# Target set selection problem for honeycomb networks 

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#### Abstract

Let $G$ be a graph with a threshold function $\theta: V(G) \rightarrow \mathbb{N}$ such that $1 \leq$ $\theta(v) \leq d_{G}(v)$ for every vertex $v$ of $G$, where $d_{G}(v)$ is the degree of $v$ in $G$. Suppose we are given a target set $S \subseteq V(G)$. The paper considers the following repetitive process on $G$. At time step 0 the vertices of $S$ are colored black and the other vertices are colored white. After that, at each time step $t>0$, the colors of white vertices (if any) are updated according to the following rule. All white vertices $v$ that have at least $\theta(v)$ black neighbors at the time step $t-1$ are colored black, and the colors of the other vertices do not change. The process runs until no more white vertices can update colors from white to black. The following optimization problem is called Target Set Selection: Finding a target set $S$ of smallest possible size such that all vertices in $G$ are black at the end of the process. Such an $S$ is called an optimal target set for $G$ under the threshold function $\theta$. We are interested in finding an optimal target set for the well-known class of honeycomb networks under an important threshold function called strict majority threshold, where $\theta(v)=\left\lceil\left(d_{G}(v)+1\right) / 2\right\rceil$ for each vertex $v$ in $G$. In a graph $G$, a feedback vertex set is a subset $S \subseteq V(G)$ such that the subgraph induced by $V(G) \backslash S$ is cycle-free. In this paper we give exact value on the size of the optimal target set for various kinds of honeycomb networks under strict majority threshold, and as a by-product we also provide a minimum feedback vertex set for different kinds regular graphs in the class of honeycomb networks.


Key words: Social networks, viral marketing, irreversible spread of influence, dynamic monopolies, target set selection, strict majority threshold, feedback vertex set, decycling, honeycomb network, hexagonal grid.

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## 1 Introduction

Most computer users connect with their friends through email, social networks and chatting applications. Recently some popular social networks, such as Facebook, YouTube, Twitter and blogs, have become one of the most important ways for companies to market themselves. A lot of companies use viral-marketing techniques to advertise their products. Viral marketing is used to quickly spread the word about their products and brands. Individual decisions are influenced by others. In viral marketing, company tries to target some small number of people to "seed" their product; advertising spread from one person to another by people talking about it - a kind of snowball effect; and then the advertising reaches nearly every potential customer.

Consider the following hypothetical scenario as a motivating example. A company wish to market a new product. The company has at hand a description of the social network $G$ formed among a sample of potential customers, where the vertices represent customers and edges connect people to their friends. The company wants to target key potential customers $S$ of the social network and persuade them into adopting the new product by handing out free samples. We assume that individuals in $S$ will be convinced to adopt the new product after they receive a free sample, and the friends of customers in $S$ would be persuaded into buying the new product, which in turn will recommend the product to other friends. The company hopes that by word-of-mouth effects, convinced vertices in $S$ can trigger a cascade of further adoptions, and finally all potential customers will be persuaded to buy the product. But now how to find a good set of potential customers $S$ to target? To study this problem, in the following we formally define it.

A graph $G$ consists of a set $V(G)$ of vertices together with a set $E(G)$ of unordered pairs of vertices called edges. We use $u v$ for an edge $\{u, v\}$. The degree of a vertex $v \in V(G)$ is the number of vertices adjacent to $v$ and is denoted by $d_{G}(v)$ (the subscript $G$ will be dropped if no confusion can arise). A person-to-person recommendation social network is usually modeled by a graph $G$ together with a threshold function $\theta: V(G) \rightarrow \mathbb{N}$ such that $1 \leq \theta(v) \leq d_{G}(v)$ for each vertex $v$ in $G$, and such a social network is denoted by $(G, \theta)$. For the sake of convenience if $\theta(v)=k$ for all vertices $v$ in $G$, then $(G, \theta)$ shall be abbreviated to $(G, k)$. In marketing setting, the threshold of a vertex (customer) $v$ represents his/her latent tendency of buying the new product when his/her neighbors (friends) do (see [31]). There are two types of important and well-studied thresholds on a graph $G$ called majority threshold and strict majority threshold (see [2, 6, 20, 21, 23, 27, 28] and references therein), which will be denoted by $\theta_{\geq}$and $\theta_{>}$respectively throughout this paper. In a majority threshold we have $\theta_{\geq}(v)=\lceil d(v) / 2\rceil$ for every vertex $v$ of $G$,
while in a strict majority threshold we have $\theta_{>}(v)=\lceil(d(v)+1) / 2\rceil$ for every vertex $v$ of $G$.

Given a vertex subset $S$ of a connected social network $(G, \theta)$. Consider the following repetitive process played on $(G, \theta)$ called activation process on $(G, \theta)$ starting from $S$. At round 0 (the beginning of the game), the vertices of $S$ are colored black and the other vertices are colored white. After that, at each round $t>0$, the colors of white vertices (if any) are updated according to the following rule:

Parallel updating rule: All white vertices $v$ that have at least $\theta(v)$ black neighbors at the previous round $t-1$ are colored black. The colors of the other vertices do not change.

The process runs until no more white vertices can update colors from white to black. The set $S$ is called a target set for $(G, \theta)$. We denote by $[S]_{\theta}^{G}$ the set of vertices that are black at the end of the process. If $F \subseteq[S]_{\theta}^{G}$, then we say that the target set $S$ influences $F$ on $(G, \theta)$. We are interested in the following optimization problem:

Target Set Selection: Finding a target set $S$ of smallest possible size such that all vertices in $(G, \theta)$ are black at the end of the activation process starting from $S$. Such an $S$ is called an optimal target set for $(G, \theta)$ and its size is denoted by $\min -\operatorname{seed}(G, \theta)$.

The theoretical investigations of certain kinds of target set selection problem were initiated by Kempe, Kleinberg, and Tardos in [18, 19], where they mainly consider probabilistic thresholds such that all thresholds are drawn randomly from a given distribution. They focused on the maximization problem - find a target set of a given size $k \in \mathbb{N}$ to maximize the expected number of black vertices at the end of the activation process.

Many authors have investigated target set selection problem with different types of thresholds and network structures in various settings and under a variety of assumptions. In a dynamic monopoly setting, Peleg [28] proved that it is NP-hard to compute the optimal target set for majority thresholds. In constant threshold setting, Dreyer and Roberts [11] showed that it is NP-hard to compute the min-seed $(G, k)$ for any $k \geq 3$, and Chen [7] showed that the target set selection problem is NP-hard when the thresholds are at most 2 , even for bounded bipartite graphs. In fact, this problem is not only NP-hard, it is also extremely hard to solve approximately. Chen [7] proved that min-seed $\left(G, \theta_{\geq}\right)$cannot be approximated within the ratio $O\left(2^{\log ^{1-\epsilon} n}\right)$ for any fixed constant $\epsilon>0$, unless $N P \subseteq \operatorname{DTIME}\left(n^{\text {polylog(n) }}\right)$, where $n=|V(G)|$.

We now turn to determine the exact value of min-seed $(G, \theta)$ for certain families of graphs $G$ under specific threshold functions $\theta$. Related results can be found in $[1,2,4,5,7,8,11,13,14,15,16,24,25,27,28,29,33]$, where min-seed $(G, \theta)$ has been investigated for different types of network structure $G$ such as bounded treewidth graphs, hexagonal grids, trees, cycle permutation graphs, generalized Petersen graphs, block-cactus graphs, chordal graphs, Hamming graphs, chordal rings, tori, meshes, butterflies, Cartesian products of two cycles.

Majority threshold model has many applications in distributed computing such as maintaining data consistency in a distributed system, fault-local mending in distributed network and overcoming failure in distributed computing [22, 23, 27, 28]. On the other hand, honeycomb networks have been suggested as an attractive architecture for interconnected networks which have been widely investigated in parallel and distributed applications (see $[3,30]$ and references therein). In this paper, we study target set selection problem under strict majority thresholds on different kinds of honeycomb networks such as honeycomb mesh $\mathrm{HM}_{t}$, honeycomb torus $\mathrm{HT}_{t}$, honeycomb rectangular torus $\operatorname{HRe} \mathrm{T}(m, n)$, honeycomb rhombic torus $\operatorname{HRoT}(m, n)$, generalized honeycomb rectangular torus $\operatorname{GHT}(m, n)$, planar hexagonal grid $\operatorname{PHG}(m, n)$, cylindrical hexagonal grid $\operatorname{CHG}(m, n)$, and toroidal hexagonal grid $\operatorname{THG}(m, n)$ (all terms will be defined in later sections).

