# Long-time behavior <br> for a hydrodynamic model on nematic liquid crystal flows with asymptotic stabilizing boundary condition and external force 

Maurizio Grasselli * and Hao Wu ${ }^{\dagger}$

August 7, 2018


#### Abstract

In this paper, we consider a simplified Ericksen-Leslie model for the nematic liquid crystal flow. The evolution system consists of the Navier-Stokes equations coupled with a convective Ginzburg-Landau type equation for the averaged molecular orientation. We suppose that the Navier-Stokes equations are characterized by a no-slip boundary condition and a timedependent external force $\mathbf{g}(t)$, while the equation for the molecular director is subject to a time-dependent Dirichlet boundary condition $\mathbf{h}(t)$. We show that, in 2 D , each global weak solution converges to a single stationary state when $\mathbf{h}(t)$ and $\mathbf{g}(t)$ converge to a timeindependent boundary datum $\mathbf{h}_{\infty}$ and $\mathbf{0}$, respectively. Estimates on the convergence rate are also obtained. In the 3D case, we prove that global weak solutions are eventually strong so that results similar to the 2 D case can be proven. We also show the existence of global strong solutions, provided that either the viscosity is large enough or the initial datum is close to a given equilibrium.


Keywords: Nematic liquid crystal flow, non-autonomous Navier-Stokes equations, timedependent Dirichlet boundary condition, long-time behavior, Lojasiewicz-Simon inequality.

AMS Subject Classification: 35B40, 35Q35, 76A15, 76D05.

## 1 Introduction

We consider the following hydrodynamical model for the flow of nematic liquid crystals

$$
\begin{align*}
\mathbf{v}_{t}+\mathbf{v} \cdot \nabla \mathbf{v}-\nu \Delta \mathbf{v}+\nabla \pi & =-\lambda \nabla \cdot(\nabla \mathbf{d} \odot \nabla \mathbf{d})+\mathbf{g}(t),  \tag{1.1}\\
\nabla \cdot \mathbf{v} & =0  \tag{1.2}\\
\mathbf{d}_{t}+\mathbf{v} \cdot \nabla \mathbf{d} & =\eta(\Delta \mathbf{d}-\mathbf{f}(\mathbf{d})), \tag{1.3}
\end{align*}
$$

in $\Omega \times \mathbb{R}^{+}$, where $\Omega \subset \mathbb{R}^{n}(n=2,3)$ is a bounded domain with sufficiently smooth boundary $\Gamma$, $\mathbf{v}=\left(v_{1}, \ldots, v_{n}\right)^{t r}$ is the velocity field of the flow and $\mathbf{d}=\left(d_{1}, \ldots, d_{n}\right)^{t r}$ represents the averaged macroscopic/continuum molecular orientations in $\mathbb{R}^{n}(n=2,3) . \pi$ is a scalar function representing the pressure (including both the hydrostatic and the induced elastic part from the orientation field). The external volume force is represented by $\mathbf{g}$. The positive constants $\nu, \lambda$ and $\eta$ stand for

[^0]viscosity, the competition between kinetic energy and potential energy, and macroscopic elastic relaxation time (Deborah number) for the molecular orientation field. $\nabla \mathbf{d} \odot \nabla \mathbf{d}$ denotes the $n \times n$ matrix whose $(i, j)$-th entry is given by $\nabla_{i} \mathbf{d} \cdot \nabla_{j} \mathbf{d}$, for $1 \leq i, j \leq n$. We assume that $\mathbf{f}(\mathbf{d})=\nabla_{\mathbf{d}} F(\mathbf{d})$ for some smooth bounded function $F: \mathbb{R}^{n} \rightarrow \mathbb{R}$. In particular, one uses the Ginzburg-Landau approximation $\mathbf{f}(\mathbf{d})=\frac{1}{\epsilon^{2}}\left(|\mathbf{d}|^{2}-1\right) \mathbf{d}$ to relax the nonlinear constraint $|\mathbf{d}|=1$ on molecule length (cf. [19, 20).

System (1.1)-(1.3) was firstly proposed in [18] as a simplified approximate system of the original Ericksen-Leslie model for the nematic liquid crystal flows (cf. [7,17]). Well-posedness of the autonomous version of system (1.1)-(1.3) (namely, with $\mathbf{g}=\mathbf{0}$, no-slip boundary condition for $\mathbf{v}$, and time-independent Dirichlet boundary condition for $\mathbf{d}$ ) has been analyzed in [20] (see also [8, 21] and, for different boundary conditions, [25). For numerical approximation we refer to [23, 26, 27. Problem (1.1)-(1.3) has also been investigated on a Riemannian manifold in [29], where the existence of a global attractor in the 2D case was proven. As far as the long-time behavior of the single trajectory is concerned, in [20], a natural question on the uniqueness of asymptotic limit for global solutions (to the autonomous system) was raised. This question was answered in [35], where it is proven that each trajectory converges to a single steady state (cf. [28,36] for some generalization). The proof is based on a suitable Łojasiewicz-Simon type inequality (see [30, cf. also [13] and references cited therein).

The technically more challenging case of time-dependent Dirichlet boundary conditions for d has been recently analyzed in [1, [5, 6, [1]. For instance, under proper assumptions on the timedependent boundary condition and assuming that $\mathbf{g}=\mathbf{0}$, the existence of global weak solution, the existence of global regular solution for viscosity coefficient big enough, and the weak/strong uniqueness were obtained in [6]. Regularity criteria for solutions in the 3D case can be found in [11. Besides, the presence of a time-dependent external force is allowed in [1] and existence of global and exponential attractors is proven in the 2D case. In this paper, we want to extend the results of [35 to the non-autonomous case treated in [1]. Thus we consider system (1.1)-(1.3) subject to the boundary conditions

$$
\begin{equation*}
\mathbf{v}(x, t)=\mathbf{0}, \quad \mathbf{d}(x, t)=\mathbf{h}(x, t), \quad(x, t) \in \Gamma \times \mathbb{R}^{+}, \tag{1.4}
\end{equation*}
$$

and the initial conditions

$$
\begin{equation*}
\left.\mathbf{v}\right|_{t=0}=\mathbf{v}_{0}(x) \text { with } \nabla \cdot \mathbf{v}_{0}=0,\left.\quad \mathbf{d}\right|_{t=0}=\mathbf{d}_{0}(x), \quad x \in \Omega . \tag{1.5}
\end{equation*}
$$

In the 2D case, we prove that each weak/strong solution converges to a single stationary state when $\mathbf{h}(t)$ and $\mathbf{g}(t)$ converge to a time-independent boundary datum $\mathbf{h}_{\infty}$ and $\mathbf{0}$, respectively. In the 3D case, we first show the eventual regularity of global weak solutions, and the existence of global strong solutions provided that either the viscosity is large enough or the initial datum is close to a given equilibrium. Then an analogous result on the long-time behavior as in 2D is also obtained. In both cases, we provide an estimate on the convergence rate.

Before ending this section, we state some key ingredients of the present paper. System (1.1)-(1.5) is non-autonomous due to the time-dependent boundary data $\mathbf{h}$ and external force g. This brings some additional difficulties into our subsequent proofs. First, in order to obtain the energy inequalities that play crucial roles in the proof of well-posedness as well as in the long-time behavior of global solutions (cf. Lemmas 2.2, 2.5, (2.6, 5.1), we have to introduce proper lifting functions (cf. (2.7) and (2.21) below). The idea was first used in [5, 6], but the lifting functions introduced in this paper are different from those in 6. This is due to the
fact that we need some specific energy inequalities which not only yield uniform estimates of the solutions, but also provide estimates of the convergence rate (cf. Section 4). The second issue regards the application of the Łojasiewicz-Simon approach (cf. 30]) which has been shown to be very useful in the study of long-time behavior of global solutions to nonlinear evolution equations (see, for instance, [12, 13, 15, 34, 35, 38] and references therein). In particular, convergent results related to various evolution equations with asymptotically autonomous source terms were established, e.g., in [4, $9,13,14$. However, our current case is much more complicated than the previous cases, because the Łojasiewicz-Simon inequality involves the vector $\mathbf{d}$ that is subject to a time-dependent boundary datum. To overcome this difficulty, we derive an extended Łojasiewicz-Simon type inequality for vector functions with arbitrary nonhomogeneous Dirichlet boundary data, which is associated with the lifted energy (cf. Corollary 3.1). This generalizes the results in [13, 35] and should have its own interest. Third, in the 3D case, we also apply the Lojasiewicz-Simon approach to prove the existence of global strong solutions provided that the initial datum is close to a local minimizer of the elastic energy and the non-autonomous terms are properly small perturbations of their asymptotic limits (cf. Section 5). Then we further discuss the stability of these energy minimizers. This extends the previous results in [20, 35] for the autonomous system, where the initial datum was required to be sufficiently close to a global energy minimizer. For the stability of the general Ericksen-Leslie system [22], we refer to the recent work [37].

The remaining part of the paper is organized as follows. The next section is devoted to report some existence and uniqueness results and basic a priori estimates for the solutions. The extended Łojasiewicz-Simon inequality we need is derived in Section 3. In Section 4 we show the convergence of each global weak/strong solution to a single steady state and provide uniform estimates on the convergence rate in 2D. Results in 3D are presented in Section 5 In particular, we study the eventual regularity of global weak solutions as well as the well-posedness when the initial data are close to local minimizers of the elastic energy. Long-time convergence of global solutions and stability of such minimizers are also proved. In the final Section 6, some useful properties of the lifting functions are reported.

## 2 Preliminaries: well-posedness and a priori estimates

Without loss of generality, from now on we set $\lambda=\eta=1$. Let us introduce the function spaces we shall work with. As usual, $L^{p}(\Omega)$ and $W^{k, p}(\Omega)$ stand for the Lebesgue and the Sobolev spaces of real valued functions, with the convention that $H^{k}(\Omega)=W^{k, 2}(\Omega)$. The spaces of vector-valued functions are denoted by bold letters, correspondingly. Without any further specification, $\|\cdot\|$ stands for the norm in $L^{2}(\Omega)$ or $\mathbf{L}^{2}(\Omega)$. This norm is induced by the scalar inner product $(u, v)=\int_{\Omega} u v d x$, where for vector valued functions the product $u v$ is replaced by the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$. We set, as usual,

$$
\mathbf{H}=\overline{\mathcal{V}}^{\mathbf{L}^{2}(\Omega)}, \quad \mathbf{V}=\overline{\mathcal{V}}^{\mathbf{H}_{0}^{1}(\Omega)}, \quad \text { where } \mathcal{V}=\left\{\mathbf{v} \in C_{0}^{\infty}\left(\Omega, \mathbb{R}^{n}\right): \nabla \cdot \mathbf{v}=0\right\}
$$

For any Banach space $B$, we denote its dual space by $B^{*}$. In particular, we denote the dual space of $\mathbf{H}_{0}^{1}(\Omega)$ by $\mathbf{H}^{-1}(\Omega)$.

In the following text, we will use the regularity result for Stokes problem (see, e.g., [33])
Lemma 2.1. For the Stokes operator $S: D(S)=\mathbf{V} \cap \mathbf{H}^{2}(\Omega) \rightarrow \mathbf{H}$ defined by

$$
S \mathbf{u}=-\Delta \mathbf{u}+\nabla \pi \in \mathbf{H}, \quad \forall \mathbf{u} \in D(S)
$$

it holds

$$
\|\mathbf{u}\|_{\mathbf{H}^{2}}+\|\pi\|_{H^{1} \backslash \mathbb{R}} \leq C\|S \mathbf{u}\|, \quad \forall \mathbf{u} \in D(S)
$$

for some positive constant $C$ only depending on $\Omega$ and the spatial dimension.
We begin to report the existence of a weak solution (see [1, Corollary 1.1, Theorem 1.4]).
Proposition 2.1. Suppose $n=2,3$. For any given $T>0$, assume

$$
\begin{align*}
& \mathbf{g} \in L^{2}\left(0, T ; \mathbf{V}^{*}\right)  \tag{2.1}\\
& \mathbf{h} \in L^{2}\left(0, T ; \mathbf{H}^{\frac{3}{2}}(\Gamma)\right),  \tag{2.2}\\
& \mathbf{h}_{t} \in L^{2}\left(0, T ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)  \tag{2.3}\\
& |\mathbf{h}|_{\mathbb{R}^{n}} \leq 1, \quad \text { a.e. on } \Gamma \times[0, T],  \tag{2.4}\\
& \left.\mathbf{d}_{0}\right|_{\Gamma}=\left.\mathbf{h}\right|_{t=0} . \tag{2.5}
\end{align*}
$$

Then for any $\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right) \in \mathbf{H} \times \mathbf{H}^{1}(\Omega)$ with $\left|\mathbf{d}_{0}\right|_{\mathbb{R}^{n}} \leq 1$ almost everywhere in $\Omega$, problem (1.1) -(1.5) admits a weak solution $(\mathbf{v}, \mathbf{d})$ such that

$$
\begin{align*}
& \mathbf{v} \in L^{\infty}(0, T ; \mathbf{H}) \cap L^{2}(0, T ; \mathbf{V}) \\
& \mathbf{d} \in L^{\infty}\left(0, T ; \mathbf{H}^{1}(\Omega)\right) \cap L^{2}\left(0, T ; \mathbf{H}^{2}(\Omega)\right), \\
& |\mathbf{d}(x, t)|_{\mathbb{R}^{n}} \leq 1, \quad \text { a.e. on } \Omega \times[0, T] \tag{2.6}
\end{align*}
$$

If $n=2$, then the weak solution $(\mathbf{v}, \mathbf{d})$ to problem (1.1) -(1.5) is unique. Moreover, we have $(\mathbf{v}, \mathbf{d}) \in C\left([0, T] ; \mathbf{H} \times \mathbf{H}^{1}(\Omega)\right)$.

Remark 2.1. The weak maximum principle (2.6) plays an important role in the analysis of system (1.1)-(1.5) (cf. [20] for the autonomous case). We recall that system (1.1)-(1.5) is a simplified version of the Ericksen-Leslie system for the liquid crystal flow of nematic type, in which the molecule is assumed to be "small" such that the stretching and rotating effects in the fluid are neglected. In particular, when the stretching effect is taken into account (cf. [31]), the weak maximum principle (2.6) fails. The lack of control of the $L_{t}^{\infty} L_{x}^{\infty}$-norm of $\mathbf{d}$ brings extra difficulties in the analysis. For instance, in this case, it is not clear how to define weak solutions (compare with [20, 22]). We refer to [2, 10, 28, 31, 36] for extensive studies (well-posedness, long-time behavior, and so on) on more general liquid crystal systems with stretching terms (see also [3, 37] for the full Ericksen-Leslie system). Whether the results obtained in this paper can be extended to those nonautonomous general liquid crystal systems involving stretching effect remains a challenging open problem.

In order to obtain proper energy inequalities for the system (1.1)-(1.5), we recall that suitable lifting functions were introduced in [5,6] to overcome the technical difficulties related to the timedependent boundary datum for $\mathbf{d}$. The first lifting function $\mathbf{d}_{E}=\mathbf{d}_{E}(x, t)$ is of elliptic type (cf. 6]):

$$
\left\{\begin{array}{l}
-\Delta \mathbf{d}_{E}=\mathbf{0}, \quad \text { in } \Omega \times \mathbb{R}^{+},  \tag{2.7}\\
\mathbf{d}_{E}=\mathbf{h}, \\
\text { on } \Gamma \times \mathbb{R}^{+} .
\end{array}\right.
$$

In particular, we define the lifting function $\mathbf{d}_{E 0}$ for the initial datum:

$$
\left\{\begin{array}{l}
-\Delta \mathbf{d}_{E 0}=\mathbf{0}, \quad \text { in } \Omega  \tag{2.8}\\
\mathbf{d}_{E 0}=\mathbf{d}_{0}, \quad \text { on } \Gamma
\end{array}\right.
$$

Set now

$$
\begin{equation*}
\widehat{\mathbf{d}}=\mathbf{d}-\mathbf{d}_{E} . \tag{2.9}
\end{equation*}
$$

Then system (1.1)-(1.5) can be rewritten into the following form:

$$
\begin{align*}
\mathbf{v}_{t}+\mathbf{v} \cdot \nabla \mathbf{v}-\nu \Delta \mathbf{v}+\nabla \pi & =-\Delta \widehat{\mathbf{d}} \cdot \nabla \mathbf{d}+\mathbf{g}(t)  \tag{2.10}\\
\nabla \cdot \mathbf{v} & =0,  \tag{2.11}\\
\widehat{\mathbf{d}}_{t}+\mathbf{v} \cdot \nabla \mathbf{d} & =\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})-\partial_{t} \mathbf{d}_{E}(t) \tag{2.12}
\end{align*}
$$

with homogeneous Dirichlet boundary conditions and initial conditions

$$
\begin{align*}
& \mathbf{v}=\mathbf{0}, \quad \widehat{\mathbf{d}}=\mathbf{0}, \quad \text { on } \Gamma \times \mathbb{R}^{+},  \tag{2.13}\\
& \left.\mathbf{v}\right|_{t=0}=\mathbf{v}_{0},\left.\quad \widehat{\mathbf{d}}\right|_{t=0}=\mathbf{d}_{0}-\mathbf{d}_{E 0}, \quad \text { in } \Omega . \tag{2.14}
\end{align*}
$$

Note that we have used the well-known identity $\nabla \cdot(\nabla \mathbf{d} \odot \nabla \mathbf{d})=\frac{1}{2} \nabla\left(|\nabla \mathbf{d}|^{2}\right)+\Delta \mathbf{d} \cdot \nabla \mathbf{d}$ to absorb the gradient term into pressure (cf. 20).

Let us introduce the lifted energy

$$
\begin{equation*}
\widehat{\mathcal{E}}(t)=\frac{1}{2}\|\mathbf{v}(t)\|^{2}+\frac{1}{2}\|\nabla \widehat{\mathbf{d}}(t)\|^{2}+\int_{\Omega} F(\mathbf{d}(t)) d x, \quad t \geq 0 . \tag{2.15}
\end{equation*}
$$

Then we can derive the basic energy inequality for system (1.1)-(1.5).
Lemma 2.2. Let the assumptions of Proposition 2.1 be satisfied for all $T>0$. Then, any weak solution which is smooth enough satisfies the following inequality for $t \geq 0$

$$
\begin{equation*}
\frac{d}{d t} \widehat{\mathcal{E}}(t)+\frac{\nu}{2}\|\nabla \mathbf{v}\|^{2}+\frac{1}{2}\|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2} \leq \frac{1}{2}\left\|\partial_{t} \mathbf{d}_{E}\right\|^{2}+C\left\|\partial_{t} \mathbf{d}_{E}\right\|+C\|\mathbf{g}\|_{\mathbf{V}^{*}}^{2}:=r(t), \tag{2.16}
\end{equation*}
$$

where $C$ is a positive constant independent of $\mathbf{v}$ and $\mathbf{d}$.
Proof. Multiplying (2.10) and (2.12) by $\mathbf{v}$ and $-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})$, respectively, integrating over $\Omega$ and adding the results together, we get

$$
\begin{align*}
& \frac{d}{d t}\left(\frac{1}{2}\|\mathbf{v}\|^{2}+\frac{1}{2}\|\nabla \widehat{\mathbf{d}}\|^{2}+\int_{\Omega} F(\mathbf{d}) d x\right)+\nu\|\nabla \mathbf{v}\|^{2}+\|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2} \\
= & \left(\partial_{t} \mathbf{d}_{E}, \Delta \widehat{\mathbf{d}}\right)+(\mathbf{g}, \mathbf{v}) . \tag{2.17}
\end{align*}
$$

In above, we have used the facts $(\mathbf{v} \cdot \nabla \mathbf{v}, \mathbf{v})=(\nabla P, \mathbf{v})=(\mathbf{v} \cdot \nabla \mathbf{d}, \mathbf{f}(\mathbf{d}))=0$ due to the impressibility condition $\nabla \cdot \mathbf{v}=0$. By the Poincaré inequality $\|\mathbf{v}\| \leq C_{P}\|\nabla \mathbf{v}\|$ and (2.6), the right-hand side of (2.17) can be estimated as follows

$$
\begin{aligned}
& \left|\left(\partial_{t} \mathbf{d}_{E}, \Delta \widehat{\mathbf{d}}\right)+(\mathbf{g}, \mathbf{v})\right| \\
\leq & \left|\left(\partial_{t} \mathbf{d}_{E}, \Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\right)\right|+\left|\left(\partial_{t} \mathbf{d}_{E}, \mathbf{f}(\mathbf{d})\right)\right|+|(\mathbf{g}, \mathbf{v})| \\
\leq & \|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|\left\|\partial_{t} \mathbf{d}_{E}\right\|+\|\mathbf{f}(\mathbf{d})\|\left\|\partial_{t} \mathbf{d}_{E}\right\|+\|\mathbf{v}\|_{\mathbf{v}}\|\mathbf{g}\| \mathbf{v}^{*} \\
\leq & \frac{\nu}{2}\|\nabla \mathbf{v}\|^{2}+\frac{1}{2}\|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}+\frac{1}{2}\left\|\partial_{t} \mathbf{d}_{E}\right\|^{2}+C\left\|\partial_{t} \mathbf{d}_{E}\right\|+C\|\mathbf{g}\|_{\mathbf{V}^{*}}^{2} .
\end{aligned}
$$

The proof is complete.
Remark 2.2. We fix the calculations in [6, Lemma 2] where the term $\left(\partial_{t} \mathbf{d}_{E}, \Delta \widehat{\mathbf{d}}\right)$ is missing. Though it does not affect the proof of existence result, it does have influence on the long-time behavior of global solutions (especially on the convergence rate).

Let us now introduce the following (Banach) spaces of translation bounded functions

$$
\begin{aligned}
L_{t b}^{q}(0,+\infty ; X):= & \left\{\mathbf{h} \in L_{l o c}^{q}([0,+\infty) ; X):\right. \\
& \left.\|\mathbf{h}\|_{L_{t b}^{q}(0,+\infty ; X)}^{q}:=\sup _{t \geq 0} \int_{t}^{t+1}\|\mathbf{h}(\tau)\|_{X}^{q} d \tau<+\infty\right\}
\end{aligned}
$$

where $X$ is a (real) Banach space and $q \in[1,+\infty)$ is given.
From the basic energy inequality (2.16), through a suitable Galerkin approximation scheme, one can derive uniform-in-time estimates for any weak solution (the proof is a minor modification of [1, Lemma 1.2, Remark 1.1]).

