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by H. J. Hupkes, and E. S. Van Vleck

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NEGATIVE DIFFUSION AND TRAVELING WAVES IN HIGH DIMENSIONAL LATTICE SYSTEMS*

H. J. HUPKES† AND E. S. VAN VLECK‡

Abstract. We consider bistable reaction diffusion systems posed on rectangular lattices in two or more spatial dimensions. The discrete diffusion term is allowed to have positive spatially periodic coefficients, and the two spatially periodic equilibria are required to be well ordered. We establish the existence of traveling wave solutions to such pure lattice systems that connect the two stable equilibria. In addition, we show that these waves can be approximated by traveling wave solutions to systems that incorporate both local and nonlocal diffusion. In certain special situations our results can also be applied to reaction diffusion systems that include (potentially large) negative coefficients. Indeed, upon splitting the lattice suitably and applying separate coordinate transformations to each sublattice, such systems can sometimes be transformed into a periodic diffusion problem that fits within our framework. In such cases, the resulting traveling structure for the original system has a separate wave profile for each sublattice and connects spatially periodic patterns that need not be well ordered. There is no direct analogue of this procedure that can be applied to reaction diffusion systems with continuous spatial variables.

Key words. traveling waves, lattice differential equations, comparison principles, negative diffusion, periodic diffusion

AMS subject classifications. 34K31, 37L60

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1. Introduction.

In this paper we consider the family of nonlocal systems

$$u_t(x,t) = \gamma u_{xx}(x,t) + \sum_{j=0}^{N} A_j[u(x+r_j,t) - u(x,t)] + f\big(u(x,t)\,;\rho\big), \tag{1.1}$$

parametrized by $\rho \in V \subset \mathbb{R}$. The diffusion constant satisfies $\gamma \geq 0$, the function u takes values in $\mathbb{R}^n$ for some $n \geq 2$, the real $(n \times n)$-matrices A_j have nonnegative entries, and the Jacobian $D_1 f(\cdot\,;\rho)$ has nonnegative off-diagonal elements. The shifts $r_0 < r_1 < \cdots < r_N$ can be taken to be both positive and negative, i.e., $r_0 < 0 < r_N$. We are interested in nonlinearities f that are bistable. In particular, writing $\mathbf{0} = (0,\ldots,0) \in \mathbb{R}^n$ and $\mathbf{1} = (1,\ldots,1) \in \mathbb{R}^n$, we assume that $f(\mathbf{0};\rho) = f(\mathbf{1};\rho) = \mathbf{0}$ are two stable equilibria for all $\rho \in V$ and that all other equilibria in the cube $[0,1]^n$ are unstable.

We are particularly interested in traveling wave solutions of (1.1) that connect the two stable equilibria. Such solutions can be written in the form $u(x,t) = \phi(x-ct)$ for some wave speed $c \in \mathbb{R}$ and some wave profile $\phi : \mathbb{R} \to \mathbb{R}^n$ that satisfies the limits

$$\lim_{\xi\to-\infty} \phi(\xi) = \mathbf{0}, \qquad \lim_{\xi\to+\infty} \phi(\xi) = \mathbf{1}. \tag{1.2}$$

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†Corresponding author. Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65203. Current address: Mathematical Institute, University of Leiden, P.O. Box 9512, 2300 RA Leiden, The Netherlands (hupkesh@missouri.edu, hhupkes@math.leidenuniv.nl).

‡Department of Mathematics, University of Kansas, Lawrence, KS 66045 (erikvv@ku.edu). This author was supported in part by the NSF through grant DMS-1115408.

It is not hard to see that this pair (ϕ, c) must satisfy the differential equation

$$-c\phi'(\xi) = \gamma\phi''(\xi) + \sum_{j=0}^{N} A_j[\phi(\xi + r_j, t) - \phi(\xi)] + f\big(\phi(\xi)\,;\rho\big). \tag{1.3}$$

Due to the presence of the shifts in the argument of ϕ that are both positive and negative, the system (1.3) is referred to as a functional differential equation of mixed type (MFDE).

Our contribution in this paper is to show that for each $\gamma \geq 0$, (1.1) has a family of traveling wave solutions, parametrized by $\rho \in V$. This family depends smoothly on the parameter ρ whenever $\gamma > 0$ or the wave speed c is nonzero. In addition, upon fixing the parameter ρ, traveling waves for (1.1) with $\gamma = 0$ can be approximated by a sequence of traveling waves for (1.1) with $\gamma = \gamma_n \downarrow 0$. As such, we generalize previous results obtained in [32, 25] for scalar versions of (1.3), i.e., where $n = 1$.

Lattice differential equations. Let us emphasize here that our interest in (1.1) is rather indirect. Indeed, our primary motivation for this paper comes from the study of differential equations posed on lattices in two or more spatial dimensions. Consider, for example, the system

$$\frac{d}{dt}u_{ij} = \alpha_{ij}[u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij}] + g(u_{ij}, \rho), \tag{1.4}$$

posed on the two-dimensional lattice $(i, j) \in \mathbb{Z}^2$. A typical smooth family of bistable nonlinearities is given by the cubics

$$g(u\,;\rho) = u(u - \rho)(1 - u) \tag{1.5}$$

with $0 < \rho < 1$. We now discuss a number of different scenarios for the diffusion coefficients α_{ij}.

Positive diffusion. In the spatially homogeneous case $\alpha_{ij} = \alpha > 0$, the lattice differential equation (LDE) (1.4) reduces to the system

$$\frac{d}{dt}u_{ij} = \alpha[u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij}] + g(u_{ij}, \rho), \tag{1.6}$$

which is often referred to as the two-dimensional discrete Nagumo equation. It has been used to describe phenomena such as phase transitions in Ising models [3] and to develop pattern recognition algorithms in image processing [11, 10]. Many authors have studied this LDE, focusing primarily on the richness of the set of equilibria [30] and the existence of traveling wave solutions [32, 43].

The LDE (1.6) with $\alpha = h^{-2}$ can be seen as the discretization of the PDE

$$\partial_t u = \Delta u + f(u) \tag{1.7}$$

on a two-dimensional grid with node spacing $h > 0$. However, the two equations are known to display significant differences in dynamical behavior, especially when $\alpha > 0$ is small and one is far away from the continuous limit. In order to illustrate this, let us consider waves that travel through the lattice in the direction $(\sigma_1, \sigma_2) = (\cos\theta, \sin\theta)$.

Substituting the Ansatz

$$u_{ij}(t) = \phi\big((i, j) \cdot (\sigma_1, \sigma_2) - ct\big) = \phi\big(i\sigma_1 + j\sigma_2 - ct\big) \tag{1.8}$$

into (1.6), we arrive at the system

$$-c\phi'(\xi) = \alpha[\phi(\xi+\sigma_1)+\phi(\xi+\sigma_2)+\phi(\xi-\sigma_1)+\phi(\xi-\sigma_2)-4\phi(\xi)]+g\big(\phi(\xi);\rho\big) = 0, \tag{1.9}$$

which is a scalar version of the MFDE (1.3) with $\gamma = 0$. As above, we require the limits

$$\lim_{\xi\to-\infty} \phi(\xi) = 0, \qquad \lim_{\xi\to+\infty} \phi(\xi) = 1. \tag{1.10}$$

Notice that the direction (σ_1, σ_2) appears explicitly in the traveling wave MFDE (1.9), which does not happen for the PDE (1.7). As a consequence, the LDE (1.6) admits spatial anisotropy in the sense that the wave speed c depends on the angle θ of propagation through the lattice. Numerical illustrations of this fact can be found in [25, 14, 27].

Notice furthermore that the traveling wave MFDE (1.9) becomes singular in the limit $c \to 0$. One of the consequences of this fact is that typically an entire range of values of ρ can exist for which the wave speed satisfies $c = 0$. This phenomenon is called propagation failure and does not occur for the PDE (1.7). It has been studied extensively in [5], where one replaces the cubic nonlinearity g by an idealized cartoon nonlinearity to obtain explicit solutions to (1.9). For each propagation angle θ, the quantity $\rho^*(\theta)$ is defined to be the supremum of values $\rho > \frac{1}{2}$ for which the wavespeed satisfies $c = 0$. It is proven that this critical value $\rho^*(\theta)$ typically satisfies $\rho^* > \frac{1}{2}$, depends continuously on θ when $\tan\theta$ is irrational, and is discontinuous when $\tan\theta$ is rational or infinite. By now there is plenty of numerical [14, 25] and theoretical [33, 21] evidence to suggest that this behavior is not just an artifact of the idealized nonlinearity g but also occurs in the case of the cubic nonlinearity (1.5).

Periodic diffusion. One of the advantages of using the discrete system (1.4) is that it is relatively easy to model spatial inhomogeneities. Many physical systems have a periodic spatial structure [17, 15, 36], so it is natural to study (1.4) with coefficients α_{ij} that vary in a periodic fashion. For example, let us suppose that $\alpha_{ij} = \alpha_o > 0$ whenever $i + j$ is odd and $\alpha_{ij} = \alpha_e > 0$ whenever $i + j$ is even, with $\alpha_o \neq \alpha_e$. Upon writing

$$u_{ij}(t) = \begin{cases} \phi_o\big(i\sigma_1 + j\sigma_2 - ct\big) & \text{for odd } i+j, \\ \phi_e\big(i\sigma_1 + j\sigma_2 - ct\big) & \text{for even } i+j, \end{cases} \tag{1.11}$$

we find the traveling wave MFDE

$$\begin{aligned} -c\phi_o'(\xi) &= \phi_e(\xi+\sigma_1) + \phi_e(\xi-\sigma_1) + \phi_e(\xi+\sigma_2) + \phi_e(\xi-\sigma_2) - 4\phi_o(\xi) + g\big(\phi_o(\xi);\rho\big), \\ -c\phi_e'(\xi) &= \phi_o(\xi+\sigma_1) + \phi_o(\xi-\sigma_1) + \phi_o(\xi+\sigma_2) + \phi_o(\xi-\sigma_2) - 4\phi_e(\xi) + g\big(\phi_e(\xi);\rho\big), \end{aligned} \tag{1.12}$$

which clearly can be written in the form (1.3) with $\gamma = 0$. Compared to (1.9), much less is known about (1.12). In section 3.1 we discuss this issue further and show how general periodic diffusion problems fit into our framework.

Negative diffusion. Although PDEs with negative diffusion are typically ill-posed, the discrete system (1.4) with $\alpha_{ij} = \alpha < 0$ does not suffer from this problem. In [39] phase transitions are discussed for a grid of particles that have visco-elastic interactions, which leads naturally to an LDE with negative diffusion. We refer to [4] for an analysis of this problem on a one-dimensional lattice. Earlier results that provide additional motivation for studying this type of antidiffusion can be found in [6, 7].

In section 3.2 we discuss a two-dimensional lattice with negative diffusion and show how the framework in this paper can be used to construct traveling waves for this system in special situations. The key observation is that splitting the lattice into odd and even sites and applying a separate rescaling to the two sublattices transforms the negative diffusion scalar system into a spatially periodic system. Whether this rescaling can be performed in such a way that the rescaled system is bistable on the cube $[0, 1]^2$ depends on the structure of the nonlinearity. If this can be done, the original system with negative diffusion has a traveling checkerboard solution, in the sense that the odd and even lattice sites each have a separate wave profile traveling with a common speed.

Continuous vs. discrete Laplacian. Let us briefly discuss our reasons for including the second derivative term in (1.3), which clearly does not appear in the traveling wave equations for the LDEs discussed above. First, as we have seen above, very interesting features of LDEs arise in the regime where waves are pinned to the lattice. Since the traveling wave systems (1.9) and (1.12) become singular as $c \to 0$, numerical methods have considerable trouble resolving the shape of the wave profiles in this regime. As illustrated in [1, 14, 25, 27], this difficulty can be overcome by adding a small second order term as in (1.3). By understanding the limit $\gamma \downarrow 0$ we can hence study how well numerical methods can resolve the fine structure of propagation failure.

Besides this technical issue, there is also a physical reason to introduce a local diffusion term in (1.1). Such a term arises naturally if we consider systems which have local as well as nonlocal interactions, and it allows us to perform continuation from systems with a continuous Laplacian to systems with a discrete Laplacian. We refer to the Frenkel–Kontorova equations [38, 37] as an example in solid-state physics where this is useful.

Existence of waves. By now, many authors have considered the existence of wave-like solutions for dissipative LDEs, using a varied palette of techniques. A significant portion of the work has focussed on spatially homogeneous LDEs with positive discrete diffusion. The seminal work of Weinberger [42] is applicable to both PDEs and LDEs and contains results on the existence of traveling waves primarily for monostable nonlinearities but also for bistable systems. Using index theory, Zinner [43] established the existence of traveling waves for the discrete Nagumo equation posed on a one-dimensional lattice. Mallet-Paret developed a linear Fredholm theory in [31] for MFDEs and employed this in [32] to obtain structural results for scalar versions of (1.1) with $\gamma = 0$. Bates, Chen, and Chmaj [2] used implicit function theorem arguments to obtain the existence of traveling waves for LDEs with long-range interactions that can be both attracting and repelling. In [23] Hupkes and Sandstede developed a version of singular perturbation theory to construct traveling waves for the two-component discrete FitzHugh-Nagumo system. In [22] modulated traveling waves were constructed using a global center manifold analysis for (1.1) with $\gamma > 0$. Finally, in a series of papers [34, 35] Shen studied scalar versions of (1.1) with $\gamma > 0$ but with a time dependent nonlinearity. She employed comparison principles to obtain existence, uniqueness, and stability results for wave-like solutions.

Main techniques. Roughly speaking, the arguments used to establish our main results can be split into two main parts. In the first part, we fix the parameter ρ and the constant $\gamma > 0$ and construct a traveling wave solution for (1.1). In the second part, we show that traveling waves persist under small perturbations of ρ and γ. This allows us to take the limit $\gamma \to 0$ and obtain families of traveling waves for (1.1) even for $\gamma = 0$.

The techniques we use to attain the first goal differ from the approach taken in [32, 25]. Indeed, the latter papers construct a global homotopy that transforms the system (1.1) into a reference system that admits explicit solutions. The problem is that this homotopy needs to be embedded into a so-called normal family that satisfies a number of detailed technical constraints. It is unclear how these conditions can be naturally generalized to higher dimensional systems.

In this paper, we avoid using any global homotopies or topological arguments and directly construct traveling waves for (1.1) with $\gamma > 0$. In particular, we do not follow the route taken in the classical papers [41, 12] where traveling waves are constructed for PDE versions of (1.1) with $\gamma > 0$ that do not contain the nonlocal terms but may include convective terms. Instead, we base our approach on the elegant techniques developed by Chen [9], who studied scalar versions of (1.1) with $\gamma > 0$ and constructed traveling waves using only comparison principles. In sections 4–7 we adapt these results for use in our higher dimensional setting. Although the main spirit of the arguments remains the same, significant modifications need to be made in order to account for the increased complexity of the cube $[0,1]^n$ that contains the dynamics of (1.1) as compared to the interval $[0,1]$.

The analysis in the second part of this paper does build upon ideas introduced in [32] for $\gamma = 0$ and [25] for $\gamma > 0$. In particular, if (ϕ, c) is a traveling wave solution to (1.1), we consider the linear operator

$$[\Lambda_{c,\gamma} v](\xi) = -\gamma v''(\xi) - cv'(\xi) - \sum_{j=0}^{N} A_j[v(\xi + r_j) - v(\xi)] - D_1 f(\phi(\xi); \rho) v(\xi) \tag{1.13}$$

associated to the linearization of (1.3). We show in section 8 that $\Lambda_{c,\gamma}$ is a Fredholm operator and has a one-dimensional kernel that is spanned by the strictly positive function ϕ'. Once established, this Krein–Rutman-type result allows the use of an implicit function theorem argument to construct a local branch of traveling wave solutions to (1.1) that depend smoothly on the parameter ρ.

The main difficulty towards understanding $\Lambda_{c,\gamma}$ is that one needs to rule out potential kernel elements that decay as $\xi \to \pm\infty$ at a rate that is faster than any exponential. Indeed, the ad hoc arguments used in [32] for this purpose cannot be immediately transferred to the high dimensional setting of (1.3). Similarly, the approach taken in [33] to prove a related Krein–Rutman result exploits special structural properties of the underlying system that are absent here.

Let us mention that recent results obtained in [8] actually cover some of the cases considered here. Indeed, in [8] the authors construct traveling wave solutions to LDEs that are posed on one-dimensional lattices and have periodic diffusion. It turns out that whenever the pair (σ_1, σ_2) is rationally related, one can construct a one-dimensional LDE covered by [8] for which the traveling wave system is equivalent to (1.12). However, the techniques used in [8] differ considerably from those used here. In particular, they work only for $\gamma = 0$ and as such cannot account for the transition $\gamma \downarrow 0$. In addition, the intricate parameter dependence of waves is not studied.

We conclude this introduction by giving a brief overview of the structure of this paper. In section 2 we state our assumptions and main results, and in section 3 we show how these results can be applied to three specific examples. In section 4 we state some basic comparison principles for (1.1). In section 5 we study spatially invariant solutions to (1.1) and analyze the separatrix that divides the basins of attraction for the two stable zeroes of f. In sections 6–7 we consider the evolution of a smooth initial condition for (1.1) with $\gamma > 0$ and prove that it converges to a traveling wave.

In section 8 we study the traveling wave system (1.3) directly. In particular, we generalize the local continuation results obtained by Mallet-Paret [32] to the current high dimensional setting. We prove our main results in section 9 and end with a discussion in section 10.

2. Main results. In this section we state our main results. We recall our main family of nonlocal systems

$$\partial_t u(x,t) = \gamma \partial_{xx} u(x,t) + \sum_{j=0}^{N} A_j [u(x+r_j,t) - u(x,t)] + f\big(u(x,t)\,;\rho\big), \tag{2.1}$$

parametrized by $\rho \in V$, where we take V to be a closed subset of $\mathbb{R}$. The diffusion constant satisfies $\gamma \geq 0$, the shifts are ordered as $r_0 < r_1 < \ldots < r_N$, and the function u takes values in $\mathbb{R}^n$ for some $n \geq 2$. For convenience, we introduce the quantities

$$r_{\min} := \min_{0 \leq j \leq N} r_j = r_0, \qquad r_{\max} := \max_{0 \leq j \leq N} r_j = r_N. \tag{2.2}$$

Before we state the rest of our assumptions on (2.1), we need to introduce some notation. First of all, we recall the shorthands $\mathbf{0} = (0,\ldots,0) \in \mathbb{R}^n$ and $\mathbf{1} = (1,\ldots,1) \in \mathbb{R}^n$. Whenever B and C are two $(p \times q)$-matrices, we use the notation $B \geq C$ to indicate that $B_{ij} \geq C_{ij}$ holds for all integers $1 \leq i \leq p$ and $1 \leq j \leq q$, while $B > C$ implies that $B_{ij} > C_{ij}$ for all such i and j. The relations $\leq$ and $<$ are defined in the analogous fashion. Obviously, all these orderings transfer naturally to vectors.

We start by stating our assumption on the matrices $\{A_j\}$. Roughly speaking, all these matrices must be nonnegative and together they must mix all the components of u. Since adding a shift $r_{N+1} = 0$ does not affect (2.1), we caution the reader that this condition should be read together with (2.5) below.

(HA) For all $0 \leq j \leq N$, the $n \times n$-matrix A_j satisfies $A_j \geq 0$. In addition, the matrix

$$\mathcal{A} := \sum_{j=0}^{N} A_j \tag{2.3}$$

is irreducible in the sense that for each pair $(i,j) \in \{1,\ldots,n\}^2$ that has $i \neq j$, there exists an integer $k \geq 2$ and a sequence $\ell_1, \ldots \ell_k$ with $\ell_1 = i$ and $\ell_k = j$ such that

$$\mathcal{A}_{\ell_1 \ell_2} \mathcal{A}_{\ell_2 \ell_3} \ldots \mathcal{A}_{\ell_{k-1} \ell_k} \neq 0. \tag{2.4}$$

Here $\mathcal{A}_{pq}$ denotes the (p,q)th element of the $(n \times n)$-matrix $\mathcal{A}$.

The following three conditions pertain to the nonlinearity f. They state that for each parameter $\rho \in V$, the function $f(\cdot\,;\rho)$ is order preserving in the terminology of [18] and bistable when restricted to a neighborhood of the cube $[0,1]^n$.

(Hf1) The function $f : \mathbb{R}^n \times V \to \mathbb{R}^n$ is C^2-smooth. In addition, for any $\rho \in V$ and $u \in \mathbb{R}^n$, there exists $\kappa = \kappa(u,\rho) > 0$ such that

$$D_1 f(u\,;\rho) \geq \mathcal{A} - \kappa(u,\rho) I. \tag{2.5}$$

(Hf2) For all $\rho \in V$, we have $f(\mathbf{0}\,;\rho) = f(\mathbf{1}\,;\rho) = \mathbf{0}$. In addition, if for some $\rho \in V$ and $\lambda \in \mathbb{C}$ we have

$$\det[D_1 f(v\,;\rho) - \lambda] = 0 \tag{2.6}$$

with either $v = \mathbf{0}$ or $v = \mathbf{1}$, then in fact $\operatorname{Re} \lambda < 0$.

(Hf3) For all $\rho \in V$, the set of vectors $q \in \mathbb{R}^n$ for which $\mathbf{0} < q < \mathbf{1}$ and $f(q;\rho) = 0$ both hold is finite. In addition, for each such q there exists a $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ such that

$$\det[D_1 f(q\,;\rho) - \lambda] = 0. \tag{2.7}$$

Our final two assumptions are technical conditions on the structure of the system (2.1). In particular, (HS1) states that any off-diagonal elements of $D_1 f - \mathcal{A}$ are either identically zero or strictly positive. This should be compared to condition (ii) in [32, sec. 2]. The second condition (HS2) states that it is impossible to rewrite (2.1) in such a way that all the shifts are either nonnegative or nonpositive. Let us emphasize that we fully expect our results to remain valid without this condition. The only reason that we include it is to keep our arguments in section 8 relatively streamlined. Indeed, the proofs in [32] often have to use separate techniques for the two special cases $r_{\min} = 0$ and $r_{\max} = 0$. In our current high dimensional setting this would become even more convoluted.

(HS1) Consider any pair $(k,l) \in \{1,\ldots,n\}^2$ with $k \neq l$. Then for each $\rho \in V$, the function

$$g(u) = f_k(u;\rho) - \mathcal{A}_{kl} u_l \tag{2.8}$$

either satisfies $\partial_{u_l} g(u;\rho) > 0$ for all $u \in \mathbb{R}^n$ or $\partial_{u_l} g(u;\rho) = 0$ for all $u \in \mathbb{R}^n$.

(HS2) Pick any $\rho \in V$ and $\sigma \in \mathbb{R}^n$ and consider the function $\widetilde{u}$ that is given by $\widetilde{u}_i(x,t) = u_i(x + \sigma_i, t)$. For any choice of $\widetilde{N} \geq 1$, $\mathcal{F} : (\mathbb{R}^n)^{\widetilde{N}+1} \to \mathbb{R}^n$ and $\widetilde{r}_0 < \widetilde{r}_1 < \cdots < \widetilde{r}_{\widetilde{N}}$ that allows us to rewrite (2.1) as

$$\partial_t \widetilde{u}(x,t) = \gamma \partial_{xx} \widetilde{u}(x,t) + \mathcal{F}\big(\widetilde{u}(x + \widetilde{r}_0), \ldots, \widetilde{u}(x + \widetilde{r}_{\widetilde{N}})\big), \tag{2.9}$$

we have $\widetilde{r}_0 < 0 < \widetilde{r}_{\widetilde{N}}$.

Our first main result states that (2.1) admits a smooth family of traveling wave solutions whenever $\gamma > 0$. It can be seen as the direct generalization of [25, Thm. 3.1], which applies only to scalar systems.

THEOREM 2.1. *Suppose that* (HA), (Hf1)–(Hf3)*, and* (HS1)–(HS2) *are all satisfied. Then for any* $\gamma > 0$*, there exist* C^1*-smooth functions* $c_\gamma : V \to \mathbb{R}$ *and* $P_\gamma : V \to W^{2,\infty}(\mathbb{R}, \mathbb{R}^n)$ *that satisfy the following properties.*

(i) *For any* $\rho \in V$*, the function* $P = P_\gamma(\rho)$ *has the limits*

$$\lim_{\xi \to -\infty} P(\xi) = \mathbf{0}, \qquad \lim_{\xi \to +\infty} P(\xi) = \mathbf{1} \tag{2.10}$$

and satisfies $P' > 0$.

(ii) *For any* $\rho \in V$*, the function*

$$u(x,t) = P_\gamma(\rho)\big(x - c_\gamma(\rho)t\big) \tag{2.11}$$

satisfies (2.1).

(iii) *Consider any* $P \in W^{2,\infty}(\mathbb{R}, \mathbb{R}^n)$ *that satisfies the limits*

$$\lim_{\xi \to -\infty} P(\xi) = \mathbf{0}, \qquad \lim_{\xi \to +\infty} P(\xi) = \mathbf{1} \tag{2.12}$$

and suppose that $u(x,t) = P(x - ct)$ *satisfies* (2.1) *for some* $\rho \in V$ *and* $c \in \mathbb{R}$. *Then we have* $c = c_\gamma(\rho)$ *and* $P(\cdot) = P_\gamma(\rho)(\cdot - \vartheta)$ *for some* $\vartheta > 0$.

Our second main result shows that the traveling waves obtained in Theorem 2.1 can be used to approximate solutions to (2.1) at the critical value $\gamma = 0$. It generalizes [25, Thm. 3.10], which as before only applies to scalar systems.

THEOREM 2.2. *Suppose that* (HA), (Hf1)–(Hf3), *and* (HS1)–(HS2) *are all satisfied. Consider two sequences $\gamma_n > 0$ and $\rho_n \in V$ that have $\gamma_n \to \gamma_*$ and $\rho_n \to \rho_*$ as $n \to \infty$ for some $\gamma_* \geq 0$ and $\rho_* \in V$. Then, possibly after passing to a subsequence, we have $c_{\gamma_n}(\rho_n) \to c_* \in \mathbb{R}$ and the limit*

$$P_*(\xi) := \lim_{n\to\infty} P_{\gamma_n}(\rho_n)(\xi) \tag{2.13}$$

exists pointwise. The function P_* is nondecreasing and satisfies the limits

$$\lim_{\xi\to-\infty} P_*(\xi) = \mathbf{0}, \qquad \lim_{\xi\to+\infty} P_*(\xi) = \mathbf{1}. \tag{2.14}$$

If either $\gamma_* > 0$ or $c_* \neq 0$, then the function $u_*(x,t) := P_*(x - c_*t)$ satisfies (2.1) *with $\gamma = \gamma_*$ and $\rho = \rho_*$. On the other hand, if $\gamma_* = 0$ and $c_* = 0$, then the time-independent function*

$$u_*(x,t) := \lim_{\xi\downarrow x} P_*(\xi) \tag{2.15}$$

satisfies (2.1) *for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$.*

Our final main result describes the structure of the family of traveling wave solutions to (2.1) at $\gamma = 0$ and should be seen as a generalization of [32, Thm. 2.1]. As in the latter result, the wave speed c is uniquely defined for all $\rho \in V$ and the accompanying wave profiles are unique whenever $c \neq 0$.

THEOREM 2.3 (cf. [32, Thm. 2.1]). *Suppose that* (HA), (Hf1)–(Hf3), *and* (HS1)–(HS2) *are all satisfied and fix $\gamma = 0$. Then there exists a continuous function $c_0 : V \to \mathbb{R}$ that satisfies the following properties.*

(i) *Writing $V_* \subset V$ for the open set, where $c_0(\rho) \neq 0$, the function c_0 is C^1-smooth on V_*.*

(ii) *There exists a C^1-smooth function $P_0 : V_* \to W^{1,\infty}(\mathbb{R}, \mathbb{R}^n)$ such that for any $\rho \in V_*$, the function $P = P_0(\rho)$ has the limits*

$$\lim_{\xi\to-\infty} P(\xi) = \mathbf{0}, \qquad \lim_{\xi\to+\infty} P(\xi) = \mathbf{1}, \tag{2.16}$$

satisfies $P' > 0$, and generates a solution to (2.1) *with $\gamma = 0$ by writing*

$$u(x,t) = P\big(x - c_0(\rho)t\big). \tag{2.17}$$

(iii) *For any $\rho \in V \setminus V_*$, there exists a nondecreasing function $P : \mathbb{R} \to \mathbb{R}^n$ that has the limits*

$$\lim_{\xi\to-\infty} P(\xi) = \mathbf{0}, \qquad \lim_{\xi\to+\infty} P(\xi) = \mathbf{1}, \tag{2.18}$$

such that the time-independent function

$$u(x,t) = P(x) \tag{2.19}$$

satisfies (2.1).

(iv) *Consider any* $c \neq 0$ *and a function* $P \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^n)$ *that satisfies the limits*

$$\lim_{\xi \to -\infty} P(\xi) = \mathbf{0}, \qquad \lim_{\xi \to +\infty} P(\xi) = \mathbf{1}. \tag{2.20}$$

Suppose that $u(x,t) = P(x - ct)$ *satisfies* (2.1) *with* $\gamma = 0$ *for some* $\rho \in V$*. Then we must have* $c = c_0(\rho)$ *and* $P(\cdot) = P_0(\rho)(\cdot - \vartheta)$ *for some* $\vartheta > 0$*. In particular, one has* $\rho \in V_*$*.*

(v) *Consider any nondecreasing function* $P : \mathbb{R} \to \mathbb{R}^n$ *that satisfies the limits*

$$\lim_{\xi \to -\infty} P(\xi) = \mathbf{0}, \qquad \lim_{\xi \to +\infty} P(\xi) = \mathbf{1}. \tag{2.21}$$

If $u(x,t) = P(x)$ *satisfies* (2.1) *with* $\gamma = 0$ *for some* $\rho \in V$*, then we must have* $\rho \in V \setminus V_*$*.*

3. Examples. In this section we illustrate our main results by considering three examples, all of which are posed on the two-dimensional spatial lattice $\mathbb{Z}^2$. We use the nearest-neighbor discrete Laplacian

$$[\Delta_+ u]_{ij} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}, \tag{3.1}$$

together with the next-nearest-neighbor version

$$[\Delta_\times u]_{ij} = u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1} - 4u_{ij}. \tag{3.2}$$

In the first example, the diffusion coefficients are positive but spatially periodic. The second example considers a system that is spatially homogeneous but that has negative nearest-neighbor diffusion. We show how the problem can be transformed into an equivalent spatially periodic system with positive diffusion coefficients. The third example builds upon the second by adding positive next-nearest-neighbor interactions. In all cases we establish that the assumptions (HA), (Hf1)–(Hf3), and (HS1)–(HS2) are all satisfied under reasonable conditions on the nonlinearity.