In Section 3 we determine the exact value of min-seed $\left(G, \theta_{>}\right)$for any honeycomb mesh $G$. In Section 4, by computing the optimal target set for a generalized honeycomb rectangular torus, we determine the exact values of min-seed $\left(G, \theta_{>}\right)$when $G$ is a honeycomb torus or a honeycomb rectangular torus or a honeycomb rhombic torus. Finally, in Section 5, we compute min-seed $\left(G, \theta_{>}\right)$for planar, cylindrical, and toroidal hexagonal grids $G$, where $\theta_{>}$denote the strict majority threshold of $G$. Our results in Section 5 are summarized in Table 1.

| Structure of $G$ | Result |
| :--- | :--- |
| Planar | $\min -\operatorname{seed}\left(G, \theta_{>}\right)=\left\lceil\frac{m n+2 m+n}{}\right\rceil-1$ |
| Cylindrical | min-seed $\left(G, \theta_{>}\right)=\left\lceil\frac{m n+2 m}{4}\right\rceil$ |
| Toroidal | $\min -\operatorname{seed}\left(G, \theta_{>}\right)=\left\lceil\frac{m n+2}{4}\right\rceil$ |

Table 1. Summary of results on min-seed $\left(G, \theta_{>}\right)$where $G$ is an $m$ by $n$ hexagonal grid with $m \geq 2, n \geq 4$, and $n$ even.

A subset $S$ of $V(G)$ is a feedback vertex set (or a decycling set) of a graph $G$ if the subgraph of $G$ induced by the vertices in $V(G) \backslash S$ is acyclic (see [12, 29] and references therein). The size of a minimum feedback vertex set in a graph $G$ is called the decycling number of $G$ and is denoted by $\nabla(G)$ (adapted from [29]).

In [2], by using feedback vertex sets for graphs, Adams, Troxell and Zinnen show lower and upper bounds for min-seed $\left(G, \theta_{\geq}\right)$when $G$ is one of the graphs planar, cylindrical, and toroidal hexagonal grids. We summarize their results in Table 2, where $\theta_{\geq}$denote the majority threshold of $G$. Since toroidal hexagonal grids are 3-regular, it can readily be seen that if $G$ is a toroidal hexagonal grid, then min$\operatorname{seed}\left(G, \theta_{\geq}\right)=\min -\operatorname{seed}\left(G, \theta_{>}\right)$. Thus our result for toroidal hexagonal grids (see Table 1) closes the gap in the corresponding result of Table 2.

| Structure of $G$ | Result |
| :--- | :--- |
| Planar | $\min -\operatorname{seed}\left(G, \theta_{\geq}\right)=\left\lceil\frac{(n-2)(m-1)}{4}\right\rceil$ |
| Cylindrical | $\min -\operatorname{seed}\left(G, \theta_{\geq}\right) \in\left\{\left\lceil\frac{(n-2) m+2}{4}\right\rceil,\left\lceil\frac{(n-2) m+2}{4}\right\rceil+1\right\}$ |
| Toroidal | min-seed $\left(G, \theta_{\geq}\right) \in\left\{\left\lceil\frac{m n+2}{4}\right\rceil,\left\lceil\frac{m n+2}{4}\right\rceil+1\right\}$ |

Table 2. Summary of results on min-seed $\left(G, \theta_{\geq}\right)$proved in [2] where $G$ is an $m$ by $n$ hexagonal grid with $m \geq 2, n \geq 4$, and $n$ even.

In [11], Dreyer and Roberts show that, for a vertex subset $S$ of a $(k+1)$-regular graph $G$, the target set $S$ can influence all vertices of $V(G) \backslash S$ in the social network $(G, k)$ if and only if $S$ is a feedback vertex set of $G$. In [2], the authors further show that if $G$ is a graph with minimum degree at least 2 , maximum degree at most 3 and $S \subseteq V(G)$, then $S$ can influence all vertices of $V(G) \backslash S$ in the social network ( $G, \theta_{\geq}$) if and only if $S$ is a feedback vertex set of $G$.

Finding a minimum feedback vertex set of a graph is quite difficult, and has been proved to be NP-complete in general [17]. However, in this paper by using the above facts, we are able to provide a minimum feedback vertex set in honeycomb torus networks, honeycomb rectangular torus networks, honeycomb rhombic torus networks, generalized honeycomb rectangular torus networks, and toroidal hexagonal grid networks.

## 2 Notations and preliminary results

In this section, we introduce the necessary notations, definitions and preliminary results which will be used through the paper. For a set $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is the graph with vertex set $S$ and edge set $\{u v \in E(G): u, v \in S\}$ and is denoted by $G[S]$. Denote by $G \backslash S$ the subgraph of $G$ induced by $V(G) \backslash S$. In order to study the optimal target sets for $(G, \theta)$ we introduce a sequential version of activation process on $(G, \theta)$, called sequential activation process in which at each round $t>0$ one employs the following sequential updating rule instead of the parallel updating rule:

Sequential updating rule: Exactly one of white vertices that have at least $\theta(v)$ black neighbors at the previous round $t-1$ is colored black. The colors of the other vertices do not change.

Given a target set $S$ for $(G, \theta)$, consider a sequential activation process starting from $S$. In this process, if $v_{1}, v_{2}, \ldots, v_{r}$ is the order that vertices in $[S]_{\theta}^{G} \backslash S$ become black, then $\left[v_{1}, v_{2}, \ldots, v_{r}\right]$ is called the convinced sequence of $S$ on $(G, \theta)$. We define an operation $\sqcup$ on convinced subsequences $\alpha=\left[v_{1}, v_{2}, \ldots, v_{r}\right]$ and $\beta=\left[u_{1}, u_{2}, \ldots, u_{s}\right]$ as follows: $\alpha \sqcup \beta=\left[v_{1}, v_{2}, \ldots, v_{r}, u_{1}, u_{2}, \ldots, u_{s}\right]$. For a list of convinced subsequences $\left\{\alpha_{i, j}\right\}_{1 \leq i \leq k, 1 \leq j \leq \ell}$, the sequences $\sqcup_{i=1}^{k} \alpha_{i, j}$ and $\sqcup_{j=1}^{\ell} \sqcup_{i=1}^{k} \alpha_{i, j}$ are defined to be

$$
\sqcup_{i=1}^{k} \alpha_{i, j}=\alpha_{1, j} \sqcup \alpha_{2, j} \sqcup \cdots \sqcup \alpha_{k, j} \text { and } \sqcup_{j=1}^{\ell} \sqcup_{i=1}^{k} \alpha_{i, j}=\sqcup_{j=1}^{\ell}\left(\sqcup_{i=1}^{k} \alpha_{i, j}\right) .
$$

A vertex-ordering $\pi$ of a graph $G$ having $n$ vertices is a numbering $\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ of $V(G)$. For an edge $v_{i} v_{j}$ with $i<j, v_{j}$ is a successor of $v_{i}$, and $v_{i}$ is a predecessor of $v_{j}$. The number of predecessors and successors of a vertex $v_{k}$ is denoted by $\operatorname{pred}_{\pi}\left(v_{k}\right)$ and $\operatorname{succ}_{\pi}\left(v_{k}\right)$, respectively. We may omit the subscript $\pi$ if the ordering is clear. The proof of the following lemma is straightforward and so is omitted. This lemma will be used frequently in the sequel, sometimes without explicit reference to it.

Lemma 1 Let $(G, \theta)$ be a connected graph $G$ with thresholds $\theta$ on the vertices of $G$. (a) An optimal target set for $(G, \theta)$ under the sequential updating rule is also an optimal target set for $(G, \theta)$ under the parallel updating rule, and vice versa. (b) Finding an optimal target set $S$ for $(G, \theta)$ is equivalent to that of finding a set $S \subseteq V(G)$ of minimum possible cardinality such that $G \backslash S$ has a vertex-ordering $\left(v_{1}, v_{2}, \ldots, v_{|V(G \backslash S)|}\right)$ with the following property: for each $1 \leq i \leq|V(G \backslash S)|$, $v_{i}$ is adjacent to at least $\theta\left(v_{i}\right)$ vertices in the set $S \cup\left\{v_{j}: j \leq i-1\right\}$.

We use similar ideas as in the proof of Theorem 1 of [33] to show the results in Lemma 2 which generalizes Lemma 4 of [9].

