h}(t_{j}) - \mathbf{h}_{\infty}\|_{\mathbf{L}^{2}(\Gamma)} \\ &\leq C \|\mathbf{d}_{\infty} - \mathbf{d}(t_{j})\|_{\mathbf{H}^{1}} + \|\mathbf{h}(t_{j}) - \mathbf{h}_{\infty}\|_{\mathbf{L}^{2}(\Gamma)}. \end{aligned}$$

Hence, letting $j \to +\infty$ in the above inequality, we deduce from (4.1) and (4.7) that $\mathbf{d}_{\infty}|_{\Gamma} = \mathbf{h}_{\infty}$. For any $\mathbf{z} \in \mathbf{H}_{0}^{1}(\Omega)$ and $j \in \mathbb{N}$, we have

$$\begin{aligned} & \left| \int_{\Omega} (-\Delta \mathbf{d}_{\infty} + \mathbf{f}(\mathbf{d}_{\infty})) \cdot \mathbf{z} dx \right| \\ \leq & \left| \int_{\Omega} (-\Delta \mathbf{d}_{\infty} + \Delta \mathbf{d}(t_{j})) \cdot \mathbf{z} dx \right| + \left| \int_{\Omega} (\mathbf{f}(\mathbf{d}_{\infty}) - \mathbf{f}(\mathbf{d}(t_{j}))) \cdot \mathbf{z} dx \right| \\ & + \left| \int_{\Omega} (-\Delta \mathbf{d}(t_{j}) + \mathbf{f}(\mathbf{d}(t_{j}))) \cdot \mathbf{z} dx \right| \\ \leq & \left\| \nabla (\mathbf{d}(t_{j}) - \mathbf{d}_{\infty}) \right\| \| \nabla \mathbf{z} \| + \left(C \| \mathbf{d}(t_{j}) - \mathbf{d}_{\infty} \|_{\mathbf{H}^{1}} + \| -\Delta \mathbf{d}(t_{j}) + \mathbf{f}(\mathbf{d}(t_{j})) \| \right) \| \mathbf{z} \|. \end{aligned}$$

Passing to the limit as $j \to +\infty$, we get

$$\int_{\Omega} (-\Delta \mathbf{d}_{\infty} + \mathbf{f}(\mathbf{d}_{\infty})) \cdot \mathbf{z} dx = 0$$

As a consequence, we see that $\mathbf{d}_{\infty} \in \mathcal{N} \cap \mathbf{H}^2(\Omega)$ solves (3.2). The proof is complete.

We can also prove the convergence of the lifted energy.

Proposition 4.2. Let the assumptions of Theorem 2.1 hold. Then the lifted energy functional $\widehat{\mathcal{E}}$ defined by (2.15) is constant on the ω -limit set $\omega(\mathbf{v}_0, \mathbf{d}_0)$. Namely, there exists a constant $\widehat{\mathcal{E}}_{\infty}$ such that $\widehat{E}(\mathbf{d}_{\infty}) \equiv \widehat{\mathcal{E}}_{\infty}$, for all $(\mathbf{0}, \mathbf{d}_{\infty})$ with $\mathbf{d}_{\infty} \in \mathcal{N} \cap \mathbf{H}^2(\Omega)$. Moreover, we have

$$\lim_{t \to +\infty} \widehat{\mathcal{E}}(t) = \widehat{\mathcal{E}}_{\infty}.$$
(4.8)

Proof. From the previous argument, we know that for arbitrary $(\mathbf{0}, \mathbf{d}_{\infty}^{(1)}), (\mathbf{0}, \mathbf{d}_{\infty}^{(2)}) \in \omega(\mathbf{v}_0, \mathbf{d}_0)$ there exist unbounded increasing sequences $\{t_j^{(1)}\}_{j=1}^{\infty}$ and $\{t_j^{(2)}\}_{j=1}^{\infty}$ such that (4.1) holds. As a result, we have

$$\lim_{j \to +\infty} \widehat{\mathcal{E}}(t_j^{(1)}) = \widehat{E}(\mathbf{d}_{\infty}^{(1)}), \quad \lim_{j \to +\infty} \widehat{\mathcal{E}}(t_j^{(2)}) = \widehat{E}(\mathbf{d}_{\infty}^{(2)}).$$

On the other hand, it follows from the basic energy inequality (2.16) that for any t' > t'' > 0,

$$|\widehat{\mathcal{E}}(t') - \widehat{\mathcal{E}}(t'')| \le \int_{t''}^{t'} r(t)dt \to 0, \quad \text{as } t', t'' \to +\infty.$$

Then by

$$|\widehat{E}(\mathbf{d}_{\infty}^{(1)}) - \widehat{E}(\mathbf{d}_{\infty}^{(2)})| \le |\widehat{\mathcal{E}}(t_j^{(1)}) - \widehat{\mathcal{E}}(t_j^{(2)})| + |\widehat{E}(\mathbf{d}_{\infty}^{(1)}) - \widehat{\mathcal{E}}(t_j^{(1)})| + |\widehat{\mathcal{E}}(t_j^{(2)}) - \widehat{E}(\mathbf{d}_{\infty}^{(2)})|,$$

letting $j \to +\infty$, we can see that $\widehat{E}(\mathbf{d}_{\infty}^{(1)}) = \widehat{E}(\mathbf{d}_{\infty}^{(2)})$. Namely, $\widehat{\mathcal{E}}$ is a constant (denoted by $\widehat{\mathcal{E}}_{\infty}$) on the ω -limit set $\omega(\mathbf{v}_0, \mathbf{d}_0)$. Moreover, for any t > 0 there exist $t_j < t_{j+1}$ such that $t \in [t_j, t_{j+1}]$ and $|\widehat{\mathcal{E}}(t) - \widehat{\mathcal{E}}_{\infty}| \leq |\widehat{\mathcal{E}}(t) - \widehat{\mathcal{E}}(t_j)| + |\widehat{\mathcal{E}}(t_j) - \widehat{\mathcal{E}}_{\infty}|$, which yields (4.8).

4.1 Convergence to equilibrium

Theorem 4.1. Let the assumptions of Theorem 2.1 hold. If, in addition, we assume (H1)–(H3), then any strong solution $(\mathbf{v}(t), \mathbf{d}(t))$ convergence to an equilibrium $(\mathbf{0}, \mathbf{d}_{\infty})$ strongly in $\mathbf{V} \times \mathbf{H}^{2}(\Omega)$ as t goes to $+\infty$.

Proof. On account of (4.2) we only need to prove that $\mathbf{d}(t)$ converges to \mathbf{d}_{∞} as $t \to +\infty$ given by (4.1). Below we adapt the idea in [4,9] to achieve our goal. Indeed, observe that we can find an integer j_0 such that for all $j \ge j_0$, $\|\mathbf{d}(t_j) - \mathbf{d}_{\infty}\|_{\mathbf{H}^1} < \frac{\beta}{3}$, where $\beta \in (0, 1)$ is the constant given in Corollary 3.1 (depending on \mathbf{d}_{∞}). Consequently, we define

$$s(t_j) = \sup\{\tau \ge t_j : \|\mathbf{d}(\tau) - \mathbf{d}_{\infty}\|_{\mathbf{H}^1} < \beta\}.$$

Since $\mathbf{d} \in C([0, +\infty); \mathbf{H}^1(\Omega))$, we can see that $s(t_j) > t_j$ for any $j \ge j_0$. By Lemma 2.2 and Proposition 4.2, we have

$$|\widehat{\mathcal{E}}(t) - \widehat{E}(\mathbf{d}_{\infty})| \ge \frac{1}{4} \min\{\nu, 1\} \int_{t}^{+\infty} \mathcal{D}^{2}(\tau) d\tau - \int_{t}^{+\infty} r(\tau) d\tau,$$

where

$$\mathcal{D}(t) = \|\nabla \mathbf{v}(t)\| + \|\Delta \widehat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))\|$$

and r is defined in (2.16) such that, thanks to (H1)-(H3), we have

$$\int_{t}^{+\infty} r(\tau) d\tau \le C(1+t)^{-1-\gamma}, \quad \forall \ t \ge 0.$$

Let the constant θ be as in Corollary 3.1 (depending on \mathbf{d}_{∞}). Using Remark 3.3, we can choose $\theta' \in (0, \theta]$ such that θ' also satisfies

$$0 < \theta' < \frac{\gamma}{2(1+\gamma)}.\tag{4.9}$$

If θ itself satisfies (4.9), we just take $\theta' = \theta$. For any fixed t_j with $j \ge j_0$, we introduce the sets

$$K_j = [t_j, s(t_j)), \quad K_j^{(1)} = \left\{ t \in K_j : \mathcal{D}(t) > (1+t)^{-(1-\theta')(1+\gamma)} \right\}, \quad K_j^{(2)} = K_j \setminus K_j^{(1)}$$

Consider the following functional on K_j

$$\Phi(t) = \widehat{\mathcal{E}}(t) - \widehat{E}(\mathbf{d}_{\infty}) + 2\int_{t}^{s(t_{j})} r(\tau)d\tau, \quad \forall \ t \in K_{j}.$$

It easily follows that

$$\lim_{j \to +\infty} \Phi(t_j) = 0. \tag{4.10}$$

Next, we have

$$\frac{d}{dt}(|\Phi(t)|^{\theta'}\operatorname{sgn}\Phi(t)) = \theta'|\Phi(t)|^{\theta'-1}\frac{d}{dt}\Phi(t)$$

$$\leq -\frac{\theta'}{4}\min\{\nu,1\}|\Phi(t)|^{\theta'-1}\mathcal{D}^{2}(t)$$

$$\leq 0,$$
(4.11)

which implies that the functional $|\Phi(t)|^{\theta'} \operatorname{sgn} \Phi(t)$ is decreasing on K_j . Keeping in mind that $\theta' \leq \theta$ and $2(1 - \theta') > 1$, we can apply Corollary 3.1 (cf. also Remark 3.3) to obtain that

$$\begin{aligned} |\Phi(t)|^{1-\theta'} &\leq |\widehat{\mathcal{E}}(t) - \widehat{\mathcal{E}}(\mathbf{d}_{\infty})|^{1-\theta'} + C\left(\int_{t}^{+\infty} r(\tau)d\tau\right)^{1-\theta'} \\ &\leq \left(\frac{1}{2}\right)^{2(1-\theta')} \|\mathbf{v}\|^{2(1-\theta')} + C\|\mathbf{h}(t) - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta'} \\ &+ C\| - \Delta\widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}} + C\left(\int_{t}^{+\infty} r(\tau)d\tau\right)^{1-\theta'} \\ &\leq C\|\nabla\mathbf{v}\| + C\| - \Delta\widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\| + C\left(\int_{t}^{+\infty} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}d\tau\right)^{1-\theta'} \\ &+ C\left(\int_{t}^{+\infty} r(\tau)d\tau\right)^{1-\theta'} \\ &\leq C\|\nabla\mathbf{v}\| + C\| - \Delta\widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\| + C(1+t)^{-(1-\theta')(1+\gamma)}. \end{aligned}$$
(4.12)