3.1. Periodic diffusion. In this example we study the system

$$\dot{u}_{ij} = \alpha_{ij}[\Delta_+ u]_{ij} + g_{ij}(u_{ij}; \rho), \qquad i, j \in \mathbb{Z}. \tag{3.3}$$

The diffusion coefficients satisfy $\alpha_{ij} > 0$, and the system is periodic in the sense that there exist integers $p \geq 1$ and $q \geq 1$ such that the identities

$$\alpha_{ij} = \alpha_{i+p,j} = \alpha_{i,j+q}, \qquad g_{ij} = g_{i+p,j} = g_{i,j+q} \tag{3.4}$$

hold for all $i, j \in \mathbb{Z}$.

Let us decompose any pair $(i,j) \in \mathbb{Z}^2$ as

$$i = i_1 p + i_2, \qquad 0 \leq i_2 < p, \qquad j = j_1 q + j_2, \qquad 0 \leq j_2 < q. \tag{3.5}$$

Introducing pq functions $v^{i_2,j_2} : \mathbb{Z}^2 \times \mathbb{R} \to \mathbb{R}$, we write

$$u_{ij}(t) = v_{ij}^{i_2,j_2}(t) \tag{3.6}$$

and look for a traveling wave solution

$$v_{ij}^{i_2,j_2}(t) = \phi_{i_2,j_2}(i\nu_1 + j\nu_2 - ct), \tag{3.7}$$

which travels through the lattice in the direction (ν_1, ν_2). Here for each pair of integers $0 \le i_2 < p$ and $0 \le j_2 < q$, the function $\phi_{i_2,j_2} : \mathbb{R} \to \mathbb{R}$ satisfies the limits

$$\lim_{\xi \to -\infty} \phi_{i_2,j_2}(\xi) = 0, \qquad \lim_{\xi \to +\infty} \phi_{i_2,j_2}(\xi) = 1. \tag{3.8}$$

The traveling wave system can be written as

$$\begin{aligned} -c\phi'_{i_2,j_2}(\xi) = \alpha_{i_2,j_2}[&\phi_{i_2+1,j_2}(\xi+\nu_1) + \phi_{i_2,j_2+1}(\xi+\nu_2) \\ &+ \phi_{i_2-1,j_2}(\xi-\nu_1) + \phi_{i_2,j_2-1}(\xi-\nu_2)] \\ &+ g_{i_2,j_2}\big(\phi_{i_2,j_2}(\xi);\rho\big) \end{aligned} \tag{3.9}$$

with the understanding that $\phi_{i_2 \pm p, j_2} = \phi_{i_2, j_2 \pm q} = \phi_{i_2,j_2}$. Upon embedding $\mathbb{R}^p \times \mathbb{R}^q$ into $\mathbb{R}^{pq}$, this can be written as an equation of the form (1.3) with $n = pq$.

The assumptions (Hf1)–(Hf3) and (HS1) can be satisfied by picking each of the nonlinearities f_{ij} to be bistable, e.g.,

$$g_{ij}(u;\rho) = u(1-u)(u-\rho), \qquad 0 < \rho < 1. \tag{3.10}$$

The irreducibility of the matrix $\mathcal{A}$ appearing in (HA) follows easily from the fact that each point in the grid $\mathbb{Z}^2$ can reach any other point by a series of vertical and horizontal jumps of unit length, mirroring the interactions encoded in the operator Δ_+. Finally, to verify (HS2) it suffices to consider $\sigma \in \mathbb{R}^p \times \mathbb{R}^q$ and look at the components of (3.9) for which σ_{i_2,j_2} is maximal and minimal. The former components are guaranteed to have at least one positive shift and the latter components have at least one negative shift.

3.2. Negative diffusion. In this example we consider a model that has repelling nearest-neighbor interactions. In particular, we consider the system

$$\dot{u}_{ij} = \alpha[\Delta_+ u]_{ij} + g\big(u_{ij};\rho\big), \qquad i,j \in \mathbb{Z} \tag{3.11}$$

with $\alpha < 0$. Let $u_{ij} = v_{ij}$ for $i+j$ even and $u_{ij} = w_{ij}$ for $i+j$ odd. Then (3.11) can be rewritten as

$$\begin{aligned} \dot{v}_{ij} &= \alpha[w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4v_{ij}] + g\big(v_{ij};\rho\big), \\ \dot{w}_{ij} &= \alpha[v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4w_{ij}] + g\big(w_{ij};\rho\big). \end{aligned} \tag{3.12}$$

The equilibrium solutions satisfy

$$0 = 4\alpha(w-v) + g\big(v;\rho\big), \qquad 0 = 4\alpha(v-w) + g\big(w;\rho\big), \tag{3.13}$$

which is the same pair of equations as encountered in the one-dimensional setting of [4] upon replacing 4α by 2α.

Picking any pair of equilibria (v_-, w_-) and (v_+, w_+), let us introduce the new variables

$$\begin{aligned} x_{ij} &= (v_{ij} - v_-)/(v_+ - v_-), \\ y_{ij} &= (w_{ij} - w_-)/(w_+ - w_-). \end{aligned} \tag{3.14}$$

Using these new variables (3.12) transforms into the system

$$\begin{aligned} \dot{x}_{ij} &= d_e[y_{i+1,j} + y_{i-1,j} + y_{i,j+1} + y_{i,j-1} - 4x_{ij}] + g_e\big(x_{ij};\rho\big), \\ \dot{y}_{ij} &= d_o[x_{i+1,j} + x_{i-1,j} + x_{i,j+1} + x_{i,j-1} - 4y_{ij}] + g_o\big(y_{ij};\rho\big) \end{aligned} \tag{3.15}$$

with modified diffusion constants

$$d_e = \alpha(w_+ - w_-)/(v_+ - v_-), \qquad d_o = \alpha(v_+ - v_-)/(w_+ - w_-) \tag{3.16}$$

and modified nonlinearities

$$\begin{aligned} g_e(x;\rho) &= (v_+ - v_-)^{-1} g\big((v_+ - v_-)x + v_-;\rho\big) \\ &\quad + \tfrac{4\alpha}{v_+ - v_-}[x\big((v_+ - v_-) - (w_+ - w_-)\big) + (v_- - w_-)], \\ g_o(y;\rho) &= (w_+ - w_-)^{-1} g\big((w_+ - w_-)y + w_-;\rho\big) \\ &\quad + \tfrac{4\alpha}{w_+ - w_-}[y\big((w_+ - w_-) - (v_+ - v_-)\big) - (v_- - w_-)]. \end{aligned} \tag{3.17}$$

In order to have $d_e, d_o > 0$ it suffices to demand $(w_+ - w_-)(v_+ - v_-) < 0$. Different choices for equilibria that satisfy this requirement are listed in the table in section 5.3 of [4] for the cubic nonlinearity $g(u;\rho) = u(1-u)(u-\rho)$.

Upon looking for a traveling wave solution

$$x_{ij}(t) = \phi_1(i\nu_1 + j\nu_2 - ct), \qquad y_{ij}(t) = \phi_2(i\nu_1 + j\nu_2 - ct), \tag{3.18}$$

we can write the resulting traveling wave system as

$$-c\phi'(\xi) = \sum_{j=0}^{3} A_j[\phi(\xi + r_j) - \phi(\xi)] + f\big(\phi(\xi);\rho\big). \tag{3.19}$$

Here the shifts are given by

$$r_0 = \nu_1, \quad r_1 = \nu_2, \quad r_2 = -\nu_1, \quad r_3 = -\nu_2, \tag{3.20}$$

while the matrices $A_j \geq 0$ are given by

$$A_0 = A_1 = A_2 = A_3 = \begin{pmatrix} 0 & d_e \\ d_0 & 0 \end{pmatrix}, \tag{3.21}$$

and the nonlinearity f is defined as

$$f\big(\phi;\rho\big) = \begin{pmatrix} -g_e(\phi_1;\rho) + 4d_e(\phi_2 - \phi_1) \\ -g_o(\phi_2;\rho) + 4d_o(\phi_1 - \phi_2) \end{pmatrix}. \tag{3.22}$$

This allows us to compute

(3.23)

$$\begin{aligned} &D_1 f(\phi;\rho) \\ &\quad = \begin{pmatrix} -(D_1 g_e(\phi_1;\rho) + 4d_e) & 4d_e \\ 4d_o & -(D_1 g_o(\phi_2;\rho) + 4d_o) \end{pmatrix} \\ &\quad = \begin{pmatrix} -[D_1 g\big((v_+ - v_-)\phi_1 + v_-;\rho\big) + 4\alpha] & 4d_e \\ 4d_o & -[D_1 g\big((w_+ - w_-)u_2 + w_-;\rho\big) + 4\alpha] \end{pmatrix}. \end{aligned}$$

Clearly the irreducibility condition on $\mathcal{A}$ is satisfied together with (Hf1), (HS1), and (HS2). In addition, the bistability criteria (Hf2)–(Hf3) can be verified by studying the table in [4, sec. 5.3]. In the bistable case, we hence see that (3.12) admits a traveling wave solution that connects (v_-, w_-) to (v_+, w_+).

3.3. Mixed diffusion. In this example we expand upon the model in section 3.2 by adding an attracting next-nearest-neighbor interaction. In particular, we consider the system

$$\dot{u}_{ij} = \alpha[\Delta_+ u]_{ij} + \beta[\Delta_\times u]_{ij} + g\big(u_{ij};\rho\big), \qquad i,j \in \mathbb{Z} \tag{3.24}$$

with $\alpha < 0 \leq \beta$. We emphasize that there is no direct condition on the size of either α or β, so our setup differs considerably from the perturbative context of [2]. We again write $u_{ij} = v_{ij}$ for $i+j$ even and $u_{ij} = w_{ij}$ for $i+j$ odd, which transforms (3.24) into

$$\begin{aligned} \dot{v}_{ij} &= \alpha[w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4v_{ij}] + \beta[\Delta_\times v]_{ij} + g\big(v_{ij};\rho\big), \\ \dot{w}_{ij} &= \alpha[v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4w_{ij}] + \beta[\Delta_\times w]_{ij} + g\big(w_{ij};\rho\big). \end{aligned} \tag{3.25}$$

Notice that the equilibrium conditions (3.13) remain unchanged. In particular, we can repeat the coordinate change (3.14) to obtain the equivalent system

$$\begin{aligned} \dot{x}_{ij} &= d_e[y_{i+1,j} + y_{i-1,j} + y_{i,j+1} + y_{i,j-1} - 4x_{ij}] + \beta[\Delta_\times x]_{ij} + g_e\big(x_{ij};\rho\big), \\ \dot{y}_{ij} &= d_o[x_{i+1,j} + x_{i-1,j} + x_{i,j+1} + x_{i,j-1} - 4y_{ij}] + \beta[\Delta_\times y]_{ij} + g_o\big(y_{ij};\rho\big), \end{aligned} \tag{3.26}$$

in which the diffusion constants d_e, d_o and the nonlinearities g_e, g_o are again given by (3.16)–(3.17).

Substitution of the traveling wave ansatz (3.18) now yields the system

$$-c\phi'(\xi) = \sum_{j=0}^{7} A_j[\phi(\xi + r_j) - \phi(\xi)] + f\big(\phi(\xi);\rho\big) \tag{3.27}$$

with shifts

$$\begin{array}{llll} r_0 = \nu_1, & r_1 = \nu_2, & r_2 = -\nu_1, & r_3 = -\nu_2, \\ r_4 = \nu_1 + \nu_2, & r_5 = \nu_1 - \nu_2, & r_6 = -\nu_1 + \nu_2, & r_7 = -\nu_1 - \nu_2 \end{array} \tag{3.28}$$

and matrices

$$A_0 = A_1 = A_2 = A_3 = \begin{pmatrix} 0 & d_e \\ d_0 & 0 \end{pmatrix}, \qquad A_4 = A_5 = A_6 = A_7 = \beta I. \tag{3.29}$$

The nonlinearity f is still given by (3.22). In particular, the presence of the next-nearest-neighbor interactions does not affect the location of the equilibria or their stability. This means that our main results are applicable to (3.24) whenever the conditions described in section 3.2 hold for (3.11).

4. Preliminary results. In this section we obtain preliminary results on the nonlinear system

$$\partial_t u(x,t) = [\mathcal{D}u](x,t) + f\big(u(x,t)\big). \tag{4.1}$$

Here we have introduced the nonlocal differential operator

$$[\mathcal{D}u](x,t) = \gamma\partial_{xx}u(x,t) + [J * u](x,t), \tag{4.2}$$

in which

$$[J * u](x,t) = \sum_{j=0}^{N} A_j[u(x + r_j, t) - u(x,t)]. \tag{4.3}$$

We impose the following condition on the nonlinearity f to reflect the fact that we have dropped the dependence on the parameter ρ.

$(\mathrm{h})_{\S 4}$ The function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions (Hf1)–(Hf3) with the understanding that $V = \{0\}$ and $f(\cdot\,;0) = f(\cdot)$.

Before we proceed, we need to fix the function space in which we will consider (4.1). To this end, we introduce the spaces

(4.4)
$$BC^0(\mathbb{R},\mathbb{R}^n) = \{u \in C(\mathbb{R},\mathbb{R}^n) \mid \|u\|_{BC^0} := \sup_{\xi\in\mathbb{R}} |u(\xi)| < \infty\},$$
$$BC^2(\mathbb{R},\mathbb{R}^n) = \{u \in C^2(\mathbb{R},\mathbb{R}^n) \mid \|u\|_{BC^2} := \max\{\|u\|_{BC^0}, \|u'\|_{BC^0}, \|u''\|_{BC^0}\} < \infty\}.$$

We also introduce the set $\mathcal{X}$ that contains all functions $u \in L^\infty(\mathbb{R}\times[0,\infty),\mathbb{R}^n)$ that satisfy the following two properties.

$(\mathrm{i})_{\mathcal{X}}$ For all $t > 0$ we have $u(\cdot,t) \in BC^2(\mathbb{R},\mathbb{R}^n)$ and $\partial_t u(\cdot,t) \in BC^0(\mathbb{R},\mathbb{R}^n)$.

$(\mathrm{ii})_{\mathcal{X}}$ As $t \downarrow 0$ we have the uniform limit

$$\sup_{x\in\mathbb{R}} \left| u(x,t) - \int_{-\infty}^{\infty} \mathcal{Z}(x-x',t)u(x',0)\,dx' \right| \to 0, \tag{4.5}$$

in which $\mathcal{Z}$ denotes the standard heat kernel

$$\mathcal{Z}(\xi,t) = \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{\xi^2}{4t}\right]. \tag{4.6}$$

In particular, functions in $\mathcal{X}$ can be spatially discontinuous at $t = 0$ and temporally discontinuous as $t \downarrow 0$. To accomodate functions that are smooth for all $t \geq 0$ we introduce the subset

$$\widehat{\mathcal{X}} = \{u \in \mathcal{X} \mid u(\cdot,0) \in BC^2(\mathbb{R},\mathbb{R}^n)\}. \tag{4.7}$$

Our first two results state a comparison and regularity principle for (4.1). The proof of the comparison principle closely follows the arguments developed in [9, Thm. 5.1].

Proposition 4.1 (cf. [9, (C2)]). *Consider the nonlinear system* (4.1) *with* $\gamma \geq 0$ *and suppose that* (HA) *and* $(\mathrm{h})_{\S 4}$ *are satisfied. Let* $u, v \in \mathcal{X}$ *be a pair of functions that satisfy the uniform bounds*

$$-\mathbf{1} \leq u(x,t) \leq \mathbf{2}, \qquad -\mathbf{1} \leq v(x,t) \leq \mathbf{2}, \qquad x \in \mathbb{R},\ t \geq 0, \tag{4.8}$$

together with the differential inequalities

(4.9)
$$\partial_t u(x,t) \geq [\mathcal{D}u](x,t) + f\big(u(x,t)\big), \qquad \partial_t v(x,t) \leq [\mathcal{D}v](x,t) + f\big(v(x,t)\big), \qquad t > 0$$

and the initial inequality

$$u(x,0) \geq v(x,0), \qquad x \in \mathbb{R}. \tag{4.10}$$

Then if $\gamma = 0$*, the inequality* $u(x,t) \geq v(x,t)$ *holds for all* $x \in \mathbb{R}$ *and* $t \geq 0$*. On the other hand, if* $\gamma > 0$*, then there exists a continuous matrix-valued function*

$$\eta_\gamma : \mathbb{R}\times(0,\infty) \to \mathbb{R}^{n\times n}_{>\mathbf{0}} \tag{4.11}$$

that does not depend on u *and* v*, such that the lower bound*

$$u(x,t) - v(x,t) \geq \eta_\gamma(x,t) \int_0^1 [u(\sigma,0) - v(\sigma,0)]d\sigma \tag{4.12}$$

holds for all $x \in \mathbb{R}$ *and* $t > 0$*.*

Proof. First assume that $\gamma \geq 0$. Upon writing $w(x,t) = u(x,t) - v(x,t)$ together with

$$\mathcal{I}(x,t) = \int_0^1 Df\big(v(x,t) + \vartheta w(x,t)\big) d\vartheta, \tag{4.13}$$

the estimate

$$\begin{aligned} \partial_t w(x,t) &\geq [\mathcal{D}w](x,t) + f\big(u(x,t)\big) - f\big(v(x,t)\big) \\ &= [\mathcal{D}w](x,t) + \mathcal{I}(x,t)w(x,t) \end{aligned} \tag{4.14}$$

holds for all $t > 0$. In order to show that $w(x,t) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}$, let us assume to the contrary that this is false. In particular, suppose that there exist $t_* > 0$, $x_* \in \mathbb{R}$ and an integer $1 \leq i \leq n$ for which $w_i(x_*, t_*) = -\vartheta < 0$. Picking $\epsilon > 0$ and $K > 0$ in such a way that $\vartheta = \epsilon e^{2Kt_*}$, we can now define

$$T := \sup\{t \geq 0 \mid w(x,t) > -\epsilon e^{2Kt}\mathbf{1} \text{ for all } x \in \mathbb{R}\}. \tag{4.15}$$

The requirement (4.5) together with the convergence $(ii)_{\mathcal{X}}$ implies that $0 < T \leq t_*$. In addition, there exists an integer $1 \leq i \leq n$ with

$$\inf_{x \in \mathbb{R}} w_i(x,T) = -\epsilon e^{2KT}, \tag{4.16}$$

since otherwise the lower bound (4.14) together with the inclusion $w(\cdot, T) \in BC^2(\mathbb{R}, \mathbb{R}^n)$ would allow the constant T to be increased. Without loss of generality we may therefore assume that $w_i(0,T) < -\frac{7}{8}e^{2KT}$.

Consider now the function

$$w^-(x,t;\sigma) = -\epsilon\left(\frac{3}{4} + \sigma z(x)\right) e^{2Kt}\mathbf{1}, \tag{4.17}$$

in which $\sigma > 0$ is a parameter and $z : \mathbb{R} \to \mathbb{R}$ is a smooth function that has $z(0) = 1$, $z(\pm\infty) = 3$, $1 \leq z \leq 3$, and $|z''| \leq 1$. Write $\sigma_* \in (\frac{1}{8}, \frac{1}{4}]$ for the minimal value of σ for which $w(x,t) \geq w^-(x,t;\sigma)$ holds for all $(x,t) \in \mathbb{R} \times [0,T]$. Since

$$w^-(\pm\infty, t; \sigma_*) = -\epsilon\left[\frac{3}{4} + 3\sigma_*\right] e^{2Kt}\mathbf{1} < -\frac{9}{8}\epsilon e^{2Kt}\mathbf{1}, \tag{4.18}$$

there exist $1 \leq i_0 \leq n$, $x_0 \in \mathbb{R}$ and $0 < t_0 \leq T$ such that $w_{i_0}(x_0, t_0) = w^-_{i_0}(x_0, t_0; \sigma_*)$. The definition of σ_* now implies that

$$\begin{aligned} \partial_t w_{i_0}(x_0,t_0) &\leq \partial_t w^-_{i_0}(x_0,t_0;\sigma_*), \\ \partial_x w_{i_0}(x_0,t_0) &= \partial_x w^-_{i_0}(x_0,t_0;\sigma_*), \\ \partial_{xx} w_{i_0}(x_0,t_0) &\geq \partial_{xx} w^-_{i_0}(x_0,t_0;\sigma_*), \end{aligned} \tag{4.19}$$

which in turns leads to the estimate

$$\begin{aligned} -\frac{7}{4}\epsilon K e^{2Kt_0} & \\ &\geq \partial_t w^-_{i_0}(x_0,t_0) \geq \partial_t w_{i_0}(x_0,t_0) \\ &\geq [\mathcal{D}w]_{i_0}(x_0,t_0) + [\mathcal{I}(x_0,t_0)w(x_0,t_0)]_{i_0} \\ &= \gamma\partial_{xx}w_{i_0}(x_0,t_0) + \sum_{j=0}^{N}[A_j w(x_0+r_j,t_0)]_{i_0} + [(\mathcal{I}(x_0,t_0)-\mathcal{A})w(x_0,t_0)]_{i_0} \\ &\geq \gamma\partial_{xx}w^-_{i_0}(x_0,t_0) + \sum_{j=0}^{N}[A_j w^-(x_0+r_j,t_0)]_{i_0} + [(\mathcal{I}(x_0,t_0)-\mathcal{A})w^-(x_0,t_0)]_{i_0}. \end{aligned} \tag{4.20}$$

In the last inequality we used the fact that all nondiagonal elements of $\mathcal{I}(x_0,t_0)-\mathcal{A}$ are nonnegative, where $\mathcal{A}$ is the matrix appearing in (HA). In particular, we obtain the bound

$$-\frac{7}{4}\epsilon K e^{2Kt_0} \geq -3\epsilon\left[\gamma + 2\sum_{j=0}^{N}|A_j| + \left\|D^2 f\right\|\right]e^{2Kt_0}. \tag{4.21}$$

This leads to a contradiction upon choosing $K \gg 1$ to be sufficiently large, showing that indeed $w(x,t) \geq \mathbf{0}$ for all $x \in \mathbb{R}$ and $t \geq 0$.

From now on, we assume that $\gamma > 0$. We pick $\kappa \gg 1$ in such a way that $\mathcal{I}(x,t) \geq \kappa I + \mathcal{A}$ holds for all $x \in \mathbb{R}$ and $t \geq 0$. Writing $\widehat{w}(x,t) = e^{\kappa t}w(x,t)$, we obtain the differential inequality

$$\partial_t\widehat{w}(x,t) \geq \gamma\partial_{xx}\widehat{w}(x,t) + \sum_{j=0}^{N}A_j\widehat{w}(x+r_j,t), \qquad t>0. \tag{4.22}$$

Similar arguments as above show that $\widehat{w}(x,t) \geq \widehat{z}(x,t) \geq \mathbf{0}$ for $(x,t) \in \mathbb{R}\times[0,\infty)$, where $\widehat{z} \in \mathcal{X}$ can be represented as

$$\begin{aligned} \widehat{z}(x,t) = &\int_{\mathbb{R}}\mathcal{Z}_\gamma(x-x',t)w(x',0)dx' \\ &+ \sum_{j=0}^{N}\int_0^t\int_{\mathbb{R}}\mathcal{Z}_\gamma(x-x',t-s)A_j\widehat{z}(x'+r_j,s)dx'ds, \end{aligned} \tag{4.23}$$

in which we have used the rescaled heat kernel

$$\mathcal{Z}_\gamma(\xi,t) = \mathcal{Z}(\xi,\gamma t). \tag{4.24}$$

Indeed, notice that $\widehat{z}(x,0) = w(x,0)$ while also

$$\partial_t\widehat{z}(x,t) = \gamma\partial_{xx}\widehat{z}(x,t) + \sum_{j=0}^{N}A_j\widehat{z}(x+r_j,t), \qquad t>0. \tag{4.25}$$

Using the fact that $\mathcal{A}^\ell > \mathbf{0}$ for some integer $\ell > 0$, one can use a standard bootstrapping argument to construct the function η_γ that satisfies the desired properties. $\square$

PROPOSITION 4.2 (cf. [9, C4]). *Suppose that* (HA) *and* $(h)_{\S 4}$ *are satisfied and consider any* $u \in \widehat{\mathcal{X}}$ *that satisfies* (4.1) *with* $\gamma > 0$ *for all* $t > 0$*. Suppose furthermore that* $\mathbf{0} \le u(x,0) \le \mathbf{1}$ *holds for all* $x \in \mathbb{R}$*. Then we have*

$$\sup_{t\ge 0} \|u(\cdot,t)\|_{BC^2} < \infty. \tag{4.26}$$

Proof. By the comparison principle we have $\mathbf{0} \le u(x,t) \le \mathbf{1}$ for all $t \ge 0$. The uniform bounds on $\partial_x u$ and $\partial_{xx} u$ can now be obtained by combining the parabolic regularity results obtained in [28, Chap. V, sec. 3, Thm. 3.1.] and [28, Chap. VII, sec. 5, Thm. 5.1.]; see also [13, Thm. A.8]. ∎

Before stating our next result, we need to introduce some notation. First of all, we note that the Perron–Frobenius theorem [16] in combination with (Hf2) implies that the largest eigenvalue λ_l of the matrix $Df(\mathbf{0})$ is simple and that we can pick $v_l \in \mathbb{R}^n$ in such a way that

$$Df(\mathbf{0})v_l = \lambda_l v_l, \qquad \lambda_l < 0, \qquad v_l > \mathbf{0}, \qquad |v_l| = 1. \tag{4.27}$$

In addition, upon writing λ_r for the largest eigenvalue of $Df(\mathbf{1})$, we can pick $v_r \in \mathbb{R}^n$ in such a way that

$$Df(\mathbf{1})v_r = \lambda_r v_r, \qquad \lambda_r < 0, \qquad v_r > \mathbf{0}, \qquad |v_r| = 1. \tag{4.28}$$

Furthermore, we introduce a C^∞-smooth function $H_+ : \mathbb{R} \to [0,1]$ that satisfies $0 \le H_+' \le 2$, $0 \le H_+'' \le 4$, $H_+(-1) = 0$ and $H_+(1) = 1$. For convenience, we also use the function $H_- = 1 - H_+$. Finally, we write

$$\mathcal{H}(\xi) = H_-(\xi)v_l + H_+(\xi)v_r. \tag{4.29}$$

Since $|v_l| = |v_r| = 1$, we see that $|\mathcal{H}(\xi)| \le 1$ and $|[\mathcal{D}\mathcal{H}](\xi)| \le \kappa_{\mathcal{H}}$ with $\kappa_{\mathcal{H}} := 4\gamma + 2n\|\mathcal{A}\|$. Throughout the remainder of this section we use these functions to construct sub- and super-solutions to (4.1) that approximate traveling waves.

PROPOSITION 4.3. *Consider the nonlinear system* (4.1) *with* $\gamma \ge 0$ *and suppose that* (HA) *and* $(h)_{\S 4}$ *are satisfied. Consider any* $u \in \widehat{X}$ *that satisfies* (4.1) *for all* $t > 0$*. In addition, suppose that* $\partial_x u(x,t) > 0$ *for all* $x \in \mathbb{R}$ *and* $t \ge 0$ *and that the following limits hold for all* $t \ge 0$*,*

$$\lim_{x\to-\infty} u(x,t) = \mathbf{0}, \qquad \lim_{x\to\infty} u(x,t) = \mathbf{1}. \tag{4.30}$$

Finally, suppose that there exists a C^1*-smooth function* $\xi : [0,\infty) \to \mathbb{R}$ *with* $\|\xi'\|_\infty < \infty$ *such that for every* $\delta > 0$*, there exist constants* $M = M(\delta) \gg 1$ *and* $\kappa = \kappa(\delta) > 0$ *that allow us to write*

$$|u(x,t)| < \delta \text{ for } x < \xi(t) - M, \qquad |\mathbf{1} - u(x,t)| < \delta \text{ for } x > \xi(t) + M \tag{4.31}$$

together with

$$\partial_x u(x,t)(t) > \kappa\mathbf{1} \text{ for } |x - \xi(t)| \le M + 2 + (r_{\max} - r_{\min}) \tag{4.32}$$

for all $t \ge 0$*.*

Then there exist constants $\sigma_1 \gg 1$ *and* $\beta > 0$ *such that for all sufficiently small* $\delta > 0$*, the functions*

$$\begin{aligned} w^+(x,t) &= u\big(x + \sigma_1\delta(1-e^{-\beta t}), t\big) + \delta e^{-\beta t}\mathcal{H}\big(x + \sigma_1\delta(1-e^{-\beta t}) - \xi(t)\big), \\ w^-(x,t) &= u\big(x - \sigma_1\delta(1-e^{-\beta t}), t\big) - \delta e^{-\beta t}\mathcal{H}\big(x - \sigma_1\delta(1-e^{-\beta t}) - \xi(t)\big) \end{aligned} \tag{4.33}$$

satisfy the differential inequalities

(4.34)
$$\partial_t w^+(x,t) \geq [\mathcal{D}w^+](x,t) + f\big(w^+(x,t)\big), \qquad \partial_t w^-(x,t) \leq [\mathcal{D}w^-](x,t) + f\big(w^-(x,t)\big)$$

for all $t > 0$.

