# A QUANTITATIVE SECOND ORDER MINIMALITY CRITERION FOR CAVITIES IN ELASTIC BODIES 

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Abstract. We consider a functional which models an elastic body with a cavity. We show that if a critical point has positive second variation then it is a strict local minimizer. We also provide a quantitative estimate.

## 1. Introduction

The role of roughness appearing onto the surfaces and interfaces of nano-structures has been proved to be of great significance in several fields such as micro-electronics, metallurgy and materials science. For instance the roughness can strongly modify the mechanical properties of multilayered structures as confirmed by the observation that dislocations, islands and cracks can be generated from a rough surface (see [10]). Many efforts have been devoted to the investigation on how to control the roughness appearing onto the surfaces and interfaces of nano-structures, leading to the study of the so-called Driven Rearrangement Instability, i.e., the morphological surfaces instability of interfaces between solids generated by elastic stress. This phenomenon has been detected, for instance, in hetero-epitaxial growth of thin films with a lattice mismatch between film and substrate and in stressed elastic solids with cavities.

The theoretical investigation of the stability of the free surface of a planar non-hydrostatically stressed solid has been performed in the pioneering papers by Asaro and Tiller [4] and Grinfeld [16]. These authors showed that the free surface is unstable with respect to a given family of sinusoidal fluctuations. They also gave a first insightful description of the phenomenon, nowadays named Asaro-Grinfeld-Tiller instability, in which a thin film growing on a flat substrate remains flat up to a critical value of the thickness, after which, the free surface becomes unstable developing corrugations and irregularities. This instability is explained as a consequence of the presence of two competing energies, usually identified with a bulk elastic energy and a surface energy. After these results the interest of the scientific community on the rigorous mathematical study of the morphological instabilities has rapidly grown. Starting from the paper [17] where Grinfeld follows the Gibbs variational approach to model the morphology of thin films, it became clear that a second order variational analysis could be successfully used. This approach has been used in the context of epitaxial growth first for a one dimensional model in [6]. Then in [5] and [12] the model introduced in [17], which is a more realistic two-dimensional model, corresponding to threedimensional configurations with planar symmetry, is studied and the problem of finding a proper functional setting is successfully addressed. This settled the framework in which a precise and detailed analysis of qualitative properties of regular equilibrium configurations has been carried out by Fusco and Morini in [14] via a second order variational analysis. Indeed they prove a sufficient condition for local minimality in terms of the positivity of second variation and provide a sufficiently complete picture of the phenomena that occur in epitaxially-growing thin films.

Such detailed analysis is instead far from being complete in the framework of stressed elastic solids with cavities. In this paper we perform a second order variational analysis for a twodimensional variational model that has been recently used to describe surface instability in morphological evolution of cavities in stressed solids (see for instance $[15,19,21]$ ) with the aim of deriving new minimality conditions for equilibria and studying their stability. The model can be

[^0]roughly described as follows. Consider a cavity in an elastic solid, that will be identified with a smooth compact set $F \subset \mathbb{R}^{2}$, starshaped with respect to the origin. The solid region is assumed to obey to the classical law of linear elasticity, so that the bulk energy can be written in the form
$$
\int_{B_{R_{0}} \backslash F} Q(E(u)) d z
$$
where $E(u)$ is the symmetric gradient of the elastic displacement $u$ and $Q$ is a bilinear form depending on the material (see Section 2 for details). The surface energy is simply assumed to be the length of the boundary of $F$. Then the energy for a regular configuration is expressed by the functional
$$
\mathcal{F}(F, u):=\int_{B_{0} \backslash F} Q(E(u)) d z+\mathcal{H}^{1}(\partial F)
$$

In this framework the shape of the void plays a key role in the evolution of cavities in stressed solid bodies, while the effects of the volume changes are negligible. Hence, one usually assumes that the void evolves preserving its volume. The equilibria are therefore identified with minimizers of $\mathcal{F}(F, u)$ under the volume constraint $|F|=d$. Since admissible configurations need not to be regular, the energy of such configurations has to be defined via a relaxation procedure. This issue, together with the study of the regularity of minima, has been addressed (even for more general functionals involving anisotropic surface energies) in [13] where, in order to keep track of the possible appearance of cracks, the relaxed functional with respect to the Hausdorff convergence has been studied. The relaxed functional can be expressed in the following form:

$$
\begin{equation*}
\mathcal{F}(F, u):=\int_{B_{0} \backslash F} Q(E(u)) d z+\mathcal{H}^{1}\left(\Gamma_{F}\right)+2 \mathcal{H}^{1}\left(\Sigma_{F}\right) \tag{1.1}
\end{equation*}
$$

where $F$ has finite perimeter, $\Gamma_{F}$ is the "regular" part of $\partial F$ and $\Sigma_{F}$ represents the cracks (see Section 2).

The main result of the paper is a quantitative minimality criterion that relies on the study of the second variation of the functional (1.1). To be more precise we prove in Theorem 5.1 that if $(F, u)$ is a smooth critical configuration and the non local quadratic form $\partial^{2} \mathcal{F}(F, u)$ associated to the second variation of $\mathcal{F}$ at $(F, u)$ is positively defined, then there exists a constant $c_{0}$ such that

$$
\begin{equation*}
\mathcal{F}(G, v)>\mathcal{F}(F, u)+c_{0}|G \Delta F|^{2} \tag{1.2}
\end{equation*}
$$

for any given admissible configuration $(G, v)$ with $G$ sufficiently close to $F$ in the Hausdorff distance and $G \neq F$. In particular this implies not only that $(F, u)$ is a strict local minimizer of (1.1) but also provides a quantitative estimate of the deviation from minimality for configurations close to $(F, u)$ in the spirit of the recent result obtained in [1]. The minimality criterion is then applied to the case of a disk subjected to radial stretching where the second variation can be explicitly estimated to prove the local and global minimality of the round configuration if the applied stress is sufficiently small.

We point out that an important open problem is how to remove the assumption of starshapedness. Indeed, even the explicit form of the relaxed functional is unknown.

We conclude by outlining the structure of the paper and making some comments about the proofs. In Section 3 we calculate the second variation of $\mathcal{F}$ at any regular configuration (see Theorem 3.3) and we exploit the volume constraint to define the associated quadratic form in a critical configuration. At the end of the section in Lemma 3.6 we prove a "weak" coercivity property of $\partial^{2} \mathcal{F}(F, u)$ in a critical point, which is the first step towards the proof of Theorem 5.1. In Section 4, as an intermediate step, we prove that the positivity of the second variation implies the local minimality among configurations $(G, v)$ for $G$ close to $F$ in the $C^{1,1}$-topology. The main point in achieving this result is to overcome the lack of $C^{1,1}$-coercivity, which would immediately imply the result. This is done by proving the stability of the weak coercivity with respect to a one-parameter perturbation of the critical configuration (see Lemma 4.7). In section 5 we exploit the regularity theory for a class of obstacle problems which arise as perturbations of (1.1) to show that the $C^{1,1}$-minimality actually implies the minimality with respect to the Hausdorff distance,
thus proving the theorem. In the last section we apply the previous analysis to the explicit case of a disk subjected to a radial stretching.

## 2. Preliminaries

In this section we fix the notation and describe precisely the required background for our analysis. We are interested in cavities identified as closed sets $F$ with $\mathcal{H}^{1}(\partial F)<+\infty$ and starshaped with respect to the origin. The fact that $F$ is starshaped allows us to describe it as a subgraph of a function. Since $F$ has finite perimeter, the function associated to its boundary turns out to have bounded pointwise total variation. This will allow us to deal with functions rather than sets.

We denote by $\mathbb{S}^{1}$ the unit circle in $\mathbb{R}^{2}$ and by $\sigma: \mathbb{R} \rightarrow \mathbb{S}^{1}$ the local diffeomorphism defined by $\sigma(\theta)=(\cos \theta, \sin \theta)$, by $\sigma^{-1}$ its local inverse and by $\sigma^{\perp}(\theta)=(\sin \theta,-\cos \theta)$ its orthogonal. We set $C_{\#}^{2}(\mathbb{R})$ to be the collection of functions in $C^{2}(\mathbb{R})$ that are $2 \pi$-periodic. In a similar way we shall define the function spaces $H_{\#}^{1}(\mathbb{R})$, etc.

With a slight abuse of notation we set (2.1)
$B V_{\#}(\mathbb{R}):=\left\{g: \mathbb{R} \rightarrow\left[0, R_{0}\right] \mid g\right.$ is upper semicontinuous, $2 \pi$-periodic and $\left.p V(g,[0,2 \pi])<\infty\right\}$,
where $p V(g,[0,2 \pi])$ is the pointwise total variation of $g$ in $[0,2 \pi]$ and $R_{0}$ is the radius of a large ball $B_{R_{0}}$. For a function $g \in B V_{\#}(\mathbb{R})$ we define the extended graph of $g$ as $\Gamma_{g} \cup \Sigma_{g}$, where

$$
\begin{equation*}
\Gamma_{g}:=\left\{\rho \sigma(\theta) \in \mathbb{R}^{2} \mid g^{-}(\theta) \leq \rho \leq g^{+}(\theta), \theta \in \mathbb{R}\right\} \tag{2.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\Sigma_{g}=\overline{\left\{\rho \sigma(\theta) \in \mathbb{R}^{2} \mid g^{+}(\theta)<\rho<g(\theta), \theta \in \mathbb{R}\right\}} \tag{2.3}
\end{equation*}
$$

Here $g^{-}(\theta):=\liminf _{\tilde{\theta} \rightarrow \theta} g(\tilde{\theta})$ and $g^{+}(\theta):=\lim _{\sup _{\tilde{\theta} \rightarrow \theta}} g(\tilde{\theta})$. We shall refer to $\Sigma_{g}$ as the set of cracks.

Let us consider a compact set $F \subset \bar{B}_{R_{0}}$ starshaped with respect to the origin. Then, for $\sigma \in \mathbb{S}^{1}$, we can write

$$
F=\left\{r \sigma(\theta) \in \mathbb{R}^{2} \mid \theta \in \mathbb{R}, 0 \leq r \leq \rho_{F}(\theta)\right\}
$$

where $\rho_{F}$ is the radial function of $F$ and is defined by

$$
\rho_{F}(\theta):=\sup \{\rho \in \mathbb{R} \mid \rho \sigma(\theta) \in F\}
$$

It is clear that $\rho_{F}: \mathbb{R} \rightarrow\left[0, R_{0}\right]$ is upper semicontinuous. Moreover we have the following result, see [13, Lemmata 2.2 and 2.3].

Lemma 2.1. Let $F \subset \bar{B}_{R_{0}}$ be a closed set starshaped with respect to the origin and let $\rho_{F}$ be the radial function of $F$. Then

$$
\partial F=\Gamma_{\rho_{F}} \cup \Sigma_{\rho_{F}} .
$$

Moreover $\mathcal{H}^{1}(\partial F)<+\infty$ if and only if $\rho_{F}$ has finite pointwise total variation.
The previous lemma rigorously shows that we may use radial functions instead of sets. Hence, for $g \in B V_{\#}(\mathbb{R})$ we set

$$
F_{g}:=\left\{\rho \sigma(\theta) \in \mathbb{R}^{2} \mid 0 \leq \rho \leq g(\theta)\right\} \quad \text { and } \quad \Omega_{g}:=B_{R_{0}} \backslash F_{g}
$$

We may think of $F_{g}$ as the void and of $\Omega_{g}$ as the elastic solid.
We can now define properly the space of admissible pairs. Given $u_{0} \in C^{\infty}\left(\mathbb{R}^{2} \backslash B_{R_{0}}\right)$ we set

$$
\begin{equation*}
X\left(u_{0}\right)=\left\{(g, v) \mid g \in B V_{\#}(\mathbb{R}), v \in H_{\mathrm{loc}}^{1}\left(\mathbb{R}^{2} \backslash F_{g} ; \mathbb{R}^{2}\right), v \equiv u_{0} \text { outside } B_{R_{0}}\right\} \tag{2.4}
\end{equation*}
$$

and we shall use the notation $X(0)$ for $u_{0} \equiv 0$. We define also the following subspaces of $X\left(u_{0}\right)$

$$
\begin{align*}
X_{\mathrm{Lip}}\left(u_{0}\right) & :=\left\{(g, v) \in X\left(u_{0}\right) \mid g \text { is Lipschitz }\right\} \\
X_{\mathrm{reg}}\left(u_{0}\right) & :=\left\{(g, v) \in X\left(u_{0}\right) \mid g \in C_{\#}^{\infty}(\mathbb{R}), v \in C^{\infty}\left(\bar{\Omega}_{g}\right)\right\} \tag{2.5}
\end{align*}
$$

We are now in position to give the proper definition of convergence in $X\left(u_{0}\right)$.
Definition 2.2. A sequence $\left(g_{n}, v_{n}\right) \subset X\left(u_{0}\right)$ is said to converge to $(g, v)$ in $X\left(u_{0}\right)$ and we write $\left(g_{n}, v_{n}\right) \xrightarrow{X}(g, v)$ if
(1) $\sup _{n \in \mathbb{N}} \mathcal{H}^{1}\left(\partial F_{g_{n}}\right)<+\infty$,
(2) ${\stackrel{n}{g_{n}}}^{n} \rightarrow F_{g}$ in Hausdorff metric,
(3) $v_{n} \rightharpoonup v$ weakly in $H^{1}\left(\omega ; \mathbb{R}^{2}\right)$ for any open set $\omega$ compactly contained in $\mathbb{R}^{2} \backslash F_{g}$.

In view of [13, Lemma 2.6], we see that $X\left(u_{0}\right)$ is closed under the convergence of Definition 2.2.
The elastic energy density is defined by $Q(E(u)):=\frac{1}{2} \mathbb{C} E(u): E(u)$, where $\mathbb{C}$ is the fourth order tensor

$$
\mathbb{C} \xi:=\left(\begin{array}{cc}
(2 \mu+\lambda) \xi_{11}+\lambda \xi_{22} & 2 \mu \xi_{12} \\
2 \mu \xi_{12} & (2 \mu+\lambda) \xi_{22}+\lambda \xi_{11}
\end{array}\right)
$$

and $E(u)$ is the symmetric gradient of $u$

$$
E(u):=\frac{1}{2}\left(D u+(D u)^{T}\right) .
$$

The constants $\mu, \lambda$ are called the Lamé coefficients and they are assumed to satisfy the following ellipticity conditions

$$
\mu>0 \text { and } \lambda>-\mu .
$$

Since $Q(\xi) \geq \min \{\mu, \mu+\lambda\}|\xi|^{2}$ for every symmetric $2 \times 2$ matrix $\xi$, the above conditions guarantee that $Q$ is coercive. We also set the ellipticity constant

$$
\eta:=\min \{\mu, \mu+\lambda\} .
$$

For a pair $(g, v) \in X_{\text {Lip }}\left(u_{0}\right)$ we may write the value of the functional (1.1) as

$$
\mathcal{F}(g, v)=\int_{\Omega_{g}} Q(E(v)) d z+\mathcal{H}^{1}\left(\Gamma_{g}\right) .
$$

Since this functional is not lower semicontinuous with respect to the convergence in $X\left(u_{0}\right)$, in order to effectively address the minimization problem we consider the relaxed functional

$$
\overline{\mathcal{F}}(g, v)=\inf \left\{\liminf _{n \rightarrow \infty} \mathcal{F}\left(g_{n}, v_{n}\right) \mid\left(g_{n}, v_{n}\right) \in X_{\text {Lip }}\left(u_{0}\right),\left(g_{n}, v_{n}\right) \xrightarrow{X}(g, v)\right\} .
$$

The following integral representation of $\overline{\mathcal{F}}$ is proved in [13, Theorem 3.1], where the more general case of anisotropic surface energy is also considered.
Theorem 2.3. Let $(g, v) \in X\left(u_{0}\right)$, then

$$
\overline{\mathcal{F}}(g, v)=\int_{\Omega_{g}} Q(E(v)) d z+\mathcal{H}^{1}\left(\Gamma_{g}\right)+2 \mathcal{H}^{1}\left(\Sigma_{g}\right) .
$$

From now on we will always deal with the relaxed functional appearing in Theorem 2.3 and with abuse of notation we will denote it simply by $\mathcal{F}(g, v)$. The minimization problem can now be properly stated as

$$
\begin{equation*}
\min \left\{\mathcal{F}(g, v)\left|(g, v) \in X\left(u_{0}\right),\left|\Omega_{g}\right|=d\right\}\right. \tag{2.6}
\end{equation*}
$$

for some given constant $d<\left|B_{R_{0}}\right|$. Existence of solutions of the problem (2.6) is then ensured by [13, Theorem 3.2].

Given $g \in B V_{\#}(\mathbb{R})$ there is one particular elastic displacement $v$ which is the minimizer of the elastic energy $\int_{\Omega_{g}} Q(E(v)) d z$ under the boundary condition $v \equiv u_{0}$ outside $B_{R_{0}}$. We call this map the elastic equilibrium associated to $g$. If $(h, u) \in X\left(u_{0}\right)$ solves (2.6) then $u$ has to be the elastic equilibrium associated with $h$.

Assume now that a solution $(h, u)$ belongs to $X_{\text {reg }}\left(u_{0}\right)$ and $h>0$, then $(h, u)$ satisfy the Euler-Lagrange equations

$$
\left\{\begin{align*}
\operatorname{div} \mathbb{C}(E(u)) & =0 & & \text { in } \Omega_{h}  \tag{2.7}\\
\mathbb{C}(E(u))[\nu] & =0 & & \text { on } \Gamma_{h} \\
Q(E(u))-k_{h} & =\text { const. } & & \text { on } \Gamma_{h},
\end{align*}\right.
$$

where $k_{h}$ is the curvature of $\Gamma_{h}$. The first two equations are standard whereas the third one is the first variation of the functional (1.1). This motivates the following definition.
Definition 2.4. A pair $(h, u) \in X_{\mathrm{reg}}\left(u_{0}\right)$ is said to be critical if it solves the equations (2.7).

We remark that if $(h, u)$ is a critical pair, then from the first two equations in (2.7) it follows that $u$ is the elastic equilibrium associated to $h$. We also point out that in the definition of a critical point we only need to assume $h$ to be smooth. Indeed, if we only assume $(h, u) \in X\left(u_{0}\right)$ and $h \in C^{\infty}(\mathbb{R})$, then it follows from the standard elliptic regularity theory (see [2]) that $u \in C^{\infty}\left(\bar{\Omega}_{h}\right)$.

The regularity for minimizers of (2.6) was studied in [12] and the following result holds. If the pair $(h, u)$ is a local minimizer of (2.6) and $0<h<R_{0}$ then there exists an open set $I \subset[0,2 \pi)$ of full measure such that $h \in C^{\infty}(I)$. In fact $h$ is even analytic in $I$. Hence our regularity assumption on a critical point in Definition 2.4 is not restrictive when $0<h<R_{0}$ and the singular set is empty.

Finally, we recall a version of the Korn's inequality which will be used throughout the paper, see e.g. [18].
Theorem 2.5 (Korn's inequality). Let $\Omega \subset \mathbb{R}^{2}$ be a bounded domain with smooth boundary and $v \in W^{1,2}\left(\Omega ; \mathbb{R}^{2}\right)$. There exists a constant $C=C(\Omega)$ such that if

$$
\int_{\Omega} D v d z=\int_{\Omega} D v^{T} d z
$$

then

$$
\int_{\Omega}|D v|^{2} d z \leq C \int_{\Omega}|E(v)|^{2} d z
$$

Moreover if $\Omega$ is an annulus $A(R, r), R>r$, the constant $C$ depends only on the ratio $r / R$ and $C \rightarrow 4$ as $r / R \rightarrow 0$.

## 3. Calculation of the second variation

The goal of this section is to calculate the second variation of the functional $\mathcal{F}$ at any point $(h, u) \in X_{\text {reg }}\left(u_{0}\right)$, where $u$ is the elastic equilibrium associated to $h$ and $0<h<R_{0}$, see formula (3.4). We then define a quadratic form for a critical pair (3.16) and give a definition of positiveness of the second variation, see Definition 3.5.

To this aim we will introduce the following notation. Given a $2 \pi$-periodic function $f: \mathbb{R} \rightarrow \mathbb{R}$ we will denote by $\underline{f}: \mathbb{R}^{2} \backslash\{0\} \rightarrow \mathbb{R}^{2}$ the map

$$
\begin{equation*}
\underline{f}(z):=f\left(\sigma^{-1}\left(\frac{z}{|z|}\right)\right) \frac{z}{|z|} \tag{3.1}
\end{equation*}
$$

For a parameter $s \in(-\varepsilon, \varepsilon)$ let $\left(h_{s}, u_{s}\right) \in X_{\text {reg }}\left(u_{0}\right)$ be a smooth perturbation of $(h, u)$, where $u_{s}$ is the elastic equilibrium associated to $h_{s}$. By smooth perturbation we mean that the function $(s, \theta) \mapsto h_{s}(\theta)$ is smooth and $\lim _{s \rightarrow 0}\left\|h_{s}-h\right\|_{C^{2}(\mathbb{R})}=0$. Moreover we set $\dot{h}_{s}=\frac{\partial}{\partial s} h_{s}, \dot{u}_{s}=\frac{\partial}{\partial s} u_{s}$ and $h_{s}^{\prime}=\frac{\partial}{\partial \theta} h_{s}$. Notations $\dot{u}, \dot{h}$ mean that we evaluate the time derivatives at $s=0$. We explicitly point out that $\dot{h}$ and $\dot{u}$ depend on $h_{s}$. Finally, for a given $h$, we define the set of functions

$$
\begin{equation*}
\mathcal{A}\left(\Omega_{h}\right):=\left\{w: \Omega_{h} \rightarrow \mathbb{R}^{2} \mid(h, w) \in X(0)\right\} \tag{3.2}
\end{equation*}
$$

Roughly this means that $w \in \mathcal{A}\left(\Omega_{h}\right)$ if $w=0$ outside $B_{R_{0}}$.
We will first write the equation for $\dot{u}$. In the following we will denote by $\tau$ the tangent unit vector to $\Gamma_{h}$ clockwise oriented and by $\nu$ the unit normal to $\Gamma_{h}$ pointing outward the set $F_{h}$.