Lemma 2.3. Let the assumptions of Proposition 2.1 hold for all $T>0$. In addition, suppose that

$$
\begin{align*}
& \mathbf{g} \in L^{2}\left(0,+\infty ; \mathbf{V}^{*}\right)  \tag{2.18}\\
& \mathbf{h} \in L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{3}{2}}(\Gamma)\right)  \tag{2.19}\\
& \mathbf{h}_{t} \in L^{2}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right) \cap L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right) \tag{2.20}
\end{align*}
$$

Then a weak solution ( $\mathbf{v}, \mathbf{d}$ ) to problem (1.1) -(1.5) given by Proposition 2.1 is a global solution on $[0,+\infty)$ and fulfills the following uniform bounds

$$
\begin{aligned}
& \|\mathbf{v}(t)\| \leq C, \quad\|\mathbf{d}(t)\|_{\mathbf{H}^{1}} \leq C, \quad \forall t \geq 0 \\
& \int_{0}^{t}\left(\nu\|\nabla \mathbf{v}(\tau)\|^{2}+\|(\Delta \mathbf{d}-\mathbf{f}(\mathbf{d}))(\tau)\|^{2}\right) d \tau \leq C, \quad \forall t \geq 0
\end{aligned}
$$

Here $C$ is a positive constant depending on $\left\|\mathbf{v}_{0}\right\|,\left\|\mathbf{d}_{0}\right\|_{\mathbf{H}^{1}},\|\mathbf{g}\|_{L^{2}\left(0,+\infty ; \mathbf{V}^{*}\right)},\|\mathbf{h}\|_{L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{3}{2}}(\Gamma)\right)}$, $\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}$ and $\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}$.

Next, we introduce the lifting function $\mathbf{d}_{P}=\mathbf{d}_{P}(x, t)$ of parabolic type, which satisfies

$$
\left\{\begin{array}{l}
\partial_{t} \mathbf{d}_{P}-\Delta \mathbf{d}_{P}=\mathbf{0}, \quad \text { in } \Omega \times \mathbb{R}^{+}  \tag{2.21}\\
\mathbf{d}_{P}=\mathbf{h}, \quad \text { on } \Gamma \times \mathbb{R}^{+} \\
\mathbf{d}_{P}(0)=\mathbf{d}_{E 0}, \quad \text { in } \Omega
\end{array}\right.
$$

The motivation of introducing the parabolic lifting function $\mathbf{d}_{P}$ is that we now have, by definition, $\Delta\left(\mathbf{d}-\mathbf{d}_{P}\right)-\left.\mathbf{f}(\mathbf{d})\right|_{\Gamma}=\mathbf{0}$. This fact is crucial when we use integration by parts to derive some higher-order differential inequalities of system (1.1)-(1.5) (cf. [6, 11]). We note that $\mathbf{d}_{P}$ in (2.21) is different from the one introduced in [6] as they have different initial values. Both choices are valid for the proof of existence result, but the current definition of $\mathbf{d}_{P}$ is necessary for the study of long-time behavior. Denote

$$
\widetilde{\mathbf{d}}=\mathbf{d}-\mathbf{d}_{P}
$$

System (1.1)-(1.5) can now be rewritten into the following form:

$$
\begin{align*}
\mathbf{v}_{t}+\mathbf{v} \cdot \nabla \mathbf{v}-\nu \Delta \mathbf{v}+\nabla P & =-\Delta \mathbf{d} \cdot \nabla \mathbf{d}+\mathbf{g}(t)  \tag{2.22}\\
\nabla \cdot \mathbf{v} & =0  \tag{2.23}\\
\widetilde{\mathbf{d}}_{t}+\mathbf{v} \cdot \nabla \mathbf{d} & =\Delta \tilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}) \tag{2.24}
\end{align*}
$$

with homogeneous Dirichlet boundary conditions and initial conditions

$$
\begin{align*}
& \mathbf{v}=\mathbf{0}, \quad \tilde{\mathbf{d}}=\mathbf{0}, \quad \text { on } \Gamma \times \mathbb{R}^{+},  \tag{2.25}\\
& \left.\mathbf{v}\right|_{t=0}=\mathbf{v}_{0},\left.\quad \tilde{\mathbf{d}}\right|_{t=0}=\mathbf{d}_{0}-\mathbf{d}_{E 0}, \quad \text { in } \Omega \tag{2.26}
\end{align*}
$$

In the sequel, we shall frequently use the following lemma (cf. [6])
Lemma 2.4. The following equivalence between norms hold

$$
\begin{aligned}
& \|\mathbf{v}\|_{\mathbf{H}^{1}} \approx\|\nabla \mathbf{v}\|, \quad\|\widetilde{\mathbf{d}}\|_{\mathbf{H}^{1}} \approx\|\nabla \widetilde{\mathbf{d}}\|, \quad \text { in } \mathbf{H}_{0}^{1}(\Omega) \\
& \|\mathbf{v}\|_{\mathbf{H}^{2}} \approx\|\Delta \mathbf{v}\|, \quad\|\widetilde{\mathbf{d}}\|_{\mathbf{H}^{2}} \approx\|\Delta \widetilde{\mathbf{d}}\|, \quad \text { in } \mathbf{H}_{0}^{1}(\Omega) \cap \mathbf{H}^{2}(\Omega) \\
& \|\widetilde{\mathbf{d}}\|_{\mathbf{H}^{3}} \approx\|\nabla(\Delta \widetilde{\mathbf{d}})\|+\|\Delta \widetilde{\mathbf{d}}\|, \quad \text { in } \mathbf{H}_{0}^{1}(\Omega) \cap \mathbf{H}^{3}(\Omega)
\end{aligned}
$$

If $\mathbf{d}$ and $\mathbf{d}_{P}$ are functions that are smooth enough and $|\mathbf{d}|_{\mathbb{R}^{n}} \leq 1,\left|\mathbf{d}_{P}\right|_{\mathbb{R}^{n}} \leq 1$, then we have

$$
\begin{aligned}
\|\Delta \mathbf{d}\| & \leq\left\|\Delta \mathbf{d}_{P}\right\|+\|\Delta \tilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+C \\
\|\nabla \Delta \mathbf{d}\| & \leq\left\|\nabla \Delta \mathbf{d}_{P}\right\|+\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|+C\|\nabla \mathbf{d}\|
\end{aligned}
$$

where $C$ is a positive constant independent of $\mathbf{d}$ and $\mathbf{d}_{P}$.
Let us introduce the quantity

$$
\mathcal{A}_{P}(t)=\|\nabla \mathbf{v}(t)\|^{2}+\|\Delta \widetilde{\mathbf{d}}(t)-\mathbf{f}(\mathbf{d}(t))\|^{2}, \quad t \geq 0
$$

Lemma 2.5. Let $n=2$ and let the assumptions of Lemma 2.3 hold. If the weak solution $(\mathbf{v}, \mathbf{d})$ is smooth enough then it satisfies the following inequality

$$
\begin{equation*}
\frac{d}{d t} \mathcal{A}_{P}(t) \leq C\left(\mathcal{A}_{P}^{2}(t)+\mathcal{A}_{P}(t)+R_{1}(t)\right) \tag{2.27}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{1}(t)=\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{4}+\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{2}+\left\|\nabla \Delta \mathbf{d}_{P}(t)\right\|^{2}+\|\mathbf{g}(t)\|^{2} \tag{2.28}
\end{equation*}
$$

Here $C$ is a positive constant depending on $\nu,\left\|\mathbf{v}_{0}\right\|,\left\|\mathbf{d}_{0}\right\|_{\mathbf{H}^{1}},\|\mathbf{g}\|_{L^{2}\left(0,+\infty ; \mathbf{V}^{*}\right)},\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)^{\prime}}$, $\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}$ and $\|\mathbf{h}\|_{L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{3}{2}}(\Gamma)\right)}$.

Proof. Taking the time derivative of $\mathcal{A}_{P}(t)$, we obtain by a direct calculation that

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t} \mathcal{A}_{P}(t)+\left(\nu\|S \mathbf{v}\|^{2}+\| \nabla\left(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}) \|^{2}\right)\right. \\
= & -(S \mathbf{v}, \mathbf{v} \cdot \nabla \mathbf{v})+(S \mathbf{v}, \mathbf{g})-(S \mathbf{v}, \Delta \mathbf{d} \cdot \nabla \mathbf{d})-(\nabla(\mathbf{v} \cdot \nabla \mathbf{d}), \nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))) \\
& -\left(\mathbf{f}^{\prime}(\mathbf{d}) \mathbf{d}_{t}, \Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\right) \\
:= & \sum_{j=1}^{5} I_{j} \tag{2.29}
\end{align*}
$$

To get this identity we have used the fact that $\Delta \widetilde{\mathbf{d}}-\left.\mathbf{f}(\mathbf{d})\right|_{\Gamma}=\mathbf{0}$ as well as $\left(S \mathbf{v}, \mathbf{v}_{t}\right)=\left(-\Delta \mathbf{v}, \mathbf{v}_{t}\right)$. It is not difficult to see that

$$
\begin{aligned}
\left|I_{1}\right| & \leq\|S \mathbf{v}\|\|\mathbf{v}\|_{\mathbf{L}^{4}}\|\nabla \mathbf{v}\|_{\mathbf{L}^{4}} \\
& \leq C\|S \mathbf{v}\|\left(\|\nabla \mathbf{v}\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\right)\left(\|\Delta \mathbf{v}\|^{\frac{1}{2}}\|\nabla \mathbf{v}\|^{\frac{1}{2}}\right)
\end{aligned}
$$

$$
\begin{gathered}
\leq C\|S \mathbf{v}\|^{\frac{3}{2}}\|\nabla \mathbf{v}\| \leq \varepsilon\|S \mathbf{v}\|^{2}+C\|\nabla \mathbf{v}\|^{2} \\
\left|I_{2}\right| \leq \varepsilon\|S \mathbf{v}\|^{2}+C\|\mathbf{g}\|^{2} .
\end{gathered}
$$

For $I_{3}$, we have

$$
\begin{aligned}
\left|I_{3}\right| & =\left|(S \mathbf{v},(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})) \cdot \nabla \mathbf{d})+(S \mathbf{v}, \mathbf{f}(\mathbf{d}) \cdot \nabla \mathbf{d})+\left(S \mathbf{v}, \partial_{t} \mathbf{d}_{P} \cdot \nabla \mathbf{d}\right)\right| \\
& \leq\|S \mathbf{v}\|\|\nabla \mathbf{d}\|_{\mathbf{L}^{4}}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{4}}+\|S \mathbf{v}\|\|\nabla \mathbf{d}\|_{\mathbf{L}^{\infty}}\left\|\partial_{t} \mathbf{d}_{P}\right\| \\
& \leq \varepsilon\|S \mathbf{v}\|^{2}+C\|\nabla \mathbf{d}\|_{\mathbf{L}^{4}}^{2}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{4}}^{2}+C\|\nabla \mathbf{d}\|_{\mathbf{L}^{\infty}}^{2}\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2} .
\end{aligned}
$$

On account of Lemma 2.3, we infer from the Sobolev embedding theorems that

$$
\begin{aligned}
&\|\nabla \mathbf{d}\|_{\mathbf{L}^{4}}^{2} \leq C\|\Delta \mathbf{d}\|\|\nabla \mathbf{d}\|+C\|\nabla \mathbf{d}\|^{2} \leq C\|\Delta \widetilde{\mathbf{d}}\|+C\left\|\partial_{t} \mathbf{d}_{P}\right\|+C \\
& \leq C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+C\left\|\partial_{t} \mathbf{d}_{P}\right\|+C \\
&\|\nabla \mathbf{d}\|_{\mathbf{L}^{\infty}}^{2} \leq C\|\nabla \Delta \mathbf{d}\|\|\nabla \mathbf{d}\|+C\|\nabla \mathbf{d}\|^{2} \\
& \leq C\left\|\nabla \Delta \mathbf{d}_{P}\right\|+C(1+\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|) \\
&\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{4}}^{2} \leq C\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\| .
\end{aligned}
$$

Using the above estimates, we obtain the estimates for $I_{3}$ and $I_{4}$ :

$$
\begin{aligned}
\left|I_{3}\right| \leq & \varepsilon\|S \mathbf{v}\|^{2}+C\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|\left(\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\left\|\partial_{t} \mathbf{d}_{P}\right\|+1\right) \\
& +C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}\left(\left\|\nabla \Delta \mathbf{d}_{P}\right\|+\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|+1\right) \\
\leq & \varepsilon\|S \mathbf{v}\|^{2}+\varepsilon\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{4}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2} \\
& +C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}\left(\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}+\left\|\nabla \Delta \mathbf{d}_{P}\right\|+1\right), \\
\left|I_{4}\right| \leq & \|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|\left(\|\nabla \mathbf{v}\|_{\mathbf{L}^{4}}\|\nabla \mathbf{d}\|_{\mathbf{L}^{4}}+\|\mathbf{v}\|_{\mathbf{L}^{\infty}}\|\mathbf{d}\|_{\mathbf{H}^{2}}\right) \\
\leq & \varepsilon\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2}+C\|\nabla \mathbf{v}\|_{\mathbf{L}^{4}}^{2}\|\nabla \mathbf{d}\|_{\mathbf{L}^{4}}^{2}+C\|\mathbf{v}\|_{\mathbf{L}^{\infty}}^{2}\left(\|\Delta \mathbf{d}\|^{2}+1\right) \\
\leq & \varepsilon\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2}+C\|\Delta \mathbf{v}\|\|\nabla \mathbf{v}\|\left(\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\left\|\partial_{t} \mathbf{d}_{P}\right\|+1\right) \\
& +C\|\Delta \mathbf{v}\|\|\mathbf{v}\|\left(\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}+\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}+1\right) \\
\leq & \varepsilon\|S \mathbf{v}\|^{2}+\varepsilon\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2}+C\|\nabla \mathbf{v}\|^{4}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{4} \\
& +C\|\nabla \mathbf{v}\|^{2}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}+\left\|\partial_{t} \mathbf{d}_{P}\right\|^{4} .
\end{aligned}
$$

We now observe that

$$
\begin{equation*}
I_{5}=-\left(\mathbf{f}^{\prime}(\mathbf{d}) \widetilde{\mathbf{d}}_{t}, \Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\right)-\left(\mathbf{f}^{\prime}(\mathbf{d}) \partial_{t} \mathbf{d}_{P}, \Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\right):=I_{5 a}+I_{5 b} . \tag{2.30}
\end{equation*}
$$

Recalling (2.24), we have

$$
\begin{align*}
\left|I_{5 a}\right| & =\left|\left(\mathbf{f}^{\prime}(\mathbf{d})(\mathbf{v} \cdot \nabla) \mathbf{d}, \Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\right)-\left(\mathbf{f}^{\prime}(\mathbf{d})(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})), \Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\right)\right| \\
& \leq\left\|\mathbf{f}^{\prime}(\mathbf{d})\right\|_{\mathbf{L}^{\infty}}\left(\|\mathbf{v}\|_{\mathbf{L}^{4}}\|\nabla \mathbf{d}\|_{\mathbf{L}^{4}}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}\right) \\
& \leq C\|\nabla \mathbf{v}\|\|\mathbf{v}\|\|\nabla \mathbf{d}\|_{\mathbf{L}^{4}}^{2}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2} \\
& \leq C\|\nabla \mathbf{v}\|^{2}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}+C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}, \\
\left|I_{5 b}\right| & \leq\left\|\mathbf{f}^{\prime}(\mathbf{d})\right\|_{\mathbf{L}^{\infty}}\left\|\partial_{t} \mathbf{d}_{P}\right\|\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\| \leq C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}+C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2} . \tag{2.31}
\end{align*}
$$

Finally, collecting the above estimates and taking $\varepsilon$ sufficiently small, we deduce that

$$
\begin{aligned}
& \frac{d}{d t} \mathcal{A}_{P}(t)+\left(\nu\|S \mathbf{v}\|^{2}+\|\nabla(\Delta \tilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2}\right) \\
\leq & C\left(\mathcal{A}_{P}^{2}(t)+\mathcal{A}_{P}(t)\right)+C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}\left(\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}+\left\|\nabla \Delta \mathbf{d}_{P}\right\|+1\right)+\|\mathbf{g}\|^{2},
\end{aligned}
$$

which easily implies the inequality (2.27).
Taking advantage of Lemmas 2.5, 6.1 and 6.2, one can deduce the following results on the regularity of weak solutions as well as the existence of strong solutions to system (1.1)-(1.5) in 2 D .

Theorem 2.1. Let $n=2$ and let the assumptions of Proposition 2.1 hold for all $T>0$. In addition, suppose that

$$
\begin{align*}
& \mathbf{g} \in L^{2}(0,+\infty ; \mathbf{H})  \tag{2.32}\\
& \mathbf{h} \in L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{5}{2}}(\Gamma)\right)  \tag{2.33}\\
& \mathbf{h}_{t} \in L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right) \cap L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right) \tag{2.34}
\end{align*}
$$

(i) System (1.1)-(1.5) admits a unique global weak solution $(\mathbf{v}, \mathbf{d})$ satisfying

$$
\begin{aligned}
& \|\mathbf{v}(t)\|_{\mathbf{v}} \leq C\left(1+t^{-1}\right), \quad\|\mathbf{d}(t)\|_{\mathbf{H}^{2}} \leq C\left(1+t^{-1}\right), \quad \forall t>0 \\
& \int_{\delta}^{t}\left(\|\mathbf{v}(\tau)\|_{\mathbf{H}^{2}}^{2}+\|\mathbf{d}(\tau)\|_{\mathbf{H}^{3}}^{2}\right) d \tau \leq C\left(1+\delta^{-1}\right) T, \quad t \in[\delta, T]
\end{aligned}
$$

where $C$ is a positive constant depending on $\nu,\left\|\mathbf{v}_{0}\right\|,\left\|\mathbf{d}_{0}\right\|_{\mathbf{H}^{1}},\|\mathbf{g}\|_{L^{2}(0,+\infty ; \mathbf{H})},\|\mathbf{h}\|_{L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{5}{2}}(\Gamma)\right)^{\prime}}$, $\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)},\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}$.
(ii) If $\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right) \in \mathbf{V} \times \mathbf{H}^{2}(\Omega)$, then problem (1.1) -(1.5) admits a unique global strong solution $(\mathbf{v}, \mathbf{d})$ satisfying

$$
\begin{align*}
& \|\mathbf{v}(t)\|_{\mathbf{V}} \leq C, \quad\|\mathbf{d}(t)\|_{\mathbf{H}^{2}} \leq C, \quad \forall t \geq 0  \tag{2.35}\\
& \int_{0}^{t}\left(\|\mathbf{v}(\tau)\|_{\mathbf{H}^{2}}^{2}+\|\mathbf{d}(\tau)\|_{\mathbf{H}^{3}}^{2}\right) d \tau \leq C T, \quad t \in[0, T] \tag{2.36}
\end{align*}
$$

where $C$ is a positive constant depending on $\nu,\left\|\mathbf{v}_{0}\right\|_{\mathbf{V}},\left\|\mathbf{d}_{0}\right\|_{\mathbf{H}^{2}},\|\mathbf{g}\|_{L^{2}(0,+\infty ; \mathbf{H})},\|\mathbf{h}\|_{L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{5}{2}}(\Gamma)\right)}$, $\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)},\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}$.

Remark 2.3. Lemma 2.3 and Theorem 2.1 still hold when $\mathbf{g}$ and $\mathbf{h}_{t}$ are translation bounded with respect to time (see [1]).

