# Unconditional convergence and optimal error estimates of a Galerkin-mixed FEM for incompressible miscible flow in porous media 

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#### Abstract

In this paper, we study the unconditional convergence and error estimates of a Galerkin-mixed FEM with the linearized semi-implicit Euler time-discrete scheme for the equations of incompressible miscible flow in porous media. We prove that the optimal $L^{2}$ error estimates hold without any time-step (convergence) conditions, while all previous works require certain time-step restrictions. Our theoretical results provide a new understanding on commonly-used linearized schemes. The proof is based on a splitting of the error into two parts: the error from the time discretization of the PDEs and the error from the finite element discretization of corresponding time-discrete PDEs. The approach used in this paper can be applied to more general nonlinear parabolic systems and many other linearized (semi)-implicit time discretizations.


## 1 Introduction

Incompressible miscible flow in porous media is governed in general by the following system of equations:

$$
\begin{align*}
& \Phi \frac{\partial c}{\partial t}-\nabla \cdot(D(\mathbf{u}) \nabla c)+\mathbf{u} \cdot \nabla c=\hat{c} q^{I}-c q^{P}  \tag{1.1}\\
& \nabla \cdot \mathbf{u}=q^{I}-q^{P}  \tag{1.2}\\
& \mathbf{u}=-\frac{k(x)}{\mu(c)} \nabla p \tag{1.3}
\end{align*}
$$

where $p$ is the pressure of the fluid mixture, $\mathbf{u}$ is the velocity and $c$ is the concentration of the first component; $k(x)$ is the permeability of the medium, $\mu(c)$ is the concentrationdependent viscosity, $\Phi$ is the porosity of the medium, $q^{I}$ and $q^{P}$ are given injection and production sources, $\hat{c}$ is the concentration of the first component in the injection source, and $D(\mathbf{u})=\left[D_{i j}(\mathbf{u})\right]_{d \times d}$ is the diffusion-dispersion tensor which may be given in different forms (see [4, 5] for details). We assume that the system is defined in a bounded smooth domain $\Omega$ in $\mathbb{R}^{d}(d=2,3)$, for $t \in[0, T]$, coupled with the initial and boundary conditions:

$$
\begin{array}{ll}
\mathbf{u} \cdot \mathbf{n}=0, \quad D(\mathbf{u}) \nabla c \cdot \mathbf{n}=0 & \text { for } x \in \partial \Omega, \quad t \in[0, T] \\
c(x, 0)=c_{0}(x) & \text { for } x \in \Omega . \tag{1.4}
\end{array}
$$

[^0]The above system was investigated extensively in the last several decades [3, 8, 17, 18, 20, due to its wide applications in various engineering areas, such as reservoir simulations and exploration of underground water, oil and gas. Existence of weak solutions of the system was obtained by Feng [20] for the 2D model and by Chen and Ewing [6] for the 3D problem. Existence of semi-classical/classical solutions is unknown so far. Numerical simulations have been done extensively with various applications, see [10, 12, 31] and the references therein. Optimal error estimates of a standard Galerkin-Galerkin method for the system (1.1)-(1.4) in two-dimensional space was obtained first by Ewing and Wheeler [19] roughly under the time-step condition $\tau=o(h)$. In their method, a linearized semi-implicit Euler scheme was used in the time direction and Galerkin FEM approximation was used both for the concentration and the pressure. Later, a Galerkin-mixed finite element method was proposed by Douglas et al [11 for this system, where a Galerkin approximation was applied for the concentration equation and a mixed approximation in the Raviart-Thomas finite element space was used for the pressure equation. A linearized semi-implicit Euler scheme, the same as that used in [19], was applied for the time discretization. Optimal error estimates were obtained under a similar time-step condition $\tau=o(h)$. There are many other numerical methods in literature for solving the equations of incompressible miscible flow, such as [39, 42] with an ELLAM in two-dimensional space, [40] with an MMOC-MFEM approximation for the 2D problem, [13, 36] with a characteristic-mixed method in two and three dimensional spaces, respectively, and [28, 29] with a collocationmixed method and a characteristic-collocation method, respectively. In all those works, error estimates were established under certain time-step conditions. Moreover, linearized semi-implicit schemes have also been analyzed with certain time-step restrictions for many other nonlinear parabolic-type systems, such as Navier-Stokes equations [1, 21, 23, 26, 27, nonlinear thermistor problems [14, 44], viscoelastic fluid flow [9, 15, 41, KdV equations [30, nonlinear Schrödinger equation [34, 38] and some other equations [2, 22, 35]. A key issue in analysis of FEMs is the boundedness of the numerical solution in $L^{\infty}$ norm or a stronger norm, which in a routine way can be estimated by the mathematical induction with an inverse inequality, such as,

$$
\begin{equation*}
\left\|u_{h}^{n}-R_{h} u\left(\cdot, t_{n}\right)\right\|_{L^{\infty}} \leq C h^{-d / 2}\left\|u_{h}^{n}-R_{h} u\left(\cdot, t_{n}\right)\right\|_{L^{2}} \leq C h^{-d / 2}\left(\tau^{m}+h^{r+1}\right), \tag{1.5}
\end{equation*}
$$

where $u_{h}^{n}$ is the finite element solution, $u$ is the exact solution, and $R_{h}$ is certain projection operator. A time-step restriction arises immediately from the above inequality. Such a timestep restriction may result in the use of a very small time step and extremely time-consuming in practical computations. The problem becomes more serious when a non-uniform mesh is used. However, we believe that those time-step restrictions may not be necessary in most cases.

In this paper, we analyze the linearized semi-implicit Euler scheme with a popular Galerkin-mixed finite element approximation in the spatial direction for the system (1.1)(1.4). We establish optimal $L^{2}$ error estimates almost without any time-step restriction (or when $h$ and $\tau$ are smaller than some positive constants, respectively). Our theoretical analysis is based on an error splitting proposed in [24] (also see [25]) for a Joule heating system with a standard Galerkin FEM. By introducing a corresponding time-discrete system, we split the numerical error into two parts, the error in the temporal direction and the error in the spatial direction. Thus with the solution $v^{n}$ of the time-discrete system, the numerical solution can be bounded by

$$
\begin{equation*}
\left\|u_{h}^{n}-R_{h} v^{n}\right\|_{L^{\infty}} \leq C h^{-d / 2}\left\|u_{h}^{n}-R_{h} v^{n}\right\|_{L^{2}} \leq C h^{-d / 2} h^{r+1} \tag{1.6}
\end{equation*}
$$

without any time-step condition. Rigorous analysis of the regularity of the solution to the time-discrete PDEs is a key to our approach. With such proved regularity, we obtain optimal and $\tau$-independent $L^{2}$-error estimates of the Galerkin-mixed FEM for the timediscrete PDEs. Our analysis presented in this paper provides a new understanding on the commonly-used linearized schemes and clears up the misgivings for the time-step size in practical computations.

The rest of the paper is organized as follows. In Section 2, we introduce the linearized semi-implicit Euler scheme with a Galerkin-mixed approximation in the spatial direction for the system (1.1)-(1.4) and present our main results. In Section 3, we present a priori estimates of the solution to the corresponding time-discrete system and the error estimate of the linearized scheme. Optimal error estimates of the fully discrete scheme in the $L^{2}$ norm are given in Section 4. In Section 5, we present numerical examples to illustrate the convergence rate and the unconditional convergence (stability) of the numerical method. Conclusions are drawn in the last section.

## 2 The Galerkin-mixed FEM and the main results

For any integer $m \geq 0$ and $1 \leq p \leq \infty$, let $W^{m, p}$ be the Sobolev space of functions defined on $\Omega$ equipped with the norm

$$
\|f\|_{W^{m, p}}=\left(\sum_{|\beta| \leq m} \int_{\Omega}\left|D^{\beta} f\right|^{p} \mathrm{~d} x\right)^{\frac{1}{p}},
$$

where

$$
D^{\beta}=\frac{\partial^{|\beta|}}{\partial x_{1}^{\beta_{1}} \cdots \partial x_{d}^{\beta_{d}}}
$$

for the multi-index $\beta=\left(\beta_{1}, \cdots, \beta_{d}\right), \beta_{1} \geq 0, \cdots, \beta_{d} \geq 0$, and $|\beta|=\beta_{1}+\cdots+\beta_{d}$. For any integer $m \geq 0$ and $0<\alpha<1$, let $C^{m+\alpha}$ denote the usual Hölder space equipped with the norm

$$
\|f\|_{C^{m+\alpha}}=\sum_{|\beta| \leq m}\left\|D^{\beta} f\right\|_{C(\bar{\Omega})}+\sum_{|\beta|=k} \sup _{x, y \in \Omega} \frac{\left|D^{\beta} f(x)-D^{\beta} f(y)\right|}{|x-y|^{\alpha}} .
$$

Let $I=(0, T)$. For any Banach space $X$, we consider functions $g: I \rightarrow X$ and define the norm

$$
\|g\|_{L^{p}(I ; X)}= \begin{cases}\left(\int_{0}^{T}\|g(t)\|_{X}^{p} d t\right)^{\frac{1}{p}}, & 1 \leq p<\infty \\ \operatorname{esssup}_{t \in I}\|g(t)\|_{X}, & p=\infty\end{cases}
$$

In addition, we define $L_{0}^{p}$ as the subspace of $L^{p}$ consisting of functions in $L^{p}$ whose integral over $\Omega$ are zeros. Finally, we denote by $H$ the space of vector-valued functions $\vec{f} \in L^{2} \times$ $L^{2} \times L^{2}$ such that $\nabla \cdot \vec{f} \in L^{2}$.

Let $\pi_{h}$ be a regular division of $\Omega$ into triangles $T_{j}, j=1, \cdots, M$, in $\mathbb{R}^{2}$ (or tetrahedra in $\mathbb{R}^{3}$ ), with $\Omega_{h}=\cup_{j} T_{j}$ and denote by $h=\max _{1 \leq j \leq M}\left\{\operatorname{diam} T_{j}\right\}$ the mesh size. For a triangle $T_{j}$ at the boundary, we define $\widetilde{T}_{j}$ as the extension of $T_{j}$ to the triangle with one curved edge (or a tetrahedra with one curved face in $\mathbb{R}^{3}$ ). For a given division of $\Omega$, we define the finite element spaces (more details are described in [25]):

$$
S_{h}=\left\{w_{h} \in L^{2}(\Omega):\left.w_{h}\right|_{\widetilde{T}_{j}} \text { is linear for each element } T_{j} \in \pi_{h} \text { and } \int_{\Omega} w_{h} d x=0\right\}
$$

$$
V_{h}=\left\{w_{h} \in C^{0}(\bar{\Omega}):\left.w_{h}\right|_{\widetilde{T}_{j}} \text { is linear for each element } T_{j} \in \pi_{h}\right\} .
$$

Let $H_{h}$ be the subspace of $H$, as introduced by Raviart and Thomas [33, 37] such that $\psi \cdot \mathbf{n}=0$ on $\partial \Omega$ and $\operatorname{div} \psi \in S_{h}$ for $\psi \in H_{h}$.