Lemma 2 Let $(G, \theta)$ be a connected graph $G$ with thresholds $\theta$ on $V(G)$ and let $\Delta$ be the maximum degree of $G$. Let $n=|V(G)|, m=|E(G)|, \delta=\max \left\{d_{G}(v)-\theta(v)\right.$ : $v \in V(G)\}, \theta_{V}=\sum_{v \in V(G)} \theta(v), \theta_{\max }=\max \{\theta(v): v \in V(G)\}$ and $\theta_{\min }=\min \{\theta(v):$ $v \in V(G)\}$. Then the following quantity $\Lambda$ is a lower bound for $\min -\operatorname{seed}(G, \theta)$ :

$$
\Lambda=\max \left\{\frac{m-(n-1) \delta}{\Delta-\delta}, \frac{\theta_{V}-m}{\theta_{\max }}, \frac{n \theta_{\min }-m}{\theta_{\min }}, \frac{\theta_{V}-(n-1) \delta}{\Delta-\delta+\theta_{\max }}, \frac{n \theta_{\min }-(n-1) \delta}{\Delta-\delta+\theta_{\min }}\right\}
$$

Proof. Let $S$ be an optimal target set for $(G, \theta)$ and let $V=V(G), s=|S|$ and $\ell=n-s$. For any two subsets $A, B \subseteq V$, let $E(A, B)$ denote the number
of edges between $A$ and $B$. Since $S$ (sequentially) influences all vertices of $(G, \theta)$, $G \backslash S$ has a vertex-ordering $\pi=\left(v_{1}, v_{2}, \ldots, v_{\ell}\right)$ with the following property: for each $1 \leq i \leq \ell, v_{i}$ is adjacent to at least $\theta\left(v_{i}\right)$ vertices in the set $S \cup\left\{v_{j}: j \leq i-1\right\}$, and hence $\operatorname{succ}_{\pi}\left(v_{i}\right) \leq d_{G}\left(v_{i}\right)-\theta\left(v_{i}\right)$. It follows that $|E(G \backslash S)|=\sum_{i=1}^{\ell-1} \operatorname{succ}_{\pi}\left(v_{i}\right) \leq$ $\sum_{i=1}^{\ell-1}\left(d_{G}\left(v_{i}\right)-\theta\left(v_{i}\right)\right) \leq(n-s-1) \delta$. Note that if $e$ is an edge in $E(G)$ but not in $E(G \backslash S)$, then $e$ has an end in $S$. This leads to $|E(G \backslash S)| \geq m-s \Delta$. Thus $m-s \Delta \leq(n-s-1) \delta$, and hence $s \geq \frac{m-(n-1) \delta}{\Delta-\delta}$. To prove the remaining part of the lemma, we see that

$$
\begin{aligned}
\min \{m, s \Delta+(n-s-1) \delta\} & \geq \min \left\{m, s \Delta+\sum_{i=1}^{\ell-1} \operatorname{succ}_{\pi}\left(v_{i}\right)\right\} \\
& \geq \min \left\{m, E\left(S,\left\{v_{j}: 1 \leq j \leq \ell\right\}\right)+\sum_{i=1}^{\ell-1} \operatorname{succ}_{\pi}\left(v_{i}\right)\right\} \\
& =\sum_{i=1}^{\ell} E\left(S \cup\left\{v_{j}: j \leq i-1\right\},\left\{v_{i}\right\}\right) \\
& \geq \sum_{i=1}^{\ell} \theta\left(v_{i}\right) \geq \max \left\{\theta_{V}-s \theta_{\max },(n-s) \theta_{\min }\right\}
\end{aligned}
$$

This implies the following four inequalities: $m \geq \theta_{V}-s \theta_{\max }, m \geq(n-s) \theta_{\min }$, $s \Delta+(n-s-1) \delta \geq \theta_{V}-s \theta_{\max }$, and $s \Delta+(n-s-1) \delta \geq(n-s) \theta_{\min }$. After simple algebraic manipulations, we obtain $s \geq \frac{\theta_{V}-m}{\theta_{\max }}, s \geq \frac{n \theta_{\min }-m}{\theta_{\min }}, s \geq \frac{\theta_{V}-(n-1) \delta}{\Delta-\delta+\theta_{\max }}$, $s \geq \frac{n \theta_{\min }-(n-1) \delta}{\Delta-\delta+\theta_{\min }}$, respectively, which complete the proof of the lemma.
We remark that the result $\min -\operatorname{seed}(G, \theta) \geq \frac{n \theta_{\min }-m}{\theta_{\min }}$ shown in Lemma 2 has already appeared in Corollary 2 of [33].

## 3 Honeycomb mesh

In this section, we determine the exact value for $\min -\operatorname{seed}\left(G, \theta_{>}\right)$where $G$ is a honeycomb mesh network with strict majority threshold function $\theta_{>}$. The honeycomb mesh of size $t$ (see [30] for a comprehensive introduction to this class of graphs and their variants), denoted by $\mathrm{HM}_{t}$ is defined inductively as follows: $\mathrm{HM}_{1}$ is a hexagon. Honeycomb mesh $\mathrm{HM}_{t}$ of size $t>1$ is obtained from $\mathrm{HM}_{t-1}$ by adding a layer of hexagons around the boundary of $\mathrm{HM}_{t-1}$. The number of vertices and edges of $\mathrm{HM}_{t}$ are $6 t^{2}$ and $9 t^{2}-3 t$, respectively. The edges of $\mathrm{HM}_{t}$ are in 3 different directions. See Figure 1 for examples of $\mathrm{HM}_{t}$ when $t=1,2,3$, where the point $O$ of $\mathrm{HM}_{t}$ is called the centre of the honeycomb mesh. Through $O$ one can draw three lines perpendicular to the three edge directions and name them as $\alpha, \beta, \gamma$ axes. These three axes will be used in Section 4 to define the honeycomb torus network introduced in [30].

Every honeycomb mesh has a nice drawing as shown in Figure 2. We call this kind of drawing the castle drawing. In Figure 2, we show an addressing scheme to describe the vertices of a honeycomb mesh which will be used in the proof of Theorem 3.


Figure 1. $\mathrm{HM}_{1}$ (left), $\mathrm{HM}_{2}$ (middle) and $\mathrm{HM}_{3}$ (right).


Figure 2. The castle drawing of a honeycomb mesh with a coordinate system on it: $\mathrm{HM}_{1}$ (lower), $\mathrm{HM}_{2}$ (middle) and $\mathrm{HM}_{3}$ (upper). In each graph, the same edge is given the same label, that is, $v_{1}^{1} v_{6}^{1}, v_{1}^{3} v_{12}^{3}, v_{1}^{5} v_{18}^{5}$ are edges.

Theorem 3 min-seed $\left(\mathrm{HM}_{t}, \theta_{>}\right)=\left(3 t^{2}+3 t\right) / 2$.
Proof. Let $G=\operatorname{HM}_{t}$ and $\theta_{\text {min }}=\min \left\{\theta_{>}(v): v \in V(G)\right\}$. Obviously, min$\operatorname{seed}\left(G, \theta_{>}\right)=\min -\operatorname{seed}(G, 2)$. Since $\mathrm{HM}_{t}$ has $6 t^{2}$ vertices and $9 t^{2}-3 t$ edges, by the result min-seed $\left(G, \theta_{>}\right) \geq \frac{|V(G)| \theta_{\min }-|E(G)|}{\theta_{\min }}$ presented in Lemma 2, we see at once that min-seed $(G, 2) \geq\left(3 t^{2}+3 t\right) / 2$.

Next we will show that min-seed $(G, 2) \leq\left(3 t^{2}+3 t\right) / 2$ by giving a target set $S$ of size $\left(3 t^{2}+3 t\right) / 2$ which influences all vertices of $V(G) \backslash S$ in $(G, 2)$. Consider the
castle drawing of $G$ with an addressing scheme on vertices, as shown in Figure 2. For each positive integer $i$, define that

$$
V_{i}= \begin{cases}\left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{3 i}^{i}\right\} & \text { if } i \text { is even } \\ \left\{v_{1}^{i}, v_{2}^{i}, \ldots, v_{3 i+3}^{i}\right\} & \text { if } i \text { is odd }\end{cases}
$$

It can be seen that $V(G)=\cup_{i=1}^{2 t-1} V_{i}$. Consider $S=\cup_{k=1}^{t}\left\{v_{1}^{2 k-1}, v_{3}^{2 k-1}, v_{5}^{2 k-1}, \ldots, v_{6 k-1}^{2 k-1}\right\}$ as a target set for $(G, 2)$ (see Figure 3 for a graphical illustration of this target set $S$ ). It is easy to check that $S$ can influence all vertices of $V(G) \backslash S$ by using the convinced sequence $\beta=\alpha_{1} \sqcup \beta_{2}$, where $\alpha_{1}=\left\{v_{2}^{1}, v_{4}^{1}, v_{6}^{1}\right\}$ and $\beta_{2}=\sqcup_{i=2}^{t}\left(\left[v_{1}^{2 i-2}, v_{2}^{2 i-2}, v_{3}^{2 i-2}, \ldots, v_{6(i-1)}^{2 i-2}\right] \sqcup\right.$ $\left[v_{2}^{2 i-1}, v_{4}^{2 i-1}, v_{6}^{2 i-1}, \ldots, v_{6 i}^{2 i-1}\right]$ ) (see Figure 1 in Appendix for a graphical illustration of this convinced sequence $\beta$ ). Since the cardinality of $S$ is $\sum_{k=1}^{t} 3 k=\frac{3 t(t+1)}{2}$, we have $\min -\operatorname{seed}\left(\mathrm{HM}_{t}, 2\right) \leq\left(3 t^{2}+3 t\right) / 2$, which completes the proof of the theorem.


Figure 3. $\mathrm{HM}_{1}$ (lower), $\mathrm{HM}_{2}$ (middle) and $\mathrm{HM}_{3}$ (upper), where the target set $S$ is the set of all black vertices.