Thus, on $K_j^{(1)}$, we have

$$|\Phi(t)|^{1-\theta'} \le C\mathcal{D}(t),$$

which together with (4.11) yields that on $K_j^{(1)}$,

$$-\frac{d}{dt}(|\Phi(t)|^{\theta'}\operatorname{sgn}\Phi(t)) \ge C\mathcal{D}(t).$$
(4.13)

As a consequence, we have

$$\int_{K_j^{(1)}} \mathcal{D}(t) dt \leq -C \int_{K_j} \frac{d}{dt} (|\Phi(t)|^{\theta'} \operatorname{sgn} \Phi(t)) dt \\
\leq C(|\Phi(t_j)|^{\theta'} + |\Phi(s(t_j))|^{\theta'}) < +\infty,$$
(4.14)

where $\Phi(s(t_j)) = 0$ if $s(t_j) = +\infty$. On the other hand, on $K_j^{(2)}$, we have

$$\int_{K_j^{(2)}} \mathcal{D}(t) dt \le C \int_{t_j}^{\infty} (1+t)^{-(1-\theta')(1+\gamma)} dt = \frac{C}{-\gamma \theta' - \theta' + \gamma} (1+t_j)^{\gamma \theta' + \theta' - \gamma}.$$
(4.15)

Here, we notice that $\gamma \theta' + \theta' - \gamma < 0$ due to (4.9). Then (4.14) and (4.15) imply that

$$\int_{K_j} \mathcal{D}(t)dt = \int_{K_j^{(1)}} \mathcal{D}(t)dt + \int_{K_j^{(2)}} \mathcal{D}(t)dt < +\infty,$$

for any j. On the other hand, it follows from (2.35) and (2.12) that

$$\begin{aligned} \|\mathbf{d}_{t}(t)\| &\leq \|\mathbf{v} \cdot \nabla \mathbf{d}\| + \|\Delta \mathbf{\hat{d}} - \mathbf{f}(\mathbf{d})\| \\ &\leq \|\mathbf{v}\|_{\mathbf{L}^{4}} \|\nabla \mathbf{d}\|_{\mathbf{L}^{4}} + \|\Delta \mathbf{\hat{d}} - \mathbf{f}(\mathbf{d})\| \\ &\leq C\mathcal{D}(t). \end{aligned}$$
(4.16)

As a consequence,

$$\int_{K_j} \|\mathbf{d}_t(t)\| dt \le C(|\Phi(t_j)|^{\theta'} + |\Phi(s(t_j))|^{\theta'}) + C(1+t_j)^{\gamma\theta'+\theta'-\gamma}.$$
(4.17)

To complete the proof, we show that

Proposition 4.3. Let the assumptions of Theorem 2.1 hold. Then there exists an integer $j_1 \ge j_0$ such that $s(t_{j_1}) = +\infty$. Thus

$$\|\mathbf{d}(t) - \mathbf{d}_{\infty}\|_{\mathbf{H}^1} < \beta, \quad \forall t \ge t_{j_1}.$$

Proof. The conclusion follows from a contradiction argument (cf. [15]). Suppose that for any $j \ge j_0$ we have $s(t_j) < +\infty$. Then, by definition, we have

$$\|\mathbf{d}(s(t_j)) - \mathbf{d}_{\infty}\|_{\mathbf{H}^1} = \beta > 0.$$

$$(4.18)$$

Besides, it follows from (4.1), (4.10) and (4.17) that

$$\begin{aligned} \|\mathbf{d}(s(t_j)) - \mathbf{d}_{\infty}\| &\leq \|\mathbf{d}(s(t_j)) - \mathbf{d}(t_j)\| + \|\mathbf{d}(t_j) - \mathbf{d}_{\infty}\| \\ &\leq \int_{t_j}^{s(t_j)} \|\mathbf{d}_t(t)\| dt + \|\mathbf{d}(t_j) - \mathbf{d}_{\infty}\| \to 0, \quad \text{as } j \to +\infty. \end{aligned}$$

Using uniform estimate (2.35) and interpolation inequality, we obtain

$$\|\mathbf{d}(s(t_j)) - \mathbf{d}_{\infty}\|_{\mathbf{H}^1}^2 \le \|\mathbf{d}(s(t_j)) - \mathbf{d}_{\infty}\|_{\mathbf{H}^2} \|\mathbf{d}(s(t_j)) - \mathbf{d}_{\infty}\| \to 0, \quad \text{as } j \to +\infty,$$

which leads a contradiction with (4.18). The proof is complete.

Due to Proposition 4.3, we have $s(t_{j_1}) = +\infty$ for some $j_1 \ge j_0$. Arguing as above, we can prove

$$\int_{t_{j_1}}^{+\infty} \|\mathbf{d}_t(t)\| dt < +\infty.$$

Thus $\mathbf{d}(t)$ converges in \mathbf{L}^2 and recalling (4.1), by compactness we conclude that

$$\lim_{t \to +\infty} \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|_{\mathbf{H}^{1}} = 0.$$
(4.19)

Finally, observe that

$$\begin{aligned} \|\Delta \mathbf{d}(t) - \Delta \mathbf{d}_{\infty}\| &= \| - \Delta \mathbf{d}(t) + \mathbf{f}(\mathbf{d}(t))\| + \|\mathbf{f}(\mathbf{d}(t)) - \mathbf{f}(\mathbf{d}_{\infty})\| \\ &\leq \| - \Delta \mathbf{d}(t) + \mathbf{f}(\mathbf{d}(t))\| + C \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|. \end{aligned}$$
(4.20)

Then (4.3) and (4.19) entail that

$$\lim_{t \to +\infty} \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|_{\mathbf{H}^2} = 0$$

and this finishes the proof of Theorem 4.1.

4.2 Convergence rate

Theorem 4.2. Let the assumptions of Theorem 2.1 hold. If, in addition, we assume (H1)-(H2) and (H4)-(H5), then we have

$$\|\mathbf{v}(t)\| + \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|_{\mathbf{H}^{1}} \le C(1+t)^{-\frac{\theta'}{1-2\theta'}}, \quad t \ge 0.$$

Moreover, if (H2) and (H5) are replaced by, respectively,

(H6) $\|\mathbf{h}_t(t)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq C(1+t)^{-1-\gamma};$ (H7) $\|\mathbf{h}(t) - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma)} \leq C(1+t)^{-1-\gamma};$

the following higher-order estimate holds

$$\|\mathbf{v}(t)\|_{\mathbf{V}} + \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|_{\mathbf{H}^2} \le C(1+t)^{-\frac{\theta'}{1-2\theta'}}, \quad t \ge 0.$$

Proof. The proof consists of several steps.

Step 1. \mathbf{L}^2 -estimate of $\mathbf{d} - \mathbf{d}_{\infty}$. This follows from an argument devised in [9]. For the readers' convenience, we sketch the proof here. From the previous argument, we only have to work on the time interval $[t_{j_1}, +\infty)$. Denote

$$\Phi(t) = \widehat{\mathcal{E}}(t) - \widehat{E}(\mathbf{d}_{\infty}) + 2\int_{t}^{+\infty} r(\tau)d\tau.$$

Since

$$\frac{d}{dt}\Phi(t) \le -\frac{\theta'}{4}\min\{\nu, 1\}\mathcal{D}^2(t) - r(t) \le 0,$$

and $\lim_{t \to +\infty} \Phi(t) = 0$, we know that $\Phi(t)$ is decreasing and $\Phi(t) \ge 0$ for $t \ge t_{j_1}$.

First, if the boundary datum \mathbf{h} and the external force \mathbf{g} become time-independent in finite time, i.e., there exists time T_0 such that for $t \geq T_0$, $\mathbf{h} = \mathbf{h}_{\infty}$ and $\mathbf{g} = \mathbf{0}$. Then the problem reduces to the autonomous system considered in [35]. Thus, below we just assume that either \mathbf{h} or \mathbf{g} does not become time-independent in finite time (namely, the system will always be non-autonomous). In this case, if there exists $t^* \geq t_{j_1}$ such that $\Phi(t^*) = 0$, then $\mathcal{D}(t) = r(t) = 0$ for all $t \geq t^*$ and this is a contradiction since r(t) cannot identically vanish from any finite time on. Therefore, we can suppose

$$\Phi(t) > 0, \quad \forall \ t \ge t_{j_1}.$$

If the open set $K_{j_1}^{(1)}$ is bounded, then there exists $t^* \ge t_{j_1}$ such that $[t^*, +\infty) \subset K_{j_1}^{(2)}$. As a result, $\mathcal{D}(t) \le (1+t)^{-(1-\theta')(1+\gamma)}$ and by (4.16), we have

$$\|\mathbf{d}(t) - \mathbf{d}_{\infty}\| \le \int_{t}^{+\infty} \|\mathbf{d}_{t}(\tau)\| d\tau \le \frac{C}{-\gamma\theta' - \theta' + \gamma} (1+t)^{\gamma\theta' + \theta' - \gamma}, \quad \forall \ t \ge t^{*}.$$

Next, we treat the case when the open set $K_{j_1}^{(1)}$ is unbounded. There exists a countable family of disjoint open sets (a_n, b_n) such that $K_{j_1}^{(1)} = \bigcup_{n=1}^{\infty} (a_n, b_n)$. On $K_{j_1}^{(1)}$, recalling (4.12), we can see that on any $(a_n, b_n) \subset K_{j_1}^{(1)}$, it holds

$$\frac{d}{dt}\Phi(t) + C\Phi^{2(1-\theta')}(t) \le 0$$

As a result, for any $t \in (a_n, b_n)$,

$$\Phi(t) \le \left[\Phi(a_n)^{2\theta'-1} + C(1-2\theta')(t-a_n)\right]^{-\frac{1}{1-2\theta'}},$$
(4.21)

where by the definition of $K_{j_1}^{(1)}$ and (4.12) we have

$$\Phi(a_n) \le C\mathcal{D}(a_n)^{\frac{1}{1-\theta'}} + C(1+a_n)^{-(1+\gamma)} = C(1+a_n)^{-1-\gamma}.$$

Using the fact $(1 + \gamma)(1 - 2\theta') > 1$ (cf. (4.9)), we can take $n^* \in \mathbb{N}$ sufficiently large such that

$$\Phi(a_{n^*})^{2\theta'-1} - C(1-2\theta')a_{n^*} \ge a_{n^*}^{(1+\gamma)(1-2\theta')} - C(1-2\theta')a_{n^*} \ge 1.$$
(4.22)