Next, we consider the 3D case. Instead of Lemma 2.5, we have the following higher-order energy inequality

Lemma 2.6. Let $n=3$ and let the assumptions of Lemma 2.3 hold. If a weak solution $(\mathbf{v}, \mathbf{d})$ is smooth enough then it satisfies the following inequality

$$
\begin{align*}
& \frac{d}{d t} \widetilde{\mathcal{A}}_{P}(t)+\left(\nu-c_{1} \widetilde{\mathcal{A}}_{P}(t)\right)\|S \mathbf{v}\|^{2}+\left(1-\frac{c_{2}}{\nu^{\frac{1}{2}}} \widetilde{\mathcal{A}}_{P}(t)\right)\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2} \\
\leq & C\left(1+\nu^{-2}\right)\left(\mathcal{A}_{P}(t)+R_{2}(t)\right), \quad t \geq 0 \tag{2.37}
\end{align*}
$$

where $\widetilde{\mathcal{A}}_{P}(t)=\mathcal{A}_{P}(t)+1$ and

$$
\begin{equation*}
R_{2}(t)=\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{2}+\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{6}+\left\|\nabla \partial_{t} \mathbf{d}_{P}(t)\right\|^{2}+\|\mathbf{g}(t)\|^{2} \tag{2.38}
\end{equation*}
$$

Here $c_{1}, c_{2}, C$ are positive constants that may depend on $\left\|\mathbf{v}_{0}\right\|,\left\|\mathbf{d}_{0}\right\|_{\mathbf{H}^{1}}$ and on $\|\mathbf{g}\|_{L^{2}\left(0,+\infty ; \mathbf{V}^{*}\right)}$, $\|\mathbf{h}\|_{L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{3}{2}}(\Gamma)\right)},\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)},\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}$, but they are independent of $\nu$.
Proof. We estimate the right-hand side of (2.29) by using the 3D version of Sobolev embedding theorems. We have

$$
\begin{aligned}
\left|I_{1}\right| \leq & \|S \mathbf{v}\|\|\mathbf{v}\|_{\mathbf{L}^{6}}\|\nabla \mathbf{v}\|_{\mathbf{L}^{3}} \\
\leq & C\|S \mathbf{v}\|\|\nabla \mathbf{v}\|\left(\|\Delta \mathbf{v}\|^{\frac{1}{2}}\|\nabla \mathbf{v}\|^{\frac{1}{2}}\right) \\
\leq & C\|S \mathbf{v}\|^{\frac{3}{2}}\|\nabla \mathbf{v}\|^{\frac{3}{2}} \leq \frac{1}{2}\|\nabla \mathbf{v}\|^{\frac{4}{3}}\|S \mathbf{v}\|^{2}+C\|\nabla \mathbf{v}\|^{2} \\
& \quad\left|I_{2}\right| \leq \frac{\nu}{8}\|S \mathbf{v}\|^{2}+\frac{2}{\nu}\|\mathbf{g}\|^{2}
\end{aligned}
$$

Recalling that $\|\mathbf{d}\|_{\mathbf{H}^{1}} \leq C$ (cf. Lemma 2.3), from the Sobolev embedding theorems as well as Agmon's inequality in dimension three, we infer

$$
\begin{aligned}
&\|\nabla \mathbf{d}\|_{\mathbf{L}^{3}} \leq C\|\Delta \mathbf{d}\|^{\frac{1}{2}}\|\nabla \mathbf{d}\|^{\frac{1}{2}}+C\|\nabla \mathbf{d}\| \leq C\left(\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\left\|\partial_{t} \mathbf{d}_{P}\right\|\right)^{\frac{1}{2}}+C \\
&\|\nabla \mathbf{d}\|_{\mathbf{L}^{6}} \leq C\|\Delta \mathbf{d}\|+C\|\nabla \mathbf{d}\| \leq C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+C\left\|\partial_{t} \mathbf{d}_{P}\right\|+C \\
&\|\nabla \mathbf{d}\|_{\mathbf{L}^{\infty}} \leq C\|\nabla \mathbf{d}\|_{\mathbf{H}^{1}}^{\frac{1}{2}}\|\nabla \mathbf{d}\|_{\mathbf{H}^{2}}^{\frac{1}{2}} \leq C\left(\|\nabla \Delta \mathbf{d}\|^{\frac{1}{2}}\|\Delta \mathbf{d}\|^{\frac{1}{2}}+\|\Delta \mathbf{d}\|+1\right) \\
& \leq C\left(\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}}\| \| \Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\left\|^{\frac{1}{2}}+\right\| \Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\left\|^{\frac{1}{2}}\right\| \nabla \Delta \mathbf{d}_{P} \|^{\frac{1}{2}}\right. \\
&+\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}}\left\|\Delta \mathbf{d}_{P}\right\|^{\frac{1}{2}}+\left\|\nabla \Delta \mathbf{d}_{P}\right\|^{\frac{1}{2}}\left\|\Delta \mathbf{d}_{P}\right\|^{\frac{1}{2}}+\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \\
&\left.+\left\|\nabla \Delta \mathbf{d}_{P}\right\|^{\frac{1}{2}}+\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\left\|\Delta \mathbf{d}_{P}\right\|+1\right) \\
&\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{3}} \leq C\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{\frac{1}{2}}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\left|I_{3}\right| \leq & \|S \mathbf{v}\|\|\nabla \mathbf{d}\|_{\mathbf{L}^{6}}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|_{\mathbf{L}^{3}}+\|S \mathbf{v}\|\|\nabla \mathbf{d}\|_{\mathbf{L}^{\infty}}\left\|\partial_{t} \mathbf{d}_{P}\right\| \\
\leq & C\|S \mathbf{v}\|\left(\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\left\|\partial_{t} \mathbf{d}_{P}\right\|+1\right)\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{\frac{1}{2}} \\
& +C\|S \mathbf{v}\|\left\|\partial_{t} \mathbf{d}_{P}\right\|\left(\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{\frac{1}{2}}+\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{\frac{1}{2}}\left\|\nabla \partial_{t} \mathbf{d}_{P}\right\|^{\frac{1}{2}}\right. \\
& +\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}}\left\|\partial_{t} \mathbf{d}_{P}\right\|^{\frac{1}{2}}+\left\|\nabla \partial_{t} \mathbf{d}_{P}\right\|^{\frac{1}{2}}\left\|\partial_{t} \mathbf{d}_{P}\right\|^{\frac{1}{2}}+\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{\frac{1}{2}} \\
& \left.+\left\|\nabla \partial_{t} \mathbf{d}_{P}\right\|^{\frac{1}{2}}+\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\left\|\partial_{t} \mathbf{d}_{P}\right\|+1\right) \\
\leq & \left(\frac{\nu}{8}+\frac{1}{2}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}\right)\|S \mathbf{v}\|^{2}+\frac{1}{8}\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2} \\
& +C\left(1+\nu^{-2}\right)\left(\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}+\left\|\partial_{t} \mathbf{d}_{P}\right\|^{6}+\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}+\left\|\nabla \Delta \mathbf{d}_{P}\right\|^{2}\right) \\
\left|I_{4}\right| \leq & \|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|\left(\|\nabla \mathbf{v}\|_{\mathbf{L}^{3}}\|\nabla \mathbf{d}\|_{\mathbf{L}^{6}}+\|\mathbf{v}\|_{\mathbf{L}^{\infty}}\|\mathbf{d}\|_{\mathbf{H}^{2}}\right) \\
\leq & C\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|\|\nabla \mathbf{v}\|^{\frac{1}{2}}\|\Delta \mathbf{v}\|^{\frac{1}{2}}\left(\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\left\|\partial_{t} \mathbf{\mathbf { d } _ { P }}\right\|+1\right) \\
& \leq\left(\frac{\nu}{8}+\frac{1}{2}\|\nabla \mathbf{v}\|^{2}\right)\|S \mathbf{v}\|^{2}+\left(\frac{1}{8}+\frac{1}{2 \nu^{\frac{1}{2}}}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}\right)\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2}
\end{aligned}
$$

$$
\begin{aligned}
+ & C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}+C\left(1+\nu^{-1}\right)\|\nabla \mathbf{v}\|^{2}+C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{4} \\
\left|I_{5 a}\right| & \leq\left\|\mathbf{f}^{\prime}(\mathbf{d})\right\|_{\mathbf{L}^{\infty}}\left(\|\mathbf{v}\|_{\mathbf{L}^{6}}\|\nabla \mathbf{d}\|_{\mathbf{L}^{3}}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}\right) \\
& \leq C\|\nabla \mathbf{v}\|^{2}\|\nabla \mathbf{d}\|_{\mathbf{L}^{3}}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2} \\
& \leq C\|\Delta \mathbf{v}\|\|\mathbf{v}\|\left(\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|+\left\|\partial_{t} \mathbf{d}_{P}\right\|+1\right)+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2} \\
& \leq \frac{\nu}{8}\|S \mathbf{v}\|^{2}+C\left(1+\nu^{-1}\right)\left(\|\nabla \mathbf{v}\|^{2}+\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}+\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}\right)
\end{aligned}
$$

We observe that $I_{5 b}$ can be estimated as in (2.31). Then, collecting all the estimates of $I_{j}$, we have

$$
\begin{aligned}
& \frac{d}{d t} \mathcal{A}_{P}(t)+\left(\nu-\|\nabla \mathbf{v}\|^{\frac{4}{3}}-\|\nabla \mathbf{v}\|^{2}\right)\|S \mathbf{v}\|^{2} \\
& +\left(1-\frac{1}{\nu^{\frac{1}{2}}}\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}\right)\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2} \\
\leq & C\left(1+\nu^{-2}\right)\left(\mathcal{A}_{P}(t)+\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}+\left\|\partial_{t} \mathbf{d}_{P}\right\|^{6}+\left\|\nabla \Delta \mathbf{d}_{P}\right\|^{2}+\|\mathbf{g}\|^{2}\right) .
\end{aligned}
$$

As a result, there exist constants $c_{1}, c_{2}>0$ independent of $\nu$ such that the following inequality holds

$$
\begin{aligned}
& \frac{d}{d t} \mathcal{A}_{P}(t)+\left(\nu-c_{1} \widetilde{\mathcal{A}}_{P}(t)\right)\|S \mathbf{v}\|^{2}+\left(1-\frac{c_{2}}{\nu^{\frac{1}{2}}} \widetilde{\mathcal{A}}_{P}(t)\right)\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2} \\
\leq & C\left(1+\nu^{-2}\right)\left(\mathcal{A}_{P}(t)+\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}+\left\|\partial_{t} \mathbf{d}_{P}\right\|^{6}+\left\|\nabla \Delta \mathbf{d}_{P}\right\|^{2}+\|\mathbf{g}\|^{2}\right),
\end{aligned}
$$

which implies (2.37).

On account of Lemma 2.6, one can deduce that system (1.1)-(1.5) admits at least one global strong solution, provided that the viscosity is large enough (see [6, Theorem 7] for the case $\mathbf{g}=\mathbf{0}$, cf. also 20,35 for the autonomous case). We just report a result under weaker assumptions than that in [6] and omit the detailed proof.

Theorem 2.2. Let $n=3$ and assume that (2.32) (2.34) and (2.4) are satisfied. For any $\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right) \in \mathbf{V} \times \mathbf{H}^{2}(\Omega)$ satisfying (2.5) and $\left|\mathbf{d}_{0}\right|_{\mathbb{R}^{3}} \leq 1$, there exists a $\nu_{0}>0$, depending on $\left\|\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right)\right\|_{\mathbf{V} \times \mathbf{H}^{2}}$ and $\|\mathbf{g}\|_{L^{2}(0,+\infty ; \mathbf{H})},\|\mathbf{h}\|_{L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{5}{2}}(\Gamma)\right)},\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)},\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}$, such that, for any $\nu \geq \nu_{0}$, problem (1.1) -(1.5) admits a global strong solution $(\mathbf{v}, \mathbf{d})$ which satisfies the same uniform estimates as in the 2D case (cf. (2.35) and (2.36)).

Remark 2.4. When $n=3$, the weak-strong uniqueness result obtained in [6, Theorem 7] still holds in our case. Thus, the global strong solution ( $\mathbf{v}, \mathbf{d}$ ) obtained in Theorem 2.2 is unique.

## 3 Extended Łojasiewicz-Simon type inequality

For all $\mathbf{d} \in \mathcal{N}:=\left\{\phi \in \mathbf{H}^{1}(\Omega):\left.\phi\right|_{\Gamma}=\mathbf{h}_{\infty}\right\}$, where $\mathbf{h}_{\infty} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ is given, we consider the functional

$$
\begin{equation*}
E(\mathbf{d})=\frac{1}{2}\|\nabla \mathbf{d}\|^{2}+\int_{\Omega} F(\mathbf{d}) d x \tag{3.1}
\end{equation*}
$$

It is straightforward to verify that

Lemma 3.1. If $\psi \in \mathbf{H}^{1}(\Omega)$ is a weak solution to the elliptic problem

$$
\left\{\begin{array}{c}
-\Delta \phi+\mathbf{f}(\phi)=\mathbf{0}  \tag{3.2}\\
\left.\phi\right|_{\Gamma}=\mathbf{h}_{\infty}
\end{array}\right.
$$

then $\psi$ is a critical point of the functional $E(\mathbf{d})$ in $\mathcal{N}$. Conversely, if $\psi$ is a critical point of the functional $E(\mathbf{d})$ in $\mathcal{N}$, then $\psi$ is a weak solution to problem (3.2).

Remark 3.1. Due to the elliptic regularity theory, if $\mathbf{h}_{\infty}$ is more regular, then $\psi$ is more regular. For instance, if $\mathbf{h}_{\infty} \in \mathbf{H}^{\frac{3}{2}}(\Gamma)$, then $\psi \in \mathbf{H}^{2}(\Omega)$.

Then we have
Lemma 3.2. Suppose that $\psi$ is a critical point of $E(\mathbf{d})$ in $\mathcal{N}$. Then there exist constants $\beta_{1}>0$, $\theta \in\left(0, \frac{1}{2}\right)$ depending on $\psi$ such that, for any $\mathbf{w} \in \mathcal{N}$ that satisfies $\|\mathbf{w}-\psi\|_{\mathbf{H}^{1}}<\beta_{1}$, there holds

$$
\begin{equation*}
\|-\Delta \mathbf{w}+\mathbf{f}(\mathbf{w})\|_{\mathbf{H}^{-1}} \geq|E(\mathbf{w})-E(\psi)|^{1-\theta} \tag{3.3}
\end{equation*}
$$

Remark 3.2. The above lemma can be viewed as an extended version of Simon's result [30] for scalar function under $L^{2}$-norm. We can refer to [13, Chapter 2, Theorem 5.2], in which the vector case subject to homogeneous Dirichlet boundary condition was considered. We observe that the result can be easily proved by modifying the argument in [13] using a simple transformation (cf. also [35, Remark 2.1]).

The Łojasiewicz-Simon type inequality (3.3) only applies to proper perturbations of the critical point of energy $E$ in the set $\mathcal{N}$ and it is not enough for our evolutionary problem (1.1)(1.5), whose boundary datum is time-dependent (not necessary in $\mathcal{N}$ ). In order to overcome this difficulty, we prove the following extended result that also involves the perturbation of boundary:

Theorem 3.1. Suppose that $\psi$ is a critical point of $E(\mathbf{d})$ in $\mathcal{N}$. Then there exists a constant $\beta \in(0,1)$ depending on $\psi$ such that, for any $\mathbf{d} \in \mathbf{H}^{1}(\Omega)$ satisfying $\|\mathbf{d}-\psi\|_{\mathbf{H}^{1}}<\beta$, there holds

$$
\begin{equation*}
C\left(\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}+\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}\right)+\|-\Delta \mathbf{d}+\mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}} \geq|E(\mathbf{d})-E(\psi)|^{1-\theta} \tag{3.4}
\end{equation*}
$$

where $\theta \in\left(0, \frac{1}{2}\right)$ is the same constant as in Lemma 3.2, while $C$ is a positive constant depending on $\psi$.

Proof. For any $\mathbf{d} \in \mathbf{H}^{1}(\Omega)$, we have that $\Delta \mathbf{d} \in \mathbf{H}^{-1}(\Omega)$. Then we consider the elliptic boundary value problem

$$
\left\{\begin{array}{l}
\Delta \mathbf{w}=\Delta \mathbf{d}  \tag{3.5}\\
\left.\mathbf{w}\right|_{\Gamma}=\mathbf{h}_{\infty}
\end{array}\right.
$$

It easily follows from the elliptic regularity theory (cf. e.g., [32, Proposition 5.1.7]) that

$$
\begin{equation*}
\|\mathbf{w}-\mathbf{d}\|_{\mathbf{H}^{1}} \leq C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \tag{3.6}
\end{equation*}
$$

which implies

$$
\begin{aligned}
\|\mathbf{w}-\psi\|_{\mathbf{H}^{1}} & \leq\|\mathbf{w}-\mathbf{d}\|_{\mathbf{H}^{1}}+\|\mathbf{d}-\psi\|_{\mathbf{H}^{1}} \\
& \leq C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}+\|\mathbf{d}-\psi\|_{\mathbf{H}^{1}}
\end{aligned}
$$

$$
\begin{equation*}
\leq C\|\mathbf{d}-\psi\|_{\mathbf{H}^{1}} . \tag{3.7}
\end{equation*}
$$

Let $\beta_{1}$ be the constant in Lemma 3.2. We infer from the above inequality that if $\beta \in(0,1)$ is chosen sufficiently small, then we have $\|\mathbf{w}-\psi\|_{\mathbf{H}^{1}}<\beta_{1}$. As a consequence of Lemma 3.2, we have

$$
\begin{equation*}
\|-\Delta \mathbf{w}+\mathbf{f}(\mathbf{w})\|_{\mathbf{H}^{-1}} \geq|E(\mathbf{w})-E(\psi)|^{1-\theta} \tag{3.8}
\end{equation*}
$$

On the other hand, by the definition of $\mathbf{w}$, we can see that

$$
\begin{align*}
|E(\mathbf{w})-E(\psi)|^{1-\theta} & \leq\|-\Delta \mathbf{w}+\mathbf{f}(\mathbf{w})\|_{\mathbf{H}^{-1}} \\
& \leq\|-\Delta \mathbf{d}+\mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}}+C\|\mathbf{f}(\mathbf{d})-\mathbf{f}(\mathbf{w})\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)} \\
& \leq\|-\Delta \mathbf{d}+\mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}}+C\|\mathbf{d}-\mathbf{w}\|_{\mathbf{H}^{1}} \\
& \leq\|-\Delta \mathbf{d}+\mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}}+C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \tag{3.9}
\end{align*}
$$

We deduce from $\theta \in\left(0, \frac{1}{2}\right)$ that

$$
\begin{equation*}
|E(\mathbf{d})-E(\psi)|^{1-\theta} \leq|E(\mathbf{w})-E(\psi)|^{1-\theta}+|E(\mathbf{d})-E(\mathbf{w})|^{1-\theta} \tag{3.10}
\end{equation*}
$$

and

$$
\begin{align*}
& |E(\mathbf{d})-E(\mathbf{w})|^{1-\theta} \\
\leq & \left(\frac{1}{2}\right)^{1-\theta}\left|\|\nabla \mathbf{d}\|^{2}-\|\nabla \mathbf{w}\|^{2}\right|^{1-\theta}+\left|\int_{\Omega}(F(\mathbf{d})-F(\mathbf{w})) d x\right|^{1-\theta} \\
\leq & C\left(\|\mathbf{d}\|_{\mathbf{H}^{1}},\|\mathbf{w}\|_{\mathbf{H}^{1}}\right)\|\mathbf{d}-\mathbf{w}\|_{\mathbf{H}^{1}}^{1-\theta} \leq C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} \tag{3.11}
\end{align*}
$$

where in (3.11) we use the facts that $\|\mathbf{d}\|_{\mathbf{H}^{1}} \leq\|\psi\|_{\mathbf{H}^{1}}+\beta$ and $\|\mathbf{w}\|_{\mathbf{H}^{1}} \leq\|\psi\|_{\mathbf{H}^{1}}+\beta_{1}$. Combining (3.9) -(3.11), we deduce (3.4).

Since the basic energy inequality (2.16) (cf. Lemma (2.2) is only valid for the lifted energy $\widehat{\mathcal{E}}$ (2.15), in order to apply the Łojasiewicz-Simon approach to our problem, we need to consider the following auxiliary functional corresponding to energy $E$ (cf. (3.1)):

$$
\begin{equation*}
\widehat{E}(\mathbf{d})=\frac{1}{2}\|\nabla \widehat{\mathbf{d}}\|^{2}+\int_{\Omega} F(\mathbf{d}) d x, \quad \forall \mathbf{d} \in \mathbf{H}^{1}(\Omega) \tag{3.12}
\end{equation*}
$$

where

$$
\widehat{\mathbf{d}}=\mathbf{d}-\mathbf{d}_{E}
$$

and $\mathbf{d}_{E}$ is the elliptic lifting function satisfying the following elliptic problem (cf. (2.7))

$$
\begin{cases}-\Delta \mathbf{d}_{E}=\mathbf{0}, & x \in \Omega  \tag{3.13}\\ \mathbf{d}_{E}=\left.\mathbf{d}\right|_{\Gamma}, & x \in \Gamma\end{cases}
$$

Then we have
Corollary 3.1. Suppose that $\psi$ is a critical point of $E(\mathbf{d})$ in $\mathcal{N}$. Then there exist constants $\beta \in(0,1)$ and $\theta \in\left(0, \frac{1}{2}\right)$ depending on $\psi$ such that, for any $\mathbf{d} \in \mathbf{H}^{1}(\Omega)$ satisfying $\|\mathbf{d}-\psi\|_{\mathbf{H}^{1}}<\beta$, there holds

$$
\begin{equation*}
C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}+\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}} \geq|\widehat{E}(\mathbf{d})-\widehat{E}(\psi)|^{1-\theta} \tag{3.14}
\end{equation*}
$$

where $C$ is a positive constant depending on $\psi$ and $\mathbf{h}_{\infty}$.