## 4 Generalized honeycomb rectangular torus

In this section, under strict majority threshold model, we study the problem of computing optimal target sets for three well-known honeycomb tori: honeycomb torus, honeycomb rectangular torus, and honeycomb rhombic torus. Actually, we will tackle this problem by considering a slightly more general class of network topologies called generalized honeycomb rectangular torus.

The honeycomb torus of size $t$ introduced in [30], denoted by $\mathrm{HT}_{t}$, is obtained from a honeycomb mesh of size $t$ by joining the pairs of degree 2 vertices in $\mathrm{HM}_{t}$ that are mirror symmetric with respect to the three axes $\alpha, \beta, \gamma$ of the $\mathrm{HM}_{t}$ (see Figure 1
for the three axes of a honeycomb mesh). Figure 4 shows how to wraparound $\mathrm{HM}_{1}$, $\mathrm{HM}_{2}$ and $\mathrm{HM}_{3}$ to obtain $\mathrm{HT}_{1}, \mathrm{HT}_{2}$ and $\mathrm{HT}_{3}$, respectively.


Figure 4. $\mathrm{HT}_{1}$ (left), $\mathrm{HT}_{2}$ (middle) and $\mathrm{HT}_{3}$ (right).
Let $m$ and $n$ be positive even integers such that $n \geq 4$. The honeycomb rectangular torus $\operatorname{HReT}(m, n)$, introduced by Stojmenovic [30] (see also [10, 26]), is the graph with the vertex set $\{(i, j): 0 \leq i<m, 0 \leq j<n\}$ such that $(i, j)$ and $(k, \ell)$ are adjacent if and only if they satisfy one of the following conditions:

1. $i=k$ and $j=\ell \pm 1 \quad(\bmod n)$;
2. $j=\ell$ and $k=i-1 \quad(\bmod m)$ if $i+j$ is even.

For example, consider Figure 5(left) which depicts $\operatorname{HReT}(4,6)$. Note that our notation for $\operatorname{HReT}(m, n)$ is slightly different from the one used by Stojmenovic in [30].


Figure 5. $\operatorname{HReT}(4,6)$ (left), $\operatorname{HRoT}(5,6)$ (middle), and $\operatorname{GHT}(4,6,2)$ (right).

Let $m$ and $n$ be positive integers such that $n$ is even. The honeycomb rhombic torus $\operatorname{HRoT}(m, n)$, introduced by Stojmenovic [30] (see also [10, 32]), is the graph with the vertex set $\{(i, j): 0 \leq i<m, 0 \leq j-i<n\}$ such that $(i, j)$ and $(k, \ell)$ are adjacent if and only if they satisfy one of the following conditions:

1. $i=k$ and $j=\ell \pm 1 \quad(\bmod n)$;
2. $j=\ell$ and $k=i-1$ if $i+j$ is even; and
3. $i=0, k=m-1$, and $\ell=j+m$ if $j$ is even.

For example, consider Figure 5(middle) which depicts $\operatorname{HRoT}(5,6)$. Note that our notation $\operatorname{HRoT}(m, n)$ for a honeycomb rhombic torus is different from the one used in $[30,32]$.

In [10] Cho and Hsu introduced a class of generalized honeycomb tori which cover the three honeycomb tori mentioned above. Let $m$ and $n$ be positive integers such that $n \geq 4$ is even. Let $d$ be any nonnegative integer such that $m-d$ is an even number. The generalized honeycomb rectangular torus (or generalized honeycomb torus), denoted by $\operatorname{GHT}(m, n, d)$ and proposed by Cho and Hsu [10], is the graph with the vertex set $\{(i, j): 0 \leq i<m, 0 \leq j<n\}$ such that $(i, j)$ and $(k, \ell)$ are adjacent if and only if they satisfy one of the following conditions:

1. $i=k$ and $j=\ell \pm 1 \quad(\bmod n)$;
2. $j=\ell$ and $k=i-1$ if $i+j$ is even; and
3. $i=0, k=m-1$, and $\ell=j+d(\bmod n)$ if $j$ is even.

For example, Figure 5(right) depicts a $\operatorname{GHT}(4,6,2)$. We remark that, in [3], the authors call $\operatorname{GHT}(m, n, d)$ the honeycomb toroidal graph.

Now, given a generalized honeycomb rectangular torus $G$, in the proof of Theorem 4, we shall show how to compute an optimal target set for $G$ under strict majority threshold model.

Theorem 4 If $G$ is a generalized honeycomb rectangular torus $\operatorname{GHT}(m, n, d)$, then $\min -\operatorname{seed}\left(G, \theta_{>}\right)=\lceil(m n+2) / 4\rceil$.

Proof. Let $G=\operatorname{GHT}(m, n, d)$ and $\delta=\max \left\{d_{G}(v)-\theta_{>}(v): v \in V(G)\right\}$. Let $\Delta$ be the maximum degree of $G$. Obviously, $G$ is a 3-regular graph. It follows that $\min -\operatorname{seed}\left(G, \theta_{>}\right)=\min -\operatorname{seed}(G, 2)$. Since $G$ has $m n$ vertices and $\frac{3 m n}{2}$ edges, by the result $\min -\operatorname{seed}\left(G, \theta_{>}\right) \geq \frac{|E(G)|-(|V(G)|-1) \delta}{\Delta-\delta}$ presented in Lemma 2, we see at once that $\min -\operatorname{seed}(G, 2) \geq\lceil(m n+2) / 4\rceil$.

Next, we shall prove that min-seed $(G, 2) \leq\lceil(m n+2) / 4\rceil$ by giving a target set $S$ for $(G, 2)$ which influences all vertices in $V(G) \backslash S$ and has cardinality $\lceil(m n+2) / 4\rceil$. Note that $n \geq 4$ is even. We let $n=4 t+r$, where $t$ is a positive integer and $r \in\{0,2\}$. The proof is divided into three cases, according to the parity of $m$ and the value of $r$.

Case 1. $m$ is even. Let $S_{1}=\cup_{j=0}^{(n-4) / 2}\{(0,2 j),(2,2 j),(4,2 j), \ldots,(m-2,2 j)\}$ and $S_{2}=\{(1, n-1),(3, n-1),(5, n-1), \ldots,(m-1, n-1)\}$. Consider $S=S_{1} \cup S_{2} \cup$ $\{(0, n-2)\}$ as a target set for $(G, 2)$ (see Figure 6 for a graphical illustration of $S$ ). Note that, in this case, $d$ is even. By the definition of $\operatorname{GHT}(m, n, d)$ and by the choice of $S$, it can be seen that if $\ell$ is even, then the vertex $(m-1, \ell)$ is adjacent to a vertex $(0, j)$ in $S$ such that $j$ is even. With this observation, it is straightforward to check that $S$ can influence all vertices of $V(G) \backslash S$ by using the convinced sequence $\alpha=\alpha_{1} \sqcup \alpha_{2}$ (see Figure 2 in Appendix for a graphical illustration of this convinced sequence $\alpha$ ), where $\alpha_{1}=[(0, n-1),(0, n-3),(0, n-5), \ldots,(0,1)] \sqcup[(1,0),(1,1),(1,2), \ldots,(1, n-2)]$ and $\alpha_{2}=\sqcup_{i=1}^{\frac{m}{2}-1}([(2 i, n-1),(2 i, n-2)] \sqcup[(2 i, n-3),(2 i, n-5),(2 i, n-7), \ldots,(2 i, 1)] \sqcup[(2 i+$ $1,0),(2 i+1,1),(2 i+1,2), \ldots,(2 i+1, n-2)])$. Since $|S|=\frac{m n}{4}+1=\lceil(m n+2) / 4\rceil$, we obtain the desired inequality min-seed $(G, 2) \leq\lceil(m n+2) / 4\rceil$.

Case 2. $m$ is odd and $r=0$. Let $S_{1}=\cup_{j=0}^{(n-4) / 2}\{(0,2 j),(2,2 j),(4,2 j), \ldots,(m-$ $3,2 j)\}, S_{2}=\{(1, n-1),(3, n-1),(5, n-1), \ldots,(m-2, n-1)\}$, and $S_{3}=\{(m-$ $1,0),(m-1,4),(m-1,8), \ldots,(m-1, n-4)\}$. Consider $S=S_{1} \cup S_{2} \cup S_{3} \cup\{(0, n-$ $2)\}$ as a target set for $(G, 2)$ (see Figure 7 for a graphical illustration of $S$ ). Note that, in this case, $d$ is odd. By the definition of $\operatorname{GHT}(m, n, d)$ and by the choice of $S$, we see that if $\ell$ is odd, then the vertex $(m-1, \ell)$ is adjacent to a vertex $(0, j)$ in $S$ such that $j$ is even. With the above in mind, it is straightforward to check that $S$ can influence all vertices of $V(G) \backslash S$ by using the convinced sequence $\alpha=\alpha_{1} \sqcup \alpha_{2} \sqcup \alpha_{3} \sqcup \alpha_{4}$ (see Figure 3 in Appendix for a graphical illustration of this convinced sequence $\alpha)$, where $\alpha_{1}=[(m-1,1),(m-1,3),(m-1,5), \ldots,(m-1, n-1)]$, $\alpha_{2}=[(m-1,2),(m-1,6),(m-1,10),(m-1,14), \ldots,(m-1, n-6),(m-1, n-2)]$, $\alpha_{3}=[(0, n-1),(0, n-3),(0, n-5), \ldots,(0,1)] \sqcup[(1,0),(1,1),(1,2), \ldots,(1, n-2)]$, and $\alpha_{4}=\sqcup_{i=1}^{\frac{m-3}{2}}([(2 i, n-1),(2 i, n-2)] \sqcup[(2 i, n-3),(2 i, n-5),(2 i, n-7), \ldots,(2 i, 1)] \sqcup$ $[(2 i+1,0),(2 i+1,1),(2 i+1,2), \ldots,(2 i+1, n-2)])$. Since $n \equiv 0 \quad(\bmod 4)$, we have $|S|=\frac{m n}{4}+1=\lceil(m n+2) / 4\rceil$, and hence min-seed $(G, 2) \leq\lceil(m n+2) / 4\rceil$.