Proposition 3.1. Let $(h, u) \in X_{\mathrm{reg}}\left(u_{0}\right)$ be such that $u$ is the elastic equilibrium associated to $h$ and $0<h<R_{0}$. Suppose $\left(h_{s}, u_{s}\right)$ is a smooth perturbation of $(h, u)$. Then the function $\dot{u} \in \mathcal{A}\left(\Omega_{h}\right)$ satisfies

$$
\begin{align*}
\int_{\Omega_{h}} \mathbb{C} E(\dot{u}): E(w) d z & =\int_{\Gamma_{h}}\langle\underline{\dot{h}}, \nu\rangle \mathbb{C} E(u): E(w) d \mathcal{H}^{1}  \tag{3.3}\\
& =-\int_{\Gamma_{h}} \operatorname{div}_{\tau}(\langle\underline{\dot{h}}, \nu\rangle \mathbb{C} E(u)) \cdot w d \mathcal{H}^{1}
\end{align*}
$$

for all $w \in \mathcal{A}\left(\Omega_{h}\right)$.

Proof. The proof is very similar to the one in [14]. Arguing as in [7, Proposition 8.1] we obtain a one parameter family of $C^{\infty}$-diffeomorphisms $\Phi_{s}(\cdot): \mathbb{R}^{2} \backslash\{0\} \rightarrow \mathbb{R}^{2} \backslash\{0\}$ such that $\Phi_{0}=$ id and $\Phi_{s}(z)=\underline{h_{s}}$ on $\partial F_{h}$.

Suppose first that $w \in \mathcal{A}\left(\Omega_{h}\right) \cap C^{\infty}\left(\bar{\Omega}_{h}\right)$. We may extend $w$ outside $\Omega_{h}$ such that $w \in \mathcal{A}\left(\Omega_{h_{s}}\right) \cap$ $C^{\infty}\left(\bar{\Omega}_{h_{s}}\right)$. Since $u_{s}$ is the elastic equilibrium in $\Omega_{h_{s}}$ we have

$$
\int_{\Omega_{h_{s}}} \mathbb{C} E\left(u_{s}\right): E(w) d z=0
$$

Differentiate this with respect to $s$ and evaluate at $s=0$ to obtain

$$
\int_{\Omega_{h}} \mathbb{C} E(\dot{u}): E(w) d z-\int_{0}^{2 \pi} \dot{h} h[\mathbb{C} E(u): E(w)](h \sigma(\theta)) d \theta=0
$$

Using the area formula and notation (3.1) we may write

$$
\int_{\Omega_{h}} \mathbb{C} E(\dot{u}): E(w) d z=\int_{\Gamma_{h}}\langle\underline{\dot{h}}, \nu\rangle \mathbb{C} E(u): E(w) d \mathcal{H}^{1}
$$

where we have used the fact that the normal can be written in polar coordinates as $\nu=\frac{h \sigma+h^{\prime} \sigma^{\perp}}{\sqrt{h^{2}+h^{\prime 2}}}$. The rest will follow by integration by parts and from the fact that $\mathbb{C} E(u)[\nu]=0$ on $\Gamma_{h}$ as in (2.7).

To obtain (3.3) for every $w \in \mathcal{A}\left(\Omega_{h}\right)$ one may use a standard approximation argument.
Remark 3.2. Notice that the equality (3.3) clearly holds also for test functions of the form $\tilde{w}(z)=w(z)+A z+b$, where $w \in \mathcal{A}\left(\Omega_{h}\right), b \in \mathbb{R}^{2}$ and $A$ is an antisymmetric matrix.

In the next theorem we derive the formula for the second variation of $\mathcal{F}$.
Theorem 3.3. Suppose that $(h, u)$ and $\left(h_{s}, u_{s}\right)$ are as in Proposition 3.1. Let $\nu$ be the outer normal of $\Gamma_{h}=\partial F_{h}, \tau$ be the tangent (positively oriented) of $\Gamma_{h}$ and $k$ be the curvature of $\Gamma_{h}$. The second variation of $\mathcal{F}$ at $(h, u)$ is

$$
\begin{align*}
\left.\frac{d^{2}}{d s^{2}} \mathcal{F}\left(h_{s}, u_{s}\right)\right|_{s=0}= & -\int_{\Omega_{h}} 2 Q(E(\dot{u})) d z+\int_{\Gamma_{h}}\left|\partial_{\tau}\langle\underline{\dot{h}}, \nu\rangle\right|^{2} d \mathcal{H}^{1} \\
& -\int_{\Gamma_{h}}\left(\partial_{\nu} Q(E(u))+k^{2}\right)\langle\underline{\dot{h}}, \nu\rangle^{2} d \mathcal{H}^{1} \\
& +\int_{\Gamma_{h}}(Q(E(u))-k) \partial_{\tau}(\langle\underline{\dot{h}}, \nu\rangle\langle\underline{\dot{h}}, \tau\rangle) d \mathcal{H}^{1}  \tag{3.4}\\
& -\int_{\Gamma_{h}}(Q(E(u))-k)\left(\frac{\langle\underline{\dot{h}}, \nu\rangle^{2}}{\langle\underline{h}, \nu\rangle}+\langle\underline{\ddot{h}}, \nu\rangle\right) d \mathcal{H}^{1}
\end{align*}
$$

Proof. We will treat the elastic and the perimeter part separately and write

$$
\mathcal{F}\left(h_{s}, u_{s}\right)=\int_{\Omega_{h_{s}}} Q\left(E\left(u_{s}\right)\right) d z+\mathcal{H}^{1}\left(\Gamma_{h_{s}}\right)=\mathcal{F}_{1}\left(h_{s}, u_{s}\right)+\mathcal{F}_{2}\left(h_{s}\right)
$$

Since $h_{s}$ is smooth, we notice that $\Sigma_{h_{s}}=\emptyset$ and denote by $\Phi_{s}$ the family of diffeomorphisms as in the proof of Proposition 3.1.
1st Variation : We start by differentiating the perimeter part $\mathcal{F}_{2}\left(h_{s}\right)$.
Since $\mathcal{H}^{1}\left(\Gamma_{h_{s}}\right)=\int_{0}^{2 \pi} \sqrt{h_{s}^{2}+h_{s}^{\prime 2}} d \theta$ we have

$$
\frac{d}{d s} \mathcal{F}_{2}\left(h_{s}\right)=\int_{0}^{2 \pi} \frac{h_{s} \dot{h}_{s}+h_{s}^{\prime} \dot{h}_{s}^{\prime}}{\sqrt{h_{s}^{2}+h_{s}^{\prime 2}}} d \theta
$$

Integrate the second term by parts and obtain

$$
\int_{0}^{2 \pi} \frac{h_{s}^{\prime} \dot{h}_{s}^{\prime}}{\sqrt{h_{s}^{2}+h_{s}^{\prime 2}}} d \theta=-\int_{0}^{2 \pi}\left(\frac{h_{s}^{\prime \prime}}{\sqrt{h_{s}^{2}+h_{s}^{\prime 2}}}-\frac{h_{s}\left(h_{s}^{\prime}\right)^{2}+\left(h_{s}^{\prime}\right)^{2} h_{s}^{\prime \prime}}{\left(h_{s}^{2}+h_{s}^{\prime 2}\right)^{3 / 2}}\right) \dot{h}_{s} d \theta
$$

Then we have

$$
\begin{align*}
\frac{d}{d s} \mathcal{F}_{2}\left(h_{s}\right) & =\int_{0}^{2 \pi} h_{s} \dot{h}_{s}\left(\frac{h_{s}^{2}+2 h_{s}^{\prime 2}-h_{s} h_{s}^{\prime \prime}}{\left(h_{s}^{2}+h_{s}^{\prime 2}\right)^{3 / 2}}\right) d \theta=\int_{0}^{2 \pi} h_{s} \dot{h}_{s} k_{s}\left(h_{s} \sigma\right) d \theta \\
& \left.=\int_{\Gamma_{h_{s}}} \underline{\left\langle\dot{h}_{s}\right.}, \nu_{h_{s}}\right\rangle k_{s} d \mathcal{H}^{1} \tag{3.5}
\end{align*}
$$

where $k_{s}=\frac{h_{s}^{2}+2 h_{s}^{\prime 2}-h_{s} h_{s}^{\prime \prime}}{\left(h_{s}^{2}+h_{s}^{\prime 2}\right)^{3 / 2}}$ is the curvature of $\Gamma_{h_{s}}$ in polar coordinates.
Let us now treat the elastic part $\mathcal{F}_{1}\left(h_{s}, u_{s}\right)$. Differentiate it with respect to $s$ and get, as in the proof of Proposition 3.1,

$$
\frac{d}{d s} \mathcal{F}_{1}\left(h_{s}, u_{s}\right)=\int_{\Omega_{h_{s}}} \mathbb{C} E\left(\dot{u}_{s}\right): E\left(u_{s}\right) d z-\int_{0}^{2 \pi} \dot{h}_{s} h_{s} Q\left(E\left(u_{s}\right)\right)\left(h_{s} \sigma\right) d \theta
$$

The first term disappears since $u_{s}$ satisfies the Euler-Lagrange equations (2.7) and $\dot{u}_{s} \in \mathcal{A}\left(\Omega_{h_{s}}\right)$. Hence, we are only left with

$$
\begin{equation*}
\frac{d}{d s} \mathcal{F}_{1}\left(h_{s}, u_{s}\right)=-\int_{0}^{2 \pi} \dot{h}_{s} h_{s} Q\left(E\left(u_{s}\right)\right)\left(h_{s} \sigma\right) d \theta=-\int_{\Gamma_{h_{s}}}\left\langle{\dot{h_{s}}}_{s}, \nu_{h_{s}}\right\rangle Q\left(E\left(u_{s}\right)\right) d \mathcal{H}^{1} \tag{3.6}
\end{equation*}
$$

Combining (3.5) and (3.6) gives the first variation of $\mathcal{F}$.
2nd Variation : We will divide the proof in two steps.
Step 1: As in [14], we begin by making a couple of general observations.
Let $d$ be the signed distance function from $\Gamma_{h}$, i.e.,

$$
d(z):= \begin{cases}-\operatorname{dist}\left(z, \Gamma_{h}\right) & \text { if } z \in F_{h} \\ \operatorname{dist}\left(z, \Gamma_{h}\right) & \text { if } z \notin F_{h}\end{cases}
$$

Since the boundary $\Gamma_{h}$ is a graph of a $C^{\infty}$-function, $d$ is $C^{1}$ function in a small tubular neighbourhood of $\Gamma_{h_{t}}$. Setting $\nu(z):=\nabla d(z)$ and $k(z):=(\operatorname{div} \nu)(z)$, we observe that on $\Gamma_{h}, \nu$ is the outer normal to $\Gamma_{h}$ and $k$ is the curvature of $\Gamma_{h}$.

First we claim that

$$
\begin{equation*}
\partial_{\nu} k=-k^{2}, \quad \text { on } \Gamma_{h} . \tag{3.7}
\end{equation*}
$$

Differentiating the identity $|\nu|=1$ with respect to $\nu$ yields $D \nu[\nu]=0$. This shows that

$$
\begin{equation*}
D \nu=D_{\tau} \nu=k \tau \otimes \tau \quad \text { and } \quad \operatorname{div} \nu=\operatorname{div}_{\tau} \nu, \quad \text { on } \Gamma_{h} . \tag{3.8}
\end{equation*}
$$

Differentiating the identity $D \nu[\nu]=0$ yields $\sum_{j=1}^{2}\left(\partial_{j k}^{2} \nu_{i} \nu_{j}+\partial_{j} \nu_{i} \partial_{k} \nu_{j}\right)=0$ for $k, i=1,2$. Hence we have

$$
\left(\partial_{\nu}(D \nu)\right)_{i k}=\sum_{j=1}^{2} \partial_{j k}^{2} \nu_{i} \nu_{j}=-\sum_{j=1}^{2} \partial_{j} \nu_{i} \partial_{k} \nu_{j}=-\left((D \nu)^{2}\right)_{i k}
$$

for $i, k=1,2$. Using the previous identity we obtain

$$
\partial_{\nu} k=\operatorname{Trace}\left(\partial_{\nu}(D \nu)\right)=-\operatorname{Trace}\left((D \nu)^{2}\right)=-k^{2} \quad \text { on } \Gamma_{h},
$$

where the last equality follows from (3.8). Hence we have (3.7).
Next we claim that

$$
\begin{equation*}
\langle\dot{\nu}, \tau\rangle=-\partial_{\tau}\langle\underline{\dot{\dot{h}}}, \nu\rangle, \quad \text { on } \Gamma_{h} . \tag{3.9}
\end{equation*}
$$

Recall that $\Phi_{s}(z): \mathbb{R}^{2} \backslash\{0\} \rightarrow \mathbb{R}^{2} \backslash\{0\}$ is a one-parameter family of $C^{\infty}$-diffeomorphisms such that $\Phi_{s}(z)=\underline{h_{s}}$ on $\Gamma_{h}$ and $\Phi_{0}=\mathrm{id}$. Notice that we have

$$
\begin{equation*}
\langle\dot{\Phi}, \nu\rangle=\langle\underline{\dot{h}}, \nu\rangle, \quad \text { on } \Gamma_{h} \tag{3.10}
\end{equation*}
$$

Differentiating $D \Phi_{s}^{-T} D \Phi_{s}^{T}[\nu]=\nu$ and calculating at $s=0$ gives $D \dot{\Phi}^{-T}[\nu]=-D \dot{\Phi}^{T}[\nu]$. Differentiate the identity

$$
\nu_{s} \circ \Phi_{s}=\frac{D \Phi_{s}^{-T}[\nu]}{\left|D \Phi_{s}^{-T}[\nu]\right|}
$$

with respect to $s$, evaluate at $s=0$ and use the previous identity to obtain

$$
\begin{equation*}
\dot{\nu}+D \nu[\dot{\Phi}]=-D \dot{\Phi}^{T}[\nu]+\left\langle D \dot{\Phi}^{T}[\nu], \nu\right\rangle \nu, \quad \text { on } \Gamma_{h} . \tag{3.11}
\end{equation*}
$$

By (3.8) we have $D \nu=D_{\tau} \nu^{T}$ on $\Gamma_{h}$. Therefore, multiplying (3.11) by $\tau$ we obtain

$$
\begin{aligned}
\langle\dot{\nu}, \tau\rangle & =-\left\langle D \dot{\Phi}^{T}[\nu], \tau\right\rangle-\langle D \nu[\dot{\Phi}], \tau\rangle \\
& =-\left\langle D \dot{\Phi}^{T}[\nu], \tau\right\rangle-\left\langle D \nu^{T}[\dot{\Phi}], \tau\right\rangle \\
& =\langle(-D\langle\dot{\Phi}, \nu\rangle), \tau\rangle=-\partial_{\tau}\langle\underline{\dot{h}}, \nu\rangle \quad \text { on } \Gamma_{h}
\end{aligned}
$$

and (3.9) is proven.
Step 2: Let us start with the perimeter part and differentiate (3.5)

$$
\begin{aligned}
\left.\frac{d^{2}}{d s^{2}} \mathcal{F}_{2}\left(h_{s}\right)\right|_{s=0}= & \overbrace{\int_{0}^{2 \pi} h \dot{h} \dot{k}(h \sigma) d \theta}^{A}+\overbrace{\int_{0}^{2 \pi} h \dot{h}^{2} \partial_{\sigma} k(h \sigma) d \theta}^{B} \\
& +\int_{0}^{2 \pi} \dot{h}^{2} k(h \sigma) d \theta+\int_{0}^{2 \pi} h \ddot{h} k(h \sigma) d \theta
\end{aligned}
$$

For the term $A$ we have that

$$
\begin{aligned}
A & =\int_{0}^{2 \pi} h \dot{h} \dot{k}(h \sigma) d \theta=\int_{\Gamma_{h}}\langle\underline{\dot{h}}, \nu\rangle \dot{k} d \mathcal{H}^{1}=\int_{\Gamma_{h}}\langle\underline{\dot{h}}, \nu\rangle \operatorname{div}_{\tau} \dot{\nu} d \mathcal{H}^{1} \\
& =-\int_{\Gamma_{h}}\langle\dot{\nu}, \tau\rangle \partial_{\tau}\langle\underline{\dot{h}}, \nu\rangle d \mathcal{H}^{1}=\int_{\Gamma_{h}}\left|\partial_{\tau}\langle\underline{\dot{h}}, \nu\rangle\right|^{2} d \mathcal{H}^{1}
\end{aligned}
$$

where we have used (3.9). For the term $B$, noticing that

$$
\partial_{\sigma} k=\frac{h}{\sqrt{h^{2}+h^{\prime 2}}} \partial_{\nu} k-\frac{h^{\prime}}{\sqrt{h^{2}+h^{\prime 2}}} \partial_{\tau} k \quad \text { and } \quad \tau=\frac{h \sigma^{\perp}-h^{\prime} \sigma}{\sqrt{h^{2}+h^{\prime 2}}}
$$

we may write

$$
\begin{aligned}
B & =\int_{0}^{2 \pi} h \dot{h}^{2} \partial_{\sigma} k(h \sigma) d \theta=\int_{\Gamma_{h}}\langle\underline{\dot{h}}, \nu\rangle^{2} \partial_{\nu} k d \mathcal{H}^{1}+\int_{\Gamma_{h}}\langle\underline{\dot{h}}, \nu\rangle\langle\underline{\dot{h}}, \tau\rangle \partial_{\tau} k d \mathcal{H}^{1} \\
& =-\int_{\Gamma_{h}}\langle\underline{\dot{h}}, \nu\rangle^{2} k^{2} d \mathcal{H}^{1}-\int_{\Gamma_{h}} k \partial_{\tau}(\langle\underline{\dot{h}}, \nu\rangle\langle\underline{\dot{h}}, \tau\rangle) d \mathcal{H}^{1}
\end{aligned}
$$

where we have used (3.7) and integration by parts. Hence, we have

$$
\begin{align*}
\left.\frac{d^{2}}{d s^{2}} \mathcal{F}_{2}\left(h_{s}\right)\right|_{s=0}= & \int_{\Gamma_{h}}\left|\partial_{\tau}\langle\underline{\dot{h}}, \nu\rangle\right|^{2} d \mathcal{H}^{1}-\int_{\Gamma_{h}}\langle\underline{\dot{h}}, \nu\rangle^{2} k^{2} d \mathcal{H}^{1}  \tag{3.12}\\
& -\int_{\Gamma_{h}} k \partial_{\tau}(\langle\underline{\dot{h}}, \nu\rangle\langle\underline{\dot{h}}, \tau\rangle) d \mathcal{H}^{1}+\int_{\Gamma_{h}} k \frac{\langle\underline{\dot{h}}, \nu\rangle^{2}}{\langle\underline{h}, \nu\rangle} d \mathcal{H}^{1}+\int_{\Gamma_{h}} k\langle\underline{\ddot{h}}, \nu\rangle d \mathcal{H}^{1} .
\end{align*}
$$

We are left with the elastic part. Differentiate (3.6) to obtain

$$
\begin{aligned}
\left.\frac{d^{2}}{d s^{2}} \mathcal{F}_{1}\left(h_{s}, u_{s}\right)\right|_{s=0}= & -\int_{0}^{2 \pi} \mathbb{C} E(\dot{u}): E(u) h \dot{h} d \theta-\int_{0}^{2 \pi} \partial_{\sigma} Q(E(u)) h \dot{h}^{2} d \theta \\
& -\int_{0}^{2 \pi} Q(E(u))\left(\dot{h}^{2}+h \ddot{h}\right) d \theta
\end{aligned}
$$

Since $\dot{u} \in \mathcal{A}\left(\Omega_{h}\right)$, we may rewrite the first term using (3.3) as follows

$$
\begin{aligned}
\int_{0}^{2 \pi} \mathbb{C} E(\dot{u}): E(u) h \dot{h} d \theta & =\int_{\Gamma_{h}}\langle\underline{\dot{h}}, \nu\rangle \mathbb{C} E(u): E(\dot{u}) d \mathcal{H}^{1} \\
& =\int_{\Omega_{h}} 2 Q(E(\dot{u})) d z
\end{aligned}
$$

For the second term, noticing that

$$
\partial_{\sigma} Q(E(u))=\frac{h}{\sqrt{h^{2}+h^{\prime 2}}} \partial_{\nu} Q(E(u))-\frac{h^{\prime}}{\sqrt{h^{2}+h^{\prime 2}}} \partial_{\tau} Q(E(u))
$$

and using integration by parts, we get

$$
\begin{aligned}
\int_{0}^{2 \pi} \partial_{\sigma} Q(E(u)) h \dot{h}^{2} d \theta & =\int_{\Gamma_{h}} \partial_{\nu} Q(E(u))\langle\dot{\underline{h}}, \nu\rangle^{2} d \mathcal{H}^{1}+\int_{\Gamma_{h}} \partial_{\tau} Q(E(u))(\langle\underline{\dot{h}}, \nu\rangle\langle\dot{\underline{h}}, \tau\rangle) d \mathcal{H}^{1} \\
& =\int_{\Gamma_{h}} \partial_{\nu} Q(E(u))\langle\dot{\underline{h}}, \nu\rangle^{2} d \mathcal{H}^{1}-\int_{\Gamma_{h}} Q(E(u)) \partial_{\tau}(\langle\underline{\dot{h}}, \nu\rangle\langle\dot{\underline{h}}, \tau\rangle) d \mathcal{H}^{1} .
\end{aligned}
$$

Finally we have that

$$
\begin{align*}
\left.\frac{d^{2}}{d s^{2}} \mathcal{F}_{1}\left(h_{s}, u_{s}\right)\right|_{s=0}= & -\int_{\Omega_{h}} 2 Q(E(\dot{u})) d z-\int_{\Gamma_{h}} \partial_{\nu} Q(E(u))\langle\underline{\dot{h}}, \nu\rangle^{2} d \mathcal{H}^{1} \\
& +\int_{\Gamma_{h}} Q(E(u)) \partial_{\tau}(\langle\underline{\dot{h}}, \nu\rangle\langle\underline{\dot{h}}, \tau\rangle) d \mathcal{H}^{1}-\int_{\Gamma_{h}} Q(E(u)) \frac{\langle\underline{\dot{h}}, \nu\rangle^{2}}{\langle\underline{h}, \nu\rangle} d \mathcal{H}^{1}  \tag{3.13}\\
& -\int_{\Gamma_{h}} Q(E(u))\langle\underline{\underline{h}}, \nu\rangle d \mathcal{H}^{1} .
\end{align*}
$$

Combining (3.13) with (3.12) yields the formula (3.4).