Therefore, we infer

$$\Phi(t) \le C(1+t)^{-\frac{1}{1-2\theta'}}, \quad \forall \ t \in (a_{n^*}, \infty) \cap K_{j_1}^{(1)}.$$

Similar to (4.13), we have (since $\Phi(t) > 0$)

$$-\frac{d}{dt}\Phi(t)^{\theta'} \ge C\mathcal{D}(t), \quad \forall \ t \in (a_{n^*}, \infty) \cap K_{j_1}^{(1)}$$

Due to (4.9), it follows that $-\gamma \theta' - \theta' + \gamma \ge \frac{\theta'}{1-2\theta'}$. Now for any $t > a_{n^*}$, we can conclude that

$$\begin{aligned} \|\mathbf{d}(t) - \mathbf{d}_{\infty}\| &\leq \int_{t}^{+\infty} \|\mathbf{d}_{t}(\tau)\| d\tau \\ &= \int_{(t,\infty)\cap K_{j_{1}}^{(1)}} \|\mathbf{d}_{t}(\tau)\| d\tau + \int_{(t,\infty)\cap K_{j_{1}}^{(2)}} \|\mathbf{d}_{t}(\tau)\| d\tau \\ &\leq C \int_{(t,\infty)\cap K_{j_{1}}^{(1)}} \mathcal{D}(\tau) d\tau + C \int_{t}^{+\infty} (1+\tau)^{-(1-\theta')(1+\gamma)} d\tau \\ &\leq C \Phi(t)^{\theta'} + C(1+t)^{\gamma \theta' + \theta' - \gamma} \\ &\leq C(1+t)^{-\frac{\theta'}{1-2\theta'}}. \end{aligned}$$

Using (2.35), after properly adjusting the constant C, we have

$$\|\mathbf{d}(t) - \mathbf{d}_{\infty}\| \le C(1+t)^{-\frac{\theta'}{1-2\theta'}}, \quad \forall t \ge 0.$$

$$(4.23)$$

Step 2. $\mathbf{H} \times \mathbf{H}^1$ -estimate. It easily from the basic energy inequality (2.16) that

$$\frac{d}{dt}y(t) + \frac{\nu}{2} \|\nabla \mathbf{v}\|^2 + \frac{1}{2} \|\Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^2 \le r(t),$$
(4.24)

where

$$y(t) = \frac{1}{2} \|\mathbf{v}(t)\|^2 + \frac{1}{2} \|\nabla(\widehat{\mathbf{d}}(t) - \widehat{\mathbf{d}}_{\infty})\|^2 + \int_{\Omega} [F(\mathbf{d})(t) - F(\mathbf{d}_{\infty}) - \mathbf{f}(\mathbf{d}_{\infty})(\mathbf{d}(t) - \mathbf{d}_{\infty})] dx$$

As in [35], using (2.35), we can show that

$$\left|\int_{\Omega} [F(\mathbf{d})(t) - F(\mathbf{d}_{\infty}) - \mathbf{f}(\mathbf{d}_{\infty})(\mathbf{d}(t) - \mathbf{d}_{\infty})]dx\right| \le C \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|^{2}.$$

Keeping in mind the definition of lifting functions, we have $\widehat{\mathbf{d}} - \widehat{\mathbf{d}}_{\infty}|_{\Gamma} = 0$ so that

$$\|\nabla(\widehat{\mathbf{d}} - \widehat{\mathbf{d}}_{\infty})\| \leq C \|\Delta(\widehat{\mathbf{d}} - \widehat{\mathbf{d}}_{\infty})\|$$

$$\leq \| -\Delta \mathbf{d} + \mathbf{f}(\mathbf{d})\| + C \|\mathbf{f}(\mathbf{d}) - \mathbf{f}(\mathbf{d}_{\infty})\|$$

$$\leq C \| -\Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\| + C \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|,$$

$$\|\nabla(\mathbf{d} - \mathbf{d}_{\infty})\| \leq \|\nabla(\widehat{\mathbf{d}} - \widehat{\mathbf{d}}_{\infty})\| + \|\nabla(\mathbf{d}_{E} - \mathbf{d}_{\infty})\|$$

$$\leq C \| -\Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\| + C \|\mathbf{d}(t) - \mathbf{d}_{\infty}\| + C \int_{t}^{+\infty} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d\tau.$$

Thus it follows that

$$y(t) \geq \frac{1}{2} \|\mathbf{v}(t)\|^{2} + \frac{1}{2} \|\nabla(\mathbf{d} - \mathbf{d}_{\infty})\|^{2} - C \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|^{2} - C \left(\int_{t}^{+\infty} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d\tau\right)^{2},$$
(4.25)

$$y(t) \leq C \|\nabla \mathbf{v}\|^2 + C \| - \Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d})\|^2 + C \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|^2.$$
(4.26)

Condition (4.9) implies that $\frac{2\theta'}{1-2\theta'} < \gamma$. Then we deduce from (4.24), (4.23), (H4)–(H5) and Lemma 6.1 that

$$\frac{d}{dt}y(t) + \alpha y(t) \le C(r(t) + \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|^2) \le C(1+t)^{-\frac{2\theta'}{1-2\theta'}},$$
(4.27)

where $\alpha > 0$ is sufficiently small. The above inequality implies that

$$y(t) \le C(1+t)^{-\frac{2\theta'}{1-2\theta'}}, \quad \forall \ t \ge 0.$$
 (4.28)

Combining it with (4.25) and recalling (H1), we get

$$\|\mathbf{v}(t)\|^{2} + \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|_{\mathbf{H}^{1}}^{2}$$

$$\leq Cy(t) + C\|\mathbf{d}(t) - \mathbf{d}_{\infty}\|^{2} + C\left(\int_{t}^{+\infty} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d\tau\right)^{2}$$

$$\leq C(1+t)^{-\frac{2\theta'}{1-2\theta'}}, \quad \forall t \geq 0.$$
(4.29)

Step 3. $\mathbf{V} \times \mathbf{H}^2$ -estimate. Taking advantage of the stronger assumptions (H6)–(H7) and (4.29), we now get a higher-order estimate. Observe first that

$$\| -\Delta \widehat{\mathbf{d}} + \mathbf{f}(\mathbf{d}) \| \le \| -\Delta \widetilde{\mathbf{d}} + \mathbf{f}(\mathbf{d}) \| + \| \Delta \mathbf{d}_P \| = \| -\Delta \widetilde{\mathbf{d}} + \mathbf{f}(\mathbf{d}) \| + \| \partial_t \mathbf{d}_P \|,$$

then we have

$$y(t) \le C \|\nabla \mathbf{v}\|^2 + C \| - \Delta \widetilde{\mathbf{d}} + \mathbf{f}(\mathbf{d})\|^2 + C \|\partial_t \mathbf{d}_P\|^2 + C \|\mathbf{d}(t) - \mathbf{d}_\infty\|^2.$$

It follows from (2.27) and (4.27) that

$$\frac{d}{dt}z(t) + \alpha_2 z(t) \le C(1+t)^{-\frac{2\theta'}{1-2\theta'}} + C(R_1(t) + \|\partial_t \mathbf{d}_P(t)\|^2),$$
(4.30)

where

$$z(t) = y(t) + \alpha_1 \mathcal{A}_P(t), \qquad (4.31)$$

and α_1 and α_2 are sufficiently small positive constants. From the definition of R_1 , (6.6) and the fact $\frac{2\theta'}{1-2\theta'} < 2 + 2\gamma$ (cf. (4.9)), we have

$$R_1(t) + \|\partial_t \mathbf{d}_P(t)\|^2 \le C(1+t)^{-\frac{2\theta'}{1-2\theta'}} + C \|\nabla \Delta \mathbf{d}_P(t)\|^2.$$
(4.32)

Hence, from (4.30) we infer that

$$z(t) \leq z(0)e^{-\alpha_{2}t} + Ce^{-\alpha_{2}t} \int_{0}^{t} e^{\alpha_{2}\tau} \left[C(1+\tau)^{-\frac{2\theta'}{1-2\theta'}} + \|\nabla\Delta\mathbf{d}_{P}(\tau)\|^{2} \right] d\tau$$

$$\leq Ce^{-\alpha_{2}t} + e^{-\alpha_{2}t} \int_{0}^{\frac{t}{2}} e^{\alpha_{2}\tau} \left[C(1+\tau)^{-\frac{2\theta'}{1-2\theta'}} + \|\nabla\Delta\mathbf{d}_{P}(\tau)\|^{2} \right] d\tau$$

$$+ e^{-\alpha_{2}t} \int_{\frac{t}{2}}^{t} e^{\alpha_{2}\tau} \left[C(1+\tau)^{-\frac{2\theta'}{1-2\theta'}} + \|\nabla\Delta\mathbf{d}_{P}(\tau)\|^{2} \right] d\tau$$

$$:= Ce^{-\alpha_{2}t} + Z_{1}(t) + Z_{2}(t). \qquad (4.33)$$

It follows from (6.4) and (H6) that

$$Z_{1}(t) \leq Ce^{-\frac{\alpha_{2}}{2}t} \int_{0}^{\frac{t}{2}} \left[C(1+\tau)^{-\frac{2\theta'}{1-2\theta'}} + \|\nabla\Delta\mathbf{d}_{P}(\tau)\|^{2} \right] d\tau$$

$$\leq Ce^{-\frac{\alpha_{2}}{2}t} \left(t + \int_{0}^{\frac{t}{2}} (1+\tau)^{-2-2\gamma} d\tau \right)$$

$$\leq C(1+t)^{-\frac{2\theta'}{1-2\theta'}}.$$

Next, by (6.7) and the fact $\frac{2\theta'}{1-2\theta'} < 1+2\gamma$, we deduce that

$$Z_{2}(t) \leq Ce^{-\alpha_{2}t} \left(1 + \frac{t}{2}\right)^{-\frac{2\theta'}{1-2\theta'}} \int_{\frac{t}{2}}^{t} e^{\alpha_{2}\tau} d\tau + C \int_{\frac{t}{2}}^{t} \|\nabla\Delta\mathbf{d}_{P}(\tau)\|^{2} d\tau$$

$$\leq C(1+t)^{-\frac{2\theta'}{1-2\theta'}} + C(1+t)^{-1-2\gamma}$$

$$\leq C(1+t)^{-\frac{2\theta'}{1-2\theta'}}.$$

As a result, we obtain that

$$z(t) \le C(1+t)^{-\frac{2\theta'}{1-2\theta'}}, \quad \forall t \ge 0.$$
 (4.34)

In particular, we have

$$\mathcal{A}_P(t) \le C(1+t)^{-\frac{2\theta'}{1-2\theta'}}, \quad t \ge 0,$$
 (4.35)

which together with (4.20) and (4.29) yields the following estimate

$$\|\mathbf{v}(t)\|_{\mathbf{V}}^2 + \|\Delta \mathbf{d}(t) - \Delta \mathbf{d}_{\infty}\|^2 \le C(1+t)^{-\frac{2\theta'}{1-2\theta'}}, \quad \forall t \ge 0.$$

Finally, using a standard elliptic estimate, we obtain (cf. (H7))

$$\|\mathbf{d}(t) - \mathbf{d}_{\infty}\|_{\mathbf{H}^{2}} \le C \|\Delta \mathbf{d}(t) - \Delta \mathbf{d}_{\infty}\| + C \|\mathbf{h}(t) - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma)} \le C(1+t)^{-\frac{\theta'}{1-2\theta'}},$$

for all $t \ge 0$ and this finishes the proof.