Case 3. $m$ is odd and $r=2$. In the following proof, the second coordinate
of a vertex $(a, b)$ in $G$ is read modulo $n$, for example we have $(m-1, d+4 t-1)=$ $(m-1, d-3)$. Let $S_{1}=\cup_{j=0}^{(n-4) / 2}\{(0,2 j),(2,2 j),(4,2 j), \ldots,(m-3,2 j)\}, S_{2}=\{(1, n-$ $1),(3, n-1),(5, n-1), \ldots,(m-2, n-1)\}$, and $S_{3}=\{(m-1, d-1),(m-1, d+$ $3),(m-1, d+7), \ldots,(m-1, d+4 t-1)\}$.

By the definition of $\operatorname{GHT}(m, n, d)$, we see that $(m-1, d),(m-1, d+2),(m-$ $1, d+4), \ldots,(m-1, d+4 t-2)$ are adjacent to vertices $(0,0),(0,2),(0,4), \ldots,(0,4 t-$ $2)$, respectively, and the vertex $(m-1, d-2)$ is adjacent to both $(m-1, d-1)$ and $(m-1, d-3)$. Note that $\{(0,0),(0,2),(0,4), \ldots,(0,4 t-2)\} \subseteq S_{1}$ and $\{(m-$ $1, d-1),(m-1, d-3)\} \subseteq S_{3}$. Consider $S=S_{1} \cup S_{2} \cup S_{3}$ as a target set for $(G, 2)$ (see Figure 8 for a graphical illustration of $S$ ). By the above observation and by the choice of $S$, it is straightforward to check that $S$ can influence all vertices of $V(G) \backslash S$ by using the convinced sequence $\alpha=\alpha_{1} \sqcup \alpha_{2} \sqcup \alpha_{3}$ (see Figure 4 in Appendix for a graphical illustration of this convinced sequence $\alpha$ ), where $\alpha_{1}=$ $[(m-1, d-2),(m-1, d),(m-1, d+2),(m-1, d+4), \ldots,(m-1, d+4 t-2)]$, $\alpha_{2}=[(m-1, d+1),(m-1, d+5),(m-1, d+9),(m-1, d+13), \ldots,(m-1, d+4 t-3)]$, and $\alpha_{3}=\sqcup_{i=0}^{(m-3) / 2}([(2 i, n-1),(2 i, n-2)] \sqcup[(2 i, n-3),(2 i, n-5),(2 i, n-7), \ldots,(2 i, 1)] \sqcup$ $[(2 i+1,0),(2 i+1,1),(2 i+1,2), \ldots,(2 i+1, n-2)])$. Since $|S|=\frac{n(m-1)}{4}+t+1=$ $\frac{m n+2}{4}=\lceil(m n+2) / 4\rceil$, we see that min-seed $(G, 2) \leq\lceil(m n+2) / 4\rceil$. This completes the proof of the theorem.


Figure 6. $\mathrm{GHT}(6,8,4)$ where the target set $S$ is the set of all black vertices.


Figure 7. GHT $(9,8,5)$ where the target set $S$ is the set of all black vertices.


Figure 8. GHT $(9,10,5)$ where the target set $S$ is the set of all black vertices.

From the definitions of the honeycomb rectangular torus, the honeycomb rhombic torus, and the generalized honeycomb rectangular torus, it can readily be seen that $\operatorname{HReT}(m, n)$ is isomorphic to $\operatorname{GHT}(m, n, 0)$ and $\operatorname{HRoT}(m, n)$ is isomorphic to $\operatorname{GHT}(m, n, m(\bmod n))$. In [10], Cho and Hsu proved that the honeycomb torus of
size $t$ is isomorphic to $\operatorname{GHT}(t, 6 t, 3 t)$. Now the following corollary follows immediately from Proposition 1 of [11], Theorem 4 and the above discussion.

Corollary 5 (1) If $G$ is a generalized honeycomb rectangular torus $\operatorname{GHT}(m, n, d)$, then the decycling number $\nabla(G)=\lceil(m n+2) / 4\rceil$. (2) If $G$ is a honeycomb torus $\mathrm{HT}_{t}$ then $\min -\operatorname{seed}\left(G, \theta_{>}\right)=\nabla(G)=\left\lceil\left(3 t^{2}+1\right) / 2\right\rceil$. (3) If $G$ is a honeycomb rectangular torus $\operatorname{HReT}(m, n)$ or a honeycomb rhombic torus $\operatorname{HRoT}(m, n)$, then min$\operatorname{seed}\left(G, \theta_{>}\right)=\nabla(G)=\lceil(m n+2) / 4\rceil$.

## 5 Hexagonal grids

In this section, under strict majority threshold model, we study the problem of computing optimal target sets for a graph $G$ which has an underlying hexagonal (or honeycomb) grid structure. Let $m$ and $n$ be two integers such that $m \geq 2, n \geq 4$, and $n$ even. An $m$ by $n$ planar hexagonal grid, denoted by $\operatorname{PHG}(m, n)$, consists of an array of $n$ rows of $m$ vertices $(x, y)$, with $0 \leq x \leq m-1,0 \leq y \leq n-1$, arranged on a standard Cartesian plane such that each vertex $(x, y)$ is adjacent to $(x, y+1)$ and, if $y$ is even (zero is considered to be even), also adjacent to $(x+1, y+1)$, provided that each coordinate is within its allowed range and no vertex of degree one is generated. As an example, $\operatorname{PHG}(5,8)$ is depicted in Figure 9.

An $m$ by $n$ cylindrical hexagonal grid $\operatorname{CHG}(m, n)$ is obtained from the $m$ by $n$ planar hexagonal grid $\operatorname{PHG}(m, n)$ by adding the edges from $(m-1, y)$ to $(0, y+1)$ for any even $y$. In other words, a $\operatorname{CHG}(m, n)$ is defined the same as a $\operatorname{PHG}(m, n)$ except that for each vertex $(x, y)$ the addition in the first coordinate is taken modulo $m$. As an example, $\operatorname{CHG}(5,8)$ is depicted in Figure 9.

An $m$ by $n$ toroidal hexagonal grid, denoted by $\operatorname{THG}(m, n)$, is defined the same as a $\operatorname{PHG}(m, n)$ except that for each vertex $(x, y)$ addition in the first coordinate is taken modulo $m$ and addition in the second coordinate is taken modulo $n$. As an example, $\operatorname{THG}(5,8)$ is depicted in Figure 9.


Figure 9. $\operatorname{PHG}(5,8)$ (left), $\operatorname{CHG}(5,8)$ (middle), and $\operatorname{THG}(5,8)$ (right).

In Theorem 3.3 of [2], Adams et al. showed that if $G$ is an $m$ by $n$ planar hexagonal grid then min-seed $\left(G, \theta_{\geq}\right)=\left\lceil\frac{(n-2)(m-1)}{4}\right\rceil$. Below we consider an $m$ by $n$ planar hexagonal grid equipped with a strict majority threshold $\theta_{>}$and determine its optimal target set.

Theorem 6 If $G$ is an $m$ by $n$ planar hexagonal grid, then min-seed $\left(G, \theta_{>}\right)=$ $\lceil(m n+2 m+n) / 4\rceil-1$.

Proof. Let $G=\operatorname{PHG}(m, n), \theta_{\min }=\min \left\{\theta_{>}(v): v \in V(G)\right\}$. Obviously, min$\operatorname{seed}\left(G, \theta_{>}\right)=\min -\operatorname{seed}(G, 2)$. Since $G$ has $m n-2$ vertices and $\frac{3 m n}{2}-\frac{n}{2}-m-2$ edges (see Lemma 3.1 of $[2]$ ), by the result min-seed $\left(G, \theta_{>}\right) \geq \frac{|V(G)| \theta_{\min }-|E(G)|}{\theta_{\min }}$ presented in Lemma 2, we see at once that min-seed $(G, 2) \geq\lceil(m n+2 m+n) / 4\rceil-1$.