In the formula (3.4) we considered any smooth perturbation $h_{s}$ of $h$. However, in order to be admissible for our minimization problem, a perturbation $h_{s}$ has to satisfy the volume constraint $\left|F_{h_{s}}\right|=\left|F_{h}\right|$, or equivalently

$$
\begin{equation*}
\int_{0}^{2 \pi} h_{s}^{2} d \theta=\int_{0}^{2 \pi} h^{2} d \theta \quad \text { for all } s>0 \tag{3.14}
\end{equation*}
$$

Remark 3.4. If $(h, u) \in X_{\text {reg }}\left(u_{0}\right)$ is a critical pair and the perturbation $\left(h_{s}\right)$ satisfies the volume constraint (3.14), then the last two terms in (3.4) vanish. Indeed one term vanishes because the term $Q(E(u))-k$ is constant on $\Gamma_{h}$ by (2.7). The second one vanishes since differentiating two times the volume constraint (3.14) with respect to $s$ we obtain

$$
\int_{\Gamma_{h}} \frac{\langle\underline{\dot{h}}, \nu\rangle^{2}}{\langle\underline{h}, \nu\rangle}+\langle\underline{\ddot{h}}, \nu\rangle d \mathcal{H}^{1}=0 .
$$

Motivated by the previous observation, for any $\psi \in H_{\#}^{1}(\mathbb{R})$ satisfying

$$
\begin{equation*}
\int_{0}^{2 \pi} h \psi d \theta=0 \tag{3.15}
\end{equation*}
$$

we define the quadratic form associated to a regular critical pair $(h, u)$

$$
\begin{align*}
\partial^{2} \mathcal{F}(h, u)[\psi]:= & -\int_{\Omega_{h}} 2 Q\left(E\left(u_{\psi}\right)\right) d z+\int_{\Gamma_{h}}\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2} d \mathcal{H}^{1}  \tag{3.16}\\
& -\int_{\Gamma_{h}}\left(\partial_{\nu} Q\left(E\left(u_{\psi}\right)\right)+k^{2}\right)\langle\underline{\psi}, \nu\rangle^{2} d \mathcal{H}^{1}
\end{align*}
$$

where $u_{\psi} \in \mathcal{A}\left(\Omega_{h}\right)$ is the unique solution to

$$
\begin{equation*}
\int_{\Omega_{h}} \mathbb{C} E\left(u_{\psi}\right): E(w) d z=-\int_{\Gamma_{h}} \operatorname{div}_{\tau}(\langle\underline{\psi}, \nu\rangle \mathbb{C} E(u)) \cdot w d \mathcal{H}^{1}, \quad \forall w \in \mathcal{A}\left(\Omega_{h}\right) \tag{3.17}
\end{equation*}
$$

We define now what we mean by the second variation of $\mathcal{F}$ being positive at a critical pair.
Definition 3.5. Suppose that $(h, u) \in X_{\text {reg }}\left(u_{0}\right)$ is a critical pair. The functional (1.1) has positive second variation at $(h, u)$ if

$$
\partial^{2} \mathcal{F}(h, u)[\psi]>0
$$

for all $\psi \in H_{\#}^{1}(\mathbb{R})$ such that $\psi \neq 0$ and satisfies (3.15).

We point out that if the second variation is positive at a critical point $(h, u)$, then the formula (3.4) and Remark 3.4 imply that for every smooth perturbation $h_{s}$ of $h$ satisfying the volume constraint $\left.\frac{d^{2}}{d s^{2}} \mathcal{F}\left(h_{s}, u_{s}\right)\right|_{s=0}>0$.

At the end of the section we prove the following compactness result.
Lemma 3.6. Suppose that a critical pair $(h, u) \in X_{\text {reg }}\left(u_{0}\right)$ is a point of positive second variation and $0<h<R_{0}$. Then there exists $c_{0}>0$ such that

$$
\partial^{2} \mathcal{F}(h, u)[\psi] \geq c_{0}\|\langle\underline{\psi}, \nu\rangle\|_{H^{1}\left(\Gamma_{h}\right)}^{2}
$$

for every $\psi \in H_{\#}^{1}(\mathbb{R})$ satisfying (3.15).
Proof. First we notice that the condition (3.15) can be written using the notation (3.1) as

$$
\begin{equation*}
\int_{\Gamma_{h}}\langle\underline{\psi}, \nu\rangle d \mathcal{H}^{1}=0 . \tag{3.18}
\end{equation*}
$$

Using the Sobolev-Poincaré inequality $\|\langle\underline{\psi}, \nu\rangle\|_{L^{2}\left(\Gamma_{h}\right)} \leq C\left\|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right\|_{L^{2}\left(\Gamma_{h}\right)}$ and (3.18) we easily see that it suffices to show that

$$
c_{0}:=\inf \left\{\partial^{2} \mathcal{F}(h, u)[\psi] \mid \psi \in H_{\#}^{1}(\mathbb{R}) \text { satisfying (3.15), } \int_{\Gamma_{h}}\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2} d \mathcal{H}^{1}=1\right\}>0
$$

Choose a sequence $\left(\psi_{n}\right)$ such that $\psi_{n}$ are smooth, satisfy (3.15), $\int_{\Gamma_{h}}\left|\partial_{\tau}\left\langle\underline{\psi_{n}}, \nu\right\rangle\right|^{2} d \mathcal{H}^{1}=1$ and

$$
\partial^{2} \mathcal{F}(h, u)\left[\psi_{n}\right] \rightarrow c_{0}
$$

By restricting to a subsequence, we may assume that $\left\langle\underline{\psi_{n}}, \nu\right\rangle \rightharpoonup f$ weakly in $H^{1}\left(\Gamma_{h}\right)$. By defining

$$
\psi(\theta):=\frac{f(h(\theta) \sigma(\theta))}{\langle\sigma, \nu\rangle}=\frac{f(h(\theta) \sigma(\theta))}{h(\theta)} \sqrt{h^{2}(\theta)+h^{2}(\theta)}
$$

we see that $f=\langle\underline{\psi}, \nu\rangle$, for some $\psi \in H_{\#}^{1}(\mathbb{R})$. Moreover since $\int_{\Gamma_{h}} f d \mathcal{H}^{1}=0$, the function $\psi$ satisfies (3.15).

Next we prove that $\mathcal{F}(h, u)$ has the following lower semicontinuity property

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \partial^{2} \mathcal{F}(h, u)\left[\psi_{n}\right] \geq \partial^{2} \mathcal{F}(h, u)[\psi] \tag{3.19}
\end{equation*}
$$

Indeed, since $\left\langle\underline{\psi_{n}}, \nu\right\rangle \rightharpoonup\langle\underline{\psi}, \nu\rangle$ weakly in $H^{1}\left(\Gamma_{h}\right)$ then $\left\langle\underline{\psi_{n}}, \nu\right\rangle \rightarrow\langle\underline{\psi}, \nu\rangle$ strongly in $L^{2}\left(\Gamma_{h}\right)$. Therefore we only need to check the convergence of the first term in (3.16).

First of all, the smoothness of $\psi_{n}$ implies that $u_{\psi_{n}}$ is smooth. Consider the domain $\tilde{\Omega}_{h}=B_{2 R_{0}} \backslash$ $F_{h}$ and the map $w_{n}(z)=u_{\psi_{n}}(z)+A_{n} z+b_{n}$, where $A_{n}$ is an antisymmetric matrix and $b_{n} \in \mathbb{R}^{2}$ is chosen such that $\int_{\tilde{\Omega}} w_{n} d z=0$. Notice that $w_{n} \in H^{1}\left(\tilde{\Omega}_{h}\right)$ and by Sobolev-Poincaré inequality it holds $\left\|w_{n}\right\|_{L^{2}\left(\tilde{\Omega}_{h}\right)} \leq C\left\|D w_{n}\right\|_{L^{2}\left(\tilde{\Omega}_{h}\right)}$. By choosing $A_{n}$ such that $\int_{\tilde{\Omega}_{h}} D w_{n} d z=\int_{\tilde{\Omega}_{h}} D w_{n}^{T} d z$ we have by Korn's inequality (Theorem 2.5) that $\left\|D w_{n}\right\|_{L^{2}\left(\tilde{\Omega}_{h}\right)} \leq C\left\|E\left(w_{n}\right)\right\|_{L^{2}\left(\tilde{\Omega}_{h}\right)}$. Moreover, since $u_{\psi_{n}} \equiv 0$ outside $B_{R_{0}}$, we have $\left\|E\left(w_{n}\right)\right\|_{L^{2}\left(\tilde{\Omega}_{h}\right)}=\left\|E\left(u_{\psi_{n}}\right)\right\|_{L^{2}\left(\Omega_{h}\right)}$. By Remark 3.2 we may use $w_{n}$ as a test function in (3.17) and using Hölder's inequality and the trace theorem we get

$$
\begin{align*}
\int_{\Omega_{h}} 2 Q\left(E\left(u_{\psi_{n}}\right)\right) d z & =-\int_{\Gamma_{h}} \operatorname{div}_{\tau}\left(\left\langle\underline{\psi_{n}}, \nu\right\rangle \mathbb{C} E(u)\right) \cdot w_{n} d \mathcal{H}^{1} \\
& \leq\left\|\left\langle\underline{\psi_{n}}, \nu\right\rangle \mathbb{C} E(u)\right\|_{H^{1}\left(\Gamma_{h}\right)}\left\|w_{n}\right\|_{L^{2}\left(\Gamma_{h}\right)}  \tag{3.20}\\
& \leq C\left\|\left\langle\underline{\psi_{n}}, \nu\right\rangle \mathbb{C} E(u)\right\|_{H^{1}\left(\Omega_{h}\right)}\left\|D w_{n}\right\|_{L^{2}\left(\tilde{\Omega}_{h}\right)} \\
& \leq C\left\|\left\langle\underline{\psi_{n}}, \nu\right\rangle \mathbb{C} E(u)\right\|_{H^{1}\left(\Omega_{h}\right)}\left\|E\left(u_{\psi_{n}}\right)\right\|_{L^{2}\left(\Omega_{h}\right)}
\end{align*}
$$

Therefore

$$
\left\|D w_{n}\right\|_{L^{2}\left(\tilde{\Omega}_{h}\right)} \leq C\left\|E\left(u_{\psi_{n}}\right)\right\|_{L^{2}\left(\Omega_{h}\right)} \leq C
$$

However, since $u_{\psi_{n}} \equiv 0$ outside $B_{R_{0}}$ we get

$$
\left|B_{2 R_{0}} \backslash B_{R_{0}}\right|\left|A_{n}\right|^{2}=\int_{B_{2 R_{0}} \backslash B_{R_{0}}}\left|D w_{n}\right|^{2} d z \leq C
$$

This implies that the matrices $A_{n}$ are bounded and therefore $\left\|D u_{\psi_{n}}\right\|_{L^{2}\left(\Omega_{h}\right)} \leq C$.

By the compactness of the trace operator we now have that $u_{\psi_{n}} \rightarrow u_{\psi}$ in $L^{2}\left(\Gamma_{h}\right)$ up to a subsequence. Use $u_{\psi_{n}}$ as a test function in (3.17) to obtain

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \int_{\Omega_{h}} 2 Q\left(E\left(u_{\psi_{n}}\right)\right) d z & =-\lim _{n \rightarrow \infty} \int_{\Gamma_{h}} \operatorname{div}_{\tau}\left(\left\langle\underline{\psi_{n}}, \nu\right\rangle \mathbb{C} E(u)\right) \cdot u_{\psi_{n}} d \mathcal{H}^{1} \\
& =-\int_{\Gamma_{h}} \operatorname{div}_{\tau}(\langle\underline{\psi}, \nu\rangle \mathbb{C} E(u)) \cdot u_{\psi} d \mathcal{H}^{1} \\
& =\int_{\Omega_{h}} 2 Q\left(E\left(u_{\psi}\right)\right) d z
\end{aligned}
$$

This proves (3.19).
The claim now follows since if $\psi \neq 0$, the lower semicontinuity (3.19) implies

$$
c_{0}=\lim _{n \rightarrow \infty} \partial^{2} \mathcal{F}(h, u)\left[\psi_{n}\right] \geq \partial^{2} \mathcal{F}(h, u)[\psi]>0
$$

On the other hand if $\psi \equiv 0$ then the constraint $\int_{\Gamma_{h}}\left|\partial_{\tau}\left\langle\underline{\psi_{n}}, \nu\right\rangle\right|^{2} d \mathcal{H}^{1}=1$ yields

$$
c_{0}=\lim _{n \rightarrow \infty} \partial^{2} \mathcal{F}(h, u)\left[\psi_{n}\right]=1
$$

## 4. $C^{1,1}$-LOCAL MINIMALITY

In this section we perform a second order analysis of the functional (1.1) with respect to $C^{1,1}$ topology in the spirit of [11]. The main result is Proposition 4.1 where it is shown that a critical point $(h, u) \in X_{\text {reg }}\left(u_{0}\right)$ with positive second variation is a strict local minimizer in the $C^{1,1}$ topology, and that the functional satisfies a growth estimate. We point out that, according to Lemma 3.6, the second variation at $(h, u)$ is coercive with respect to a norm which is weaker than the $C^{1,1}$-norm. Therefore the local minimality does not follow directly from Lemma 3.6. The idea is to prove a coercivity bound in a whole $C^{1,1}$-neighborhood of the critical point, which is carried out in Lemma 4.7. The main difficulty is to control the bulk energy, which will be done by using regularity theory for linear elliptic systems. We prove the main result first without worrying about the technicalities. All the technical lemmata are proven later in the section.

Proposition 4.1. Suppose that the critical pair $(h, u) \in X_{\text {reg }}\left(u_{0}\right)$ is a point of positive second variation such that $0<h<R_{0}$. There exists $\delta>0$ such that for any admissible pair $(g, v) \in X\left(u_{0}\right)$ with $g \in C_{\#}^{1,1}(\mathbb{R}),\|g\|_{L^{2}([0,2 \pi))}=\|h\|_{L^{2}([0,2 \pi))}$ and $\|h-g\|_{C^{1,1}(\mathbb{R})} \leq \delta$ we have

$$
\mathcal{F}(g, v) \geq \mathcal{F}(h, u)+c_{1}\|h-g\|_{L^{2}([0,2 \pi))}^{2} .
$$

Proof. Assume first that $g \in C_{\#}^{\infty}(\mathbb{R})$ and $\|h-g\|_{C^{2}(\mathbb{R})} \leq \delta$. By scaling we may assume that $\|h\|_{L^{2}([0,2 \pi))}=\left(\int_{0}^{2 \pi} h^{2} d \theta\right)^{\frac{1}{2}}=1$. We define

$$
g_{t}:=\frac{h+t(g-h)}{\|h+t(g-h)\|_{L^{2}}}
$$

so that $g_{t}$ satisfies the volume constraint, and set

$$
f(t):=\mathcal{F}\left(g_{t}, v_{t}\right),
$$

where $v_{t}$ are the elastic equilibria associated to $g_{t}$. We calculate $\frac{d^{2}}{d t^{2}} \mathcal{F}\left(g_{t}, v_{t}\right)$ for every $t \in[0,1)$ by applying the formula (3.4) to $\left(g_{t}\right)_{s}=g_{t+s}$ of $g_{t}$ and get

$$
\begin{align*}
f^{\prime \prime}(t)=\frac{d^{2}}{d t^{2}} \mathcal{F}\left(g_{t}, v_{t}\right)= & -\int_{\Omega_{g_{t}}} 2 Q\left(E\left(\dot{v}_{t}\right)\right) d z+\int_{\Gamma_{g_{t}}}\left|\partial_{\tau_{t}}\left\langle\underline{\dot{g}_{t}}, \nu_{t}\right\rangle\right|^{2} d \mathcal{H}^{1} \\
& -\int_{\Gamma_{g_{t}}}\left(\partial_{\nu_{t}} Q\left(E\left(v_{t}\right)\right)+k_{t}^{2}\right)\left\langle\underline{\dot{g}_{t}}, \nu_{t}\right\rangle^{2} d \mathcal{H}^{1} \\
& \left.+\int_{\Gamma_{g_{t}}}\left(Q\left(E\left(v_{t}\right)\right)-k_{t}\right) \partial_{\tau_{t}}\left(\underline{\left\langle\dot{g}_{t}\right.}, \nu_{t}\right\rangle\left\langle\underline{\dot{g}_{t}}, \tau_{t}\right\rangle\right) d \mathcal{H}^{1}  \tag{4.1}\\
& -\int_{\Gamma_{g_{t}}}\left(Q\left(E\left(v_{t}\right)\right)-k_{t}\right)\left(\frac{\left\langle\underline{\dot{g}_{t}}, \nu_{t}\right\rangle^{2}}{\left\langle\underline{g_{t}}, \nu_{t}\right\rangle}+\left\langle\underline{\ddot{g}_{t}}, \nu_{t}\right\rangle\right) d \mathcal{H}^{1} .
\end{align*}
$$

Here $\nu_{t}$ is the outer normal, $\tau_{t}$ the tangent, $k_{t}$ the curvature of $\Gamma_{g_{t}}$ and $\dot{v}_{t}$ is the unique solution to

$$
\int_{\Omega_{g_{t}}} \mathbb{C} E\left(\dot{v}_{t}\right): E(w) d z=-\int_{\Gamma_{g_{t}}} \operatorname{div}_{\tau_{t}}\left(\left\langle\underline{\dot{g}_{t}}, \nu_{t}\right\rangle \mathbb{C} E\left(v_{t}\right)\right) \cdot w d \mathcal{H}^{1}, \quad \forall w \in \mathcal{A}\left(\Omega_{g_{t}}\right)
$$

Remark 3.4 and Lemma 3.6 yield

$$
f^{\prime \prime}(0)=\left.\frac{d^{2}}{d t^{2}} \mathcal{F}\left(g_{t}, v_{t}\right)\right|_{t=0}=\partial^{2} \mathcal{F}(h, u)[\dot{g}] \geq c_{0}\|\langle\underline{\dot{g}}, \nu\rangle\|_{H^{1}\left(\Gamma_{h}\right)}^{2}
$$

It will be shown later in Lemma 4.7 that, when $\delta>0$ is chosen to be small enough, the previous inequality implies

$$
\begin{equation*}
f^{\prime \prime}(t)=\frac{d^{2}}{d t^{2}} \mathcal{F}\left(g_{t}, v_{t}\right) \geq \frac{c_{0}}{2}\left\|\left\langle\underline{\dot{g}_{t}}, \nu_{t}\right\rangle\right\|_{H^{1}\left(\Gamma_{g_{t}}\right)}^{2} \quad \text { for all } t \in[0,1) \tag{4.2}
\end{equation*}
$$

It is now clear that $\left\|\left\langle\underline{\dot{g}_{t}}, \nu_{t}\right\rangle\right\|_{H^{1}\left(\Gamma_{g_{t}}\right)}^{2} \geq c\left\|\dot{g}_{t}\right\|_{L^{2}([0,2 \pi))}^{2}$ holds for all $t \in[0,1]$. Since $\int_{0}^{2 \pi} g^{2} d \theta=$ $\int_{0}^{2 \pi} h^{2} d \theta$ we have $\int_{0}^{2 \pi}(h-g)^{2} d \theta=2 \int_{0}^{2 \pi} h(h-g) d \theta$ and therefore

$$
\begin{equation*}
\left\|\dot{g}_{t}\right\|_{L^{2}([0,2 \pi))}^{2}=\frac{1}{\|h+t(g-h)\|_{L^{2}}^{4}}\left(\int_{0}^{2 \pi}(h-g)^{2} d \theta-\frac{1}{4}\left(\int_{0}^{2 \pi}(h-g)^{2} d \theta\right)^{2}\right) \tag{4.3}
\end{equation*}
$$

Since $\int_{0}^{2 \pi}(h-g)^{2} d \theta$ is very small we obtain from (4.3) that

$$
\begin{equation*}
\left\|\dot{g}_{t}\right\|_{L^{2}([0,2 \pi))}^{2} \geq \frac{1}{2}\|h-g\|_{L^{2}([0,2 \pi))}^{2} \tag{4.4}
\end{equation*}
$$

From (4.2) and (4.4) we conclude that $f^{\prime \prime}(t) \geq \tilde{c}\|h-g\|_{L^{2}}^{2}$. Since $(h, u)$ is a critical pair we have $f^{\prime}(0)=0$ and therefore

$$
\begin{aligned}
\mathcal{F}(g, v)-\mathcal{F}(h, u) & =f(1)-f(0)=\int_{0}^{1}(1-t) f^{\prime \prime}(t) d t \\
& \geq \tilde{c}\|h-g\|_{L^{2}([0,2 \pi))}^{2} \int_{0}^{1}(1-t) d t \\
& =\frac{\tilde{c}}{2}\|h-g\|_{L^{2}([0,2 \pi))}^{2}
\end{aligned}
$$

which proves the claim when $g$ is smooth.
When $g \in C_{\#}^{1,1,}(\mathbb{R})$ the claim follows by using a standard approximation.
It remains to prove (4.2). The proof is based on a compactness argument and for that we have to study the continuity of the second variation formula (3.4). To control the boundary terms in (3.4) we need fractional Sobolev spaces whose definition and basic properties are recalled here. The function $h$ is as in Proposition 4.1 and $\Gamma_{h}$ is its graph.

Definition 4.2. For $0<s<1$ and $1<p<\infty$ we define the fractional Sobolev space $W^{s, p}\left(\Gamma_{h}\right)$ as the set of those functions $v \in L^{p}\left(\Gamma_{h}\right)$ for which the Gagliardo seminorm is finite, i.e.

$$
\begin{equation*}
[v]_{s, p ; \Gamma_{h}}=\left(\int_{\Gamma_{h}} \int_{\Gamma_{h}} \frac{|v(z)-v(w)|^{p}}{|z-w|^{1+s p}} d \mathcal{H}^{1}(w) d \mathcal{H}^{1}(z)\right)^{1 / p}<\infty \tag{4.5}
\end{equation*}
$$

The fractional Sobolev norm is defined as $\|v\|_{W^{s, p}\left(\Gamma_{h}\right)}:=\|v\|_{L^{p}\left(\Gamma_{h}\right)}+[v]_{s, p ; \Gamma_{h}}$.
The space $W^{-s, p}\left(\Gamma_{h}\right)$ is the dual space of $W^{s, p}\left(\Gamma_{h}\right)$ and the dual norm of a function $v$ is defined as

$$
\|v\|_{W^{-s, p}\left(\Gamma_{h}\right)}:=\sup \left\{\int_{\Gamma_{h}} v u d \mathcal{H}^{1}(z) \mid\|u\|_{W^{s, p}\left(\partial F_{h}\right)} \leq 1\right\}
$$

We also use the notation $H^{s}\left(\Gamma_{h}\right)$ for $W^{s, 2}\left(\Gamma_{h}\right)$ for $-1<s<1$ and the convention $W^{0, p}\left(\Gamma_{h}\right):=$ $L^{p}\left(\Gamma_{h}\right)$. By Jensen's inequality we have the following classical embedding theorem.
Theorem 4.3. Let $-1 \leq t \leq s \leq 1, q \geq p$ such that $s-1 / p \geq t-1 / q$. Then there is a constant $C$ depending on $t, s, p, q$ and on the $C^{1}$-norm of $h$ such that

$$
\|v\|_{W^{t, q}\left(\Gamma_{h}\right)} \leq C\|v\|_{W^{s, p}\left(\Gamma_{h}\right)}
$$

We also have the following trace theorem.
Theorem 4.4. If $p>1$ there exists a continuous linear operator $T: W^{1, p}\left(\Omega_{h}\right) \rightarrow W^{1-1 / p, p}\left(\Gamma_{h}\right)$ such that $T v=\left.v\right|_{\Gamma_{h}}$ whenever $v$ is continuous on $\bar{\Omega}_{h}$. The norm of $T$ depends on the $C^{1}$-norm of $h$ and $\gamma$.