5 Long-time behavior in 3D

As in the classical Navier–Stokes case (see [16]), we can prove the eventual regularity of any global weak solution. Thus the convergence results can also be extended to the 3D case. Indeed, comparing with Lemma 2.6, we derive first an alternative higher-order energy inequality.

Lemma 5.1. Let the assumptions of Proposition 2.1 hold for all T > 0. Suppose, in addition, that (2.32)–(2.34) are satisfied. If a weak solution (\mathbf{v}, \mathbf{d}) is smooth enough then it fulfills the following inequality

$$\frac{d}{dt}\mathcal{A}_P(t) + \nu \|S\mathbf{v}\|^2 + \|\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))\|^2 \le C_*(\mathcal{A}_P^3(t) + \mathcal{A}_P(t) + R_3(t)),$$
(5.1)

where

$$R_{3}(t) = \|\partial_{t}\mathbf{d}_{P}(t)\|^{6} + \|\partial_{t}\mathbf{d}_{P}(t)\|^{2} + \|\nabla\Delta\mathbf{d}_{P}(t)\|^{2} + \|\mathbf{g}(t)\|^{2},$$
(5.2)

for all $t \ge 0$. Here C_* is a positive constant that may depend on ν , $\|\mathbf{v}_0\|$, $\|\mathbf{d}_0\|_{\mathbf{H}^1}$, $\|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{V}^*)}$, $\|\mathbf{h}\|_{L^2_{tb}(0,+\infty;\mathbf{H}^{\frac{3}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^2(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^1(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}$.

Proof. We reconsider the estimates in the proof of Lemma 2.6. Recalling (2.29) and (2.30), thanks to the Young inequality, it is not difficult to obtain that

$$\begin{aligned} |I_1| &\leq \|S\mathbf{v}\| \|\mathbf{v}\|_{\mathbf{L}^6} \|\nabla \mathbf{v}\|_{\mathbf{L}^3} \leq C \|S\mathbf{v}\|^{\frac{3}{2}} \|\nabla \mathbf{v}\|^{\frac{3}{2}} \leq \varepsilon \|S\mathbf{v}\|^2 + C \|\nabla \mathbf{v}\|^6, \\ |I_2| &\leq \varepsilon \|S\mathbf{v}\|^2 + C \|\mathbf{g}\|^2, \end{aligned}$$

$$|I_3| \leq \varepsilon ||S\mathbf{v}||^2 + \varepsilon ||\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))||^2 + C ||\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})||^6 + C ||\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})||^2 + C ||\partial_t \mathbf{d}_P||^6 + C ||\partial_t \mathbf{d}_P||^2 + C ||\nabla \Delta \mathbf{d}_P||^2,$$

$$|I_4| \leq \varepsilon ||S\mathbf{v}||^2 + \varepsilon ||\nabla(\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d}))||^2 + C ||\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})||^6 + C ||\nabla \mathbf{v}||^6 + C ||\nabla \mathbf{v}||^2 + C ||\partial_t \mathbf{d}_P||^6,$$

$$|I_{5a}| \le \varepsilon \|S\mathbf{v}\|^2 + C \|\Delta \widetilde{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^2 + C \|\nabla \mathbf{v}\|^2 + C \|\partial_t \mathbf{d}_P\|^2.$$

In addition, I_{5b} can be exactly estimated as (2.31). Collecting all the estimates, and taking ε to be sufficiently small, we obtain our conclusion (5.1).

Then we prove the following sufficient condition for the existence of global strong solution in 3D.

Proposition 5.1. Suppose that the assumptions of Proposition 2.1 and (2.32)–(2.34) are satisfied. In addition, assume that $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$. If there exists a sufficiently small $\varepsilon_0 \in (0, 1]$ such that

$$\int_{0}^{+\infty} (\nu \|\nabla \mathbf{v}(t)\|^{2} + \|\Delta \widehat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))\|^{2}) dt \le \varepsilon_{0}.$$
(5.3)

then problem (1.1)–(1.5) admits a unique global strong solution (\mathbf{v}, \mathbf{d}) in $\Omega \times (0, +\infty)$, provided that $\|\mathbf{h}_t\|_{L^2(0, +\infty; \mathbf{H}^{\frac{1}{2}}(\Gamma))}$ is small enough.

Proof. For simplicity, we give a formal proof. To make it rigorous we should work within a proper approximation scheme (see, for instance, [1, 6]). Let $L_i > 0$ (i = 1, 2, 3, 4, 5) be the constants such that

$$\|\mathbf{v}_0\| + \|\mathbf{d}_0\|_{\mathbf{H}^1} \leq L_1, \tag{5.4}$$

$$\left\|\mathbf{h}_{t}\right\|_{L^{2}(0,+\infty;\mathbf{H}^{\frac{1}{2}}(\Gamma))} \leq L_{2},\tag{5.5}$$

$$\|\mathbf{h}\|_{L^{2}_{th}(0,+\infty;\mathbf{H}^{\frac{3}{2}}(\Gamma))} \leq L_{3}, \tag{5.6}$$

$$\left\|\mathbf{h}_{t}\right\|_{L^{1}(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))} \leq L_{4}, \tag{5.7}$$

$$\|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{V}^*)} \leq L_5.$$
(5.8)

It follows from the basic energy inequality (2.16) that

$$\widehat{\mathcal{E}}(t) + \frac{1}{2} \int_0^t \left(\nu \|\nabla \mathbf{v}\|^2 + \|\Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^2 \right) d\tau \le \widehat{\mathcal{E}}(0) + \int_0^{+\infty} r(t) dt, \quad \forall t \ge 0.$$
(5.9)

Then, by definition of $\widehat{\mathcal{E}}$ and Lemma 6.1, we have

$$\|\mathbf{v}(t)\| + \|\mathbf{d}(t)\|_{\mathbf{H}^1} \le C_1, \quad \forall t \ge 0,$$
 (5.10)

$$\int_{0}^{+\infty} \left(\nu \|\nabla \mathbf{v}\|^{2} + \|\Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^{2} \right) dt \le C_{2},$$
(5.11)

where the constants C_1, C_2 depend on $L_1, ..., L_5$ and Ω .

Let K > 0 be such that

$$\nu \|\nabla \mathbf{v}_0\|^2 + \|\Delta \widetilde{\mathbf{d}}(0) - f(\mathbf{d}_0)\|^2 \le K.$$
(5.12)

Keeping Lemma 5.1 in mind and arguing as in [20], we consider the following Cauchy problem

$$\frac{d}{dt}Y(t) = C_*(Y(t)^3 + Y(t)) + C_*R_3(t), \quad Y(0) = \max\left\{1, \nu^{-1}\right\} K \ge \mathcal{A}_P(0).$$
(5.13)

We denote by $I = [0, T_{max})$ the (right) maximal interval for the existence of a (nonnegative) solution Y(t) so that $\lim_{t \to T_{max}^-} Y(t) = +\infty$. It easily follows from (5.1) and the comparison principle that $0 \leq \mathcal{A}_P(t) \leq Y(t)$, for any $t \in I$. Consequently, $\mathcal{A}_P(t)$ is finite on I. We deduce from Lemma 6.2 that

$$\int_0^{+\infty} R_3(t)dt \le C_3,$$

where C_3 is a constant depending on Ω , $\|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{H})}$ and L_2 . Besides, we note that T_{max} is determined by Y(0), C_* and C_3 such that $T_{max} = T_{max}(Y(0), C_*, C_3)$ is increasing when $Y(0) \ge 0$ is decreasing. Taking $t_0 = \frac{1}{2}T_{max} > 0$, then it follows that Y(t) (as well as $\mathcal{A}_P(t)$) is uniformly bounded on $[0, t_0]$. This easily implies the local existence of a unique strong solution to problem (1.1)-(1.5) (at least) on $[0, t_0]$ (actually on $[0, T_{max})$, but we lose uniform estimates on such maximal interval).

By Lemma 6.2 (cf. (6.15)), we have

$$\sup_{t \ge 0} \|\Delta(\mathbf{d}_P(t) - \mathbf{d}_E(t))\|^2 \le c \|\mathbf{h}_t\|_{L^2(0, +\infty; \mathbf{H}^{\frac{1}{2}}(\Gamma))}^2,$$
(5.14)

where c is a constant that depends only on Ω . Set now

$$\bar{\varepsilon}_0 = \min\left\{1, \frac{t_0 K}{8}\right\}, \quad L_6 = \min\left\{1, L_2^2, \frac{K}{4c}\right\}.$$
 (5.15)

From the assumption, there exists a small constant $\varepsilon_0 \leq \overline{\varepsilon}_0$ such that (5.3) is satisfied. Therefore, we can find $t_* \in [\frac{t_0}{2}, t_0]$ such that

$$\nu \|\nabla \mathbf{v}(t_*)\|^2 + \|\Delta \widehat{\mathbf{d}}(t_*) - \mathbf{f}(\mathbf{d}(t_*))\|^2 \le 2\bar{\varepsilon}_0 t_0^{-1}.$$

Moreover, if we further assume

$$\|\mathbf{h}_t\|_{L^2(0,+\infty;\mathbf{H}^{\frac{1}{2}}(\Gamma))}^2 \le L_6,$$

then by (5.14) we obtain

$$\begin{aligned} \mathcal{A}_{P}(t_{*}) &\leq \nu \|\nabla \mathbf{v}(t_{*})\|^{2} + \|\Delta \mathbf{d}(t_{*}) - \mathbf{f}(\mathbf{d}(t_{*}))\|^{2} \\ &\leq \nu \|\nabla \mathbf{v}(t_{*})\|^{2} + 2\|\Delta \mathbf{\hat{d}}(t_{*}) - \mathbf{f}(\mathbf{d}(t_{*}))\|^{2} + 2\|\Delta (\mathbf{d}_{P}(t_{*}) - \mathbf{d}_{E}(t_{*}))\|^{2} \\ &\leq \nu \|\nabla \mathbf{v}(t_{*})\|^{2} + 2\|\Delta \mathbf{\hat{d}}(t_{*}) - \mathbf{f}(\mathbf{d}(t_{*}))\|^{2} + 2c\|\mathbf{h}_{t}\|^{2}_{L^{2}(0, +\infty; \mathbf{H}^{\frac{1}{2}}(\Gamma))} \\ &\leq 4\bar{\varepsilon}_{0}t_{0}^{-1} + 2cL_{6} \leq K \\ &\leq \max\left\{1, \nu^{-1}\right\}K = Y(0). \end{aligned}$$

Taking t_* as the initial time for the ordinary differential equation (5.13), we infer from the above argument that $\mathcal{A}_P(t)$ is uniformly bounded at least on $[0, \frac{3t_0}{2}] \subset [0, t_* + t_0]$. Moreover, its bound only depends on Ω , ν , L_1, \ldots, L_6 , C_* and t_0 . Then by an iterative argument we can show that $\mathcal{A}_P(t)$ is uniformly bounded for all $t \geq 0$ and this enable us to extend the local strong solution to the whole time interval $[0, +\infty)$. The proof is complete.