Next we will show that min-seed $(G, 2) \leq\lceil(m n+2 m+n) / 4\rceil-1$ by giving a target set $S$ for $(G, 2)$ which influences all vertices of $V(G) \backslash S$ and has $|S|=$ $\lceil(m n+2 m+n) / 4\rceil-1$. Note that $n$ is even and $n \geq 4$. Let $n=4 t+r$, where $t$ is a positive integer and $r \in\{0,2\}$. The proof is divided into three cases, according to the value of $r$ and the parity of $m$.

Case 1. $r=2$. In this case, consider $S=\{(0,2 i) \mid 0 \leq i \leq 2 t\} \cup\{(j, 4+4 k) \mid 1 \leq$ $j \leq m-1,0 \leq k \leq t-1\} \cup\{(1,0),(2,0),(3,0), \ldots,(m-2,0)\} \cup\{(m-1,1)\}$ as a target set for $(G, 2)$ (see Figure 10 for a graphical illustration of $S$ ). It is straightforward to check that $S$ can influence all vertices of $V(G) \backslash S$ in $(G, 2)$ by using the convinced sequence $\alpha=\alpha_{1} \sqcup \alpha_{2} \sqcup \alpha_{3} \sqcup \alpha_{4}$ (see Figure 5 in Appendix for a graphical illustration of this convinced sequence $\alpha$ ), where
$\alpha_{1}=[(0,1),(1,1),(2,1), \ldots,(m-2,1)]$,
$\alpha_{2}=\sqcup_{k=0}^{t-2}[(0,5+4 k),(1,5+4 k),(2,5+4 k), \ldots,(m-1,5+4 k)]$,
$\alpha_{3}=\sqcup_{k=0}^{t-1}\left([(0,3+4 k),(1,3+4 k),(1,2+4 k)] \sqcup\left(\sqcup_{j=2}^{m-1}[(j, 3+4 k),(j, 2+4 k)]\right)\right)$, and $\alpha_{4}=[(1, n-1),(2, n-1),(3, n-1), \ldots,(m-1, n-1)]$.
Since $|S|=(2 t+1)+(m-1) t+(m-1)=\lceil(m n+2 m+n) / 4\rceil-1$, we see that $\min -\operatorname{seed}(G, 2) \leq\lceil(m n+2 m+n) / 4\rceil-1$.

Case 2. $r=0$ and $m$ is even. Consider $S=\{(0,2 i) \mid 0 \leq i \leq 2 t-1\} \cup\{(j, 2+$ $4 k) \mid 1 \leq j \leq m-1,0 \leq k \leq t-1\} \cup\{(2,0),(4,0),(6,0), \ldots,(m-2,0)\}$ as a target set for $(G, 2)$ (see Figure 11 for a graphical illustration of $S$ ). It is straightforward to check that $S$ can influence all vertices of $V(G) \backslash S$ in $(G, 2)$ by using the convinced sequence $\alpha=\alpha_{1} \sqcup \alpha_{2} \sqcup \alpha_{3} \sqcup \alpha_{4} \sqcup \alpha_{5}$ (see Figure 6 in Appendix for a graphical illustration of this convinced sequence $\alpha$ ), where
$\alpha_{1}=[(0,1),(1,1),(2,1), \ldots,(m-1,1)]$,
$\alpha_{2}=[(1,0),(3,0),(5,0), \ldots,(m-3,0)]$,
$\alpha_{3}=\sqcup_{k=0}^{t-2}[(0,3+4 k),(1,3+4 k),(2,3+4 k), \ldots,(m-1,3+4 k)]$,
$\alpha_{4}=\sqcup_{k=0}^{t-2}\left([(0,5+4 k),(1,5+4 k),(1,4+4 k)] \sqcup\left(\sqcup_{j=2}^{m-1}[(j, 5+4 k),(j, 4+4 k)]\right)\right)$, and $\alpha_{5}=[(1, n-1),(2, n-1),(3, n-1), \ldots,(m-1, n-1)]$.
Since $|S|=2 t+(m-1) t+\left(\frac{m}{2}-1\right)=\lceil(m n+2 m+n) / 4\rceil-1$, we see that min$\operatorname{seed}(G, 2) \leq\lceil(m n+2 m+n) / 4\rceil-1$.

Case 3. $r=0$ and $m$ is odd. Consider $S=\{(0,2 i) \mid 0 \leq i \leq 2 t-1\} \cup\{(j, 2+$ $4 k) \mid 1 \leq j \leq m-1,0 \leq k \leq t-1\} \cup\{(2,0),(4,0),(6,0), \ldots,(m-3,0)\} \cup\{(m-2,0)\}$ as a target set for $(G, 2)$ (see Figure 12 for a graphical illustration of $S$ ). It is straightforward to check that $S$ can influence all vertices of $V(G) \backslash S$ by using the convinced sequence $\alpha=\alpha_{1} \sqcup \alpha_{2} \sqcup \alpha_{3} \sqcup \alpha_{4} \sqcup \alpha_{5}$ (see Figure 7 in Appendix for a graphical illustration of this convinced sequence $\alpha$ ), where
$\alpha_{1}=[(0,1),(1,1),(2,1), \ldots,(m-1,1)]$,
$\alpha_{2}=[(1,0),(3,0),(5,0), \ldots,(m-4,0)]$,
$\alpha_{3}=\sqcup_{k=0}^{t-2}[(0,3+4 k),(1,3+4 k),(2,3+4 k), \ldots,(m-1,3+4 k)]$,
$\alpha_{4}=\sqcup_{k=0}^{t-2}\left([(0,5+4 k),(1,5+4 k),(1,4+4 k)] \sqcup\left(\sqcup_{j=2}^{m-1}[(j, 5+4 k),(j, 4+4 k)]\right)\right)$, and
$\alpha_{5}=[(1, n-1),(2, n-1),(3, n-1), \ldots,(m-1, n-1)]$.
Since $|S|=2 t+(m-1) t+\left(\frac{m-1}{2}\right)=\lceil(m n+2 m+n) / 4\rceil-1$, we see that min$\operatorname{seed}(G, 2) \leq\lceil(m n+2 m+n) / 4\rceil-1$.


Figure 10. $\operatorname{PHG}(8,6)$ (left), $\operatorname{PHG}(8,10)$ (middle), and $\operatorname{PHG}(8,14)$ (right) where the target set $S$ is the set of all black vertices.


Figure 11. $\operatorname{PHG}(4,16)$ (left), $\operatorname{PHG}(6,16)$ (middle), and $\operatorname{PHG}(8,16)$ (right) where the target set $S$ is the set of all black vertices.


Figure 12. $\operatorname{PHG}(5,16)$ (left), $\operatorname{PHG}(7,16)$ (middle), and $\operatorname{PHG}(9,16)$ (right) where the target set $S$ is the set of all black vertices.

In Theorems 4.1 and 4.2 of [2], Adams et al. showed that if $G$ is an $m$ by $n$ cylindrical hexagonal grid, then min-seed $\left(G, \theta_{\geq}\right) \in\left\{\left\lceil\frac{(n-2) m+2}{4}\right\rceil,\left\lceil\frac{(n-2) m+2}{4}\right\rceil+1\right\}$. Below we consider an $m$ by $n$ cylindrical hexagonal grid equipped with a strict majority threshold $\theta_{>}$and determine its optimal target set.

Theorem 7 If $G$ is an $m$ by $n$ cylindrical hexagonal grid, then $\min -\operatorname{seed}\left(G, \theta_{>}\right)=$ $\lceil(m n+2 m) / 4\rceil$.

Proof. Let $G=\operatorname{CHG}(m, n), \theta_{\min }=\min \left\{\theta_{>}(v): v \in V(G)\right\}$. Obviously, min$\operatorname{seed}\left(G, \theta_{>}\right)=\min -\operatorname{seed}(G, 2)$. Since $G$ has $m n$ vertices and $\frac{3 m n}{2}-m$ edges (see Lemma 4.1 of $[2]$ ), by the result $\min -\operatorname{seed}\left(G, \theta_{>}\right) \geq \frac{|V(G)| \theta_{\min }-|E(G)|}{\theta_{\text {min }}}$ presented in Lemma 2, we see at once that min-seed $(G, 2) \geq\lceil(m n+2 m) / 4\rceil$.

Next we will show that min-seed $(G, 2) \leq\lceil(m n+2 m) / 4\rceil$ by giving a target set $S$ for $(G, 2)$ which influences all vertices of $V(G) \backslash S$ and has $|S|=\lceil(m n+2 m) / 4\rceil$. Notice that $n$ is even. Let $n=4 t+r$ where $t$ is a positive integer and $r \in\{0,2\}$. The proof is divided into three cases, according to the value of $r$ and the parity of $m$.