Proof. From the definition of $\widehat{E}(\mathbf{d})$, we set, for $\psi \in \mathcal{N}$,

$$
\begin{equation*}
\widehat{E}(\psi)=\frac{1}{2}\|\nabla \widehat{\psi}\|^{2}+\int_{\Omega} F(\psi) d x \tag{3.15}
\end{equation*}
$$

where $\widehat{\psi}=\psi-\psi_{E}$ and $\psi_{E}$ satisfies

$$
\begin{cases}-\Delta \psi_{E}=\mathbf{0}, & x \in \Omega,  \tag{3.16}\\ \psi_{E}=\mathbf{h}_{\infty}, & x \in \Gamma\end{cases}
$$

A direct calculation yields that

$$
\begin{aligned}
& \widehat{E}(\mathbf{d})=E(\mathbf{d})+\frac{1}{2}\left\|\nabla \mathbf{d}_{E}\right\|^{2}-\int_{\Omega} \nabla \mathbf{d}: \nabla \mathbf{d}_{E} d x \\
& \widehat{E}(\psi)=E(\psi)+\frac{1}{2}\left\|\nabla \psi_{E}\right\|^{2}-\int_{\Omega} \nabla \psi: \nabla \psi_{E} d x
\end{aligned}
$$

where we used the notation $\mathbf{A}: \mathbf{B}=\sum_{i, j=1}^{n} A_{i j} B_{i j}$. Theorem 3.1 implies that there exist constants $\beta \in(0,1)$ and $\theta \in\left(0, \frac{1}{2}\right)$, such that for any $\mathbf{d} \in \mathbf{H}^{1}(\Omega)$ satisfying $\|\mathbf{d}-\psi\|_{\mathbf{H}^{1}}<\beta$, (3.4) holds. Next, we proceed to estimate the quantity $|\widehat{E}(\mathbf{d})-\widehat{E}(\psi)|^{1-\theta}$

$$
\begin{align*}
& |\widehat{E}(\mathbf{d})-\widehat{E}(\psi)|^{1-\theta} \\
\leq & |E(\mathbf{d})-E(\psi)|^{1-\theta}+\left(\frac{1}{2}\right)^{1-\theta}\left|\int_{\Omega} \nabla\left(\mathbf{d}_{E}-\psi_{E}\right): \nabla\left(\mathbf{d}_{E}+\psi_{E}\right) d x\right|^{1-\theta} \\
& +\left|\int_{\Omega}\left(\nabla \mathbf{d}: \nabla \mathbf{d}_{E}-\nabla \psi: \nabla \psi_{E}\right) d x\right|^{1-\theta} \\
:= & J_{1}+J_{2}+J_{3} . \tag{3.17}
\end{align*}
$$

The estimate for $J_{1}$ follows from (3.4). Since $\|\mathbf{d}-\psi\|_{\mathbf{H}^{1}}<\beta<1$, then $\|\mathbf{d}\|_{\mathbf{H}^{1}} \leq\|\psi\|_{\mathbf{H}^{1}}+1$. For $J_{2}$, we infer from the elliptic estimate (cf. [32, Proposition 5.1.7]) that

$$
\begin{align*}
J_{2} & \leq C\left\|\nabla\left(\mathbf{d}_{E}-\psi_{E}\right)\right\|^{1-\theta}\left\|\nabla\left(\mathbf{d}_{E}+\psi_{E}\right)\right\|^{1-\theta} \\
& \leq C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}\left(\left\|\left.\mathbf{d}\right|_{\Gamma}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}+\left\|\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\right)^{1-\theta} \\
& \leq C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}\left(\|\mathbf{d}\|_{\mathbf{H}^{1}(\Omega)}+\left\|\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}\right)^{1-\theta} \\
& \leq C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} . \tag{3.18}
\end{align*}
$$

Recalling the function $\mathbf{w}$ introduced in (3.5), we estimate $J_{3}$ as follows

$$
\begin{align*}
J_{3}= & \left|\int_{\Omega}\left[\nabla(\mathbf{d}-\mathbf{w}): \nabla \mathbf{d}_{E}+\nabla \mathbf{w}: \nabla\left(\mathbf{d}_{E}-\psi_{E}\right)+\nabla(\mathbf{w}-\psi): \nabla \psi_{E}\right] d x\right|^{1-\theta} \\
\leq & \left|\int_{\Omega} \nabla(\mathbf{d}-\mathbf{w}): \nabla \mathbf{d}_{E} d x\right|^{1-\theta}+\left|\int_{\Omega} \nabla \mathbf{w}: \nabla\left(\mathbf{d}_{E}-\psi_{E}\right)\right|^{1-\theta} \\
& +\left|\int_{\Omega} \nabla(\mathbf{w}-\psi): \nabla \psi_{E} d x\right|^{1-\theta} \\
:= & J_{3 a}+J_{3 b}+J_{3 c} . \tag{3.19}
\end{align*}
$$

Using (3.6) and (3.7) and the fact $\|\mathbf{d}-\psi\|_{\mathbf{H}^{1}}<\beta$, we observe that

$$
J_{3 a} \leq\|\nabla(\mathbf{d}-\mathbf{w})\|^{1-\theta}\left\|\nabla \mathbf{d}_{E}\right\|^{1-\theta}
$$

$$
\begin{align*}
\leq & C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}\left\|\left.\mathbf{d}\right|_{\Gamma}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} \\
\leq & C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}\|\mathbf{d}\|_{\mathbf{H}^{1}(\Omega)}^{1-\theta} \\
\leq & C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta},  \tag{3.20}\\
J_{3 b} \leq & \|\nabla \mathbf{w}\|^{1-\theta}\left\|\nabla\left(\mathbf{d}_{E}-\psi_{E}\right)\right\|^{1-\theta} \\
\leq & C\left(\|\psi\|_{\mathbf{H}^{1}}+C \beta\right)^{1-\theta}\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} \\
\leq & C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta} . \tag{3.21}
\end{align*}
$$

For $J_{3 c}$, using integration by parts and noticing that $\Delta \psi_{E}=\mathbf{0},\left.(\mathbf{w}-\psi)\right|_{\Gamma}=\mathbf{0}$, we obtain

$$
\begin{equation*}
\int_{\Omega} \nabla(\mathbf{w}-\psi): \nabla \psi_{E} d x=-\int_{\Omega}(\mathbf{w}-\psi) \cdot \Delta \psi_{E} d x+\left.\int_{\Gamma}(\mathbf{w}-\psi)\right|_{\Gamma} \cdot \partial_{\mathbf{n}} \psi_{E} d S=0 \tag{3.22}
\end{equation*}
$$

where $\mathbf{n}$ is the unit outer normal to the boundary $\Gamma$. Thus (3.22) implies that

$$
\begin{equation*}
J_{3 c}=0 . \tag{3.23}
\end{equation*}
$$

Finally, since $1-\theta \in(0,1)$, we have $\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq C\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta}$. In summary, we can conclude from (3.4), (3.17)-(3.23), and $\Delta \widehat{\mathbf{d}}=\Delta \mathrm{d}$ that (3.14) holds. The proof is complete.

Remark 3.3. If $\theta \in\left(0, \frac{1}{2}\right)$ is such that (3.14) holds, then, for all $\theta^{\prime} \in(0, \theta)$ and any $\mathbf{d} \in \mathbf{H}^{1}(\Omega)$ satisfying $\|\mathbf{d}-\psi\|_{\mathbf{H}^{1}}<\beta$, we still have

$$
\begin{equation*}
C\left(\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{1-\theta^{\prime}}+\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}}\right) \geq|\widehat{E}(\mathbf{d})-\widehat{E}(\psi)|^{1-\theta^{\prime}}, \tag{3.24}
\end{equation*}
$$

where $C$ is a (properly adjusted) positive constant depending on $\psi$ and $\mathbf{h}_{\infty}$. To see this, we first notice that, since $2>\frac{1-\theta^{\prime}}{1-\theta}>1$, for any $a, b \geq 0$, it holds $(a+b)^{\frac{1-\theta^{\prime}}{1-\theta}} \leq 2\left(a^{\frac{1-\theta^{\prime}}{1-\theta}}+b^{\frac{1-\theta^{\prime}}{1-\theta}}\right)$. Then it follows from (3.14) that

$$
\begin{aligned}
& |\widehat{E}(\mathbf{d})-\widehat{E}(\psi)|^{1-\theta^{\prime}}=\left(|\widehat{E}(\mathbf{d})-\widehat{E}(\psi)|^{1-\theta}\right)^{\frac{1-\theta^{\prime}}{1-\theta}} \\
\leq & C^{\frac{1-\theta^{\prime}}{1-\theta}}\left(\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}(\Gamma)}}^{1-\theta}+\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}}\right)^{\frac{1-\theta^{\prime}}{1-\theta}} \\
\leq & C\left(\left\|\left.\mathbf{d}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}(\Gamma)}}^{1-\theta^{\prime}}+\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}}\right) .
\end{aligned}
$$

## 4 Long-time behavior in 2D

In this section, we focus on the case $n=2$. In order to study the long-time behavior of global solutions to problem (1.1)-(1.5), we need some decay conditions on the time-dependent external force $\mathbf{g}$ and boundary data $\mathbf{h}$, namely,
(H1) $\int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d \tau \leq C(1+t)^{-1-\gamma} ;$
(H2) $\int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}(\Gamma)}}^{2} d \tau \leq C(1+t)^{-1-\gamma}$;
(H3) $\int_{t}^{+\infty}\|\mathbf{g}(\tau)\|^{2} d \tau \leq C(1+t)^{-1-\gamma}$;
(H4) $\|\mathbf{g}(t)\|^{2} \leq C(1+t)^{-2-\gamma}$;
(H5) $\left\|\mathbf{h}_{t}(t)\right\|_{\mathbf{L}^{2}(\Gamma)} \leq C(1+t)^{-1-\gamma}$;
for all $t \geq 0$. Here $C$ and $\gamma$ are given positive constants. We also note that (H4) entails (H3).
Since in the 2D case weak solutions become strong for positive times (cf. Theorem 2.1), we can confine ourselves to consider strong solutions. We recall that, for any given global strong solution ( $\mathbf{v}, \mathbf{d}$ ), we have the uniform estimate (2.35). It follows that the $\omega$-limit set of the corresponding initial datum $\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right)$ is non-empty. Namely, for any unbounded increasing sequence $\left\{t_{n}\right\}_{n=1}^{\infty}$, there are functions $\mathbf{v}_{\infty} \in \mathbf{V}$ and $\mathbf{d}_{\infty} \in \mathbf{H}^{2}(\Omega)$ such that, up to a subsequence $\left\{t_{j}\right\}_{j=1}^{\infty} \subset\left\{t_{n}\right\}_{n=1}^{\infty}$, we have

$$
\begin{equation*}
\lim _{j \rightarrow+\infty}\left\|\mathbf{v}\left(t_{j}\right)-\mathbf{v}_{\infty}\right\|=0, \quad \lim _{j \rightarrow+\infty}\left\|\mathbf{d}\left(t_{j}\right)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}=0 \tag{4.1}
\end{equation*}
$$

Next, we characterize the structure of the $\omega$-limit set. In order to do that, we first recall a technical lemma (see [38, Lemma 6.2.1])

Lemma 4.1. Let $T$ be given with $0<T \leq+\infty$. Suppose that $y$ and $h$ are nonnegative continuous functions defined on $[0, T]$ and satisfy the following conditions: $\frac{d y}{d t} \leq c_{1} y^{2}+c_{2}+h$, with $\int_{0}^{T} y(t) d t \leq c_{3}, \int_{0}^{T} h(t) d t \leq c_{4}$, where $c_{i}(i=1,2,3,4)$ are given nonnegative constants. Then for any $\rho \in(0, T)$, the following estimates holds: $y(t+\rho) \leq\left(\frac{c_{3}}{\rho}+c_{2} \rho+c_{4}\right) e^{c_{1} c_{3}}$, for all $t \in[0, T-\rho]$. Furthermore, if $T=+\infty$, then $\lim _{t \rightarrow+\infty} y(t)=0$.

Proposition 4.1. Let the assumptions of Theorem 2.1 hold. Then the $\omega$-limit set $\omega\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right)$ is a subset of

$$
\mathcal{S}=\left\{(\mathbf{0}, \mathbf{u}): \mathbf{u} \in \mathcal{N} \cap \mathbf{H}^{2}(\Omega) \text { such that }-\Delta \mathbf{u}+\mathbf{f}(\mathbf{u})=0 \text { in } \Omega\right\}
$$

Moreover, we have

$$
\begin{gather*}
\lim _{t \rightarrow+\infty}\|\mathbf{v}(t)\| \mathbf{v}=0  \tag{4.2}\\
\lim _{t \rightarrow+\infty}\|-\Delta \mathbf{d}(t)+\mathbf{f}(\mathbf{d}(t))\|=0 \tag{4.3}
\end{gather*}
$$

Proof. It follows from Lemma 2.2 that

$$
\int_{0}^{+\infty}\|\nabla \mathbf{v}(t)\|^{2}+\|\Delta \widehat{\mathbf{d}}(t)-\mathbf{f}(\mathbf{d}(t))\|^{2} d t<+\infty
$$

which together with the definition of $\mathcal{A}_{P}$ and (6.3) yields

$$
\begin{equation*}
\int_{0}^{+\infty} \mathcal{A}_{P}(t) d t \leq \int_{0}^{+\infty}\left(\|\nabla \mathbf{v}(t)\|^{2}+2\|(\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))(t)\|^{2}+2\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{2}\right) d t<+\infty \tag{4.4}
\end{equation*}
$$

Using Lemma 2.5 and Lemma 4.1, we can see that

$$
\lim _{t \rightarrow+\infty} \mathcal{A}_{P}(t)=0
$$

which implies $\lim _{t \rightarrow+\infty}\|\nabla \mathbf{v}(t)\|=0$. Hence, for any $\left(\mathbf{v}_{\infty}, \mathbf{d}_{\infty}\right) \in \omega\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right)$, we have $\mathbf{v}_{\infty}=\mathbf{0}$. On the other hand, by definition of $\mathcal{A}_{P}$, (4.4) also yields that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty}\|-\Delta \widetilde{\mathbf{d}}(t)+\mathbf{f}(\mathbf{d}(t))\|=0 \tag{4.5}
\end{equation*}
$$

From Lemma 6.2, we have $\lim _{t \rightarrow+\infty}\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|=0$ (cf. (6.5)). As a result, it follows from the inequality

$$
\begin{equation*}
0 \leq\|-\Delta \mathbf{d}(t)+\mathbf{f}(\mathbf{d}(t))\| \leq\|-\Delta \widetilde{\mathbf{d}}(t)+\mathbf{f}(\mathbf{d}(t))\|+\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|, \quad \forall t \geq 0 \tag{4.6}
\end{equation*}
$$

that (4.3) holds. Concerning the limit function $\mathbf{d}_{\infty}$, we infer from (2.35) that $\mathbf{d}_{\infty} \in \mathbf{H}^{2}(\Omega)$ and (4.1) holds. We now check the boundary condition for $\mathbf{d}_{\infty}$. Since $\mathbf{h}_{t} \in L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right), \mathbf{h}(t)$ strongly converges to a certain function $\mathbf{h}_{\infty} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ as time goes to infinity with a controlled rate, namely,

$$
\begin{equation*}
\left\|\mathbf{h}(t)-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \leq \int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} d \tau \rightarrow 0, \quad \text { as } t \rightarrow+\infty \tag{4.7}
\end{equation*}
$$

On the other hand, we infer from (2.33) and (2.34) that $\mathbf{h} \in L^{\infty}\left(0,+\infty ; \mathbf{H}^{\frac{3}{2}}(\Gamma)\right)$. Consequently, $\mathbf{h}_{\infty} \in \mathbf{H}^{\frac{3}{2}}(\Gamma)$ and $\mathbf{h}$ weakly-star converges to $\mathbf{h}_{\infty}$ in $L^{\infty}\left(0,+\infty ; \mathbf{H}^{\frac{3}{2}}(\Gamma)\right)$. By interpolation, we have $\lim _{t \rightarrow+\infty}\left\|\mathbf{h}(t)-\mathbf{h}_{\infty}\right\|_{\mathbf{L}^{2}(\Gamma)}=0$. Thus, from the asymptotic behavior of the boundary datum $\mathbf{h}$, we have for any $j \in \mathbb{N}$,

$$
\begin{aligned}
\left\|\mathbf{d}_{\infty} \mid \Gamma-\mathbf{h}_{\infty}\right\|_{\mathbf{L}^{2}(\Gamma)} & \leq\left\|\left.\mathbf{d}_{\infty}\right|_{\Gamma}-\mathbf{h}\left(t_{j}\right)\right\|_{\mathbf{L}^{2}(\Gamma)}+\left\|\mathbf{h}\left(t_{j}\right)-\mathbf{h}_{\infty}\right\|_{\mathbf{L}^{2}(\Gamma)} \\
& \leq C\left\|\mathbf{d}_{\infty}-\mathbf{d}\left(t_{j}\right)\right\|_{\mathbf{H}^{1}}+\left\|\mathbf{h}\left(t_{j}\right)-\mathbf{h}_{\infty}\right\|_{\mathbf{L}^{2}(\Gamma)}
\end{aligned}
$$

Hence, letting $j \rightarrow+\infty$ in the above inequality, we deduce from (4.1) and (4.7) that $\left.\mathbf{d}_{\infty}\right|_{\Gamma}=\mathbf{h}_{\infty}$. For any $\mathbf{z} \in \mathbf{H}_{0}^{1}(\Omega)$ and $j \in \mathbb{N}$, we have

$$
\begin{aligned}
& \left|\int_{\Omega}\left(-\Delta \mathbf{d}_{\infty}+\mathbf{f}\left(\mathbf{d}_{\infty}\right)\right) \cdot \mathbf{z} d x\right| \\
\leq & \left|\int_{\Omega}\left(-\Delta \mathbf{d}_{\infty}+\Delta \mathbf{d}\left(t_{j}\right)\right) \cdot \mathbf{z} d x\right|+\left|\int_{\Omega}\left(\mathbf{f}\left(\mathbf{d}_{\infty}\right)-\mathbf{f}\left(\mathbf{d}\left(t_{j}\right)\right)\right) \cdot \mathbf{z} d x\right| \\
& +\left|\int_{\Omega}\left(-\Delta \mathbf{d}\left(t_{j}\right)+\mathbf{f}\left(\mathbf{d}\left(t_{j}\right)\right)\right) \cdot \mathbf{z} d x\right| \\
\leq & \left\|\nabla\left(\mathbf{d}\left(t_{j}\right)-\mathbf{d}_{\infty}\right)\right\|\|\nabla \mathbf{z}\|+\left(C\left\|\mathbf{d}\left(t_{j}\right)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}+\left\|-\Delta \mathbf{d}\left(t_{j}\right)+\mathbf{f}\left(\mathbf{d}\left(t_{j}\right)\right)\right\|\right)\|\mathbf{z}\| .
\end{aligned}
$$

Passing to the limit as $j \rightarrow+\infty$, we get

$$
\int_{\Omega}\left(-\Delta \mathbf{d}_{\infty}+\mathbf{f}\left(\mathbf{d}_{\infty}\right)\right) \cdot \mathbf{z} d x=0
$$

As a consequence, we see that $\mathbf{d}_{\infty} \in \mathcal{N} \cap \mathbf{H}^{2}(\Omega)$ solves (3.2). The proof is complete.
We can also prove the convergence of the lifted energy.
Proposition 4.2. Let the assumptions of Theorem 2.1 hold. Then the lifted energy functional $\widehat{\mathcal{E}}$ defined by (2.15) is constant on the $\omega$-limit set $\omega\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right)$. Namely, there exists a constant $\widehat{\mathcal{E}}_{\infty}$ such that $\widehat{E}\left(\mathbf{d}_{\infty}\right) \equiv \widehat{\mathcal{E}}_{\infty}$, for all $\left(\mathbf{0}, \mathbf{d}_{\infty}\right)$ with $\mathbf{d}_{\infty} \in \mathcal{N} \cap \mathbf{H}^{2}(\Omega)$. Moreover, we have

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \widehat{\mathcal{E}}(t)=\widehat{\mathcal{E}}_{\infty} \tag{4.8}
\end{equation*}
$$

Proof. From the previous argument, we know that for $\operatorname{arbitrary}\left(\mathbf{0}, \mathbf{d}_{\infty}^{(1)}\right),\left(\mathbf{0}, \mathbf{d}_{\infty}^{(2)}\right) \in \omega\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right)$ there exist unbounded increasing sequences $\left\{t_{j}^{(1)}\right\}_{j=1}^{\infty}$ and $\left\{t_{j}^{(2)}\right\}_{j=1}^{\infty}$ such that (4.1) holds. As a result, we have

$$
\lim _{j \rightarrow+\infty} \widehat{\mathcal{E}}\left(t_{j}^{(1)}\right)=\widehat{E}\left(\mathbf{d}_{\infty}^{(1)}\right), \quad \lim _{j \rightarrow+\infty} \widehat{\mathcal{E}}\left(t_{j}^{(2)}\right)=\widehat{E}\left(\mathbf{d}_{\infty}^{(2)}\right)
$$

On the other hand, it follows from the basic energy inequality (2.16) that for any $t^{\prime}>t^{\prime \prime}>0$,

$$
\left|\widehat{\mathcal{E}}\left(t^{\prime}\right)-\widehat{\mathcal{E}}\left(t^{\prime \prime}\right)\right| \leq \int_{t^{\prime \prime}}^{t^{\prime}} r(t) d t \rightarrow 0, \quad \text { as } t^{\prime}, t^{\prime \prime} \rightarrow+\infty
$$

Then by

$$
\left|\widehat{E}\left(\mathbf{d}_{\infty}^{(1)}\right)-\widehat{E}\left(\mathbf{d}_{\infty}^{(2)}\right)\right| \leq\left|\widehat{\mathcal{E}}\left(t_{j}^{(1)}\right)-\widehat{\mathcal{E}}\left(t_{j}^{(2)}\right)\right|+\left|\widehat{E}\left(\mathbf{d}_{\infty}^{(1)}\right)-\widehat{\mathcal{E}}\left(t_{j}^{(1)}\right)\right|+\left|\widehat{\mathcal{E}}\left(t_{j}^{(2)}\right)-\widehat{E}\left(\mathbf{d}_{\infty}^{(2)}\right)\right|
$$

letting $j \rightarrow+\infty$, we can see that $\widehat{E}\left(\mathbf{d}_{\infty}^{(1)}\right)=\widehat{E}\left(\mathbf{d}_{\infty}^{(2)}\right)$. Namely, $\widehat{\mathcal{E}}$ is a constant (denoted by $\widehat{\mathcal{E}}_{\infty}$ ) on the $\omega$-limit set $\omega\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right)$. Moreover, for any $t>0$ there exist $t_{j}<t_{j+1}$ such that $t \in\left[t_{j}, t_{j+1}\right]$ and $\left|\widehat{\mathcal{E}}(t)-\widehat{\mathcal{E}}_{\infty}\right| \leq\left|\widehat{\mathcal{E}}(t)-\widehat{\mathcal{E}}\left(t_{j}\right)\right|+\left|\widehat{\mathcal{E}}\left(t_{j}\right)-\widehat{\mathcal{E}}_{\infty}\right|$, which yields (4.8).

### 4.1 Convergence to equilibrium

Theorem 4.1. Let the assumptions of Theorem 2.1 hold. If, in addition, we assume (H1)-(H3), then any strong solution $(\mathbf{v}(t), \mathbf{d}(t))$ convergence to an equilibrium $\left(\mathbf{0}, \mathbf{d}_{\infty}\right)$ strongly in $\mathbf{V} \times \mathbf{H}^{2}(\Omega)$ as $t$ goes to $+\infty$.