In the rest part of this paper, we assume that the solution to the initial-boundary value problem (1.1)-(1.4) exists and satisfies

$$
\begin{align*}
& \|p\|_{L^{\infty}\left(I ; H^{3}\right)}+\|\mathbf{u}\|_{L^{\infty}\left(I ; H^{2}\right)}+\left\|\mathbf{u}_{t}\right\|_{L^{2}\left(I ; W^{1,3 / 2}\right)}+\|c\|_{L^{\infty}\left(I ; W^{2, s}\right)} \\
& +\left\|c_{t}\right\|_{L^{\infty}\left(I ; H^{2}\right)}+\left\|c_{t}\right\|_{L^{s}\left(I ; W^{1, s}\right)}+\left\|c_{t t}\right\|_{L^{s}\left(I ; L^{s}\right)} \leq C \tag{2.1}
\end{align*}
$$

for some $s>d$ and

$$
\begin{equation*}
\left\|q^{I}\right\|_{H^{1}},\left\|q^{P}\right\|_{H^{1}} \leq C . \tag{2.2}
\end{equation*}
$$

Correspondingly, we assume that the permeability $k \in W^{2, \infty}(\Omega)$ and satisfies

$$
\begin{equation*}
k_{0}^{-1} \leq k(x) \leq k_{0} \quad \text { for } \quad x \in \Omega, \tag{2.3}
\end{equation*}
$$

the concentration-dependent viscosity $\mu \in C^{1}(\mathbb{R})$ and satisfies

$$
\begin{equation*}
\mu_{0}^{-1} \leq \mu(s) \leq \mu_{0} \quad \text { for } \quad s \in \mathbb{R} \tag{2.4}
\end{equation*}
$$

for some positive constant $\mu_{0}$. Moreover, we assume that the diffusion-dispersion tensor is given by $D(\mathbf{u})=\Phi d_{m} I+D^{*}(\mathbf{u})$, where $d_{m}>0, D^{*}(\mathbf{u})=d_{1}(\mathbf{u}) I+d_{2}(\mathbf{u})(\mathbf{u} \otimes \mathbf{u})$ is symmetric and positive definite and $\partial_{u_{i}} D \in L^{\infty}, \partial_{u_{i} u_{j}}^{2} D \in L^{\infty}$ [5]. For the initial-boundary value problem (1.1)-(1.4) to be well-posed, we require

$$
\begin{equation*}
\int_{\Omega} q^{I} \mathrm{~d} x=\int_{\Omega} q^{P} \mathrm{~d} x . \tag{2.5}
\end{equation*}
$$

Let $\left\{t_{n}\right\}_{n=0}^{N}$ be a uniform partition of the time interval $[0, T]$ with $\tau=T / N$ and denote

$$
p^{n}(x)=p\left(x, t_{n}\right), \quad \mathbf{u}^{n}(x)=\mathbf{u}\left(x, t_{n}\right), \quad c^{n}(x)=c\left(x, t_{n}\right) .
$$

For any sequence of functions $\left\{f^{n}\right\}_{n=0}^{N}$, we define

$$
D_{t} f^{n+1}=\frac{f^{n+1}-f^{n}}{\tau} .
$$

The fully discrete mixed finite element scheme is to find $P_{h}^{n} \in S_{h} /\{$ constant $\}, U_{h}^{n} \in H_{h}$ and $\mathcal{C}_{h}^{n} \in V_{h}, n=0,1, \cdots, N$, such that for all $v_{h} \in H_{h}, \varphi_{h} \in S_{h}$ and $\phi_{h} \in V_{h}$,

$$
\begin{align*}
& \left(\frac{\mu\left(\mathcal{C}_{h}^{n}\right)}{k(x)} U_{h}^{n+1}, v_{h}\right)=-\left(P_{h}^{n+1}, \nabla \cdot v_{h}\right),  \tag{2.6}\\
& \left(\nabla \cdot U_{h}^{n+1}, \varphi_{h}\right)=\left(q^{I}-q^{P}, \varphi_{h}\right),  \tag{2.7}\\
& \left(\Phi D_{t} \mathcal{C}_{h}^{n+1}, \phi_{h}\right)+\left(D\left(U_{h}^{n+1}\right) \nabla \mathcal{C}_{h}^{n+1}, \nabla \phi_{h}\right) \\
& \quad+\left(U_{h}^{n+1} \cdot \nabla \mathcal{C}_{h}^{n+1}, \phi_{h}\right)=\left(\hat{c}^{I} q^{I}-\mathcal{C}_{h}^{n+1} q^{P}, \phi_{h}\right) \tag{2.8}
\end{align*}
$$

where the initial data $\mathcal{C}_{h}^{0}$ is the Lagrangian piecewise linear interpolation of $c^{0}$.
In this paper, we denote by $C$ a generic positive constant and by $\epsilon$ a generic small positive constant, which are independent of $n, h$ and $\tau$. We present our main results in the following theorem.

Theorem 2.1 Suppose that the initial-boundary value problem (1.1)-(1.4) has a unique solution ( $p, \mathbf{u}, c$ ) which satisfies (2.1). Then there exist positive constants $h_{0}$ and $\tau_{0}$ such that when $h<h_{0}$ and $\tau<\tau_{0}$, the finite element system (2.6)-(2.8) admits a unique solution $\left(P_{h}^{n}, U_{h}^{n}, \mathcal{C}_{h}^{n}\right), n=1, \cdots, N$, which satisfies that

$$
\begin{equation*}
\max _{1 \leq n \leq N}\left\|P_{h}^{n}-p^{n}\right\|_{L^{2}}+\max _{1 \leq n \leq N}\left\|U_{h}^{n}-\mathbf{u}^{n}\right\|_{L^{2}}+\max _{1 \leq n \leq N}\left\|\mathcal{C}_{h}^{n}-c^{n}\right\|_{L^{2}} \leq C\left(\tau+h^{2}\right) \tag{2.9}
\end{equation*}
$$

We will present the proof of Theorem 2.1 in the next two sections. The key to our proof is the following error splitting

$$
\begin{aligned}
& \left\|U_{h}^{n}-\mathbf{u}^{n}\right\|_{L^{2}} \leq\left\|e_{u}^{n}\right\|_{L^{2}}+\left\|U^{n}-U_{h}^{n}\right\|_{L^{2}} \\
& \left\|P_{h}^{n}-p^{n}\right\|_{L^{2}} \leq\left\|e_{p}^{n}\right\|_{L^{2}}+\left\|P^{n}-P_{h}^{n}\right\|_{L^{2}} \\
& \left\|\mathcal{C}_{h}^{n}-c^{n}\right\|_{L^{2}} \leq\left\|e_{c}^{n}\right\|_{L^{2}}+\left\|\mathcal{C}^{n}-\mathcal{C}_{h}^{n}\right\|_{L^{2}}
\end{aligned}
$$

where

$$
\begin{aligned}
e_{p}^{n} & =P^{n}-p^{n} \\
e_{u}^{n} & =U^{n}-\mathbf{u}^{n} \\
e_{c}^{n} & =\mathcal{C}^{n}-c^{n}
\end{aligned}
$$

and $\left(P^{n}, U^{n}, \mathcal{C}^{n}\right)$ is the solution of a time-discrete system defined in next section.

## 3 Error estimates for time-discrete system

We define the time-discrete solution $\left(P^{n}, U^{n}, \mathcal{C}^{n}\right)$ by the following elliptic system:

$$
\begin{align*}
& U^{n+1}=-\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \nabla P^{n+1}  \tag{3.1}\\
& \nabla \cdot U^{n+1}=q^{I}-q^{P}  \tag{3.2}\\
& \Phi D_{t} \mathcal{C}^{n+1}-\nabla \cdot\left(D\left(U^{n+1}\right) \nabla \mathcal{C}^{n+1}\right)+U^{n+1} \cdot \nabla \mathcal{C}^{n+1}=\hat{c} q^{I}-\mathcal{C}^{n+1} q^{P} \tag{3.3}
\end{align*}
$$

for $x \in \Omega$ and $t \in[0, T]$, with the initial and boundary conditions

$$
\begin{array}{ll}
U^{n+1} \cdot \mathbf{n}=0, \quad D\left(U^{n+1}\right) \nabla \mathcal{C}^{n+1} \cdot \mathbf{n}=0 & \text { for } x \in \partial \Omega, \quad t \in[0, T] \\
\mathcal{C}^{0}(x)=c_{0}(x) & \text { for } x \in \Omega \tag{3.4}
\end{array}
$$

The condition $\int_{\Omega} P^{n+1} d x=0$ is enforced for the uniqueness of solution.
In this section, we prove the existence, uniqueness and regularity of the solution of the above time-discrete system.