Proof. We will only consider the function w^+, as the statements concerning w^- can be handled in a similar fashion. For convenience, we introduce the shorthand $y = x + \sigma_1\delta(1 - e^{-\beta t})$ and compute

(4.35)
$$\begin{aligned}\partial_t w^+(x,t) = \partial_t u(y,t) &+ \beta\sigma_1\delta e^{-\beta t}\partial_x u(y,t) + \delta e^{-\beta t}\big(\beta\delta\sigma_1 e^{-\beta t} - \xi'(t)\big)\mathcal{H}'\big(y - \xi(t)\big)\\ &- \beta\delta e^{-\beta t}\mathcal{H}\big(y - \xi(t)\big).\end{aligned}$$

In particular, upon writing

$$\mathcal{J}^+(x,t) = \partial_t w^+(x,t) - [\mathcal{D}w^+](x,t) - f\big(w^+(x,t)\big), \tag{4.36}$$

we may compute

(4.37)
$$\begin{aligned}\mathcal{J}^+(x,t) &= [\mathcal{D}u](y,t) + f\big(u(y,t)\big) - [\mathcal{D}w^+](x,t) - f\big(w^+(x,t)\big)\\ &\quad + \beta\sigma_1\delta e^{-\beta t}\partial_x u(y,t) + \delta e^{-\beta t}\big(\beta\delta\sigma_1 e^{-\beta t} - \xi'(t)\big)\mathcal{H}'\big(y - \xi(t)\big)\\ &\quad - \beta\delta e^{-\beta t}\mathcal{H}\big(y - \xi(t)\big)\\ &= f\big(u(y,t)\big) - f\Big(u(y,t) + \delta e^{-\beta t}\mathcal{H}\big(y - \xi(t)\big)\Big) - \delta e^{-\beta t}[\mathcal{D}\mathcal{H}]\big(y - \xi(t)\big)\\ &\quad + \beta\sigma_1\delta e^{-\beta t}\partial_x u(y,t) + \delta e^{-\beta t}\big(\beta\delta\sigma_1 e^{-\beta t} - \xi'(t)\big)\mathcal{H}'\big(y - \xi(t)\big)\\ &\quad - \beta\delta e^{-\beta t}\mathcal{H}\big(y - \xi(t)\big).\end{aligned}$$

Pick $\delta_0 > 0$ and $\beta > 0$ to be sufficiently small to ensure that $Df(u)v_r \leq -2\beta v_r$ holds for all u that have $|u - \mathbf{1}| < \delta_0$, while also $Df(u)v_l \leq -2\beta v_l$ for all u that have $|u| < \delta_0$.

Restricting our attention to the setting $y \geq M(\delta_0) + \xi(t) + 1 - r_{\min}$, we see that

$$[\mathcal{D}\mathcal{H}]\big(y - \xi(t)\big) = 0, \qquad \mathcal{H}'\big(y - \xi(t)\big) = 0, \qquad \mathcal{H}\big(y - \xi(t)\big) = v_r, \tag{4.38}$$

which implies that

$$\mathcal{J}^+(x,t) \quad \geq \quad \mathcal{J}_0^+(x,t) := f\big(u(y,t)\big) - f\big(u(y,t) + \delta e^{-\beta t}v_r\big) - \beta\delta e^{-\beta t}v_r. \tag{4.39}$$

We may now estimate

$$\big|\mathcal{J}_0^+(x,t) + \delta e^{-\beta t}Df\big(u(y,t)\big)v_r + \beta\delta v_r e^{-\beta t}\big| \leq \frac{1}{2}\left\|D^2 f\right\|\delta^2 e^{-2\beta t}\left|v_r\right|^2. \tag{4.40}$$

In particular, by choosing a sufficiently small $\delta > 0$, our choice of $\beta > 0$ ensures that

$$\mathcal{J}^+(x,t) \geq \frac{1}{2}\beta\delta v_r e^{-\beta t} > \mathbf{0}. \tag{4.41}$$

A similar estimate can be obtained for $y \leq \xi(t) - M(\delta_0) - 1 - r_{\max}$.

We now turn to the case that $|y - \xi(t)| \leq M(\delta_0) + 2 + r_{\max} - r_{\min}$, which allows us to estimate

(4.42)
$$\begin{aligned}\big|\mathcal{J}^+(x,t) - \beta\sigma_1\delta e^{-\beta t}\partial_x u(y,t)\big| &\leq \|Df\|\,\delta e^{-\beta t} + \delta e^{-\beta t}\kappa_{\mathcal{H}}\\ &\quad + 2\delta^2\beta\sigma_1 e^{-2\beta t} + \delta\,\|\xi'\|\,e^{-\beta t} + \beta\delta e^{-\beta t}\\ &= \delta e^{-\beta t}\big[\,\|Df\| + \kappa_{\mathcal{H}} + 2\delta\beta\sigma_1 e^{-\beta t} + \|\xi'\| + \beta\big].\end{aligned}$$

In particular, upon choosing

$$\sigma_1 = 4\beta^{-1}\kappa(\delta_0)^{-1}\big[\,\|Df\| + \kappa_{\mathcal{H}} + \|\xi'\| + \beta\big] \tag{4.43}$$

and subsequently restricting δ to ensure that

$$\delta \leq \frac{1}{8}\kappa(\delta_0), \tag{4.44}$$

the desired conclusion $\mathcal{J}^+(x,t) > \mathbf{0}$ follows easily. ∎

COROLLARY 4.4. *Consider the setting of Proposition* 4.3. *There exist constants* $\sigma_2 \gg 1$, $\sigma_3 > 0$, *and* $\beta > 0$ *such that for any sufficiently small* $\delta > 0$ *and any pair* $w^{\pm} \in \mathcal{X}$ *that satisfies* (4.1) *together with the initial bounds*

$$w^+(x,0) \leq u(x,0) + \delta\mathbf{1}, \qquad w^-(x,0) \geq u(x,0) - \delta\mathbf{1}, \tag{4.45}$$

the inequalities

$$\begin{aligned} w^+(x,t) &\leq u\big(x + \sigma_2\delta(1-e^{-\beta t}), t\big) + \sigma_3\delta e^{-\beta t},\\ w^-(x,t) &\geq u\big(x - \sigma_2\delta(1-e^{-\beta t}), t\big) - \sigma_3\delta e^{-\beta t} \end{aligned} \tag{4.46}$$

hold for all $t \geq 0$.

COROLLARY 4.5. *Consider the system* (4.1) *with* $\gamma \geq 0$ *and suppose that* (HA) *and* $(h)_{\S 4}$ *are satisfied. Suppose furthermore that there exists a pair* $(P,c) \in BC^2(\mathbb{R},\mathbb{R}^n) \times \mathbb{R}$ *that satisfies the limits*

$$\lim_{\xi\to-\infty} P(\xi) = \mathbf{0}, \qquad \lim_{\xi\to+\infty} P(\xi) = \mathbf{1} \tag{4.47}$$

has $P'(\xi) > \mathbf{0}$ *for all* $\xi \in \mathbb{R}$ *and yields a solution to* (4.1) *upon writing* $u(x,t) = P(x - ct)$.

Then there exist constants $\sigma_2 \gg 1$, $\sigma_3 > 0$ *and* $\beta > 0$ *such that for any sufficiently small* $\delta > 0$ *and any pair* $w^{\pm} \in \mathcal{X}$ *that satisfies* (4.1) *together with the initial bounds*

$$w^+(x,0) \leq P(x) + \delta\mathbf{1}, \qquad w^-(x,0) \geq P(x) - \delta\mathbf{1}, \tag{4.48}$$

the inequalities

$$\begin{aligned} w^+(x,t) &\leq P\big(x + \sigma_2\delta(1-e^{-\beta t}) - ct\big) + \sigma_3\delta e^{-\beta t},\\ w^-(x,t) &\geq P\big(x - \sigma_2\delta(1-e^{-\beta t}) - ct\big) - \sigma_3\delta e^{-\beta t}, \end{aligned} \tag{4.49}$$

hold for all $t \geq 0$.

5. Spatially invariant solutions. Throughout this section, we study the class of spatially invariant solutions to our main nonlinear system (2.1). In particular, we consider the initial value problem

$$u'(t) = f\big(u(t)\big), \qquad u(0) = u_0 \in \mathbb{R}^n \tag{5.1}$$

and impose the following condition on the nonlinearity f to reflect the fact that we have dropped the dependence on the parameter ρ.

$(\mathrm{h})_{\S 5}$ The function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions (Hf1)–(Hf3) for some irreducible matrix $\mathcal{A} \geq \mathbf{0} \in \mathbb{R}^{n\times n}$ with the understanding that $V = \{0\}$ and $f(\cdot\,;0) = f(\cdot)$.

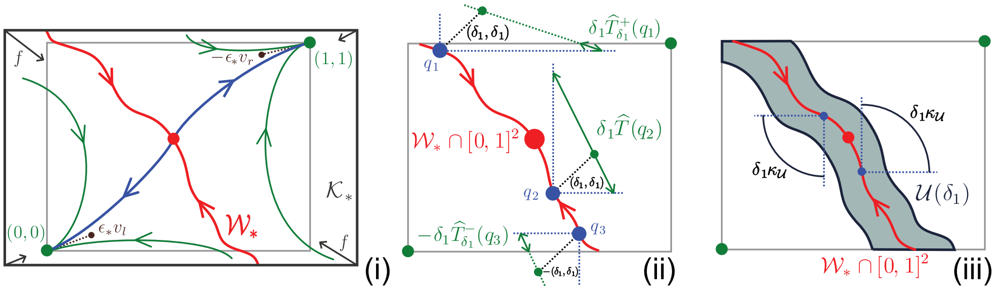

FIG. 1. *Panel* (i) *illustrates the definitions of* $\mathcal{K}_*$ *and* $\mathcal{W}_*$ *and depicts a number of trajectories under the flow* Φ. *Panel* (ii) *highlights the relation between the definitions of the tangent spaces* $\widehat{T}(q)$ *and* $\widehat{T}^{\pm}_{\delta_1}(q)$. *Finally, panel* (iii) *represents the tubular neighborhood* $\mathcal{U}(\delta_1)$ *and the constant* $\kappa_{\mathcal{U}}$.

We use the notation $u(t) = \Phi(t; u_0)$ to refer to the unique solution of the initial value problem (5.1). In addition, we are interested in the linearized problem

$$v'(t) = Df\big(\Phi(t; u_0)\big)v(t), \qquad v(0) = v_0 \in \mathbb{R}^n \tag{5.2}$$

for any $u_0 \in \mathbb{R}^n$ and write $v(t) = \Psi(t; u_0)v_0$ to refer to the solution of this system.

The eigenvectors $v_l > \mathbf{0}$ and $v_r > \mathbf{0}$ introduced in (4.27)–(4.28) can be used to introduce a convenient forward-invariant set for (5.1) that is slightly larger than the cube $[0, 1]^n$.

PROPOSITION 5.1. *Consider the nonlinear ODE* (5.1) *and suppose that* $(\mathrm{h})_{\S 5}$ *is satisfied. Then there exists* $\epsilon_* > 0$ *such that for each* $0 < \epsilon \leq 2\epsilon_*$ *the set*

$$\mathcal{K}(\epsilon) = \{u \in \mathbb{R}^n \mid -\epsilon v_l \leq u \leq \mathbf{1} + \epsilon v_r\} \tag{5.3}$$

satisfies $\Phi\big(t; \mathcal{K}(\epsilon)\big) \subset \mathcal{K}(\epsilon)$ *for all* $t \geq 0$. *In addition, if* $f(q) = \mathbf{0}$ *for some* $q \in \mathcal{K}(\epsilon_*) \setminus \{\mathbf{0}, \mathbf{1}\}$, *then in fact* $\mathbf{0} < q < \mathbf{1}$.

Using the constant $\epsilon_* > 0$ introduced above, we write $\mathcal{K}_* = \mathcal{K}(\epsilon_*)$. We recall that the ω-limit set for any $u \in \mathbb{R}^n$ is defined by

$$\omega^+(u) = \left\{ v \in \mathbb{R}^n \mid \text{ there exists a sequence } t_k \to \infty \text{ with } \lim_{k\to\infty} \Phi(t_k; u) = v \right\}. \tag{5.4}$$

Note that (Hf2) implies that both $\mathbf{0}$ and $\mathbf{1}$ are stable. In particular, if $\mathbf{0} \in \omega^+(u)$ for some $u \in \mathbb{R}^n$, then in fact we have $\lim_{t\to\infty} \Phi(t; u) = \mathbf{0}$ with a similar statement for $\mathbf{1}$. A second consequence of (Hf2) is that the sets

$$\mathcal{B}(\mathbf{0}) = \{u \in \mathcal{K}_* \mid \omega^+(u) = \{\mathbf{0}\}\}, \qquad \mathcal{B}(\mathbf{1}) = \{u \in \mathcal{K}_* \mid \omega^+(u) = \{\mathbf{1}\}\} \tag{5.5}$$

are both open in $\mathcal{K}_*$. Our main focus in this section is the separatrix that divides $\mathcal{B}(\mathbf{0})$ and $\mathcal{B}(\mathbf{1})$. In particular, we introduce the set

$$\mathcal{W}_* = \{u \in \mathcal{K}_* \text{ for which } \{\mathbf{0}, \mathbf{1}\} \cap \omega^+(u) = \emptyset\}, \tag{5.6}$$

illustrated in Figure 1(i). In addition, for any $q \in \mathcal{W}_*$, we introduce the suggestively named space

$$T(q) = \mathbf{0} \cup \{v \in \mathbb{R}^n \mid \Psi(t; q)v \notin \mathbb{R}^n_{\geq \mathbf{0}} \cup \mathbb{R}^n_{\leq \mathbf{0}} \text{ for all } t \geq 0\} \tag{5.7}$$

and write

$$T(\mathcal{W}_*) = \{(q, \psi) \mid q \in \mathcal{W}_* \text{ and } \psi \in T(q)\}. \tag{5.8}$$

Our first main result summarizes some useful properties of the separatrix $\mathcal{W}_*$ and validates the notation used in the definitions above.

PROPOSITION 5.2. *Consider the nonlinear ODE* (5.1) *and suppose that* $(\mathrm{h})_{\S 5}$ *is satisfied. Then the following properties hold.*

(i) *The set $\mathcal{W}_*$ is compact and satisfies $\Phi(t; \mathcal{W}_*) \subset \mathcal{W}_*$ for every $t \geq 0$.*

(ii) *Consider any continuous path $\Gamma : [0,1] \to \mathcal{K}_*$ that has $\Gamma(0) \in \mathcal{B}(\mathbf{0})$, $\Gamma(1) \in \mathcal{B}(\mathbf{1})$, and*

$$\Gamma(t_1) \leq \Gamma(t_2), \qquad \Gamma(t_1) \neq \Gamma(t_2) \tag{5.9}$$

for all $0 \leq t_1 < t_2 \leq 1$. Then there is precisely one $0 \leq t_* \leq 1$ such that $\Gamma(t_*) \in \mathcal{W}_*$.

(iii) *The set $\mathcal{W}_*$ is an $(n-1)$-dimensional submanifold of $\mathcal{K}_*$ that is C^1-smooth. For any $q \in \mathcal{W}_*$, the tangent space to $\mathcal{W}_*$ at q is given by $T(q)$.*

(iv) *There exist constants $K > 0$ and $\alpha > 0$ such that for all $q \in \mathcal{W}_*$ and $\psi \in T(q)$ we have*

$$|\Psi(t; q)\psi| \leq K e^{-\alpha t} |\psi| \, |\Psi(t; q)\mathbf{1}| \, . \tag{5.10}$$

(v) *For every $\epsilon > 0$ there exists $\vartheta = \vartheta(\epsilon) > 0$ such that*

$$|\Psi(t; q)\mathbf{1}| \geq \vartheta e^{-\epsilon t} \tag{5.11}$$

holds for all $q \in \mathcal{W}_*$ and all $t \geq 0$.

Our next point of concern is the construction of a tubular neighborhood around the separatrix $\mathcal{W}_*$. To this end, we pick any $q \in \mathcal{W}_*$ and consider the following subset of $T(q)$:

$$\widehat{T}(q) = \{\psi \in T(q) \mid \mathbf{1} + \psi \geq \mathbf{0}\}. \tag{5.12}$$

In addition, for any $\delta_1 > 0$ and $q \in \mathcal{W}_* \cap [0,1]^n$, we consider the restricted sets

$$\begin{aligned} \widehat{T}^-_{\delta_1}(q) &= \{\psi \in \widehat{T}(q) \mid q - \delta_1[\mathbf{1} + \psi] \in [0,1]^n\}, \\ \widehat{T}^+_{\delta_1}(q) &= \{\psi \in \widehat{T}(q) \mid q + \delta_1[\mathbf{1} + \psi] \in [0,1]^n\}, \end{aligned} \tag{5.13}$$

as illustrated in Figure 1(ii). For any $\delta_1 > 0$, these sets allow us to define the regions

$$\begin{aligned} \mathcal{U}^-(\delta_1) &= \{u \in [0,1]^n \mid u \leq q + \delta_1[\mathbf{1} + \psi] \text{ for some } q \in \mathcal{W}_* \cap [0,1]^n, \psi \in \widehat{T}^+_{\delta_1}(q)\}, \\ \mathcal{U}^+(\delta_1) &= \{u \in [0,1]^n \mid u \geq q - \delta_1[\mathbf{1} + \psi] \text{ for some } q \in \mathcal{W}_* \cap [0,1]^n, \psi \in \widehat{T}^-_{\delta_1}(q)\}, \end{aligned} \tag{5.14}$$

together with the tubular neighborhood

$$\mathcal{U}(\delta_1) = \mathcal{U}^-(\delta_1) \cap \mathcal{U}^+(\delta_1) \subset [0,1]^n \tag{5.15}$$

depicted in Figure 1(iii). The final two main results of this section establish some useful properties of this tubular neighborhood that will play an important role in the construction of sub- and supersolutions for (2.1).

PROPOSITION 5.3. *Consider the nonlinear ODE* (5.1) *and suppose that* $(\mathrm{h})_{\S 5}$ *is satisfied. Then the following properties hold.*

(i) *Pick a sufficiently small* $\delta_1 > 0$ *and consider any continuous path* $\Gamma : [0,1] \to [0,1]^n$ *that has* $\Gamma(0) = \mathbf{0}$, $\Gamma(1) = \mathbf{1}$ *and* $\Gamma(t_1) \le \Gamma(t_2)$ *for all* $0 \le t_1 \le t_2 \le 1$. *Then there exists* $t_l < t_\diamond < t_r$ *such that*

$$\Gamma(t_\diamond) = q \in \mathcal{W}_* \cap [0,1]^n \tag{5.16}$$

together with

$$\Gamma(t_l) = q - \delta_1[\mathbf{1} + \psi_l], \qquad \Gamma(t_r) = q + \delta_1[\mathbf{1} + \psi_r] \tag{5.17}$$

for some $\psi_l \in \widehat{T}^-_{\delta_1}(q)$ *and* $\psi_r \in \widehat{T}^+_{\delta_1}(q)$.

(ii) *For any sufficiently small* $\delta_1 > 0$, *there exist constants* $\vartheta = \vartheta(\delta_1) > 0$ *and* $T = T(\delta_1) \gg 1$ *so that for every* $q \in \mathcal{W}_* \cap [0,1]^n$ *and every pair* $\psi_\pm \in \widehat{T}^\pm_{\delta_1}(q)$ *there exist two functions*

$$\phi^\pm_{\delta_1}(t) = \phi^\pm_{\delta_1}(t; q, \psi_\pm) \in C^1([0,\infty), \mathbb{R}^n) \tag{5.18}$$

that satisfy the initial conditions

$$\phi^-_{\delta_1}(0) = q - \delta_1[\mathbf{1} + \psi_-], \qquad \phi^+_{\delta_1}(0) = q + \delta_1[\mathbf{1} + \psi_+], \tag{5.19}$$

together with the estimates

$$\mathbf{0} \le \phi^-_{\delta_1}(t) \le \delta_1 \mathbf{1}, \qquad (1-\delta_1)\mathbf{1} \le \phi^+_{\delta_1}(t) \le \mathbf{1}, \qquad t \ge T, \tag{5.20}$$

and the differential inequalities

$$\frac{d}{dt}\phi^-_{\delta_1}(t) - f\big(\phi^-_{\delta_1}(t)\big) > \vartheta \mathbf{1}, \qquad \frac{d}{dt}\phi^+_{\delta_1}(t) - f\big(\phi^+_{\delta_1}(t)\big) < -\vartheta \mathbf{1}, \qquad t \ge 0. \tag{5.21}$$

PROPOSITION 5.4. *Consider the nonlinear ODE* (5.1) *and suppose that* $(\mathrm{h})_{\S 5}$ *is satisfied. Then there exists a constant* $\kappa_{\mathcal{U}}$ *such that for any* $\delta_1 > 0$, *any* $q \in \mathcal{W}_* \cap [0,1]^n$, *and any*

$$v \in \mathbb{R}^n_{\le \mathbf{0}} \cup \mathbb{R}^n_{\ge \mathbf{0}}, \qquad |v| \ge \kappa_{\mathcal{U}}\, \delta_1, \tag{5.22}$$

we have $q + v \notin \mathcal{U}(\delta_1)$.

Throughout the remainder of this section we treat $(\mathrm{h})_{\S 5}$ as a standing assumption and provide the proofs of Propositions 5.1–5.4. We start by establishing that the vector field of (5.1) points inwards on the boundary of $\mathcal{K}(\epsilon)$.

Proof of Proposition 5.1. Since $v_l > 0$ and $Df(\mathbf{0})v_l = \lambda_l v_l$ for $\lambda_l < 0$, we can pick $\epsilon > 0$ to be sufficiently small to ensure that $Df(-t\epsilon v_l)v_l \le \frac{\lambda}{2} v_l$ for all $0 \le t \le 1$. This implies that

$$f(-\epsilon v_l) = -\epsilon \int_0^1 Df(-t\epsilon v_l) v_l > -\epsilon \frac{\lambda}{2} v_l > \mathbf{0}. \tag{5.23}$$

Similarly, we can ensure that $f(\mathbf{1} + \epsilon v_r) < \mathbf{0}$. Now, consider any $u \in \partial\mathcal{K}(\epsilon)$. Suppose that for some integer $1 \le i \le n$ we have $u_i = -\epsilon(v_l)_i$. We may then compute

$$\begin{aligned} f(u)_i &= f(-\epsilon v_l)_i + \textstyle\sum_{j \ne i} \int_0^1 \partial_j f_i(-\epsilon(1-t)v_l + tu)(u_j + \epsilon(v_l)_j)\, dt \\ &\ge f(-\epsilon v_l)_i + \textstyle\sum_{j \ne i} \mathcal{A}_{ij}(u_j + \epsilon(v_l)_j) > 0. \end{aligned} \tag{5.24}$$

A similar argument shows that $f(u)_i < 0$ if $u_i = 1 + \epsilon(v_r)_i$. In particular, the vector field f points inwards on $\partial\mathcal{K}(\epsilon)$, establishing that $\mathcal{K}(\epsilon)$ is forward invariant under the flow Φ.

We now turn to the claim concerning the equilibria. Let us first show that any $q \in \partial[0,1]^n \setminus \{\mathbf{0}, \mathbf{1}\}$ must have $f(q) \neq \mathbf{0}$. Assuming to the contrary that $f(q) = \mathbf{0}$, we introduce the three sets

$$\Sigma_1 = \{i \mid q_i = 0\}, \qquad \Sigma_2 = \{i \mid q_i = 1\}, \qquad \Sigma_3 = \{j \mid 0 < q_j < 1\} \tag{5.25}$$

and observe that either Σ_1 or Σ_2 is nonempty. If Σ_1 is nonempty, then for every $i \in \Sigma_1$ we can write

$$0 = f(q)_i = \sum_{j \in \Sigma_2 \cup \Sigma_3} \int_0^1 \partial_j f_i(tq) q_j \, dt \geq \sum_{j \in \Sigma_2 \cup \Sigma_3} \mathcal{A}_{ij} q_j \geq 0, \tag{5.26}$$

which shows that $\mathcal{A}_{ij} = 0$ whenever $i \in \Sigma_1$ and $j \in \Sigma_2 \cup \Sigma_3$. Since both these sets are nonempty, this contradicts the irreducibility of $\mathcal{A}$. A similar contradiction can be obtained if Σ_2 is nonempty.

To complete the proof, let us suppose that there exists a sequence $\epsilon_k \to 0$ and $q_k \in \mathcal{K}(\epsilon_k) \setminus [0,1]^n$ with $f(q_k) = \mathbf{0}$. After passing to a subsequence, we must have $q_k \to q_* \in \partial[0,1]^n$ with $f(q_*) = 0$, which implies that $q_* \in \{\mathbf{0}, \mathbf{1}\}$. This is impossible due to the stability assumption (Hf2) on these zeroes. ∎

Proof of Proposition 5.2(i). The compactness of $\mathcal{W}_*$ is a consequence of the disjoint union

$$\mathcal{K}_* = \mathcal{B}(\mathbf{0}) \cup \mathcal{B}(\mathbf{1}) \cup \mathcal{W}_*. \tag{5.27}$$

In addition, the nature of ω-limit sets implies that $\mathcal{W}_*$ inherits the forward invariance of $\mathcal{K}_*$. ∎

In order to prove item (ii) of Proposition 5.2, we need to understand the topology of $\mathcal{W}_*$. In particular, we show that $\mathcal{W}_*$ is completely unordered.

LEMMA 5.5. *For any pair $p, q \in \mathcal{W}_*$ that has $p \neq q$, neither of the two inequalities $p \leq q$ and $q \leq p$ can hold.*

Proof. Without loss of generality, let us suppose that $p \leq q$. The comparison principle now implies that for any $t > 0$ we have

$$\Phi(t;p) < \Phi(t;q). \tag{5.28}$$

Pick any $t_* > 0$ and consider the ray

$$L = \{u \in \mathbb{R}^n \mid u = \vartheta\Phi(t_*;p) + (1-\vartheta)\Phi(t_*;q) \text{ with } 0 < \vartheta < 1\}. \tag{5.29}$$

A result due to Hirsch [19, Lem. 4.3] states that the set of $u \in L$ that do not converge to an equilibrium is at most countable. Therefore, since the set of equilibria in $\mathcal{K}_*$ is finite, there exist $u_1, u_2 \in L$ with $u_1 < u_2$ that both converge to the same equilibrium q_∞. Now, we must have $q_\infty \neq \mathbf{0}$ and $q_\infty \neq \mathbf{1}$ since otherwise $\Phi(t;p) \to \mathbf{0}$ or $\Phi(t;q) \to \mathbf{1}$ as $t \to \infty$. In particular, by Proposition 5.1 and (Hf3) the equilibrium q_∞ must be an unstable equilibrium. Obviously, u_1 and u_2 both lie on the center-stable manifold $\mathcal{W}^{cs}(q_\infty)$ and $\Phi(t;u_1) < \Phi(t;u_2)$ for all $t \geq 0$.

Let us write $\lambda_\infty > 0$ for the largest eigenvalue of $Df(q_\infty)$ and $v_\infty > \mathbf{0}$ for an associated eigenvector. In addition, we write $\mathcal{V}^{cs} \subset \mathbb{R}^n$ for the subspace spanned by the generalized eigenvectors of $Df(q_\infty)$ that are associated to eigenvalues that have $\mathrm{Re}\,\lambda \le 0$. We claim that any nonzero $v \in \mathcal{V}^{cs}$ cannot have $v \ge \mathbf{0}$ or $v \le \mathbf{0}$. Indeed, if this is the case, then by the comparison principle we have $\Psi(t;q_\infty)v > \mathbf{0}$ for every $t > 0$, which allows us to pick t_0 and $\epsilon > 0$ with $\Psi(t_0;q_\infty)v > \epsilon v_\infty$. This implies that $\Psi(t+t_0;q_\infty)v > \epsilon e^{\lambda_\infty t} v_\infty$, which gives a contradiction. In particular, there exists $C > 0$ such that for any nonzero $v \in \mathcal{V}^{cs}$ we have

$$v + |v|\, C\mathbf{1} \notin \mathbb{R}^n_{\ge \mathbf{0}}, \qquad v - |v|\, C\mathbf{1} \notin \mathbb{R}^n_{\le \mathbf{0}}. \tag{5.30}$$

In the vicinity of q_∞, the center-stable manifold $\mathcal{W}^{cs}(q_\infty)$ can be written as a graph over $\mathcal{V}^{cs}$. However, in view of (5.30) this contradicts the fact that $\Phi(t;u_1) < \Phi(t;u_2)$ must hold for all $t \ge 0$. ∎

Proof of Proposition 5.2(ii). Write $\Gamma_* = \{\Gamma(t)\}_{t=0}^1$ and note that Γ_* is a closed subset of $\mathcal{K}_*$. The existence of t_* follows from the fact that the nonempty sets $\mathcal{B}(\mathbf{0}) \cap \Gamma_*$ and $\mathcal{B}(\mathbf{1}) \cap \Gamma_*$ are both open in Γ_*, which means they cannot cover Γ_* together. The uniqueness of t_* follows from Lemma 5.5. ∎

We now set out to address the smoothness of the manifold $\mathcal{W}_*$. To this end, we pick any $u \in \mathbb{R}^n$ and introduce the hyperplane

$$\mathcal{V}_u = \{v \in \mathbb{R}^n \mid \langle v, \mathbf{1}\rangle = \langle u, \mathbf{1}\rangle\}, \tag{5.31}$$

in which $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. In particular, $\mathcal{V}_u$ contains u and is perpendicular to $\mathbf{1}$. For any $\delta > 0$, we also introduce the open subset

$$\mathcal{V}_{u,\delta} = \{v \in \mathcal{V}_u \mid |v - u| < \delta\}. \tag{5.32}$$

As a first step, we modify an argument due to Hirsch [18] which allows us to show that $\mathcal{W}_*$ is a Lipschitz-smooth manifold of dimension $n-1$.