Proof. On account of (4.2) we only need to prove that $\mathbf{d}(t)$ converges to $\mathbf{d}_{\infty}$ as $t \rightarrow+\infty$ given by (4.1). Below we adapt the idea in [4, 9] to achieve our goal. Indeed, observe that we can find an integer $j_{0}$ such that for all $j \geq j_{0},\left\|\mathbf{d}\left(t_{j}\right)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}<\frac{\beta}{3}$, where $\beta \in(0,1)$ is the constant given in Corollary 3.1 (depending on $\mathbf{d}_{\infty}$ ). Consequently, we define

$$
s\left(t_{j}\right)=\sup \left\{\tau \geq t_{j}:\left\|\mathbf{d}(\tau)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}<\beta\right\}
$$

Since $\mathbf{d} \in C\left([0,+\infty) ; \mathbf{H}^{1}(\Omega)\right)$, we can see that $s\left(t_{j}\right)>t_{j}$ for any $j \geq j_{0}$. By Lemma 2.2 and Proposition 4.2, we have

$$
\left|\widehat{\mathcal{E}}(t)-\widehat{E}\left(\mathbf{d}_{\infty}\right)\right| \geq \frac{1}{4} \min \{\nu, 1\} \int_{t}^{+\infty} \mathcal{D}^{2}(\tau) d \tau-\int_{t}^{+\infty} r(\tau) d \tau
$$

where

$$
\mathcal{D}(t)=\|\nabla \mathbf{v}(t)\|+\|\Delta \widehat{\mathbf{d}}(t)-\mathbf{f}(\mathbf{d}(t))\|,
$$

and $r$ is defined in (2.16) such that, thanks to (H1)-(H3), we have

$$
\int_{t}^{+\infty} r(\tau) d \tau \leq C(1+t)^{-1-\gamma}, \quad \forall t \geq 0
$$

Let the constant $\theta$ be as in Corollary 3.1 (depending on $\mathbf{d}_{\infty}$ ). Using Remark 3.3, we can choose $\theta^{\prime} \in(0, \theta]$ such that $\theta^{\prime}$ also satisfies

$$
\begin{equation*}
0<\theta^{\prime}<\frac{\gamma}{2(1+\gamma)} \tag{4.9}
\end{equation*}
$$

If $\theta$ itself satisfies (4.9), we just take $\theta^{\prime}=\theta$. For any fixed $t_{j}$ with $j \geq j_{0}$, we introduce the sets

$$
K_{j}=\left[t_{j}, s\left(t_{j}\right)\right), \quad K_{j}^{(1)}=\left\{t \in K_{j}: \mathcal{D}(t)>(1+t)^{-\left(1-\theta^{\prime}\right)(1+\gamma)}\right\}, \quad K_{j}^{(2)}=K_{j} \backslash K_{j}^{(1)}
$$

Consider the following functional on $K_{j}$

$$
\Phi(t)=\widehat{\mathcal{E}}(t)-\widehat{E}\left(\mathbf{d}_{\infty}\right)+2 \int_{t}^{s\left(t_{j}\right)} r(\tau) d \tau, \quad \forall t \in K_{j} .
$$

It easily follows that

$$
\begin{equation*}
\lim _{j \rightarrow+\infty} \Phi\left(t_{j}\right)=0 \tag{4.10}
\end{equation*}
$$

Next, we have

$$
\begin{align*}
\frac{d}{d t}\left(|\Phi(t)|^{\theta^{\prime}} \operatorname{sgn} \Phi(t)\right) & =\theta^{\prime}|\Phi(t)|^{\theta^{\prime}-1} \frac{d}{d t} \Phi(t) \\
& \leq-\frac{\theta^{\prime}}{4} \min \{\nu, 1\}|\Phi(t)|^{\theta^{\prime}-1} \mathcal{D}^{2}(t) \\
& \leq 0, \tag{4.11}
\end{align*}
$$

which implies that the functional $|\Phi(t)|^{\theta^{\prime}} \operatorname{sgn} \Phi(t)$ is decreasing on $K_{j}$. Keeping in mind that $\theta^{\prime} \leq \theta$ and $2\left(1-\theta^{\prime}\right)>1$, we can apply Corollary 3.1 (cf. also Remark 3.3) to obtain that

$$
\begin{align*}
|\Phi(t)|^{1-\theta^{\prime}} \leq & \left|\widehat{\mathcal{E}}(t)-\widehat{E}\left(\mathbf{d}_{\infty}\right)\right|^{1-\theta^{\prime}}+C\left(\int_{t}^{+\infty} r(\tau) d \tau\right)^{1-\theta^{\prime}} \\
\leq & \left(\frac{1}{2}\right)^{2\left(1-\theta^{\prime}\right)}\|\mathbf{v}\|^{2\left(1-\theta^{\prime}\right)}+C\left\|\mathbf{h}(t)-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{\theta^{\prime}}{2}}(\Gamma)}^{1-\theta^{\prime}} \\
& +C\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|_{\mathbf{H}^{-1}}+C\left(\int_{t}^{+\infty} r(\tau) d \tau\right)^{1-\theta^{\prime}} \\
\leq & C\|\nabla \mathbf{v}\|+C\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|+C\left(\int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d \tau\right)^{1-\theta^{\prime}} \\
& +C\left(\int_{t}^{+\infty} r(\tau) d \tau\right)^{1-\theta^{\prime}} \\
\leq & C\|\nabla \mathbf{v}\|+C\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|+C(1+t)^{-\left(1-\theta^{\prime}\right)(1+\gamma)} . \tag{4.12}
\end{align*}
$$

Thus, on $K_{j}^{(1)}$, we have

$$
|\Phi(t)|^{1-\theta^{\prime}} \leq C \mathcal{D}(t)
$$

which together with (4.11) yields that on $K_{j}^{(1)}$,

$$
\begin{equation*}
-\frac{d}{d t}\left(|\Phi(t)|^{\theta^{\prime}} \operatorname{sgn} \Phi(t)\right) \geq C \mathcal{D}(t) \tag{4.13}
\end{equation*}
$$

As a consequence, we have

$$
\begin{align*}
\int_{K_{j}^{(1)}} \mathcal{D}(t) d t & \leq-C \int_{K_{j}} \frac{d}{d t}\left(|\Phi(t)|^{\theta^{\prime}} \operatorname{sgn} \Phi(t)\right) d t \\
& \leq C\left(\left|\Phi\left(t_{j}\right)\right|^{\theta^{\prime}}+\left|\Phi\left(s\left(t_{j}\right)\right)\right|^{\theta^{\prime}}\right)<+\infty \tag{4.14}
\end{align*}
$$

where $\Phi\left(s\left(t_{j}\right)\right)=0$ if $s\left(t_{j}\right)=+\infty$. On the other hand, on $K_{j}^{(2)}$, we have

$$
\begin{equation*}
\int_{K_{j}^{(2)}} \mathcal{D}(t) d t \leq C \int_{t_{j}}^{\infty}(1+t)^{-\left(1-\theta^{\prime}\right)(1+\gamma)} d t=\frac{C}{-\gamma \theta^{\prime}-\theta^{\prime}+\gamma}\left(1+t_{j}\right)^{\gamma^{\theta^{\prime}+\theta^{\prime}-\gamma}} \tag{4.15}
\end{equation*}
$$

Here, we notice that $\gamma \theta^{\prime}+\theta^{\prime}-\gamma<0$ due to (4.9). Then (4.14) and (4.15) imply that

$$
\int_{K_{j}} \mathcal{D}(t) d t=\int_{K_{j}^{(1)}} \mathcal{D}(t) d t+\int_{K_{j}^{(2)}} \mathcal{D}(t) d t<+\infty,
$$

for any $j$. On the other hand, it follows from (2.35) and (2.12) that

$$
\begin{align*}
\left\|\mathbf{d}_{t}(t)\right\| & \leq\|\mathbf{v} \cdot \nabla \mathbf{d}\|+\|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\| \\
& \leq\|\mathbf{v}\|_{\mathbf{L}^{4}}\|\nabla \mathbf{d}\|_{\mathbf{L}^{4}}+\|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\| \\
& \leq C \mathcal{D}(t) \tag{4.16}
\end{align*}
$$

As a consequence,

$$
\begin{equation*}
\int_{K_{j}}\left\|\mathbf{d}_{t}(t)\right\| d t \leq C\left(\left|\Phi\left(t_{j}\right)\right|^{\theta^{\prime}}+\left|\Phi\left(s\left(t_{j}\right)\right)\right|^{\theta^{\prime}}\right)+C\left(1+t_{j}\right)^{\gamma \theta^{\prime}+\theta^{\prime}-\gamma} \tag{4.17}
\end{equation*}
$$

To complete the proof, we show that
Proposition 4.3. Let the assumptions of Theorem 2.1] hold. Then there exists an integer $j_{1} \geq j_{0}$ such that $s\left(t_{j_{1}}\right)=+\infty$. Thus

$$
\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}<\beta, \quad \forall t \geq t_{j_{1}}
$$

Proof. The conclusion follows from a contradiction argument (cf. [15]). Suppose that for any $j \geq j_{0}$ we have $s\left(t_{j}\right)<+\infty$. Then, by definition, we have

$$
\begin{equation*}
\left\|\mathbf{d}\left(s\left(t_{j}\right)\right)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}=\beta>0 \tag{4.18}
\end{equation*}
$$

Besides, it follows from (4.1), (4.10) and (4.17) that

$$
\begin{aligned}
\left\|\mathbf{d}\left(s\left(t_{j}\right)\right)-\mathbf{d}_{\infty}\right\| & \leq\left\|\mathbf{d}\left(s\left(t_{j}\right)\right)-\mathbf{d}\left(t_{j}\right)\right\|+\left\|\mathbf{d}\left(t_{j}\right)-\mathbf{d}_{\infty}\right\| \\
& \leq \int_{t_{j}}^{s\left(t_{j}\right)}\left\|\mathbf{d}_{t}(t)\right\| d t+\left\|\mathbf{d}\left(t_{j}\right)-\mathbf{d}_{\infty}\right\| \rightarrow 0, \quad \text { as } j \rightarrow+\infty
\end{aligned}
$$

Using uniform estimate (2.35) and interpolation inequality, we obtain

$$
\left\|\mathbf{d}\left(s\left(t_{j}\right)\right)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}^{2} \leq\left\|\mathbf{d}\left(s\left(t_{j}\right)\right)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{2}}\left\|\mathbf{d}\left(s\left(t_{j}\right)\right)-\mathbf{d}_{\infty}\right\| \rightarrow 0, \quad \text { as } j \rightarrow+\infty
$$

which leads a contradiction with (4.18). The proof is complete.
Due to Proposition 4.3, we have $s\left(t_{j_{1}}\right)=+\infty$ for some $j_{1} \geq j_{0}$. Arguing as above, we can prove

$$
\int_{t_{j_{1}}}^{+\infty}\left\|\mathbf{d}_{t}(t)\right\| d t<+\infty
$$

Thus $\mathbf{d}(t)$ converges in $\mathbf{L}^{2}$ and recalling (4.1), by compactness we conclude that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty}\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}=0 \tag{4.19}
\end{equation*}
$$

Finally, observe that

$$
\begin{align*}
\left\|\Delta \mathbf{d}(t)-\Delta \mathbf{d}_{\infty}\right\| & =\|-\Delta \mathbf{d}(t)+\mathbf{f}(\mathbf{d}(t))\|+\left\|\mathbf{f}(\mathbf{d}(t))-\mathbf{f}\left(\mathbf{d}_{\infty}\right)\right\| \\
& \leq\|-\Delta \mathbf{d}(t)+\mathbf{f}(\mathbf{d}(t))\|+C\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\| \tag{4.20}
\end{align*}
$$

Then (4.3) and (4.19) entail that

$$
\lim _{t \rightarrow+\infty}\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{2}}=0
$$

and this finishes the proof of Theorem 4.1.

### 4.2 Convergence rate

Theorem 4.2. Let the assumptions of Theorem 2.11 hold. If, in addition, we assume (H1)-(H2) and (H4)-(H5), then we have

$$
\|\mathbf{v}(t)\|+\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}} \leq C(1+t)^{-\frac{\theta^{\prime}}{1-2 \theta^{\prime}}}, \quad t \geq 0
$$

Moreover, if (H2) and (H5) are replaced by, respectively,
(H6) $\left\|\mathbf{h}_{t}(t)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq C(1+t)^{-1-\gamma}$;
(H7) $\left\|\mathbf{h}(t)-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{3}{2}(\Gamma)}} \leq C(1+t)^{-1-\gamma} ;$
the following higher-order estimate holds

$$
\|\mathbf{v}(t)\|_{\mathbf{v}}+\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{2}} \leq C(1+t)^{-\frac{\theta^{\prime}}{1-2 \theta^{\prime}}}, \quad t \geq 0
$$

Proof. The proof consists of several steps.
Step 1. $\mathbf{L}^{2}$-estimate of $\mathbf{d}-\mathbf{d}_{\infty}$. This follows from an argument devised in [9]. For the readers' convenience, we sketch the proof here. From the previous argument, we only have to work on the time interval $\left[t_{j_{1}},+\infty\right)$. Denote

$$
\Phi(t)=\widehat{\mathcal{E}}(t)-\widehat{E}\left(\mathbf{d}_{\infty}\right)+2 \int_{t}^{+\infty} r(\tau) d \tau
$$

Since

$$
\frac{d}{d t} \Phi(t) \leq-\frac{\theta^{\prime}}{4} \min \{\nu, 1\} \mathcal{D}^{2}(t)-r(t) \leq 0,
$$

and $\lim _{t \rightarrow+\infty} \Phi(t)=0$, we know that $\Phi(t)$ is decreasing and $\Phi(t) \geq 0$ for $t \geq t_{j_{1}}$.
First, if the boundary datum $\mathbf{h}$ and the external force $\mathbf{g}$ become time-independent in finite time, i.e., there exists time $T_{0}$ such that for $t \geq T_{0}, \mathbf{h}=\mathbf{h}_{\infty}$ and $\mathbf{g}=\mathbf{0}$. Then the problem reduces to the autonomous system considered in [35]. Thus, below we just assume that either $\mathbf{h}$ or $\mathbf{g}$ does not become time-independent in finite time (namely, the system will always be non-autonomous). In this case, if there exists $t^{*} \geq t_{j_{1}}$ such that $\Phi\left(t^{*}\right)=0$, then $\mathcal{D}(t)=r(t)=0$ for all $t \geq t^{*}$ and this is a contradiction since $r(t)$ cannot identically vanish from any finite time on. Therefore, we can suppose

$$
\Phi(t)>0, \quad \forall t \geq t_{j_{1}} .
$$

If the open set $K_{j_{1}}^{(1)}$ is bounded, then there exists $t^{*} \geq t_{j_{1}}$ such that $\left[t^{*},+\infty\right) \subset K_{j_{1}}^{(2)}$. As a result, $\mathcal{D}(t) \leq(1+t)^{-\left(1-\theta^{\prime}\right)(1+\gamma)}$ and by (4.16), we have

$$
\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\| \leq \int_{t}^{+\infty}\left\|\mathbf{d}_{t}(\tau)\right\| d \tau \leq \frac{C}{-\gamma \theta^{\prime}-\theta^{\prime}+\gamma}(1+t)^{\gamma^{\theta^{\prime}+\theta^{\prime}-\gamma}}, \quad \forall t \geq t^{*}
$$

Next, we treat the case when the open set $K_{j_{1}}^{(1)}$ is unbounded. There exists a countable family of disjoint open sets $\left(a_{n}, b_{n}\right)$ such that $K_{j_{1}}^{(1)}=\cup_{n=1}^{\infty}\left(a_{n}, b_{n}\right)$. On $K_{j_{1}}^{(1)}$, recalling (4.12), we can see that on any $\left(a_{n}, b_{n}\right) \subset K_{j_{1}}^{(1)}$, it holds

$$
\frac{d}{d t} \Phi(t)+C \Phi^{2\left(1-\theta^{\prime}\right)}(t) \leq 0 .
$$

As a result, for any $t \in\left(a_{n}, b_{n}\right)$,

$$
\begin{equation*}
\Phi(t) \leq\left[\Phi\left(a_{n}\right)^{2 \theta^{\prime}-1}+C\left(1-2 \theta^{\prime}\right)\left(t-a_{n}\right)\right]^{-\frac{1}{1-2 \theta^{\prime}}}, \tag{4.21}
\end{equation*}
$$

where by the definition of $K_{j_{1}}^{(1)}$ and (4.12) we have

$$
\Phi\left(a_{n}\right) \leq C \mathcal{D}\left(a_{n}\right)^{\frac{1}{1-\theta^{\prime}}}+C\left(1+a_{n}\right)^{-(1+\gamma)}=C\left(1+a_{n}\right)^{-1-\gamma} .
$$

Using the fact $(1+\gamma)\left(1-2 \theta^{\prime}\right)>1$ (cf. (4.9)), we can take $n^{*} \in \mathbb{N}$ sufficiently large such that

$$
\begin{equation*}
\Phi\left(a_{n^{*}}\right)^{2 \theta^{\prime}-1}-C\left(1-2 \theta^{\prime}\right) a_{n^{*}} \geq a_{n^{*}}^{(1+\gamma)\left(1-2 \theta^{\prime}\right)}-C\left(1-2 \theta^{\prime}\right) a_{n^{*}} \geq 1 . \tag{4.22}
\end{equation*}
$$

Therefore, we infer

$$
\Phi(t) \leq C(1+t)^{-\frac{1}{1-2 \theta^{\prime}}}, \quad \forall t \in\left(a_{n^{*}}, \infty\right) \cap K_{j_{1}}^{(1)} .
$$

Similar to (4.13), we have (since $\Phi(t)>0$ )

$$
-\frac{d}{d t} \Phi(t)^{\theta^{\prime}} \geq C \mathcal{D}(t), \quad \forall t \in\left(a_{n^{*}}, \infty\right) \cap K_{j_{1}}^{(1)} .
$$

Due to (4.9), it follows that $-\gamma \theta^{\prime}-\theta^{\prime}+\gamma \geq \frac{\theta^{\prime}}{1-2 \theta^{\prime}}$. Now for any $t>a_{n^{*}}$, we can conclude that

$$
\begin{aligned}
\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\| & \leq \int_{t}^{+\infty}\left\|\mathbf{d}_{t}(\tau)\right\| d \tau \\
& =\int_{(t, \infty) \cap K_{j_{1}}^{(1)}}\left\|\mathbf{d}_{t}(\tau)\right\| d \tau+\int_{(t, \infty) \cap K_{j_{1}}^{(2)}}\left\|\mathbf{d}_{t}(\tau)\right\| d \tau \\
& \leq C \int_{(t, \infty) \cap K_{j_{1}}^{(1)}} \mathcal{D}(\tau) d \tau+C \int_{t}^{+\infty}(1+\tau)^{-\left(1-\theta^{\prime}\right)(1+\gamma)} d \tau \\
& \leq C \Phi(t)^{\theta^{\prime}}+C(1+t)^{\gamma \theta^{\prime}+\theta^{\prime}-\gamma} \\
& \leq C(1+t)^{-\frac{\theta^{\prime}}{1-2 \theta^{\prime}}} .
\end{aligned}
$$

Using (2.35), after properly adjusting the constant $C$, we have

$$
\begin{equation*}
\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\| \leq C(1+t)^{-\frac{\theta^{\prime}}{1-2 \theta^{\prime}}}, \quad \forall t \geq 0 \tag{4.23}
\end{equation*}
$$

Step 2. $\mathbf{H} \times \mathbf{H}^{1}$-estimate. It easily from the basic energy inequality (2.16) that

$$
\begin{equation*}
\frac{d}{d t} y(t)+\frac{\nu}{2}\|\nabla \mathbf{v}\|^{2}+\frac{1}{2}\|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2} \leq r(t) \tag{4.24}
\end{equation*}
$$

where

$$
y(t)=\frac{1}{2}\|\mathbf{v}(t)\|^{2}+\frac{1}{2}\left\|\nabla\left(\widehat{\mathbf{d}}(t)-\widehat{\mathbf{d}}_{\infty}\right)\right\|^{2}+\int_{\Omega}\left[F(\mathbf{d})(t)-F\left(\mathbf{d}_{\infty}\right)-\mathbf{f}\left(\mathbf{d}_{\infty}\right)\left(\mathbf{d}(t)-\mathbf{d}_{\infty}\right)\right] d x .
$$

As in [35, using (2.35), we can show that

$$
\left|\int_{\Omega}\left[F(\mathbf{d})(t)-F\left(\mathbf{d}_{\infty}\right)-\mathbf{f}\left(\mathbf{d}_{\infty}\right)\left(\mathbf{d}(t)-\mathbf{d}_{\infty}\right)\right] d x\right| \leq C\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|^{2} .
$$

Keeping in mind the definition of lifting functions, we have $\widehat{\mathbf{d}}-\left.\widehat{\mathbf{d}}_{\infty}\right|_{\Gamma}=0$ so that

$$
\left\|\nabla\left(\widehat{\mathbf{d}}-\widehat{\mathbf{d}}_{\infty}\right)\right\| \leq C\left\|\Delta\left(\widehat{\mathbf{d}}-\widehat{\mathbf{d}}_{\infty}\right)\right\|
$$

$$
\begin{aligned}
& \leq\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|+C\left\|\mathbf{f}(\mathbf{d})-\mathbf{f}\left(\mathbf{d}_{\infty}\right)\right\| \\
& \leq C\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|+C\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|,
\end{aligned}
$$

$$
\begin{aligned}
\left\|\nabla\left(\mathbf{d}-\mathbf{d}_{\infty}\right)\right\| & \leq\left\|\nabla\left(\widehat{\mathbf{d}}-\widehat{\mathbf{d}}_{\infty}\right)\right\|+\left\|\nabla\left(\mathbf{d}_{E}-\mathbf{d}_{\infty}\right)\right\| \\
& \leq C\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|+C\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|+C \int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d \tau .
\end{aligned}
$$

Thus it follows that

$$
\begin{align*}
y(t) \geq & \frac{1}{2}\|\mathbf{v}(t)\|^{2}+\frac{1}{2}\left\|\nabla\left(\mathbf{d}-\mathbf{d}_{\infty}\right)\right\|^{2}-C\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|^{2} \\
& -C\left(\int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d \tau\right)^{2}  \tag{4.25}\\
y(t) \leq & C\|\nabla \mathbf{v}\|^{2}+C\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|^{2}+C\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|^{2} . \tag{4.26}
\end{align*}
$$

Condition (4.9) implies that $\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}<\gamma$. Then we deduce from (4.24), (4.23), (H4)-(H5) and Lemma 6.1] that

$$
\begin{equation*}
\frac{d}{d t} y(t)+\alpha y(t) \leq C\left(r(t)+\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|^{2}\right) \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}, \tag{4.27}
\end{equation*}
$$

where $\alpha>0$ is sufficiently small. The above inequality implies that

$$
\begin{equation*}
y(t) \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}, \quad \forall t \geq 0 . \tag{4.28}
\end{equation*}
$$

Combining it with (4.25) and recalling (H1), we get

$$
\begin{align*}
& \|\mathbf{v}(t)\|^{2}+\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}^{2} \\
\leq & C y(t)+C\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|^{2}+C\left(\int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d \tau\right)^{2} \\
\leq & C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}, \quad \forall t \geq 0 . \tag{4.29}
\end{align*}
$$

Step 3. $\mathbf{V} \times \mathbf{H}^{2}$-estimate. Taking advantage of the stronger assumptions (H6)-(H7) and (4.29), we now get a higher-order estimate. Observe first that

$$
\|-\Delta \widehat{\mathbf{d}}+\mathbf{f}(\mathbf{d})\| \leq\|-\Delta \widetilde{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|+\left\|\Delta \mathbf{d}_{P}\right\|=\|-\Delta \widetilde{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|+\left\|\partial_{t} \mathbf{d}_{P}\right\|
$$

then we have

$$
y(t) \leq C\|\nabla \mathbf{v}\|^{2}+C\|-\Delta \widetilde{\mathbf{d}}+\mathbf{f}(\mathbf{d})\|^{2}+C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}+C\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|^{2} .
$$