Theorem 3.1 Suppose that the initial-boundary value problem (1.1)-(1.4) has a unique solution ( $p, \mathbf{u}, c$ ) which satisfies (2.1). Then there exists a positive constant $\tau_{0}$ such that when $\tau<\tau_{0}$, the time-discrete system (3.1)-(3.4) admits a unique solution $\left(P^{n}, U^{n}, \mathcal{C}^{n}\right)$, $n=1, \cdots, N$, which satisfies

$$
\begin{equation*}
\left\|P^{n}\right\|_{H^{2}}+\left\|U^{n}\right\|_{H^{2}}+\left\|\mathcal{C}^{n}\right\|_{W^{2, s}}+\left\|D_{t} \mathcal{C}^{n}\right\|_{L^{s}}+\left\|\nabla \mathcal{C}^{n}\right\|_{L^{\infty}} \tag{3.5}
\end{equation*}
$$

$$
+\left(\sum_{n=1}^{N} \tau\left\|D_{t} U^{n}\right\|_{L^{3}}^{2}\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{N} \tau\left\|\nabla D_{t} U^{n}\right\|_{L^{3 / 2}}^{2}\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{N} \tau\left\|D_{t} C^{n}\right\|_{H^{2}}^{2}\right)^{\frac{1}{2}} \leq C
$$

and

$$
\begin{equation*}
\max _{1 \leq n \leq N}\left\|e_{p}^{n}\right\|_{L^{s}}+\max _{1 \leq n \leq N}\left\|e_{u}^{n}\right\|_{L^{s}}+\max _{1 \leq n \leq N}\left\|e_{c}^{n}\right\|_{L^{s}} \leq C \tau \tag{3.6}
\end{equation*}
$$

Proof It suffices to establish the estimates presented in (3.5). With such estimates, existence and uniqueness of solution follow a routine way. We observe that $e_{p}^{n}, e_{u}^{n}$ and $e_{c}^{n}$ satisfy the following equations

$$
\begin{align*}
& -\nabla \cdot\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \nabla e_{p}^{n+1}\right)=\nabla \cdot\left[\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}-\frac{k(x)}{\mu\left(c^{n}\right)}\right) \nabla p^{n+1}\right]  \tag{3.7}\\
& e_{u}^{n+1}=-\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \nabla e_{p}^{n+1}-\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}-\frac{k(x)}{\mu\left(c^{n}\right)}\right) \nabla p^{n+1}  \tag{3.8}\\
& \Phi D_{t} e_{c}^{n+1}-\nabla \cdot\left(D\left(U^{n+1}\right) \nabla e_{c}^{n+1}\right)+U^{n+1} \cdot \nabla e_{c}^{n+1} \\
& =\nabla \cdot\left(\left(D\left(U^{n+1}\right)-D\left(\mathbf{u}^{n+1}\right)\right) \nabla c^{n+1}\right)-\left(U^{n+1}-\mathbf{u}^{n+1}\right) \cdot \nabla c^{n+1}-e_{c}^{n+1} q^{P}+\mathcal{E}^{n+1} \tag{3.9}
\end{align*}
$$

for $x \in \Omega$ and $t \in[0, T]$, with the initial and boundary conditions

$$
\begin{array}{ll}
\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \nabla e_{p}^{n+1} \cdot \mathbf{n}=0, \quad D\left(U^{n+1}\right) \nabla e_{c}^{n+1} \cdot \mathbf{n}=0, & \text { for } x \in \partial \Omega, \quad t \in[0, T]  \tag{3.10}\\
e_{c}^{0}(x)=0, & \text { for } x \in \Omega
\end{array}
$$

where

$$
\mathcal{E}^{n+1}=\Phi\left(c_{t}^{n+1}-D_{t} c^{n+1}\right)
$$

is the truncation error due to the discretization in the time direction. From the regularity assumption for $c$ in (2.1), we see that

$$
\begin{array}{ll}
\left\|\mathcal{E}^{n+1}\right\|_{L^{2}} \leq C, & \sum_{n=0}^{N-1} \tau\left\|\mathcal{E}^{n+1}\right\|_{L^{2}}^{2} \leq C \tau^{2} \\
\left\|\mathcal{E}^{n+1}\right\|_{L^{s}} \leq C, & \sum_{n=0}^{N-1} \tau\left\|\mathcal{E}^{n+1}\right\|_{L^{s}}^{s} \leq C \tau^{s} \tag{3.12}
\end{array}
$$

To prove the error estimate (3.6), we multiply (3.7) by $e_{p}^{n+1}$ to get

$$
\begin{equation*}
\left\|\nabla e_{p}^{n+1}\right\|_{L^{2}} \leq\left\|\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}-\frac{k(x)}{\mu\left(c^{n}\right)}\right) \nabla p^{n+1}\right\|_{L^{2}} \leq C\left\|e_{c}^{n}\right\|_{L^{2}}\left\|\nabla p^{n+1}\right\|_{L^{\infty}} \leq C\left\|e_{c}^{n}\right\|_{L^{2}} \tag{3.13}
\end{equation*}
$$

which together with (3.8) implies that

$$
\begin{equation*}
\left\|e_{u}^{n+1}\right\|_{L^{2}} \leq C\left\|\nabla e_{p}^{n+1}\right\|_{L^{2}}+C\left\|e_{c}^{n}\right\|_{L^{2}}\left\|\nabla p^{n+1}\right\|_{L^{\infty}} \leq C\left\|e_{c}^{n}\right\|_{L^{2}} \tag{3.14}
\end{equation*}
$$

Then we multiply (3.9) by $e_{c}^{n+1}$ to get

$$
\frac{1}{2} D_{t}\left(\Phi\left\|e_{c}^{n+1}\right\|_{L^{2}}^{2}\right)+\left\|\sqrt{D\left(U^{n+1}\right)} \nabla e_{c}^{n+1}\right\|_{L^{2}}^{2}
$$

$$
\begin{align*}
& \leq C\left\|e^{n+1}\right\|_{L^{4}}^{2}\left\|q^{I}-q^{P}\right\|_{L^{2}}+C\left\|e_{u}^{n+1}\right\|_{L^{2}}\left\|\nabla e_{c}^{n+1}\right\|_{L^{2}}\left\|\nabla c^{n+1}\right\|_{L^{\infty}} \\
& \quad+C\left\|e_{u}^{n+1}\right\|_{L^{2}}\left\|e_{c}^{n+1}\right\|_{L^{2}}\left\|\nabla c^{n+1}\right\|_{L^{\infty}}+C\left\|e_{c}^{n+1}\right\|_{L^{4}}^{2}\left\|q^{P}\right\|_{L^{2}}+C\left\|\mathcal{E}^{n+1}\right\|_{L^{2}}\left\|e_{c}^{n+1}\right\|_{H^{1}} \\
& \leq C\left\|e_{c}^{n+1}\right\|_{L^{4}}^{2}+C\left\|e_{c}^{n+1}\right\|_{H^{1}}\left\|e_{c}^{n}\right\|_{L^{2}}+C \epsilon^{-1}\left\|\mathcal{E}^{n+1}\right\|_{L^{2}}^{2}+\epsilon\left\|e_{c}^{n+1}\right\|_{H^{1}}^{2} \\
& \leq \epsilon\left\|\nabla e_{c}^{n+1}\right\|_{L^{2}}^{2}+C_{\epsilon}\left\|e_{c}^{n+1}\right\|_{L^{2}}^{2}+C\left\|e_{c}^{n}\right\|_{L^{2}}^{2}+C\left\|\mathcal{E}^{n+1}\right\|_{L^{2}}^{2} \tag{3.15}
\end{align*}
$$

where we have used the inequalities

$$
\left|\left(U^{n+1} \cdot \nabla e_{c}^{n+1}, e_{c}^{n+1}\right)\right|=\left|\left(\nabla \cdot U^{n+1},\left|e_{c}^{n+1}\right|^{2}\right)\right| \leq\left\|e^{n+1}\right\|_{L^{4}}^{2}\left\|q^{I}-q^{P}\right\|_{L^{2}}
$$

and

$$
\left\|e_{c}^{n+1}\right\|_{L^{4}}^{2} \leq \epsilon\left\|\nabla e_{c}^{n+1}\right\|_{L^{2}}^{2}+C_{\epsilon}\left\|e_{c}^{n+1}\right\|_{L^{2}}^{2}
$$

The square root of $D\left(U^{n+1}\right)$ exists because $D\left(U^{n+1}\right)$ is a symmetric and positive definite matrix. With Gronwall's inequality and (3.11), (3.15) reduces to

$$
\begin{equation*}
\left\|e_{c}^{n+1}\right\|_{L^{2}}^{2}+\sum_{n=0}^{N-1} \tau\left\|\nabla e_{c}^{n+1}\right\|_{L^{2}}^{2} \leq C \tau^{2} \tag{3.16}
\end{equation*}
$$

provided $\tau<\tau_{1}$ for some positive constant $\tau_{1}$. The above inequality implies that

$$
\begin{equation*}
\left\|D_{t} e_{c}^{n+1}\right\|_{L^{2}} \leq C \tag{3.17}
\end{equation*}
$$

From (3.13)-(3.14), we derive that

$$
\begin{equation*}
\left\|e_{u}^{n+1}\right\|_{L^{2}}+\left\|\nabla e_{p}^{n+1}\right\|_{L^{2}} \leq C \tau \tag{3.18}
\end{equation*}
$$

To prove (3.5), we rewrite (3.9) as

$$
\begin{align*}
& \Phi D_{t} e_{c}^{n+1}-\nabla \cdot\left(D\left(\mathbf{u}^{n+1}\right) \nabla e_{c}^{n+1}\right)  \tag{3.19}\\
& =-\nabla \cdot\left(\left(D\left(\mathbf{u}^{n+1}\right)-D\left(U^{n+1}\right)\right) \nabla e_{c}^{n+1}\right)-\left(U^{n+1}-\mathbf{u}^{n+1}\right) \cdot \nabla e_{c}^{n+1}-\mathbf{u}^{n+1} \cdot \nabla e_{c}^{n+1} \\
& \quad+\nabla \cdot\left(\left(D\left(U^{n+1}\right)-D\left(\mathbf{u}^{n+1}\right)\right) \nabla c^{n+1}\right)-\left(U^{n+1}-\mathbf{u}^{n+1}\right) \cdot \nabla c^{n+1}-e_{c}^{n+1} q^{P}+\mathcal{E}^{n+1}
\end{align*}
$$

Since $\mathbf{u}^{n+1} \in C^{\alpha}(\bar{\Omega})$, using the $W^{1,6}$ estimates of elliptic equations for $e_{c}$ [7], we get

$$
\begin{aligned}
\left\|\nabla e_{c}^{n+1}\right\|_{L^{6}} \leq & C\left\|D_{t} e_{c}^{n+1}\right\|_{L^{2}}+C\left\|e_{u}^{n+1}\right\|_{L^{\infty}}\left\|\nabla e_{c}^{n+1}\right\|_{L^{6}}+C\left\|\mathbf{u}^{n+1}\right\|_{L^{\infty}}\left\|\nabla e_{c}^{n+1}\right\|_{L^{2}} \\
& +C\left\|e_{u}^{n+1}\right\|_{L^{\infty}}\left\|\nabla c^{n+1}\right\|_{L^{6}}+C\left\|q^{P}\right\|_{L^{6}}\left\|e_{c}^{n+1}\right\|_{L^{3}}+C\left\|\mathcal{E}^{n+1}\right\|_{L^{2}}
\end{aligned}
$$

which further reduces to

$$
\begin{equation*}
\left\|\nabla e_{c}^{n+1}\right\|_{L^{6}} \leq C_{0}\left\|e_{u}^{n+1}\right\|_{L^{\infty}}\left(C_{0}+\left\|\nabla e_{c}^{n+1}\right\|_{L^{6}}\right)+C_{0} \tag{3.20}
\end{equation*}
$$

for some positive constant $C_{0}$ independent of $n, \tau, h$.
On the other hand, we can rewrite the equation (3.7) into the following form:

$$
\begin{equation*}
-\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \Delta e_{p}^{n+1}=\nabla\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}\right) \cdot \nabla e_{p}^{n+1}+\nabla \cdot\left[\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}-\frac{k(x)}{\mu\left(c^{n}\right)}\right) \nabla p^{n+1}\right] \tag{3.21}
\end{equation*}
$$