The next lemma will be used frequently.
Lemma 4.5. Let $-1<s<1$ and suppose that $v$ is a smooth function on $\Gamma_{h}$. Then the following hold.
(i) If $a \in C^{1}\left(\Gamma_{h}\right)$ then

$$
\|a v\|_{W^{s, p}\left(\Gamma_{h}\right)} \leq C\|a\|_{C^{1}\left(\Gamma_{h}\right)}\|v\|_{W^{s, p}\left(\Gamma_{h}\right)}
$$

where the constant $C$ depends on $p, s$ and the $C^{1}$-norm of $h$.
(ii) If $\Psi: \Gamma_{h} \rightarrow \Psi\left(\Gamma_{h}\right)$ is a $C^{1}$-diffeomorphism, then

$$
\left\|v \circ \Psi^{-1}\right\|_{W^{s, p}\left(\Psi\left(\Gamma_{h}\right)\right)} \leq C\|v\|_{W^{s, p}\left(\Gamma_{h}\right)}
$$

where the constant $C$ depends on $p, s$ and the $C^{1}$-norms of $h, \Psi$ and $\Psi^{-1}$.
We will also need to control the regularity of the elastic equilibrium. To this aim, the following elliptic estimate turns out to be useful, see [14, Lemma 4.1].
Lemma 4.6. Suppose $(g, v) \in X(0)$ is such that $\gamma \leq g \leq R_{0}-\gamma, g \in C_{\#}^{2}(\mathbb{R})$ and $v \in \mathcal{A}\left(\Omega_{g}\right)$ satisfies

$$
\begin{equation*}
\int_{\Omega_{g}} \mathbb{C} E(v): E(w) d z=\int_{\Omega_{g}} f: E(w) d z \quad \text { for every } w \in \mathcal{A}\left(\Omega_{g}\right) \tag{4.6}
\end{equation*}
$$

where $f \in C^{1}\left(\bar{\Omega}_{g} ; \mathbb{M}^{2 \times 2}\right)$. Then for any $p>2$ we have the following estimate

$$
\begin{align*}
\|E(v)\|_{W^{1, p}\left(\Omega_{g} ; \mathbb{M}^{2 \times 2}\right)} & +\|\nabla \mathbb{C} E(v)\|_{H^{-\frac{1}{2}}\left(\Gamma_{g} ; \mathbb{T}\right)} \\
& \leq C\left(\|E(v)\|_{L^{2}\left(\Omega_{g} ; \mathbb{M}^{2 \times 2}\right)}+\|f\|_{C^{1}\left(\bar{\Omega}_{g} ; \mathbb{M}^{2 \times 2}\right)}\right) \tag{4.7}
\end{align*}
$$

where $\mathbb{T}$ denotes the space of third order tensors and the constant $C$ depends on $\gamma, p$ and the $C^{2}$-norm of $g$.

We are now in position to give the proof of the inequality (4.2). To control the bulk energy we use techniques developed in [14]. The main difference is that we use directly elliptic regularity rather than dealing with eigenvalues of compact operators.

Lemma 4.7. Suppose that a critical pair $(h, u) \in X\left(u_{0}\right)$ is a point of positive second variation with $0<h<R_{0}$ and $\|h\|_{L^{2}}=1$. Then there exists $\delta>0$ such that for any admissible pair $(g, v) \in X_{\mathrm{reg}}\left(u_{0}\right)$ with $\|g\|_{L^{2}}=1$ and $\|h-g\|_{C^{2}(\mathbb{R})} \leq \delta$ we have for

$$
g_{t}=\frac{h+t(g-h)}{\|h+t(g-h)\|_{L^{2}}}
$$

that

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}} \mathcal{F}\left(g_{t}, v_{t}\right) \geq \frac{c_{0}}{2}\left\|\left\langle\underline{\dot{g}_{t}}, \nu_{t}\right\rangle\right\|_{H^{1}\left(\Gamma_{g_{t}}\right)}^{2} \quad \text { for all } t \in[0,1] \tag{4.8}
\end{equation*}
$$

where $v_{t}$ is the elastic equilibrium associated to $g_{t}$. The constant $c_{0}$ is from Lemma 3.6.
Proof. Choose $\gamma>0$ such that $\gamma \leq h \leq R_{0}-\gamma$. Suppose that the claim is not true and there are pairs $\left(g_{n}, v_{n}\right) \in X_{\text {reg }}\left(u_{0}\right)$ and $t_{n} \in[0,1]$ with

$$
\left\|h-g_{n}\right\|_{C^{2}(\mathbb{R})} \rightarrow 0
$$

for which the claim doesn't hold. Denoting

$$
\dot{g}_{n}:=\left.\frac{\partial}{\partial t}\right|_{t=t_{n}}\left(\frac{h+t\left(g_{n}-h\right)}{\left\|h+t\left(g_{n}-h\right)\right\|_{L^{2}}}\right) \quad \text { and } \quad \ddot{g}_{n}:=\left.\frac{\partial^{2}}{\partial t^{2}}\right|_{t=t_{n}} \quad\left(\frac{h+t\left(g_{n}-h\right)}{\left\|h+t\left(g_{n}-h\right)\right\|_{L^{2}}}\right)
$$

this implies

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\mathcal{F}^{\prime \prime}\left(g_{n}, v_{n}\right)}{\left\|\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle\right\|_{H^{1}\left(\Gamma_{g_{n}}\right)}^{2}} \leq \frac{c_{0}}{2} \tag{4.9}
\end{equation*}
$$

where by the notation $\mathcal{F}^{\prime \prime}\left(g_{n}, v_{n}\right)$ we mean

$$
\begin{align*}
\mathcal{F}^{\prime \prime}\left(g_{n}, v_{n}\right)= & -\int_{\Omega_{g_{n}}} 2 Q\left(E\left(\dot{v}_{n}\right)\right) d z+\int_{\Gamma_{g_{n}}}\left|\partial_{\tau_{n}}\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle\right|^{2} d \mathcal{H}^{1} \\
& -\int_{\Gamma_{g_{n}}}\left(\partial_{\nu_{n}} Q\left(E\left(v_{n}\right)\right)+k_{n}^{2}\right)\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle^{2} d \mathcal{H}^{1} \\
& +\int_{\Gamma_{g_{n}}}\left(Q\left(E\left(v_{n}\right)\right)-k_{n}\right) \partial_{\tau_{n}}\left(\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle\left\langle\underline{\dot{g}_{n}}, \tau_{n}\right\rangle\right) d \mathcal{H}^{1}  \tag{4.10}\\
& -\int_{\Gamma_{g_{n}}}\left(Q\left(E\left(v_{n}\right)\right)-k_{n}\right)\left(\frac{\left\langle\underline{\dot{g}}_{n}, \nu_{n}\right\rangle^{2}}{\left\langle\underline{g_{n}}, \nu_{n}\right\rangle}+\left\langle\underline{\ddot{g}_{n}}, \nu_{n}\right\rangle\right) d \mathcal{H}^{1} \\
& =I_{1}+I_{2}+I_{3}+I_{4}+I_{5} .
\end{align*}
$$

Here $\nu_{n}$ is the outer normal to $F_{g_{n}}, \tau_{n}$ and $k_{n}$ are the tangent vector and the curvature of $\Gamma_{g_{n}}$ and $\dot{v}_{n}$ is the unique solution to

$$
\begin{equation*}
\int_{\Omega_{g_{n}}} \mathbb{C} E\left(\dot{v}_{n}\right): E(w) d z=-\int_{\Gamma_{g_{n}}} \operatorname{div}_{\tau_{n}}\left(\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle \mathbb{C} E\left(v_{n}\right)\right) \cdot w d \mathcal{H}^{1}, \quad \forall w \in \mathcal{A}\left(\Omega_{g_{t}}\right) \tag{4.11}
\end{equation*}
$$

As in Proposition 3.1 we find $C^{\infty}$-diffeomorphisms $\Psi_{n}: \bar{\Omega}_{h} \rightarrow \bar{\Omega}_{g_{n}}$ such that $\Psi_{n}: \Gamma_{h} \rightarrow \Gamma_{g_{n}}$ and

$$
\left\|\Psi_{n}-i d\right\|_{C^{2}\left(\bar{\Omega}_{h} ; \mathbb{R}^{2}\right)} \leq C\left\|h-g_{n}\right\|_{C^{2}(\mathbb{R})}
$$

The goal is to examine the contribution of each term in (4.10) to the limit (4.9). We begin by proving that the contribution of $I_{4}$ and $I_{5}$ to (4.9) is zero.

Notice that the $C^{2}$-convergence of $g_{n}$ implies $k_{n} \circ \Psi_{n} \rightarrow k$ in $L^{\infty}\left(\Gamma_{h}\right)$. Moreover, since $v_{n}$ solves the first two equations in (2.7) and $\sup _{n}\left\|g_{n}\right\|_{C^{2}([0,2 \pi))} \leq C$, we have by a Schauder type estimate for Lamé system, see [14], that there is $\alpha \in(0,1)$ such that

$$
\begin{equation*}
\sup _{n}\left\|v_{n}\right\|_{C^{1, \alpha}\left(\bar{\Omega}_{n}^{\prime} ; \mathbb{R}^{2}\right)}<\infty, \quad \text { for } \Omega_{n}^{\prime}=B_{R_{0}-\gamma} \backslash F_{g_{n}} \tag{4.12}
\end{equation*}
$$

Next we prove the following elliptic estimate

$$
\begin{align*}
&\left\|E\left(u \circ \Psi_{n}^{-1}\right)-E\left(v_{n}\right)\right\|_{W^{1, p}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)}+\| \nabla \mathbb{C} E\left(u \circ \Psi_{n}^{-1}\right)-\nabla \mathbb{C} E\left(v_{n}\right) \|_{H^{-\frac{1}{2}}\left(\Gamma_{g_{n}} ; \mathbb{T}\right)}  \tag{4.13}\\
& \leq C\left\|h-g_{n}\right\|_{C^{2}(\mathbb{R})},
\end{align*}
$$

where $p>2$ and $C$ depends on $\gamma, p$ and the $C^{2}$-norms of $h$ and $u$. Indeed by the equations (2.7) satisfied by $u$ and $v_{n}$ and a standard change of variables we obtain

$$
\begin{equation*}
\int_{\Omega_{g_{n}}} \mathbb{C}\left(E\left(u \circ \Psi_{n}^{-1}\right)-E\left(v_{n}\right)\right): E(w) d z=\int_{\Omega_{g_{n}}} f_{n}: E(w) d z, \quad \forall w \in \mathcal{A}\left(\Omega_{g_{n}}\right) \tag{4.14}
\end{equation*}
$$

where $f_{n} \in C^{1}\left(\bar{\Omega}_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)$ satisfies

$$
\left\|f_{n}\right\|_{C^{1}\left(\bar{\Omega}_{g_{n}}\right)} \leq C\left\|h-g_{n}\right\|_{C^{2}(\mathbb{R})}
$$

for $C$ depending only on the $C^{2}$-norm of $u$. Lemma 4.6 yields the estimate

$$
\begin{aligned}
\left\|E\left(u \circ \Psi_{n}^{-1}\right)-E\left(v_{n}\right)\right\|_{W^{1, p}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)}+\left\|\nabla \mathbb{C} E\left(u \circ \Psi_{n}^{-1}\right)-\nabla \mathbb{C} E\left(v_{n}\right)\right\|_{H^{-\frac{1}{2}}\left(\Gamma_{g_{n}} ; \mathbb{T}\right)} \\
\leq C\left(\left\|E\left(u \circ \Psi_{n}^{-1}\right)-E\left(v_{n}\right)\right\|_{L^{2}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)}+\left\|h-g_{n}\right\|_{C^{2}(\mathbb{R})}\right)
\end{aligned}
$$

On the other hand, using $w=u \circ \Psi_{n}^{-1}-v_{n}$ as a test function in (4.14), we obtain

$$
\left\|E\left(u \circ \Psi_{n}^{-1}\right)-E\left(v_{n}\right)\right\|_{L^{2}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)} \leq C\left\|f_{n}\right\|_{L^{2}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)}
$$

This concludes the proof of (4.13).
By the trace theorem 4.4, Lemma 4.5 and (4.13) we obtain

$$
\begin{aligned}
\left\|E\left(v_{n} \circ \Psi_{n}\right)-E(u)\right\|_{H^{\frac{1}{2}}\left(\Gamma_{h} ; \mathbb{M}^{2 \times 2}\right)} & \leq C\left\|E\left(v_{n} \circ \Psi_{n}\right)-E(u)\right\|_{H^{1}\left(\Omega_{h} ; \mathbb{M}^{2 \times 2}\right)} \\
& \leq C\left\|g_{n}-h\right\|_{C^{2}(\mathbb{R})}
\end{aligned}
$$

This estimate together with (4.12) implies $v_{n} \circ \Psi_{n} \rightarrow u$ in $C^{1, \alpha}$. In particular, we have that

$$
\left(Q\left(E\left(v_{n}\right)\right)-k_{n}\right) \circ \Psi_{n} \rightarrow Q(E(u))-k \equiv \lambda
$$

uniformly, where $\lambda$ is a Lagrange multiplier. We may use this to estimate the term $I_{5}$ in (4.10). By explicit calculations one easily obtains that $\left\|\left\langle\underline{\ddot{g}_{n}}, \nu_{n}\right\rangle\right\|_{L^{1}} \leq C\left\|\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle\right\|_{L^{2}}^{2}$ and recalling that the functions $\dot{g}_{n}$ and $\ddot{g}_{n}$ satisfy the volume constraint, as in Remark 3.4,

$$
\int_{\Gamma_{g_{n}}} \frac{\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle^{2}}{\left\langle\underline{g_{n}}, \nu_{n}\right\rangle}+\left\langle\underline{\ddot{g}_{n}}, \nu_{n}\right\rangle d \mathcal{H}^{1}=0
$$

we get

$$
\begin{aligned}
\int_{\Gamma_{g_{n}}} & \left(Q\left(E\left(v_{n}\right)\right)-k_{n}\right)\left(\frac{\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle^{2}}{\left\langle\underline{g_{n}}, \nu_{n}\right\rangle}+\left\langle\underline{\ddot{g}_{n}}, \nu_{n}\right\rangle\right) d \mathcal{H}^{1} \\
& =\int_{\Gamma_{g_{n}}}\left(Q\left(E\left(v_{n}\right)\right)-k_{n}-\lambda\right)\left(\frac{\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle^{2}}{\left\langle\underline{g_{n}}, \nu_{n}\right\rangle}+\left\langle\underline{\ddot{g}_{n}}, \nu_{n}\right\rangle\right) d \mathcal{H}^{1} \\
& \leq C\left\|Q\left(E\left(v_{n}\right)\right)-k_{n}-\lambda\right\|_{L^{\infty}\left(\Gamma_{g_{n}}\right)}\left\|\left\langle\underline{\left\langle g_{n}\right.}, \nu_{n}\right\rangle\right\|_{L^{2}\left(\Gamma_{g_{n}}\right)}^{2} .
\end{aligned}
$$

Using the polar decomposition we have $\nu_{n}=\frac{g_{n} \sigma+g_{n}^{\prime} \sigma^{\perp}}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}}$ and $\tau_{n}=\frac{g_{n} \sigma^{\perp}-g_{n}^{\prime} \sigma}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}}$. Since $\left\langle\underline{\dot{g}_{n}}, \tau_{n}\right\rangle(z)=$ $\frac{\left\langle z, \tau_{n}\right\rangle}{\left\langle z, \nu_{n}\right\rangle}\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle(z)$ and $\left\|\frac{\left\langle z, \tau_{n}\right\rangle}{\left\langle z, \nu_{n}\right\rangle}\right\|_{H^{1}\left(\Gamma_{g_{n}}\right)} \leq C$ we have as above that

$$
\begin{aligned}
& \int_{\Gamma_{g_{n}}}\left(Q\left(E\left(v_{n}\right)\right)-k_{n}\right) \partial_{\tau_{n}}\left(\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle\left\langle\underline{\dot{g}_{n}}, \tau_{n}\right\rangle\right) d \mathcal{H}^{1} \\
& \quad \leq C\left\|Q\left(E\left(v_{n}\right)\right)-k_{n}-\lambda\right\|_{L^{\infty}\left(\Gamma_{g_{n}}\right)}\left\|\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle\right\|_{H^{1}\left(\Gamma_{g_{n}}\right)}^{2}
\end{aligned}
$$

Hence the contribution of the terms $I_{4}$ and $I_{5}$ to the limit (4.9) is zero.
The remaining terms $I_{1}, I_{2}$ and $I_{3}$ form a quadratic form. The goal is to show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\mathcal{F}^{\prime \prime}\left(g_{n}, v_{n}\right)}{\left\|\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle\right\|_{H^{1}\left(\Gamma_{g_{n}}\right)}^{2}}=\lim _{n \rightarrow \infty} \frac{\partial^{2} \mathcal{F}(h, u)\left[\underline{\psi_{n}}\right]}{\left\|\left\langle\underline{\psi_{n}}, \nu\right\rangle\right\|_{H^{1}\left(\Gamma_{h}\right)}^{2}} \tag{4.15}
\end{equation*}
$$

where

$$
\psi_{n}=\frac{\dot{g}_{n} g_{n}}{h}
$$

and $u_{\psi_{n}}$ solves

$$
\begin{equation*}
\int_{\Omega_{h}} \mathbb{C} E\left(u_{\psi_{n}}\right): E(w) d z=-\int_{\Gamma_{h}} \operatorname{div}_{\tau}\left(\left\langle\underline{\psi_{n}}, \nu\right\rangle \mathbb{C} E(u)\right) \cdot w d \mathcal{H}^{1}, \quad \forall w \in \mathcal{A}\left(\Omega_{h}\right) \tag{4.16}
\end{equation*}
$$

Notice that $\psi_{n}$ satisfies the volume constraint $\int_{0}^{2 \pi} h \psi_{n} d \theta=0$ then we may use Lemma 3.6 to conclude that

$$
\lim _{n \rightarrow \infty} \frac{\partial^{2} \mathcal{F}(h, u)\left[\psi_{n}\right]}{\left\|\left\langle\underline{\psi_{n}}, \nu\right\rangle\right\|_{H^{1}\left(\Gamma_{h}\right)}^{2}} \geq c_{0},
$$

which then contradicts (4.9) and proves the claim.
To show (4.15) we will compare the contribution of each term in the quadratic form
$\partial^{2} \mathcal{F}(h, u)\left[\underline{\psi_{n}}\right]=-\int_{\Omega_{h}} 2 Q\left(E\left(u_{\psi_{n}}\right)\right) d z+\int_{\Gamma_{h}}\left|\partial_{\tau}\left\langle\underline{\psi_{n}}, \nu\right\rangle\right|^{2} d \mathcal{H}^{1}-\int_{\Gamma_{h}}\left(\partial_{\nu} Q(E(u))+k^{2}\right)\left\langle\underline{\psi_{n}}, \nu\right\rangle^{2} d \mathcal{H}^{1}$ with respect to the one given by $I_{1}, I_{2}$ and $I_{3}$ in (4.10).