A consequence of the above proposition is the eventual regularity of global weak solutions.

Theorem 5.1. Suppose that the assumptions of Proposition 2.1 and (2.32)–(2.34) are satisfied. Let (\mathbf{v}, \mathbf{d}) be a global weak solution to (1.1)–(1.5). Then there exists a large time $T^* \in (0, +\infty)$ such that (\mathbf{v}, \mathbf{d}) is a strong solution on $(T^*, +\infty)$.

Proof. Let $L_1, L_2, L_3, L_4, L_5 > 0$ be the constants as in the proof of Proposition 5.2. For a weak solution (\mathbf{v}, \mathbf{d}) , we still have the uniform estimates (5.10) and (5.11). Considering the ODE problem (5.13), we can fix the constants $\bar{\varepsilon}_0$, L_6 and t_0 . Taking $\varepsilon_0 = \bar{\varepsilon}_0$, we observe that there must exist a sufficiently large $T_1 > 0$ such that

$$\int_{T_1}^{+\infty} \left(\nu \|\nabla \mathbf{v}\|^2 + \|\Delta \widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})\|^2 \right) dt \leq \varepsilon_0,$$
(5.16)

$$\|\Delta \mathbf{d}_P(t) - \Delta \mathbf{d}_E(t)\| \leq L_6, \quad \forall t \ge [T_1, +\infty), \tag{5.17}$$

where for the second inequality we have used Lemma 6.2(i) and the fact that $\partial_t \mathbf{d}_P(t) = \Delta \mathbf{d}_P(t) - \Delta \mathbf{d}_E(t)$. Also, (5.16) implies that there is $T^* \in [T_1, T_1 + 2t_0]$ such that

$$\nu \|\nabla \mathbf{v}(T_*)\|^2 + \|\Delta \widehat{\mathbf{d}}(T_*) - \mathbf{f}(\mathbf{d}(T_*))\|^2 \le \frac{\overline{\varepsilon}_0}{t_0}.$$
(5.18)

As a result,

$$\nu \|\nabla \mathbf{v}(T_*)\|^2 + \|\Delta \widetilde{\mathbf{d}}(T_*) - \mathbf{f}(\mathbf{d}(T_*))\|^2$$

$$\leq \nu \|\nabla \mathbf{v}(T_*)\|^2 + 2\|\Delta \widehat{\mathbf{d}}(T_*) - \mathbf{f}(\mathbf{d}(T_*))\|^2 + 2\|\Delta (\mathbf{d}_P(T_*) - \mathbf{d}_E(T_*))\|^2$$

$$\leq \frac{2\overline{\varepsilon}_0}{t_0} + 2cL_6$$

$$\leq K.$$

Taking T^* as the initial time, then we can apply Proposition 5.1 to conclude that problem (1.1)–(1.5) admits a unique global strong solution $(\mathbf{v}', \mathbf{d}')$. By the weak/strong uniqueness result [6, Theorem 7], we see that (\mathbf{v}, \mathbf{d}) coincides with $(\mathbf{v}', \mathbf{d}')$ on $[T^*, +\infty)$. The proof is complete.

Thanks to the eventual regularity result we can argue as in the previous section to prove the following result

Theorem 5.2. Suppose that the assumptions of Theorem 5.1 hold. Then any global weak solution given by Proposition 2.1 converges in $\mathbf{V} \times \mathbf{H}^2(\Omega)$ to a single equilibrium $(\mathbf{0}, \mathbf{d}_{\infty})$ with estimates on the convergence rate similar to the 2D case, provided that \mathbf{g} and \mathbf{h} fulfill the corresponding hypotheses (H1)–(H7) as in Theorems 4.1 and 4.2.

Remark 5.1. We recall that there exists a (unique) global strong solution when the viscosity is large enough (cf. Theorem 2.2). Consequently, due to Lemma 2.6, all the results proven in Section 4 (i.e., Theorem 4.1 and Theorem 4.2) still hold with the same assumptions on the data. The related proofs just require some minor modifications.

The existence of a global strong solution is also ensured (with no restrictions on the fluid viscosity) when the initial data are close to a given equilibrium and the time dependent boundary data satisfies suitable bounds. First, recall that the basic energy inequality (2.16) implies (cf. (5.9))

$$\int_0^t (\nu \|\nabla \mathbf{v}(t)\|^2 + \|\Delta \widehat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))\|^2) dt \le 2(\widehat{\mathcal{E}}(0) - \widehat{\mathcal{E}}(t)) + 2\int_0^{+\infty} r(t) dt$$

and

$$\int_{0}^{+\infty} r(t)dt \le C_r \left(\|\mathbf{h}_t\|_{L^2(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}^2 + \|\mathbf{h}_t\|_{L^1(0,+\infty;\mathbf{H}^{-\frac{1}{2}}(\Gamma))}^2 + \|\mathbf{g}\|_{L^2(0,+\infty;\mathbf{V}^*)}^2 \right), \quad (5.19)$$

where C_r is a universal constant. Then we can easily deduce from Proposition 5.1 that if the lifted energy stays sufficiently close to its initial state, then system (1.1)–(1.5) admits a unique global strong solution (cf. [20] for the autonomous case).

Proposition 5.2. Assume (2.32)–(2.34) and (2.4) hold. Moreover, suppose that $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$ satisfying (2.5) and $|\mathbf{d}_0|_{\mathbb{R}^3} \leq 1$. If there exists a sufficiently small $\varepsilon_0 \in (0, 1]$ such that

$$\widehat{\mathcal{E}}(t) \ge \widehat{\mathcal{E}}(0) - \varepsilon_0, \quad \forall t \ge 0,$$
(5.20)

where $\widehat{\mathcal{E}}$ is the lifted energy defined by (2.15), then problem (1.1)–(1.5) admits a unique global strong solution (\mathbf{v}, \mathbf{d}) in $\Omega \times (0, +\infty)$, provided that $\|\mathbf{h}_t\|_{L^2(0, +\infty; \mathbf{H}^{\frac{1}{2}}(\Gamma))}$, $\|\mathbf{h}_t\|_{L^1(0, +\infty; \mathbf{H}^{-\frac{1}{2}}(\Gamma))}$ and $\|\mathbf{g}\|_{L^2(0, +\infty; \mathbf{V}^*)}$ are small enough.

Let us assume that for all $t \ge 0$ (comparing with assumptions (H1), (H4), (H5))

(H1')
$$\int_{t}^{+\infty} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d\tau \leq M_{1}(1+t)^{-1-\gamma};$$

(H4')
$$\|\mathbf{g}(t)\|^2 \le M_2(1+t)^{-2-\gamma};$$

(H5') $\|\mathbf{h}_t(t)\|_{\mathbf{L}^2(\Gamma)} \le M_3(1+t)^{-1-\gamma}.$

Here M_j , j = 1, 2, 3 and γ are positive constants. γ characterizes the decay rate of nonautonomous terms, while M_j control their magnitude.

In spirit of Proposition 5.2, in what follows, we prove the global existence of a strong solution that originates near a local minimizer of the lifted energy with suitably small perturbations in terms of the nonautonomous terms **h** and **g** (namely, the magnitudes M_j should be sufficiently small).

Theorem 5.3. Suppose that (2.32)–(2.34) and (2.4) hold, the constant $\gamma > 1$. Moreover, assume that $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbf{V} \times \mathbf{H}^2(\Omega)$ satisfies (2.5) and $|\mathbf{d}_0|_{\mathbb{R}^3} \leq 1$. Denote by \mathbf{d}_E^* the unique solution to

$$\begin{cases} -\Delta \mathbf{d}_E^* = \mathbf{0}, & x \in \Omega, \\ \mathbf{d}_E^* = \mathbf{h}_{\infty}, & x \in \Gamma, \end{cases}$$
(5.21)

and set

$$\mathscr{E}(\mathbf{d}) = \frac{1}{2} \|\nabla(\mathbf{d} - \mathbf{d}_E^*)\|^2 + \int_{\Omega} F(\mathbf{d}) dx, \quad \forall \mathbf{d} \in \mathcal{N}.$$

Let $\mathbf{d}^* \in \mathcal{N} \cap \mathbf{H}^2(\Omega)$ be a local minimizer of $\mathscr{E}(\mathbf{d})$ in the sense that $\mathscr{E}(\mathbf{d}) \geq \mathscr{E}(\mathbf{d}^*)$ for all $\mathbf{d} \in \mathcal{N}$ satisfying $\|\mathbf{d} - \mathbf{d}^*\|_{\mathbf{H}^1} < \delta$, where $\delta > 0$ is a certain small constant. Suppose also that the initial data \mathbf{v}_0 and \mathbf{d}_0 satisfy

$$\|\mathbf{v}_0\|_{\mathbf{V}} \le 1, \quad \|\mathbf{d}_0 - \mathbf{d}^*\|_{\mathbf{H}^2} \le 1.$$
 (5.22)

Then there exist positive constants $\sigma_1, \sigma_2, M_1, M_2, M_3, L_0$, which are sufficiently small and may depend on the system coefficients, on Ω and on \mathbf{d}^* , such that if the initial data $(\mathbf{v}_0, \mathbf{d}_0)$ and \mathbf{h} also fulfill

$$\|\mathbf{v}_0\| \le \sigma_1, \quad \|\mathbf{d}_0 - \mathbf{d}^*\|_{\mathbf{H}^1} \le \sigma_2, \quad \|\mathbf{h}_t\|_{L^2(0, +\infty; \mathbf{H}^{\frac{1}{2}}(\Gamma))}^2 \le L_0,$$

and (H1'), (H4'), (H5') hold with such M_j , j = 1, 2, 3, then problem (1.1)–(1.5) admits a unique global strong solution (**v**, **d**).