Since we have

$$
\left\|\nabla\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}\right) \cdot \nabla e_{p}^{n+1}\right\|_{L^{6}} \leq\left\|\nabla\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}\right)\right\|_{L^{6}}\left\|\nabla e_{p}^{n+1}\right\|_{L^{\infty}}
$$

$$
\leq\left(C+C\left\|\nabla \mathcal{C}^{n}\right\|_{L^{6}}\right)\left\|\nabla e_{p}^{n+1}\right\|_{L^{\infty}}
$$

and

$$
\begin{aligned}
& \left\|\nabla \cdot\left[\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}-\frac{k(x)}{\mu\left(c^{n}\right)}\right) \nabla p^{n+1}\right]\right\|_{L^{6}} \\
& \leq\left\|\nabla\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}-\frac{k(x)}{\mu\left(c^{n}\right)}\right) \cdot \nabla p^{n+1}\right\|_{L^{6}}+\left\|\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)}-\frac{k(x)}{\mu\left(c^{n}\right)}\right) \nabla \cdot \nabla p^{n+1}\right\|_{L^{6}} \\
& \leq\left\|\nabla e_{c}^{n}\right\|_{L^{6}}\left\|\nabla p^{n+1}\right\|_{L^{\infty}}+\left\|e_{c}^{n}\right\|_{L^{\infty}}\left\|p^{n+1}\right\|_{W^{2,6}},
\end{aligned}
$$

by the $W^{2,6}$ estimates of elliptic equations [7], we derive from (3.21) that

$$
\begin{align*}
\left\|e_{p}^{n+1}\right\|_{W^{2,6}} \leq & \left(C+C\left\|\nabla \mathcal{C}^{n}\right\|_{L^{6}}\right)\left\|\nabla e_{p}^{n+1}\right\|_{L^{\infty}}+C\left\|\nabla e_{c}^{n}\right\|_{L^{6}}+C\left\|e_{c}^{n}\right\|_{L^{\infty}} \\
\leq & \left(C+C\left\|\nabla e_{c}^{n}\right\|_{L^{6}}\right)\left(\epsilon\left\|e_{p}^{n+1}\right\|_{W^{2,6}}+C_{\epsilon}\left\|e_{p}\right\|_{L^{2}}\right) \\
& +(C+\epsilon)\left\|\nabla e_{c}^{n}\right\|_{L^{6}}+\left(C+C_{\epsilon}\right)\left\|e_{c}^{n}\right\|_{L^{2}} \\
\leq & \left(C+C\left\|\nabla e_{c}^{n}\right\|_{L^{6}}\right) \epsilon\left\|e_{p}^{n+1}\right\|_{W^{2,6}}+C_{\epsilon}+C_{\epsilon}\left\|\nabla e_{c}^{n}\right\|_{L^{6}} \tag{3.22}
\end{align*}
$$

From (3.8) we observe that

$$
\begin{align*}
\left\|e_{u}^{n+1}\right\|_{W^{1,6}} & \leq\left(C+C\left\|e_{c}^{n}\right\|_{W^{1,6}}\right)\left(\left\|\nabla e_{p}^{n+1}\right\|_{L^{\infty}}+\left\|\nabla p^{n+1}\right\|_{L^{\infty}}\right)+C\left\|e_{p}^{n+1}\right\|_{W^{2,6}} \\
& \leq\left(C+C\left\|\nabla e_{c}^{n}\right\|_{L^{6}}\right) \epsilon\left\|e_{p}^{n+1}\right\|_{W^{2,6}}+C_{\epsilon}+C_{\epsilon}\left\|\nabla e_{c}^{n}\right\|_{L^{6}}, \tag{3.23}
\end{align*}
$$

and by the Sobolev interpolation inequality, we have

$$
\begin{equation*}
\left\|e_{u}^{n+1}\right\|_{L^{\infty}} \leq C\left\|e_{u}^{n+1}\right\|_{L^{2}}^{1 / 4}\left\|e_{u}^{n+1}\right\|_{W^{1,6}}^{3 / 4} \leq C \tau^{1 / 4}\left\|e_{u}^{n+1}\right\|_{W^{1,6}}^{3 / 4} \tag{3.24}
\end{equation*}
$$

With the estimates (3.20), (3.22), (3.23) and (3.24), we now apply mathematical induction to prove

$$
\begin{equation*}
\left\|\nabla e_{c}^{n}\right\|_{L^{6}} \leq 4 C_{0} \tag{3.25}
\end{equation*}
$$

for $n=1,2, \cdots, N$. Clearly, the above inequality holds when $n=0$. With the above inequalities, by choosing a proper small $\epsilon=\epsilon\left(C_{0}\right)$, we derive from (3.22)-(3.24) that

$$
\begin{equation*}
\left\|e_{u}^{n+1}\right\|_{L^{\infty}} \leq C_{1} \tau^{1 / 4} \tag{3.26}
\end{equation*}
$$

where $C_{1}$ may depend on $C_{0}$. Hence, there exists a positive constant $\tau_{0}=\tau_{0}\left(C_{1}\right)$ such that when $\tau<\tau_{0}$, we have

$$
C_{0}\left\|e_{u}^{n+1}\right\|_{L^{\infty}} \leq 1 / 2
$$

Substituting the above inequality into (3.20) gives

$$
\left\|\nabla e_{c}^{n+1}\right\|_{L^{6}} \leq 4 C_{0}
$$

By mathematical induction, (3.25) holds for $1 \leq n \leq N$. From (3.22) and (3.23), we also get

$$
\begin{equation*}
\max _{1 \leq n \leq N}\left\|e_{p}^{n+1}\right\|_{W^{2,6}}+\max _{1 \leq n \leq N}\left\|e_{u}^{n+1}\right\|_{W^{1,6}} \leq C \tag{3.27}
\end{equation*}
$$

By a similar approach as (3.23), we can prove that $\left\|\nabla e_{u}^{n+1}\right\|_{L^{3 / 2}} \leq C\left(\left\|e_{c}^{n}\right\|_{L^{2}}+\left\|\nabla e_{c}^{n}\right\|_{L^{3 / 2}}\right)$ and so

$$
\begin{aligned}
& \left\|\nabla D_{t} U^{n+1}\right\|_{L^{3 / 2}} \leq\left\|\nabla D_{t} e_{u}^{n+1}\right\|_{L^{3 / 2}}+\left\|\nabla D_{t} \mathbf{u}^{n+1}\right\|_{L^{3 / 2}} \\
& \leq C \tau^{-1}\left\|\nabla e_{u}^{n+1}\right\|_{L^{3 / 2}}+C \tau^{-1}\left\|\nabla e_{u}^{n}\right\|_{L^{3 / 2}}+\left\|\nabla D_{t} \mathbf{u}^{n+1}\right\|_{L^{3 / 2}} \\
& \leq C \tau^{-1}\left(\left\|e_{c}^{n}\right\|_{L^{2}}+\left\|\nabla e_{c}^{n}\right\|_{L^{3 / 2}}\right)+C \tau^{-1}\left(\left\|e_{c}^{n-1}\right\|_{L^{2}}+\left\|\nabla e_{c}^{n-1}\right\|_{L^{3 / 2}}\right)+\left\|\nabla D_{t} \mathbf{u}^{n+1}\right\|_{L^{3 / 2}},
\end{aligned}
$$

and from (3.16) we see that

$$
\begin{equation*}
\left(\sum_{n=1}^{N-1} \tau\left\|\nabla D_{t} U^{n+1}\right\|_{L^{3 / 2}}^{2}\right)^{\frac{1}{2}} \leq C \tag{3.28}
\end{equation*}
$$

Since $W^{1,3 / 2} \hookrightarrow L^{3}$ in $\mathbb{R}^{d}(d=2,3)$, it follows that

$$
\left\|D_{t} U^{n+1}\right\|_{L^{3}} \leq C\left\|D_{t} U^{n+1}\right\|_{L^{2}}+C\left\|\nabla D_{t} U^{n+1}\right\|_{L^{3 / 2}}
$$

From (3.18) and (3.28), we see that

$$
\begin{equation*}
\left(\sum_{n=1}^{N-1} \tau\left\|D_{t} U^{n+1}\right\|_{L^{3}}^{2}\right)^{\frac{1}{2}} \leq C . \tag{3.29}
\end{equation*}
$$

For the $H^{2}$ regularity of $\mathcal{C}^{n+1}$, we rewrite (3.3) as

$$
\begin{equation*}
-\nabla \cdot\left(D\left(U^{n+1}\right) \nabla \mathcal{C}^{n+1}\right)+U^{n+1} \cdot \nabla \mathcal{C}^{n+1}=-\Phi D_{t} \mathcal{C}^{n+1}+\hat{c} q^{I}-\mathcal{C}^{n+1} q^{P} \tag{3.30}
\end{equation*}
$$

With (3.17), (3.25) and (3.27), we see that the right-hand side above is bounded in $L^{2}$ and so we can apply the $H^{2}$ estimates for the elliptic equation [16] to obtain

$$
\begin{equation*}
\left\|\mathcal{C}^{n+1}\right\|_{H^{2}} \leq C \tag{3.31}
\end{equation*}
$$

We rewrite the equations (3.1)-(3.2) as

$$
-\nabla \cdot\left(\frac{k(x)}{\mu\left(C^{n}\right)} \nabla P^{n+1}\right)=q^{I}-q^{P}
$$

and apply the $H^{3}$ estimates of elliptic equations [16] to the above equation. Then we get

$$
\begin{equation*}
\left\|e_{p}^{n+1}\right\|_{H^{3}}+\left\|e_{u}^{n+1}\right\|_{H^{2}} \leq C \tag{3.32}
\end{equation*}
$$

Note that $H^{2} \hookrightarrow C^{\alpha}$ for some $\alpha>0$. With the Hölder regularity of $\mathcal{C}^{n}$, applying the $W^{1, s}$ estimates to (3.7) and using (3.8), it is not difficult to see that

$$
\begin{equation*}
\left\|e_{u}^{n+1}\right\|_{L^{s}} \leq C\left\|e_{c}^{n}\right\|_{L^{s}} \tag{3.33}
\end{equation*}
$$