We first point out that since $g_{n} \rightarrow h$ in $C^{2}$ we have that $\nu_{n} \circ \Psi_{n} \rightarrow \nu$ and $\tau_{n} \circ \Psi_{n} \rightarrow \tau$ in $C^{1}\left(\Gamma_{h}\right)$. Therefore from the definition of $\psi_{n}$ we get

$$
\lim _{n \rightarrow \infty} \frac{\left\|\left\langle\underline{\psi_{n}}, \nu\right\rangle\right\|_{H^{1}\left(\Gamma_{h}\right)}}{\left\|\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle\right\|_{H^{1}\left(\Gamma_{g_{n}}\right)}}=1
$$

and the convergence of $I_{2}$,

$$
\lim _{n \rightarrow \infty} \frac{\int_{\Gamma_{h}}\left|\partial_{\tau}\left\langle\underline{\psi_{n}}, \nu\right\rangle\right|^{2} d \mathcal{H}^{1}}{\int_{\Gamma_{g_{n}}}\left|\partial_{\tau_{n}}\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle\right|^{2} d \mathcal{H}^{1}}=1
$$

The convergence of $I_{1}$ follows from the equations (4.11) and (4.16). Indeed, by using a standard change of variables, these equations yield
$\int_{\Omega_{g_{n}}}\left(\mathbb{C} E\left(u_{\psi_{n}} \circ \Psi_{n}^{-1}\right)-\mathbb{C} E\left(\dot{v}_{n}\right)\right): E(w) d z=\int_{\Omega_{g_{n}}}\left(\tilde{f}_{n} E\left(u_{\psi_{n}} \circ \Psi_{n}^{-1}\right)\right): E(w) d z+\int_{\Gamma_{g_{n}}} d_{n} \cdot w d \mathcal{H}^{1}$ for any $w \in \mathcal{A}\left(\Omega_{g_{n}}\right)$. Here

$$
d_{n}=\operatorname{div}_{\tau}\left(\left\langle\underline{\psi_{n}}, \nu\right\rangle \mathbb{C} E(u)\right) \circ \Psi_{n}^{-1}\left|D_{\tau_{n}} \Psi_{n}^{-1}\right|-\operatorname{div}_{\tau_{n}}\left(\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle \mathbb{C} E\left(v_{n}\right)\right)
$$

and $\tilde{f}_{n} \in L^{2}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)$. For $\tilde{f}_{n}$ we have

$$
\begin{equation*}
\left\|\tilde{f}_{n}\right\|_{L^{\infty}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)} \rightarrow 0 \tag{4.18}
\end{equation*}
$$

By the estimate (4.13) we get

$$
\left\|\nabla \mathbb{C} E\left(u \circ \Psi_{n}^{-1}\right)-\nabla \mathbb{C} E\left(v_{n}\right)\right\|_{H^{-\frac{1}{2}}\left(\Gamma_{g_{n}} ; \mathbb{T}\right)} \rightarrow 0
$$

Therefore, by Lemma 4.5 , the choice of $\Psi_{n}$ and from $\left\|\frac{\psi_{n}}{\bar{g}_{n}}-1\right\|_{C^{1}(\mathbb{R})} \rightarrow 0$ we have that

$$
\begin{equation*}
\left.\left\|d_{n}\right\|_{H^{-\frac{1}{2}}\left(\Gamma_{g_{n}} ; \mathbb{R}^{2}\right)} \| \underline{\underline{\dot{g}_{n}}}, \nu_{n}\right\rangle \|_{H^{1}\left(\Gamma_{g_{n}}\right)}^{-1} \rightarrow 0 . \tag{4.19}
\end{equation*}
$$

Choose

$$
w(z)=\left(u_{\psi_{n}} \circ \Psi_{n}^{-1}-\dot{v}_{n}\right)(z)+A z+b
$$

as a test function in (4.17) where $A$ is antisymmetric and $b$ is a vector. This yields

$$
\begin{align*}
\int_{\Omega_{g_{n}}} Q\left(E \left(u_{\psi_{n}} \circ\right.\right. & \left.\left.\Psi_{n}^{-1}-\dot{v}_{n}\right)\right) d z  \tag{4.20}\\
\leq & \left.C\left\|\tilde{f}_{n}\right\|_{L^{\infty}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)}\left\|E\left(u_{\psi_{n}}\right)\right\|_{L^{2}\left(\Omega_{h} ; \mathbb{M}^{2 \times 2}\right)}\left\|E\left(u_{\psi_{n}} \circ \Psi_{n}^{-1}-\dot{v}_{n}\right)\right\|_{L^{2}\left(\Omega_{g_{n}} ; \mathbb{M}^{2} \times 2\right.}\right) \\
& +\left\|d_{n}\right\|_{H^{-\frac{1}{2}}\left(\Gamma_{g_{n}} ; \mathbb{R}^{2}\right)}\|w\|_{H^{\frac{1}{2}}\left(\Gamma_{g_{n}} ; \mathbb{R}^{2}\right)}
\end{align*}
$$

By Theorem 4.4 we get that

$$
\|w\|_{H^{\frac{1}{2}}\left(\Gamma_{g_{n}} ; \mathbb{R}^{2}\right)} \leq C\|w\|_{H^{1}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)} .
$$

As in the proof of Lemma 3.6 we choose $A$ such that

$$
\|w\|_{H^{1}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)} \leq C\|E(w)\|_{L^{2}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)}=C\left\|E\left(u_{\psi_{n}} \circ \Psi_{n}^{-1}-\dot{v}_{n}\right)\right\|_{L^{2}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)}
$$

by Korn's and Poincaré's inequalities (choose $b$ accordingly). The two previous inequalities and (4.20) yield
$\left\|E\left(u_{\psi_{n}} \circ \Psi_{n}^{-1}-\dot{v}_{n}\right)\right\|_{L^{2}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)} \leq C\left(\left\|\tilde{f}_{n}\right\|_{L^{\infty}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)}\left\|E\left(u_{\psi_{n}}\right)\right\|_{L^{2}\left(\Omega_{h} ; \mathbb{M}^{2 \times 2}\right)}+\left\|d_{n}\right\|_{H^{-\frac{1}{2}}\left(\Gamma_{g_{n}} ; \mathbb{R}^{2}\right)}\right)$.
Arguing as in (3.20) we may estimate

$$
\left\|E\left(u_{\psi_{n}}\right)\right\|_{L^{2}\left(\Omega_{h} ; \mathbb{M}^{2 \times 2}\right)} \leq C\left\|\left\langle\underline{\psi_{n}}, \nu\right\rangle\right\|_{H^{1}\left(\Gamma_{h}\right)} .
$$

Therefore using (4.18) and (4.19) we deduce that

$$
\frac{\left\|E\left(u_{\psi_{n}}\right)\right\|_{L^{2}\left(\Omega_{h} ; \mathbb{M}^{2 \times 2}\right)}^{2}-\left\|E\left(\dot{v}_{n}\right)\right\|_{L^{2}\left(\Omega_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)}^{2}}{\left.\| \underline{\dot{g}_{n}}, \nu_{n}\right\rangle \|_{H^{1}\left(\Gamma_{g_{n}}\right)}^{2}} \rightarrow 0 .
$$

This proves the convergence of $I_{1}$.
We are left with the term $I_{3}$ in (4.10). We need to show that

$$
\left.\left|\int_{\Gamma_{g_{n}}}\left(\partial_{\nu_{n}} Q\left(E\left(v_{n}\right)\right)+k_{n}^{2}\right)\left\langle\underline{\dot{g}_{n}}, \nu_{n}\right\rangle^{2} d \mathcal{H}^{1}-\int_{\Gamma_{h}}\left(\partial_{\nu} Q(E(u))+k^{2}\right)\left\langle\underline{\psi_{n}}, \nu\right\rangle^{2} d \mathcal{H}^{1}\right| \| \underline{\dot{g}_{n}}, \nu_{n}\right\rangle \|_{H^{1}\left(\Gamma_{g_{n}}\right)}^{-2} \rightarrow 0 .
$$

Due to the $C^{2}$-convergence of $g_{n}$ and the $C^{1}$-convergence of $\frac{\psi_{n}}{g_{n}}$ we just need to show

$$
\begin{equation*}
\left\|\partial_{\nu_{n}} Q\left(E\left(v_{n}\right)\right) \circ \Psi_{n}-\partial_{\nu} Q(E(u))\right\|_{H^{-\frac{1}{2}}\left(\Gamma_{h}\right)} \rightarrow 0 \tag{4.21}
\end{equation*}
$$

This will be done as [14, Proposition 4.5]. For every $\varphi \in H^{\frac{1}{2}}\left(\Gamma_{h}\right)$ we have

$$
\begin{aligned}
& \int_{\Gamma_{h}}\left(\frac{\partial}{\partial x_{1}} Q\left(E\left(v_{n}\right)\right) \circ \Psi_{n}-\frac{\partial}{\partial x_{1}} Q(E(u))\right) \varphi d \mathcal{H}^{1} \\
& =\int_{\Gamma_{h}}\left(\mathbb{C} E\left(\frac{\partial v_{n}}{\partial x_{1}}\right) \circ \Psi_{n}-\mathbb{C} E\left(\frac{\partial u}{\partial x_{1}}\right)\right):\left(E\left(v_{n}\right) \circ \Psi_{n}\right) \varphi d \mathcal{H}^{1} \\
& \quad+\int_{\Gamma_{h}} \mathbb{C} E\left(\frac{\partial u}{\partial x_{1}}\right):\left(E\left(v_{n}\right) \circ \Psi_{n}-E(u)\right) \varphi d \mathcal{H}^{1} \\
& \leq\left\|\left(\nabla \mathbb{C} E\left(v_{n}\right)\right) \circ \Psi_{n}-\nabla \mathbb{C} E(u)\right\|_{H^{-\frac{1}{2}}\left(\Gamma_{h} ; \mathbb{T}\right)}\left\|\left(E\left(v_{n}\right) \circ \Psi_{n}\right) \varphi\right\|_{\left.H^{\frac{1}{2}}\left(\Gamma_{h} ; \mathbb{M}^{2 \times 2}\right)\right)} \\
& \quad+C\left\|E\left(v_{n}\right) \circ \Psi_{n}-E(u)\right\|_{L^{2}\left(\Gamma_{h} ; \mathbb{M}^{2} \times 2\right.}\|\varphi\|_{L^{2}\left(\Gamma_{h}\right)}
\end{aligned}
$$

where the constant depends on $C^{2}$-norms of $u$ and $h$. Fix $p>2$. By the definition of Gagliardo seminorm, Hölder's inequality, Theorem 4.3 and Theorem 4.4, we obtain

$$
\begin{aligned}
& \left\|\left(E\left(v_{n}\right) \circ \Psi_{n}\right) \varphi\right\|_{\left.H^{\frac{1}{2}}\left(\Gamma_{h} ; \mathbb{M}^{2 \times 2}\right)\right)} \\
& \quad \leq C\left\|\left(E\left(v_{n}\right) \circ \Psi_{n}\right)\right\|_{\left.L^{\infty}\left(\Gamma_{h} ; \mathbb{M}^{2 \times 2}\right)\right)}\|\varphi\|_{H^{\frac{1}{2}}\left(\Gamma_{h}\right)}+C\left\|\left(E\left(v_{n}\right) \circ \Psi_{n}\right)\right\|_{\left.W^{\frac{p+2}{2 p}, \frac{2 p}{p-2}}\left(\Gamma_{h} ; \mathbb{M}^{2 \times 2}\right)\right)}\|\varphi\|_{L^{p}\left(\Gamma_{h}\right)} \\
& \quad \leq C\left(\left\|\left(E\left(v_{n}\right) \circ \Psi_{n}\right)\right\|_{\left.L^{\infty}\left(\Gamma_{h} ; \mathbb{M}^{2 \times 2}\right)\right)}+\left\|\left(E\left(v_{n}\right) \circ \Psi_{n}\right)\right\|_{W^{1, \frac{2 p}{p-2}}\left(\Omega_{h} ; \mathbb{M}^{2 \times 2}\right)}\right)\|\varphi\|_{H^{\frac{1}{2}}\left(\Gamma_{h}\right)} .
\end{aligned}
$$

By repeating the previous argument for $\frac{\partial}{\partial x_{2}}$ we obtain by (4.12) and (4.13) that

$$
\begin{align*}
& \left\|\nabla Q\left(E\left(v_{n}\right)\right) \circ \Psi_{n}-\nabla Q(E(u))\right\|_{H^{-\frac{1}{2}}\left(\Gamma_{h} ; \mathbb{R}^{2}\right)} \\
& \quad \leq C\left(\left\|\left(\nabla \mathbb{C} E\left(v_{n}\right)\right) \circ \Psi_{n}-\nabla \mathbb{C} E(u)\right\|_{H^{-\frac{1}{2}}\left(\Gamma_{h} ; \mathbb{T}\right)}+\left\|E\left(v_{n}\right) \circ \Psi_{n}-E(u)\right\|_{L^{2}\left(\Gamma_{h} ; \mathbb{M}^{2 \times 2}\right)}\right)  \tag{4.22}\\
& \quad \leq C\left\|g_{n}-h\right\|_{C^{2}(\mathbb{R})} .
\end{align*}
$$

Since $\nu_{n} \circ \Psi_{n} \rightarrow \nu$ in $C^{1}$, (4.22) implies (4.21). This concludes the convergence of the term $I_{3}$ and completes the proof.

## 5. Local minimality

This section is devoted to prove the main result of the paper, the local minimality criterion. Namely, we show that if a critical point $(h, u) \in X_{\text {reg }}\left(u_{0}\right)$ has positive second variation, then it is a strict local minimizer in the Hausdorff distance of sets and a quantitative estimate in terms of the measure of the symmetric difference between the minimum and a competitor holds. Due to the sharp quantitative isoperimetric inequality, the exponent 2 in (5.1) is optimal.

Theorem 5.1. Suppose that $(h, u) \in X_{\mathrm{reg}}\left(u_{0}\right)$ is a critical pair for $\mathcal{F}$ with $0<h<R_{0}$. If the second variation of $\mathcal{F}$ is positive at $(h, u)$, then there is $\delta>0$ such that for any $(g, v) \in X\left(u_{0}\right)$ with $\left|\Omega_{g}\right|=\left|\Omega_{h}\right|$ and $0<d_{\mathcal{H}}\left(\Gamma_{g} \cup \Sigma_{g}, \Gamma_{h}\right) \leq \delta$ it holds that

$$
\begin{equation*}
\mathcal{F}(g, v)>\mathcal{F}(h, u)+c\left|\Omega_{g} \Delta \Omega_{h}\right|^{2} \tag{5.1}
\end{equation*}
$$

for some $c>0$.
The proof is based on a contradiction argument and follows some ideas contained in [14], [9] and [1]. Assume, for a contradiction, that $\left(h_{n}, u_{n}\right)$ is a sequence satisfying

$$
\mathcal{F}\left(h_{n}, u_{n}\right) \leq \mathcal{F}(h, u)+c_{0}\left|\Omega_{h_{n}} \Delta \Omega_{h}\right|^{2} \quad \text { and } \quad 0<d_{\mathcal{H}}\left(\Gamma_{h_{n}} \cup \Sigma_{h_{n}}, \Gamma_{h}\right) \leq \frac{1}{n}
$$

The idea is to replace $\left(h_{n}, u_{n}\right)$ with the minimizer $\left(g_{n}, v_{n}\right)$ of an auxiliary constrained-penalized problem, and to prove that the $\left(g_{n}, v_{n}\right)$ are sufficiently regular to apply the $C^{1,1}$-minimality criterion to get a contradiction. As auxiliary problem we choose

$$
\min \left\{\mathcal{F}(g, v)+\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right|+\sqrt{\left(\left|\Omega_{g} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}}:(g, v) \in X\left(u_{0}\right), g \leq h+\frac{1}{n}\right\}
$$

where the second penalization term will provide the quantitative estimate in (5.1) and the obstacle $g \leq h+1 / n$ plays a key role in proving the regularity of $\left(g_{n}, v_{n}\right)$.

The regularity proof is divided in three steps. In Lemma 5.6 we prove that $g_{n}$ is Lipschitz using some geometrical arguments. Then, in Lemma 5.7, we show that $g_{n}$ is a quasiminimizer for the area functional which in turns implies its $C^{1, \alpha}$-regularity. Finally, we deduce the $C^{1,1}$-regularity in Lemma 5.8, by using the Euler-Lagrange equation for $\left(g_{n}, v_{n}\right)$.

The following isoperimetric-type result will be used frequently in this section. The proof can be found in [1, Lemma 4.1].

## Lemma 5.2.

(i) Let $f \in C_{\#}^{\infty}(\mathbb{R})$ be non-negative and let $g \in B V_{\#}(\mathbb{R})$, then there exists a constant $C$, depending only on $f$, such that

$$
\mathcal{H}^{1}\left(\Gamma_{g}\right)-\mathcal{H}^{1}\left(\Gamma_{f}\right) \geq-C\left|\Omega_{g} \Delta \Omega_{f}\right|
$$

(ii) Suppose $D$ is a set of finite perimeter. Then

$$
P\left(D \cup B_{r}(x)\right)-P\left(B_{r}(x)\right) \geq \frac{1}{r}|D|,
$$

where $P$ stands for the perimeter.
We will also need the following property of concave functions.
Lemma 5.3. Suppose that $f_{n} \in C^{1}(\mathbb{R})$ and $f \in C^{1}(\mathbb{R})$ are such that $\left\|f_{n}-f\right\|_{L^{\infty}(\mathbb{R})} \rightarrow 0$. If the $f_{n}$ are concave then

$$
\left\|f_{n}-f\right\|_{C_{l o c}^{1}(\mathbb{R})} \rightarrow 0
$$

Proof. Let $R>0$ and fix $\varepsilon>0$. Since $f \in C^{1}(\mathbb{R})$ we find $\delta>0$ such that

$$
f(\delta+x)-f(x) \geq f^{\prime}(x) \delta-\varepsilon \delta
$$

for every $|x| \leq R$. On the other hand, since the $f_{n}$ are concave, we have

$$
\frac{f_{n}(\delta+x)-f_{n}(x)}{\delta} \leq f_{n}^{\prime}(x)
$$

Hence

$$
f^{\prime}(x)-f_{n}^{\prime}(x) \leq \frac{f(\delta+x)-f_{n}(\delta+x)-\left(f(x)-f_{n}(x)\right)}{\delta}+\varepsilon \leq 2 \varepsilon
$$

when $n$ is large. The reverse inequality $f_{n}^{\prime}(x)-f^{\prime}(x) \leq 2 \varepsilon$ follows from a similar argument.
We begin the study of the properties of solutions of the auxiliary problem by proving an exterior ball condition.

Theorem 5.4. Let $h \in C_{\#}^{\infty}(\mathbb{R})$ such that $0<h<R_{0}$. Then for every $c, \varepsilon \in[0,1]$ and $n \in \mathbb{N}$ every solution of the problem

$$
\begin{equation*}
\min \left\{\mathcal{F}(g, v)+\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right|+c \sqrt{\left(\left|\Omega_{g} \Delta \Omega_{h}\right|-\varepsilon\right)^{2}+\varepsilon}:(g, v) \in X\left(u_{0}\right), g \leq h+\frac{1}{n}\right\} \tag{5.2}
\end{equation*}
$$

satisfies the following uniform exterior ball condition: for every $z \in \partial F_{g}$ and for every $r<$ $\min \left\{1 /(\Lambda+1), 1 /\left\|\kappa_{h}\right\|_{\infty}\right\}$, where $\kappa_{h}$ is the curvature of $\Gamma_{h}$, there exists $z_{0}$ such that $B_{r}\left(z_{0}\right) \subset \mathbb{R}^{2} \backslash F_{g}$ and $\partial B_{r}\left(z_{0}\right) \cap\left(\Gamma_{g} \cup \Sigma_{g}\right)=\{z\}$.

Figure 5.1.


Proof. The proof follows the argument from [14, Lemma 6.7]. Recall that $\partial F_{g}=\Gamma_{g} \cup \Sigma_{g}$. Given a ball $B_{r}\left(z_{0}\right)$ define the half circle $S_{r}\left(z_{0}\right)=\partial B_{r}\left(z_{0}\right) \cap\left\{z \in \mathbb{R}^{2}:\left\langle z-z_{0}, z_{0}\right\rangle<0\right\}$. Assume that there exists a ball $B_{r}\left(z_{0}\right) \subset \mathbb{R}^{2} \backslash F_{g}$ such that $S_{r}\left(z_{0}\right)$ intersects $\Gamma_{g} \cup \Sigma_{g}$ in two different points $z_{1}=\left(\rho_{1}, \theta_{1}\right)$ and $z_{2}=\left(\rho_{2}, \theta_{2}\right)$. When $r<1 /\left\|\kappa_{h}\right\|_{\infty}$ it is clear that the arc $f=f(\theta)$ of $S_{r}\left(z_{0}\right)$ connecting $z_{1}$ and $z_{2}$ satisfies $f(\theta) \leq h(\theta)+\frac{1}{n}$ for $\theta \in\left(\theta_{1}, \theta_{2}\right)$. Let $\tilde{g}$ be defined by $\tilde{g}=f$ for $\theta \in\left(\theta_{1}, \theta_{2}\right)$ and $\tilde{g}=g$ otherwise. Denote by $\tilde{f}$ the arc of $\Gamma_{g} \cup \Sigma_{g}$ connecting $z_{1}$ and $z_{2}$ and by $D$ the region enclosed by $f \cup \tilde{f}$, see Figure 5.1.

Notice that

Since $\Omega_{\tilde{g}} \Delta \Omega_{g}=D$ and $\Omega_{\tilde{g}} \subset \Omega_{g}$ we see that

$$
\begin{align*}
& \mathcal{F}(\tilde{g}, v)+\Lambda| | \Omega_{\tilde{g}}\left|-\left|\Omega_{h}\right|\right|+c \sqrt{\left(\left|\Omega_{\tilde{g}} \Delta \Omega_{h}\right|-\varepsilon\right)^{2}+\varepsilon} \\
& \leq \mathcal{F}(g, v)+\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right|+c \sqrt{\left(\left|\Omega_{g} \Delta \Omega_{h}\right|-\varepsilon\right)^{2}+\varepsilon}+\mathcal{H}^{1}(f)-\mathcal{H}^{1}(\tilde{f})+(\Lambda+1)|D| \tag{5.4}
\end{align*}
$$

Moreover from Lemma 5.2 we infer that

$$
\mathcal{H}^{1}(f)-\mathcal{H}^{1}(\tilde{f}) \leq P\left(B_{r}\left(z_{0}\right)\right)-P\left(D \cup B_{r}\left(z_{0}\right)\right) \leq-\frac{1}{r}|D|
$$

Hence, since $r<1 /(\Lambda+1)$, the inequality (5.4) contradicts the minimality of $(g, v)$. The conclusion now follows arguing as [8, Lemma 2] or [12, Proposition 3.3, Step 2].
Lemma 5.5. Let $h, c, \varepsilon$ and $n$ be as in the previous theorem. Suppose $(g, v) \in X\left(u_{0}\right)$ is any minimizer of (5.2). Then there exists $\Lambda_{0}>0$, independent of $c, \varepsilon$ and $n$, such that if $\Lambda \geq \Lambda_{0}$ then $\left|\Omega_{g}\right| \geq\left|\Omega_{h}\right|$.

Proof. We argue by contradiction supposing that $\left|\Omega_{g}\right|<\left|\Omega_{h}\right|$ for every $\Lambda>0$. We observe that there exists $0<r<1$ such that, if we define $\Omega_{g}^{r}=B_{R_{0}} \backslash r F_{g}$, we have $\left|\Omega_{g}^{r}\right|=\left|\Omega_{h}\right|$. Moreover, since

$$
\left|\Omega_{g}^{r}\right|=\pi R_{0}^{2}-\frac{r^{2}}{2} \int_{0}^{2 \pi} g^{2} d \theta
$$

we get

$$
r=\left(\frac{\pi R_{0}^{2}-\left|\Omega_{h}\right|}{\pi R_{0}^{2}-\left|\Omega_{g}\right|}\right)^{\frac{1}{2}}<1
$$

Clearly $\Omega_{g}^{r}=\Omega_{g_{r}}$ for $g_{r}(\theta)=r g(\theta)$. Define the function $v_{r}: \Omega_{g_{r}} \rightarrow \mathbb{R}^{2}$ as

$$
v_{r}(z)= \begin{cases}u_{0}\left(\frac{z}{|z|} R_{0}\right) & \text { if } r R_{0} \leq|z| \leq R_{0} \\ v\left(\frac{z}{r}\right) & \text { if } g_{r}\left(\frac{z}{|z|}\right) \leq|z|<r R_{0}\end{cases}
$$

Since $\Omega_{g_{r}} \supset \Omega_{g}$, we see that $\left|\Omega_{g_{r}} \Delta \Omega_{g}\right|=\left|\Omega_{h}\right|-\left|\Omega_{g}\right|$. Using the inequality (5.3) we have, for $\Lambda$ sufficiently large, that

$$
\begin{aligned}
\mathcal{F}\left(g_{r}, v_{r}\right) & +\Lambda| | \Omega_{g_{r}}\left|-\left|\Omega_{h}\right|\right|+c \sqrt{\left(\left|\Omega_{g_{r}} \Delta \Omega_{h}\right|-\varepsilon\right)^{2}+\varepsilon} \\
-\mathcal{F}(g, v) & -\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right|-c \sqrt{\left(\left|\Omega_{g} \Delta \Omega_{h}\right|-\varepsilon\right)^{2}+\varepsilon} \\
& \leq \int_{r R_{0} \leq|z| \leq R_{0}} Q\left(E\left(v_{r}\right)\right) d z-\Lambda\left(\left|\Omega_{h}\right|-\left|\Omega_{g}\right|\right)+c\left|\Omega_{g_{r}} \Delta \Omega_{g}\right| \\
& \leq C(1-r)-(\Lambda-1)\left(\left|\Omega_{h}\right|-\left|\Omega_{g}\right|\right) \\
& \leq C\left(\left|\Omega_{h}\right|-\left|\Omega_{g}\right|\right)-(\Lambda-1)\left(\left|\Omega_{h}\right|-\left|\Omega_{g}\right|\right)<0
\end{aligned}
$$

which contradicts the minimality of $(g, v)$.
In the following we study convergence properties of solutions for the constrained obstacle problem (5.2).