Proof. Without loss of generality, we assume $\delta \in (0, 1]$. In the subsequent proof, C_i $(i \in \mathbb{N})$ stand for positive constants that only depend on Ω , ν , γ and \mathbf{d}^* . Under the current assumption (5.22) on the initial data, it is not difficult to see that the constants L_1 and K in (5.4) and (5.12) depend on \mathbf{d}^* only. We just take $L_2 = L_3 = L_4 = L_5 = 1$ in (5.6) for the sake of simplicity. Then we have the uniform estimate (cf. (5.10))

$$\|\mathbf{v}(t)\| + \|\mathbf{d}(t)\|_{\mathbf{H}^1} \le C_1, \quad t \ge 0.$$

Arguing as in the proof of Proposition 5.1, we find that problem (1.1)–(1.5) admits a unique strong solution (at least) on $[0, t_0]$, whose $\mathbf{V} \times \mathbf{H}^2$ norm is uniformly bounded on $[0, t_0]$:

$$\|\mathbf{v}(t)\|_{\mathbf{V}} + \|\mathbf{d}(t)\|_{\mathbf{H}^2} \le C_3, \quad t \in [0, t_0].$$
(5.23)

Besides, we can also fix the constants $\bar{\varepsilon}_0$ and L_6 (see (5.15)). In the subsequent proof, we just take

$$\varepsilon_0 = \bar{\varepsilon}_0, \quad L_0 = L_6.$$

It follows from (5.19) that

$$\int_{0}^{+\infty} r(t)dt \le C_r C_s (M_1 + M_2 + M_3^2) \le \frac{\varepsilon_0}{4},$$

provided that $M_1, M_2, M_3 > 0$ are assumed to be properly small and satisfying

$$M_1 + M_2 + M_3^2 \le \frac{\varepsilon_0}{4C_r C_s}$$

where C_s is a universal constant due to the Sobolev embedding. Hence, according to Propositions 5.1 and 5.2, in order to prove the existence of global strong solution, we only have to verify that

$$\widehat{\mathcal{E}}(t) - \widehat{\mathcal{E}}(0) \ge -\frac{\varepsilon_0}{2}, \quad \forall t \ge 0.$$
 (5.24)

First, we notice that (recalling (2.7), (2.8) and (3.12))

$$\widehat{\mathcal{E}}(0) - \widehat{\mathcal{E}}(t) \leq \frac{1}{2} \|\mathbf{v}_{0}\|^{2} + \widehat{E}(\mathbf{d}_{0}) - \widehat{E}(\mathbf{d}(t)) = \frac{1}{2} \|\mathbf{v}_{0}\|^{2} + \frac{1}{2} \|\nabla(\mathbf{d}_{0} - \mathbf{d}_{E0})\|^{2} - \frac{1}{2} \|\nabla(\mathbf{d}(t) - \mathbf{d}_{E})\|^{2} + \int_{\Omega} F(\mathbf{d}_{0}) - F(\mathbf{d}(t)) dx \leq \frac{1}{2} \|\mathbf{v}_{0}\|^{2} + C_{4}(\|\mathbf{d}_{0} - \mathbf{d}(t)\|_{\mathbf{H}^{1}} + \|\mathbf{d}_{E0} - \mathbf{d}_{E}\|_{\mathbf{H}^{1}}).$$
(5.25)

On the other hand, thanks to standard elliptic estimates, we have

$$\begin{aligned} \|\mathbf{d}_{E0} - \mathbf{d}_{E}\|_{\mathbf{H}^{1}} &\leq c \|\mathbf{d}_{0}|_{\Gamma} - \mathbf{h}(t)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\ &\leq c \|\mathbf{d}_{0}|_{\Gamma} - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} + c \|\mathbf{h}_{\infty} - \mathbf{h}(t)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\ &\leq c \|\mathbf{d}_{0} - \mathbf{d}^{*}\|_{\mathbf{H}^{1}} + c \int_{t}^{+\infty} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d\tau \\ &\leq c\sigma_{2} + cM_{1}, \quad \forall t \geq 0. \end{aligned}$$
(5.26)

Let

$$\sigma_1 \le \min\left\{1, \frac{\sqrt{\varepsilon_0}}{2}\right\}, \quad \sigma_2 \le \frac{\varepsilon_0}{8C_4} \min\{1, c^{-1}\}, \quad M_1 \le \min\left\{1, \frac{\varepsilon_0}{8C_4c}\right\}.$$
(5.27)

Due to (5.25) and (5.26), in order to prove (5.24), we only have to verify

$$\|\mathbf{d}_0 - \mathbf{d}(t)\|_{\mathbf{H}^1} \le \frac{\varepsilon_0}{8C_4}, \quad \forall \ t \ge 0.$$
(5.28)

Since $\mathbf{d}^* \in \mathcal{N} \cap \mathbf{H}^2(\Omega)$ is the local minimizer of \mathscr{E} , it is easily to verify that \mathbf{d}^* satisfies (3.2) and thus is the critical point of E. As a consequence, Corollary 3.1 holds for \mathbf{d}^* with constants θ, β determined by \mathbf{d}^* . By (4.9), θ' can be determined by θ and γ . In addition, we further choose θ' smaller if necessary such that (recall that $\gamma > 1$)

$$\theta' \le \frac{\gamma - 1}{2\gamma}.\tag{5.29}$$

Let us define

$$\varpi = \min\left\{\frac{\beta}{2}, \ \frac{\delta}{2}, \ \frac{\varepsilon_0}{10C_4}\right\},\tag{5.30}$$

and set

$$\bar{t}_0 = \sup\{t \in [0, t_0], \|\mathbf{d}(t) - \mathbf{d}^*\|_{\mathbf{H}^1} < \varpi, \forall s \in [0, t)\}$$

If we assume

$$\sigma_2 \le \frac{1}{4}\varpi,\tag{5.31}$$

then by the continuity of $\mathbf{d}(t)$ in $\mathbf{H}^1(\Omega)$, we have $\bar{t}_0 > 0$. Next, we shall prove that $\bar{t}_0 > t_0$ by contradiction. We introduce the auxiliary functional

$$\Psi_1(t) = \widehat{\mathcal{E}}(t) - \widehat{E}(\mathbf{d}^*) + 2\int_t^{+\infty} r(\tau)d\tau,$$

and the function

$$\bar{\mathbf{d}}(t) = \mathbf{d}(t) - \mathbf{d}_E + \mathbf{d}_E^*$$

It easily follows that

$$\Psi_{1}(t) \geq \widehat{E}(\mathbf{d}(t)) - \widehat{E}(\mathbf{d}^{*}) = \widehat{E}(\mathbf{d}(t)) - \mathscr{E}(\bar{\mathbf{d}}(t)) + \mathscr{E}(\bar{\mathbf{d}}(t)) - \widehat{E}(\mathbf{d}^{*})$$
$$= \int_{\Omega} F(\mathbf{d}(t)) - F(\bar{\mathbf{d}}(t)) dx + \mathscr{E}(\bar{\mathbf{d}}(t)) - \widehat{E}(\mathbf{d}^{*}).$$
(5.32)

By definition, $\bar{\mathbf{d}}(t) \in \mathcal{N}$. Moreover, on $[0, \bar{t}_0]$,

$$\|\mathbf{d}(t) - \mathbf{d}^*\|_{\mathbf{H}^1} \le \|\mathbf{d}(t) - \mathbf{d}^*\|_{\mathbf{H}^1} + \|\mathbf{d}_E - \mathbf{d}^*_E\|_{\mathbf{H}^1}$$

$$\le \quad \varpi + c\|\mathbf{h}(t) - \mathbf{h}_{\infty}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \le \frac{\delta}{2} + c\int_t^{+\infty} \|\mathbf{h}_t(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d\tau$$

$$\le \quad \frac{\delta}{2} + cM_1.$$

Taking

$$M_1 \le \min\left\{1, \frac{\delta}{4c}\right\},\tag{5.33}$$

then we have $\|\bar{\mathbf{d}}(t) - \mathbf{d}^*\|_{\mathbf{H}^1} \leq \delta$. Since \mathbf{d}^* is a local minimizer of \mathscr{E} , we see that

$$\mathscr{E}(\bar{\mathbf{d}}(t)) - \widehat{E}(\mathbf{d}^*) = \mathscr{E}(\bar{\mathbf{d}}(t)) - \mathscr{E}(\mathbf{d}^*) \ge 0, \quad t \in [0, \bar{t}_0].$$
(5.34)

On the other hand, since $|\mathbf{d}(t)|_{\mathbb{R}^3} \leq 1$ and $|\bar{\mathbf{d}}(t)|_{\mathbb{R}^3} \leq 3$ (this is due to the maximum principle (2.6)), we infer from the standard elliptic estimate and (H5') that

$$\left| \int_{\Omega} F(\mathbf{d}(t)) - F(\bar{\mathbf{d}}(t)) dx \right| \leq C_{5} \| - \mathbf{d}_{E} + \mathbf{d}_{E}^{*} \|$$

$$\leq C_{5} c \int_{t}^{+\infty} \| \mathbf{h}_{t}(\tau) \|_{\mathbf{L}^{2}(\Gamma)} d\tau$$

$$\leq C_{5} c M_{3} \gamma^{-1} (1+t)^{-\gamma}. \tag{5.35}$$

Let us introduce now two further functions

$$z(t) = (C_5c+1)M_3\gamma^{-1}(1+t)^{-\gamma}, \quad \Psi(t) = \Psi_1(t) + z(t).$$

We deduce from (5.32)–(5.35) that

$$\Psi(t) \ge M_3 \gamma^{-1} (1+t)^{-\gamma} > 0, \quad t \in [0, \bar{t}_0],$$

and by the basic energy inequality (2.16)

$$\frac{d}{dt}\Psi(t) = \frac{d}{dt}\widehat{\mathcal{E}}(t) - 2r(t) - (C_5c+1)M_3(1+t)^{-1-\gamma} \\
\leq -\frac{1}{4}\min\{\nu,1\}\mathcal{D}^2(t) - (C_5c+1)M_3(1+t)^{-1-\gamma} \\
\leq -C_6\left(\mathcal{D}(t) + M_3^{\frac{1}{2}}(1+t)^{-\frac{1+\gamma}{2}}\right)^2,$$

where $\mathcal{D}(t) = \|\nabla \mathbf{v}(t)\| + \|\Delta \hat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))\|$. Arguing as to get (4.12), using Remark 3.3 and assumptions (H1'), (H4'), we deduce

$$\Psi(t)^{1-\theta'} \le C_7 \left(\mathcal{D}(t) + (M_1 + M_2)(1+t)^{-(1-\theta')(1+\gamma)} + M_3^{1-\theta'}(1+t)^{-(1-\theta')\gamma} \right).$$