Multiplying (3.9) by $\left|e_{c}^{n+1}\right|^{s-2} e_{c}^{n+1}$ and using (3.33), one can derive that

$$
\begin{aligned}
& \int_{\Omega} s^{-1} \Phi D_{t}\left|e_{c}^{n+1}\right|^{s} \mathrm{~d} x+(s-1) \int_{\Omega}\left|e_{c}^{n+1}\right|^{s-2} D\left(U^{n+1}\right) \nabla e_{c}^{n+1} \cdot \nabla e_{c}^{n+1} \mathrm{~d} x \\
& \leq \int_{\Omega} \int_{\Omega} s^{-1}\left(q^{I}-q^{P}\right)\left|e_{c}^{n+1}\right|^{s} \mathrm{~d} x+\left\|\nabla c^{n+1}\right\|_{L^{\infty}}\left\|U^{n+1}-\mathbf{u}^{n+1}\right\|_{L^{s}}\left\|e_{c}^{n+1}\right\|_{L^{s}}^{s-1}
\end{aligned}
$$

$$
\begin{aligned}
& +\left\|q^{P}\right\|_{L^{\infty}}\left\|e_{c}^{n+1}\right\|_{L^{s}}+\left\|\mathcal{E}^{n+1}\right\|_{W^{-1, s}}\left\|e_{c}^{n+1}\right\|_{L^{s}}^{\frac{s-2}{2}}\left\|\left|e_{c}^{n+1}\right|^{s-2} \nabla e_{c}^{n+1}\right\|_{L^{2}} \\
& +(s-1)\left\|D\left(U^{n+1}\right)-D\left(\mathbf{u}^{n+1}\right)\right\|_{L^{s}}\left\|\nabla c^{n+1}\right\|_{L^{\infty}}\left\|e_{c}^{n+1}\right\|_{L^{s}}^{\frac{s-2}{2}}\left\|\left|e_{c}^{n+1}\right|^{s-2} \nabla e_{c}^{n+1}\right\|_{L^{2}} \\
\leq & \left(C+\left\|\mathcal{E}^{n+1}\right\|_{W^{-1, s}}^{s}\right) \epsilon^{-1}\left(\left\|e_{c}^{n+1}\right\|_{L^{s}}^{s}+\left\|e_{c}^{n}\right\|_{L^{s}}^{s}\right)+\epsilon(s-1) \int_{\Omega}\left|e_{c}^{n+1}\right|^{s-2}\left|\nabla e_{c}^{n+1}\right|^{2} \mathrm{~d} x
\end{aligned}
$$

Choosing a proper small $\epsilon$ and using Gronwall's inequality lead to

$$
\left\|e_{c}^{n+1}\right\|_{L^{s}} \leq C \tau
$$

where we have used (3.12). It follows that

$$
\begin{equation*}
\left\|D_{t} e_{c}^{n+1}\right\|_{L^{s}} \leq C \tag{3.34}
\end{equation*}
$$

With the above estimate, by applying the $W^{2, s}$ estimate to (3.30), we obtain

$$
\begin{equation*}
\left\|e_{c}^{n+1}\right\|_{W^{2, s}} \leq C \tag{3.35}
\end{equation*}
$$

and by the Sobolev embedding theorem we get $\left\|\nabla e_{c}^{n+1}\right\|_{L^{\infty}} \leq C\left\|e_{c}^{n+1}\right\|_{W^{2, s}} \leq C$. By applying the $W^{1, s}$ estimates to the elliptic equation (3.7) and using (3.8), we obtain

$$
\begin{equation*}
\left\|\nabla e_{p}^{n+1}\right\|_{L^{s}}+\left\|e_{u}^{n+1}\right\|_{L^{s}} \leq C \tau \tag{3.36}
\end{equation*}
$$

Finally, we multiply (3.9) by $-\nabla \cdot\left(D\left(U^{n+1}\right) \nabla e_{c}^{n+1}\right)$ and summing up the results for $n=0,1, \cdots, N-1$. Then we get

$$
\begin{align*}
& D_{t}\left(D\left(U^{n+1}\right) \nabla e_{c}^{n+1}, \nabla e_{c}^{n+1}\right)+\left\|\nabla \cdot\left(D\left(U^{n+1}\right) \nabla e_{c}^{n+1}\right)\right\|_{L^{2}}^{2}  \tag{3.37}\\
& \leq C\left\|D_{t} D\left(U^{n+1}\right)\right\|_{L^{3}}\left\|\nabla e_{c}^{n+1}\right\|_{L^{2}}\left\|\nabla e_{c}^{n+1}\right\|_{L^{6}}+C\left\|e_{u}^{n+1}\right\|_{H^{1}}^{2}\left\|\nabla c^{n+1}\right\|_{L^{\infty}}^{2} \\
& \quad+C\left\|e_{u}^{n+1}\right\|_{L^{6}}^{2}\left\|c^{n+1}\right\|_{W^{2,3}}^{2}+C\left\|e_{c}^{n+1}\right\|_{H^{1}}^{2}+C\left\|\mathcal{E}^{n+1}\right\|_{L^{2}}^{2} \\
& \leq C_{\epsilon}\left\|D_{t} D\left(U^{n+1}\right)\right\|_{L^{3}}^{2}\left\|\nabla e_{c}^{n+1}\right\|_{L^{2}}^{2}+\epsilon\left\|e_{c}^{n+1}\right\|_{W^{1,6}}^{2}+C\left(\left\|e_{u}^{n+1}\right\|_{H^{1}}^{2}+\left\|e_{c}^{n+1}\right\|_{H^{1}}^{2}+\left\|\mathcal{E}^{n+1}\right\|_{L^{2}}^{2}\right)
\end{align*}
$$

Since by the Sobolev inequality and the theory of elliptic equations we have

$$
\left\|e_{c}^{n+1}\right\|_{W^{1,6}} \leq C\left\|e_{c}^{n+1}\right\|_{H^{2}} \leq C\left\|\nabla \cdot\left(D\left(U^{n+1}\right) \nabla e_{c}^{n+1}\right)\right\|_{L^{2}}
$$

and by the $H^{1}$ estimates of the equation (3.7) we have

$$
\left\|e_{u}^{n+1}\right\|_{H^{1}} \leq C\left\|e_{c}^{n+1}\right\|_{H^{1}}\left\|\nabla p^{n+1}\right\|_{L^{\infty}}+C\left\|e_{c}^{n+1}\right\|_{L^{3}}\left\|p^{n+1}\right\|_{W^{2,6}} \leq C\left\|e_{c}^{n+1}\right\|_{H^{1}},
$$

by choosing a small $\epsilon$, the inequality (3.37) reduces to

$$
\begin{aligned}
& D_{t}\left(D\left(U^{n+1}\right) \nabla e_{c}^{n+1}, \nabla e_{c}^{n+1}\right)+\frac{1}{2}\left\|\nabla \cdot\left(D\left(U^{n+1}\right) \nabla e_{c}^{n+1}\right)\right\|_{L^{2}}^{2} \\
& \leq C\left\|D_{t} D\left(U^{n+1}\right)\right\|_{L^{3}}^{2}\left\|\nabla e_{c}^{n+1}\right\|_{L^{2}}^{2}+C\left(\left\|\nabla e_{c}^{n+1}\right\|_{L^{2}}^{2}+\left\|e_{c}^{n+1}\right\|_{L^{2}}^{2}+\left\|\mathcal{E}^{n+1}\right\|_{L^{2}}^{2}\right) \\
& \leq\left(C\left\|D_{t} D\left(U^{n+1}\right)\right\|_{L^{3}}^{2}+C\right)\left(D\left(U^{n+1}\right) \nabla e_{c}^{n+1}, \nabla e_{c}^{n+1}\right)+C\left(\left\|e_{c}^{n+1}\right\|_{L^{2}}^{2}+\left\|\mathcal{E}^{n+1}\right\|_{L^{2}}^{2}\right)
\end{aligned}
$$

By applying Gronwall's inequality, using (3.11), (3.16) and (3.29), we get

$$
\max _{1 \leq n \leq N}\left\|\nabla e_{c}^{n}\right\|_{L^{2}}^{2}+\sum_{n=1}^{N} \tau\left\|e_{c}^{n}\right\|_{H^{2}}^{2} \leq C \tau^{2}
$$

In particular, the above inequality implies that

$$
\sum_{n=1}^{N} \tau\left\|D_{t} \mathcal{C}^{n}\right\|_{H^{2}}^{2} \leq \sum_{n=1}^{N} \tau\left\|D_{t} c^{n}\right\|_{H^{2}}^{2}+\sum_{n=1}^{N} \tau\left\|D_{t} e_{c}^{n}\right\|_{H^{2}}^{2} \leq C+C \tau^{-2} \sum_{n=1}^{N} \tau\left\|e_{c}^{n}\right\|_{H^{2}}^{2} \leq C
$$

The proof of Theorem 3.1 is complete.

## 4 Error estimates of the fully-discrete system

To provide optimal error estimates for the fully discrete scheme (2.6) $-(2.8)$, we define three projections below.