It follows from (2.27) and (4.27) that

$$
\begin{equation*}
\frac{d}{d t} z(t)+\alpha_{2} z(t) \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}+C\left(R_{1}(t)+\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{2}\right) \tag{4.30}
\end{equation*}
$$

where

$$
\begin{equation*}
z(t)=y(t)+\alpha_{1} \mathcal{A}_{P}(t), \tag{4.31}
\end{equation*}
$$

and $\alpha_{1}$ and $\alpha_{2}$ are sufficiently small positive constants. From the definition of $R_{1}$, (6.6) and the fact $\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}<2+2 \gamma$ (cf. (4.9)), we have

$$
\begin{equation*}
R_{1}(t)+\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{2} \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}+C\left\|\nabla \Delta \mathbf{d}_{P}(t)\right\|^{2} . \tag{4.32}
\end{equation*}
$$

Hence, from (4.30) we infer that

$$
\begin{align*}
z(t) \leq & z(0) e^{-\alpha_{2} t}+C e^{-\alpha_{2} t} \int_{0}^{t} e^{\alpha_{2} \tau}\left[C(1+\tau)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}+\left\|\nabla \Delta \mathbf{d}_{P}(\tau)\right\|^{2}\right] d \tau \\
\leq & C e^{-\alpha_{2} t}+e^{-\alpha_{2} t} \int_{0}^{\frac{t}{2}} e^{\alpha_{2} \tau}\left[C(1+\tau)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}+\left\|\nabla \Delta \mathbf{d}_{P}(\tau)\right\|^{2}\right] d \tau \\
& +e^{-\alpha_{2} t} \int_{\frac{t}{2}}^{t} e^{\alpha_{2} \tau}\left[C(1+\tau)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}+\left\|\nabla \Delta \mathbf{d}_{P}(\tau)\right\|^{2}\right] d \tau \\
:= & C e^{-\alpha_{2} t}+Z_{1}(t)+Z_{2}(t) \tag{4.33}
\end{align*}
$$

It follows from (6.4) and (H6) that

$$
\begin{aligned}
Z_{1}(t) & \leq C e^{-\frac{\alpha_{2}}{2} t} \int_{0}^{\frac{t}{2}}\left[C(1+\tau)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}+\left\|\nabla \Delta \mathbf{d}_{P}(\tau)\right\|^{2}\right] d \tau \\
& \leq C e^{-\frac{\alpha_{2}}{2} t}\left(t+\int_{0}^{\frac{t}{2}}(1+\tau)^{-2-2 \gamma} d \tau\right) \\
& \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}
\end{aligned}
$$

Next, by (6.7) and the fact $\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}<1+2 \gamma$, we deduce that

$$
\begin{aligned}
Z_{2}(t) & \leq C e^{-\alpha_{2} t}\left(1+\frac{t}{2}\right)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}} \int_{\frac{t}{2}}^{t} e^{\alpha_{2} \tau} d \tau+C \int_{\frac{t}{2}}^{t}\left\|\nabla \Delta \mathbf{d}_{P}(\tau)\right\|^{2} d \tau \\
& \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}+C(1+t)^{-1-2 \gamma} \\
& \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}
\end{aligned}
$$

As a result, we obtain that

$$
\begin{equation*}
z(t) \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}, \quad \forall t \geq 0 \tag{4.34}
\end{equation*}
$$

In particular, we have

$$
\begin{equation*}
\mathcal{A}_{P}(t) \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}, \quad t \geq 0 \tag{4.35}
\end{equation*}
$$

which together with (4.20) and (4.29) yields the following estimate

$$
\|\mathbf{v}(t)\|_{\mathbf{V}}^{2}+\left\|\Delta \mathbf{d}(t)-\Delta \mathbf{d}_{\infty}\right\|^{2} \leq C(1+t)^{-\frac{2 \theta^{\prime}}{1-2 \theta^{\prime}}}, \quad \forall t \geq 0
$$

Finally, using a standard elliptic estimate, we obtain (cf. (H7))

$$
\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{2}} \leq C\left\|\Delta \mathbf{d}(t)-\Delta \mathbf{d}_{\infty}\right\|+C\left\|\mathbf{h}(t)-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma)} \leq C(1+t)^{-\frac{\theta^{\prime}}{1-2 \theta^{\prime}}}
$$

for all $t \geq 0$ and this finishes the proof.

## 5 Long-time behavior in 3D

As in the classical Navier-Stokes case (see [16]), we can prove the eventual regularity of any global weak solution. Thus the convergence results can also be extended to the 3D case. Indeed, comparing with Lemma 2.6, we derive first an alternative higher-order energy inequality.

Lemma 5.1. Let the assumptions of Proposition 2.1 hold for all $T>0$. Suppose, in addition, that (2.32) -(2.34) are satisfied. If a weak solution (v, d) is smooth enough then it fulfills the following inequality

$$
\begin{equation*}
\frac{d}{d t} \mathcal{A}_{P}(t)+\nu\|S \mathbf{v}\|^{2}+\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2} \leq C_{*}\left(\mathcal{A}_{P}^{3}(t)+\mathcal{A}_{P}(t)+R_{3}(t)\right) \tag{5.1}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{3}(t)=\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{6}+\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{2}+\left\|\nabla \Delta \mathbf{d}_{P}(t)\right\|^{2}+\|\mathbf{g}(t)\|^{2} \tag{5.2}
\end{equation*}
$$

for all $t \geq 0$. Here $C_{*}$ is a positive constant that may depend on $\nu,\left\|\mathbf{v}_{0}\right\|,\left\|\mathbf{d}_{0}\right\|_{\mathbf{H}^{1}},\|\mathbf{g}\|_{L^{2}\left(0,+\infty ; \mathbf{V}^{*}\right)}$, $\|\mathbf{h}\|_{L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{3}{2}}(\Gamma)\right)},\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)},\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}$.
Proof. We reconsider the estimates in the proof of Lemma 2.6. Recalling (2.29) and (2.30), thanks to the Young inequality, it is not difficult to obtain that

$$
\begin{gathered}
\left|I_{1}\right| \leq\|S \mathbf{v}\|\|\mathbf{v}\|_{\mathbf{L}^{6}}\|\nabla \mathbf{v}\|_{\mathbf{L}^{3}} \leq C\|S \mathbf{v}\|^{\frac{3}{2}}\|\nabla \mathbf{v}\|^{\frac{3}{2}} \leq \varepsilon\|S \mathbf{v}\|^{2}+C\|\nabla \mathbf{v}\|^{6} \\
\left|I_{2}\right| \leq \varepsilon\|S \mathbf{v}\|^{2}+C\|\mathbf{g}\|^{2} \\
\left|I_{3}\right| \leq \begin{array}{c}
\varepsilon\|S \mathbf{v}\|^{2}+\varepsilon\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{6}+C\|\Delta \tilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2} \\
+C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{6}+C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}+C\left\|\nabla \Delta \mathbf{d}_{P}\right\|^{2}
\end{array} \\
\left|I_{4}\right| \leq \varepsilon\|S \mathbf{v}\|^{2}+\varepsilon\|\nabla(\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\|^{2}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{6} \\
+C\|\nabla \mathbf{v}\|^{6}+C\|\nabla \mathbf{v}\|^{2}+C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{6}, \\
\left|I_{5 a}\right| \leq \varepsilon\|S \mathbf{v}\|^{2}+C\|\Delta \widetilde{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}+C\|\nabla \mathbf{v}\|^{2}+C\left\|\partial_{t} \mathbf{d}_{P}\right\|^{2}
\end{gathered}
$$

In addition, $I_{5 b}$ can be exactly estimated as (2.31). Collecting all the estimates, and taking $\varepsilon$ to be sufficiently small, we obtain our conclusion (5.1).

Then we prove the following sufficient condition for the existence of global strong solution in 3D.

Proposition 5.1. Suppose that the assumptions of Proposition 2.1 and (2.32) -(2.34) are satisfied. In addition, assume that $\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right) \in \mathbf{V} \times \mathbf{H}^{2}(\Omega)$. If there exists a sufficiently small $\varepsilon_{0} \in(0,1]$ such that

$$
\begin{equation*}
\int_{0}^{+\infty}\left(\nu\|\nabla \mathbf{v}(t)\|^{2}+\|\Delta \widehat{\mathbf{d}}(t)-\mathbf{f}(\mathbf{d}(t))\|^{2}\right) d t \leq \varepsilon_{0} \tag{5.3}
\end{equation*}
$$

then problem (1.1) -(1.5) admits a unique global strong solution $(\mathbf{v}, \mathbf{d})$ in $\Omega \times(0,+\infty)$, provided that $\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)}$ is small enough.
Proof. For simplicity, we give a formal proof. To make it rigorous we should work within a proper approximation scheme (see, for instance, [1, 6]). Let $L_{i}>0(i=1,2,3,4,5)$ be the constants such that

$$
\begin{align*}
\left\|\mathbf{v}_{0}\right\|+\left\|\mathbf{d}_{0}\right\|_{\mathbf{H}^{1}} & \leq L_{1}  \tag{5.4}\\
\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)} & \leq L_{2}  \tag{5.5}\\
\|\mathbf{h}\|_{L_{t b}^{2}\left(0,+\infty ; \mathbf{H}^{\frac{3}{2}}(\Gamma)\right)} & \leq L_{3} \tag{5.6}
\end{align*}
$$

$$
\begin{align*}
\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)} & \leq L_{4},  \tag{5.7}\\
\|\mathbf{g}\|_{L^{2}\left(0,+\infty ; \mathbf{V}^{*}\right)} & \leq L_{5} . \tag{5.8}
\end{align*}
$$

It follows from the basic energy inequality (2.16) that

$$
\begin{equation*}
\widehat{\mathcal{E}}(t)+\frac{1}{2} \int_{0}^{t}\left(\nu\|\nabla \mathbf{v}\|^{2}+\|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}\right) d \tau \leq \widehat{\mathcal{E}}(0)+\int_{0}^{+\infty} r(t) d t, \quad \forall t \geq 0 \tag{5.9}
\end{equation*}
$$

Then, by definition of $\widehat{\mathcal{E}}$ and Lemma6.1, we have

$$
\begin{align*}
& \|\mathbf{v}(t)\|+\|\mathbf{d}(t)\|_{\mathbf{H}^{1}} \leq C_{1}, \quad \forall t \geq 0  \tag{5.10}\\
& \int_{0}^{+\infty}\left(\nu\|\nabla \mathbf{v}\|^{2}+\|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}\right) d t \leq C_{2} \tag{5.11}
\end{align*}
$$

where the constants $C_{1}, C_{2}$ depend on $L_{1}, \ldots, L_{5}$ and $\Omega$.
Let $K>0$ be such that

$$
\begin{equation*}
\nu\left\|\nabla \mathbf{v}_{0}\right\|^{2}+\left\|\Delta \widetilde{\mathbf{d}}(0)-f\left(\mathbf{d}_{0}\right)\right\|^{2} \leq K . \tag{5.12}
\end{equation*}
$$

Keeping Lemma 5.1 in mind and arguing as in [20], we consider the following Cauchy problem

$$
\begin{equation*}
\frac{d}{d t} Y(t)=C_{*}\left(Y(t)^{3}+Y(t)\right)+C_{*} R_{3}(t), \quad Y(0)=\max \left\{1, \nu^{-1}\right\} K \geq \mathcal{A}_{P}(0) \tag{5.13}
\end{equation*}
$$

We denote by $I=\left[0, T_{\max }\right.$ ) the (right) maximal interval for the existence of a (nonnegative) solution $Y(t)$ so that $\lim _{t \rightarrow T_{\max }} Y(t)=+\infty$. It easily follows from (5.1) and the comparison principle that $0 \leq \mathcal{A}_{P}(t) \leq Y(t)$, for any $t \in I$. Consequently, $\mathcal{A}_{P}(t)$ is finite on $I$. We deduce from Lemma 6.2 that

$$
\int_{0}^{+\infty} R_{3}(t) d t \leq C_{3}
$$

where $C_{3}$ is a constant depending on $\Omega,\|\mathbf{g}\|_{L^{2}(0,+\infty ; \mathbf{H})}$ and $L_{2}$. Besides, we note that $T_{\max }$ is determined by $Y(0), C_{*}$ and $C_{3}$ such that $T_{\max }=T_{\max }\left(Y(0), C_{*}, C_{3}\right)$ is increasing when $Y(0) \geq 0$ is decreasing. Taking $t_{0}=\frac{1}{2} T_{\max }>0$, then it follows that $Y(t)$ (as well as $\mathcal{A}_{P}(t)$ ) is uniformly bounded on $\left[0, t_{0}\right]$. This easily implies the local existence of a unique strong solution to problem (1.1)-(1.5) (at least) on $\left[0, t_{0}\right]$ (actually on $\left[0, T_{\max }\right.$ ), but we lose uniform estimates on such maximal interval).

By Lemma 6.2 (cf. (6.15)), we have

$$
\begin{equation*}
\sup _{t \geq 0}\left\|\Delta\left(\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right)\right\|^{2} \leq c\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)}^{2}, \tag{5.14}
\end{equation*}
$$

where $c$ is a constant that depends only on $\Omega$. Set now

$$
\begin{equation*}
\bar{\varepsilon}_{0}=\min \left\{1, \frac{t_{0} K}{8}\right\}, \quad L_{6}=\min \left\{1, L_{2}^{2}, \frac{K}{4 c}\right\} . \tag{5.15}
\end{equation*}
$$

From the assumption, there exists a small constant $\varepsilon_{0} \leq \bar{\varepsilon}_{0}$ such that (5.3) is satisfied. Therefore, we can find $t_{*} \in\left[\frac{t_{0}}{2}, t_{0}\right]$ such that

$$
\nu\left\|\nabla \mathbf{v}\left(t_{*}\right)\right\|^{2}+\left\|\Delta \widehat{\mathbf{d}}\left(t_{*}\right)-\mathbf{f}\left(\mathbf{d}\left(t_{*}\right)\right)\right\|^{2} \leq 2 \bar{\varepsilon}_{0} t_{0}^{-1} .
$$

Moreover, if we further assume

$$
\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)}^{2} \leq L_{6},
$$

then by (5.14) we obtain

$$
\begin{aligned}
\mathcal{A}_{P}\left(t_{*}\right) & \leq \nu\left\|\nabla \mathbf{v}\left(t_{*}\right)\right\|^{2}+\left\|\Delta \widetilde{\mathbf{d}}\left(t_{*}\right)-\mathbf{f}\left(\mathbf{d}\left(t_{*}\right)\right)\right\|^{2} \\
& \leq \nu\left\|\nabla \mathbf{v}\left(t_{*}\right)\right\|^{2}+2\left\|\Delta \widehat{\mathbf{d}}\left(t_{*}\right)-\mathbf{f}\left(\mathbf{d}\left(t_{*}\right)\right)\right\|^{2}+2\left\|\Delta\left(\mathbf{d}_{P}\left(t_{*}\right)-\mathbf{d}_{E}\left(t_{*}\right)\right)\right\|^{2} \\
& \leq \nu\left\|\nabla \mathbf{v}\left(t_{*}\right)\right\|^{2}+2\left\|\Delta \widehat{\mathbf{d}}\left(t_{*}\right)-\mathbf{f}\left(\mathbf{d}\left(t_{*}\right)\right)\right\|^{2}+2 c\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)}^{2} \\
& \leq 4 \bar{\varepsilon}_{0} t_{0}^{-1}+2 c L_{6} \leq K \\
& \leq \max \left\{1, \nu^{-1}\right\} K=Y(0) .
\end{aligned}
$$

Taking $t_{*}$ as the initial time for the ordinary differential equation (5.13), we infer from the above argument that $\mathcal{A}_{P}(t)$ is uniformly bounded at least on $\left[0, \frac{3 t_{0}}{2}\right] \subset\left[0, t_{*}+t_{0}\right]$. Moreover, its bound only depends on $\Omega, \nu, L_{1}, \ldots, L_{6}, C_{*}$ and $t_{0}$. Then by an iterative argument we can show that $\mathcal{A}_{P}(t)$ is uniformly bounded for all $t \geq 0$ and this enable us to extend the local strong solution to the whole time interval $[0,+\infty)$. The proof is complete.

A consequence of the above proposition is the eventual regularity of global weak solutions.
Theorem 5.1. Suppose that the assumptions of Proposition 2.1 and (2.32)-(2.34) are satisfied. Let $(\mathbf{v}, \mathbf{d})$ be a global weak solution to (1.1)-(1.5). Then there exists a large time $T^{*} \in(0,+\infty)$ such that $(\mathbf{v}, \mathbf{d})$ is a strong solution on $\left(T^{*},+\infty\right)$.

Proof. Let $L_{1}, L_{2}, L_{3}, L_{4}, L_{5}>0$ be the constants as in the proof of Proposition 5.2, For a weak solution ( $\mathbf{v}, \mathbf{d}$ ), we still have the uniform estimates (5.10) and (5.11). Considering the ODE problem (5.13), we can fix the constants $\bar{\varepsilon}_{0}, L_{6}$ and $t_{0}$. Taking $\varepsilon_{0}=\bar{\varepsilon}_{0}$, we observe that there must exist a sufficiently large $T_{1}>0$ such that

$$
\begin{align*}
\int_{T_{1}}^{+\infty}\left(\nu\|\nabla \mathbf{v}\|^{2}+\|\Delta \widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\|^{2}\right) d t & \leq \varepsilon_{0}  \tag{5.16}\\
\left\|\Delta \mathbf{d}_{P}(t)-\Delta \mathbf{d}_{E}(t)\right\| & \leq L_{6}, \quad \forall t \geq\left[T_{1},+\infty\right) \tag{5.17}
\end{align*}
$$

where for the second inequality we have used Lemmar $6.2(\mathrm{i})$ and the fact that $\partial_{t} \mathbf{d}_{P}(t)=\Delta \mathbf{d}_{P}(t)-$ $\Delta \mathbf{d}_{E}(t)$. Also, (5.16) implies that there is $T^{*} \in\left[T_{1}, T_{1}+2 t_{0}\right]$ such that

$$
\begin{equation*}
\nu\left\|\nabla \mathbf{v}\left(T_{*}\right)\right\|^{2}+\left\|\Delta \widehat{\mathbf{d}}\left(T_{*}\right)-\mathbf{f}\left(\mathbf{d}\left(T_{*}\right)\right)\right\|^{2} \leq \frac{\bar{\varepsilon}_{0}}{t_{0}} . \tag{5.18}
\end{equation*}
$$

As a result,

$$
\begin{aligned}
& \nu\left\|\nabla \mathbf{v}\left(T_{*}\right)\right\|^{2}+\left\|\Delta \widetilde{\mathbf{d}}\left(T_{*}\right)-\mathbf{f}\left(\mathbf{d}\left(T_{*}\right)\right)\right\|^{2} \\
\leq & \nu\left\|\nabla \mathbf{v}\left(T_{*}\right)\right\|^{2}+2\left\|\Delta \widehat{\mathbf{d}}\left(T_{*}\right)-\mathbf{f}\left(\mathbf{d}\left(T_{*}\right)\right)\right\|^{2}+2\left\|\Delta\left(\mathbf{d}_{P}\left(T_{*}\right)-\mathbf{d}_{E}\left(T_{*}\right)\right)\right\|^{2} \\
\leq & \frac{2 \bar{\varepsilon}_{0}}{t_{0}}+2 c L_{6} \\
\leq & K .
\end{aligned}
$$

Taking $T^{*}$ as the initial time, then we can apply Proposition 5.1 to conclude that problem (1.1)-(1.5) admits a unique global strong solution $\left(\mathbf{v}^{\prime}, \mathbf{d}^{\prime}\right)$. By the weak/strong uniqueness result [6. Theorem 7], we see that $(\mathbf{v}, \mathbf{d})$ coincides with $\left(\mathbf{v}^{\prime}, \mathbf{d}^{\prime}\right)$ on $\left[T^{*},+\infty\right)$. The proof is complete.

Thanks to the eventual regularity result we can argue as in the previous section to prove the following result

Theorem 5.2. Suppose that the assumptions of Theorem 5.1 hold. Then any global weak solution given by Proposition 2.1 converges in $\mathbf{V} \times \mathbf{H}^{2}(\Omega)$ to a single equilibrium $\left(\mathbf{0}, \mathbf{d}_{\infty}\right)$ with estimates on the convergence rate similar to the $2 D$ case, provided that $\mathbf{g}$ and $\mathbf{h}$ fulfill the corresponding hypotheses (H1)-(H7) as in Theorems 4.1 and 4.2.

Remark 5.1. We recall that there exists a (unique) global strong solution when the viscosity is large enough (cf. Theorem 2.2). Consequently, due to Lemma 2.6, all the results proven in Section 4 (i.e., Theorem 4.1 and Theorem 4.2) still hold with the same assumptions on the data. The related proofs just require some minor modifications.

The existence of a global strong solution is also ensured (with no restrictions on the fluid viscosity) when the initial data are close to a given equilibrium and the time dependent boundary data satisfies suitable bounds. First, recall that the basic energy inequality (2.16) implies (cf. (5.9))

$$
\int_{0}^{t}\left(\nu\|\nabla \mathbf{v}(t)\|^{2}+\|\Delta \widehat{\mathbf{d}}(t)-\mathbf{f}(\mathbf{d}(t))\|^{2}\right) d t \leq 2(\widehat{\mathcal{E}}(0)-\widehat{\mathcal{E}}(t))+2 \int_{0}^{+\infty} r(t) d t
$$

and

$$
\begin{equation*}
\int_{0}^{+\infty} r(t) d t \leq C_{r}\left(\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}^{2}+\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}+\|\mathbf{g}\|_{L^{2}\left(0,+\infty ; \mathbf{V}^{*}\right)}^{2}\right) \tag{5.19}
\end{equation*}
$$

where $C_{r}$ is a universal constant. Then we can easily deduce from Proposition 5.1 that if the lifted energy stays sufficiently close to its initial state, then system (1.1)-(1.5) admits a unique global strong solution (cf. [20] for the autonomous case).