Assuming

$$M_1 \le \frac{1}{2}M_3^{\frac{1}{2}}, \quad M_2 \le \frac{1}{2}M_3^{\frac{1}{2}}, \quad M_3 \le 1,$$
 (5.36)

we can see that

$$\Psi(t)^{1-\theta'} \le C_7 \left(\mathcal{D}(t) + 2M_3^{\frac{1}{2}} (1+t)^{-(1-\theta')\gamma} \right).$$

As a result, we find

$$-\frac{d}{dt}\Psi(t)^{\theta'} = -\theta'\Psi(t)^{\theta'-1}\frac{d}{dt}\Psi(t)$$

$$\geq \frac{C_6\left(\mathcal{D}(t) + M_3^{\frac{1}{2}}(1+t)^{-\frac{1+\gamma}{2}}\right)^2}{C_7\left(\mathcal{D}(t) + M_3^{\frac{1}{2}}(1+t)^{-(1-\theta')\gamma}\right)}$$

$$\geq C_8\left(\mathcal{D}(t) + M_3^{\frac{1}{2}}(1+t)^{-\frac{1+\gamma}{2}}\right), \qquad (5.37)$$

where we have used the fact that $\frac{1+\gamma}{2} \leq (1-\theta')\gamma$ (cf. (5.29)). It follows from (4.16), (5.23), (5.36), (5.37), assumptions (H1'), (H4'), (H5') and the definition of Ψ that

$$\int_{0}^{\bar{t}_{0}} \|\mathbf{d}_{t}(t)\| dt \leq C_{9} \Psi(0)^{\theta'}$$

$$\leq C_{10} \left(\|\mathbf{v}_{0}\|^{2} + \|\mathbf{d}_{0} - \mathbf{d}^{*}\|_{\mathbf{H}^{1}} + \|\mathbf{d}_{E0} - \mathbf{d}^{*}_{E}\|_{\mathbf{H}^{1}} + \int_{0}^{+\infty} r(t) dt + z(0) \right)^{\theta'}$$

$$\leq C_{11} \left(\|\mathbf{v}_{0}\|^{2} + \|\mathbf{d}_{0} - \mathbf{d}^{*}\|_{\mathbf{H}^{1}} + M_{3}^{\frac{1}{2}} \right)^{\theta'}.$$
(5.38)

By (5.23), (5.38) and an interpolation inequality, we get

$$\begin{aligned} \|\mathbf{d}(\bar{t}_{0}) - \mathbf{d}^{*}\|_{\mathbf{H}^{1}} \\ &\leq \|\mathbf{d}(\bar{t}_{0}) - \mathbf{d}_{0}\|_{\mathbf{H}^{1}} + \|\mathbf{d}_{0} - \mathbf{d}^{*}\|_{\mathbf{H}^{1}} \\ &\leq C_{12}(\|\mathbf{d}(\bar{t}_{0})\|_{\mathbf{H}^{2}} + \|\mathbf{d}_{0}\|_{\mathbf{H}^{2}})^{\frac{1}{2}} \|\mathbf{d}(\bar{t}_{0}) - \mathbf{d}_{0}\|^{\frac{1}{2}} + \|\mathbf{d}_{0} - \mathbf{d}^{*}\|_{\mathbf{H}^{1}} \\ &\leq C_{13}\left(\|\mathbf{v}_{0}\|^{\theta'} + \|\mathbf{d}_{0} - \mathbf{d}^{*}\|_{\mathbf{H}^{1}}^{\frac{\theta'}{2}} + M_{3}^{\frac{\theta'}{4}}\right) + \|\mathbf{d}_{0} - \mathbf{d}^{*}\|_{\mathbf{H}^{1}}. \end{aligned}$$
(5.39)

Taking now

$$\sigma_1 \le \min\left\{1, \frac{\sqrt{\varepsilon_0}}{2}, \left(\frac{\varpi}{6C_{13}}\right)^{\frac{1}{\theta'}}\right\}, \quad \sigma_2 \le \min\left\{1, \frac{1}{4}\varpi, \left(\frac{\varpi}{6C_{13}}\right)^{\frac{2}{\theta'}}\right\}, \tag{5.40}$$

$$M_3 \le \min\left\{1, \left(\frac{\varpi}{6C_{13}}\right)^{\frac{4}{\theta'}}\right\},\tag{5.41}$$

we infer from (5.39) that

$$\|\mathbf{d}(\bar{t}_0) - \mathbf{d}^*\|_{\mathbf{H}^1} \le \frac{3}{4}\varpi < \varpi.$$

This leads to a contradiction with the definition of \bar{t}_0 . As a result, we have $\bar{t}_0 > t_0$, and

$$\|\mathbf{d}_0 - \mathbf{d}(t)\|_{\mathbf{H}^1} \leq \|\mathbf{d}_0 - \mathbf{d}^*\|_{\mathbf{H}^1} + \|\mathbf{d}^* - \mathbf{d}(t)\|_{\mathbf{H}^1}$$

$$\leq \sigma_2 + \varpi \leq \frac{5}{4} \varpi \leq \frac{\varepsilon_0}{8C_4}, \quad \forall \ t \in [0, t_0].$$
(5.42)

Thus, (5.24) holds on $[0, t_0]$, which implies

$$\int_0^{t_0} (\nu \|\nabla \mathbf{v}(t)\|^2 + \|\Delta \widehat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))\|^2) dt \le \varepsilon_0$$

As in Proposition 5.1, there exists $t_* \in [\frac{t_0}{2}, t_0]$ such that

$$\nu \|\nabla \mathbf{v}(t_*)\|^2 + \|\Delta \widehat{\mathbf{d}}(t_*) - \mathbf{f}(\mathbf{d}(t_*))\|^2 \le 2\varepsilon_0 t_0^{-1},$$

and again we have $\mathcal{A}_P(t_*) \leq \max\{1, \nu^{-1}\} K$. Taking t_* as the initial time for the Cauchy problem (5.13), we can extend the (unique) strong solution to $[0, \frac{3}{2}t_0]$ and its $\mathbf{V} \times \mathbf{H}^2$ -norm is uniformly bounded by the same constant C_3 as on $[0, t_0]$. Repeating the above argument in $[0, \frac{3}{2}t_0]$, we can verify that (5.24) still holds. By iteration we can show that (5.24) holds for all $t \geq 0$. Hence, our conclusion follows from Proposition 5.2.

Finally, we can conclude with the following local stability result:

Theorem 5.4. Let the assumptions of Theorem 5.3 hold. Then any global strong solution given by Theorem 5.3 converges in $\mathbf{V} \times \mathbf{H}^2(\Omega)$ to a single equilibrium $(\mathbf{0}, \mathbf{d}_{\infty})$ with $\mathbf{d}_{\infty} \in \mathcal{N} \cap \mathbf{H}^2(\Omega)$ such that $\mathscr{E}(\mathbf{d}_{\infty}) = \mathscr{E}(\mathbf{d}^*)$. In addition, convergence rate estimates similar to the 2D case hold provided that \mathbf{g} and \mathbf{h} fulfill the corresponding hypotheses (i.e., assumptions (H1), (H4), (H5) are replaced by (H1'), (H4'), (H5'), respectively). Indeed, the local energy minimizer \mathbf{d}^* is (locally) Lyapunov stable, and in particular, if \mathbf{d}^* is an isolated local minimizer of \mathscr{E} , then it is (locally) asymptotically stable.

Proof. Arguing as in Section 4 we still find

$$\lim_{t \to +\infty} \left(\|\mathbf{v}(t)\|_{\mathbf{V}} + \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|_{\mathbf{H}^2} \right) = 0,$$
(5.43)

for some $\mathbf{d}_{\infty} \in \mathcal{N} \cap \mathbf{H}^2(\Omega)$. The estimate on the convergence rates can be obtained following the proof of Theorem 4.2.

Recalling the proof of Theorem 5.3, we actually showed that

$$\|\mathbf{d}(t) - \mathbf{d}^*\|_{\mathbf{H}^1} \le \varpi, \quad \forall t \ge 0, \tag{5.44}$$

which implies that (let t be large)

$$\|\mathbf{d}_{\infty} - \mathbf{d}^*\|_{\mathbf{H}^1} \le \|\mathbf{d}(t) - \mathbf{d}_{\infty}\|_{\mathbf{H}^1} + \|\mathbf{d}(t) - \mathbf{d}^*\|_{\mathbf{H}^1} < 2\varpi \le \min\{\beta, \delta\}.$$

Taking $\mathbf{d} = \mathbf{d}_{\infty}$ and $\psi = \mathbf{d}^*$ in Corollary 3.1, we see that

$$|\mathscr{E}(\mathbf{d}_{\infty}) - \mathscr{E}(\mathbf{d}^*)|^{1-\theta} = |\widehat{E}(\mathbf{d}_{\infty}) - \widehat{E}(\mathbf{d}^*)|^{1-\theta} \le \| -\Delta\widehat{\mathbf{d}}^* + \mathbf{f}(\mathbf{d}^*)\| = 0.$$

Since $\|\mathbf{d}_{\infty} - \mathbf{d}^*\|_{\mathbf{H}^1} \leq \delta$, \mathbf{d}_{∞} is also an energy minimizer of \mathscr{E} .

Moreover, the proof of Theorem 5.3 implies that, for arbitrary (small) $\epsilon > 0$, if we replace the choice of ϖ (5.30) by

$$\varpi_1 = \min\left\{\epsilon, \frac{\beta}{2}, \ \frac{\delta}{2}, \ \frac{\varepsilon_0}{10C_4}\right\},\tag{5.45}$$

then we are able to choose the constants σ_i , M_j sufficiently small in a similar manner such that (5.43) and (5.44) hold with ϖ being replaced by ϖ_1 (and thus (5.44) holds for ϵ). This yields the (locally) Lyapunov stability of \mathbf{d}^* . Finally, it is easy to see that if \mathbf{d}^* is an isolated local minimizer, then $\mathbf{d}_{\infty} = \mathbf{d}^*$ and \mathbf{d}^* is asymptotically stable. The proof is complete.

6 Appendix

We report some properties of the lifting functions \mathbf{d}_E and \mathbf{d}_P (cf. (2.7) and (2.21)) that have been used in the previous sections. Below we denote by c a generic positive constant which depends on n and Ω at most.