Case 1. $r=2$. Consider $S=\{(0,2 i) \mid 0 \leq i \leq 2 t\} \cup\{(j, 4 k) \mid 1 \leq j \leq m-2,0 \leq$ $k \leq t\} \cup\{(m-1, n-2)\}$ as a target set for $(G, 2)$ (see Figure 13 for a graphical illustration of $S$ ). It is straightforward to check that $S$ can influence all vertices of $V(G) \backslash S$ by using the convinced sequence $\alpha=\alpha_{1} \sqcup \alpha_{2} \sqcup \alpha_{3} \sqcup \alpha_{4}$ (see Figure 8 in Appendix for a graphical illustration of this convinced sequence $\alpha$ ), where
$\alpha_{1}=\sqcup_{k=0}^{t-1}[(0,1+4 k),(1,1+4 k),(2,1+4 k), \ldots,(m-2,1+4 k)]$,
$\alpha_{2}=\sqcup_{k=0}^{t-1}\left([(0,3+4 k),(1,3+4 k),(1,2+4 k)] \sqcup\left(\sqcup_{j=2}^{m-2}[(j, 3+4 k),(j, 2+4 k)]\right)\right)$,
$\alpha_{3}=[(0, n-1),(1, n-1),(2, n-1), \ldots,(m-1, n-1)]$, and
$\alpha_{4}=[(m-1, n-3),(m-1, n-4),(m-1, n-5), \ldots,(m-1,0)]$.
Since $|S|=(2 t+1)+(m-2)(t+1)+1=\lceil(m n+2 m) / 4\rceil$, we see that min$\operatorname{seed}(G, 2) \leq\lceil(m n+2 m) / 4\rceil$.

Case 2. $r=0$ and $m$ is even. Consider $S=\{(0,2 i) \mid 0 \leq i \leq 2 t-1\} \cup\{(j, 2+$ $4 k) \mid 1 \leq j \leq m-2,0 \leq k \leq t-1\} \cup\{(2,0),(4,0),(6,0), \ldots,(m-2,0)\} \cup\{(m-1, n-2)\}$ as a target set for $(G, 2)$ (see Figure 14 for a graphical illustration of $S$ ). It is straightforward to check that $S$ can influence all vertices of $V(G) \backslash S$ by using the convinced sequence $\alpha=\alpha_{1} \sqcup \alpha_{2} \sqcup \alpha_{3} \sqcup \alpha_{4} \sqcup \beta_{1} \sqcup \beta_{2}$ (see Figure 9 in Appendix for a graphical illustration of this convinced sequence $\alpha$ ), where
$\alpha_{1}=\sqcup_{k=0}^{t-2}[(0,3+4 k),(1,3+4 k),(2,3+4 k), \ldots,(m-2,3+4 k)]$,
$\alpha_{2}=\sqcup_{k=0}^{t-2}\left([(0,5+4 k),(1,5+4 k),(1,4+4 k)] \sqcup\left(\sqcup_{j=2}^{m-2}[(j, 5+4 k),(j, 4+4 k)]\right)\right)$,
$\alpha_{3}=[(0, n-1),(1, n-1),(2, n-1), \ldots,(m-1, n-1)]$,
$\alpha_{4}=[(m-1, n-3),(m-1, n-4),(m-1, n-5), \ldots,(m-1,2)]$,
$\beta_{1}=[(0,1),(1,1),(2,1), \ldots,(m-1,1)]$, and
$\beta_{2}=[(1,0),(3,0),(5,0), \ldots,(m-1,0)]$.
Since $|S|=2 t+(m-2) t+\frac{m-2}{2}+1=\lceil(m n+2 m) / 4\rceil$, we see that min-seed $(G, 2) \leq$ $\lceil(m n+2 m) / 4\rceil$.

Case 3. $r=0$ and $m$ is odd. Consider $S=\{(0,2 i) \mid 0 \leq i \leq 2 t-1\} \cup$ $\{(j, 2+4 k) \mid 1 \leq j \leq m-2,0 \leq k \leq t-1\} \cup\{(2,0),(4,0),(6,0), \ldots,(m-3,0)\} \cup$ $\{(m-1,1),(m-1, n-2)\}$ as a target set for $(G, 2)$ (see Figure 15 for a graphical illustration of $S$ ). It is straightforward to check that $S$ can influence all vertices of
$V(G) \backslash S$ by using the convinced sequence $\alpha=\alpha_{1} \sqcup \alpha_{2} \sqcup \alpha_{3} \sqcup \alpha_{4} \sqcup \beta_{1} \sqcup \beta_{2} \sqcup \beta_{3}$ (see Figure 10 in Appendix for a graphical illustration of this convinced sequence $\alpha$ ), where
$\alpha_{1}=\sqcup_{k=0}^{t-2}[(0,3+4 k),(1,3+4 k),(2,3+4 k), \ldots,(m-2,3+4 k)]$,
$\alpha_{2}=\sqcup_{k=0}^{t-2}\left([(0,5+4 k),(1,5+4 k),(1,4+4 k)] \sqcup\left(\sqcup_{j=2}^{m-2}[(j, 5+4 k),(j, 4+4 k)]\right)\right)$,
$\alpha_{3}=[(0, n-1),(1, n-1),(2, n-1), \ldots,(m-1, n-1)]$,
$\alpha_{4}=[(m-1, n-3),(m-1, n-4),(m-1, n-5), \ldots,(m-1,2)]$,
$\beta_{1}=[(0,1),(1,1),(2,1), \ldots,(m-2,1)]$,
$\beta_{2}=[(1,0),(3,0),(5,0), \ldots,(m-4,0)]$, and
$\beta_{3}=[(m-2,0),(m-1,0)]$.
Since $|S|=2 t+(m-2) t+\frac{m-3}{2}+2=\lceil(m n+2 m) / 4\rceil$, we see that $\min -\operatorname{seed}(G, 2) \leq$ $\lceil(m n+2 m) / 4\rceil$.


Figure 13. $\operatorname{CHG}(8,6)$ (left), $\mathrm{CHG}(8,10)$ (middle), and $\mathrm{CHG}(8,14)$ (right) where the target set $S$ is the set of all black vertices.


Figure 14. $\operatorname{CHG}(6,16)$ (left), $\operatorname{CHG}(8,16)$ (middle), and $\operatorname{CHG}(10,16)$ (right) where the target set $S$ is the set of all black vertices.


Figure 15. $\operatorname{CHG}(5,16)$ (left), $\operatorname{CHG}(7,16)$ (middle), and $\operatorname{CHG}(9,16)$ (right) where the target set $S$ is the set of all black vertices.

In Theorems 5.1 and 5.2 of [2], Adams et al. showed that if $G$ is an $m$ by $n$ toroidal hexagonal grid then min-seed $\left(G, \theta_{\geq}\right) \in\left\{\left\lceil\frac{m n+2}{4}\right\rceil,\left\lceil\frac{m n+2}{4}\right\rceil+1\right\}$. Below we consider an $m$ by $n$ toroidal hexagonal grid equipped with a strict majority threshold $\theta_{>}$and determine its optimal target set. Since $\operatorname{THG}(m, n)$ is 3-regular, it can be seen that if $G$ is a toroidal hexagonal grid then min-seed $\left(G, \theta_{\geq}\right)=\min -\operatorname{seed}\left(G, \theta_{>}\right)$. Thus our result in Theorem 8 closes the gap in the corresponding result proved by Adams et al. (see Table 2).

Theorem 8 If $G$ is an $m$ by $n$ toroidal hexagonal grid, then min-seed $\left(G, \theta_{>}\right)=$ $\min -\operatorname{seed}\left(G, \theta_{\geq}\right)=\nabla(G)=\lceil(m n+2) / 4\rceil$.

Proof. Let $G=\operatorname{THG}(m, n)$. To prove min-seed $\left(G, \theta_{>}\right)=\lceil(m n+2) / 4\rceil$ we show that $G$ is isomorphic to the honeycomb rhombic torus $\operatorname{HRoT}(m, n)$. Let $f$ be a function from the vertex set of $\operatorname{HRoT}(m, n)$ to the vertex set of $G$ such that $f(i, j)=$ ( $m-1-i, j-i$ ). It is straightforward to check that $f$ is a bijection and preserves edges. Since both $\operatorname{HRoT}(m, n)$ and $G$ have $\frac{3 m n}{2}$ edges, $f$ also preserves non-edges. Therefore $f$ is an isomorphism from $\operatorname{HRoT}(m, n)$ to $G$. It follows that, by Proposition 1 of [11] and Corollary 5, min-seed $\left(G, \theta_{>}\right)=\nabla(G)=\lceil(m n+2) / 4\rceil$.