Proposition 5.2. Assume (2.32) -(2.34) and (2.4) hold. Moreover, suppose that $\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right) \in$ $\mathbf{V} \times \mathbf{H}^{2}(\Omega)$ satisfying (2.5) and $\left|\mathbf{d}_{0}\right|_{\mathbb{R}^{3}} \leq 1$. If there exists a sufficiently small $\varepsilon_{0} \in(0,1]$ such that

$$
\begin{equation*}
\widehat{\mathcal{E}}(t) \geq \widehat{\mathcal{E}}(0)-\varepsilon_{0}, \quad \forall t \geq 0 \tag{5.20}
\end{equation*}
$$

where $\widehat{\mathcal{E}}$ is the lifted energy defined by (2.15), then problem (1.1) -(1.5) admits a unique global strong solution $(\mathbf{v}, \mathbf{d})$ in $\Omega \times(0,+\infty)$, provided that $\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)},\left\|\mathbf{h}_{t}\right\|_{L^{1}\left(0,+\infty ; \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)}$ and $\|\mathbf{g}\|_{L^{2}\left(0,+\infty ; \mathbf{V}^{*}\right)}$ are small enough.

Let us assume that for all $t \geq 0$ (comparing with assumptions (H1), (H4), (H5))
$\left(\mathrm{H}^{\prime}\right) \int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d \tau \leq M_{1}(1+t)^{-1-\gamma} ;$
$\left(\mathrm{H} 4^{\prime}\right)\|\mathbf{g}(t)\|^{2} \leq M_{2}(1+t)^{-2-\gamma} ;$
$\left(\mathrm{H}^{\prime}\right) \quad\left\|\mathbf{h}_{t}(t)\right\|_{\mathbf{L}^{2}(\Gamma)} \leq M_{3}(1+t)^{-1-\gamma}$.
Here $M_{j}, j=1,2,3$ and $\gamma$ are positive constants. $\gamma$ characterizes the decay rate of nonautonomous terms, while $M_{j}$ control their magnitude.

In spirit of Proposition5.2, in what follows, we prove the global existence of a strong solution that originates near a local minimizer of the lifted energy with suitably small perturbations in terms of the nonautonomous terms $\mathbf{h}$ and $\mathbf{g}$ (namely, the magnitudes $M_{j}$ should be sufficiently small).

Theorem 5.3. Suppose that (2.32) -(2.34) and (2.4) hold, the constant $\gamma>1$. Moreover, assume that $\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right) \in \mathbf{V} \times \mathbf{H}^{2}(\Omega)$ satisfies (2.5) and $\left|\mathbf{d}_{0}\right|_{\mathbb{R}^{3}} \leq 1$. Denote by $\mathbf{d}_{E}^{*}$ the unique solution to

$$
\left\{\begin{array}{l}
-\Delta \mathbf{d}_{E}^{*}=\mathbf{0}, \quad x \in \Omega  \tag{5.21}\\
\mathbf{d}_{E}^{*}=\mathbf{h}_{\infty}, \quad x \in \Gamma
\end{array}\right.
$$

and set

$$
\mathscr{E}(\mathbf{d})=\frac{1}{2}\left\|\nabla\left(\mathbf{d}-\mathbf{d}_{E}^{*}\right)\right\|^{2}+\int_{\Omega} F(\mathbf{d}) d x, \quad \forall \mathbf{d} \in \mathcal{N}
$$

Let $\mathbf{d}^{*} \in \mathcal{N} \cap \mathbf{H}^{2}(\Omega)$ be a local minimizer of $\mathscr{E}(\mathbf{d})$ in the sense that $\mathscr{E}(\mathbf{d}) \geq \mathscr{E}\left(\mathbf{d}^{*}\right)$ for all $\mathbf{d} \in \mathcal{N}$ satisfying $\left\|\mathbf{d}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}<\delta$, where $\delta>0$ is a certain small constant. Suppose also that the initial data $\mathbf{v}_{0}$ and $\mathbf{d}_{0}$ satisfy

$$
\begin{equation*}
\left\|\mathbf{v}_{0}\right\|_{\mathbf{V}} \leq 1, \quad\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{2}} \leq 1 \tag{5.22}
\end{equation*}
$$

Then there exist positive constants $\sigma_{1}, \sigma_{2}, M_{1}, M_{2}, M_{3}, L_{0}$, which are sufficiently small and may depend on the system coefficients, on $\Omega$ and on $\mathbf{d}^{*}$, such that if the initial data $\left(\mathbf{v}_{0}, \mathbf{d}_{0}\right)$ and $\mathbf{h}$ also fulfill

$$
\left\|\mathbf{v}_{0}\right\| \leq \sigma_{1}, \quad\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} \leq \sigma_{2}, \quad\left\|\mathbf{h}_{t}\right\|_{L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)}^{2} \leq L_{0}
$$

and (H1'), (H4'), (H5') hold with such $M_{j}, j=1,2,3$, then problem (1.1) -(1.5) admits a unique global strong solution $(\mathbf{v}, \mathbf{d})$.

Proof. Without loss of generality, we assume $\delta \in(0,1]$. In the subsequent proof, $C_{i}(i \in \mathbb{N})$ stand for positive constants that only depend on $\Omega, \nu, \gamma$ and $\mathbf{d}^{*}$. Under the current assumption (5.22) on the initial data, it is not difficult to see that the constants $L_{1}$ and $K$ in (5.4) and (5.12) depend on $\mathbf{d}^{*}$ only. We just take $L_{2}=L_{3}=L_{4}=L_{5}=1$ in (5.6) for the sake of simplicity. Then we have the uniform estimate (cf. (5.10))

$$
\|\mathbf{v}(t)\|+\|\mathbf{d}(t)\|_{\mathbf{H}^{1}} \leq C_{1}, \quad t \geq 0
$$

Arguing as in the proof of Proposition 5.1, we find that problem (1.1)-(1.5) admits a unique strong solution (at least) on $\left[0, t_{0}\right]$, whose $\mathbf{V} \times \mathbf{H}^{2}$ norm is uniformly bounded on $\left[0, t_{0}\right]$ :

$$
\begin{equation*}
\|\mathbf{v}(t)\|_{\mathbf{V}}+\|\mathbf{d}(t)\|_{\mathbf{H}^{2}} \leq C_{3}, \quad t \in\left[0, t_{0}\right] \tag{5.23}
\end{equation*}
$$

Besides, we can also fix the constants $\bar{\varepsilon}_{0}$ and $L_{6}$ (see (5.15)). In the subsequent proof, we just take

$$
\varepsilon_{0}=\bar{\varepsilon}_{0}, \quad L_{0}=L_{6}
$$

It follows from (5.19) that

$$
\int_{0}^{+\infty} r(t) d t \leq C_{r} C_{s}\left(M_{1}+M_{2}+M_{3}^{2}\right) \leq \frac{\varepsilon_{0}}{4}
$$

provided that $M_{1}, M_{2}, M_{3}>0$ are assumed to be properly small and satisfying

$$
M_{1}+M_{2}+M_{3}^{2} \leq \frac{\varepsilon_{0}}{4 C_{r} C_{s}}
$$

where $C_{s}$ is a universal constant due to the Sobolev embedding. Hence, according to Propositions 5.1 and 5.2, in order to prove the existence of global strong solution, we only have to verify that

$$
\begin{equation*}
\widehat{\mathcal{E}}(t)-\widehat{\mathcal{E}}(0) \geq-\frac{\varepsilon_{0}}{2}, \quad \forall t \geq 0 \tag{5.24}
\end{equation*}
$$

First, we notice that (recalling (2.7), (2.8) and (3.12))

$$
\begin{align*}
& \widehat{\mathcal{E}}(0)-\widehat{\mathcal{E}}(t) \\
\leq & \frac{1}{2}\left\|\mathbf{v}_{0}\right\|^{2}+\widehat{E}\left(\mathbf{d}_{0}\right)-\widehat{E}(\mathbf{d}(t)) \\
= & \frac{1}{2}\left\|\mathbf{v}_{0}\right\|^{2}+\frac{1}{2}\left\|\nabla\left(\mathbf{d}_{0}-\mathbf{d}_{E 0}\right)\right\|^{2}-\frac{1}{2}\left\|\nabla\left(\mathbf{d}(t)-\mathbf{d}_{E}\right)\right\|^{2}+\int_{\Omega} F\left(\mathbf{d}_{0}\right)-F(\mathbf{d}(t)) d x \\
\leq & \frac{1}{2}\left\|\mathbf{v}_{0}\right\|^{2}+C_{4}\left(\left\|\mathbf{d}_{0}-\mathbf{d}(t)\right\|_{\mathbf{H}^{1}}+\left\|\mathbf{d}_{E 0}-\mathbf{d}_{E}\right\|_{\mathbf{H}^{1}}\right) \tag{5.25}
\end{align*}
$$

On the other hand, thanks to standard elliptic estimates, we have

$$
\begin{align*}
\left\|\mathbf{d}_{E 0}-\mathbf{d}_{E}\right\|_{\mathbf{H}^{1}} & \leq c\left\|\left.\mathbf{d}_{0}\right|_{\Gamma}-\mathbf{h}(t)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\
& \leq c\left\|\left.\mathbf{d}_{0}\right|_{\Gamma}-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}+c\left\|\mathbf{h}_{\infty}-\mathbf{h}(t)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\
& \leq c\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}+c \int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d \tau \\
& \leq c \sigma_{2}+c M_{1}, \quad \forall t \geq 0 . \tag{5.26}
\end{align*}
$$

Let

$$
\begin{equation*}
\sigma_{1} \leq \min \left\{1, \frac{\sqrt{\varepsilon_{0}}}{2}\right\}, \quad \sigma_{2} \leq \frac{\varepsilon_{0}}{8 C_{4}} \min \left\{1, c^{-1}\right\}, \quad M_{1} \leq \min \left\{1, \frac{\varepsilon_{0}}{8 C_{4} c}\right\} \tag{5.27}
\end{equation*}
$$

Due to (5.25) and (5.26), in order to prove (5.24), we only have to verify

$$
\begin{equation*}
\left\|\mathbf{d}_{0}-\mathbf{d}(t)\right\|_{\mathbf{H}^{1}} \leq \frac{\varepsilon_{0}}{8 C_{4}}, \quad \forall t \geq 0 \tag{5.28}
\end{equation*}
$$

Since $\mathbf{d}^{*} \in \mathcal{N} \cap \mathbf{H}^{2}(\Omega)$ is the local minimizer of $\mathscr{E}$, it is easily to verify that $\mathbf{d}^{*}$ satisfies (3.2) and thus is the critical point of $E$. As a consequence, Corollary 3.1 holds for $\mathbf{d}^{*}$ with constants $\theta, \beta$ determined by $\mathbf{d}^{*}$. By (4.9), $\theta^{\prime}$ can be determined by $\theta$ and $\gamma$. In addition, we further choose $\theta^{\prime}$ smaller if necessary such that (recall that $\gamma>1$ )

$$
\begin{equation*}
\theta^{\prime} \leq \frac{\gamma-1}{2 \gamma} \tag{5.29}
\end{equation*}
$$

Let us define

$$
\begin{equation*}
\varpi=\min \left\{\frac{\beta}{2}, \frac{\delta}{2}, \frac{\varepsilon_{0}}{10 C_{4}}\right\} \tag{5.30}
\end{equation*}
$$

and set

$$
\bar{t}_{0}=\sup \left\{t \in\left[0, t_{0}\right],\left\|\mathbf{d}(t)-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}<\varpi, \forall s \in[0, t)\right\}
$$

If we assume

$$
\begin{equation*}
\sigma_{2} \leq \frac{1}{4} \varpi \tag{5.31}
\end{equation*}
$$

then by the continuity of $\mathbf{d}(t)$ in $\mathbf{H}^{1}(\Omega)$, we have $\bar{t}_{0}>0$. Next, we shall prove that $\bar{t}_{0}>t_{0}$ by contradiction. We introduce the auxiliary functional

$$
\Psi_{1}(t)=\widehat{\mathcal{E}}(t)-\widehat{E}\left(\mathbf{d}^{*}\right)+2 \int_{t}^{+\infty} r(\tau) d \tau
$$

and the function

$$
\overline{\mathbf{d}}(t)=\mathbf{d}(t)-\mathbf{d}_{E}+\mathbf{d}_{E}^{*} .
$$

It easily follows that

$$
\begin{align*}
\Psi_{1}(t) & \geq \widehat{E}(\mathbf{d}(t))-\widehat{E}\left(\mathbf{d}^{*}\right)=\widehat{E}(\mathbf{d}(t))-\mathscr{E}(\overline{\mathbf{d}}(t))+\mathscr{E}(\overline{\mathbf{d}}(t))-\widehat{E}\left(\mathbf{d}^{*}\right) \\
& =\int_{\Omega} F(\mathbf{d}(t))-F(\overline{\mathbf{d}}(t)) d x+\mathscr{E}(\overline{\mathbf{d}}(t))-\widehat{E}\left(\mathbf{d}^{*}\right) \tag{5.32}
\end{align*}
$$

By definition, $\overline{\mathbf{d}}(t) \in \mathcal{N}$. Moreover, on $\left[0, \bar{t}_{0}\right]$,

$$
\begin{aligned}
& \left\|\overline{\mathbf{d}}(t)-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} \leq\left\|\mathbf{d}(t)-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}+\left\|\mathbf{d}_{E}-\mathbf{d}_{E}^{*}\right\|_{\mathbf{H}^{1}} \\
\leq & \varpi+c\left\|\mathbf{h}(t)-\mathbf{h}_{\infty}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq \frac{\delta}{2}+c \int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} d \tau \\
\leq & \frac{\delta}{2}+c M_{1}
\end{aligned}
$$

Taking

$$
\begin{equation*}
M_{1} \leq \min \left\{1, \frac{\delta}{4 c}\right\} \tag{5.33}
\end{equation*}
$$

then we have $\left\|\overline{\mathbf{d}}(t)-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} \leq \delta$. Since $\mathbf{d}^{*}$ is a local minimizer of $\mathscr{E}$, we see that

$$
\begin{equation*}
\mathscr{E}(\overline{\mathbf{d}}(t))-\widehat{E}\left(\mathbf{d}^{*}\right)=\mathscr{E}(\overline{\mathbf{d}}(t))-\mathscr{E}\left(\mathbf{d}^{*}\right) \geq 0, \quad t \in\left[0, \bar{t}_{0}\right] \tag{5.34}
\end{equation*}
$$

On the other hand, since $|\mathbf{d}(t)|_{\mathbb{R}^{3}} \leq 1$ and $|\overline{\mathbf{d}}(t)|_{\mathbb{R}^{3}} \leq 3$ (this is due to the maximum principle (2.6)), we infer from the standard elliptic estimate and (H5') that

$$
\begin{align*}
\left|\int_{\Omega} F(\mathbf{d}(t))-F(\overline{\mathbf{d}}(t)) d x\right| & \leq C_{5}\left\|-\mathbf{d}_{E}+\mathbf{d}_{E}^{*}\right\| \\
& \leq C_{5} c \int_{t}^{+\infty}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{L}^{2}(\Gamma)} d \tau \\
& \leq C_{5} c M_{3} \gamma^{-1}(1+t)^{-\gamma} \tag{5.35}
\end{align*}
$$

Let us introduce now two further functions

$$
z(t)=\left(C_{5} c+1\right) M_{3} \gamma^{-1}(1+t)^{-\gamma}, \quad \Psi(t)=\Psi_{1}(t)+z(t)
$$

We deduce from (5.32) -(5.35) that

$$
\Psi(t) \geq M_{3} \gamma^{-1}(1+t)^{-\gamma}>0, \quad t \in\left[0, \bar{t}_{0}\right]
$$

and by the basic energy inequality (2.16)

$$
\begin{aligned}
\frac{d}{d t} \Psi(t) & =\frac{d}{d t} \widehat{\mathcal{E}}(t)-2 r(t)-\left(C_{5} c+1\right) M_{3}(1+t)^{-1-\gamma} \\
& \leq-\frac{1}{4} \min \{\nu, 1\} \mathcal{D}^{2}(t)-\left(C_{5} c+1\right) M_{3}(1+t)^{-1-\gamma} \\
& \leq-C_{6}\left(\mathcal{D}(t)+M_{3}^{\frac{1}{2}}(1+t)^{-\frac{1+\gamma}{2}}\right)^{2}
\end{aligned}
$$

where $\mathcal{D}(t)=\|\nabla \mathbf{v}(t)\|+\|\Delta \widehat{\mathbf{d}}(t)-\mathbf{f}(\mathbf{d}(t))\|$. Arguing as to get (4.12), using Remark 3.3 and assumptions (H1'), (H4'), we deduce

$$
\Psi(t)^{1-\theta^{\prime}} \leq C_{7}\left(\mathcal{D}(t)+\left(M_{1}+M_{2}\right)(1+t)^{-\left(1-\theta^{\prime}\right)(1+\gamma)}+M_{3}^{1-\theta^{\prime}}(1+t)^{-\left(1-\theta^{\prime}\right) \gamma}\right)
$$

Assuming

$$
\begin{equation*}
M_{1} \leq \frac{1}{2} M_{3}^{\frac{1}{2}}, \quad M_{2} \leq \frac{1}{2} M_{3}^{\frac{1}{2}}, \quad M_{3} \leq 1, \tag{5.36}
\end{equation*}
$$

we can see that

$$
\Psi(t)^{1-\theta^{\prime}} \leq C_{7}\left(\mathcal{D}(t)+2 M_{3}^{\frac{1}{2}}(1+t)^{-\left(1-\theta^{\prime}\right) \gamma}\right)
$$

As a result, we find

$$
\begin{align*}
-\frac{d}{d t} \Psi(t)^{\theta^{\prime}} & =-\theta^{\prime} \Psi(t)^{\theta^{\prime}-1} \frac{d}{d t} \Psi(t) \\
& \geq \frac{C_{6}\left(\mathcal{D}(t)+M_{3}^{\frac{1}{2}}(1+t)^{-\frac{1+\gamma}{2}}\right)^{2}}{C_{7}\left(\mathcal{D}(t)+M_{3}^{\frac{1}{2}}(1+t)^{-\left(1-\theta^{\prime}\right) \gamma}\right)} \\
& \geq C_{8}\left(\mathcal{D}(t)+M_{3}^{\frac{1}{2}}(1+t)^{-\frac{1+\gamma}{2}}\right) \tag{5.37}
\end{align*}
$$

where we have used the fact that $\frac{1+\gamma}{2} \leq\left(1-\theta^{\prime}\right) \gamma$ (cf. (5.29)). It follows from (4.16), (5.23), (5.36), (5.37), assumptions ( $\mathrm{H} 1^{\prime}$ ), ( $\mathrm{H} 4^{\prime}$ ), ( $\mathrm{H} 5^{\prime}$ ) and the definition of $\Psi$ that

$$
\begin{align*}
& \int_{0}^{\tau_{0}}\left\|\mathbf{d}_{t}(t)\right\| d t \leq C_{9} \Psi(0)^{\theta^{\prime}} \\
\leq & C_{10}\left(\left\|\mathbf{v}_{0}\right\|^{2}+\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}+\left\|\mathbf{d}_{E 0}-\mathbf{d}_{E}^{*}\right\|_{\mathbf{H}^{1}}+\int_{0}^{+\infty} r(t) d t+z(0)\right)^{\theta^{\prime}} \\
\leq & C_{11}\left(\left\|\mathbf{v}_{0}\right\|^{2}+\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}+M_{3}^{\frac{1}{2}}\right)^{\theta^{\prime}} \tag{5.38}
\end{align*}
$$

By (5.23), (55.38) and an interpolation inequality, we get

$$
\begin{align*}
& \left\|\mathbf{d}\left(\bar{t}_{0}\right)-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} \\
\leq & \left\|\mathbf{d}\left(\bar{t}_{0}\right)-\mathbf{d}_{0}\right\|_{\mathbf{H}^{1}}+\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} \\
\leq & C_{12}\left(\left\|\mathbf{d}\left(\bar{t}_{0}\right)\right\|_{\mathbf{H}^{2}}+\left\|\mathbf{d}_{0}\right\|_{\mathbf{H}^{2}}^{\frac{1}{2}}\left\|\mathbf{d}\left(\bar{t}_{0}\right)-\mathbf{d}_{0}\right\|^{\frac{1}{2}}+\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}\right. \\
\leq & C_{13}\left(\left\|\mathbf{v}_{0}\right\|^{\theta^{\prime}}+\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}^{\frac{\theta^{\prime}}{2}}+M_{3}^{\frac{\theta^{\prime}}{4}}\right)+\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} . \tag{5.39}
\end{align*}
$$

Taking now

$$
\begin{align*}
& \sigma_{1} \leq \min \left\{1, \frac{\sqrt{\varepsilon_{0}}}{2},\left(\frac{\varpi}{6 C_{13}}\right)^{\frac{1}{\theta^{\prime}}}\right\}, \quad \sigma_{2} \leq \min \left\{1, \frac{1}{4} \varpi,\left(\frac{\varpi}{6 C_{13}}\right)^{\frac{2}{\theta^{\prime}}}\right\},  \tag{5.40}\\
& M_{3} \leq \min \left\{1,\left(\frac{\varpi}{6 C_{13}}\right)^{\frac{4}{\theta^{\prime}}}\right\}, \tag{5.41}
\end{align*}
$$

we infer from (5.39) that

$$
\left\|\mathbf{d}\left(\bar{t}_{0}\right)-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} \leq \frac{3}{4} \varpi<\varpi
$$

This leads to a contradiction with the definition of $\bar{t}_{0}$. As a result, we have $\bar{t}_{0}>t_{0}$, and

$$
\left\|\mathbf{d}_{0}-\mathbf{d}(t)\right\|_{\mathbf{H}^{1}} \leq\left\|\mathbf{d}_{0}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}+\left\|\mathbf{d}^{*}-\mathbf{d}(t)\right\|_{\mathbf{H}^{1}}
$$

$$
\begin{equation*}
\leq \sigma_{2}+\varpi \leq \frac{5}{4} \varpi \leq \frac{\varepsilon_{0}}{8 C_{4}}, \quad \forall t \in\left[0, t_{0}\right] \tag{5.42}
\end{equation*}
$$

Thus, (5.24) holds on $\left[0, t_{0}\right]$, which implies

$$
\int_{0}^{t_{0}}\left(\nu\|\nabla \mathbf{v}(t)\|^{2}+\|\Delta \widehat{\mathbf{d}}(t)-\mathbf{f}(\mathbf{d}(t))\|^{2}\right) d t \leq \varepsilon_{0}
$$

As in Proposition [5.1, there exists $t_{*} \in\left[\frac{t_{0}}{2}, t_{0}\right]$ such that

$$
\nu\left\|\nabla \mathbf{v}\left(t_{*}\right)\right\|^{2}+\left\|\Delta \widehat{\mathbf{d}}\left(t_{*}\right)-\mathbf{f}\left(\mathbf{d}\left(t_{*}\right)\right)\right\|^{2} \leq 2 \varepsilon_{0} t_{0}^{-1}
$$

and again we have $\mathcal{A}_{P}\left(t_{*}\right) \leq \max \left\{1, \nu^{-1}\right\} K$. Taking $t_{*}$ as the initial time for the Cauchy problem (5.13), we can extend the (unique) strong solution to [0, $\left.\frac{3}{2} t_{0}\right]$ and its $\mathbf{V} \times \mathbf{H}^{2}$-norm is uniformly bounded by the same constant $C_{3}$ as on $\left[0, t_{0}\right]$. Repeating the above argument in $\left[0, \frac{3}{2} t_{0}\right]$, we can verify that (5.24) still holds. By iteration we can show that (5.24) holds for all $t \geq 0$. Hence, our conclusion follows from Proposition 5.2,

Finally, we can conclude with the following local stability result:
Theorem 5.4. Let the assumptions of Theorem 5.3 hold. Then any global strong solution given by Theorem 5.3 converges in $\mathbf{V} \times \mathbf{H}^{2}(\Omega)$ to a single equilibrium $\left(\mathbf{0}, \mathbf{d}_{\infty}\right)$ with $\mathbf{d}_{\infty} \in \mathcal{N} \cap \mathbf{H}^{2}(\Omega)$ such that $\mathscr{E}\left(\mathbf{d}_{\infty}\right)=\mathscr{E}\left(\mathbf{d}^{*}\right)$. In addition, convergence rate estimates similar to the $2 D$ case hold provided that $\mathbf{g}$ and $\mathbf{h}$ fulfill the corresponding hypotheses (i.e., assumptions (H1), (H4), (H5) are replaced by (H1'), (H4'), (H5'), respectively). Indeed, the local energy minimizer $\mathbf{d}^{*}$ is (locally) Lyapunov stable, and in particular, if $\mathbf{d}^{*}$ is an isolated local minimizer of $\mathscr{E}$, then it is (locally) asymptotically stable.