Lemma 5.6. Let $h$ be as in Theorem 5.4. Assume $g_{n} \in B V_{\#}(\mathbb{R})$ is such that $g_{n} \leq h+1 / n$ and it satisfies the uniform exterior ball condition. If

$$
\begin{equation*}
g_{n} \rightarrow h \text { in } L^{1} \text { and } \lim _{n \rightarrow \infty} \mathcal{H}^{1}\left(\Gamma_{g_{n}} \cup \Sigma_{g_{n}}\right)=\mathcal{H}^{1}\left(\Gamma_{h}\right) \tag{5.5}
\end{equation*}
$$

then $g_{n} \rightarrow h$ in $L^{\infty}$. Moreover, for $n$ sufficiently large, the $g_{n}$ are uniformly Lipschitz continuous.
Proof. Here we follow an argument from [14, Theorem 6.9, Steps 1 and 2]. We claim that

$$
\sup _{\mathbb{R}}\left|g_{n}-h\right| \rightarrow 0 \text { as } n \rightarrow+\infty
$$

Let us first note that $\Gamma_{g_{n}} \cup \Sigma_{g_{n}}$ is a connected compact set. Up to a subsequence, we can assume that $\Gamma_{g_{n}} \cup \Sigma_{g_{n}}$ converges in the Hausdorff distance to some compact connected set $K$. The
continuity of $h$ and condition (5.5) imply that $\Gamma_{h} \subset K$. By Gołąb's semicontinuity Theorem (see, e.g. [3, Theorem 4.4.17]) and assumption (5.5) we see that

$$
\mathcal{H}^{1}\left(\Gamma_{h}\right) \leq \mathcal{H}^{1}(K) \leq \lim _{n \rightarrow+\infty} \mathcal{H}^{1}\left(\Gamma_{g_{n}} \cup \Sigma_{g_{n}}\right)=\mathcal{H}^{1}\left(\Gamma_{h}\right)
$$

This implies that $\mathcal{H}^{1}\left(K \backslash \Gamma_{h}\right)=0$. Since $K$ is connected, it follows from a density lower bound (see, e.g. [3, Lemma 4.4.5]) that $K=\Gamma_{h}$. Now the claim follows from the definition of the Hausdorff metric and from the continuity of $h$.

From the previous claim we see that, for $n$ sufficiently large, it holds $\gamma \leq g_{n} \leq R_{0}-\gamma$ for some $\gamma>0$ small, independent from $n$. Hence, since the polar coordinates mapping is a $C^{\infty}$-local diffeomorphism far from the origin, the representation in polar coordinates of $F_{g_{n}}$ (still denoted by $F_{g_{n}}$ ) satisfies the uniform exterior ball condition up to changing the radius $r$ to $\tilde{r} \in(0,1)$ by a factor depending only on $\gamma$. Now we prove that $g_{n}$ are $L$-Lipschitz with $L \leq \frac{8}{\tilde{r}}\|h\|_{C^{1}(\mathbb{R})}$.

We argue by contradiction and assume that there exists $\theta$ and $\theta_{k} \rightarrow \theta$ such that

$$
\lim _{k \rightarrow \infty} \frac{\left|g_{n}\left(\theta_{k}\right)-g_{n}(\theta)\right|}{\left|\theta_{k}-\theta\right|} \geq \frac{8}{\tilde{r}}\|h\|_{C^{1}(\mathbb{R})}
$$

and set $z=\left(\theta, g_{n}(\theta)\right)$. Without loss of generality we may assume that the sequence $\left\{\theta_{k}\right\}_{k} \in \mathbb{N}$ is monotone and $g_{n}\left(\theta_{k}\right)$ is increasing. By the uniform exterior ball condition we find a ball $B_{\tilde{r}}\left(z_{0}\right) \subset \mathbb{R}^{2} \backslash F_{g_{n}}$ such that $\partial B_{\tilde{r}}\left(z_{0}\right) \cap\left(\Gamma_{g_{n}} \cup \Sigma_{g_{n}}\right)=\{z\}$ and

$$
z_{0}=z+\tilde{r}\left(\frac{M}{\sqrt{1+M^{2}}}, \frac{1}{\sqrt{1+M^{2}}}\right), \quad \text { for } \quad M \geq \frac{4}{\tilde{r}}\|h\|_{C^{1}(\mathbb{R})}
$$

Let $z^{\prime} \in \partial B_{\tilde{r}}\left(z_{0}\right)$ such that

$$
z^{\prime}=z_{0}-\tilde{r}\left(\frac{\sqrt{M^{2}-3}}{\sqrt{1+M^{2}}}, \frac{2}{\sqrt{1+M^{2}}}\right)
$$

We write $z^{\prime}=: z+\tilde{r}\left(w_{1}, w_{2}\right)$ with

$$
w_{1}=\frac{M-\sqrt{M^{2}-3}}{\sqrt{1+M^{2}}}>0 \quad \text { and } \quad w_{2}=\frac{-1}{\sqrt{1+M^{2}}}<0
$$

and since $B_{\tilde{r}}\left(z_{0}\right) \subset \mathbb{R}^{2} \backslash F_{g_{n}}$ we have $g_{n}\left(\theta+\tilde{r} w_{1}\right) \leq g_{n}(\theta)+\tilde{r} w_{2}$. Setting $\delta_{n}=\sup _{\mathbb{R}}\left|h-g_{n}\right|$ and recalling $\|h\|_{C^{1}(\mathbb{R})} \leq M / 4$ we get

$$
h\left(\theta+\tilde{r} w_{1}\right) \geq h(\theta)-\frac{\tilde{r} M}{4} w_{1} \geq g_{n}(\theta)-\delta_{n}-\frac{\tilde{r} M}{4} w_{1}
$$

Therefore we deduce

$$
\begin{aligned}
h\left(\theta+\tilde{r} w_{1}\right)-g_{n}\left(\theta+\tilde{r} w_{1}\right) & \geq-\delta_{n}-\tilde{r}\left(\frac{M}{4} w_{1}-w_{2}\right) \\
& =-\delta_{n}+\frac{\tilde{r}}{\sqrt{1+M^{2}}}\left(1-\frac{M}{4}\left(M-\sqrt{M^{2}-3}\right)\right) \\
& =-\delta_{n}+\frac{\tilde{r}}{\sqrt{1+M^{2}}}\left(1-\frac{3 M}{4\left(M+\sqrt{M^{2}-3}\right)}\right)>\delta_{n}
\end{aligned}
$$

where the last inequality, which holds for $n$ sufficiently large, gives a contradiction.
In the next lemma we show the $C^{1, \alpha}$-regularity of the minimizer for the penalized obstacle problem.

Lemma 5.7. Let $h$ be as in Theorem 5.4 and $\left(g_{n}, v_{n}\right) \in X\left(u_{0}\right)$ be any minimizer of the problem

$$
\begin{equation*}
\min \left\{\mathcal{F}(g, v)+\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right|+c \sqrt{\left(\left|\Omega_{g} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}}:(g, v) \in X\left(u_{0}\right), g \leq h+\frac{1}{n}\right\} \tag{5.6}
\end{equation*}
$$

where $c \in[0,1]$ and $\varepsilon_{n} \rightarrow 0$. Assume also that $g_{n} \rightarrow h$ in $L^{1}$ and that

$$
\lim _{n \rightarrow \infty} \mathcal{H}^{1}\left(\Gamma_{g_{n}} \cup \Sigma_{g_{n}}\right)=\mathcal{H}^{1}\left(\Gamma_{h}\right) \quad \text { and } \quad \sup _{n} \int_{\Omega_{g_{n}}} Q\left(E\left(v_{n}\right)\right) d z<+\infty
$$

Then for all $\alpha \in\left(0, \frac{1}{2}\right)$ and for $n$ large enough $g_{n} \in C^{1, \alpha}(\mathbb{R})$, the sequence $\left\{\nabla v_{n}\right\}$ is equibounded in $C^{0, \alpha}\left(\bar{\Omega}_{g_{n}} ; \mathbb{M}^{2 \times 2}\right)$, and $g_{n} \rightarrow h$ in $C^{1, \alpha}(\mathbb{R})$.
Proof. From Lemma 5.6 we infer that $g_{n}$ is sufficiently regular to ensure a decay estimate for $\nabla v_{n}$. Indeed, for $z_{0} \in \Gamma_{g_{n}}$ there exist $c_{n}>0$, a radius $r_{n}$ and an exponent $\alpha_{n} \in(0,1 / 2)$ such that

$$
\int_{B_{r}\left(z_{0}\right) \cap \Omega_{g_{n}}}\left|\nabla v_{n}\right|^{2} \leq c_{n} r^{1+2 \alpha_{n}},
$$

for every $r<r_{n}$. This follows from the fact that $v_{n}$ minimizes the elastic energy in $\Omega_{g_{n}}$ and the boundary $\Gamma_{g_{n}}$ is Lipschitz, see Theorem 3.13 in [12].

Since $g_{n}$ is Lipschitz, we may extend $v_{n}$ in $B_{r}\left(z_{0}\right)$ such that

$$
\begin{equation*}
\int_{B_{r}\left(z_{0}\right)}\left|\nabla \tilde{v}_{n}\right|^{2} \leq c_{n} r^{1+2 \alpha_{n}} \tag{5.7}
\end{equation*}
$$

where $\tilde{v}_{n}$ stands for the extension.
For $r<r_{n}$, denote by $z_{r}^{\prime}$ and $z_{r}^{\prime \prime}$ the two points on $\Gamma_{g_{n}} \cap \partial B_{r}\left(z_{0}\right)$ such that the open sub-arcs of $\Gamma_{g_{n}}$ with end points $z_{r}^{\prime}, z_{0}$ and $z_{r}^{\prime \prime}$, $z_{0}$ are contained in $\Gamma_{g_{n}} \cap \partial B_{r}\left(z_{0}\right)$. Setting $z_{r}^{\prime}=g_{n}\left(\theta_{r}^{\prime}\right) \sigma\left(\theta_{r}^{\prime}\right)$ and $z_{r}^{\prime \prime}=g_{n}\left(\theta_{r}^{\prime \prime}\right) \sigma\left(\theta_{r}^{\prime \prime}\right)$, denote by $l$ the line segment joining $z_{r}^{\prime}$ and $z_{r}^{\prime \prime}$ and define

$$
\tilde{g}_{n}(\theta):= \begin{cases}g_{n}(\theta) & \theta \in[0,2 \pi) \backslash\left(\theta_{r}^{\prime}, \theta_{r}^{\prime \prime}\right) \\ \min \left\{h(\theta)+\frac{1}{n}, l(\theta)\right\} & \theta \in\left(\theta_{r}^{\prime}, \theta_{r}^{\prime \prime}\right)\end{cases}
$$

where $l(\theta)$ is the polar representation of $l$.
By (5.7) and by the minimality of the pair $\left(g_{n}, v_{n}\right)$ we have

$$
\begin{equation*}
\mathcal{H}^{1}\left(\Gamma_{g_{n}} \cap B_{r}\left(z_{0}\right)\right)-\mathcal{H}^{1}\left(\Gamma_{\tilde{g}_{n}} \cap B_{r}\left(z_{0}\right)\right) \leq C_{n} r^{1+2 \alpha_{n}} . \tag{5.8}
\end{equation*}
$$

Indeed we can estimate

$$
\begin{aligned}
0 \geq & \mathcal{F}\left(g_{n}, v_{n}\right)-\mathcal{F}\left(\tilde{g}_{n}, \tilde{v}_{n}\right)+\Lambda\left(| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|\right|-\left|\left|\Omega_{\tilde{g}_{n}}\right|-\left|\Omega_{h}\right|\right|\right) \\
& +c\left(\sqrt{\left(\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}}-\sqrt{\left(\left|\Omega_{\tilde{g}_{n}} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}}\right) \\
\geq & \mathcal{H}^{1}\left(\Gamma_{g_{n}} \cap B_{r}\left(z_{0}\right)\right)-\mathcal{H}^{1}\left(\Gamma_{\tilde{g}_{n}} \cap B_{r}\left(z_{0}\right)\right)-\int_{B_{r}\left(z_{0}\right)} Q\left(E\left(\tilde{v}_{n}\right)\right) d z-(\Lambda+1) \pi r^{2} \\
\geq & \mathcal{H}^{1}\left(\Gamma_{g_{n}} \cap B_{r}\left(z_{0}\right)\right)-\mathcal{H}^{1}\left(\Gamma_{\tilde{g}_{n}} \cap B_{r}\left(z_{0}\right)\right)-C_{n} r^{1+2 \alpha_{n}}
\end{aligned}
$$

We will show later that

$$
\begin{equation*}
\mathcal{H}^{1}\left(\Gamma_{\tilde{g}_{n}} \cap B_{r}\left(z_{0}\right)\right)-\mathcal{H}^{1}(l) \leq C r^{2} . \tag{5.9}
\end{equation*}
$$

Now the inequality (5.9) together with (5.8) gives us

$$
\begin{equation*}
\mathcal{H}^{1}\left(\Gamma_{g_{n}} \cap B_{r}\left(z_{0}\right)\right)-\mathcal{H}^{1}(l) \leq C r^{1+2 \alpha_{n}} \tag{5.10}
\end{equation*}
$$

and the desired $C^{1, \alpha}$-regularity follows from a classical result for quasiminimizers of the area functional (see Theorem 1 in [20]) once we observe that

$$
\mathcal{H}^{1}(l)=\inf \left\{P\left(F ; B_{r}\left(z_{0}\right)\right): F \Delta \Omega_{g_{n}} \Subset B_{r}\left(z_{0}\right)\right\}
$$

The proof of (5.9) is a consequence of the $C^{2}$-regularity of $h$ and goes as follows (see Figure 5.2):

$$
\begin{aligned}
\mathcal{H}^{1}\left(\Gamma_{\tilde{g}_{n}} \cap B_{r}\left(z_{0}\right)\right)-\mathcal{H}^{1}(l) & \leq \int_{\theta_{r}^{\prime}}^{\theta_{r}^{\prime \prime}} \sqrt{\left(\tilde{g}_{n}(\theta)\right)^{2}+\left(\tilde{g}_{n}^{\prime}(\theta)\right)^{2}}-\sqrt{(l(\theta))^{2}+\left(l^{\prime}(\theta)\right)^{2}} d \theta \\
& \leq \frac{1}{\gamma} \int_{\theta_{r}^{\prime}}^{\theta_{r}^{\prime \prime}}\left(\tilde{g}_{n}^{2}-l^{2}\right) d \theta+\frac{1}{\gamma} \int_{\theta_{r}^{\prime}}^{\theta_{r}^{\prime \prime}}\left(\tilde{g}_{n}^{\prime}+l^{\prime}\right)\left(\tilde{g}_{n}^{\prime}-l^{\prime}\right) d \theta \\
& \leq \frac{1}{\gamma}\left|B_{r}\left(z_{0}\right)\right|+\frac{C}{\gamma} \int_{\theta_{r}^{\prime}}^{\theta_{r}^{\prime \prime}}\left|\tilde{g}_{n}^{\prime}-l^{\prime}\right| d \theta
\end{aligned}
$$

where $C$ depends on the Lipschitz norm of $\tilde{g}_{n}$ and $l$ in the interval $\left(\theta_{r}^{\prime}, \theta_{r}^{\prime \prime}\right)$ and $\gamma$ is a positive constant with $\gamma<\min _{\mathbb{R}} h$.

Figure 5.2.

-0

To estimate the last term we first note that either the set $\{h+1 / n<l\}$ is empty or there exists $\theta_{0} \in\left(\theta_{r}^{\prime}, \theta_{r}^{\prime \prime}\right)$ such that $\tilde{g}_{n}^{\prime}\left(\theta_{0}\right)-l^{\prime}\left(\theta_{0}\right)=0$ and using a second order Taylor expansion around $\theta_{0}$ we easily get

$$
\int_{\theta_{r}^{\prime}}^{\theta_{r}^{\prime \prime}}\left|\tilde{g}_{n}^{\prime}-l^{\prime}\right| d \theta \leq C r^{2}
$$

where $C$ depends on the $C^{2}$-norm of $h$.
Now we claim that $g_{n}$ converges to $h$ in the $C^{1}$-norm. As in the proof of Lemma 5.6 we will work in the plane $(\theta, \rho)$ and we recall that the subgraph of $g_{n}$, still denoted by $F_{g_{n}}$, satisfies the uniform exterior ball condition. From the $C^{1}$-regularity and the uniform Lipschitz estimate, in the Lemma 5.6, we obtain $\sup _{n}\left\|g_{n}\right\|_{C^{1}}<\infty$. Hence, from the uniform exterior ball condition we conclude that at every point there exists a parabola touching $g_{n}$ from above. In other words, there is $C>0$ such that for every $\theta_{0}$ it holds for $P(\theta)=g_{n}\left(\theta_{0}\right)+g_{n}^{\prime}\left(\theta_{0}\right)\left(\theta-\theta_{0}\right)+C\left(\theta-\theta_{0}\right)^{2}$ that

$$
\min _{\theta}\left(P-g_{n}\right)=\left(P-g_{n}\right)\left(\theta_{0}\right)=0
$$

This implies that the $g_{n}$ are uniformly semiconcave, i.e., for every $n$ the function

$$
\theta \mapsto g_{n}(\theta)-C \theta^{2}
$$

is concave. We may now use Lemma 5.3 to conclude the desired $C^{1}$-convergence of $g_{n}$.
The convergence of $g_{n}$ to $h$ in $C^{1}$-norm allows us to use a blow-up method (see [14, Theorem 6.10]) to infer the uniform estimate

$$
\begin{equation*}
\int_{B_{r}\left(z_{0}\right)}\left|\nabla v_{n}\right|^{2} \leq c_{0} r^{1+2 \sigma} \tag{5.11}
\end{equation*}
$$

for any $\sigma \in(1 / 2,1)$ and for all $r<r_{0}$ where $c_{0}$ and $r_{0}$ are independent of $n$.
Once we have (5.11), we can repeat the argument used to prove (5.10), replacing (5.7) by (5.11), to infer

$$
\mathcal{H}^{1}\left(\Gamma_{g_{n}} \cap B_{r}\left(z_{0}\right)\right)-\mathcal{H}^{1}(l) \leq C r^{1+2 \sigma}
$$

This implies a uniform estimate for the $C^{1, \alpha}$-norms of $g_{n}$ for $\alpha \in(0,1 / 2)$ (see for instance [9, Proposition 2.2]). The $C^{1, \alpha}$-convergence of $g_{n}$ now follows by a compactness argument.

To conclude the proof we have just to observe that, since $v_{n}$ is a solution of the Lamé system in $\Omega_{g_{n}}$, we can apply the elliptic estimates provided in [14, Proposition 8.9] to deduce that $\nabla v_{n}$ is uniformly bounded in $C^{0, \alpha}\left(\bar{\Omega}_{g_{n}}, \mathbb{R}^{2} \times \mathbb{R}^{2}\right)$ for all $\alpha \in(0,1 / 2)$.