LEMMA 5.6 (cf. [18, Thm. 3.1]). *Consider any $q \in \mathcal{W}_*$ for which $q \notin \partial\mathcal{K}_*$. Then there exists a constant $\delta > 0$ and a Lipschitz-smooth function $\rho = \rho_q : \mathcal{V}_{q,\delta} \to \mathbb{R}$ such that*

$$v + \rho(v)\mathbf{1} \in \mathcal{W}_* \tag{5.33}$$

for all $v \in \mathcal{V}_{q,\delta}$.

Proof. Pick $\epsilon > 0$ to be sufficiently small to ensure that the two points $q_\pm := q \pm \epsilon\mathbf{1}$ satisfy $q_\pm \in \mathcal{K}_*$ but $q_\pm \notin \partial\mathcal{K}_*$. Lemma 5.5 implies that $q_- \in \mathcal{B}(\mathbf{0})$ and $q_+ \in \mathcal{B}(\mathbf{1})$. Since both these basins of attraction are open, there exists $\delta > 0$ such that $\mathcal{V}_{q_-,\delta} \subset \mathcal{B}(\mathbf{0})$ and $\mathcal{V}_{q_+,\delta} \subset \mathcal{B}(\mathbf{1})$. Proposition 5.2(ii) now implies that for every pair $v_\pm \in \mathcal{V}_{q_\pm,\delta}$ that is related by $v_+ - v_- = 2\epsilon\mathbf{1}$, the line between v_- and v_+ has a unique intersection with $\mathcal{W}_*$. This intersection point can be used to define $\rho(v)$ for $v = \frac{1}{2}v_- + \frac{1}{2}v_+ \in \mathcal{V}_{q,\delta}$.

To see that ρ is Lipschitz continuous, consider two sequences v_k, $\widetilde{v}_k$ in $\mathcal{V}_{q,\delta}$ that have

$$|\rho(v_k) - \rho(\widetilde{v}_k)| \,/\, |v_k - \widetilde{v}_k| \to \infty \text{ as } k \to \infty. \tag{5.34}$$

Write $\pi : \mathbb{R}^n \to \mathcal{V}_q$ for the linear projection onto $\mathcal{V}_q$ along $\mathbf{1}$. Upon defining

$$w_k = v_k + \rho(v_k)\mathbf{1}, \qquad \widetilde{w}_k = \widetilde{v}_k + \rho(\widetilde{v}_k)\mathbf{1}, \tag{5.35}$$

we obviously have

$$v_k = \pi w_k, \qquad \widetilde{v}_k = \pi \widetilde{w}_k. \tag{5.36}$$

In addition, we can compute

$$|w_k - \widetilde{w}_k| \,/\, |v_k - \widetilde{v}_k| \geq \Big| |\rho(v_k)\mathbf{1} - \rho(\widetilde{v}_k)\mathbf{1}| - |v_k - \widetilde{v}_k| \Big| / \, |v_k - \widetilde{v}_k| \to \infty, \tag{5.37}$$

which implies that

$$|w_k - \widetilde{w}_k| \,/\, |v_k - \widetilde{v}_k| \to \infty \text{ as } k \to \infty. \tag{5.38}$$

Upon writing $\alpha_k = [w_k - \widetilde{w}_k] / \, |w_k - \widetilde{w}_k|$, this shows that

$$|\alpha_k| \,/\, |\pi \alpha_k| = 1 / \, |\pi \alpha_k| \to \infty \text{ as } k \to \infty, \tag{5.39}$$

which means that $\pi\alpha_k \to 0$ as $k \to \infty$. Switching v_k and $\widetilde{v}_k$ for the appropriate values of k, this implies that $\alpha_k \to \mathbf{1}/\,|\mathbf{1}|$ as $k \to \infty$, which shows that $\alpha_{k_*} > \mathbf{0}$ for some integer $k_* > 0$. However, the resulting inequality $w_{k_*} > \widetilde{w}_{k_*}$ contradicts Lemma 5.5. ∎

Before we can obtain additional smoothness properties for the separatrix $\mathcal{W}_*$, we need to develop some preliminary results for the tangent space $T(\mathcal{W}_*)$. In particular, we set out to prove part (iv) of Proposition 5.2, which provides an exponential separation for the linearized flow Ψ acting on $T(q)$ and on the perpendicular direction $\mathbf{1}$.

LEMMA 5.7. *The set $T(\mathcal{W}_*) \cap (\mathcal{W}_* \times \mathbb{S}^{n-1})$ is compact in $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Consider any sequence $\{(q_k, \psi_k)\} \in T(\mathcal{W}_*)$ that has $|\psi_k| = 1$ for all $k \in \mathbb{N}$. Passing to a subsequence, we find $q_k \to q_* \in \mathcal{W}_*$ and $\psi_k \to \psi_* \in \mathbb{S}^{n-1}$ as $k \to \infty$, and it suffices to show that $\psi_* \in T(q_*)$. If not, there exists $T > 0$ such that

$$\Psi(T; q_*)\psi_* \in \mathbb{R}^n_{\geq \mathbf{0}} \cup \mathbb{R}^n_{\leq \mathbf{0}}. \tag{5.40}$$

The proof of the comparison principle in Proposition 4.1 now implies that for all $t > 0$ we actually have

$$\Psi(T + t; q_*)\psi_* \in \mathbb{R}^n_{> \mathbf{0}} \cup \mathbb{R}^n_{< \mathbf{0}}. \tag{5.41}$$

Basic continuity properties can now be used to show that for all q sufficiently close to q_* and all ψ sufficiently close to ψ_* we have

$$\Psi(T + t; q)\psi \in \mathbb{R}^n_{> \mathbf{0}} \cup \mathbb{R}^n_{< \mathbf{0}}, \tag{5.42}$$

which leads to a contradiction. ∎

LEMMA 5.8. *There exists $\delta_* > 0$ such that*

$$\Psi(t; q)\mathbf{1} \geq \delta_* \, |\Psi(t; q)\mathbf{1}| \, \mathbf{1} \tag{5.43}$$

holds for all $q \in \mathcal{W}_*$.

Proof. Fixing $q \in \mathcal{W}_*$, let us consider the function

$$g(t) = \Psi(t; q)\mathbf{1} / \, |\Psi(t; q)\mathbf{1}| \, . \tag{5.44}$$

Upon writing $q(t) = \Phi(t\,; q)$, a short computation shows that we may write $g'(t) = \mathcal{G}\big(t, g(t)\big)$ after introducing the function

$$\mathcal{G}\big(t, g\big) = Df\big(q(t)\big)g - g\langle Df\big(q(t)\big)g, g\mathbf{1}\rangle. \tag{5.45}$$

By construction, we have $g(t) \in \mathbb{S}^{n-1} \cap \mathbb{R}^n_{> \mathbf{0}}$ for all $t \geq 0$. Let us suppose that we have a sequence $t_k \to \infty$ with $g(t_k) \to \partial\mathbb{R}^n_{\geq \mathbf{0}}$. By compactness, we may pass to a subsequence for which $g(t_k) \to g_*$ for some $g_* \in \partial \mathbb{R}^n_{\geq \mathbf{0}} \cap \mathbb{S}^{n-1}$. Arguing similarly as in the proof of Proposition 5.1, the conditions (Hf1) and (HA) imply that there exists at least one integer $1 \leq i \leq n$ with $(g_*)_i = 0$ and $\mathcal{G}_i(t, g_*) > \vartheta > 0$ for all $t \geq 0$. Using the fact that $q(t)$ remains in the compact set $\mathcal{W}_*$ for all $t \geq 0$, we hence see that there exists $\delta > 0$ such that $\mathcal{G}_i(t, g) > \frac{1}{2}\vartheta > 0$ whenever $|g - g_*| < \delta$. This however precludes $g(t)$ from approaching g_* and hence leads to a contradiction. □

Proof of Proposition 5.2(iv). For any $q \in \mathcal{W}_*$ and $v \in T(q) \cap \mathbb{S}^{n-1}$, we introduce the two functions $\psi_v(t) = \Psi(t; q)v$ and $\phi_v(t) = \Psi(t; q)\mathrm{Abs}(v)$, where $\mathrm{Abs}(v) \in \mathbb{R}^n_{\geq \mathbf{0}}$ is the vector given by $\mathrm{Abs}(v)_i = |v_i|$. Remembering that we cannot have $v \geq \mathbf{0}$ or $v \leq \mathbf{0}$, the comparion principle now implies that for all $t > 0$ we have

$$-\phi_{v,q}(t) < \psi_{v,q}(t) < \phi_{v,q}(t). \tag{5.46}$$

Pick any $T_* > 0$. We now claim that there exists $0 < \vartheta < 1$ such that for all $(q, v) \in T(\mathcal{W}_*)$ with $|v| = 1$, we have

$$-\vartheta\phi_{v,q}(T_*) \leq \psi_{v,q}(T_*) \leq \vartheta\phi_{v,q}(T_*). \tag{5.47}$$

If not, there exist sequences $(q_k, v_k) \in T(\mathcal{W}_*)$, $i_k \in \{1, \ldots, n\}$ and $0 < \vartheta_k < 1$ with $|v_k| = 1$ and $\vartheta_k \to 1$ such that

$$|\psi_{v_k,q_k}(T_*)_{i_k}| > \vartheta_k\phi_{v_k,q_k}(T_*)_{i_k}. \tag{5.48}$$

Lemma 5.7 shows that after passing to a subsequence, we have $q_k \to q_* \in \mathcal{W}_*$, $v_k \to v_* \in T(q_*)$, and $i_k \to i_*$. Continuity properties of Ψ now imply that

$$|\psi_{v_*,q_*}(T_*)_{i_*}| = \phi_{v_*,q_*}(T_*)_{i_*}, \tag{5.49}$$

which gives a contradiction. Using the fact that $\mathrm{Abs}(\psi_{v,q}(T_*)) \leq \vartheta\phi_{v,q}(T_*)$, we may iterate (5.47) to obtain

$$-\vartheta^k\Psi(kT_*; q)\mathbf{1} \leq -\vartheta^k\phi_{v,q}(kT_*) \leq \psi_{v,q}(kT_*) \leq \vartheta^k\phi_{v,q}(kT_*) \leq \vartheta^k\Psi(kT_*; q)\mathbf{1}, \tag{5.50}$$

which suffices to complete the proof. □

In order to establish that the separatrix $\mathcal{W}_*$ is C^1-smooth, we need to study the smoothness of the map $v \mapsto \rho_q(v)$ introduced in Lemma 5.6. In particular, we show that the sets $T(q)$ are in fact vector spaces that can be used to describe the derivatives of the map ρ_q.

LEMMA 5.9. *Pick any $q \in \mathcal{W}_*$ and consider $\psi_1, \psi_2 \in T(q)$. If either $\psi_1 \leq \psi_2$ or $\psi_1 \geq \psi_2$ holds, then in fact $\psi_1 = \psi_2$.*

Proof. Let us suppose for concreteness that $\psi_1 \leq \psi_2$ but $\psi_1 \neq \psi_2$. For all $t > 0$, the comparison principle now implies that

$$\Psi(t; q)\psi_1 < \Psi(t; q)\psi_2. \tag{5.51}$$

In particular, there exist $t_* > 0$ and $\epsilon > 0$ such that

$$\Psi(t_*; q)[\psi_1 + \epsilon\mathbf{1}] < \Psi(t_*; q)\psi_2. \tag{5.52}$$

Lemma 5.8 and Proposition 5.2(iv) together imply that for sufficiently large $T > 0$ we have

$$\Psi(t_* + T; q)[\psi_1 + \epsilon\mathbf{1}] > \mathbf{0}. \tag{5.53}$$

This however implies that also $\Psi(t_* + T; q)\psi_2 > \mathbf{0}$, which contradicts the fact that $\psi_2 \in T(q)$. ∎

LEMMA 5.10. *Recall the hyperplane $\mathcal{V}_0$ defined in* (5.31). *For each $q \in \mathcal{W}_*$, there exists a bounded linear map $\tau_q : \mathcal{V}_0 \to \mathbb{R}$ such that for any $v \in \mathcal{V}_0$ we have*

$$v + (\tau_q v)\mathbf{1} \in T(q). \tag{5.54}$$

In particular, the space $T(q)$ is an $(n-1)$-dimensional vector space.

Proof. We first show that $T(q)$ is a vector space. Observe that the definition (5.7) directly implies that for any $\lambda \in \mathbb{R}$ we have $\lambda\psi \in T(q)$ whenever $\psi \in T(q)$. Suppose now that there exist $\psi_1, \psi_2 \in T(q)$ with $\psi_1 + \psi_2 \notin T(q)$. This implies that there exists $t_* > 0$ such that

$$\Psi(t_*; q)\psi_1 + \Psi(t_*; q)\psi_2 \geq \mathbf{0}, \tag{5.55}$$

possibly after switching $\psi_1 \mapsto -\psi_1$ and $\psi_2 \mapsto -\psi_2$. In particular, we have $\Psi(t_*; q)\psi_1 \geq -\Psi(t_*; q)\psi_2$. This however contradicts Lemma 5.9 since both $\Psi(t_*; q)\psi_1$ and $\Psi(t_*; q)\psi_2$ are contained in $T\big(\Phi(t_*; q)\big)$.

Let us now consider the open sets

$$\begin{aligned} V_+(q) &= \{\psi \in \mathbb{R}^n \mid \Psi(t_*; q)\psi > \mathbf{0} \text{ for some } t_* \geq 0\}, \\ V_-(q) &= \{\psi \in \mathbb{R}^n \mid \Psi(t_*; q)\psi < \mathbf{0} \text{ for some } t_* \geq 0\}. \end{aligned} \tag{5.56}$$

Pick any $v \in \mathcal{V}_0$. By choosing $\lambda = 2\,|v|$, we can ensure that $v \pm \lambda\mathbf{1} \in V_\pm(q)$. The nonordering of $T(q)$ now implies that there exists precisely one $\tau \in (-\lambda, \lambda)$ such that $v + \tau\mathbf{1} \in T(q)$, which can be used to define the value $\tau_q v$. ∎

LEMMA 5.11. *Consider any $q \in \mathcal{W}_*$ for which $q \notin \partial\mathcal{K}_*$. The function $\rho = \rho_q : \mathcal{V}_{q,\delta} \to \mathbb{R}$ defined in Lemma* 5.6 *is C^1-smooth with*

$$D\rho(v) = \tau_{q(v)}, \qquad q(v) = v + \rho(v)\mathbf{1}. \tag{5.57}$$

Proof. We start by showing that ρ is differentiable at q. Pick any $v_0 \in \mathcal{V}_0$ with $|v| = 1$. Let h_k be a sequence of real numbers with $h_k \to 0$ and consider the sequence

$$\alpha_k := \frac{1}{h_k}[\rho(q + h_k v_0) - \rho(q)] = \frac{1}{h_k}\rho(q + h_k v_0), \tag{5.58}$$

where we used $\rho(q) = 0$. The Lipschitz continuity of g implies that α_k is bounded. It hence suffices to show that for any convergent subsequence $\alpha_k \to \alpha_*$ we in fact have $\alpha_* = \tau_q v_0$. Suppose therefore that $\alpha_* \neq \tau_q v_0$ and introduce the vectors

$$v_k = q + h_k v_0 \in \mathcal{V}_{q,\delta}, \qquad w_k = v_k + \rho(v_k)\mathbf{1} \in \mathcal{W}_*. \tag{5.59}$$

By construction, we have

$$w_k = q + h_k[v_0 + \alpha_k\mathbf{1}]. \tag{5.60}$$

Upon writing

$$z_k(t) := \Phi(t; w_k) - \Phi(t; q), \tag{5.61}$$

together with $q(t) = \Phi(t; q)$, we may compute

$$\begin{aligned} z_k'(t) &= \left[\int_0^1 Df\big(q(t) + s z_k(t)\big)\, ds\right] z_k(t) \\ &= Df\big(q(t)\big) z_k(t) + \mathcal{N}\big(t, z_k(t)\big), \end{aligned} \tag{5.62}$$

in which we have $\mathcal{N}(t,z) = O(|z|^2)$ and $D_2\mathcal{N}(t,z) = O(|z|)$ as $z \to 0$, uniformly for $t \geq 0$. In particular, we may write

$$z_k(t) = \Psi(t;q)z_k(0) + \Psi(t;q)\int_0^t \Psi\big(-s;q(s)\big)\mathcal{N}\big(s, z_k(s)\big)\,ds. \tag{5.63}$$

Notice that

$$z_k(0) = h_k[v_0 + \alpha_k] = h_k\psi + \vartheta h_k \mathbf{1} + o(h_k) \text{ as } k \to \infty \tag{5.64}$$

with $\psi = v_0 + (\tau_q v_0)\mathbf{1} \in T(q)$ and $\vartheta = \alpha_* - \tau_q v_0 \neq 0$. In particular, there exists $t_* > 0$ such that

$$\Psi(t_*;q)[\psi + \vartheta\mathbf{1}] \in \mathbb{R}^n_{>\mathbf{0}} \cup \mathbb{R}^n_{<\mathbf{0}}. \tag{5.65}$$

This means that for sufficiently large k we must have either $z_k(t_*) > \mathbf{0}$ or $z_k(t_*) < \mathbf{0}$, which violates the nonordering property of $\mathcal{W}_*$ established in Lemma 5.5. Similar arguments can be used to show that ρ is differentiable at all points $v \in \mathcal{V}_{q,\delta}$.

To see that $(q,v) \mapsto \tau_q v$ is continuous, consider a sequence $v_k \to v_* \in \mathcal{V}_0$ and $q_k \to q_* \in \mathcal{W}_*$. Writing $\psi_k = v_k + (\tau_{q_k} v_k)\mathbf{1} \in T(q_k)$, we observe that the sequence $\{\psi_k\}$ is bounded since $\{v_k\}$ is bounded and $\|\tau_{q_k}\| \leq 2$. Consider an arbitrary convergent subsequence $\psi_k \to \psi_* \in \mathbb{R}^n$. Recalling the linear projection $\pi : \mathbb{R}^n \to \mathcal{V}_0$ onto $\mathcal{V}_0$ along $\mathbf{1}$, we note that $\pi\psi_k = v_k$, which in turn implies that $\pi\psi_* = v_*$. Since $T(\mathcal{W}_*)$ is closed, we have $\psi_* \in T(q_*)$, which shows that

$$\psi_* = \pi\psi_* + (\tau_{q_*}\pi\psi_*)\mathbf{1} = v_* + (\tau_{q_*} v_*)\mathbf{1}, \tag{5.66}$$

as desired. ∎

We now proceed to establish part (v) of Proposition 5.2. The main idea is that $\Psi(t;q)\mathbf{1}$ cannot decay exponentially as $t \to \infty$, since a nonlinear argument would then allow us to show that $\Phi(t;q+\epsilon\mathbf{1})$ cannot converge to $\mathbf{1}$ as $t \to \infty$ for all small $\epsilon > 0$.

LEMMA 5.12. *For every $K > 0$ and $\epsilon > 0$, there exists a constant T_* such that for every $q \in \mathcal{W}_*$ we have*

$$|\Psi(t_*;q)\mathbf{1}| \geq Ke^{-\epsilon t_*} \tag{5.67}$$

for some $t_* = t_*(q)$ that has $0 \leq t_* \leq T_*$.

Proof. Arguing to the contrary, there exist two constants $K_* > 0$ and $\epsilon_* > 0$ together with two sequences $T_k \to \infty$ and $q_k \in \mathcal{W}_*$ such that

$$|\Psi(t;q_k)\mathbf{1}| < K_* e^{-\epsilon_* t} \text{ for all } 0 \leq t \leq T_k. \tag{5.68}$$

After passing to a subsequence, we have $q_k \to q_* \in \mathcal{W}_*$ as $k \to \infty$ and by continuity also

$$|\Psi(t;q_*)\mathbf{1}| \leq K_* e^{-\epsilon_* t} \text{ for all } t \geq 0. \tag{5.69}$$

In order to show that this cannot happen, we will construct a supersolution to the nonlinear ODE (5.1). In particular, we write $q_*(t) = \Phi(t;q_*)$ and consider the function

$$u^+(t) = q_*(t) + \delta_1(1 + \delta_1 Ct)\Psi(t;q_*)\mathbf{1}, \tag{5.70}$$

in which the constants $C \gg 1$ and $\delta_1 > 0$ remain to be determined. Upon writing

$$\mathcal{J}^+(t) = \frac{d}{dt}u^+(t) - f\big(u^+(t)\big), \tag{5.71}$$

we may compute

$$\begin{aligned}\mathcal{J}^+(t) &= f\big(q_*(t)\big) + Df\big(q_*(t)\big)\delta_1(1+\delta_1 Ct)\Psi(t;q_*)\mathbf{1} + \delta_1^2 C\Psi(t;q_*)\mathbf{1} \\ &\quad - f\big(q_*(t) + \delta_1(1+\delta_1 Ct)\Psi(t;q_*)\mathbf{1}\big) \\ &= -\big[f\big(q_*(t) + \delta_1(1+\delta_1 Ct)\Psi(t;q_*)\mathbf{1}\big) - f\big(q_*(t)\big) \\ &\qquad - Df\big(q_*(t)\big)\delta_1(1+\delta_1 Ct)\Psi(t;q_*)\mathbf{1}\big] \\ &\quad + \delta_1^2 C\Psi(t;q_*)\mathbf{1}.\end{aligned} \tag{5.72}$$

In particular, we find that

$$\left|\mathcal{J}^+(t) - \delta_1^2 C\Psi(t;q_*)\mathbf{1}\right| \quad \le \quad \tfrac{1}{2}\left\|D^2 f\right\| \delta_1^2\big(1+\delta_1 Ct)^2 \left|\Psi(t;q_*)\mathbf{1}\right|^2. \tag{5.73}$$

In view of Lemma 5.8, it is possible to choose $C \gg 1$ in such a way that we have

$$C\Psi(t;q_*)\mathbf{1} \ge 2K_* \left\|D^2 f\right\| \left|\Psi(t;q_*)\mathbf{1}\right| \mathbf{1} \tag{5.74}$$

for all $t \ge 0$. In addition, the assumption (5.69) allows us to choose $\delta_1 > 0$ in such a way that

$$(1+\delta_1 Ct)^2 \left|\Psi(t;q_*)\mathbf{1}\right| \le 2K_* \tag{5.75}$$

for all $t \ge 0$. These choices ensure that for all $t \ge 0$ we have

$$\left|\mathcal{J}^+(t) - \delta_1^2 C\Psi(t;q_*)\mathbf{1}\right| \mathbf{1} \le \frac{1}{2}\delta_1^2 C\Psi(t;q_*)\mathbf{1} \tag{5.76}$$

and hence $\mathcal{J}^+(t) \ge \mathbf{0}$. In particular, $u^+(t)$ is a supersolution for (5.1), which means that for all $t \ge 0$ we have

$$u^+(t) \ge \Phi\big(t;u^+(0)\big) > q_*(t). \tag{5.77}$$

However, after possibly decreasing the size of $\delta_1 > 0$ and increasing the size of $\epsilon_* > 0$ that appears in the definition of $\mathcal{W}_*$, we see that $\Phi\big(t;u^+(0)\big) \to \mathbf{1}$ as $t \to \infty$. This is precluded by the definition (5.70), which requires $u^+(t) - q_*(t) \to 0$ as $t \to \infty$. ☐

Proof of Proposition 5.2(v). Recall the constant $\delta_* > 0$ from Lemma 5.8 and pick $K > 0$ in such a way that $K\delta_* > 1$. Recall the constant $T_* = T_*(K,\epsilon)$ from Lemma 5.12 and choose $\vartheta > 0$ to ensure that

$$\left|\Psi(t;q)\mathbf{1}\right| \ge \vartheta \text{ for all } q \in \mathcal{W}_* \text{ and } 0 \le t \le T_*, \tag{5.78}$$

which is possible by compactness. For every $q \in \mathcal{W}_*$ we may estimate

$$\Psi(t_*(q);q)\mathbf{1} \ge \delta_* \left|\Psi(t_*(q);q)\mathbf{1}\right| \mathbf{1} \ge K\delta_* e^{-\epsilon t_*(q)}\mathbf{1} \ge e^{-\epsilon t_*(q)}\mathbf{1}. \tag{5.79}$$

In particular, for any $t \ge 0$ there is a chain $0 := t_0 < t_1 < \cdots < t_\ell$ with

$$t_i - t_{i-1} \le T_*, \qquad t - t_\ell \le T_*, \qquad \Psi(t_i;q)\mathbf{1} \ge e^{-\epsilon t_i}\mathbf{1} \tag{5.80}$$

for all $1 \le i \le \ell$. This implies the desired conclusion

$$\left|\Psi(t;q)\mathbf{1}\right| \ge e^{-\epsilon t_\ell}\vartheta \ge e^{-\epsilon t}\vartheta. \qquad ☐ \tag{5.81}$$

In the final part of this section, we provide proofs for Propositions 5.3–5.4. We start by establishing some basic properties of the restriction spaces $\widehat{T}(q)$ and $\widehat{T}^{\pm}_{\delta_1}(q)$, which will be used to construct the functions $\phi^{\pm}_{\delta_1}$ mentioned in part (ii) of Proposition 5.3.

LEMMA 5.13. *The spaces $\widehat{T}(q)$ and $\widehat{T}^{\pm}_{\delta_1}(q)$ satisfy the following properties.*

(i) *There exists a constant $C \gg 1$ such that*

$$1 \le |\mathbf{1} + \psi| \le C \tag{5.82}$$

holds for any $q \in \mathcal{W}_*$ and any $\psi \in \widehat{T}(q)$.

(ii) *There exists a constant $\vartheta > 0$ such that for any $q \in \mathcal{W}_*$ and any $v \ge \mathbf{0}$, the inequalities*

$$|q + v - \widetilde{q}| \ge \vartheta\, |v|\,, \qquad |q - v - \widetilde{q}| \ge \vartheta\, |v| \tag{5.83}$$

hold for all $\widetilde{q} \in \mathcal{W}_*$.

(iii) *For all sufficiently small $\delta_1 > 0$, there exists a constant $\epsilon = \epsilon(\delta_1) > 0$ such that for any $q \in \mathcal{W}_* \cap [0,1]^n$ and any pair $\psi_\pm \in \widehat{T}^{\pm}_{\delta_1}(q)$, the vectors*

$$u_- = q - \delta_1[\mathbf{1} + \psi_-] + \epsilon\mathbf{1}, \qquad u_+ = q + \delta_1[\mathbf{1} + \psi_+] - \epsilon\mathbf{1} \tag{5.84}$$

satisfy the inequalities

$$\mathbf{0} \le u_- < q_-, \qquad q_+ < u_+ \le \mathbf{1} \tag{5.85}$$

for some pair $q_\pm \in \mathcal{W}_*$. In particular, we have the limits

$$\lim_{t\to\infty} \Phi(t; u_-) = \mathbf{0}, \qquad \lim_{t\to\infty} \Phi(t; u_+) = \mathbf{1}. \tag{5.86}$$

Proof. The lower bound in (i) is trivial, since we cannot have $\psi \le \mathbf{0}$. The upper bound in (i) follows from the fact that the function

$$\mathcal{G} : T(\mathcal{W}_*) \cap \big(\mathcal{W}_* \times \mathbb{S}^{n-1}\big) \to \mathbb{R} \tag{5.87}$$

defined by

$$\mathcal{G}(q, \psi) = \max_{1 \le i \le n} \{\psi_i\} / \min_{1 \le i \le n} \{\psi_i\} < 0 \tag{5.88}$$

is well defined and continuous.

Restricting ourselves to sufficiently small $v \in \mathbb{R}^n_{\ge \mathbf{0}}$, the statement in (ii) follows from the compactness of $\mathcal{T}(\mathcal{W}_*)$ together with the fact that any $\psi \in T(q)$ cannot have $\psi \le \mathbf{0}$ or $\psi \ge \mathbf{0}$. For large $|v|$, we can use the compactness of $\mathcal{W}_*$ together with the fact that $q \pm v \notin \mathcal{W}_*$. Finally, the statements in (iii) follow directly from (i) and (ii). ∎

LEMMA 5.14. *There exists a constant $K > 0$ such that for any pair $w_\pm \in \mathcal{K}_*$ that has $w_- < w_+$, the function*

$$\phi(t) = \phi(t; w_-, w_+) = e^{-Kt}\Phi(t; w_-) + (1 - e^{-Kt})\Phi(t; w_+) \tag{5.89}$$

satisfies the estimate

$$\phi'(t) - f\big(\phi(t)\big) \ge \frac{1}{2} K e^{-Kt}\big[\Phi(t; w_+) - \Phi(t; w_-)\big] > \mathbf{0} \tag{5.90}$$

for all $t \ge 0$.