Let $\Pi_{h}: L^{2}(\Omega) \rightarrow S_{h}$ be the $L^{2}$ projection defined by

$$
\left(\Pi_{h} \phi, \chi\right)=(\phi, \chi), \quad \text { for all } \phi \in L^{2} \text { and } \chi \in S_{h}
$$

For any fixed integer $n \geq 1$, let $\Pi_{h}^{n}: H^{1}(\Omega) \rightarrow V_{h}$ be a projection defined by the following elliptic problem,

$$
\begin{equation*}
\left(D\left(U^{n}\right) \nabla\left(v-\Pi_{h}^{n} v\right), \nabla \phi_{h}\right)=0, \quad \text { for all } \quad \phi_{h} \in V_{h}, \quad v \in H^{1}(\Omega) \tag{4.1}
\end{equation*}
$$

with $\int_{\Omega}\left(v-\Pi_{h}^{n} v\right) d x=0$, and we define $\Pi_{h}^{0}:=\Pi_{h}^{1}$. Moreover, let $Q_{h}: H \rightarrow H_{h}$ be a projection such that 37

$$
\begin{equation*}
\left(\nabla \cdot\left(w-Q_{h} w\right), \chi\right)=0, \quad \text { for all } \quad \chi \in S_{h}, w \in H \tag{4.2}
\end{equation*}
$$

with a slight modification on the triangles/tetrahadons near the boundary.
By the theory of Galerkin and mixed finite element methods for linear elliptic problems [32, 37], with the regularity $U^{n} \in H^{2}(\Omega)$, we have

$$
\begin{align*}
& \left\|v-\Pi_{h} v\right\|_{L^{2}}+h\left\|v-\Pi_{h} v\right\|_{H^{1}} \leq C h^{2}\|v\|_{H^{2}}, \quad \text { for all } v \in H^{2}(\Omega) \\
& \left\|v-\Pi_{h}^{n+1} v\right\|_{L^{p}}+h\left\|v-\Pi_{h}^{n+1} v\right\|_{W^{1, p}} \leq C h^{2}\|v\|_{W^{2, p}}, \quad \text { for all } v \in W^{2, p}(\Omega), \\
& \left\|w-Q_{h} w\right\|_{L^{2}}+h\left\|w-Q_{h} w\right\|_{H^{1}} \leq C h^{2}\|w\|_{H^{2}}, \quad \text { for all } w \in H \tag{4.3}
\end{align*}
$$

Previous works on Galerkin (or mixed) FEM for the nonlinear parabolic system (1.1)(1.3) required the estimate

$$
\begin{equation*}
\left\|\partial_{t}\left(c^{n+1}-\widetilde{\Pi}_{h}^{n+1} c^{n+1}\right)\right\|_{L^{2}} \leq C h^{2} \tag{4.4}
\end{equation*}
$$

for an elliptic projection operator $\widetilde{\Pi}_{h}^{n+1}$ defined by

$$
\begin{equation*}
\left(D(\mathbf{u}) \nabla\left(v-\widetilde{\Pi}_{h} v\right), \nabla \phi_{h}\right)=0, \quad \text { for all } \phi_{h} \in V_{h}, \quad v \in H^{1}(\Omega) \tag{4.5}
\end{equation*}
$$

The inequality (4.4) was proved by Wheeler 43] based on the regularity assumption $\left\|\nabla D(\mathbf{u})_{t}\right\|_{L^{\infty}} \leq$ $C$. In the following lemma, we will prove an analogues inequality:

$$
\begin{equation*}
\left(\sum_{n=0}^{N-1} \tau\left\|D_{t}\left(\mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{H^{-1}}^{2}\right)^{1 / 2} \leq C h^{2} \tag{4.6}
\end{equation*}
$$

based on weaker regularity of $U^{n}$ proved in the last section. The above inequality is necessary to obtain optimal $L^{2}$ error estimates.

Lemma 4.1 With the regularity assumption (2.1), the regularity of $\left(\mathcal{C}^{n+1}, U^{n+1}\right)$ given in (3.5), and the error estimates given in (3.6), the estimate (4.6) holds.

Proof We only prove the lemma for the 3D problem (with $s>3$ in Theorem 3.1). The 2D problem can be handled similarly. Note that

$$
\begin{equation*}
\left(D\left(U^{n+1}\right) \nabla\left(\mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right), \nabla \phi_{h}\right)=0 \tag{4.7}
\end{equation*}
$$

$$
\begin{equation*}
\left(D\left(U^{n}\right) \nabla\left(\mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right), \nabla \phi_{h}\right)=0 \tag{4.8}
\end{equation*}
$$

The difference of the above two equations gives

$$
\begin{aligned}
& \left(D\left(U^{n+1}\right) \nabla\left(\Pi_{h}^{n} \mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right), \nabla \phi_{h}\right) \\
& +\left(\left(D\left(U^{n+1}\right)-D\left(U^{n}\right)\right) \nabla\left(\mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right), \nabla \phi_{h}\right)=0
\end{aligned}
$$

By the $W^{1, p}$ estimates of elliptic projections [32], we have

$$
\begin{aligned}
& \left\|\nabla\left(\Pi_{h}^{n} \mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{6 / 5}} \leq C\left\|\left(D\left(U^{n+1}\right)-D\left(U^{n}\right)\right) \nabla\left(\mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right)\right\|_{L^{6 / 5}} \\
& \leq C\left\|D\left(U^{n+1}\right)-D\left(U^{n}\right)\right\|_{L^{2}}\left\|\nabla\left(\mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right)\right\|_{L^{3}} \leq C\left\|D\left(U^{n+1}\right)-D\left(U^{n}\right)\right\|_{L^{2}} h
\end{aligned}
$$

For any $\varphi \in H^{1}(\Omega)$ with $\int_{\Omega} \varphi \mathrm{d} x=0$, let $\psi$ be the solution of the equation

$$
-\nabla \cdot\left(D\left(U^{n+1}\right) \nabla \psi\right)=\varphi
$$

with the boundary condition $\nabla \psi \cdot \mathbf{n}=0$ on $\partial \Omega$. Since $U^{n+1}$ is uniformly bounded in $H^{2}(\Omega)$, i.e. $\left\|U^{n+1}\right\|_{H^{2}} \leq C$, it is easy to check that

$$
\|\psi\|_{H^{3}} \leq C\|\varphi\|_{H^{1}}
$$

By the boundary condition $U^{n+1} \cdot \mathbf{n}=U^{n} \cdot \mathbf{n}=0$ and the expression of the function $D(\mathbf{u})$, we see that $D\left(U^{n+1}\right) \nabla \psi \cdot \mathbf{n}=D\left(U^{n}\right) \nabla \psi \cdot \mathbf{n}=0$ on $\partial \Omega$. We see that for $n \geq 1$,

$$
\begin{aligned}
& \left.\Pi_{h}^{n} \mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}, \varphi\right) \\
= & \left(D\left(U^{n+1}\right) \nabla\left(\Pi_{h}^{n} \mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right), \nabla \psi\right) \\
= & \left(D\left(U^{n+1}\right) \nabla\left(\Pi_{h}^{n} \mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right), \nabla\left(\psi-P_{h} \psi\right)\right) \\
& -\left(\left(D\left(U^{n+1}\right)-D\left(U^{n}\right)\right) \nabla\left(\mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right), \nabla\left(P_{h} \psi-\psi\right)\right) \\
& -\left(\left(D\left(U^{n+1}\right)-D\left(U^{n}\right)\right) \nabla\left(\mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right), \nabla \psi\right) \\
\leq & C\left\|D\left(U^{n+1}\right)-D\left(U^{n}\right)\right\|_{L^{2}} h\left\|\nabla\left(\psi-P_{h} \psi\right)\right\|_{L^{6}} \\
& +\left\|\mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right\|_{L^{3}}\left\|\nabla \cdot\left(\left(D\left(U^{n+1}\right)-D\left(U^{n}\right)\right) \nabla \psi\right)\right\|_{L^{3 / 2}} \\
\leq & C\left\|D\left(U^{n+1}\right)-D\left(U^{n}\right)\right\|_{L^{2}} h^{2}\|\psi\|_{W^{2,6}} \\
& +C h^{2}\left\|\mathcal{C}^{n+1}\right\|_{W^{2,3}}\left(\left\|\nabla\left(D\left(U^{n+1}\right)-D\left(U^{n}\right)\right)\right\|_{L^{3 / 2}}\|\nabla \psi\|_{L^{\infty}}+\left\|U^{n+1}-U^{n}\right\|_{L^{2}}\|\psi\|_{W^{2,6}}\right) \\
\leq & C\left(\left\|D_{t} U^{n+1}\right\|_{L^{2}}+C\left\|\nabla D_{t} U^{n+1}\right\|_{L^{3 / 2}}+\left\|\nabla U^{n+1} D_{t} U^{n+1}\right\|_{L^{3 / 2}}\right) \tau h^{2}\|\psi\|_{H^{3}} \\
\leq & \left.C\left(\left\|D_{t} U^{n+1}\right\|_{L^{2}}+C\left\|\nabla D_{t} U^{n+1}\right\|_{L^{3 / 2}}+\left\|\nabla U^{n+1} D_{t} U^{n+1}\right\|_{L^{3 / 2}}\right)\right) \tau h^{2}\|\varphi\|_{H^{1}} .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \left\|\Pi_{h}^{n} \mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{H^{-1}} \\
& \left.\leq C\left(\left\|D_{t} U^{n+1}\right\|_{L^{2}}+C\left\|\nabla D_{t} U^{n+1}\right\|_{L^{3 / 2}}+\left\|\nabla U^{n+1} D_{t} U^{n+1}\right\|_{L^{3 / 2}}\right)\right) \tau h^{2}
\end{aligned}
$$

By (3.6), we have

$$
\left\|D_{t} U^{n+1}\right\|_{L^{2}} \leq\left\|D_{t} e_{u}^{n+1}\right\|_{L^{2}}+\left\|D_{t} \mathbf{u}^{n+1}\right\|_{L^{2}} \leq C+\left\|D_{t} \mathbf{u}^{n+1}\right\|_{L^{2}}
$$

$$
\left\|\nabla U^{n+1} D_{t} U^{n+1}\right\|_{L^{3 / 2}} \leq\left\|\nabla U^{n+1}\right\|_{L^{6}}\left\|D_{t} U^{n+1}\right\|_{L^{2}} \leq C+C\left\|D_{t} \mathbf{u}^{n+1}\right\|_{L^{2}}
$$

With the regularity assumption (2.1) on $\mathbf{u}$ and the estimate (3.6), we derive that

$$
\left(\sum_{n=0}^{N-1} \tau\left\|\Pi_{h}^{n} \mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{H^{-1}}^{2}\right)^{\frac{1}{2}} \leq C \tau h^{2}
$$