Lemma 5.8. Let $(h, u) \in X_{\text {reg }}\left(u_{0}\right)$ be a critical point of $\mathcal{F}$ such that $0<h<R_{0}$, and let $\left(g_{n}, v_{n}\right)$ be as in the previous lemma with $\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|=o\left(\sqrt{\varepsilon_{n}}\right)$ if $\varepsilon_{n}$ is not identically zero and $\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|=o(1)$ if $\varepsilon_{n}=0$ for all $n$. Suppose that $\nabla v_{n} \rightharpoonup \nabla u$ weakly in $L_{\mathrm{loc}}^{2}\left(\Omega_{h} ; \mathbb{R}^{2} \times \mathbb{R}^{2}\right)$ and

$$
\lim _{n \rightarrow \infty} \int_{\Omega_{g_{n}}} Q\left(E\left(v_{n}\right)\right) d z=\int_{\Omega_{h}} Q(E(u)) d z
$$

Then $g_{n} \in C^{1,1}(\mathbb{R})$ and $g_{n} \rightarrow h$ in $C^{1,1}(\mathbb{R})$, for $n$ sufficiently large.
Proof. From Lemma 5.7 we know that $g_{n} \rightarrow h$ in $C^{1, \alpha}(\mathbb{R})$. Therefore for large $n$ there exist diffeomorphisms $\Phi_{n}: \bar{\Omega}_{g_{n}} \rightarrow \bar{\Omega}_{h}$ such that $\Phi_{n} \rightarrow i d$ in $C^{1, \alpha}$. Let $B_{R}$ be any ball of radius $R \in\left(R_{0}-\max _{\mathbb{R}} h, R_{0}\right)$. Since

$$
\sup _{n \in \mathbb{N}}\left\{\left\|v_{n}\right\|_{C^{1, \alpha}\left(\bar{\Omega}_{g_{n}}\right)}\right\}<\infty
$$

by the convergence $\nabla v_{n} \rightharpoonup \nabla u$ we have that

$$
\begin{equation*}
\nabla v_{n} \circ \Phi_{n}^{-1} \rightarrow \nabla u \quad \text { in } C^{0, \alpha}\left(\bar{\Omega}_{h} \cap B_{R} ; \mathbb{M}^{2 \times 2}\right) \tag{5.12}
\end{equation*}
$$

To prove the claim set $I_{n}:=\left\{\theta \in[0,2 \pi] \mid g_{n}(\theta)<h(\theta)+1 / n=: h_{n}(\theta)\right\}$. Since $I_{n}$ is open, we may write $I_{n}=\bigcup_{i=1}^{\infty}\left(a_{i}^{n}, b_{i}^{n}\right)$. Notice that

$$
\begin{equation*}
g_{n}^{\prime}(\theta)=h_{n}^{\prime}(\theta)=h^{\prime}(\theta) \quad \text { on }[0,2 \pi] \backslash I_{n} \tag{5.13}
\end{equation*}
$$

If $I_{n}$ is empty, the claim is trivial. Therefore we may assume that $I_{n} \neq \emptyset$. Since $g_{n} \in C^{1, \alpha}(\mathbb{R})$, we can write the Euler-Lagrange equation for $\left(g_{n}, v_{n}\right)$ in the weak sense:

$$
\begin{equation*}
k_{g_{n}}(\theta)=Q\left(E\left(v_{n}\right)\right)\left(\theta, g_{n}(\theta)\right)+\beta_{n}\left(\theta, g_{n}(\theta)\right)+\lambda_{n}, \quad \theta \in I_{n} \tag{5.14}
\end{equation*}
$$

Here

$$
\beta_{n}=\frac{\Lambda\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|}{\sqrt{\left(\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}}} \operatorname{sign}\left(\chi_{\Omega_{h}}-\chi_{\Omega_{g_{n}}}\right)
$$

and $\lambda_{n}$ is some Lagrange multiplier. Notice that from the assumptions it follows that

$$
\begin{equation*}
\left|\beta_{n}\right|=\frac{\Lambda\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|}{\sqrt{\left(\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}}} \leq \Lambda \frac{\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|}{\sqrt{\varepsilon_{n}}} \rightarrow 0 \tag{5.15}
\end{equation*}
$$

Recall the Euler-Lagrange equation for $(h, u)$

$$
\begin{equation*}
k_{h}(\theta)=Q(E(u))(\theta, h(\theta))+\lambda_{\infty} \tag{5.16}
\end{equation*}
$$

We will show that $\lambda_{n} \rightarrow \lambda_{\infty}$. Notice that for the curvature in polar coordinates it holds that

$$
k_{g_{n}} g_{n}=\frac{g_{n}^{2}+2 g_{n}^{\prime 2}-g_{n} g_{n}^{\prime \prime}}{\left(g_{n}^{2}+g_{n}^{\prime 2}\right)^{\frac{3}{2}}} g_{n}=-\left(\frac{g_{n}^{\prime}}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}}\right)^{\prime}+\frac{g_{n}}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}}
$$

Hence, multiplying (5.14) by $g_{n}$, integrating over $I_{n}$ and using (5.16) yield

$$
\begin{aligned}
\int_{I_{n}} & {\left[Q\left(E\left(v_{n}\right)\right)\left(\theta, g_{n}(\theta)\right)+\beta_{n}\left(\theta, g_{n}(\theta)\right)+\lambda_{n}\right] g_{n} d \theta=\int_{I_{n}} k_{g_{n}} g_{n} d \theta } \\
& =\int_{I_{n}}-\left(\frac{g_{n}^{\prime}}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}}\right)^{\prime}+\frac{g_{n}}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}} d \theta \\
& =\sum_{i=1}^{\infty}-\left(\frac{g_{n}^{\prime}\left(b_{i}^{n}\right)}{\sqrt{g_{n}^{2}\left(b_{i}^{n}\right)+g_{n}^{\prime 2}\left(b_{i}^{n}\right)}}-\frac{g_{n}^{\prime}\left(a_{i}^{n}\right)}{\sqrt{g_{n}^{2}\left(a_{i}^{n}\right)+g_{n}^{\prime 2}\left(a_{i}^{n}\right)}}\right)+\int_{a_{i}^{n}}^{b_{i}^{n}} \frac{g_{n}}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}} d \theta \\
& =\sum_{i=1}^{\infty}-\left(\frac{h_{n}^{\prime}\left(b_{i}^{n}\right)}{\sqrt{h_{n}^{2}\left(b_{i}^{n}\right)+h_{n}^{\prime 2}\left(b_{i}^{n}\right)}}-\frac{h_{n}^{\prime}\left(a_{i}^{n}\right)}{\sqrt{h_{n}^{2}\left(a_{i}^{n}\right)+h_{n}^{\prime 2}\left(a_{i}^{n}\right)}}\right)+\int_{a_{i}^{n}}^{b_{i}^{n}} \frac{g_{n}}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}} d \theta \\
& =\int_{I_{n}} k_{h_{n}} h_{n} d \theta+\int_{I_{n}} \frac{g_{n}}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}}-\frac{h_{n}}{\sqrt{h_{n}^{2}+h_{n}^{\prime 2}}} d \theta \\
& =\int_{I_{n}}\left[Q(E(u))(\theta, h(\theta))+\lambda_{\infty}\right] h d \theta+\int_{I_{n}}\left(k_{h_{n}} h_{n}-k_{h} h\right)+\frac{g_{n}}{\sqrt{g_{n}^{2}+g_{n}^{\prime 2}}}-\frac{h_{n}}{\sqrt{h_{n}^{2}+h_{n}^{\prime 2}}} d \theta .
\end{aligned}
$$

Recall that $h_{n}=h+1 / n$. Therefore by (5.12), (5.15) and the previous calculations we conclude that

$$
\lim _{n \rightarrow \infty} \frac{1}{\left|I_{n}\right|} \int_{I_{n}} \lambda_{n} g_{n}-\lambda_{\infty} h d \theta=0
$$

which clearly implies $\lambda_{n} \rightarrow \lambda_{\infty}$.
From (5.13) and (5.14) we conclude that $g_{n} \in C^{1,1}(\mathbb{R})$. Moreover by the equations (5.13), (5.14) and (5.16) together with the convergences (5.12), (5.15) and $\lambda_{n} \rightarrow \lambda_{\infty}$ we conclude that

$$
k_{g_{n}} \rightarrow k_{h} \quad \text { in } L^{\infty}
$$

This in turn gives us the convergence

$$
g_{n}^{\prime \prime} \rightarrow h^{\prime \prime} \quad \text { in } L^{\infty}
$$

Now we are in position to prove the main theorem of this section.
Proof of Theorem 5.1. Step 1: We show first that $(h, u)$ is a strict local minimizer, i.e., we prove the claim without the estimate on the right-hand side of (5.1).

Observe that from the results of the previous section we may assume that $(h, u)$ is a $C^{1,1}$-local minimizer. The result will follow once we prove that the $C^{1,1}$-local minimality implies the local minimality. Arguing by contradiction let us assume that for any $n \in \mathbb{N}$ there exist $\left(h_{n}, u_{n}\right) \in X\left(u_{0}\right)$ with $\left|\Omega_{h_{n}}\right|=\left|\Omega_{h}\right|$ such that

$$
\mathcal{F}\left(h_{n}, u_{n}\right) \leq \mathcal{F}(h, u) \quad \text { and } \quad 0<d_{\mathcal{H}}\left(\Gamma_{h_{n}} \cup \Sigma_{h_{n}}, \Gamma_{h}\right) \leq \frac{1}{n}
$$

Consider the sequence $\left(g_{n}, v_{n}\right) \in X\left(u_{0}\right)$ of minimizers of the following penalized obstacle problem

$$
\min \left\{\mathcal{F}(g, v)+\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right|:(g, v) \in X\left(u_{0}\right), g \leq h+\frac{1}{n}\right\}
$$

for some large $\Lambda$. Since $\left(h_{n}, u_{n}\right)$ and $(h, u)$ are clearly competitors, we have that

$$
\mathcal{F}\left(g_{n}, v_{n}\right) \leq \mathcal{F}\left(h_{n}, u_{n}\right) \leq \mathcal{F}(h, u)
$$

By the contradiction assumption we may assume that $\left(h_{n}, u_{n}\right) \neq(h, u)$.
By the compactness property of $X\left(u_{0}\right)$ there exists $(g, v)$ such that, up to subsequences, $\left(g_{n}, v_{n}\right) \rightarrow(g, v)$ in $X\left(u_{0}\right)$. Let $(f, w) \in X\left(u_{0}\right)$ with $f \leq h$, by the lower semicontinuity of $\mathcal{F}$ and the minimality of $\left(g_{n}, v_{n}\right)$, we get

$$
\begin{align*}
\mathcal{F}(g, v)+\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right| & \leq \liminf _{n \rightarrow \infty}\left[\mathcal{F}\left(g_{n}, v_{n}\right)+\Lambda| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|\right|\right]  \tag{5.17}\\
& \leq \mathcal{F}(f, w)+\Lambda| | \Omega_{f}\left|-\left|\Omega_{h}\right|\right|
\end{align*}
$$

Choosing $(f, w)=(h, v)$ in the previous inequality, we obtain that

$$
\begin{equation*}
\mathcal{H}^{1}\left(\Gamma_{g}\right)+\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right| \leq \mathcal{H}^{1}\left(\Gamma_{h}\right) \tag{5.18}
\end{equation*}
$$

When $\Lambda$ is sufficiently large, (5.18) and Lemma 5.2 imply that $g=h$. Moreover, we observe that from (5.17) it follows that $(h, v)$ minimizes $\mathcal{F}$ in the class of all $(f, w) \in X\left(u_{0}\right)$ with $f=h$. In particular $v$ must coincide with the elastic equilibrium $u$.

Choosing $(f, w)=(h, u)$ in (5.17), using the lower semicontinuity of $g \mapsto \mathcal{H}^{1}\left(\Gamma_{g}\right)$ with respect to the $L^{1}$-convergence and the lower semicontinuity of the elastic energy with respect to the weak $H^{1}$-convergence, we deduce

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \mathcal{H}^{1}\left(\Gamma_{g_{n}} \cup \Sigma_{g_{n}}\right)=\mathcal{H}^{1}\left(\Gamma_{h}\right) \\
\lim _{n \rightarrow \infty} \int_{\Omega_{g_{n}}} Q\left(E\left(v_{n}\right)\right) d z=\int_{\Omega_{h}} Q(E(u)) d z
\end{gathered}
$$

From Lemma 5.8 we get $g_{n} \rightarrow h$ in $C^{1,1}(\mathbb{R})$.
We only need to modify $g_{n}$ such that it satisfies the volume constraint. We simply define $\tilde{g}_{n}(\theta):=g_{n}(\theta)+\delta_{n}$ where $\delta_{n}$ are chosen so that $\left|\Omega_{\tilde{g}_{n}}\right|=\left|\Omega_{h}\right|$. By Lemma 5.5 it holds $\left|\Omega_{g_{n}}\right| \geq\left|\Omega_{h}\right|$
and therefore $\delta_{n} \geq 0$ and $\Omega_{\tilde{g}_{n}} \subset \Omega_{g_{n}}$. Hence $v_{n}$ is well defined in $\Omega_{\tilde{g}_{n}}$ and $\left(\tilde{g}_{n}, v_{n}\right)$ is an admissible pair.

Since $h>0$ and $g_{n} \rightarrow h$ uniformly, we have $g_{n}>\gamma$ for some $\gamma>0$ independent from $n$ and $\delta_{n} \rightarrow 0$. We may estimate

$$
\begin{aligned}
\mathcal{H}^{1}\left(\Gamma_{\tilde{g}_{n}}\right)-\mathcal{H}^{1}\left(\Gamma_{g_{n}}\right) & =\int_{0}^{2 \pi} \sqrt{\left(g_{n}+\delta_{n}\right)^{2}+g_{n}^{\prime 2}}-\sqrt{g_{n}^{2}+g_{n}^{\prime 2}} d \theta \\
& \leq \frac{1}{\gamma} \int_{0}^{2 \pi} 2 g_{n} \delta_{n}+\delta_{n}^{2} d \theta
\end{aligned}
$$

and

$$
\left|\left|\Omega_{\tilde{g}_{n}}\right|-\left|\Omega_{g_{n}}\right|\right|=\frac{1}{2} \int_{0}^{2 \pi}\left(g_{n}+\delta_{n}\right)^{2}-g_{n}^{2} d \theta=\frac{1}{2} \int_{0}^{2 \pi} 2 g_{n} \delta_{n}+\delta_{n}^{2} d \theta
$$

Therefore whenever $\Lambda \geq \frac{2}{\gamma}$ we have

$$
\begin{equation*}
\mathcal{H}^{1}\left(\Gamma_{\tilde{g}_{n}}\right)-\mathcal{H}^{1}\left(\Gamma_{g_{n}}\right) \leq \Lambda| | \Omega_{\tilde{g}_{n}}\left|-\left|\Omega_{g_{n}}\right|\right| . \tag{5.19}
\end{equation*}
$$

The claim now follows, since by the choice of $\tilde{g}_{n}$ and by (5.19) we have

$$
\begin{aligned}
\mathcal{F}\left(\tilde{g}_{n}, v_{n}\right) & =\mathcal{F}\left(\tilde{g}_{n}, v_{n}\right)+\Lambda| | \Omega_{\tilde{g}_{n}}\left|-\left|\Omega_{h}\right|\right| \\
& \leq \mathcal{F}\left(g_{n}, v_{n}\right)+\Lambda| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|\right| \leq \mathcal{F}\left(h_{n}, u_{n}\right) \leq \mathcal{F}(h, u)
\end{aligned}
$$

This contradicts the fact that $(h, u)$ is a strict $C^{1,1}$-local minimizer.
Step 2: We will now prove the theorem. The proof is very similar to the first step. Arguing by contradiction we assume that there are $\left(h_{n}, u_{n}\right) \in X\left(u_{0}\right)$ with $\left|\Omega_{h_{n}}\right|=\left|\Omega_{h}\right|$ such that

$$
\mathcal{F}\left(h_{n}, u_{n}\right) \leq \mathcal{F}(h, u)+c_{0}\left|\Omega_{h_{n}} \Delta \Omega_{h}\right|^{2} \quad \text { and } \quad 0<d_{\mathcal{H}}\left(\Gamma_{h_{n}} \cup \Sigma_{g_{n}}, \Gamma_{h}\right) \leq \frac{1}{n}
$$

Denote $\varepsilon_{n}:=\left|\Omega_{h_{n}} \Delta \Omega_{h}\right|$. Notice that $d_{\mathcal{H}}\left(\Gamma_{h_{n}} \cup \Sigma_{g_{n}}, \Gamma_{h}\right) \rightarrow 0$ implies $\chi_{\Omega_{h_{n}}} \rightarrow \chi_{\Omega_{h}}$ in $L^{1}$ and therefore $\varepsilon_{n} \rightarrow 0$.

This time we replace the contradicting sequence $\left(h_{n}, u_{n}\right)$ by $\left(g_{n}, v_{n}\right) \in X\left(u_{0}\right)$ which minimizes

$$
\min \left\{\mathcal{F}(g, v)+\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right|+\sqrt{\left(\left|\Omega_{g} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}}:(g, v) \in X\left(u_{0}\right), g \leq h+\frac{1}{n}\right\}
$$

By compactness we may assume that, up to a subsequence, $\left(g_{n}, v_{n}\right) \rightarrow(g, v)$ in $X\left(u_{0}\right)$. By a completely similar argument as in Step 1 we conclude that $(g, v)=(h, u)$ whenever $\Lambda$ is sufficiently large. Moreover, we have that

$$
\begin{gathered}
\lim _{n \rightarrow \infty} \mathcal{H}^{1}\left(\Gamma_{g_{n}} \cup \Sigma_{g_{n}}\right)=\mathcal{H}^{1}\left(\Gamma_{h}\right), \\
\lim _{n \rightarrow \infty} \int_{\Omega_{g_{n}}} Q\left(E\left(v_{n}\right)\right) d z=\int_{\Omega_{h}} Q(E(u)) d z
\end{gathered}
$$

To conclude that $g_{n} \rightarrow h$ in $C^{1,1}(\mathbb{R})$, we will prove that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|}{\varepsilon_{n}}=1 \tag{5.20}
\end{equation*}
$$

and apply Lemma 5.8.
Suppose that (5.20) were false. Then there exists $c>0$ such that $\left|\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|-\varepsilon_{n}\right| \geq c \varepsilon_{n}$. Using the minimality of $\left(g_{n}, v_{n}\right)$ and the contradiction assumption for $\left(h_{n}, u_{n}\right)$, we obtain

$$
\begin{align*}
\mathcal{F}\left(g_{n}, v_{n}\right)+ & \Lambda\left|\left|\Omega_{g_{n}}\right|-\left|\Omega_{h}\right|\right|+\sqrt{\left(\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}} \\
& \leq \mathcal{F}\left(h_{n}, u_{n}\right)+\sqrt{\varepsilon_{n}}  \tag{5.21}\\
& <\mathcal{F}(h, u)+c_{0} \varepsilon_{n}^{2}+\sqrt{\varepsilon_{n}} .
\end{align*}
$$

Now we observe that from [13, Proposition 6.1], for $\Lambda$ sufficiently large, $(h, u)$ is also a minimizer of the penalized problem

$$
\mathcal{F}(g, v)+\Lambda| | \Omega_{g}\left|-\left|\Omega_{h}\right|\right| .
$$

Hence we have

$$
\begin{equation*}
\mathcal{F}(h, u) \leq \mathcal{F}\left(g_{n}, v_{n}\right)+\Lambda| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|\right| . \tag{5.22}
\end{equation*}
$$

Combining (5.21) and (5.22) we get

$$
\sqrt{c^{2} \varepsilon_{n}^{2}+\varepsilon_{n}} \leq \sqrt{\left(\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}}<c_{0} \varepsilon_{n}^{2}+\sqrt{\varepsilon_{n}}
$$

which is a contradiction since $\varepsilon_{n} \rightarrow 0$ proving (5.20).
Arguing as in (5.21) and by using (5.20) we obtain

$$
\begin{align*}
\mathcal{F}\left(g_{n}, v_{n}\right)+\Lambda| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|\right| & \leq \mathcal{F}\left(h_{n}, u_{n}\right)+\sqrt{\varepsilon_{n}}-\sqrt{\left(\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|-\varepsilon_{n}\right)^{2}+\varepsilon_{n}} \\
& <\mathcal{F}(h, u)+c_{0} \varepsilon_{n}^{2}  \tag{5.23}\\
& \leq \mathcal{F}(h, u)+2 c_{0}\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|^{2}
\end{align*}
$$

when $n$ is large.
As in Step 1 define $\tilde{g}_{n}(\theta):=g_{n}(\theta)+\delta_{n}$ where $\delta_{n} \geq 0$ are such that $\left|\Omega_{\tilde{g}_{n}}\right|=\left|\Omega_{h}\right|$. By choosing $\Lambda$ large enough we have

$$
\begin{equation*}
\mathcal{H}^{1}\left(\Gamma_{\tilde{g}_{n}}\right)-\mathcal{H}^{1}\left(\Gamma_{g_{n}}\right) \leq \frac{\Lambda}{2}| | \Omega_{\tilde{g}_{n}}\left|-\left|\Omega_{g_{n}}\right|\right| . \tag{5.24}
\end{equation*}
$$

Therefore since

$$
\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|^{2} \leq 2\left|\Omega_{\tilde{g}_{n}} \Delta \Omega_{h}\right|^{2}+2\left|\Omega_{\tilde{g}_{n}} \Delta \Omega_{g_{n}}\right|^{2}=2\left|\Omega_{\tilde{g}_{n}} \Delta \Omega_{h}\right|^{2}+2| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|^{2}\right.
$$

we have by (5.23) and (5.24) that

$$
\begin{aligned}
\mathcal{F}\left(\tilde{g}_{n}, v_{n}\right) & \leq \mathcal{F}\left(g_{n}, v_{n}\right)+\frac{\Lambda}{2}| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|\right| \\
& <\mathcal{F}(h, u)+2 c_{0}\left|\Omega_{g_{n}} \Delta \Omega_{h}\right|^{2}-\frac{\Lambda}{2}| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|\right| \\
& \leq \mathcal{F}(h, u)+4 c_{0}\left|\Omega_{\tilde{g}_{n}} \Delta \Omega_{h}\right|^{2}-\frac{\Lambda}{2}| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|\right|+4 c_{0}| | \Omega_{g_{n}}\left|-\left|\Omega_{h}\right|\right|^{2} \\
& \leq \mathcal{F}(h, u)+4 c_{0}\left|\Omega_{\tilde{g}_{n}} \Delta \Omega_{h}\right|^{2},
\end{aligned}
$$

when $n$ is sufficiently large. This contradicts Proposition 4.1 when $c_{0}$ is chosen to be small enough.

## 6. The case of the disk

In this section we consider the particular case when a radial stretching is applied to a material with round cavity $F=\bar{B}_{r}$. We prove that the disk remains stable under small radial stretching. This result is similar to the case of flat configuration in [14]. The main difference to the flat case, where the minimal shape is a rectangle, is that the curvature of the disk is nonzero and therefore the second variation formula becomes considerably more complicated. Instead of trying to explicitly write the second variation, we use fine estimates to find a range of stability.

The Dirichlet boundary condition has the form of radial stretching,

$$
\begin{equation*}
u_{0}(\rho \sigma(\theta))=\alpha R_{0} \sigma(\theta) \quad \text { for } \rho \geq R_{0} \tag{6.1}
\end{equation*}
$$

where $\alpha \in \mathbb{R}$ is some constant. The region occupied by the elastic material is the annulus $A\left(R_{0}, r\right):=B_{R_{0}} \backslash \bar{B}_{r}$. For $u_{0}$ as above we say that $(h, u) \in X\left(u_{0}\right)$ is a round configuration if $h(\theta) \equiv r$ and $u$ is the elastic equilibrium associated to $h$.

For the next theorem we define

$$
\beta(t):=1+\frac{\mu+\lambda}{\mu} \frac{t^{2}}{R_{0}^{2}}
$$

Recall also the definition of the ellipticity constant $\eta=\min \{\mu, \mu+\lambda\}$.