Proof. Writing $\mathcal{J}(t) = \phi'(t) - f\big(\phi(t)\big)$, we can compute

$$\begin{aligned} \mathcal{J}(t) = {} & K e^{-Kt}\big[\Phi(t; w_+) - \Phi(t; w_-)\big] \\ & + e^{-Kt}\big[f\big(\Phi(t; w_-)\big) - f\big(\phi(t)\big)\big] + (1 - e^{-Kt})\big[f\big(\Phi(t; w_+)\big) - f\big(\phi(t)\big)\big]. \end{aligned} \tag{5.91}$$

This allows us to estimate

$$
\begin{aligned}
&\big|\mathcal{J}(t) - Ke^{-Kt}\big[\Phi(t;w_+) - \Phi(t;w_-)\big]\big| \\
&\quad \le e^{-Kt}\,\|Df\|\,|\phi(t) - \Phi(t;w_-)| \\
&\qquad + (1-e^{-Kt})\,\|Df\|\,|\phi(t) - \Phi(t;w_+)| \\
&\quad \le e^{-Kt}\,\|Df\|\,(1-e^{-Kt})\,|\Phi(t;w_+) - \Phi(t;w_-)| \\
&\qquad + (1-e^{-Kt})\,\|Df\|\,e^{-Kt}\,|\Phi(t;w_+) - \Phi(t;w_-)| \\
&\quad \le 2e^{-Kt}\,\|Df\|\,|\Phi(t;w_+) - \Phi(t;w_-)|\,.
\end{aligned} \tag{5.92}
$$

The desired bound (5.90) now follows upon choosing $K = 4\,\|Df\|$. ∎

Proof of Proposition 5.3(i). The existence of $t_\diamond$ follows directly from Proposition 5.2(ii). The existence of t_l and t_r follows from the fact that the $(n-1)$-dimensional space $T(q)$ can be written as a graph over the plane $\mathcal{V}_0$, which is perpendicular to $\mathbf{1}$. ∎

Proof of Proposition 5.3(ii). We restrict ourselves to constructing the function $\phi^-_{\delta_1}(t)$. Recall the eigenvalue $\lambda_l < 0$ and the eigenvector $v_l \geq \mathbf{0}$ for $Df(\mathbf{0})$ that were defined in (4.27). Note that there exists a positive cone $\mathcal{C} \subset \mathbb{R}^n_{\geq \mathbf{0}}$ together with a constant $\kappa > 0$ such that $v_l \in \mathrm{int}(K)$ while

$$f(u) \leq -\frac{1}{2}\,|\lambda_l|\,u \tag{5.93}$$

for any $u \in \mathcal{C}_\kappa$, in which

$$\mathcal{C}_\kappa = \{u \in \mathcal{C} \mid |u| \leq \kappa\}. \tag{5.94}$$

Since λ_l is a simple eigenvalue for $Df(0)$ and v_l is the only eigenvector of $Df(0)$ in $\mathbb{R}^n_{\geq 0}$, it is possible to choose a second cone $\mathcal{C}'$ and constant $\kappa' > 0$ in such a way that

$$v_l \in \mathrm{int}(\mathcal{C}') \subset \mathcal{C}' \subset \mathcal{C}, \qquad \kappa' < \kappa, \tag{5.95}$$

both hold, together with the trapping bound

$$\Phi(t;u') \in \mathcal{C}_\kappa \text{ for all } t \geq 0 \text{ and } u' \in \mathcal{C}_{\kappa'}. \tag{5.96}$$

For any $\delta_1 > 0$, $q \in \mathcal{W}_* \cap [0,1]^n$ and $\psi \in \widehat{T}^-_{\delta_1}(q)$, we now introduce the pair of vectors

$$w_- = w_-(\delta_1, q, \psi) = q - \delta_1[\mathbf{1} + \psi], \qquad w_+ = w_+(\delta_1, q, \psi) = q - \delta_1[\mathbf{1} + \psi] + \epsilon(\delta_1)\mathbf{1}, \tag{5.97}$$

using the quantity $\epsilon(\delta_1)$ defined in Lemma 5.13(iii). Since both $w_\pm \in \mathcal{B}(\mathbf{0})$ and $w_\pm \geq \mathbf{0}$, we find that there exists a time T such that $\Phi(t^\pm_*; w_\pm) \in \mathcal{C}'_{\kappa'}$ for some pair $0 \leq t^\pm_* \leq T$. By compactness, this time $T = T(\delta_1)$ can be chosen to be independent of the pair (q, ψ).

We now construct $\phi^-_{\delta_1}$ by recalling the function ϕ from Lemma 5.14 and writing

$$\phi^-_{\delta_1}(t) = \phi(\gamma_{\delta_1}(t); w_-, w_+), \tag{5.98}$$

where $\gamma_{\delta_1} : [0,\infty) \to [0,\infty)$ is a C^1-smooth function that has $\gamma_{\delta_1}(t) = t$ for all $0 \leq t \leq T(\delta_1)$, together with

$$0 < \gamma'_{\delta_1}(t) \leq 1, \qquad T(\delta_1) \leq \gamma_{\delta_1}(t) \leq T(\delta_1) + 1, \qquad t \geq T(\delta_1). \tag{5.99}$$

Notice that

$$\frac{d}{dt}\phi^-_{\delta_1}(t) - f\big(\phi^-_{\delta_1}(t)\big) = \gamma'_\delta(t)\phi'(\gamma_\delta(t); w_-, w_+) - f\big(\phi(\gamma_\delta(t); w_-, w_+)\big). \tag{5.100}$$

By compactness, there exists a constant $\nu_1 = \nu_1(\delta_1)$ such that

$$\Phi(t; w_+) - \Phi(t; w_-) > \nu_1 \mathbf{1}, \qquad 0 \le t \le T(\delta_1) + 1, \tag{5.101}$$

independent of the pair (q, ψ). In particular, for some constant $\nu_2 = \nu_2(\delta_1)$ we have

$$\phi'(\gamma_\delta(t); w_-, w_+) - f\big(\phi(\gamma_\delta(t); w_-, w_+)\big) > \nu_2 \mathbf{1}, \qquad t \ge 0. \tag{5.102}$$

In addition, there exists $\nu_3 = \nu_3(\delta_1)$ such that

$$-f\big(\phi(\gamma_\delta(t); w_-, w_+)\big) > \nu_3 \mathbf{1}, \qquad t \ge T(\delta_1), \tag{5.103}$$

since $\Phi(t; w_\pm) \in \mathcal{C}_\kappa$ for all $t \ge T(\delta_1)$ and $\phi(\gamma_\delta(t); w_-, w_+)$ is bounded away from zero uniformly for the choice of (q, ψ). In particular, for all $t \ge T(\delta_1)$ we have

$$\begin{aligned} \frac{d}{dt}\phi^-_{\delta_1}(t) - f\big(\phi^-_{\delta_1}(t)\big) &= \gamma'_\delta(t)\big[\phi'(\gamma_\delta(t); w_-, w_+) - f\big(\phi(\gamma_\delta(t); w_-, w_+)\big)\big] \\ &\quad - [1 - \gamma'_\delta(t)] f\big(\phi(\gamma_\delta(t); w_-, w_+)\big) \\ &> \gamma'_\delta(t)\nu_2 \mathbf{1} + [1 - \gamma'_\delta(t)]\nu_3 \mathbf{1}, \end{aligned} \tag{5.104}$$

which completes the proof. ◻

Proof of Proposition 5.4. Recall the constants $C \gg 1$ and $\vartheta > 0$ appearing in Lemma 5.13. Items (i) and (ii) of this lemma imply that it suffices to choose $\kappa_{\mathcal{U}} = C(1 + \vartheta^{-1})$. ◻

6. Existence of traveling waves—initial estimates. In this section we return to the nonlinear system

$$\partial_t u(x, t) = [\mathcal{D}u](x, t) + f\big(u(x, t)\big), \tag{6.1}$$

in which the nonlocal differential operator $\mathcal{D}$ is defined in (4.2). Throughout this section we restrict ourselves to the setting $\gamma > 0$. In addition to the condition $(\mathrm{h})_{\S 4}$, we need to impose the following assumption on the separatrix $\mathcal{W}_*$ introduced in section 5.

(H$\mathcal{W}$) There exist constants $\epsilon > 0$ and ϑ such that the inequality

$$|\Psi(t; q)\mathbf{1}| \ge \vartheta e^{\epsilon t} \tag{6.2}$$

holds for all $q \in \mathcal{W}_*$ and $t \ge 0$.

This condition is slightly stronger than the statement in Proposition 5.2(v). However, in what follows we will use the fact that arbitrarily small perturbations of the system (6.1) are sufficient to ensure that (H$\mathcal{W}$) does in fact hold.

In order to show that (6.1) admits a traveling wave solution, we will consider the long-term behavior of the function $u_* \in \widehat{\mathcal{X}}$ that satisfies (6.1) for all $t > 0$ and has the initial profile

$$u_*(x, 0) = \frac{1}{2}\big(1 + \tanh(x)\big)\mathbf{1}. \tag{6.3}$$

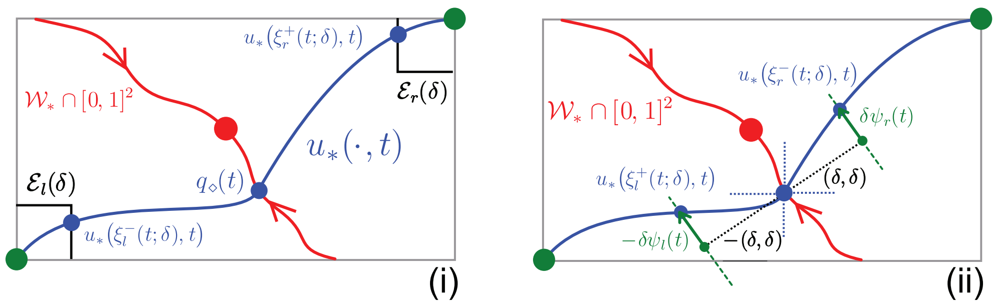

FIG. 2. *Panel* (i) *illustrates the definitions of* $\xi_l^-(t;\delta)$ *and* $\xi_r^-(t;\delta)$*, the spatial coordinates where* $u_*(\cdot,t)$ *crosses* $\mathcal{E}_l(\delta)$ *and* $\mathcal{E}_r(\delta)$*. Panel* (ii) *zooms in near* $q_\diamond(t)$ *and illustrates the definitions of* $\xi_l^+(t;\delta)$ *and* $\xi_r^-(t;\delta)$*, the spatial coordinates between which* $u_*(\cdot,t)$ *is guaranteed to be inside the tubular neighborhood* $\mathcal{U}(\delta)$*.*

Notice that $u_*(\cdot,0)$ is strictly increasing, while $\lim_{x\to-\infty} u_*(x,0) = \mathbf{0}$ and $\lim_{x\to+\infty} u_*(x,0) = \mathbf{1}$. Our first main result in this section shows that these properties persist for all $t > 0$. In particular, upon introducing the spaces

$$\begin{aligned} \mathcal{E}_l(\delta) &= \{0 < v \le \delta\mathbf{1} \text{ for which } v_i = \delta \text{ for some } 1 \le i \le n\}, \\ \mathcal{E}_r(\delta) &= \{(1-\delta)\mathbf{1} \le v < \mathbf{1} \text{ for which } v_i = (1-\delta) \text{ for some } 1 \le i \le n\}, \end{aligned} \tag{6.4}$$

we see that for each $t > 0$, the function $u_*(\cdot,t)$ has unique intersection points with $\mathcal{E}_l(\delta)$ and $\mathcal{E}_r(\delta)$ whenever $\delta > 0$ is sufficiently small. Our second main result states that the distance between these intersection points can be uniformly bounded for $t \ge 0$. This key property allows the use of compactness arguments in section 7 to show that u_* converges to a traveling wave.

PROPOSITION 6.1. *Consider the system* (6.1) *with* $\gamma > 0$ *and suppose that* (HA), $(\mathrm{h})_{\S 4}$, *and* (H$\mathcal{W}$) *are all satisfied. Then the function* u_* *satisfies the following properties.*

(i) *For each* $t \ge 0$*, the function* $u_*(\cdot,t)$ *is strictly increasing and satisfies the limits*

$$\lim_{x\to-\infty} u_*(x,t) = \mathbf{0}, \qquad \lim_{x\to\infty} u_*(x,t) = \mathbf{1}. \tag{6.5}$$

(ii) *Pick a sufficiently small* $\delta > 0$*. For every* $t \ge 0$*, there exist unique quantities*

$$\xi_l^-(t;\delta) < \xi_l^+(t;\delta) < \xi_\diamond(t) < \xi_r^-(t;\delta) < \xi_r^+(t;\delta) \tag{6.6}$$

with the property that

$$u_*(\xi_l^-,t) \in \mathcal{E}_l(\delta), \qquad u_*(\xi_r^+,t) \in \mathcal{E}_r(\delta) \tag{6.7}$$

together with

$$\begin{aligned} u_*(\xi_\diamond,t) &= q_\diamond \in \mathcal{W}_* \cap [0,1]^n, \\ u_*(\xi_l^+,t) &= q_\diamond - \delta[\mathbf{1}+\psi_l], \\ u_*(\xi_r^-,t) &= q_\diamond + \delta[\mathbf{1}+\psi_r] \end{aligned} \tag{6.8}$$

for some pair $\psi_l \in \widehat{T}_\delta^-(q_\diamond)$ *and* $\psi_r \in \widehat{T}_\delta^+(q_\diamond)$*; see Figure* 2.

(iii) *For each sufficiently small* $\delta > 0$*, there exist constants* $\epsilon = \epsilon(\delta) > 0$*,* $C = C(\delta) \gg 1$*, and* $T = T(\delta) \gg 1$ *such that for all* $t \geq \tau \geq 0$ *we have*

$$\begin{aligned} \xi_r^-(t;\delta) &\leq \xi_r^+(\tau;\delta) + 2\epsilon^{-1} + C(t-\tau), \\ \xi_l^+(t;\delta) &\geq \xi_l^-(\tau;\delta) - 2\epsilon^{-1} - C(t-\tau), \end{aligned} \tag{6.9}$$

while for all $\tau \geq 0$ *and* $t \geq \tau + T$ *we have*

$$\begin{aligned} \xi_r^+(t;\delta) &\leq \xi_r^-(\tau;\delta) + 2\epsilon^{-1} + C(t-\tau), \\ \xi_l^-(t;\delta) &\geq \xi_l^+(\tau;\delta) - 2\epsilon^{-1} - C(t-\tau). \end{aligned} \tag{6.10}$$

(iv) *There exists a constant* $\delta > 0$ *and a constant* $h_1 \gg 1$ *such that for all* $t \geq 0$ *we have*

$$\xi_r^-(t;\delta) - \xi_l^+(t;\delta) \leq h_1. \tag{6.11}$$

Corollary 6.2. *Consider the setting of Proposition* 6.1*. For every sufficiently small* $\delta > 0$ *there exists* $m_1(\delta) \gg 1$ *such that for all* $t \geq 0$ *we have*

$$\xi_r^+(t;\delta) - \xi_l^-(t;\delta) \leq m_1(\delta). \tag{6.12}$$

Proof. Pick a sufficiently small $\delta > 0$ and pick $t > T = T(\delta)$. We may then compute

$$\begin{aligned} \xi_r^+(t;\delta) - \xi_l^-(t;\delta) &\leq \xi_r^-(t-T;\delta) - \xi_l^+(t-T;\delta) + 4\epsilon^{-1} + 2CT \\ &\leq h_1 + 4\epsilon^{-1} + 2CT. \end{aligned} \tag{6.13}$$

For $0 \leq t \leq T$, one can use the continuity of the quantities ξ_r^+ and ξ_l^- with respect to t. $\square$

Throughout the remainder of this section, we treat (HA), $(\mathrm{h})_{\S 4}$, and (H$\mathcal{W}$) as standing assumptions and fix $\gamma > 0$. Roughly speaking, our approach towards establishing Proposition 6.1 is to adapt Lemmas 3.2 and 4.3 from [9] to our higher dimensional setting. The chief obstacle is that we need to accomodate for the flow along the separatrix $\mathcal{W}_*$. Indeed, in the scalar context of [9] this flow is trivial as the separatrix consists of a single point.

Lemma 6.3 (cf. [9, Lem. 3.2]). *Recall the functions* $\phi_\delta^\pm$ *defined in Proposition* 5.3 *and the functions* $H_\pm$ *defined in section* 4*. For any sufficiently small* $\delta > 0$*, there exist constants* $\epsilon = \epsilon(\delta) > 0$ *and* $C = C(\delta) \gg 1$ *such that for any* $q \in \mathcal{W}_* \cap [0,1]^n$*, any pair* $\psi_\pm \in \widehat{T}_\delta^\pm(q)$*, and any* $\theta \geq 0$*, the functions*

$$\begin{aligned} w^+(x,t) &= (\mathbf{1} + \delta v_r) H_+\big(1 + \epsilon(x+Ct)\big) + \phi_\delta^-(t+\theta; q, \psi_-) H_-\big(1 + \epsilon(x+Ct)\big), \\ w^-(x,t) &= -\delta v_l H_-\big(\epsilon(x-Ct) - 1\big) + \phi_\delta^+(t+\theta; q, \psi_+) H_+\big(\epsilon(x-Ct) - 1\big) \end{aligned} \tag{6.14}$$

satisfy the differential inequalities (4.34).

Proof. We will prove the statement only for w^+ and $\theta = 0$, the arguments for w^- and $\theta > 0$ being analogous. Writing $y = 1 + \epsilon(x + Ct)$, we compute

$$\partial_t w^+(x,t) = \epsilon C(\mathbf{1} + \delta v_r) H_+'(y) + [\phi_\delta^-]'(t) H_-(y) + \epsilon C \phi_\delta^-(t) H_-'(y). \tag{6.15}$$

In particular, upon writing

$$\mathcal{J}^+(x,t) = \partial_t w^+(x,t) - [\mathcal{D}w^+](x,t) - f\big(w^+(x,t)\big), \tag{6.16}$$

we obtain

(6.17)
$$\begin{aligned}\mathcal{J}^+(x,t) &= \epsilon C(\mathbf{1}+\delta v_r)H_+'(y) + [\phi_\delta^-]'(t)H_-(y) + \epsilon C\phi_\delta^-(t)H_-'(y)\\ &\quad - [\mathcal{D}(\mathbf{1}+\delta v_r)H_+](y) - [\mathcal{D}\phi_\delta^-(t)H_-](y)\\ &\quad - f\big((\mathbf{1}+\delta v_r)H_+(y) + \phi_\delta^-(t)H_-(y)\big)\\ &= \epsilon C\big(\mathbf{1}+\delta v_r - \phi_\delta^-(t)\big)H_+'(y) - f\big((\mathbf{1}+\delta v_r)H_+(y) + \phi_\delta^-(t)H_-(y)\big)\\ &\quad - [\mathcal{D}(\mathbf{1}+\delta v_r)H_+](y) - [\mathcal{D}\phi_\delta^-(t)H_-](y) + [\phi_\delta^-]'(t)H_-(y).\end{aligned}$$

Possibly after decreasing the constant $\epsilon(\delta)$ in Lemma 5.13, we have $\mathbf{1}+\delta v_r - \phi_\delta^-(t) > \frac{1}{2}\delta v_r > \mathbf{0}$ for all $t \geq 0$. In addition, using the fact that the differential operator $\mathcal{D}$ vanishes on constant functions, it is not hard to see that

$$[\mathcal{D}(\mathbf{1}+\delta v_r)H_+](y) + [\mathcal{D}\phi_\delta^-(t)H_-](y) \to 0 \text{ as } \epsilon \to 0, \tag{6.18}$$

uniformly for $t \geq 0$ and $y \in \mathbb{R}$. Finally, the inequality (5.21) implies that there exist $\kappa = \kappa(\delta) > 0$ and $\nu_1 = \nu_1(\delta) > 0$ such that

$$[\phi_\delta^-]'(t)H_-(y) - f\big((\mathbf{1}+\delta v_r)H_+(y) + \phi_\delta^-(t)H_-(y)\big) > \nu_1\mathbf{1} \tag{6.19}$$

whenever $H_+(y) \leq \kappa$ or $H_-(y) \leq \kappa$. In particular, for all such y we can arrange for $\mathcal{J}^+(x,t) \geq \mathbf{0}$ by picking a sufficiently small $\epsilon > 0$.

On the other hand, there exists $\nu_2 = \nu_2(\delta) > 0$ such that $H_+'(y) \geq \nu_2$ for all $y \in \mathbb{R}$ for which $\kappa \leq H_+(y) \leq 1-\kappa$. Choosing $C \gg 1$ to be sufficiently large ensures that also $\mathcal{J}^+(x,t) \geq \mathbf{0}$ for these values of y. ◻

Proof of Proposition 6.1(i). At $t = 0$, the statements follow directly from our choice (6.3) for $u_*(x,0)$. The fact that $u_*(\cdot,t)$ is strictly increasing for $t > 0$ follows from the comparison principle and the fact that $u_*(\cdot,0)$ is strictly increasing. For each fixed $t > 0$, the limits (6.5) can be obtained by studying the functions $w^\pm$ constructed in Lemma 6.3 and taking the limits $\delta \to 0$ and $\theta \to \infty$. ◻

Proof of Proposition 6.1(ii). The existence of $\xi_\diamond$ and $q_\diamond$ follows from Proposition 5.2(ii). The existence of ξ_l^+ and ξ_r^- follows from Proposition 5.3(i), while the existence of ξ_l^- and ξ_r^+ follows from the limits (6.5). The uniqueness of all these quantities follows from the fact that $u_*(\cdot,t)$ is strictly increasing for all $t \geq 0$. ◻

Proof of Proposition 6.1(iii). We first focus on the bound (6.9) for $\xi_r^-(t;\delta)$. By choosing $\delta > 0$ to be sufficiently small, we can ensure that there exists a $\theta > 0$ and a pair $q \in \mathcal{W}_*$ and $\psi_+ \in \widehat{T}_\delta^+(q)$ such that $\phi_\delta^+(\theta;q,\psi_+) \leq u_*(\xi_r^+(\tau,\delta),\tau)$ while also $\phi_\delta^+(\theta+t;q,\psi_+) \notin \mathcal{U}(\delta)$ for all $t \geq 0$. In particular, recalling the function $w^-(x,t) = w^-(x,t;\delta,q,\psi_+,\theta)$ from Lemma 6.3, we see that

$$u_*(x,\tau) \geq w^-\big(x + C\tau - \xi_r^+(\tau,\delta),\tau\big). \tag{6.20}$$

Now, for all $t \geq \tau$ we have

$$w^-(2\epsilon^{-1} + Ct, t) = \phi_{\delta_1}^+(t+\theta;q,\psi_+) \notin \mathcal{U}(\delta), \tag{6.21}$$

which by the comparison principle implies that

$$\xi_r^-(t,\delta) \leq 2\epsilon^{-1} + Ct - C\tau + \xi_r^+(\tau,\delta), \tag{6.22}$$

as desired. The bound for ξ_l^+ follows in a similar fashion.

We now turn to the bound (6.10) for $\xi_l^-(t;\delta)$. Write $q = q_\diamond(\tau)$ and $\psi_- = \psi_l(\tau)$ and recall the function $w^+(x,t) = w^+(x,t;\delta,q,\psi_-,0)$ from Lemma 6.3. For all $x \in \mathbb{R}$, we have

$$u_*(x,\tau) \le w^+\big(x - \xi_l^+(\tau,\delta), 0\big). \tag{6.23}$$

Notice furthermore that

$$w^+(-2\epsilon^{-1} - C(t-\tau), t-\tau) = \phi_\delta^-(t-\tau; q, \psi_-). \tag{6.24}$$

Recall the constant $T = T(\delta)$ introduced in Proposition 5.3(ii). Since $\phi_\delta^-(t-\tau;q,\psi_-) \le \delta\mathbf{1}$ whenever $t \ge \tau + T$, the comparison principle implies that for all such t we have

$$\xi_l^-(t,\delta) \ge -2\epsilon^{-1} - C(t-\tau) + \xi_l^+(\tau,\delta), \tag{6.25}$$

as desired. The bound for ξ_r^+ follows in a similar fashion. ∎

In order to establish item (iv) of Proposition 6.1, we need to understand the flow of (6.1) near the separatrix $\mathcal{W}_*$. The condition (H$\mathcal{W}$) roughly states that this separatrix is repulsive. Since solutions to (6.1) that have small spatial derivatives locally tend to follow the flow of the ODE (5.1), it is reasonable to expect that $u_*(\cdot,t)$ cannot become very flat near the separatrix.

In order to make this precise, we pick $q \in \mathcal{W}_*$ and introduce the notation

$$B_q(t) = Df\big(\Phi(t;q)\big). \tag{6.26}$$

Before considering the full nonlinear system (6.1), we focus on the linearized system

$$\partial_t v(x,t) = [\mathcal{D}v](x,t) + B_q(t)v(x,t) \tag{6.27}$$

in the next series of results. We use the notation H_0 to refer to the Heaviside function defined by

$$H_0(x) = 0 \text{ for } x < 0, \qquad H_0(x) = 1 \text{ for } x \ge 0. \tag{6.28}$$

Lemma 6.4. *For all sufficiently large $T \gg 1$, there exists $\xi = \xi(T) \gg 1$ such that for any $q \in \mathcal{W}_*$ and any $\psi \in \widehat{T}(q)$, the function $w \in \mathcal{X}$ that satisfies the linear PDE* (6.27) *with the initial condition*

$$w(x,0) = H_0(x)[\mathbf{1} + \psi] \tag{6.29}$$

satisfies the bound

$$w(x,T) \ge \frac{4}{5}\Psi(T;q)\mathbf{1}, \qquad x \ge \xi, \tag{6.30}$$

together with

$$w(x,T) \le \frac{1}{5}\Psi(T;q)\mathbf{1}, \qquad x \le -\xi. \tag{6.31}$$

Proof. Pick any $q \in \mathcal{W}_*$ and $\psi \in \widehat{T}(q)$ and consider the function

$$w^-(x,t) = H_+(-1+\epsilon x)\Psi\big(t;q\big)(\mathbf{1}+\psi) - \nu t\Psi\big(t;q\big)\mathbf{1}, \tag{6.32}$$

where $\epsilon > 0$ and $\nu = \nu(\epsilon) > 0$ remain to be determined. Upon writing

$$\mathcal{J}^-(x,t) = \partial_t w^-(x,t) - [\mathcal{D}w^-](x,t) - B_q(t)w^-(x,t) \tag{6.33}$$

and introducing the new variable $y = -1 + \epsilon x$, we may compute

$$\begin{aligned}\mathcal{J}^-(x,t) &= H_+(y)B_q(t)\Psi(t;q)(\mathbf{1}+\psi) - \nu\Psi(t;q)\mathbf{1} - \nu t B_q(t)\Psi(t;q)\mathbf{1} \\ &\qquad - [\mathcal{D}\Psi(t;q)(\mathbf{1}+\psi)H_+](y) \\ &\qquad - H_+(y)B_q(t)\Psi(t;q)(\mathbf{1}+\psi) + \nu t B_q(t)\Psi(t;q)\mathbf{1} \\ &= -\nu\Psi(t;q)\mathbf{1} - [\mathcal{D}\Psi(t;q)(\mathbf{1}+\psi)H_+](y).\end{aligned} \tag{6.34}$$

By Proposition 5.2(iv), Lemma 5.13, and the assumption (H$\mathcal{W}$), there exist constants $K \gg 1$ and $\vartheta > 0$ such that

$$K\Psi(t;q)\mathbf{1} > \Psi(t;q)(\mathbf{1}+\psi) + \vartheta\mathbf{1} \tag{6.35}$$

for all $t \geq 0$, $q \in \mathcal{W}_*$, and $\psi \in \widehat{T}(q)$. In particular, since

$$[\mathcal{D}vH_+](y) \to 0 \text{ as } \epsilon \to 0 \tag{6.36}$$

uniformly for $v \in \mathbb{S}^{n-1}$, we can choose $\nu(\epsilon) > 0$ in such a way that $\mathcal{J}^-(x,t) \leq 0$ holds for all $x \in \mathbb{R}$ and $t \geq 0$, while also $\nu(\epsilon) \to 0$ as $\epsilon \to 0$. In particular, by the comparison principle we have $w(x,t) \geq w^-(x,t)$ for all $x \in \mathbb{R}$ and $t \geq 0$.

Recall the constant δ_* from Lemma 5.8. Upon choosing a sufficiently large $T \gg 1$, Proposition 5.2(iv) implies that

$$|\Psi(T;q)\psi| \leq \frac{1}{10}\delta_* \,|\Psi(T;q)\mathbf{1}| \tag{6.37}$$

for all $q \in \mathcal{W}_*$ and $\psi \in \widehat{T}(q)$. We now choose $\epsilon_* > 0$ in such a way that $\nu(\epsilon_*)T \leq \frac{1}{10}$ and write $\xi = 2\epsilon_*^{-1}$. For any $x \geq \xi$, we can now use Lemma 5.8 to estimate

$$\begin{aligned}w^-(x,T) &= \Psi(T;q)(\mathbf{1}+\psi) - T\nu(\epsilon_*)\Psi(T;q)\mathbf{1} \\ &\geq \Psi(T;q)\mathbf{1} - \tfrac{1}{10}\delta_* \,|\Psi(T;q)\mathbf{1}|\,\mathbf{1} - \tfrac{1}{10}\Psi(T;q)\mathbf{1} \\ &\geq \Psi(T;q)\mathbf{1} - \tfrac{1}{10}\Psi(T;q)\mathbf{1} - \tfrac{1}{10}\Psi(T;q)\mathbf{1} \\ &= \tfrac{4}{5}\Psi(T;q)\mathbf{1}.\end{aligned} \tag{6.38}$$

The lower bound (6.31) can be obtained in a similar fashion by studying the function

$$w^+(x,t) = H_+(1+\epsilon x)\Psi\big(t;q\big)(\mathbf{1}+\psi) + \nu t\Psi\big(t;q\big)\mathbf{1}. \tag{6.39}$$ ◻

LEMMA 6.5. *There exists a constant $C \gg 1$ such that for any $q \in \mathcal{W}_*$ and any $\psi \in \widehat{T}(q)$, the function $w \in \mathcal{X}$ that satisfies the linear PDE (6.27) with the initial condition*

$$w(x,0) = H_0(x)[\mathbf{1}+\psi] \tag{6.40}$$

satisfies the bound

$$|\partial_x w(x,t)| \leq Ct^{-1/2}\,|\Psi(t;q)\mathbf{1}| \tag{6.41}$$

for all $x \in \mathbb{R}$ and $t > 0$.