Since

$$
\begin{aligned}
& \left\|D_{t}\left(\mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{H^{-1}} \\
& \leq\left\|D_{t} \mathcal{C}^{n+1}-\Pi_{h}^{n} D_{t} \mathcal{C}^{n+1}\right\|_{H^{-1}}+\tau^{-1}\left\|\Pi_{h}^{n+1} \mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right\|_{H^{-1}} \\
& \leq C\left\|D_{t} \mathcal{C}^{n+1}\right\|_{H^{2}} h^{2}+\tau^{-1}\left\|\Pi_{h}^{n+1} \mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right\|_{H^{-1}} \\
& \leq C\left\|D_{t} e_{c}^{n+1}\right\|_{H^{2}} h^{2}+C\left\|D_{t} c^{n+1}\right\|_{H^{2}} h^{2}+\tau^{-1}\left\|\Pi_{h}^{n+1} \mathcal{C}^{n+1}-\Pi_{h}^{n} \mathcal{C}^{n+1}\right\|_{H^{-1}}
\end{aligned}
$$

with (3.5) and (2.1), (4.6) follows immediately.
The proof of Lemma 4.1 is complete.
Theorem 4.1 Suppose that the initial-boundary value problem (1.1)-(1.4) has a unique solution ( $p, \mathbf{u}, c$ ) which satisfies (2.1). Then there exist positive constants $h_{0}$ and $\tau_{0}$ such that when $h<h_{0}$ and $\tau<\tau_{0}$, the finite element system (2.6)-(2.7) admits a unique solution $\left(P_{h}^{n}, U_{h}^{n}, \mathcal{C}_{h}^{n}\right), n=1, \cdots, N$, which satisfies

$$
\max _{1 \leq n \leq N}\left\|P_{h}^{n}-P^{n}\right\|_{L^{2}}+\max _{1 \leq n \leq N}\left\|U_{h}^{n}-U^{n}\right\|_{L^{2}}+\max _{1 \leq n \leq N}\left\|\mathcal{C}_{h}^{n}-\mathcal{C}^{n}\right\|_{L^{2}} \leq C h^{2} .
$$

where $\left\{\left(P^{n}, U^{n}, \mathcal{C}^{n}\right)\right\}_{n=1}^{N}$ is the solution of the time-discrete system (3.1)-(3.4).
Proof The solution to the time-discrete system (3.1)-(3.4) satisfies

$$
\begin{align*}
& \left(\frac{\mu\left(\mathcal{C}^{n}\right)}{k(x)} U^{n+1}, v_{h}\right)=-\left(P^{n+1}, \nabla \cdot v_{h}\right)  \tag{4.9}\\
& \left(\nabla \cdot U^{n+1}, \varphi_{h}\right)=\left(q^{I}-q^{P}, \varphi_{h}\right)  \tag{4.10}\\
& \left(\Phi D_{t} \mathcal{C}^{n+1}, \phi_{h}\right)+\left(D\left(U^{n+1}\right) \nabla \mathcal{C}^{n+1}, \nabla \phi_{h}\right) \\
& \quad+\left(U^{n+1} \cdot \nabla \mathcal{C}^{n+1}, \phi_{h}\right)=\left(\hat{c} q^{I}-\mathcal{C}^{n+1} q^{P}, \phi_{h}\right) . \tag{4.11}
\end{align*}
$$

for any $v_{h} \in H_{h}, \varphi_{h} \in S_{h}$ and $\phi_{h} \in V_{h}$. The above equations with the finite element system (2.6)-(2.8) imply that the error functions $P_{h}^{n+1}-\Pi_{h} P^{n+1}, U_{h}^{n+1}-U^{n+1}, \mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}$ satisfy

$$
\begin{align*}
& \left(\frac{\mu\left(\mathcal{C}_{h}^{n}\right)}{k(x)} U_{h}^{n+1}-\frac{\mu\left(\mathcal{C}^{n}\right)}{k(x)} U^{n+1}, v_{h}\right)=-\left(P_{h}^{n+1}-\Pi_{h} P^{n+1}, \nabla \cdot v_{h}\right),  \tag{4.12}\\
& \left(\nabla \cdot\left(U_{h}^{n+1}-U^{n+1}\right), \varphi_{h}\right)=0  \tag{4.13}\\
& \left(\Phi D_{t}\left(\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right), \phi_{h}\right)+\left(D\left(U_{h}^{n+1}\right) \nabla\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right), \nabla \phi_{h}\right) \\
& =-\left(U^{n+1} \cdot \nabla\left(\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right), \phi_{h}\right)-\left(\left(U_{h}^{n+1}-U^{n+1}\right) \cdot \nabla \mathcal{C}_{h}^{n+1}, \phi_{h}\right) \\
& \quad-\left(\left(\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right) q^{P}, \phi_{h}\right)+\left(\left(D\left(U^{n+1}\right)-D\left(U_{h}^{n+1}\right)\right) \nabla \Pi_{h}^{n+1} \mathcal{C}^{n+1}, \nabla \phi_{h}\right) . \tag{4.14}
\end{align*}
$$

First, we present an upper bound for the error function $\left\|U_{h}^{n+1}-U^{n+1}\right\|_{L^{2}}$ in terms of $\left\|\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right\|_{L^{2}}$. By the definition of the projection operator $Q_{h}$ in (4.2), from (4.13) we see that

$$
\left(\nabla \cdot\left(U_{h}^{n+1}-Q_{h} U^{n+1}\right), \varphi_{h}\right)=0, \quad \text { for all } \varphi_{h} \in S_{h}
$$

which implies that $\nabla \cdot\left(U_{h}^{n+1}-Q_{h} U^{n+1}\right)=0$ in $\Omega$. Taking $v_{h}=U_{h}^{n+1}-Q_{h} U^{n+1}$ in (4.12), we get

$$
\begin{aligned}
& \left(\frac{\mu\left(\mathcal{C}_{h}^{n}\right)}{k(x)}\left(U_{h}^{n+1}-Q_{h} U^{n+1}\right)+\frac{\mu\left(\mathcal{C}_{h}^{n}\right)}{k(x)}\left(Q_{h} U^{n+1}-U^{n+1}\right)\right. \\
& \left.\quad+\left(\frac{\mu\left(\mathcal{C}_{h}^{n}\right)}{k(x)}-\frac{\mu\left(\mathcal{C}^{n}\right)}{k(x)}\right) U^{n+1}, U_{h}^{n+1}-Q_{h} U^{n+1}\right)=0
\end{aligned}
$$

which further implies that

$$
\left\|U_{h}^{n+1}-Q_{h} U^{n+1}\right\|_{L^{2}} \leq C\left\|Q_{h} U^{n+1}-U^{n+1}\right\|_{L^{2}}+C\left\|\mathcal{C}_{h}^{n}-\mathcal{C}^{n}\right\|_{L^{2}}
$$

With (4.3), the above inequality reduces to

$$
\begin{equation*}
\left\|U_{h}^{n+1}-U^{n+1}\right\|_{L^{2}} \leq C h^{2}+C\left\|\mathcal{C}_{h}^{n}-\mathcal{C}^{n}\right\|_{L^{2}} \tag{4.15}
\end{equation*}
$$

Secondly, we take $\phi_{h}=\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}$ in (4.14) and get

$$
\begin{align*}
& \left.\frac{1}{2} D_{t}\left\|\sqrt{\Phi}\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{2}}^{2}+\| \sqrt{D\left(U_{h}^{n+1}\right.}\right) \nabla\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right) \|_{L^{2}}^{2} \\
& \leq\left(\Phi D_{t}\left(\mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right), \phi_{h}-\frac{1}{\Omega \mid} \int_{\Omega} \phi_{h} \mathrm{~d} x\right) \\
& \quad+C\left\|q^{I}-q^{P}\right\|_{L^{3}}\left\|\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{6}}\left\|\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right\|_{L^{2}} \\
& \quad+C\left\|U^{n+1}\right\|_{L^{\infty}}\left\|\nabla\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{2}}\left\|\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right\|_{L^{2}} \\
& \quad+C\left\|\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{6}}\left\|U_{h}^{n+1}-U^{n+1}\right\|_{L^{2}}\left\|\nabla\left(\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{3}} \\
& \left.\quad+\left\|U_{h}^{n+1}-U^{n+1}\right\|_{L^{2}}\left\|\nabla\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{3}}\right) \\
& \quad+C\left\|q^{P}\right\|_{L^{3}}\left(\left\|\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{3}}^{2}+\left\|\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{6}}\left\|\mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{2}}\right) \\
& \quad+C\left\|\nabla \Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{\infty}}\left\|\nabla\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{2}}\left\|U_{h}^{n+1}-U^{n+1}\right\|_{L^{2}} \\
& \leq\left(\epsilon+C h^{-d / 6}\left\|U_{h}^{n+1}-U^{n+1}\right\|_{L^{2}}\right)\left\|\nabla\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{2}}^{2} \\
& \quad+C \epsilon^{-1}\left\|D_{t}\left(\mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{H^{-1}}^{2}+C \epsilon^{-1}\left\|U_{h}^{n+1}-U^{n+1}\right\|_{L^{2}}^{2} \\
& \quad+C \epsilon^{-1}\left\|\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{2}}^{2}++C \epsilon^{-1} h^{4}, \tag{4.16}
\end{align*}
$$

where we have used (4.3) and the following estimate:

$$
\begin{aligned}
& \left|\left(U^{n+1} \cdot \nabla\left(\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right), \mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right| \\
& \quad=\left|\left(\left(q^{I}-q^{P}\right)\left(\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right), \mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right| \\
& \quad+\left|\left(U^{n+1}\left(\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right), \nabla\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right)\right| \\
& \quad \leq C\left\|q^{I}-q^{P}\right\|_{L^{3}}\left\|\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{6}}\left\|\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right\|_{L^{2}} \\
& \quad+C\left\|U^{n+1}\right\|_{L^{\infty}}\left\|\nabla\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{2}}\left\|\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right\|_{L^{2}} .
\end{aligned}
$$

From (4.15) we observe that (4.16) reduces to

$$
\begin{aligned}
& D_{t}\left\|\sqrt{\Phi}\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{2}}^{2}+\frac{1}{2}\left\|\sqrt{D\left(U_{h}^{n+1}\right)} \nabla\left(\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{2}}^{2} \\
& \leq C h^{4}+C h^{-1 / 2}\left\|\mathcal{C}_{h}^{n}-\Pi_{h}^{n} \mathcal{C}^{n}\right\|_{L^{2}}\left\|\nabla\left(\mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{L^{2}}^{2} \\
& +C\left(\left\|\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{2}}^{2}+\left\|\mathcal{C}_{h}^{n}-\Pi_{h}^{n} \mathcal{C}^{n}\right\|_{L^{2}}^{2}\right)+C\left\|D_{t}\left(\mathcal{C}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right)\right\|_{H^{-1}}^{2}
\end{aligned}
$$

By applying Gronwall's inequality (with mathematical induction on $\left\|\mathcal{C}_{h}^{n}-\Pi_{h}^{n} \mathcal{C}^{n}\right\|_{L^{2}} \leq h$ ), we deduce that there exists a positive constant $h_{0}$ such that when $h<h_{0}$ we have

$$
\begin{equation*}
\left\|\mathcal{C}_{h}^{n+1}-\Pi_{h}^{n+1} \mathcal{C}^{n+1}\right\|_{L^{2}} \leq C h^{2} \tag{4.17}
\end{equation*}
$$

From (4.3) and (4.15), we further get

$$
\begin{align*}
& \left\|U_{h}^{n+1}-U^{n+1}\right\|_{L^{2}} \leq C h^{2}  \tag{4.18}\\
& \left\|\mathcal{C}_{h}^{n+1}-\mathcal{C}^{n+1}\right\|_{L^{2}} \leq C h^{2} \tag{4.19}
\end{align*}
$$