Proof. Arguing as in Section 4 we still find

$$
\begin{equation*}
\lim _{t \rightarrow+\infty}\left(\|\mathbf{v}(t)\|_{\mathbf{V}}+\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{2}}\right)=0 \tag{5.43}
\end{equation*}
$$

for some $\mathbf{d}_{\infty} \in \mathcal{N} \cap \mathbf{H}^{2}(\Omega)$. The estimate on the convergence rates can be obtained following the proof of Theorem 4.2.

Recalling the proof of Theorem 5.3, we actually showed that

$$
\begin{equation*}
\left\|\mathbf{d}(t)-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} \leq \varpi, \quad \forall t \geq 0 \tag{5.44}
\end{equation*}
$$

which implies that (let $t$ be large)

$$
\left\|\mathbf{d}_{\infty}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} \leq\left\|\mathbf{d}(t)-\mathbf{d}_{\infty}\right\|_{\mathbf{H}^{1}}+\left\|\mathbf{d}(t)-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}}<2 \varpi \leq \min \{\beta, \delta\}
$$

Taking $\mathbf{d}=\mathbf{d}_{\infty}$ and $\psi=\mathbf{d}^{*}$ in Corollary 3.1, we see that

$$
\left|\mathscr{E}\left(\mathbf{d}_{\infty}\right)-\mathscr{E}\left(\mathbf{d}^{*}\right)\right|^{1-\theta}=\left|\widehat{E}\left(\mathbf{d}_{\infty}\right)-\widehat{E}\left(\mathbf{d}^{*}\right)\right|^{1-\theta} \leq\left\|-\Delta \widehat{\mathbf{d}}^{*}+\mathbf{f}\left(\mathbf{d}^{*}\right)\right\|=0
$$

Since $\left\|\mathbf{d}_{\infty}-\mathbf{d}^{*}\right\|_{\mathbf{H}^{1}} \leq \delta, \mathbf{d}_{\infty}$ is also an energy minimizer of $\mathscr{E}$.
Moreover, the proof of Theorem 5.3 implies that, for arbitrary (small) $\epsilon>0$, if we replace the choice of $\varpi(5.30)$ by

$$
\begin{equation*}
\varpi_{1}=\min \left\{\epsilon, \frac{\beta}{2}, \frac{\delta}{2}, \frac{\varepsilon_{0}}{10 C_{4}}\right\} \tag{5.45}
\end{equation*}
$$

then we are able to choose the constants $\sigma_{i}, M_{j}$ sufficiently small in a similar manner such that (5.43) and (5.44) hold with $\varpi$ being replaced by $\varpi_{1}$ (and thus (5.44) holds for $\epsilon$ ). This yields the (locally) Lyapunov stability of $\mathbf{d}^{*}$. Finally, it is easy to see that if $\mathbf{d}^{*}$ is an isolated local minimizer, then $\mathbf{d}_{\infty}=\mathbf{d}^{*}$ and $\mathbf{d}^{*}$ is asymptotically stable. The proof is complete.

## 6 Appendix

We report some properties of the lifting functions $\mathbf{d}_{E}$ and $\mathbf{d}_{P}$ (cf. (2.7) and (2.21)) that have been used in the previous sections. Below we denote by $c$ a generic positive constant which depends on $n$ and $\Omega$ at most.

Lemma 6.1. For any $t \geq 0$, and $k=0,1,2, \ldots, j=0,1$, we have
(i) $\left\|\partial_{t}^{j} \mathbf{d}_{E}(t)\right\|_{\mathbf{H}^{k}} \leq c\left\|\partial_{t}^{j} \mathbf{h}(t)\right\|_{\mathbf{H}^{k-\frac{1}{2}}(\Gamma)}$;
(ii) $\left\|\mathbf{d}_{E}(t)-\mathbf{d}_{*}\right\|_{\mathbf{H}^{k}} \leq c\left\|\mathbf{h}(t)-\mathbf{h}_{*}\right\|_{\mathbf{H}^{k-\frac{1}{2}}(\Gamma)}$, where $\mathbf{d}_{*}$ is the unique solution to

$$
\left\{\begin{array}{l}
-\Delta \mathbf{d}_{*}=\mathbf{0}, \quad x \in \Omega  \tag{6.1}\\
\mathbf{d}_{*}=\mathbf{h}_{*}, \quad x \in \Gamma
\end{array}\right.
$$

Proof. The conclusion follows from the classical elliptic regularity theory (cf., e.g., [24, 32]).
Lemma 6.2. Let $\mathbf{d}_{0} \in \mathbf{H}^{2}(\Omega)$ with $\left|\mathbf{d}_{0}\right|_{\mathbb{R}^{n}} \leq 1$. Suppose that $\mathbf{h}$ satisfy (2.4) -(2.5) and $\mathbf{h}_{t} \in$ $L_{l o c}^{2}\left([0,+\infty) ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)$. Then, for any $t>0$, the following estimates hold

$$
\begin{align*}
\left\|\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right\|_{\mathbf{H}^{1}}^{2} & \leq c e^{-t} \int_{0}^{t} e^{\tau}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}^{2} d \tau  \tag{6.2}\\
\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{2}+\left\|\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right\|_{\mathbf{H}^{2}}^{2} & \leq c \int_{0}^{t}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{2} d \tau  \tag{6.3}\\
\int_{0}^{t}\left\|\nabla \Delta \mathbf{d}_{P}\right\|^{2} d \tau & \leq c \int_{0}^{t}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{2} d \tau \tag{6.4}
\end{align*}
$$

In addition, we have
(i) if $\mathbf{h}_{t} \in L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)$ then

$$
\begin{equation*}
\lim _{t \rightarrow+\infty}\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|=0 \tag{6.5}
\end{equation*}
$$

(ii) if $\mathbf{h}_{t}$ satisfies (H6) then, for all $t \geq 0$,

$$
\begin{align*}
\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|^{2}+\left\|\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right\|_{\mathbf{H}^{2}}^{2} & \leq c(1+t)^{-2-2 \gamma}  \tag{6.6}\\
\int_{\frac{t}{2}}^{t}\left\|\nabla \Delta \mathbf{d}_{P}(\tau)\right\|^{2} d \tau & \leq c(1+t)^{-1-2 \gamma} \tag{6.7}
\end{align*}
$$

Proof. It follows from (2.7) and (2.21) that

$$
\left\{\begin{array}{l}
-\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)=-\partial_{t} \mathbf{d}_{P}, \quad \text { in } \Omega \times \mathbb{R}^{+}  \tag{6.8}\\
\mathbf{d}_{P}-\mathbf{d}_{E}=\mathbf{0}, \quad \text { on } \Gamma \times \mathbb{R}^{+},
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\partial_{t}\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)-\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)=-\partial_{t} \mathbf{d}_{E}, \quad \text { in } \Omega \times \mathbb{R}^{+}  \tag{6.9}\\
\mathbf{d}_{P}-\mathbf{d}_{E}=\mathbf{0}, \quad \text { on } \Gamma \times \mathbb{R}^{+} \\
\mathbf{d}_{P}-\left.\mathbf{d}_{E}\right|_{t=0}=\mathbf{0}, \quad \text { in } \Omega
\end{array}\right.
$$

Multiplying the first equation in (6.9) by $\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)-\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)$, integrating by parts and using the Poincaré inequality, we obtain

$$
\frac{1}{2} \frac{d}{d t}\left(\left\|\mathbf{d}_{P}-\mathbf{d}_{E}\right\|^{2}+\left\|\nabla\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2}\right)+\left\|\nabla\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2}+\left\|\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2}
$$

$$
\begin{align*}
& \leq\left(\left\|\mathbf{d}_{P}-\mathbf{d}_{E}\right\|+\left\|\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|\right)\left\|\partial_{t} \mathbf{d}_{E}\right\| \\
& \leq\left(C_{P}\left\|\nabla\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|+\left\|\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|\right)\left\|\partial_{t} \mathbf{d}_{E}\right\| \\
& \leq \frac{1}{2}\left\|\nabla\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2}+\frac{1}{2}\left\|\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2}+\left(\frac{1}{2} C_{P}^{2}+\frac{1}{2}\right)\left\|\partial_{t} \mathbf{d}_{E}\right\|^{2} \tag{6.10}
\end{align*}
$$

which, together with Lemma 6.1 implies

$$
\begin{align*}
\left\|\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right\|_{\mathbf{H}^{1}}^{2} & \leq c e^{-c_{1} t} \int_{0}^{t} e^{c_{1} \tau}\left\|\partial_{t} \mathbf{d}_{E}(\tau)\right\|^{2} d \tau \\
& \leq c e^{-c_{1} t} \int_{0}^{t} e^{c_{1} \tau}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}^{2} d \tau \tag{6.11}
\end{align*}
$$

that is, (6.2).
Applying now the Laplacian to the first equation in (6.9), we get

$$
\left\{\begin{array}{l}
\partial_{t} \Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)-\Delta^{2}\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)=\mathbf{0}, \quad \text { in } \Omega \times \mathbb{R}^{+}  \tag{6.12}\\
\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)=\mathbf{h}_{t}, \quad \text { on } \Gamma \times \mathbb{R}^{+}, \\
\left.\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right|_{t=0}=\mathbf{0}, \quad \text { in } \Omega
\end{array}\right.
$$

Multiplying the first equation of (6.12) by $\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)$ and integrating by parts, we get

$$
\begin{align*}
& \frac{1}{2} \frac{d}{d t}\left\|\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2}+\left\|\nabla \Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2} \\
\leq & \left\|\partial_{\mathbf{n}} \Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}\left\|\mathbf{h}_{t}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\
\leq & c\left\|\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|_{\mathbf{H}^{1}}\left\|\mathbf{h}_{t}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \\
\leq & \frac{1}{2}\left(\left\|\nabla \Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2}+\left\|\Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2}\right)+c\left\|\mathbf{h}_{t}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{2} . \tag{6.13}
\end{align*}
$$

Hence, from (6.10) and (6.13) we infer

$$
\begin{equation*}
\frac{d}{d t}\left\|\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right\|_{\mathbf{H}^{2}}^{2}+c_{2}\left(\left\|\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right\|_{\mathbf{H}^{2}}^{2}+\left\|\nabla \Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)\right\|^{2}\right) \leq c\left\|\mathbf{h}_{t}\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{2} \tag{6.14}
\end{equation*}
$$

which entails (6.4) and

$$
\begin{equation*}
\left\|\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right\|_{\mathbf{H}^{2}}^{2} \leq c \int_{0}^{t}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{2} d \tau \tag{6.15}
\end{equation*}
$$

Thus (6.3) follows from (6.15) and the fact $\left\|\partial_{t} \mathbf{d}_{P}(t)\right\|=\left\|\Delta \mathbf{d}_{P}(t)\right\|$.
Now if $\mathbf{h}_{t} \in L^{2}\left(0,+\infty ; \mathbf{H}^{\frac{1}{2}}(\Gamma)\right)$, we infer from (6.10) that

$$
\begin{align*}
\int_{0}^{+\infty}\left\|\Delta\left(\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right)\right\|^{2} d t & \leq c \int_{0}^{+\infty}\left\|\partial_{t} \mathbf{d}_{E}(t)\right\|^{2} d t \\
& \leq c \int_{0}^{+\infty}\left\|\mathbf{h}_{t}(t)\right\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}^{2} d t<+\infty \tag{6.16}
\end{align*}
$$

Then it follows from (6.13), (6.16) and Lemma 4.1 that

$$
\lim _{t \rightarrow+\infty}\left\|\Delta\left(\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right)\right\|^{2}=0
$$

which implies (6.5).

Furthermore, if (H6) holds, then (6.14) implies that (cf., e.g., [34])

$$
\left\|\mathbf{d}_{P}(t)-\mathbf{d}_{E}(t)\right\|_{\mathbf{H}^{2}}^{2} \leq c(1+t)^{-2-2 \gamma}, \quad \forall t \geq 0
$$

Using (6.14) once more, we have

$$
\begin{aligned}
& \int_{\frac{t}{2}}^{t}\left\|\nabla \Delta \mathbf{d}_{P}(\tau)\right\|^{2} d \tau=\int_{\frac{t}{2}}^{t}\left\|\nabla \Delta\left(\mathbf{d}_{P}-\mathbf{d}_{E}\right)(\tau)\right\|^{2} d \tau \\
\leq & c\left\|\mathbf{d}_{P}\left(\frac{t}{2}\right)-\mathbf{d}_{E}\left(\frac{t}{2}\right)\right\|_{\mathbf{H}^{2}}^{2}+c \int_{\frac{t}{2}}^{t}\left\|\mathbf{h}_{t}(\tau)\right\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}^{2} d \tau \\
\leq & c\left(1+\frac{t}{2}\right)^{-2-2 \gamma}+\frac{c}{1+2 \gamma}\left(1+\frac{t}{2}\right)^{-1-2 \gamma} \\
\leq & c(1+t)^{-1-2 \gamma}, \quad \forall t \geq 0
\end{aligned}
$$

and this gives (6.7). The proof is complete.

Acknowledgments. The authors would like to thank the referees for their helpful comments and suggestions on an earlier version of this paper. This work originated from a visit of the first author to the Fudan University whose hospitality is gratefully acknowledged. The second author was partially supported by NSF of China 11001058, Specialized Research Fund for the Doctoral Program of Higher Education and "Chen Guang" project supported by Shanghai Municipal Education Commission and Shanghai Education Development Foundation.

## References

[1] S. Bosia, Well-posedness and long term behavior of a simplified Ericksen-Leslie nonautonomous system for nematic liquid crystal flow, Comm. Pure Appl. Anal., 11 (2012), 407-441.
[2] C. Cavaterra and E. Rocca, On a 3D isothermal model for nematic liquid crystals accounting for stretching terms, Z. Angew. Math. Phys., 64 (2013), 69-82.
[3] C. Cavaterra, E. Rocca and H. Wu, Global weak solution and blow-up criterion of the general Ericksen-Leslie system for nematic liquid crystal flows, preprint 2012, arXiv:1212.0043.
[4] R. Chill and M.A. Jendoubi, Convergence to steady states in asymptotically autonomous semilinear evolution equations, Nonlinear Anal., 53 (2003), 1017-1039.
[5] B. Climent-Ezquerra, F. Guillén-González and M.A. Rojas-Medar, Reproductivity for a nematic liquid crystal model, Z. Angew. Math. Phys., 576 (2006), 984-998.
[6] B. Climent-Ezquerra, F. Guillén-González and M. Jesus Moreno-Iraberte, Regularity and time-periodicity for a nematic liquid crystal model, Nonlinear Anal., 71 (2009), 539-549.
[7] J.L. Ericksen, Conservation laws for liquid crystals, Trans. Soc. Rheol., 5 (1961), 22-34.
[8] J. Fan and T. Ozawa, Regularity criteria for a simplified Ericksen-Leslie system modeling the flow of liquid crystals, Discrete Contin. Dyn. Syst., 25 (2009), 859-867.
[9] M. Grasselli, H. Petzeltová and G. Schimperna, Convergence to stationary solutions for a parabolic-hyperbolic phase-field system, Commun. Pure Appl. Anal., 5 (2006), 827-838.
[10] M. Grasselli and H. Wu, Finite-dimensional global attractor for a system modeling the 2D nematic liquid crystal flow, Z. Angew. Math. Phys., 62 (2011), 979-992.
[11] F. Guillén-González, M.A. Rodríguez-Bellido and M.A. Rojas-Medar, Sufficient conditions for regularity and uniqueness of a 3D nematic liquid crystal model, Math. Nachr., 282 (2009), 846-867.
[12] A. Haraux and M.A. Jendoubi, Decay estimates to equilibrium for some evolution equations with an analytic nonlinearity, Asymptot. Anal., 26 (2001), 21-36.
[13] S.-Z. Huang, Gradient Inequalities, with Applications to Asymptotic Behavior and Stability of Gradient-like Systems, Mathematical Surveys and Monographs, 126, AMS, Providence, RI, 2006.
[14] S.-Z. Huang and P. Takác, Convergence in gradient-like systems which are asymptotically autonomous and analytic, Nonlinear Anal., 46 (2001), 675-698.
[15] M.A. Jendoubi, A simple unified approach to some convergence theorem of L. Simon, J. Funct. Anal., 153 (1998), 187-202.
[16] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math., 63 (1934), 193-248.
[17] F.M. Leslie, Theory of flow phenomena in liquid crystals, in Advances in Liquid Crystals, 4, 1-81, Academic Press, New York, 1979.
[18] F.-H. Lin, Nonlinear theory of defects in nematic liquid crystals: Phase transitions and flow phenomena, Comm. Pure Appl. Math., 42 (1989), 789-814.
[19] F.-H. Lin, J.-Y. Lin and C.-Y. Wang, Liquid crystal flows in two dimensions, Arch. Ration. Mech. Anal., 197 (2010), 297-336.
[20] F.-H. Lin and C. Liu, Nonparabolic dissipative system modeling the flow of liquid crystals, Comm. Pure Appl. Math., 48 (1995), 501-537.
[21] F.-H. Lin and C. Liu, Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals, Discrete Contin. Dyn. Syst., 2 (1996), 1-23.
[22] F.-H. Lin and C. Liu, Existence of solutions for the Ericksen-Leslie system, Arch. Ration. Mech. Anal., 154(2) (2000), 135-156.
[23] P. Lin and C. Liu, Simulations of singularity dynamics in liquid crystal flows: a $C^{0}$ finite element approach, J. Comput. Phys., 215 (2006), 348-362.
[24] J.-L. Lions and E. Magenes, Nonhomogeneous boundary value problems and applications, 1, Springer-Verlag, New York, 1972.
[25] C. Liu and J. Shen, On liquid crystal flows with free-slip boundary conditions, Discrete Contin. Dyn. Syst., 7 (2001), 307-318.
[26] C. Liu and N.J. Walkington, Approximation of liquid crystal flows, SIAM J. Numerical Analysis, 37 (2000), 725-741.
[27] C. Liu and N.J. Walkington, Mixed methods for the approximation of liquid crystal flows, Math. Model. Numer. Anal., 36 (2002), 205-222.
[28] H. Petzeltová, E. Rocca and G. Schimperna, On the long-time behavior of some mathematical models for nematic liquid crystals, Calc. Var. Partial Differential Equationas, (2012), online first, DOI: 10.1007/s00526-012-0496-1.
[29] S. Shkoller, Well-posedness and global attractors for liquid crystals on Riemannian manifolds, Comm. Partial Differential Equations, 27 (2001), 1103-1137.
[30] L. Simon, Asymptotics for a class of nonlinear evolution equation with applications to geometric problems, Ann. Math. (2), 118 (1983), 525-571.
[31] H. Sun and C. Liu, On energetic variational approaches in modeling the nematic liquid crystal flows, Discrete Contin. Dyn. Syst., 23 (2009), 455-475.
[32] M. Taylor, Partial Differential Equations, Vol. I, Applied Math. Sciences, 115, SpringerVerlag, New York, 1996.
[33] R. Temam, Navier-Stokes Equations and Nonlinear Functional Analysis, Second edition, CBMS-NSF Reg. Conf. Ser. Appl. Math., 66, SIAM, Philadelphia, PA, 1995.
[34] H. Wu, M. Grasselli and S. Zheng, Convergence to equilibrium for a parabolic-hyperbolic phase-field system with Neumann boundary conditions, Math. Models Methods Appl. Sci., 17 (2007), 1-29.
[35] H. Wu, Long-time behavior for nonlinear hydrodynamic system modeling the nematic liquid crystal flows, Discrete Contin. Dyn. Syst., 26 (2010), 379-396.
[36] H. Wu, X. Xu and C. Liu, Asymptotic behavior for a nematic liquid crystal model with different kinematic transport properties, Calc. Var. Partial Differential Equations, 45(3\&4) (2012), 319-345.
[37] H. Wu, X. Xu and C. Liu, On the general Ericksen-Leslie system: Parodi's relation, well-posedness and stability, Arch. Rational Mech. Anal., (2012), online first, DOI: 10.1007/s00205-012-0588-2.
[38] S. Zheng, Nonlinear Evolution Equations, Pitman series Monographs and Survey in Pure and Applied Mathematics, 133, Chapman \& Hall/CRC, Boca Raton, Florida, 2004.


[^0]:    *Dipartimento di Matematica, Politecnico di Milano, Milano 20133, Italy, maurizio.grasselli@polimi.it
    ${ }^{\dagger}$ Corresponding author. School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, China, haowufd@yahoo.com