Proof. We write $y(\cdot,t)$ for the Fourier transform of $\partial_x w(\cdot,t)$, i.e.,

$$y(\nu,t) = \int_{-\infty}^{\infty} e^{-i\nu x}\partial_x w(x,t)\,dx. \tag{6.42}$$

Fixing $\nu \in \mathbb{R}$, a short computation shows that the function $y(\nu,\cdot)$ satisfies the ODE

$$\partial_t y(\nu,t) = \left[-\gamma\nu^2 + \sum_{j=0}^{N}(e^{i\nu r_j}-1)A_j + B_q(t)\right] y(\nu,t) \tag{6.43}$$

for $t \geq 0$ with the initial condition

$$y(\nu,0) = \mathbf{1} + \psi. \tag{6.44}$$

Let us now consider the nonlocal system

$$\partial_t v(x,t) = -\gamma\nu^2 v(x,t) + \sum_{j=0}^{N} A_j[v(x+r_j,t) - v(x,t)] + B_q(t)v(x,t). \tag{6.45}$$

Upon writing

$$v(x,t) = e^{i\nu x}y(\nu,t), \tag{6.46}$$

one readily sees that v and hence also $\widetilde{v}(x,t) := \operatorname{Re} v(x,t)$ solve (6.45). In view of the initial estimate

$$-\big(\mathbf{1}+\psi\big) \leq \widetilde{v}(x,0) = \cos(\nu x)[\mathbf{1}+\psi] \leq \mathbf{1}+\psi, \tag{6.47}$$

the comparison principle implies that

$$|\widetilde{v}(x,0)| \leq e^{-\gamma\nu^2 t}\Psi(t;q)[\mathbf{1}+\psi]. \tag{6.48}$$

A similar result holds for the imaginary part of $v(x,t)$, which in view of Proposition 5.2(iv) and Lemma 5.13 yields the estimate

$$|y(\nu,t)| \leq 2e^{-\gamma\nu^2 t}\Psi(t;q)[\mathbf{1}+\psi] \leq C_1 e^{-\gamma\nu^2 t}\Psi(t;q)\mathbf{1} \tag{6.49}$$

for some $C_1 \gg 1$. In particular, we may compute

$$|\partial_x(x,t)| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\nu x}y(\nu,t)d\nu\right| \leq \frac{C_1}{2\pi}\left[\int_{-\infty}^{\infty} e^{-\gamma\nu^2 t}d\nu\right]|\Psi(t;q)\mathbf{1}|\,, \tag{6.50}$$

which establishes the desired bound. ∎

LEMMA 6.6. *Recall the constant $\kappa_{\mathcal{U}}$ that appears in Proposition 5.4. There exists a constant $T_* \gg 1$ such that for any $q \in \mathcal{W}_*$ and any pair $\psi_v, \psi_w \in \widehat{T}(q)$, there exists $\xi_* = \xi_*(q;\psi_v,\psi_w) \in \mathbb{R}$ such that the solutions $v_{\mathrm{I}}, w_{\mathrm{I}} \in \mathcal{X}$ to the linear system (6.27) with the initial conditions*

$$v_{\mathrm{I}}(x,0) = H_0(x)[\mathbf{1}+\psi_v], \qquad w_{\mathrm{I}}(x,0) = -H_0(-x)[\mathbf{1}+\psi_w] \tag{6.51}$$

satisfy the inequalities

$$|v_{\mathrm{I}}(\xi_*,T_*)| \geq 3\kappa_{\mathcal{U}}, \qquad |w_{\mathrm{I}}(\xi_*,T_*)| \geq 3\kappa_{\mathcal{U}}. \tag{6.52}$$

Proof. First, we claim that for all $t \geq 0$ we have the limits

$$\lim_{x\to-\infty} v_{\mathrm{I}}(x,t) = \mathbf{0}, \qquad \lim_{x\to+\infty} v_{\mathrm{I}}(x,t) = \Psi(t;q)[\mathbf{1}+\psi_v], \tag{6.53}$$

together with their analogues

$$\lim_{x\to-\infty} w_{\mathrm{I}}(x,t) = -\Psi(t;q)[\mathbf{1}+\psi_w], \qquad \lim_{x\to+\infty} w_{\mathrm{I}}(x,t) = \mathbf{0}. \tag{6.54}$$

Indeed, the comparison principle shows that $\mathbf{0} \leq v_I(x,t) \leq \Psi(t;q)[\mathbf{1}+\psi_v]$ for all $x \in \mathbb{R}$ and $t \geq 0$. The limits in (6.53) can now be read off from the subsolution $w^-(x,t)$ and the supersolution $w^+(x,t)$ constructed in (6.32) and (6.39). The limits for w_{I} follow after the replacements $w_{\mathrm{I}} \mapsto -w_{\mathrm{I}}$ and $x \mapsto -x$.

Upon writing

$$\widetilde{w}_{\mathrm{I}}(x,t) = w_{\mathrm{I}}(x,t) + \Psi(t;q_0)[\mathbf{1}+\psi_w], \tag{6.55}$$

Lemma 6.4 implies that after picking a sufficiently large $T \gg 1$, we have

$$v_{\mathrm{I}}(\xi(T),T) \geq \frac{4}{5}\Psi(T;q)\mathbf{1}, \qquad \widetilde{w}_{\mathrm{I}}(-\xi(T),T) \leq \frac{1}{5}\Psi(T;q)\mathbf{1}. \tag{6.56}$$

Possibly after increasing T, we can ensure that

$$\Psi(T;q)[\mathbf{1}+\psi_w] \leq \frac{6}{5}\Psi(T;q)\mathbf{1}. \tag{6.57}$$

Using the fact that both $v_{\mathrm{I}}(\cdot,T)$ and $\widetilde{w}_{\mathrm{I}}(\cdot,T)$ are nondecreasing functions, we see that the inequalities

$$v_{\mathrm{I}}(x,T) \geq \frac{4}{5}H_0\big(x-\xi(T)\big)\Psi(T;q)\mathbf{1}, \qquad \widetilde{w}_{\mathrm{I}}(x,T) \leq \frac{1}{5}\Psi(T;q)\mathbf{1}+H_0\big(x+\xi(T)\big)\Psi(T;q)\mathbf{1} \tag{6.58}$$

hold for all $x \in \mathbb{R}$. In particular, we obtain

$$\widetilde{w}_{\mathrm{I}}(x,T) \leq \frac{1}{5}\Psi(T;q)\mathbf{1} + \frac{5}{4}v_{\mathrm{I}}(x+2\xi(T),T). \tag{6.59}$$

The comparison principle now implies that for all $t \geq T$ we have

$$\begin{aligned} w_{\mathrm{I}}(x,t) &= \widetilde{w}_{\mathrm{I}}(x,t) - \Psi(t;q)[\mathbf{1}+\psi_w] \\ &\leq \frac{1}{5}\Psi(t;q)\mathbf{1} + \frac{5}{4}v_I(x+2\xi(T),t) - \Psi(t;q)\mathbf{1} - \Psi(t;q)\psi_w \\ &\leq -\frac{4}{5}\Psi(t;q)\mathbf{1} + \frac{5}{4}v_I(x,t) + \frac{5}{2}C\xi(T)t^{-1/2}\left|\Psi(t;q)\mathbf{1}\right|\mathbf{1} + \left|\Psi(t;q)\psi_w\right|\mathbf{1}. \end{aligned} \tag{6.60}$$

In view of (H$\mathcal{W}$), it is possible to pick $T_* \gg T$ in such a way that

$$-\frac{4}{5}\Psi(T_*;q)\mathbf{1}+\frac{15}{4}\kappa_{\mathcal{U}}\mathbf{1}+\frac{5}{2}\xi(T)CT_*^{-1/2}\left|\Psi(T_*;q)\mathbf{1}\right|\mathbf{1}+\left|\Psi(T_*;q)\psi_w\right|\mathbf{1} \leq -3\kappa_{\mathcal{U}}\mathbf{1} \tag{6.61}$$

holds for all $q \in \mathcal{W}_*$ and $\psi_w \in \widehat{T}(q)$. The requirements in the statement of this result can now be satisfied by choosing $\xi_* = \xi_*(q,\psi_v)$ to ensure that

$$|v_{\mathrm{I}}(\xi_*,T_*)| = 3\kappa_{\mathcal{U}}. \qquad \square \tag{6.62}$$

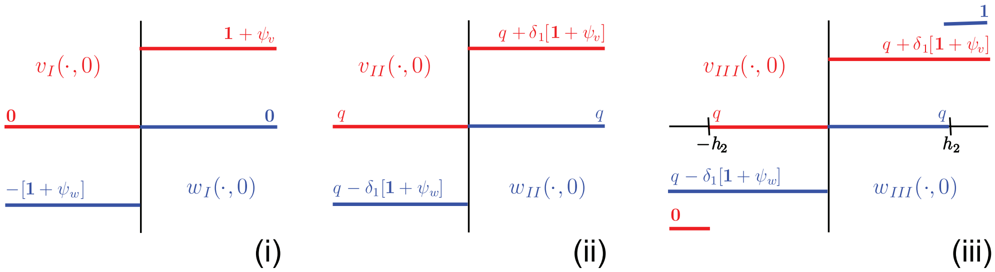

FIG. 3. *Panels* (i)–(iii) *illustrate the initial conditions for the functions* v_{I} *through* v_{III} *and* w_{I} *through* w_{III} *described in Lemmas* 6.6–6.8.

We are now ready to turn to the full nonlinear system (6.1). We use the linear solutions v_{I} and w_{I} defined above to obtain upper and lower bounds on solutions to (6.1) that have increasingly intricate initial conditions; see Figure 3.

LEMMA 6.7. *Recall the constants T_* and $\xi_* = \xi_*(q,\psi_v,\psi_w)$ introduced in Lemma* 6.6. *There exists a constant $\delta_1 > 0$ such that for any $q \in \mathcal{W}_* \cap [0,1]^n$, $\psi_v \in \widehat{T}^+_{\delta_1}(q)$, and $\psi_w \in \widehat{T}^-_{\delta_1}(q)$, the solutions $v_{\mathrm{II}}, w_{\mathrm{II}} \in \mathcal{X}$ to the nonlinear system* (4.1) *with the initial conditions*

$$v_{\mathrm{II}}(x,0) = q + \delta_1 H_0(x)[\mathbf{1}+\psi_v], \qquad w_{\mathrm{II}}(x,0) = q - \delta_1 H_0(-x)[\mathbf{1}+\psi_w] \tag{6.63}$$

satisfy the inequalities

$$v_{\mathrm{II}}(\xi_*,T_*) \geq \Phi(t;q), \qquad w_{\mathrm{II}}(\xi_*,T_*) \leq \Phi(t;q), \tag{6.64}$$

together with the estimates

$$|v_{\mathrm{II}}(\xi_*,T_*) - \Phi(t;q)| \geq 2\delta_1\kappa_{\mathcal{U}}, \qquad |w_{\mathrm{II}}(\xi_*,T_*) - \Phi(t;q)| \geq 2\delta_1\kappa_{\mathcal{U}}. \tag{6.65}$$

Proof. We set out to construct a subsolution for v_{II}. To this end, we recall the function $v_{\mathrm{I}}(x,t) = v_{\mathrm{I}}(x,t;q,\psi_v)$ from Lemma 6.6 and write

$$v^-(x,t) = \Phi(t;q) + \delta_1 v_{\mathrm{I}}(x,t) - \delta_1^2 Ct\Psi(t;q)\mathbf{1}, \tag{6.66}$$

together with

$$\mathcal{J}^-(x,t) = \partial_t v^-(x,t) - [\mathcal{D}v^-](x,t) - f\big(v^-(x,t)\big). \tag{6.67}$$

Writing $q(t) = \Phi(t;q)$, we compute

$$\begin{aligned}\mathcal{J}^-(x,t) &= f\big(q(t)\big) + \delta_1[\mathcal{D}v_{\mathrm{I}}](x,t) + \delta_1 B_q(t)v_{\mathrm{I}}(x,t) - \delta_1^2 C\Psi(t;q)\mathbf{1} - \delta_1^2 CtB_q(t)\Psi(t;q)\mathbf{1}\\ &\quad - \delta_1[\mathcal{D}v_{\mathrm{I}}](x,t) - f\big(q(t) + \delta_1 v_{\mathrm{I}}(x,t) - \delta_1^2 Ct\Psi(t;q)\mathbf{1}\big)\\ &= -\big[f\big(q(t) + \delta_1 v_{\mathrm{I}}(x,t) - \delta_1^2 Ct\Psi(t;q)\mathbf{1}\big) - f\big(q(t)\big)\\ &\quad - Df(q(t))[\delta_1 v_{\mathrm{I}}(x,t) - \delta_1^2 Ct\Psi(t;q)\mathbf{1}]\big] - \delta_1^2 C\Psi(t;q)\mathbf{1}.\end{aligned} \tag{6.68}$$

In particular, we see that

$$\big|\mathcal{J}^-(x,t) + \delta_1^2 C\Psi(t;q_0)\mathbf{1}\big| \quad\leq\quad \tfrac{1}{2}\left\|D^2 f\right\| \delta_1^2\big(\,|v_{\mathrm{I}}(x,t)| + \delta_1 Ct\,|\Psi(t;q)\mathbf{1}|\,\big)^2. \tag{6.69}$$

We now choose $C \gg 1$ in such a way that

$$C\Psi(t;q)\mathbf{1} \geq 4\left\|D^2 f\right\| |v_{\mathrm{I}}(x,t)|^2 \mathbf{1} \tag{6.70}$$

holds for all $x \in \mathbb{R}$ and $0 \leq t \leq T_*$. In addition, we choose δ_1 to be sufficiently small to ensure that

$$4\delta_1^2 C t^2 |\Psi(t;q)\mathbf{1}|^2 \mathbf{1} \leq \Psi(t;q)\mathbf{1} \tag{6.71}$$

for all $0 \leq t \leq T_*$. This ensures that for all such t and x we have

$$\left|\mathcal{J}^-(x,t) + \delta_1^2 C\Psi(t;q_0)\mathbf{1}\right| \mathbf{1} \quad \leq \quad \tfrac{1}{2}\delta_1^2 C\Psi(t;q_0)\mathbf{1}, \tag{6.72}$$

which shows that $\mathcal{J}^-(x,t) \leq 0$ for all $x \in \mathbb{R}$ and $0 \leq t \leq T_*$. Since $v^-(x,0) = v_{\mathrm{II}}(x,0)$, the comparison principle implies that $v_{\mathrm{II}}(x,T_*) \geq v^-(x,T_*)$ for all $x \in \mathbb{R}$. By further decreasing δ_1 to ensure that

$$\delta_1 C T_* |\Psi(T_*;q)\mathbf{1}| \leq \kappa_{\mathcal{U}}, \tag{6.73}$$

the first estimate in (6.65) can be obtained. The estimate for w_{II} can be obtained in a similar fashion. ▯

LEMMA 6.8. *Recall the constants T_* and $\xi_* = \xi_*(q,\psi_v,\psi_w)$ introduced in Lemma* 6.6, *together with the constant $\delta_1 > 0$ introduced in Lemma* 6.7. *There exists a constant $h_2 \gg 1$ such that for any $q \in \mathcal{W}_* \cap [0,1]^n$, any $\psi_v \in \widehat{T}^+_{\delta_1}(q)$, and any $\psi_w \in \widehat{T}^-_{\delta_1}(q)$, the solutions $v_{\mathrm{III}}, w_{\mathrm{III}} \in \mathcal{X}$ to the nonlinear system* (4.1) *with the initial conditions*

$$\begin{aligned} v_{\mathrm{III}}(x,0) &= qH_0(x+h_2) + \delta_1 H_0(x)[\mathbf{1}+\psi_v], \\ w_{\mathrm{III}}(x,0) &= \big(1 - H_0(x-h_2)\big)q + \mathbf{1}H_0(x-h_2) - \delta_1 H_0(-x)[\mathbf{1}+\psi_w] \end{aligned} \tag{6.74}$$

satisfy the inequalities

$$v_{\mathrm{III}}(\xi_*,T_*) \geq \Phi(t;q), \qquad w_{\mathrm{III}}(\xi_*,T_*) \leq \Phi(t;q), \tag{6.75}$$

together with the estimates

$$|v_{\mathrm{III}}(\xi_*,T_*) - \Phi(t;q)| \geq \delta_1\kappa_{\mathcal{U}}, \qquad |w_{\mathrm{III}}(\xi_*,T_*) - \Phi(t;q)| \geq \delta_1\kappa_{\mathcal{U}}. \tag{6.76}$$

Proof. For any $\nu_3 > 0$, we introduce the $C^1([0,\infty),\mathbb{R})$ function

$$g_{\nu_3}(t) = \begin{cases} te^{-\nu_3 t} & \text{for } 0 \leq t \leq \nu_3^{-1}, \\ \nu_3^{-1}e^{-1} & \text{for } t \geq \nu_3^{-1}. \end{cases} \tag{6.77}$$

Notice that $g'_{\nu_3}(t) \geq 0$ for $t \geq 0$, together with $g_{\nu_3}(0) = 0$, $g'_{\nu_3}(0) = 1$ and $0 \leq g_{\nu_3}(t) \leq \nu_3^{-1}e^{-1}$. We again write $q(t) = \Phi(t;q)$ and consider the function

$$v^-(x,t) = v_{\mathrm{II}}(x,t) - q(t)H_-\big(1+\epsilon(x-\xi_*-C_1(t-T_*))\big) - \kappa_2 te^{\nu_2 t}\mathbf{1} - C_3 g_{\nu_3}(t)\mathbf{1}, \tag{6.78}$$

in which the constants $\epsilon > 0$, $\nu_2 > 0$, $\nu_3 \gg 1$, $C_1 \gg 1$, $\kappa_2 > 0$, and $C_3 \gg 1$ remain to be determined. As before, we set out to show that $\mathcal{J}^-(x,t) \leq 0$ for all $x \in \mathbb{R}$ and $0 \leq t \leq T_*$, in which

$$\mathcal{J}^-(x,t) = \partial_t v^-(x,t) - [\mathcal{D}v^-](x,t) - f\big(v^-(x,t)\big). \tag{6.79}$$

Upon writing $y = 1 + \epsilon\big(x - \xi_* - C_1(t - T_*)\big)$, we may compute

(6.80)

$$\begin{aligned}
&\mathcal{J}^-(x,t)\\
&\quad = [\mathcal{D}v_{\mathrm{II}}](x,t) + f\big(v_{\mathrm{II}}(x,t)\big) - f\big(q(t)\big)H^-(y) + C_1\epsilon q(t)H'_-(y) - \kappa_2 e^{\nu_2 t}\mathbf{1}\\
&\qquad - \nu_2\kappa_2 t e^{\nu_2 t}\mathbf{1} - C_3 g'_{\nu_3}(t)\mathbf{1} - [\mathcal{D}v_{\mathrm{II}}](x,t) + [\mathcal{D}q(t)H_-](y)\\
&\qquad - f\big(v_{\mathrm{II}}(x,t) - q(t)H_-(y) - \kappa_2 t e^{\nu_2 t}\mathbf{1} - C_3 g_{\nu_3}(t)\mathbf{1}\big)\\
&\quad = f(v_{\mathrm{II}}(x,t)) - f\big(v_{\mathrm{II}}(x,t) - q(t)H_-(y) - \kappa_2 t e^{\nu_2 t}\mathbf{1} - C_3 g_{\nu_3}(t)\mathbf{1}\big) - f(q(t))H_-(y)\\
&\qquad - \kappa_2(1 + \nu_2 t)e^{\nu_2 t}\mathbf{1} - C_3 g'_{\nu_3}(t)\mathbf{1} - C_1\epsilon q(t)H'_+(y) + [\mathcal{D}q(t)H_-](y).
\end{aligned}$$

Before we proceed further, we claim that for all $0 \le t \le T_*$ we have the limit

$$\lim_{x\to-\infty} v_{\mathrm{II}}(x,t) = q(t). \tag{6.81}$$

Indeed, after the substitution $C \mapsto -C$ in (6.66), the function v^- constructed there is in fact a supersolution for v_{II}. The limit (6.81) now follows from the limits (6.53).

For any $\vartheta \in (0,1)$, we write y_ϑ for the unique $y \in \mathbb{R}$ that has $H_-(y) = \vartheta$ and introduce the constant

$$M(\epsilon) = \sup_{0\le t\le T_*,\, y\le y_{1-\epsilon}} |v_{\mathrm{II}}(x,t) - q(t)|\,. \tag{6.82}$$

Since $x = x(y,t) = \epsilon^{-1}y + \xi_* + C_1(t - T_*)$ and $y_{1-\epsilon} \to -1$ as $\epsilon \downarrow 0$, the limit (6.81) implies that $M(\epsilon) \to 0$ as $\epsilon \downarrow 0$.

Let us now introduce the notation

$$\mathcal{G}(x,t) = \left|\mathcal{J}^-(x,t) + \kappa_2(1+\nu_2 t)e^{\nu_2 t}\mathbf{1} + C_3 g'_{\nu_3}(t)\mathbf{1} + C_1\epsilon q(t)H'_+(y)\right|. \tag{6.83}$$

Whenever $y \le y_\epsilon$ and $0 \le t \le T_*$, we can use $f(\mathbf{0}) = \mathbf{0}$ to obtain the estimate

(6.84)

$$\begin{aligned}
\mathcal{G}(x,t) \le\ & |f(v_{\mathrm{II}}(x,t)) - f(q(t))| + |1 - H_-(y)|\,|f(q(t))|\\
& + \|Df\|\left[\,|v_{\mathrm{II}}(x,t) - q(t)| + |(1 - H_-(y))q(t)| + |\kappa_2 t e^{\nu_2 t}\mathbf{1}| + |C_3 g_{\nu_3}(t)\mathbf{1}|\,\right]\\
& + |[\mathcal{D}q(t)H_-](y)|\\
\le\ & 2\|Df\|\,M(\epsilon) + \epsilon(|f(q(t))| + \|Df\|\,|q(t)|)\\
& + |[\mathcal{D}q(t)H_-](y)| + \kappa_2 t e^{\nu_2 t}\|Df\|\,|\mathbf{1}| + C_3 g_{\nu_3}(t)\,\|Df\|\,|\mathbf{1}|\,.
\end{aligned}$$

We now fix $\nu_2 = 2\|Df\|\,|\mathbf{1}|$. In addition, we choose

(6.85)

$$\kappa_2 = \kappa_2(\epsilon) = 8\|Df\|\,M(\epsilon) + 4\epsilon \sup_{q\in\mathcal{W}_*}[|f(q)| + \|Df\|\,|q|] + 4\sup_{q\in\mathcal{W}_*,\,y\in\mathbb{R}}|[\mathcal{D}qH_-](y)|$$

and remark that $\kappa_2(\epsilon) \to 0$ as $\epsilon \downarrow 0$. By appropriately restricting $\epsilon > 0$ we can hence ensure that

$$\kappa_2(\epsilon)T_* e^{\nu_2 T_*} \le \frac{1}{2}\delta_1\kappa_{\mathcal{U}}. \tag{6.86}$$

With these restrictions in place, we obtain the estimate

$$\mathcal{G}(x,t) \le \kappa_2 t e^{\nu_2 t}\|Df\|\,|\mathbf{1}| + \frac{1}{4}\kappa_2 + C_3\frac{1}{\nu_3}e^{-1}\|Df\|\,|\mathbf{1}|\,. \tag{6.87}$$

We now pick $C_3 \gg 1$ in such a way that

$$C_3 \geq 4 \sup_{q \in \mathcal{W}_*} [|f(q)| + \|Df\| \, |q|] \tag{6.88}$$

and $\nu_3 \gg 1$ to ensure that

$$\nu_3 \geq 4\kappa_2^{-1} C_3 e^{-1} \|Df\| \, |\mathbf{1}| , \qquad C_3 \nu_3^{-1} \leq \frac{1}{2} \delta_1 \kappa_{\mathcal{U}}. \tag{6.89}$$

The first condition on ν_3 ensures that

$$\mathcal{G}(x,t) \quad \leq \quad \tfrac{1}{2}\kappa_2 \nu_2 t e^{\nu_2 t} + \tfrac{1}{2}\kappa_2, \tag{6.90}$$

which in turn implies that $\mathcal{J}^-(x,t) \leq 0$ whenever $y \leq y_\epsilon$ and $0 \leq t \leq T_*$.

Whenever $y \geq y_\epsilon$, we can use the inequality $H_-(y) \leq \epsilon$ to estimate

$$\begin{aligned} \mathcal{G}(x,t) &\leq \epsilon \, |f(q(t))| + \|Df\| \left[\epsilon \, |q(t)| + \kappa_2 t e^{\nu_2 t} \mathbf{1} + C_3 g_{\nu_3}(t) \mathbf{1} \right] \\ &\quad + |[\mathcal{D}q(t) H_-](y)| \\ &\leq \epsilon \big(\, |f(q(t))| + \|Df\| \, |q(t)| \, \big) + |[\mathcal{D}q(t)H_-](y)| + [\kappa_2 t e^{\nu_2 t} + C_3 g_{\nu_3}(t)] \, \|Df\| \, |\mathbf{1}| \, . \end{aligned} \tag{6.91}$$

This estimate is stronger than (6.84), so we do not have to consider it further.

It remains to consider the case that $y_{1-\epsilon} < y < y_\epsilon$ and $0 \leq t \leq T_*$. In this case we may estimate

$$\begin{aligned} \mathcal{G}(x,t) &\leq H_-(y) \, |f(q(t))| + \|Df\| \left[H_-(y) \, |q(t)| + \kappa_2 t e^{\nu_2 t} \mathbf{1} + C_3 g_{\nu_3}(t) \mathbf{1} \right] \\ &\quad + |[\mathcal{D}q(t)H_-](y)| \\ &\leq H_-(y) \, |f(q(t))| + \|Df\| \, H_-(y) \, |q(t)| \\ &\quad + |[\mathcal{D}q(t)H_-](y)| + \kappa_2 t e^{\nu_2 t} \|Df\| \, |\mathbf{1}| + C_3 \frac{1}{\nu_3} e^{-1} \|Df\| \, |\mathbf{1}| \, . \end{aligned} \tag{6.92}$$

Our restrictions on κ_2, ν_2, and ν_3 now yield

$$\begin{aligned} \mathcal{G}(x,t) &\leq H_-(y) \, |f(q(t))| + \|Df\| \, H_-(y) \, |q(t)| \\ &\quad + \frac{1}{2} \kappa_2 \nu_2 t e^{\nu_2 t} + \frac{1}{2} \kappa_2. \end{aligned} \tag{6.93}$$

In addition, the restriction on C_3 implies that there exists $t_* > 0$ such that

$$C_3 g'_{\nu_3}(t) \geq 2 \, |f(q(t))| + 2 \, \|Df\| \, |q(t)| \tag{6.94}$$

holds for all $0 \leq t \leq t_*$. This shows that $\mathcal{J}^-(x,t) \leq 0$ for $0 \leq t \leq t_*$.

Now, there exists $\vartheta > 0$ such that

$$q(t) \geq \vartheta \mathbf{1} \tag{6.95}$$

holds for all $t \geq t_*$, independent of the choice of $q \in \mathcal{W}_* \cap [0,1]^n$. In particular, we can choose $C_1 = C_1(\epsilon) \gg 1$ in such a way that

$$C_1 \epsilon q(t) H'_+(y) \geq H_-(y) \sup_{q \in \mathcal{W}_*} \left[\, |f(q)| + \|Df\| \, |q| \, \right] \mathbf{1} \tag{6.96}$$

and hence $\mathcal{J}^-(x,t) \leq 0$ holds for all $y_{1-\epsilon} < y < y_\epsilon$ and $t_* \leq t \leq T_*$.

To complete the proof, we can pick

$$h_2 \geq 2\epsilon^{-1} - \xi_* + C_1(\epsilon)T_*, \tag{6.97}$$

which ensures that

$$v_{\mathrm{III}}(x,0) \geq v^-(x,0), \qquad x \in \mathbb{R}. \tag{6.98}$$

The desired inequality (6.76) for v_{III} now follows from the comparison principle together with the choices (6.86) and (6.89). The inequality for w_{III} can be established in an analogous fashion. $\square$

Proof of Proposition 6.1(iv). Recall the constants T_* and $\xi_* = \xi_*(q, \psi_v, \psi_w)$ introduced in Lemma 6.6 and the constant $\delta_1 > 0$ introduced in Lemma 6.7. In addition, recall the constant $h_2 \gg 1$ and the functions v_{III}, w_{III} introduced in Lemma 6.8.

For any $t_0 \geq 0$, we set out to show that

$$\xi_r^-(t_0 + T_*, \delta_1) - \xi_l^+(t_0 + T_*, \delta_1) \leq \max\{\xi_r^-(t_0, \delta_1) - \xi_l^+(t_0, \delta_1), 2h_2\}. \tag{6.99}$$

Without loss of generality, we will assume that $\xi_\diamond(t_0) = 0$. We write

$$h_+ = \max\{\xi_r^-(t_0, \delta_1), h_2\}, \qquad h_- = \min\{\xi_l^+(t_0, \delta_1), -h_2\}, \tag{6.100}$$

which shows that for all $x \in \mathbb{R}$ we have

$$u_*(x, t_0) \geq v_{\mathrm{III}}(x - h_+, 0\,; q_\diamond(t_0), \psi_r(t_0)), \qquad u_*(x, t_0) \leq w_{\mathrm{III}}(x - h_-, 0\,; q_\diamond(t_0), \psi_l(t_0)). \tag{6.101}$$

In particular, the comparison principle implies that

$$\begin{aligned} u_*(\xi_*, t_0 + T_*) &\geq v_{\mathrm{III}}(\xi_* - h_+, T_*; q_\diamond(t_0), \psi_r(t_0)), \\ u_*(\xi_*, t_0 + T_*) &\leq w_{\mathrm{III}}(\xi_* - h_-, T_*; q_\diamond(t_0), \psi_l(t_0)). \end{aligned} \tag{6.102}$$

Using the definition of $\kappa_{\mathcal{U}}$, we see that

$$u(\xi_* + h_+, t_0 + T_*) \in \mathcal{U}^+(\delta_1) \backslash \mathcal{U}(\delta_1), \qquad u(\xi_* + h_-, t_0 + T_*) \in \mathcal{U}^-(\delta_1) \backslash \mathcal{U}(\delta_1). \tag{6.103}$$

In particular, we find that

$$\xi_r^-(t_0 + T_*, \delta_1) \leq \xi_* + h_+, \qquad \xi_l^+(t_0 + T_*, \delta_1) \geq \xi_* + h_-, \tag{6.104}$$

which shows that

$$\xi_r^-(t_0 + T_*, \delta_1) - \xi_l^+(t_0 + T_*, \delta_1) \leq h_+ - h_- \leq \max\{2h_2, \xi_r^-(t_0, \delta_1) - \xi_l^+(t_0, \delta_1)\}, \tag{6.105}$$

as desired. In order to establish the uniform bound (6.11), it now suffices to note that for all $0 \leq t \leq T_*$, we have

$$\begin{aligned} \xi_r^-(t, \delta_1) - \xi_l(t, \delta_1) &\leq \xi_r^+(0; \delta) - \xi_l^-(0; \delta) + 4\epsilon^{-1} + Ct \\ &\leq \xi_r^+(0; \delta) - \xi_l^-(0; \delta) + 4\epsilon^{-1} + CT_*, \end{aligned} \tag{6.106}$$

by Proposition 6.1(iii). $\square$

7. Existence of traveling waves—convergence. The preparations in section 6 allow us to return to the nonlinear system

$$\partial_t u(x,t) = [\mathcal{D}u](x,t) + f\big(u(x,t)\big) \tag{7.1}$$

and establish the existence of traveling waves. In particular, in this section we set out to prove the following result.