Finally, we estimate the error $\left\|P_{h}-P^{n+1}\right\|_{L^{2}}$. We redefine $g$ to be the solution to the equation

$$
-\nabla \cdot\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \nabla g\right)=P_{h}^{n+1}-\Pi_{h} P^{n+1}
$$

with the boundary condition $\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \nabla g \cdot \mathbf{n}=0$ on $\partial \Omega$. Easy to check that

$$
\|g\|_{H^{2}} \leq C\left\|P_{h}^{n+1}-\Pi_{h} P^{n+1}\right\|_{L^{2}}
$$

Let

$$
v_{h}=Q_{h}\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \nabla g\right)
$$

Then

$$
\left(\varphi, \nabla \cdot v_{h}\right)=-\left(\varphi, P_{h}^{n+1}-\Pi_{h} P^{n+1}\right), \quad \varphi \in S_{h}
$$

and from (4.12) we obtain

$$
\begin{aligned}
\left\|P_{h}^{n+1}-\Pi_{h} P^{n+1}\right\|_{L^{2}}^{2} & =\left(\frac{\mu\left(\mathcal{C}_{h}^{n}\right)}{k(x)} U_{h}^{n+1}-\frac{\mu\left(\mathcal{C}^{n}\right)}{k(x)} U^{n+1}, Q_{h}\left(\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \nabla g\right)\right) \\
& \leq C\left(\left\|\mathcal{C}_{h}^{n}-\mathcal{C}^{n}\right\|_{L^{2}}+\left\|U_{h}^{n+1}-U^{n+1}\right\|_{L^{2}}\right)\left\|\frac{k(x)}{\mu\left(\mathcal{C}^{n}\right)} \nabla g\right\|_{H^{1}} \\
& \leq C h^{2}\left\|P_{h}^{n+1}-\Pi_{h} P^{n+1}\right\|_{L^{2}}
\end{aligned}
$$

which implies that

$$
\left\|P_{h}^{n+1}-\Pi_{h} P^{n+1}\right\|_{L^{2}} \leq C h^{2}
$$

The proof of Theorem 4.1 is complete.
Combining Theorems 3.1 and Theorem 4.1, we complete the proof of Theorem 2.1,

## 5 Numerical examples

In this section, we present two numerical examples to confirm our theoretical analysis. All computations are performed by using the software FreeFEM ++ .
Example 5.1 We rewrite the system (1.1)-(1.4) by

$$
\begin{align*}
& \frac{\partial c}{\partial t}-\nabla \cdot(D(\mathbf{u}) \nabla c)+\mathbf{u} \cdot \nabla c=g  \tag{5.1}\\
& \nabla \cdot \mathbf{u}=f  \tag{5.2}\\
& \mathbf{u}=-\frac{1}{\mu(c)} \nabla p \tag{5.3}
\end{align*}
$$

where $\Omega=(0,1) \times(0,1)$ and $D(\mathbf{u})=1+|\mathbf{u}|^{2} /(1+|\mathbf{u}|)$ and $\mu(c)=1+c^{2}$. The functions $f$ and $g$ are chosen corresponding to the exact solution

$$
\begin{align*}
p & =1+1000 x^{2}(1-x)^{3} y^{2}(1-y)^{3} t^{2} e^{t}  \tag{5.4}\\
\mathbf{u} & =-\frac{1}{\mu(c)} \nabla p  \tag{5.5}\\
c & =0.1+50 x^{2}(1-x)^{2} y^{2}(1-y)^{2} t e^{t} \tag{5.6}
\end{align*}
$$

Clearly, the boundary condition (1.4) is satisfied.
A uniform triangular partition with $M+1$ nodes in each direction is used to generate the FEM mesh (with $h=1 / M$ ). We solve the system by the proposed method up to the time $t=1$. To illustrate our error estimates, numerical errors with $\tau=8 h^{2}$ are presented in Table 1. from which we can see that the $L^{2}$ errors are proportional to $O\left(h^{2}\right)$. To demonstrate the unconditionaly convergence (stability) of the numerical method, we solve the system with a fixed $\tau$ and several different spatial mesh size $h$. We present numerical errors in Table 2. We can observe from Table 2 that numerical errors behave like $O(\tau)$ as $h / \tau \rightarrow 0$. This implies that the time-step conditions are not necessary.

Example 5.2 We consider the equations (5.1)-(5.3) in a circle centered at $(0.5,0.5)$ with the radius 0.5 and with inhomogeneous Neumann boundary conditions correspondingly to the exact solution given in (5.4)-(5.6). The mesh generated here consists of $M$ boundary points with $M=32,64,128$, respectively, as shown in Figure 1, Numerical errors with fixed $\tau$ and several different $h$ are presented in Table 3, which also show clearly that no time-step condition is needed.


Figure 1: The FEM meshes with $M=32, M=64$ and $M=128$.

Table 1: Errors of the Galerkin-mixed FEM in $L^{2}$ norm.

| $\tau$ | $h$ | $\left\\|U_{h}^{N}-\mathbf{u}\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ | $\left\\|\mathcal{C}_{h}^{N}-c\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ |
| :--- | :--- | :---: | :---: |
| $1 / 8$ | $1 / 8$ | $2.024 \mathrm{E}-01$ | $7.114 \mathrm{E}-02$ |
| $1 / 32$ | $1 / 16$ | $5.264 \mathrm{E}-02$ | $1.713 \mathrm{E}-02$ |
| $1 / 128$ | $1 / 32$ | $1.333 \mathrm{E}-02$ | $4.070 \mathrm{E}-03$ |
| convergence rate |  | 1.98 | 2.07 |

Table 2: Errors of the Galerkin-mixed FEM with fixed $\tau$ and refined $h$.

| $\tau=0.05$ | $h$ | $\left\\|U_{h}^{N}-\mathbf{u}\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ | $\left\\|\mathcal{C}_{h}^{N}-c\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ |
| :---: | :---: | :---: | :---: |
|  | $1 / 8$ | $1.955 \mathrm{E}-01$ | $4.748 \mathrm{E}-02$ |
|  | $1 / 16$ | $5.531 \mathrm{E}-02$ | $2.081 \mathrm{E}-02$ |
|  | $1 / 32$ | $2.409 \mathrm{E}-02$ | $1.077 \mathrm{E}-02$ |
|  | $1 / 64$ | $1.998 \mathrm{E}-02$ | $8.243 \mathrm{E}-03$ |
| $\tau=0.1$ | $h$ | $\left\\|U_{h}^{N}-\mathbf{u}\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ | $\left\\|\mathcal{C}_{h}^{N}-c\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ |
|  | $1 / 8$ | $1.998 \mathrm{E}-01$ | $6.216 \mathrm{E}-02$ |
|  | $1 / 16$ | $6.577 \mathrm{E}-02$ | $3.348 \mathrm{E}-02$ |
|  | $1 / 32$ | $4.168 \mathrm{E}-02$ | $2.240 \mathrm{E}-02$ |
|  | $1 / 64$ | $3.910 \mathrm{E}-02$ | $1.961 \mathrm{E}-02$ |
| $\tau=0.25$ | $h$ | $\left\\|U_{h}^{N}-\mathbf{u}\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ | $\left\\|\mathcal{C}_{h}^{N}-c\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ |
|  | $1 / 8$ | $2.195 \mathrm{E}-01$ | $1.336 \mathrm{E}-01$ |
|  | $1 / 16$ | $1.088 \mathrm{E}-01$ | $9.885 \mathrm{E}-02$ |
|  | $1 / 32$ | $9.491 \mathrm{E}-02$ | $8.426 \mathrm{E}-02$ |
|  | $1 / 64$ | $9.349 \mathrm{E}-02$ | $8.062 \mathrm{E}-02$ |

Table 3: Errors of the Galerkin-mixed FEM with fixed $\tau$ and refined $h$.

| $\tau=0.05$ | $M$ | $\left\\|U_{h}^{N}-\mathbf{u}\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ | $\left\\|\mathcal{C}_{h}^{N}-c\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ |
| :---: | :---: | :---: | :---: |
|  | 32 | $7.105 \mathrm{E}-02$ | $1.445 \mathrm{E}-02$ |
|  | 64 | $2.526 \mathrm{E}-02$ | $4.022 \mathrm{E}-03$ |
|  | 128 | $1.523 \mathrm{E}-02$ | $7.754 \mathrm{E}-04$ |
| $\tau=0.1$ | $M$ | $\left\\|U_{h}^{N}-\mathbf{u}\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ | $\left\\|\mathcal{C}_{h}^{N}-c\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ |
|  | 32 | $7.560 \mathrm{E}-02$ | $1.569 \mathrm{E}-02$ |
|  | 64 | $3.523 \mathrm{E}-02$ | $4.340 \mathrm{E}-03$ |
|  | 128 | $2.869 \mathrm{E}-02$ | $1.248 \mathrm{E}-03$ |
| $\tau=0.25$ | $M$ | $\left\\|U_{h}^{N}-\mathbf{u}\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ | $\left\\|\mathcal{C}_{h}^{N}-c\left(\cdot, t_{N}\right)\right\\|_{L^{2}}$ |
|  | 32 | $9.719 \mathrm{E}-02$ | $2.900 \mathrm{E}-02$ |
|  | 64 | $6.960 \mathrm{E}-02$ | $1.429 \mathrm{E}-02$ |
|  | 128 | $6.632 \mathrm{E}-02$ | $7.940 \mathrm{E}-03$ |

## 6 Conclusions

We have studied error analysis for a nonlinear and strongly coupled parabolic system from incompressible miscible flow in porous media with a commonly-used Galerkin-mixed FEM and linearized semi-implicit Euler scheme. Optimal $L^{2}$ error estimates were obtained almost without any time-step condition, while all previous works imposed certain restriction for the time-step size. The unconditional error analysis presented in this paper can be extended to models with other boundary conditions and numerical methods with high-order approximations, while here we only focus our analysis on the problem with a homogeneous boundary condition and the lowest-order Galerkin-mixed FEM. In fact, we have proved the error estimates:

$$
\left\|U_{h}^{n}-U^{n}\right\|_{L^{2}} \leq C h^{2}, \quad\left\|U^{n}-u^{n}\right\|_{L^{2}} \leq C \tau
$$

We can see from our proof that the two inequalities also hold for higher-order finite element methods. The inequalities imply the boundedness of numerical solution. With some more precise analysis for the time-discrete system, optimal error estimates of high-order Galerkin type methods can be obtained in the traditional way. Also we believe that the idea of the error splitting coupled with the regularity analysis of the time-discrete PDEs can be applied to many other nonlinear parabolic PDEs and time discretizations to obtain optimal error estimates unconditionally.

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