## References

[1] E. Ackerman, O. Ben-Zwi, G. Wolfovitz, Combinatorial model and bounds for target set selection, Theoretical Computer Science 411 (2010) 4017-4022.
[2] S. S. Adams, D. S. Troxell, S. L. Zinnen, Dynamic monopolies and feedback vertex sets in hexagonal grids, Computers and Mathematics with Applications 62 (2011) 4049-4057.
[3] B. Alspach, M. Dean, Honeycomb toroidal graphs are Caley graphs, Information Processing Letters 109 (2009) 705-708.
[4] O. Ben-Zwi, D. Hermelin, D. Lokshtanov, I. Newman, Treewidth governs the complexity of target set selection, Discrete Optimization 8 (2011) 87-96.
[5] E. Berger, Dynamic monopolies of constant size, J. Combin. Theory Ser. B, 83 (2001) 191-200.
[6] C. L. Chang, Y. D. Lyuu, Bounding the number of tolerable faults in majoritybased systems, Lecture Notes in Computer Science 6078 (2010) 109-119.
[7] N. Chen On the approximability of influence in social networks, SIAM J. Discrete Math. 23 (2009) 1400-1415.
[8] C-Y Chiang, L-H Huang, B-J Li, Jiaojiao Wu, H-G Yeh, Some Results on the Target Set Selection Problem, submitted. Also in: arXiv:1111.6685v1.
[9] C-Y Chiang, L-H Huang, W-T Huang, H-G Yeh, The Target Set Selection Problem on Cycle Permutation Graphs, Generalized Petersen Graphs and Torus Cordalis, submitted. Also in: arXiv:1112.1313v1.
[10] H-J Cho, L-Y Hsu, Generalized honeycomb torus, Information Processing Letters 86 (2003) 185-190.
[11] P. A. Dreyer, F. S. Roberts, Irreversible $k$-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion, Discrete Applied Math. 157 (2009) 1615-1627.
[12] P. Festa, P. M. Pardalos, M. G. C. Resende, Feedback set problems, in Handbook of Combinatorial Optimization, Supplement Vol. A, Kluwer Academic Publishers, (2000) 209-259.
[13] P. Flocchini, F. Geurts, N. Santoro, Optimal irreversible dynamos in chordal rings, Discrete Applied Mathematics 113 (2001) 23-42.
[14] P. Flocchini, R. Královič, P. Ruźička, A. Roncato, N. Santoro. On time versus size for monotone dynamic monopolies in regular topologies, Journal of Discrete Algorithms 1 (2003) 129-150.
[15] P. Flocchini, E. Lodi, F. Luccio, L. Pagli, N. Santoro, Dynamic monopolies in tori, Discrete Applied Mathematics 137 (2004) 197-212.
[16] P. Flocchini, Contamination and Decontamination in Majority-Based Systems, Journal of Cellular Automata 4 (2009) 183-200.
[17] R. M. Karp, Reducibility among combinatorial problems, Complexity of Computer Computations 40 (1972) 85-103.
[18] D. Kempe, J. Kleinberg, E. Tardos, Maximizing the spread of influence through a social network, in Proceedings of the 9th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 137-146, 2003.
[19] D. Kempe, J. Kleinberg, E. Tardos, Influential nodes in a diffusion model for social networks, in: Proceedings of the 32th International Colloquium on Automata, Languages and Programming, pages 1127-1138, 2005.
[20] K. Khoshkhah, H. Soltani, M. Zaker, On dynamic monopolies of graphs: The average and strict majority thresholds, Discrete Optimization (2012), doi:10.1016/j.disopt.2012.02.001
[21] D. M. Kilgour, Approval balloting for muti-winner elections, in J. Laslier \& M. R. Sanver (Eds.), Handbook on approval voting, pages 105-124, Heidelberg, Springer.
[22] S. Kutten, D. Peleg, Fault-local distributed mending, in: Proceedings of the 36th IEEE Symposium on Foundations of Computer Science, 1995.
[23] N. Linial, D. Peleg, Y. Rabinovich, M. Saks, Sphere packing and local majorities in graphs, in: Second ISTCS, IEEE Computer Society Press, Silver Spring, MD, pages 141-149, 1993.
[24] F. Luccio, Almost exact minimum feedback vertex set in meshes and butterflies, Information Processing Letters 66 (1998) 59-64.
[25] F. Luccio, L. Pagli, H. Sanossian. Irreversible dynamos in butterflies, 6th Int. Coll. on Structural Information and Communication Complexity (SIROCCO), pages 204-218, 1999.
[26] B. Parhami, D-M Kwai, A Unified Formulation of Honeycomb and Diamond Networks, IEEE Transactions on paralel and distributed systems 12 (2001) 7480.
[27] D. Peleg, Size bounds for dynamic monopolies, Discrete Applied Mathematics 86 (1998) 263-273.
[28] D. Peleg, Local majorities, coalitions and monopolies in graphs: a review, Theoretical Computer Science 282 (2002) 231-257.
[29] D. A. Pike, Y. Zou, Decycling Cartesian products of two cycles, SIAM Journal on Discrete Mathematics, 19 (2005) 651-663.
[30] I. Stojmenovic, Honeycomb networks: Topological properties and communication algorithms, IEEE Trans. Parallel Distributed Systems, 8 (1997) 1036-1042.
[31] D. J. Watts, The accidental influentials, Harvard Business Review, 85 (2007) 22-23.
[32] X. Yang, The diameter of honeycomb rhombic tori, Applied Mathematics Letters 17 (2004) 167-172.
[33] M. Zaker, On dynamic monopolies of graphs with general thresholds, Discrete Mathematics 312 (2012) 1136-1143.

Appendix: [Not for publication - for referees' information only]


Figure 1. $\mathrm{HM}_{4}$ and its target set $S$. For $i=2,3,4$, convinced subsequences $\alpha_{2 i-2}=$ $\left[v_{1}^{2 i-2}, v_{2}^{2 i-2}, v_{3}^{2 i-2}, \ldots, v_{6(i-1)}^{2 i-2}\right]$ and $\alpha_{2 i-1}=\left[v_{2}^{2 i-1}, v_{4}^{2 i-1}, v_{6}^{2 i-1}, \ldots, v_{6 i}^{2 i-1}\right]$ are illustrated by colored directed paths. $\beta=\sqcup_{k=1}^{7} \alpha_{k}$.


Figure 2. $\operatorname{GHT}(6,8,4)$ and its target sets $S$. Convinced subsequences $\alpha_{1}, \alpha_{2}$ are illustrated by colored directed paths and $\alpha=\alpha_{1} \sqcup \alpha_{2}$.


Figure 3. $\operatorname{GHT}(9,8,5)$ and its target sets $S$. Convinced subsequences $\alpha_{1}, \ldots, \alpha_{4}$ are illustrated by colored directed paths and $\alpha=\sqcup_{i=1}^{4} \alpha_{i}$.


Figure 4. $\operatorname{GHT}(9,10,5)$ and its target sets $S$. Convinced subsequences $\alpha_{1}, \alpha_{2}, \alpha_{3}$ are illustrated by colored directed paths and $\alpha=\sqcup_{i=1}^{3} \alpha_{i}$.


Figure 5. $\operatorname{PHG}(8,6), \operatorname{PHG}(8,10), \operatorname{PHG}(8,14)$ and their target sets $S$. Convinced subsequences $\alpha_{1}, \ldots, \alpha_{4}$ are illustrated by colored directed paths and $\alpha=$ $\sqcup_{k=1}^{4} \alpha_{k}$.


Figure 6. $\operatorname{PHG}(4,16), \operatorname{PHG}(6,16), \operatorname{PHG}(8,16)$ and their target sets $S$. Convinced subsequences $\alpha_{1}, \ldots, \alpha_{5}$ are illustrated by colored directed paths and $\alpha=$ $\sqcup_{k=1}^{5} \alpha_{k}$.


Figure 7. $\operatorname{PHG}(5,16), \operatorname{PHG}(7,16), \operatorname{PHG}(9,16)$ and their target sets $S$. Convinced subsequences $\alpha_{1}, \ldots, \alpha_{5}$ are illustrated by colored directed paths and $\alpha=$ $\sqcup_{k=1}^{5} \alpha_{k}$.


Figure 8. $\operatorname{CHG}(8,6), \operatorname{CHG}(8,10), \operatorname{CHG}(8,14)$ and their target sets $S$. Convinced subsequences $\alpha_{1}, \ldots, \alpha_{4}$ are illustrated by colored directed paths and $\alpha=$ $\sqcup_{k=1}^{4} \alpha_{k}$.


Figure 9. $\operatorname{CHG}(6,16), \operatorname{CHG}(8,16), \operatorname{CHG}(10,16)$ and their target sets $S$. Convinced subsequences $\alpha_{1}, \ldots, \alpha_{4}$ and $\beta_{1}, \beta_{2}$ are illustrated by colored directed paths and $\alpha=\left(\sqcup_{k=1}^{4} \alpha_{k}\right) \sqcup \beta_{1} \sqcup \beta_{2}$.


Figure 10. $\operatorname{CHG}(5,16), \mathrm{CHG}(7,16), \mathrm{CHG}(9,16)$ and their target sets $S$. Convinced subsequences $\alpha_{1}, \ldots, \alpha_{4}$ and $\beta_{1}, \beta_{2}, \beta_{3}$ are illustrated by colored directed paths and $\alpha=\left(\sqcup_{k=1}^{4} \alpha_{k}\right) \sqcup \beta_{1} \sqcup \beta_{2} \sqcup \beta_{3}$.


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