PROPOSITION 7.1. *Consider the nonlinear system* (7.1) *with* $\gamma > 0$ *and suppose that* (HA), $(\mathrm{h})_{\S 4}$, *and* (H$\mathcal{W}$) *are all satisfied. Then there exists a constant* $c \in \mathbb{R}$ *and a function* $P \in W^{2,\infty}(\mathbb{R}, \mathbb{R}^n)$ *that satisfies the limits*

$$\lim_{\xi \to -\infty} P(\xi) = \mathbf{0}, \qquad \lim_{\xi \to \infty} P(\xi) = \mathbf{1}, \tag{7.2}$$

has $P' > \mathbf{0}$, *and yields a solution to* (7.1) *upon writing*

$$u(x,t) = P\big(x - ct\big). \tag{7.3}$$

Our approach towards proving the above result closely follows the arguments used in steps 3 and 4 of [9, sec. 4]. In particular, we consider the evolution of the solution $u_* \in \widehat{\mathcal{X}}$ to (7.1) that has the smooth initial profile (6.3). Combining regularity results with the comparison principle, it is possible to show that in an appropriate comoving frame u_* converges temporally to a function U, which in turn must be the profile of a traveling wave solution to (7.1).

Throughout the remainder of this section we fix $\gamma > 0$ and treat (HA), $(\mathrm{h})_{\S 4}$, and (H$\mathcal{W}$) as standing assumptions. We also recall the functions $\xi_\diamond$, $\xi_l^\pm$, and $\xi_r^\pm$ defined in Proposition 6.1.

LEMMA 7.2. *For every* $M > 0$ *there exists a constant* $\eta_M > 0$ *such that*

$$\partial_x u_*(x + \xi_\diamond(t), t) \geq \eta_M \mathbf{1} \tag{7.4}$$

holds for all $t \geq 0$ *and* $-M \leq x \leq M$.

Proof. Applying Proposition 4.1 to the functions $u(x,t) = u_*(x+h, t+\tau)$ and $v(x,t) = u_*(x, t+\tau)$ and subsequently taking the limit $h \to 0$ shows that

$$\partial_x u_*(x, t+\tau) \geq \eta_\gamma(x-y,t) \int_y^{y+1} \partial_x u_*(\sigma,\tau)\, d\sigma \tag{7.5}$$

holds for all $t > 0$, $\tau \geq 0$, and $x, y \in \mathbb{R}$. In view of Lemma 5.13(ii), there exists a constant $\nu_1 > 0$ such that for all $\tau \geq 0$ we have

$$\big|u_*\big(\xi_r^-(\tau;\delta_1),\tau\big) - u_*\big(\xi_l^+(\tau;\delta_1),\tau\big)\big| \geq \nu_1. \tag{7.6}$$

In particular, there exists a constant $\nu_2 > 0$ such that for all $\tau \geq 0$, there exists $y_\tau \in \mathbb{R}$ that satisfies

$$\xi_\diamond(\tau) - h_1 \leq \xi_l^+(\tau,\delta_1) \leq y_\tau \leq \xi_r^-(\tau,\delta_1) - 1 \leq \xi_\diamond(\tau) + h_1 - 1, \tag{7.7}$$

together with an integer $1 \leq i_\tau \leq n$ so that

$$\int_{y_\tau}^{y_\tau+1} [\partial_x u_*(\sigma,\tau)]_{i_\tau}\, d\sigma \geq \nu_2. \tag{7.8}$$

This means that for all $1 \le j \le n$ and $\tau \ge 0$ we have

$$[\partial_x u_*(x, \tau+1)]_j \ge \eta_\gamma(x - y_\tau, 1)_{j i_\tau} \nu_2 > 0. \tag{7.9}$$

Notice that (6.9) and Corollary 6.2 imply that

$$\begin{aligned} |\xi_\diamond(\tau+1) - \xi_\diamond(\tau)| &\le \xi_r^+(\tau, \delta_1) - \xi_l^-(\tau, \delta_1) + 4\epsilon^{-1}(\delta_1) + 2C(\delta_1) \\ &\le m_1(\delta_1) + 4\epsilon^{-1}(\delta_1) + 2C(\delta_1). \end{aligned} \tag{7.10}$$

In particular, for each $M > 0$, the quantity $x - y_\tau$ appearing in (7.9) can be uniformly bounded for all $\tau \ge 0$ and $x \in \mathbb{R}$ that have $|x - \xi_\diamond(\tau+1)| < M$.

In order to complete the proof, it now suffices to establish (7.4) for $0 \le t \le 1$. This can be achieved by using regularity and the fact that $\partial_x u_*(x,t) > \mathbf{0}$ for all $x \in \mathbb{R}$ and $t \ge 0$. ∎

LEMMA 7.3. *There exists a C^1-smooth function $U : \mathbb{R} \to \mathbb{R}^n$ and a sequence $t_j \to \infty$ such that*

$$u_*\big(\cdot + \xi_\diamond(t_j), t_j\big) \to U, \qquad j \to \infty, \tag{7.11}$$

where the convergence is in the space $BC^0(\mathbb{R}, \mathbb{R}^n)$. In addition, we have the inequality $U'(\xi) > \mathbf{0}$ for all $\xi \in \mathbb{R}$ together with the limits

$$\lim_{\xi \to -\infty} U(\xi) = \mathbf{0}, \qquad \lim_{\xi \to \infty} U(\xi) = \mathbf{1}. \tag{7.12}$$

Proof. Since the family $\{u_*(\cdot + \xi_\diamond(t), t)\}_{t \ge 0}$ consists of strictly increasing bounded functions, there exists a sequence $t_j \to \infty$ and a nondecreasing function U such that the pointwise convergence

$$u_*(\xi + \xi_\diamond(t_j), t_j) \to U(\xi), \qquad j \to \infty \tag{7.13}$$

holds for all $\xi \in \mathbb{R}$. Obviously, the bounds $\mathbf{0} \le U \le \mathbf{1}$ carry over from the corresponding bounds for u_*. In addition, we have $U(0) \in \mathcal{W}_* \cap [0,1]^n$ by compactness. In view of Corollary 6.2, the limits (7.12) can be obtained by observing that for all $\delta > 0$ we have

$$U\big(-m_1(\delta)\big) \le \delta \mathbf{1}, \qquad U\big(m_1(\delta)\big) \ge (1-\delta)\mathbf{1}. \tag{7.14}$$

Proposition 4.2 implies that there exists $C_1 \gg 1$ such that $|\partial_x u_*(x,t)| \le C_1$ for all $x \in \mathbb{R}$ and $t \ge 0$. Combining this estimate with Lemma 7.2, we obtain the inequality

$$\eta_{|\xi|+1} h \mathbf{1} \le U(\xi + h) - U(\xi) \le C_1 h \mathbf{1} \tag{7.15}$$

for all $0 \le h \le 1$. Applying Corollary 6.2, we see that the convergence $u_*(\cdot + \xi_\diamond(t_j), t_j) \to U$ holds in $BC^0(\mathbb{R}, \mathbb{R}^n)$. Another application of Proposition 4.2 yields a uniform bound on $\partial_{xx} u_*$, which combined with the Ascoli–Arzela theorem shows that in fact $U \in C^1(\mathbb{R}, \mathbb{R}^n)$. Finally, the lower bound in (7.15) yields $U'(\xi) > \mathbf{0}$ for all $\xi \in \mathbb{R}$. ∎

We now introduce the function $\widetilde{U} \in \widehat{\mathcal{X}}$ that solves the nonlinear system (7.1) with the initial condition

$$\widetilde{U}(x, 0) = U(x). \tag{7.16}$$

The uniform convergence (7.11) implies that for every $\delta > 0$, we have the inequalities

$$u_*(x + \xi_\diamond(t_j), t_j) - \delta\mathbf{1} < U(x) < u_*(x + \xi_\diamond(t_j), t_j) + \delta\mathbf{1} \tag{7.17}$$

for all sufficiently large integers j. In view of Lemma 7.2, all the conditions of Corollary 4.4 are satisfied, which shows that for all sufficiently large j and all $t \geq 0$ we have

$$\begin{aligned} \widetilde{U}(x,t) &\geq u_*(x + \xi_\diamond(t_j) + \sigma_2\delta(1 - e^{-\beta t}), t + t_j) - \sigma_3\delta e^{-\beta t}\mathbf{1}, \\ \widetilde{U}(x,t) &\leq u_*(x + \xi_\diamond(t_j) + \sigma_2\delta(1 - e^{-\beta t}), t + t_j) + \sigma_3\delta e^{-\beta t}\mathbf{1}. \end{aligned} \tag{7.18}$$

Sending $j \to \infty$ and subsequently $\delta \to 0$, we find

$$\limsup_{j\to\infty} u_*(x + \xi_\diamond(t_j), t_j + t) \leq \widetilde{U}(x,t) \leq \liminf_{j\to\infty} u_*(x + \xi_\diamond(t_j), t_j + t) \tag{7.19}$$

for all $x \in \mathbb{R}$ and $t \geq 0$. Using similar arguments as in the proof of Lemma 7.3, this shows that

$$u_*(\cdot + \xi_\diamond(t_j), t_j + t) \to \widetilde{U}(\cdot, t), \qquad j \to \infty, \tag{7.20}$$

where the convergence is in the space $BC^0(\mathbb{R}, \mathbb{R}^n)$.

LEMMA 7.4. *For all $t \geq 0$, we have the limits*

$$\lim_{|x|\to\pm\infty} \partial_x \widetilde{U}(x,t) = 0. \tag{7.21}$$

Proof. There exists $C > 0$ such that the estimate

$$\|\phi'\|^2_{C([0,1],\mathbb{R}^n)} \leq C\,\|\phi\|_{C([0,1],\mathbb{R}^n)}\,\|\phi\|_{C^2([0,1],\mathbb{R}^n)} \tag{7.22}$$

holds for any $\phi \in C^2([0,1], \mathbb{R}^n)$. Proposition 4.2 provides uniform bounds on $\partial_{xx}u_*$, which can be combined with Corollary 6.2 to yield

$$\lim_{x\to\infty} \sup_{|\xi|\geq x,\, t\geq 1} \partial_x u_*(\xi + \xi_\diamond(t), t) = 0. \tag{7.23}$$

In particular, an application of Ascoli–Arzela shows that for each $t \geq 0$, the convergence (7.20) holds in $BC^1(\mathbb{R}, \mathbb{R}^n)$. The limits (7.21) now follow from (7.23). ☐

Corollary 6.2 implies that there exist $\delta_0 > 0$ and $m_0 \gg 1$ such that

$$u_*(x - m_0, 1) - \delta_0\mathbf{1} \leq u_*(x, 0) \leq u_*(x + m_0, 1) + \delta_0\mathbf{1}. \tag{7.24}$$

Corollary 4.4 hence yields the estimates

$$\begin{aligned} u_*(x,t) &\geq u_*(x - m_0 - \sigma_2\delta_0(1 - e^{-\beta t}), t + 1) - \sigma_3\delta_0 e^{-\beta t}\mathbf{1}, \\ u_*(x,t) &\leq u_*(x + m_0 + \sigma_2\delta_0(1 - e^{-\beta t}), t + 1) + \sigma_3\delta_0 e^{-\beta t}\mathbf{1}. \end{aligned} \tag{7.25}$$

Setting $t = t_j$ and sending $j \to \infty$, we can use the convergence (7.20) to obtain

$$\widetilde{U}(x - m_0 - \sigma_2\delta_0, 1) \leq U(x) \leq \widetilde{U}(x + m_0 + \sigma_2\delta_0, 1) \tag{7.26}$$

for all $x \in \mathbb{R}$. This allows us to define two constants $\xi_* < \xi^*$ with

$$\xi_* = \sup\{\xi \mid \widetilde{U}(\cdot + \xi, 1) \leq U(\cdot)\}, \qquad \xi^* = \inf\{\xi \mid \widetilde{U}(\cdot + \xi, 1) \geq U(\cdot)\}. \tag{7.27}$$

LEMMA 7.5. *We have* $\xi_* = \xi^*$.

Proof. Assume to the contrary that $\xi^* > \xi_*$. Then we have $\widetilde{U}(x+\xi_*,1) \leq U(x)$ for all $x \in \mathbb{R}$, but $\widetilde{U}(\cdot+\xi_*,1) \neq U$. This means that $\widetilde{U}(x+\xi_*,2) < \widetilde{U}(x,1)$ for all $x \in \mathbb{R}$. In particular, for any $M \gg 1$ there exists $h = h(M)$ such that

$$\widetilde{U}(x+\xi_*+2\sigma_2 h,2) < \widetilde{U}(x,1) \tag{7.28}$$

holds for all $x \in [-M,M]$. In view of the bound (7.21) on $\partial_x \widetilde{U}$, we can ensure that the inequality

$$\widetilde{U}(x+\xi_*+\sigma_2(2+\sigma_3)h,2) - h\mathbf{1} < \widetilde{U}(x,1) \tag{7.29}$$

holds for all $x \in \mathbb{R}$ by fixing a sufficiently large $M \gg 1$ and picking $h = h(M)$.

The uniform convergence (7.11) implies that for all sufficiently large integers j and all $x \in \mathbb{R}$ we have

$$u_*(x,t_j) - \frac{h}{8} \leq U\big(x-\xi_\diamond(t_j)\big) \leq u_*(x,t_j) + \frac{h}{8}, \tag{7.30}$$

which in turn implies that for all $t \geq 0$ and $x \in \mathbb{R}$ we have

$$u_*\left(x-\frac{h}{8}\sigma_2, t+t_j\right) - \frac{h}{8}\sigma_3\mathbf{1} \leq \widetilde{U}(x-\xi_\diamond(t_j),t) \leq u_*\left(x+\frac{h}{8}\sigma_2, t+t_j\right) + \frac{h}{8}\sigma_3\mathbf{1}. \tag{7.31}$$

Combining this with (7.29) implies that

$$u_*\left(x+\xi_*+\sigma_2\left(\frac{7}{4}+\sigma_3\right)h, t_j+2\right) - \left(1+\frac{1}{4}\sigma_3\right)h\mathbf{1} \leq u_*(x,t_j+1). \tag{7.32}$$

A final application of the comparison principle now shows that for all $t \geq t_j+1$ and all $x \in \mathbb{R}$ we have

$$u_*\left(x+\xi_*+\frac{3}{4}\sigma_2(1+\sigma_3)h, t+1\right) - \sigma_3\left(1+\frac{1}{4}\sigma_3\right)he^{-\beta t}\mathbf{1} \leq u_*(x,t). \tag{7.33}$$

Writing $x = \xi + \xi_\diamond(t_k)$ together with $t = t_k$ and subsequently sending $k \to \infty$, we may use (7.20) to obtain

$$\widetilde{U}\left(\xi+\xi_*+\frac{3}{4}\sigma_2(1+\sigma_3)h, 1\right) \leq U(\xi), \tag{7.34}$$

which contradicts the definition of ξ_*. ☐

Proof of Proposition 7.1. The argument used in the proof of Lemma 7.5 can be repeated for any $1 \leq t \leq 2$, which implies that for all $1 \leq t \leq 2$ we have

$$\widetilde{U}(x,t) = U\big(x-c(t)\big) \tag{7.35}$$

for some function $c(t)$. Since $\widetilde{U}$ satisfies (7.1), one sees that $c(t)$ must be a constant, which implies that $\widetilde{U}$ is a traveling wave solution to (7.1). ☐

8. Persistence of traveling waves. In this section, we turn our attention directly to the traveling wave MFDE

$$-\gamma u''(\xi) - cu'(\xi) = \sum_{j=0}^{N} A_j[u(\xi + r_j) - u(\xi)] + f\big(u(\xi)\big). \tag{8.1}$$

In order to reflect the fact that we have dropped the dependence of the nonlinearity on ρ, we impose the following condition.

$(\mathrm{h})_{\S 8}$ The conditions (HA), (Hf1)–(Hf3), and (HS1)–(HS2) are all satisfied with the understanding that $V = \{0\}$ and $f(\cdot\,; 0) = f(\cdot)$.

The main goal is to show that the techniques developed in [32, 25] for scalar versions of (8.1) can be adapted to the current high dimensional setting. Although the broad ideas used in [32] continue to work, there are important technical details that need to be addressed. Briefly stated, the problem is that unlike scalars, nonzero matrices cannot necessarily be inverted.

Our first main result states that (8.1) cannot simultaneously have heteroclinic solutions that connect the two stable equilibria to an unstable equilibrium. This is an essential ingredient towards understanding the limiting behavior of wave profiles as system parameters are changed.

PROPOSITION 8.1 (cf. [32, Lem. 7.1]). *Consider the nonlinear MFDE* (8.1) *with* $\gamma \geq 0$ *and suppose that* $(\mathrm{h})_{\S 8}$ *is satisfied. Consider any* $q_* \in \mathbb{R}^n$ *that has* $\mathbf{0} < q_* < \mathbf{1}$ *together with* $f(q_*) = 0$. *Then there do not simultaneously exist nondecreasing solutions* u_- *and* u_+ *to* (8.1) *that have*

$$\lim_{\xi \to -\infty} u_-(\xi) = \mathbf{0}, \qquad \lim_{\xi \to +\infty} u_-(\xi) = q_*, \qquad \lim_{\xi \to -\infty} u_+(\xi) = q_*, \qquad \lim_{\xi \to +\infty} u_+(\xi) = \mathbf{1}. \tag{8.2}$$

Our second main result concerns the linearization of (8.1) around a solution $u = P$, which we write as

$$-\gamma v''(\xi) - cv'(\xi) = \sum_{j=0}^{N} A_j[v(\xi + r_j) - v(\xi)] + Df\big(P(\xi)\big)v(\xi). \tag{8.3}$$

For convenience, write $s_\gamma = 1$ if $\gamma = 0$ and $s_\gamma = 2$ if $\gamma > 0$. We introduce the operator

$$\Lambda_{c,\gamma} : W^{s_\gamma,\infty}(\mathbb{R}, \mathbb{R}^n) \to L^\infty(\mathbb{R}, \mathbb{R}^n) \tag{8.4}$$

associated to the linear MFDE (8.3) that acts as

$$[\Lambda_{c,\gamma} v](\xi) = -\gamma v''(\xi) - cv'(\xi) - \sum_{j=0}^{N} A_j[v(\xi + r_j) - v(\xi)] - Df\big(P(\xi)\big)v(\xi). \tag{8.5}$$

We also introduce the formal adjoint $\Lambda^*_{c,\gamma} : W^{s_\gamma,\infty}(\mathbb{R}, \mathbb{R}^n) \to L^\infty(\mathbb{R}, \mathbb{R}^n)$ that acts as

$$[\Lambda^*_{c,\gamma} v](\xi) = -\gamma v''(\xi) + cv'(\xi) - \sum_{j=0}^{N} A_j[v(\xi - r_j) - v(\xi)] - Df\big(P(\xi)\big)v(\xi). \tag{8.6}$$

Our second main result gives conditions under which $\Lambda_{c,\gamma}$ is a Fredholm operator with a one-dimensional kernel and zero index. As in [32], this result allows us to use the implicit function theorem to show that solutions to the nonlinear system (8.1) persist under small changes of system parameters.

PROPOSITION 8.2 (cf. [32, Thm. 4.1]). *Consider the linear MFDE* (8.3) *with* $\gamma \geq 0$ *and* $\gamma + |c| > 0$ *and suppose that* $(\mathrm{h})_{\S 8}$ *is satisfied. Suppose furthermore that for some* $\alpha > 0$ *the function* $P \in BC(\mathbb{R}, \mathbb{R}^n)$ *has the asymptotics*

$$(8.7)\quad |P(\xi)| = O\big(e^{-\alpha|\xi|}\big), \qquad \xi \to -\infty, \qquad |P(\xi) - \mathbf{1}| = O\big(e^{-\alpha|\xi|}\big), \qquad \xi \to +\infty.$$

Finally, suppose that that there exists a nontrivial solution $p \in W^{s_\gamma,\infty}(\mathbb{R}, \mathbb{R}^n)$ *to* (8.3) *that has* $p(\xi) \geq \mathbf{0}$ *for all* $\xi \in \mathbb{R}$. *Then the operator* $\Lambda_{c,\gamma}$ *is a Fredholm operator with*

$$(8.8)\qquad \dim \operatorname{Ker}(\Lambda_{c,\gamma}) = \dim \operatorname{Ker}(\Lambda^*_{c,\gamma}) = \operatorname{codim} \operatorname{Range}(\Lambda_{c,\gamma}) = 1.$$

In addition, the element $p \in \operatorname{Ker}(\Lambda_{c,\gamma})$ *satisfies* $p(\xi) > \mathbf{0}$ *for all* $\xi \in \mathbb{R}$ *and there exists* $p_* \in \operatorname{Ker}(\Lambda^*_{c,\gamma})$ *that has* $p_*(\xi) > \mathbf{0}$ *for all* $\xi \in \mathbb{R}$.

The crucial ingredient in the proof of these two results is a detailed understanding of the asymptotic behavior of solutions to (8.1). One expects that if a solution approaches an equilibrium q, the asymptotic behavior can be understood by studying the autonomous system

$$(8.9)\qquad -cv'(\xi) = \gamma v''(\xi) + \sum_{j=0}^{N} A_j[v(\xi + r_j) - v(\xi)] + Df(q)v(\xi).$$

In particular, in the first part of this section we analyze the characteristic function

$$(8.10)\qquad \Delta_{c,\gamma,q}(z) = -\gamma z^2 - cz - \sum_{j=0}^{N} A_j(e^{zr_j} - 1) - Df(q)$$

and look for pairs $\lambda \in \mathbb{C}$, $w \in \mathbb{C}^n$ that have $\Delta_{c,\gamma,q}(\lambda)w = 0$. Indeed, any such pair yields a solution to (8.9) upon writing $v(\xi) = e^{\lambda\xi}w$, and one hopes that the leading order behavior of solutions to the nonlinear system (8.1) can be expressed in terms of such eigensolutions. A result along these lines can be found in [31, Prop. 7.2]. However, when dealing with MFDEs, there is a possibility that solutions approach their limits at a rate that is faster than *any* exponential. In the second part of this section, we will develop comparison principles and find a specific restatement of (8.3) that will allow us to rule out this pathological possibility.

Our analysis of the characteristic function (8.10) is aided considerably by earlier work in [8]. In particular, upon writing

$$(8.11)\qquad A_q(\lambda) = \sum_{j=0}^{N} A_j(e^{\lambda r_j} - 1) + Df(q),$$

the authors studied the eigenvalue problem

$$(8.12)\qquad \mu v = A_q(\lambda)v, \qquad v \geq \mathbf{0},$$

which is closely related to (8.10). By Perron–Frobenius [16], this problem has a unique solution pair $\mu = \mu_q(\lambda)$, $v = v_q(\lambda) > \mathbf{0}$ for each $\lambda \in \mathbb{R}$. The results in [8] state that μ_q is analytic and strictly convex with $\mu_q(\lambda) \to \infty$ as $\lambda \to \pm\infty$. In particular, for any $0 < t < 1$ and $\lambda_1 \neq \lambda_2$ we have the inequality

$$(8.13)\qquad \mu_q(t\lambda_1 + (1-t)\lambda_2) < t\mu_q(\lambda_1) + (1-t)\mu_q(\lambda_2).$$

Upon introducing the polynomial $\psi_{c,\gamma}(\lambda) = -\gamma\lambda^2 - c\lambda$, we see that any $\lambda \in \mathbb{R}$ that solves

$$(8.14)\qquad \psi_{c,\gamma}(\lambda) = \mu_q(\lambda)$$

automatically has $\Delta_{c,\gamma,q}(\lambda)v_q(\lambda) = 0$. Vice versa, if $\Delta_{c,\gamma,q}(\lambda)v = 0$ for some nonzero $v \geq \mathbf{0}$, then (8.14) must be satisfied. Since $\psi''(\lambda) \leq 0$ and μ is strictly convex, (8.14) has at most two real solutions. The next three results explore the relation between the functions μ_q and $\Delta_{c,\gamma,q}$.

LEMMA 8.3. *Suppose that* $(\mathrm{h})_{\S 8}$ *is satisfied and pick any solution to* $f(q) = \mathbf{0}$ *for which the equation* $\det[Df(q) - \lambda I] = 0$ *has no solutions with* $\mathrm{Re}\,\lambda \geq 0$. *Then* (8.14) *with* $\gamma \geq 0$ *has precisely two real solutions* $\lambda^- < 0 < \lambda^+$.

Proof. This follows from the fact that our assumption on q implies that $\mu_q(0) < 0$ while on the other hand $\psi_{c,\gamma}(0) = 0$. ◻

LEMMA 8.4. *Suppose that* $(\mathrm{h})_{\S 8}$ *is satisfied and pick any solution to* $f(q) = \mathbf{0}$ *for which the equation* $\det[Df(q) - \lambda I] = 0$ *has at least one solution with* $\mathrm{Re}\,\lambda > 0$. *Then for any pair* $\lambda^- < 0 < \lambda^+$, (8.14) *with* $\gamma \geq 0$ *cannot be satisfied for both* $\lambda = \lambda^{\pm}$.

Proof. This follows from the fact that our assumption on q implies that $\mu_q(0) > 0$ while again $\psi_{c,\gamma}(0) = 0$. ◻

LEMMA 8.5. *Suppose that* $(\mathrm{h})_{\S 8}$ *is satisfied, pick any* $q \in [0, 1]^n$ *for which* $f(q) = \mathbf{0}$, *and consider the autonomous sytem* (8.9) *with* $\gamma \geq 0$. *Suppose that* (8.14) *has two distinct solutions* $\lambda^- < \lambda^+$. *Then the characteristic equation* $\det \Delta_{c,\gamma,q}(z) = 0$ *has two simple roots at* $z = \lambda^{\pm}$. *In addition, consider any* $z \in \mathbb{C} \setminus \{\lambda^-, \lambda^+\}$ *for which* $\det \Delta_{c,\gamma,q}(z) = 0$. *Then either* $\mathrm{Re}\, z \leq \lambda^-$ *or* $\mathrm{Re}\, z \geq \lambda^+$ *must hold, where equality is only possible if* $\gamma = c = 0$. *If in fact* $\mathrm{Im}\, z = 0$, *then we cannot have* $\Delta_{c,\gamma,q}(z)v = \mathbf{0}$ *for any nonzero* $v \in \mathbb{R}^n_{\geq \mathbf{0}}$.

Proof. For convenience, we introduce the shorthand $\psi(\lambda) = \psi_{c,\gamma}(\lambda)$. Note that $\lambda^{\pm}$ are simple roots to (8.14), which means $\psi'(\lambda^{\pm}) \neq \mu_q'(\lambda^{\pm})$. In order to show that $z = \lambda^{\pm}$ are also simple roots of $\det \Delta_{c,\gamma,q}(z) = 0$, we must show that

$$\frac{d}{dz} \det \Delta_{c,\gamma,q}(z)_{|z=\lambda^{\pm}} \neq 0. \tag{8.15}$$

To see this, we introduce the function

$$\mathcal{F}(\psi, z) = \det[\psi I - A_q(z)], \tag{8.16}$$

which clearly satisfies $\mathcal{F}(\mu_q(\lambda), \lambda) = 0$ for all $\lambda \in \mathbb{R}$. In addition, since $\mu_q(\lambda)$ is a simple eigenvalue for $A_q(\lambda)$ we must have $D_1\mathcal{F}(\mu_q(\lambda), \lambda) \neq 0$. The implicit function theorem now yields

$$\mu_q'(\lambda) = -D_2\mathcal{F}\big(\mu_q(\lambda), \lambda\big)/D_1\mathcal{F}\big(\mu_q(\lambda), \lambda\big). \tag{8.17}$$

Upon writing $\mathcal{G}(z) = \det \Delta_{c,\gamma,q}(z)$, we obviously have $\mathcal{G}(z) = \mathcal{F}(\psi(z), z)$. We may hence compute

$$\mathcal{G}'(z) = D_1\mathcal{F}(\psi(z), z)\psi'(z) + D_2\mathcal{F}(\psi(z), z), \tag{8.18}$$

which yields

$$\begin{aligned} \mathcal{G}'(\lambda^{\pm}) &= D_1\mathcal{F}\big(\psi(\lambda^{\pm}), \lambda^{\pm}\big)\psi'(\lambda^{\pm}) + D_2\mathcal{F}\big(\psi(\lambda^{\pm}), \lambda^{\pm}\big) \\ &= D_1\mathcal{F}\big(\mu_q(\lambda^{\pm}), \lambda^{\pm}\big)\psi'(\lambda^{\pm}) + D_2\mathcal{F}\big(\mu_q(\lambda^{\pm}), \lambda^{\pm}\big) \\ &= D_1\mathcal{F}\big(\mu_q(\lambda^{\pm}), \lambda^{\pm}\big)[\psi'(\lambda^{\pm}) - \mu_q'(\lambda^{\pm})]. \end{aligned} \tag{8.19}$$

In particular, we have $\mathcal{G}'(\lambda^{\pm}) \neq 0$, as desired.