Theorem 6.1. Let

$$
r_{0}:=\sup \left\{t \leq R_{0} \left\lvert\,\left(1+t^{2}\right) \log \left(\frac{R_{0}}{t}\right) \geq \frac{\eta}{4 \mu}\right.\right\}
$$

and define the function $G: \mathbb{R} \rightarrow\left[-\infty, R_{0}\right)$ as

$$
G(\alpha):=\sup \left\{t \leq R_{0} \left\lvert\, t \log \left(\frac{R_{0}}{t}\right) \beta^{2}(t) \geq \frac{\eta}{32(\mu+\lambda)^{2} \alpha^{2}}\right.\right\}
$$

If $r \in\left(r_{0}, R_{0}\right)$ and $\alpha \in \mathbb{R}$ satisfy

$$
\begin{equation*}
r>G(\alpha) \tag{6.2}
\end{equation*}
$$

then the round configuration is a strict local minimizer of $\mathcal{F}$ under the volume constraint.
The elastic equilibrium $u$ can be explicitly calculated. Indeed, because of the symmetry we can write

$$
u(\rho \sigma(\theta))=f(\rho) \sigma(\theta)
$$

and applying the first equation in (2.7) we have

$$
f^{\prime \prime}(\rho)+\frac{f^{\prime}(\rho)}{\rho}-\frac{f(\rho)}{\rho^{2}}=0
$$

This can be easily solved

$$
f(\rho)=\frac{a}{\rho}+b \rho,
$$

for some $a, b \in \mathbb{R}$. To find $a$ and $b$ observe that
$\mathbb{C} E(u)=2 \mu\left(\begin{array}{ll}f^{\prime}(\rho) \cos ^{2} \theta+\frac{f(\rho)}{\rho} \sin ^{2} \theta & \left(f^{\prime}(\rho)-\frac{f(\rho)}{\rho}\right) \sin \theta \cos \theta \\ \left(f^{\prime}(\rho)-\frac{f(\rho)}{\rho}\right) \sin \theta \cos \theta & f^{\prime}(\rho) \sin ^{2} \theta+\frac{f(\rho)}{\rho} \cos ^{2} \theta\end{array}\right)+\lambda\left(f^{\prime}(\rho)+\frac{f(\rho)}{\rho}\right)\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$.
Therefore, the second equation in (2.7) gives

$$
(2 \mu+\lambda) f^{\prime}(r)+\lambda \frac{f(r)}{r}=0
$$

This and the Dirichlet condition (6.1) yield

$$
\begin{equation*}
\frac{a}{r^{2}}=\frac{\mu+\lambda}{\mu} b \quad \text { and } \quad b=\frac{\alpha}{\beta(r)} \tag{6.4}
\end{equation*}
$$

It is trivial to check that the round configuration is a critical point of $\mathcal{F}$. To prove Theorem 6.1 we need to show that the round configuration is a point of positive second variation. To this aim, let us explicitly write the quadratic form (3.16). By (6.3) and (6.4) we have

$$
\mathbb{C} E(u)=4 b(\mu+\lambda)\left(\begin{array}{cc}
\sin ^{2} \theta & -\sin \theta \cos \theta \\
-\sin \theta \cos \theta & \cos ^{2} \theta
\end{array}\right)=4 b(\mu+\lambda) \tau \otimes \tau
$$

on the boundary $\partial B_{r}$. Hence, for $\psi \in H_{\#}^{1}(\mathbb{R})$, we have

$$
\operatorname{div}_{\tau}(\langle\underline{\psi}, \nu\rangle \mathbb{C} E(u))=4 b(\mu+\lambda)\left(-\langle\underline{\psi}, \nu\rangle \nu+\partial_{\tau}\langle\underline{\psi}, \nu\rangle \tau\right)
$$

and the equation (3.17) for $u_{\psi}$ becomes

$$
\begin{equation*}
\int_{A(R, r)} \mathbb{C} E\left(u_{\psi}\right): E(w) d z=-4 b(\mu+\lambda) \int_{\partial B_{r}}\left(-\langle\underline{\psi}, \nu\rangle\langle w, \nu\rangle+\left(\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right)\langle w, \tau\rangle\right) d \mathcal{H}^{1} \tag{6.5}
\end{equation*}
$$

Moreover, in the case of a round configuration the elastic energy is

$$
\begin{equation*}
Q(E(u))=2(\mu+\lambda) b^{2}+2 \mu \frac{a^{2}}{\rho^{4}} \tag{6.6}
\end{equation*}
$$

and therefore, by (6.4), we get

$$
\partial_{\nu} Q(E(u))=-\frac{8(\mu+\lambda)^{2}}{\mu} \frac{b^{2}}{r} \quad \text { on } \quad \partial B_{r}
$$

Hence, (3.16) becomes

$$
\begin{align*}
\partial^{2} \mathcal{F}(h, u)[\psi]= & -\int_{A(R, r)} 2 Q\left(E\left(u_{\psi}\right)\right) d z+\int_{\partial B_{r}}\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2} d \mathcal{H}^{1} \\
& +\int_{\partial B_{r}}\left(\frac{8(\mu+\lambda)^{2}}{\mu} \frac{b^{2}}{r}-\frac{1}{r^{2}}\right)\langle\underline{\psi}, \nu\rangle^{2} d \mathcal{H}^{1} \tag{6.7}
\end{align*}
$$

where $u_{\psi} \in \mathcal{A}\left(B_{R} \backslash \bar{B}_{r}\right)$ solves (6.5), and $\psi$ satisfies $\int_{0}^{2 \pi} \psi d \theta=0$.
Now the goal is to prove that (6.7) is positive whenever the assumptions of Theorem 6.1 are satisfied. The main obstacle is to bound the first term in (6.7) which will be done by using the equation (6.5). To this aim we need the following simple lemma, which we prove to keep track of the optimal constant.

Lemma 6.2. Suppose that $v \in W^{1,2}\left(A\left(R_{0}, r\right) ; \mathbb{R}^{2}\right)$ is a continuous map with $v=0$ on $\partial B_{R_{0}}$ and $A$ is a matrix. Then for $w(z)=v(z)+A z$ we have that

$$
\int_{\partial B_{r}}|w|^{2} d \mathcal{H}^{1} \leq r \log \left(\frac{R_{0}}{r}\right) \int_{A\left(R_{0}, r\right)}\left|D v-\frac{r}{R_{0}-r} A\right|^{2} d z
$$

Proof. Consider $w$ in polar coordinates. Fix an angle $\theta$ and integrate over $\left[r, R_{0}\right]$

$$
A R_{0} \sigma(\theta)-w(r \sigma(\theta))=\int_{r}^{R_{0}} D w(\rho \sigma(\theta)) \sigma(\theta) d \rho
$$

which implies

$$
|w(r \sigma(\theta))| \leq \int_{r}^{R_{0}}\left|D v(\sigma(\theta))-\frac{r}{R_{0}-r} A\right| d \rho
$$

Integrate over $\theta$ and use Hölder's inequality to obtain

$$
\begin{aligned}
\int_{0}^{2 \pi}|w(\rho \sigma(\theta))|^{2} d \theta & \leq \int_{0}^{2 \pi}\left(\int_{r}^{R_{0}}\left|D v(\rho \sigma(\theta))-\frac{r}{R_{0}-r} A\right| d \rho\right)^{2} d \theta \\
& \leq \int_{0}^{2 \pi}\left(\int_{r}^{R_{0}} \frac{1}{\rho} d \rho \cdot \int_{r}^{R_{0}}\left|D v(\rho \sigma(\theta))-\frac{r}{R_{0}-r} A\right|^{2} \rho d \rho\right) d \theta \\
& =\log \left(\frac{R_{0}}{r}\right) \int_{A\left(R_{0}, r\right)}\left|D v-\frac{r}{R_{0}-r} A\right|^{2} d z
\end{aligned}
$$

The inequality follows from $\int_{\partial B_{r}}|w|^{2} d \mathcal{H}^{1}=r \int_{0}^{2 \pi}|w(r, \theta)|^{2} d \theta$.
Proof of Theorem 6.1. As we stated before, by the local minimality criterion it is enough to prove that the second variation of $\mathcal{F}$ at $(h, u)$ is positive. Suppose that $\psi \in H_{\#}^{1}(\mathbb{R})$ satisfies $\int_{0}^{2 \pi} \psi d \theta=0$ and $\psi \neq 0$. Without loss of generality we may assume $\psi$ to be smooth. To estimate the first term in (6.7) we claim that

$$
\begin{equation*}
2 \int_{A\left(R_{0}, r\right)} Q\left(E\left(u_{\psi}\right)\right) d z \leq \frac{32(\mu+\lambda)^{2} b^{2}}{\eta} r \log \left(\frac{R_{0}}{r}\right) \int_{\partial B_{r}}\langle\underline{\psi}, \nu\rangle^{2}+\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2} d \mathcal{H}^{1} \tag{6.8}
\end{equation*}
$$

To this aim, choose $w(z)=u_{\psi}(z)+A z$ as a test function in (6.5) where $A$ is antisymmetric, to obtain

$$
\begin{align*}
& 2 \int_{A\left(R_{0}, r\right)} Q\left(E\left(u_{\psi}\right)\right) d z=-4 b(\mu+\lambda) \int_{\partial B_{r}}\left(-\langle\underline{\psi}, \nu\rangle\langle w, \nu\rangle+\partial_{\tau}\langle\underline{\psi}, \nu\rangle\langle w, \tau\rangle\right) d \mathcal{H}^{1} \\
& \leq 4 b(\mu+\lambda)\left(\int_{\partial B_{r}}\langle\underline{\psi}, \nu\rangle^{2}+\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2} d \mathcal{H}^{1}\right)^{1 / 2}\left(\int_{\partial B_{r}}|w|^{2} d \mathcal{H}^{1}\right)^{1 / 2} \tag{6.9}
\end{align*}
$$

Apply Lemma 6.2 to $w$ to get

$$
\begin{equation*}
\int_{\partial B_{r}}|w|^{2} d \mathcal{H}^{1} \leq r \log \left(\frac{R_{0}}{r}\right) \int_{A\left(R_{0}, r\right)}\left|D u_{\psi}-\frac{r}{R-r} A\right|^{2} d z \tag{6.10}
\end{equation*}
$$

Let $R_{k} \rightarrow \infty$ and for every $k$ choose an antisymmetric $A_{k}$ such that

$$
\int_{A\left(R_{k}, r\right)} D u_{\psi}-\frac{r}{R-r} A_{k} d z=\int_{A\left(R_{k}, r\right)} D u_{\psi}^{T}+\frac{r}{R-r} A_{k} d z
$$

By Theorem 2.5 we get

$$
\int_{A\left(R_{k}, r\right)}\left|D u_{\psi}-\frac{r}{R-r} A_{k}\right|^{2} d z \leq C_{k} \int_{A\left(R_{k}, r\right)}\left|E\left(u_{\psi}\right)\right|^{2} d z=C_{k} \int_{A(R, r)}\left|E\left(u_{\psi}\right)\right|^{2} d z
$$

Together with (6.10) this yields

$$
\int_{\partial B_{r}}|w|^{2} d \mathcal{H}^{1} \leq r \log \left(\frac{R_{0}}{r}\right) C_{k} \int_{A\left(R_{0}, r\right)}\left|E\left(u_{\psi}\right)\right|^{2} d z
$$

Since $C_{k} \rightarrow 4$ as $R_{k} \rightarrow \infty$ we have that

$$
\int_{\partial B_{r}}|w|^{2} d \mathcal{H}^{1} \leq \frac{4 r}{\eta} \log \left(\frac{R_{0}}{r}\right) \int_{A\left(R_{0}, r\right)} Q\left(E\left(u_{\psi}\right)\right) d z
$$

Now (6.8) follows from (6.9) and from the previous inequality.
We estimate (6.7) by using (6.8) and obtain

$$
\begin{align*}
\partial^{2} \mathcal{F}(h, u)[\psi] \geq & -32 \eta^{-1}(\mu+\lambda)^{2} b^{2} r \log \left(\frac{R_{0}}{r}\right) \int_{\partial B_{r}}\langle\underline{\psi}, \nu\rangle^{2}+\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2} d \mathcal{H}^{1} \\
& +\int_{\partial B_{r}}\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2} d \mathcal{H}^{1}+\int_{\partial B_{r}}\left(\frac{8(\mu+\lambda)^{2}}{\mu} \frac{b^{2}}{r}-\frac{1}{r^{2}}\right)\langle\underline{\psi}, \nu\rangle^{2} d \mathcal{H}^{1} \\
= & \int_{\partial B_{r}}\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2}-\frac{1}{r^{2}}\langle\underline{\psi}, \nu\rangle^{2} d \mathcal{H}^{1}  \tag{6.11}\\
& -32 \eta^{-1}(\mu+\lambda)^{2} b^{2} r \log \left(\frac{R_{0}}{r}\right) \int_{\partial B_{r}}\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2} d \mathcal{H}^{1} \\
& +\left(\frac{r}{\mu}-4 \eta^{-1} r^{3} \log \left(\frac{R_{0}}{r}\right)\right) 8(\mu+\lambda)^{2} b^{2} \int_{\partial B_{r}} \frac{1}{r^{2}}\langle\underline{\psi}, \nu\rangle^{2} d \mathcal{H}^{1} .
\end{align*}
$$

Let us first treat the last term in (6.11). For every $r>r_{0}$ we have that

$$
\partial^{2} \mathcal{F}(h, u)[\psi]>\left(1-32 \eta^{-1}(\mu+\lambda)^{2} b^{2} r \log \left(\frac{R_{0}}{r}\right)\right) \int_{\partial B_{r}}\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2}-\frac{1}{r^{2}}\langle\underline{\psi}, \nu\rangle^{2} d \mathcal{H}^{1} .
$$

Furthermore, if (6.2) is satisfied, then

$$
1-32 \eta^{-1}(\mu+\lambda)^{2} b^{2} r \log \left(\frac{R_{0}}{r}\right)>0
$$

By the definition (3.1) we see that $\langle\underline{\psi}, \nu\rangle=\psi\left(\sigma^{-1}\left(\frac{z}{|z|}\right)\right)$. Hence, by the Wirtinger's inequality, we get

$$
\int_{\partial B_{r}}\left|\partial_{\tau}\langle\underline{\psi}, \nu\rangle\right|^{2}-\frac{1}{r^{2}}\langle\underline{\psi}, \nu\rangle^{2} d \mathcal{H}^{1}=\frac{1}{r} \int_{0}^{2 \pi}\left|\psi^{\prime}(\theta)\right|^{2}-|\psi(\theta)|^{2} d \theta \geq 0
$$

which concludes the proof.
At the end of the section we study the global minimality of the round configuration. We begin with the following remark.

Remark 6.3. Suppose that $R_{0}$ and $r_{0}$ are as in Theorem 6.1 and fix $\alpha \in \mathbb{R}$ and a small $\varepsilon>0$. Then for every $r \in\left[r_{0}+\varepsilon, R_{0}\right]$ such that $r \geq G(\alpha)+\varepsilon$ the proof above actually gives

$$
\partial^{2} \mathcal{F}(h, u)[\psi] \geq c_{2} \int_{0}^{2 \pi}\left|\psi^{\prime}(\theta)\right|^{2}-|\psi(\theta)|^{2} d \theta+c_{1} \int_{0}^{2 \pi}|\psi(\theta)|^{2} d \theta
$$

for some small $0<c_{1}<c_{2}$, independent of $r$. Using the Wirtinger's inequality we get

$$
\partial^{2} \mathcal{F}(h, u)[\psi] \geq c_{0}\|\psi\|_{H^{1}([0,2 \pi))}^{2}
$$

for $c_{0}$ depending only on $R_{0}, r_{0}, \alpha$ and $\varepsilon$. This is a uniform version of Lemma 3.6.

We can use this uniform bound of the constant $c_{0}$ to prove a uniform local $C^{1,1}$-minimality of the round configuration for $r \in\left[r_{0}+\varepsilon, R_{0}\right]$ with $r \geq G(\alpha)+\varepsilon$. Indeed, arguing as in Proposition 4.1 and in Lemma 4.7 we conclude that there is $\delta>0$ such that for any $(g, v) \in X\left(u_{0}\right)$ with $\left|F_{g}\right|=\left|B_{r}\right|$ and $\|g-r\|_{C^{1,1}(\mathbb{R})} \leq \delta$ it holds

$$
\mathcal{F}(g, v) \geq \mathcal{F}\left(r, u_{r}\right)
$$

where $u_{r}$ stands for the elastic equilibrium associated to the disk $B_{r}$.
The previous remark enables us to prove the global minimality of the disk when the volume of the annulus is small.

Proposition 6.4. Suppose that $R_{0}$ is the radius of the large ball and $u_{0}$ is the Dirichlet boundary conditions as in (6.1) with fixed $\alpha>0$. There exists $r_{g l o b}<R_{0}$ such that for every $r \in\left(r_{g l o b}, R_{0}\right)$ the round configuration, with a disk $B_{r}$, is a global minimizer of $\mathcal{F}$ under the volume constraint.

Proof. We argue by contradiction and assume that there exist a sequence of radii $r_{n} \nearrow R_{0}$ and a sequence $\left(k_{n}, w_{n}\right) \in X\left(u_{0}\right)$ of minimizers of $\mathcal{F}$ under the volume constraint $\left|\Omega_{k_{n}}\right|=\left|A\left(R_{0}, r_{n}\right)\right|$ such that

$$
\mathcal{F}\left(k_{n}, w_{n}\right)<\mathcal{F}\left(r_{n}, u_{n}\right),
$$

where $u_{n}$ stands for the elastic equilibrium relative to $r_{n}$. Since $\left(k_{n}, w_{n}\right)$ minimizes $\mathcal{F}$ we immediately have that $\mathcal{H}^{1}\left(\Gamma_{k_{n}} \cup \Sigma_{k_{n}}\right) \rightarrow 2 \pi R_{0}$. Therefore, since $F_{k_{n}}$ is connected, we deduce that $\varepsilon_{n}:=d_{\mathcal{H}}\left(\Gamma_{k_{n}} \cup \Sigma_{k_{n}}, \Gamma_{r_{n}}\right) \rightarrow 0$ as $n \rightarrow \infty$.

We may calculate the elastic equilibrium

$$
u_{n}(\rho, \theta)=\left(\frac{a_{n}}{\rho}+b_{n} \rho\right) \sigma(\theta)
$$

where

$$
b_{n}=\left(1+\frac{\mu+\lambda}{\mu} \frac{r_{n}^{2}}{R_{0}^{2}}\right)^{-1} \alpha \quad \text { and } \quad a_{n}=\frac{\mu+\lambda}{\mu} r_{n}^{2} b_{n}
$$

By Remark 6.3 we have that for large $n$ it holds

$$
\partial^{2} \mathcal{F}\left(r_{n}, u_{n}\right)[\psi] \geq c_{0}\|\psi\|_{H^{1}\left(\partial B_{r_{n}}\right)}^{2}
$$

for $\int_{0}^{2 \pi} \psi d \theta=0$, where $c_{0}$ is independent of $n$.
We note that $u_{n}$ is also the elastic equilibrium in the annulus $A\left(R, r_{n}\right)$, for any $R>R_{0}$, with respect to its own boundary conditions on $\partial B_{R}, v(R, \theta)=u_{n}(R, \theta)$. For $R>R_{0}$ we define

$$
\mathcal{F}_{R}(g, v)=\int_{B_{R} \backslash F_{g}} Q(E(v)) d z+\mathcal{H}^{1}\left(\Gamma_{g}\right)+2 \mathcal{H}^{1}\left(\Sigma_{g}\right)
$$

and

$$
X_{R}\left(u_{n}\right)=\left\{(g, v) \mid g \in B V_{\#}(\mathbb{R}), v \in H_{\mathrm{loc}}^{1}\left(\mathbb{R}^{2} \backslash F_{g} ; \mathbb{R}^{2}\right), v=u_{n} \text { outside } B_{R}\right\}
$$

Consider the estimate (6.11) for $\partial^{2} \mathcal{F}_{R_{1}}\left(r_{n}, u_{n}\right)[\psi]$, i.e., replace $R_{0}, r$ and $b$ by $R_{1}, r_{n}$ and $b_{n}$. By continuity we may choose $R_{1}$ close to $R_{0}$ such that

$$
\partial^{2} \mathcal{F}_{R_{1}}\left(r_{n}, u_{n}\right)[\psi] \geq \frac{c_{0}}{2}\|\psi\|_{H^{1}\left(\partial B_{r_{n}}\right)}^{2}
$$

for $\int_{0}^{2 \pi} \psi d \theta=0$. Arguing as in Remark 6.3 we conclude that $\left(r_{n}, u_{n}\right)$ is a local $C^{1,1}$-minimizer of $\mathcal{F}_{R_{1}}$ uniformly in $n$, i.e., there is $\delta>0$, independent of $n$, such that for any $(g, v) \in X_{R_{1}}\left(u_{n}\right)$, with $\left\|g-r_{n}\right\|_{C^{1,1}(\mathbb{R})}<\delta$, it holds

$$
\begin{equation*}
\mathcal{F}_{R_{1}}(g, v) \geq \mathcal{F}_{R_{1}}\left(r_{n}, u_{n}\right) \tag{6.12}
\end{equation*}
$$

Define

$$
\tilde{w}_{n}(z):= \begin{cases}w_{n}(z) & \text { if } z \in \bar{B}_{R_{0}} \backslash F_{k_{n}} \\ u_{n}(z) & \text { if } z \in A\left(R_{1}, R_{0}\right)\end{cases}
$$

By the assumption on $\left(k_{n}, w_{n}\right)$ it holds

$$
\begin{equation*}
\mathcal{F}_{R_{1}}\left(k_{n}, \tilde{w}_{n}\right)<\mathcal{F}_{R_{1}}\left(r_{n}, u_{n}\right) \tag{6.13}
\end{equation*}
$$

Suppose that $\left(g_{n}, v_{n}\right)$ is a solution of the problem

$$
\min \left\{\mathcal{F}_{R_{1}}(g, v)+\Lambda| | F_{g}\left|-\left|B_{r_{n}}\right|\right|:(g, v) \in X_{R_{1}}\left(u_{n}\right), g \leq r_{n}+\varepsilon_{n}\right\}
$$

where $\Lambda$ is large. Arguing as in Lemma 5.6, Lemma 5.7 and Lemma 5.8 we conclude that $g_{n} \rightarrow R_{0}$ in $C^{1,1}(\mathbb{R})$. In particular, $\left\|g_{n}-r_{n}\right\|_{C^{1,1}(\mathbb{R})} \rightarrow 0$.

By the minimality of $\left(g_{n}, v_{n}\right)$ we have that $\mathcal{F}_{R_{1}}\left(g_{n}, v_{n}\right)+\Lambda| | F_{g}\left|-\left|B_{r_{n}}\right|\right| \leq \mathcal{F}_{R_{1}}\left(k_{n}, \tilde{w}_{n}\right)$. Defining $\tilde{g}_{n}=g_{n}+\delta_{n}$ such that $\left|F_{\tilde{g}_{n}}\right|=\left|B_{r_{n}}\right|$ we obtain, as in (5.19), that

$$
\begin{equation*}
\mathcal{F}_{R_{1}}\left(\tilde{g}_{n}, v_{n}\right) \leq \mathcal{F}_{R_{1}}\left(g_{n}, v_{n}\right)+\Lambda| | F_{g}\left|-\left|B_{r_{n}}\right|\right| \leq \mathcal{F}_{R_{1}}\left(k_{n}, \tilde{w}_{n}\right) \tag{6.14}
\end{equation*}
$$

when $\Lambda$ is large enough. Moreover $\delta_{n} \rightarrow 0$. Hence $\left\|\tilde{g}_{n}-r_{n}\right\|_{C^{1,1}(\mathbb{R})} \rightarrow 0$ and therefore (6.12), (6.13) and (6.14) imply

$$
\mathcal{F}_{R_{1}}\left(r_{n}, u_{n}\right) \leq \mathcal{F}_{R_{1}}\left(\tilde{g}_{n}, v_{n}\right) \leq \mathcal{F}_{R_{1}}\left(k_{n}, \tilde{w}_{n}\right)<\mathcal{F}_{R_{1}}\left(r_{n}, u_{n}\right),
$$

which is a contradiction.

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