Lemma 6.1. For any $t \ge 0$, and k = 0, 1, 2, ..., j = 0, 1, we have (i) $\|\partial_t^j \mathbf{d}_E(t)\|_{\mathbf{H}^k} \le c \|\partial_t^j \mathbf{h}(t)\|_{\mathbf{H}^{k-\frac{1}{2}}(\Gamma)};$ (ii) $\|\mathbf{d}_E(t) - \mathbf{d}_*\|_{\mathbf{H}^k} \le c \|\mathbf{h}(t) - \mathbf{h}_*\|_{\mathbf{H}^{k-\frac{1}{2}}(\Gamma)},$ where \mathbf{d}_* is the unique solution to

$$\begin{cases} -\Delta \mathbf{d}_* = \mathbf{0}, & x \in \Omega, \\ \mathbf{d}_* = \mathbf{h}_*, & x \in \Gamma. \end{cases}$$
(6.1)

Proof. The conclusion follows from the classical elliptic regularity theory (cf., e.g., [24, 32]).

Lemma 6.2. Let $\mathbf{d}_0 \in \mathbf{H}^2(\Omega)$ with $|\mathbf{d}_0|_{\mathbb{R}^n} \leq 1$. Suppose that \mathbf{h} satisfy (2.4)–(2.5) and $\mathbf{h}_t \in L^2_{loc}([0, +\infty); \mathbf{H}^{\frac{1}{2}}(\Gamma))$. Then, for any t > 0, the following estimates hold

$$\|\mathbf{d}_{P}(t) - \mathbf{d}_{E}(t)\|_{\mathbf{H}^{1}}^{2} \leq c e^{-t} \int_{0}^{t} e^{\tau} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}^{2} d\tau, \qquad (6.2)$$

$$\|\partial_t \mathbf{d}_P(t)\|^2 + \|\mathbf{d}_P(t) - \mathbf{d}_E(t)\|_{\mathbf{H}^2}^2 \leq c \int_0^t \|\mathbf{h}_t(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^2 d\tau,$$
(6.3)

$$\int_0^t \|\nabla \Delta \mathbf{d}_P\|^2 d\tau \leq c \int_0^t \|\mathbf{h}_t(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^2 d\tau.$$
(6.4)

In addition, we have

(*i*) if $\mathbf{h}_t \in L^2(0, +\infty; \mathbf{H}^{\frac{1}{2}}(\Gamma))$ then

$$\lim_{t \to +\infty} \|\partial_t \mathbf{d}_P(t)\| = 0, \tag{6.5}$$

(ii) if \mathbf{h}_t satisfies (H6) then, for all $t \geq 0$,

$$\|\partial_t \mathbf{d}_P(t)\|^2 + \|\mathbf{d}_P(t) - \mathbf{d}_E(t)\|_{\mathbf{H}^2}^2 \leq c(1+t)^{-2-2\gamma}, \tag{6.6}$$

$$\int_{\frac{t}{2}}^{t} \|\nabla \Delta \mathbf{d}_P(\tau)\|^2 d\tau \leq c(1+t)^{-1-2\gamma}.$$
(6.7)

Proof. It follows from (2.7) and (2.21) that

$$\begin{cases} -\Delta(\mathbf{d}_P - \mathbf{d}_E) = -\partial_t \mathbf{d}_P, & \text{in } \Omega \times \mathbb{R}^+, \\ \mathbf{d}_P - \mathbf{d}_E = \mathbf{0}, & \text{on } \Gamma \times \mathbb{R}^+, \end{cases}$$
(6.8)

and

$$\begin{cases} \partial_t (\mathbf{d}_P - \mathbf{d}_E) - \Delta (\mathbf{d}_P - \mathbf{d}_E) = -\partial_t \mathbf{d}_E, & \text{in } \Omega \times \mathbb{R}^+, \\ \mathbf{d}_P - \mathbf{d}_E = \mathbf{0}, & \text{on } \Gamma \times \mathbb{R}^+, \\ \mathbf{d}_P - \mathbf{d}_E|_{t=0} = \mathbf{0}, & \text{in } \Omega. \end{cases}$$
(6.9)

Multiplying the first equation in (6.9) by $(\mathbf{d}_P - \mathbf{d}_E) - \Delta(\mathbf{d}_P - \mathbf{d}_E)$, integrating by parts and using the Poincaré inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{d}_P - \mathbf{d}_E\|^2 + \|\nabla(\mathbf{d}_P - \mathbf{d}_E)\|^2) + \|\nabla(\mathbf{d}_P - \mathbf{d}_E)\|^2 + \|\Delta(\mathbf{d}_P - \mathbf{d}_E)\|^2$$

$$\leq (\|\mathbf{d}_{P} - \mathbf{d}_{E}\| + \|\Delta(\mathbf{d}_{P} - \mathbf{d}_{E})\|)\|\partial_{t}\mathbf{d}_{E}\|$$

$$\leq (C_{P}\|\nabla(\mathbf{d}_{P} - \mathbf{d}_{E})\| + \|\Delta(\mathbf{d}_{P} - \mathbf{d}_{E})\|)\|\partial_{t}\mathbf{d}_{E}\|$$

$$\leq \frac{1}{2}\|\nabla(\mathbf{d}_{P} - \mathbf{d}_{E})\|^{2} + \frac{1}{2}\|\Delta(\mathbf{d}_{P} - \mathbf{d}_{E})\|^{2} + \left(\frac{1}{2}C_{P}^{2} + \frac{1}{2}\right)\|\partial_{t}\mathbf{d}_{E}\|^{2},$$
(6.10)

which, together with Lemma 6.1, implies

$$\|\mathbf{d}_{P}(t) - \mathbf{d}_{E}(t)\|_{\mathbf{H}^{1}}^{2} \leq c e^{-c_{1}t} \int_{0}^{t} e^{c_{1}\tau} \|\partial_{t}\mathbf{d}_{E}(\tau)\|^{2} d\tau$$

$$\leq c e^{-c_{1}t} \int_{0}^{t} e^{c_{1}\tau} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}^{2} d\tau, \qquad (6.11)$$

that is, (6.2).

Applying now the Laplacian to the first equation in (6.9), we get

$$\begin{cases} \partial_t \Delta (\mathbf{d}_P - \mathbf{d}_E) - \Delta^2 (\mathbf{d}_P - \mathbf{d}_E) = \mathbf{0}, & \text{in } \Omega \times \mathbb{R}^+, \\ \Delta (\mathbf{d}_P - \mathbf{d}_E) = \mathbf{h}_t, & \text{on } \Gamma \times \mathbb{R}^+, \\ \Delta (\mathbf{d}_P - \mathbf{d}_E)|_{t=0} = \mathbf{0}, & \text{in } \Omega. \end{cases}$$
(6.12)

Multiplying the first equation of (6.12) by $\Delta(\mathbf{d}_P - \mathbf{d}_E)$ and integrating by parts, we get

$$\frac{1}{2} \frac{d}{dt} \|\Delta(\mathbf{d}_P - \mathbf{d}_E)\|^2 + \|\nabla\Delta(\mathbf{d}_P - \mathbf{d}_E)\|^2$$

$$\leq \|\partial_{\mathbf{n}}\Delta(\mathbf{d}_P - \mathbf{d}_E)\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \|\mathbf{h}_t\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}$$

$$\leq c \|\Delta(\mathbf{d}_P - \mathbf{d}_E)\|_{\mathbf{H}^1} \|\mathbf{h}_t\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}$$

$$\leq \frac{1}{2} (\|\nabla\Delta(\mathbf{d}_P - \mathbf{d}_E)\|^2 + \|\Delta(\mathbf{d}_P - \mathbf{d}_E)\|^2) + c\|\mathbf{h}_t\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^2.$$
(6.13)

Hence, from (6.10) and (6.13) we infer

$$\frac{d}{dt} \|\mathbf{d}_P(t) - \mathbf{d}_E(t)\|_{\mathbf{H}^2}^2 + c_2(\|\mathbf{d}_P(t) - \mathbf{d}_E(t)\|_{\mathbf{H}^2}^2 + \|\nabla\Delta(\mathbf{d}_P - \mathbf{d}_E)\|^2) \le c\|\mathbf{h}_t\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^2, \quad (6.14)$$

which entails (6.4) and

$$\|\mathbf{d}_{P}(t) - \mathbf{d}_{E}(t)\|_{\mathbf{H}^{2}}^{2} \le c \int_{0}^{t} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{2} d\tau.$$
(6.15)

Thus (6.3) follows from (6.15) and the fact $\|\partial_t \mathbf{d}_P(t)\| = \|\Delta \mathbf{d}_P(t)\|$.

Now if $\mathbf{h}_t \in L^2(0, +\infty; \mathbf{H}^{\frac{1}{2}}(\Gamma))$, we infer from (6.10) that

$$\int_{0}^{+\infty} \|\Delta(\mathbf{d}_{P}(t) - \mathbf{d}_{E}(t))\|^{2} dt \leq c \int_{0}^{+\infty} \|\partial_{t} \mathbf{d}_{E}(t)\|^{2} dt \\
\leq c \int_{0}^{+\infty} \|\mathbf{h}_{t}(t)\|^{2}_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} dt < +\infty.$$
(6.16)

Then it follows from (6.13), (6.16) and Lemma 4.1 that

$$\lim_{t \to +\infty} \|\Delta(\mathbf{d}_P(t) - \mathbf{d}_E(t))\|^2 = 0,$$

which implies (6.5).

Furthermore, if (H6) holds, then (6.14) implies that (cf., e.g., [34])

$$\|\mathbf{d}_P(t) - \mathbf{d}_E(t)\|_{\mathbf{H}^2}^2 \le c(1+t)^{-2-2\gamma}, \quad \forall \ t \ge 0.$$

Using (6.14) once more, we have

$$\begin{split} \int_{\frac{t}{2}}^{t} \|\nabla\Delta\mathbf{d}_{P}(\tau)\|^{2} d\tau &= \int_{\frac{t}{2}}^{t} \|\nabla\Delta(\mathbf{d}_{P} - \mathbf{d}_{E})(\tau)\|^{2} d\tau \\ &\leq c \left\|\mathbf{d}_{P}\left(\frac{t}{2}\right) - \mathbf{d}_{E}\left(\frac{t}{2}\right)\right\|_{\mathbf{H}^{2}}^{2} + c \int_{\frac{t}{2}}^{t} \|\mathbf{h}_{t}(\tau)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{2} d\tau \\ &\leq c \left(1 + \frac{t}{2}\right)^{-2 - 2\gamma} + \frac{c}{1 + 2\gamma} \left(1 + \frac{t}{2}\right)^{-1 - 2\gamma} \\ &\leq c \left(1 + t\right)^{-1 - 2\gamma}, \quad \forall t \geq 0, \end{split}$$

and this gives (6.7). The proof is complete.

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