Let us now consider a pair $(z, v_z) \in \mathbb{C} \times \mathbb{C}^n$ that has $\Delta_{c,\gamma,q}(z)v_z = 0$. In addition, let us suppose that $\lambda^- \le \operatorname{Re} z \le \lambda^+$ but $z \ne \lambda^\pm$. Upon writing $z = \lambda + i\nu$ with $\lambda, \nu \in \mathbb{R}$, let us consider the nonlocal system

$$\begin{aligned}\partial_t v(x,t) &= \gamma \partial_{xx} v(x,t) + (\gamma\lambda^2 + c\lambda)v(x,t)\\ &\quad + \sum_{j=0}^{N} A_j[e^{\lambda r_j} v(x + r_j, t) - v(x,t)] + Df(q)v(x,t).\end{aligned} \tag{8.20}$$

A short calculation shows that the two functions

$$\begin{aligned} v(x,t) &= \operatorname{Re} e^{i\nu\left(x-(c+2\gamma\lambda)t\right)} v_z,\\ w(x,t) &= e^{(\mu_q(\lambda)+\gamma\lambda^2+c\lambda)t} v_q(\lambda)\end{aligned} \tag{8.21}$$

both satisfy (8.20). Since $v_q(\lambda) > \mathbf{0}$, there exists $\kappa > 0$ such that

$$-\kappa w(x,t) \le v(x,t) \le \kappa w(x,t) \tag{8.22}$$

holds for all $x \in \mathbb{R}$ and $t \ge 0$. If $\lambda^- < \lambda < \lambda^+$, then $w(x,t) \to 0$ as $t \to \infty$, which contradicts the fact that $\|v(\cdot,t)\|_\infty$ does not decay. Let us therefore suppose that either $\lambda = \lambda^\pm$. In this case $w(x,t) = v_q(\lambda)$ is constant in time and space. However, by appropriately choosing κ we can ensure that $v(x,0) \le \kappa w(x,0)$ with equality $v_{i_*}(x_*,0) = \kappa w_{i_*}(x_*,0)$ for some (but not all) $x_* \in \mathbb{R}$ and $1 \le i_* \le n$. If $\gamma > 0$, the comparison principle stated in Proposition 4.1 implies that $v(x,t) < \kappa w(x,t)$ for all $t > 0$, which contradicts the fact that

$$v_{i_*}\big((c+2\gamma\lambda)t + x_*, t\big) = v_{i_*}(x_*,0) = \kappa w_{i_*}(x_*,0) = \kappa w_{i_*}(x_*,t). \tag{8.23}$$

If $\gamma = 0$, we can only conclude that $v(x,t) \le \kappa w(x,t)$ for all $t > 0$. If, however, $c \ne 0$ also, then for all small $t > 0$ we have

$$v_{i_*}(x_*,t) < v_{i_*}(x_*,0) = \kappa w_{i_*}(x_*,t) \tag{8.24}$$

but also

$$v_{i_*}(x_*, 2\pi\,|\nu c|^{-1}) = \kappa w_i(x_*, 2\pi\,|\nu c|^{-1}). \tag{8.25}$$

This violates the uniqueness of solutions to (8.20) with $\gamma = 0$ and hence completes our proof. $\square$

We now turn our attention to the linear system

$$-\gamma v''(\xi) - cv'(\xi) = \sum_{j=0}^{N} A_j v(\xi + r_j) + B(\xi)v(\xi), \tag{8.26}$$

together with its inhomogeneous counterpart

$$-\gamma v''(\xi) - cv'(\xi) = \sum_{j=0}^{N} A_j v(\xi + r_j) + B(\xi)v(\xi) + h(\xi). \tag{8.27}$$

We remark that (8.26) reduces to (8.3) upon writing $B(\xi) = Df(P(\xi)) - \mathcal{A}$. Alternatively, if u_1 and u_2 both satisfy (8.1), the difference $v = u_1 - u_2$ satisfies (8.26) with coefficients

$$B(\xi) = \int_0^1 \big[Df\big(u_2(\xi) + \sigma(u_1(\xi) - u_2(\xi))\big) - \mathcal{A}\big]\, d\sigma. \tag{8.28}$$

This motivates the following condition on the function B.

(hb) We have $B \in L^\infty(\mathbb{R}, \mathbb{R}^{n\times n})$ and there exists $\kappa > 0$ such that $B(\xi) + \kappa I \geq 0$ for all $\xi \in \mathbb{R}$. In addition, for any pair $(i,j) \in \{1, \dots, n\}^2$ with $i \neq j$ we either have $B_{ij}(\xi) = 0$ for all $\xi \in \mathbb{R}$ or $B_{ij}(\xi) > \kappa_{ij} > 0$ for all $\xi \in \mathbb{R}$.

In order to generalize the results in [32], we need to exploit some freedom that we have in the formulation of the MFDE (8.26) that is not present in the scalar case. In particular, for any $v \in L^\infty(\mathbb{R}, \mathbb{R}^n)$ and $\sigma \in \mathbb{R}^n$, we introduce the new function v^σ that has

$$v_i^\sigma(\xi) = v_i(\xi + \sigma_i). \tag{8.29}$$

A short calculation shows that the homogeneous system (8.26) is equivalent to the system

$$-\gamma D_{\xi\xi} v^\sigma(\xi) - c D_\xi v^\sigma(\xi) = [J^\sigma v^\sigma](\xi) + B^\sigma_{\mathrm{diag}}(\xi) v^\sigma(\xi), \tag{8.30}$$

in which we have introduced the matrix-valued function

$$[B^\sigma_{\mathrm{diag}}]_{ik}(\xi) = B_{ii}(\xi + \sigma_i)\delta_{ik}, \tag{8.31}$$

together with the operator

$$[J^\sigma v]_i(\xi) = \sum_{j=0}^{N}\sum_{k=0}^{n} [A_j]_{ik} v_k(\xi + r_j + \sigma_i - \sigma_k) + \sum_{k\neq i} B_{ik}(\xi + \sigma_i) v_k(\xi + \sigma_i - \sigma_k). \tag{8.32}$$

For convenience, we introduce the index set

$$\mathcal{I} = \{0, \dots, N+1\} \times \{1, \dots, n\}^2 \tag{8.33}$$

and rewrite the operator J^σ as

$$[J^\sigma v]_i(\xi) = \sum_{(j,k,l)\in\mathcal{I}} \delta_{ik} \beta^\sigma_{jkl}(\xi) v_l(\xi + r_j + \sigma_k - \sigma_l), \tag{8.34}$$

using appropriately defined scalar functions $\{\beta^\sigma_{jkl}\}$.

The assumption (hb) implies that there exist constants $\{\alpha^\pm_{jkl}\}$ that do not depend on σ such that the inequalities

$$0 \leq \alpha^-_{jkl} \leq \beta^\sigma_{jkl}(\xi) \leq \alpha^+_{jkl}, \qquad (j,k,l) \in \mathcal{I}, \tag{8.35}$$

hold for all $\sigma \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$. Furthermore, (hb) implies that the constants can be chosen in such a way that $\alpha^-_{jkl} = 0$ automatically implies that also $\alpha^+_{jkl} = 0$. We now introduce the sets

$$\begin{aligned} \mathcal{I}^\sigma_- &= \{(j,k,l) \in \mathcal{I} \mid \alpha^-_{jkl} > 0 \text{ and } r_j + \sigma_k - \sigma_l < 0\}, \\ \mathcal{I}^\sigma_0 &= \{(j,k,l) \in \mathcal{I} \mid \alpha^-_{jkl} > 0 \text{ and } r_j + \sigma_k - \sigma_l = 0\}, \\ \mathcal{I}^\sigma_+ &= \{(j,k,l) \in \mathcal{I} \mid \alpha^-_{jkl} > 0 \text{ and } r_j + \sigma_k - \sigma_l > 0\}. \end{aligned} \tag{8.36}$$

In addition, we introduce the quantities

$$r^\sigma_{\min} = \min_{(j,k,l)\in\mathcal{I}^\sigma_-} r_j + \sigma_k - \sigma_l, \qquad r^\sigma_{\max} = \max_{(j,k,l)\in\mathcal{I}^\sigma_+} r_j + \sigma_k - \sigma_l \tag{8.37}$$

with the understanding that extrema over empty sets are taken to be zero. Finally, we introduce the sets

$$\begin{aligned} \Sigma_-^\sigma &= \{l \in \{1, \dots, n\} \text{ for which } (j,k,l) \notin \mathcal{I}_-^\sigma \text{ for all } 0 \le j \le N+1 \text{ and } 1 \le k \le n\}, \\ \Sigma_+^\sigma &= \{l \in \{1, \dots, n\} \text{ for which } (j,k,l) \notin \mathcal{I}_+^\sigma \text{ for all } 0 \le j \le N+1 \text{ and } 1 \le k \le n\}, \end{aligned} \tag{8.38}$$

together with the pair of projection operators

$$\pi_\pm^\sigma : \mathbb{R}^n \to \mathbb{R}^n, \qquad [\pi_\pm^\sigma v]_i = \begin{cases} v_i & i \in \Sigma_\pm^\sigma, \\ 0, & i \notin \Sigma_\pm^\sigma. \end{cases} \tag{8.39}$$

For some of our results we will need to impose the following condition, which should be compared to (HS2).

(hs) For all $\sigma \in \mathbb{R}^n$ we have $\mathcal{I}_-^\sigma \neq \emptyset$ and $\mathcal{I}_+^\sigma \neq \emptyset$.

LEMMA 8.6. *Suppose that* (hb) *is satisfied. There exists $\sigma_* \in \mathbb{R}^n$ such that for every $(j,k,l) \in \mathcal{I}_-^{\sigma_*}$ we have $(j',k',k) \in \mathcal{I}_-^{\sigma_*}$ for some pair $0 \le j' \le N+1$ and $1 \le k' \le n$.*

Proof. We consider the weighted graph

$$\widehat{\mathcal{G}} = \big(\mathcal{V}(\widehat{\mathcal{G}}), \mathcal{E}(\widehat{\mathcal{G}}), w_{\mathcal{E}}^\sigma\big) \tag{8.40}$$

with vertices

$$\mathcal{V}(\widehat{\mathcal{G}}) = \{1, \dots, n\} \tag{8.41}$$

and (repeated) directed edges

$$\mathcal{E}(\widehat{\mathcal{G}}) = \{(j,k,l) \in \mathcal{I} \mid \alpha_{jkl}^- > 0\}, \qquad w_{\mathcal{E}}^\sigma(j,k,l) = r_j + \sigma_k - \sigma_l \tag{8.42}$$

with the understanding that the edge (j,k,l) points from k to l. Stated in terms of this graph, we need to prove that every vertex that has an outgoing edge with negative weight must also have an incoming edge with negative weight.

For each σ, we write $\mathcal{L}(\sigma) \subset \mathcal{V}(\widehat{\mathcal{G}})$ for the set of vertices that are part of a directed loop with negative weight, i.e., we say $k \in \mathcal{L}(\sigma)$ if for some $\ell \ge 2$ there exists a sequence

$$k_1, \dots, k_\ell, \qquad k_1 = k_\ell = k \tag{8.43}$$

together with a sequence

$$j_1, \dots, j_{\ell-1} \tag{8.44}$$

such that for all $1 \le i \le \ell - 1$ we have $(j_i, k_i, k_{i+1}) \in \mathcal{I}_-^\sigma$.

In addition, we write $\mathcal{C}(\sigma) \subset \mathcal{V}(\widehat{\mathcal{G}})$ for the set of vertices that are reachable from $\mathcal{L}(\sigma)$ via negative weight edges. More precisely, we say $k \in \mathcal{C}(\sigma)$ if for some $\ell \ge 1$ there exists a sequence

$$k_1, \dots, k_\ell, \qquad k_1 \in \mathcal{L}(\sigma), \qquad k_\ell = k \tag{8.45}$$

together with a sequence

$$j_1, \dots, j_{\ell-1} \tag{8.46}$$

such that for all $1 \le i \le \ell - 1$ we have $(j_i, k_i, k_{i+1}) \in \mathcal{I}_-^\sigma$. Obviously, we have $\mathcal{L}(\sigma) \subset \mathcal{C}(\sigma)$.

We remark that it suffices to find σ_* such that $k \in \mathcal{C}(\sigma_*)$ holds whenever $(j,k,l) \in \mathcal{I}_-^{\sigma_*}$. We therefore write

$$\mathcal{P}(\sigma) = \{(j,k,l) \in \mathcal{I}_-^\sigma \mid k \notin \mathcal{C}(\sigma)\} \tag{8.47}$$

for the set of problematic edges. In addition, we write

$$\mathcal{P}_{\min}(\sigma) = \{(j,k,l) \in \mathcal{P}(\sigma) \mid w_\mathcal{E}^\sigma(j',k',l') \ge w_\mathcal{E}^\sigma(j,k,l) \text{ for all } (j',k',l') \in \mathcal{P}(\sigma)\} \tag{8.48}$$

for the set of minimally weighted problematic edges, together with

$$\mathcal{V}_{\min}(\sigma) = \{k \in \mathcal{V}(\widehat{\mathcal{G}}) \mid \exists (j,k,l) \in \mathcal{P}_{\min}(\sigma)\} \tag{8.49}$$

for the set of vertices with outgoing minimally weighted problematic edges.

We start at $\sigma_\diamond = \mathbf{0}$. If $\mathcal{P}(\sigma_\diamond) = \emptyset$, we are done. If not, we write

$$\sigma_k(t) = \begin{cases} [\sigma_\diamond]_k & \text{if } k \notin \mathcal{V}_{\min}(\sigma_\diamond), \\ [\sigma_\diamond]_k + t\ell[k] & \text{if } k \in \mathcal{V}_{\min}(\sigma_\diamond), \end{cases} \tag{8.50}$$

where $\ell[k] \ge 1$ is the length of the longest chain of directed edges in $\mathcal{P}_{\min}(\sigma_\diamond)$ that originates from k. This integer is well defined because $\mathcal{P}_{\min}(\sigma_\diamond)$ can contain no loops.

Our choice of $\sigma(t)$ ensures that the weights of edges in $\mathcal{P}_{\min}$ are strictly increasing as t increases. In particular, we may write $t_* > 0$ for the first time $t > 0$ for which either $w_\mathcal{E}^{\sigma(t)}(j,k,l) \ge 0$ for all $(j,k,l) \in \mathcal{P}_{\min}(\sigma_\diamond)$ or for which there exists $(j,k,l) \in \mathcal{E}(\widehat{\mathcal{G}})$ with either $k \notin \mathcal{C}(\sigma_\diamond)$ or $l \notin \mathcal{C}(\sigma_\diamond)$ such that

$$w_\mathcal{E}^{\sigma(t)}(j,k,l) \le w_\mathcal{E}^{\sigma(t)}(j',k',l') \text{ for all } (j',k',l') \in \mathcal{P}_{\min}(\sigma_\diamond). \tag{8.51}$$

Varying t does not affect the weights of edges between elements of $\mathcal{C}(\sigma_\diamond)$. In particular, we have

$$\mathcal{C}(\sigma_\diamond) \subset \mathcal{C}\big(\sigma(t_*)\big). \tag{8.52}$$

We can hence set $\sigma_\diamond = \sigma(t_*)$ and repeat the process. Since the minimum weight of the edges in $\mathcal{P}(\sigma_\diamond)$ increases with each step by an amount that is bounded away from zero, we will have $\mathcal{P}(\sigma_\diamond) = \emptyset$ after a finite number of steps. ∎

LEMMA 8.7. *Suppose that* (hb) *and* (hs) *are satisfied. There exists* $\sigma_{**} \in \mathbb{R}$ *such that for every* $1 \le l \le n$ *there exists a pair* $0 \le j \le N+1$ *and* $1 \le k \le n$ *for which* $(j,k,l) \in \mathcal{I}_-^{\sigma_{**}}$. *In particular,* $\Sigma_-^{\sigma_{**}} = \emptyset$.

Proof. We continue using the setup developed in the proof of Lemma 8.6 and write σ_* for the vector constructed there. The assumption (hs) implies that $\mathcal{C}(\sigma_*)$ is nonempty. We write $\mathcal{C}^c(\sigma_*)$ for the complement of this set. If $\mathcal{C}^c(\sigma_*)$ is empty, there is nothing to prove. We now introduce the function $\sigma(t)$ by writing

$$\sigma_k(t) = \begin{cases} [\sigma_*]_k & \text{if } k \in \mathcal{C}(\sigma_*), \\ [\sigma_*]_k + t & \text{if } k \in \mathcal{C}^c(\sigma_*). \end{cases} \tag{8.53}$$

Notice that at $t = 0$, all edges between $\mathcal{C}(\sigma_*)$ and $\mathcal{C}^c(\sigma_*)$ have nonnegative weight. In addition, the weights of edges internal to $\mathcal{C}(\sigma_*)$ and $\mathcal{C}^c(\sigma_*)$ remain unchanged upon changing t. However, the weights of edges pointing from $\mathcal{C}(\sigma_*)$ to $\mathcal{C}^c(\sigma_*)$ decrease as t increases, while the weights of edges that point from $\mathcal{C}^c(\sigma_*)$ to $\mathcal{C}(\sigma_*)$ increase as t increases. In particular, there exists $t_\diamond > 0$ for which

$$\mathcal{C}(\sigma_*) \subset \mathcal{C}(\sigma(t_\diamond)), \qquad \mathcal{C}(\sigma_*) \neq \mathcal{C}(\sigma(t_\diamond)). \tag{8.54}$$

This process can be repeated as often as needed to find $\sigma_{**} \in \mathbb{R}^n$ for which $\mathcal{C}^c(\sigma_{**}) = \emptyset$. ◻

With these preparations in hand, we are ready to formulate two comparison principles for (8.27). These should be seen as the analogue of results obtained in [32, sec. 3] for $\gamma = 0$ and [25] for $\gamma > 0$. The latter case is easier to handle because the comparison principle from section 4 can be invoked.

PROPOSITION 8.8. *Consider the inhomogeneous system* (8.27) *with* $h(\xi) \geq \mathbf{0}$ *and suppose that* (HA), (hb), *and* (hs) *are satisfied. Fix* $\gamma \geq 0$ *and* $c \in \mathbb{R}$ *with either* $\gamma > 0$ *or* $c \neq 0$. *Consider any function* $v \in W^{s_\gamma,\infty}(\mathbb{R}, \mathbb{R}^n)$ *that has* $v(\xi) \geq \mathbf{0}$ *for all* $\xi \in \mathbb{R}$ *and satisfies* (8.27) *for all* $\xi \in \mathbb{R}$. *If there exists a pair* $(i_0, \xi_0) \in \{1, \ldots, n\} \times \mathbb{R}$ *with* $v_{i_0}(\xi_0) = 0$, *then in fact* $v(\xi) = \mathbf{0}$ *for all* $\xi \in \mathbb{R}$.

PROPOSITION 8.9. *Consider the homogeneous system* (8.26) *with* $\gamma = c = 0$ *and suppose that* (HA), (hb), *and* (hs) *are satisfied. Consider any function*

$$v \in L^\infty(\mathbb{R}, \mathbb{R}^n) \tag{8.55}$$

that satisfies (8.26) *with* $\gamma = c = 0$ *for all* $\xi \in \mathbb{R}$ *and has* $v(\xi) \geq \mathbf{0}$ *for all* $\xi \in \mathbb{R}$. *Suppose furthermore that there exists* $i_0 \in \{1, \ldots, n\}$ *and* $\tau \in \mathbb{R}$ *for which*

$$v_{i_0}(\xi) = 0, \qquad \xi \geq \tau. \tag{8.56}$$

Then we have $v(\xi) = \mathbf{0}$ *for all* $\xi \in \mathbb{R}$.

LEMMA 8.10 (cf. [25, Cor. A.7]). *If* $\gamma > 0$, *all the statements in Proposition* 8.8 *are valid, even if (hs) is not satisfied.*

Proof. Since $h(\xi) \geq 0$, we observe that $\overline{v}(x,t) = v(x - ct)$ satisfies the differential inequality

$$\begin{aligned} \partial_t \overline{v}(x,t) &= \gamma \partial_{xx} \overline{v}(x,t) + \sum_{j=0}^{N} A_j \overline{v}(\xi + r_j, t) + B(x - ct)\overline{v}(x,t) + h(x - ct) \\ &\geq \gamma \partial_{xx} \overline{v}(x,t) + \sum_{j=0}^{N} A_j \overline{v}(\xi + r_j, t) + B(x - ct)\overline{v}(x,t), \end{aligned} \tag{8.57}$$

which is covered by the comparison principle stated in Proposition 4.1. In particular, if v does not vanish everywhere, we must have $\overline{v}(x,t) > \mathbf{0}$ for all $x \in \mathbb{R}$ and $t > 0$. This contradicts the fact that $\overline{v}_{i_0}(\xi_0 + ct, t) = v_{i_0}(\xi_0) = 0$. ◻

LEMMA 8.11 (cf. [32, Lem. 3.1]). *Consider the inhomogeneous system* (8.27) *with* $h(\xi) \geq \mathbf{0}$ *and suppose that* (HA) *and* (hb) *are satisfied. Fix* $\gamma = 0$ *and* $c \neq 0$ *and consider any function* $v \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^n)$ *that has* $v(\xi) \geq \mathbf{0}$ *for all* $\xi \in \mathbb{R}$ *and satisfies* (8.27) *for all* $\xi \in \mathbb{R}$. *Suppose furthermore that there exists a pair* $(i_0, \xi_0) \in \{1, \ldots, n\} \times \mathbb{R}$ *such that* $v_{i_0}(\xi_0) = 0$. *Then if* $c < 0$, *we have*

$$v(\xi) = \mathbf{0}, \qquad \xi \leq \xi_0 + n r_{\min}, \tag{8.58}$$

while if $c > 0$ *we have*

$$v(\xi) = \mathbf{0}, \qquad \xi \geq \xi_0 + n r_{\max}. \tag{8.59}$$

Proof. We restrict ourselves to the case $c > 0$. Notice that for any $1 \le i \le n$ and $\xi_* \in \mathbb{R}$ for which $v_i(\xi_*) = 0$, the system (8.27) implies that $v_i'(\xi_*) \le 0$. In particular, a standard differential inequality now implies that $v_i(\xi) = 0$ for all $\xi \ge \xi_*$. In addition, a necessary condition for $v_i'(\xi_*) = 0$ is that

$$[A_j]_{i\ell} v_\ell(\xi_* + r_j) = 0, \qquad 1 \le \ell \le n, \qquad 0 \le j \le N. \tag{8.60}$$

Using condition (HA) repeatedly, we now see that for all $1 \le \ell \le n$ there exists $\xi_\ell \le \xi_* + n r_{\max}$ for which $v_\ell(\xi_\ell) = 0$, which completes the proof. ∎

Lemma 8.12 (cf. [32, Lem. 3.3]). *Suppose that* (HA) *and* (hb) *are satisfied and fix $\tau \in \mathbb{R}$, $\gamma \ge 0$, and $c \in \mathbb{R}$ with $\gamma + |c| > 0$. Recall the vector $\sigma_* \in \mathbb{R}^n$ defined in Lemma* 8.6. *Consider any function*

$$v^{\sigma_*} \in L^\infty((-\infty, \tau + r^{\sigma_*}_{\max}], \mathbb{R}^n) \tag{8.61}$$

that satisfies the homogeneous system (8.30) *for all $\xi \le \tau$ and suppose that*

$$v^{\sigma_*}(\xi) = \mathbf{0}, \qquad \tau + r^{\sigma_*}_{\min} \le \xi \le \tau + r^{\sigma_*}_{\max}. \tag{8.62}$$

Suppose furthermore that $v^{\sigma_*}(\xi_*) \neq \mathbf{0}$ for some $\xi_* < \tau + r^{\sigma_*}_{\min}$. Then there exist two pairs

$$(i_-, \xi_-) \in \{1, \dots, n\} \times (-\infty, \tau + r^{\sigma_*}_{\min}], \qquad (i_+, \xi_+) \in \{1, \dots, n\} \times (-\infty, \tau + r^{\sigma_*}_{\min}] \tag{8.63}$$

that have $|\xi_+ - \xi_-| \le |r^{\sigma_*}_{\min}|$ together with

$$v_{i_-}(\xi_-) < 0 < v_{i_+}(\xi_+). \tag{8.64}$$

Proof. Without loss of generality, we suppose that $\tau = 0$ and $\sigma_* = \mathbf{0}$. It suffices to show that there exists $\delta > 0$ such that (8.62) together with the inequality

$$v(\xi) \le \mathbf{0}, \qquad 2r_{\min} \le \xi \le r_{\min}, \tag{8.65}$$

automatically implies that

$$v_l(\xi) = 0, \qquad -\delta + r_{\min} \le \xi \le r_{\min}, \qquad 1 \le l \le n. \tag{8.66}$$

We pick $\delta > 0$ in such a way that $r_j \le -2\delta$ whenever $r_j < 0$. Let us now consider any $(j, k, l) \in \mathcal{I}^0_-$ and any $\xi_* \in \mathbb{R}$ that has

$$-\delta + r_{\min} \le \xi_* + r_j \le r_{\min}. \tag{8.67}$$

Our choice of $\delta > 0$ shows that $r_{\min} \le \xi_* \le 0$, which means $v'(\xi) = v''(\xi) = \mathbf{0}$ and $v(\xi + r) = \mathbf{0}$ whenever $0 \le r \le r_{\max}$. In particular, (8.30) now implies that

$$v_l(\xi_* + r_j) = 0. \tag{8.68}$$

In particular, we have established (8.66) for all $l \notin \Sigma^0_-$.

Upon writing $w(\xi) = \pi^0_- v(\xi)$ and viewing this as an element of $\mathbb{R}^m$ with $m = \#\Sigma^0_-$, the properties described in Lemma 8.6 allow us to write

$$-\gamma w''(\xi) - c w'(\xi) = L_+(\xi) \mathrm{ev}^+_\xi w + g(\xi), \tag{8.69}$$

where $L_+(\xi)$ is a linear operator mapping $C([0, r_{\max}], \mathbb{R}^m)$ into $\mathbb{R}^m$ and $[\mathrm{ev}_\xi^+ w](\theta) = w(\xi + \theta)$ for $0 \le \theta \le r_{\max}^\sigma$. The function $g(\xi)$ incorporates the contributions from $(I - \pi_-^0)v$ and satisfies

$$g(\xi) = 0, \qquad -\delta + r_{\min} \le \xi \le r_{\min}. \tag{8.70}$$

In particular, the uniqueness of solutions to advanced equations now implies that also $w(\xi) = 0$ for $-\delta + r_{\min} \le \xi \le r_{\min}$, as desired. ∎

LEMMA 8.13. *Consider the inhomogeneous linear system* (8.27) *with* $h(\xi) \ge \mathbf{0}$ *and suppose that* (HA), (hb)*, and* (hs) *are satisfied. Fix* $\tau \in \mathbb{R}$*,* $\gamma = 0$*, and* $c \neq 0$*. Consider any function* $v \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^n)$ *that satisfies* (8.27) *for all* $\xi \in \mathbb{R}$ *and suppose that*

$$v(\xi) = \mathbf{0}, \qquad \xi \ge \tau. \tag{8.71}$$

Suppose furthermore that $v(\xi) \ge \mathbf{0}$ *for all* $\xi \in \mathbb{R}$*. Then we have* $v(\xi) = \mathbf{0}$ *for all* $\xi \in \mathbb{R}$*.*

Proof. Recall the vector $\sigma_{**} \in \mathbb{R}^n$ defined in Lemma 8.7. Since v vanishes on a half line, it it clear that $v^{\sigma_{**}}$ vanishes on an interval of length $r_{\max}^{\sigma_{**}} - r_{\min}^{\sigma_{**}}$. We can now proceed as in the first part of the proof of Lemma 8.12, noting that in this case $\Sigma_-^{\sigma_{**}} = \emptyset$. ∎

Proof of Propostion 8.8. If $\gamma > 0$, the claim follows from Lemma 8.10. If $\gamma = 0$, then Lemma 8.11 implies that v vanishes on an entire half line. Possibly after substituting $\xi \mapsto -\xi$, Lemma 8.13 can be used to extend this conclusion to the full line. ∎

LEMMA 8.14 (cf. [32, Lem. 3.3]). *Suppose that* (HA), (hb)*, and* (hs) *are satisfied and fix* $\tau \in \mathbb{R}$*. Recall the vector* $\sigma_{**} \in \mathbb{R}^n$ *defined in Lemma* 8.7*. Consider any function*

$$v^{\sigma_{**}} \in L^\infty((-\infty, \tau + r_{\max}^{\sigma_{**}}], \mathbb{R}^n) \tag{8.72}$$

that satisfies the homogeneous system (8.30) *with* $\gamma = c = 0$ *for all* $\xi \le \tau$ *and suppose that*

$$v^{\sigma_{**}}(\xi) = \mathbf{0}, \qquad \tau + r_{\min}^{\sigma_{**}} \le \xi \le \tau + r_{\max}^{\sigma_{**}}. \tag{8.73}$$

Suppose furthermore that $v^{\sigma_{**}}(\xi_*) \neq \mathbf{0}$ *for some* $\xi_* < \tau + r_{\min}^{\sigma_{**}}$*. Then there exist two pairs*

$$(i_-, \xi_-) \in \{1, \ldots, n\} \times (-\infty, \tau + r_{\min}^{\sigma_{**}}], \qquad (i_+, \xi_+) \in \{1, \ldots, n\} \times (-\infty, \tau + r_{\min}^{\sigma_{**}}] \tag{8.74}$$

that have $|\xi_+ - \xi_-| \le |r_{\min}^{\sigma_{**}}|$ *together with*

$$v_{i_-}^{\sigma_{**}}(\xi_-) < 0 < v_{i_+}^{\sigma_{**}}(\xi_+). \tag{8.75}$$

Proof. We can proceed as in the first part of the proof of Lemma 8.12, noting that (hs) again implies that $\Sigma_-^{\sigma_{**}} = \emptyset$. ∎

Proof of Proposition 8.9. Picking any pair $(j, l_0) \in \{1, \ldots, n\}^2$ for which $[A_j]_{i_0 l_0} > 0$, we must have $v_{l_0}(\xi) = 0$ for all $\xi \ge \tau + r_j$. In view of the irreducibility assumption (HA), we can repeat this argument to show that $v(\xi) = \mathbf{0}$ for all $\xi \ge \tau + n r_{\max}$. We can now apply Lemma 8.14 to conclude that in fact $v(\xi) = \mathbf{0}$ for all $\xi \in \mathbb{R}$. ∎

As a final preparation before we turn to the proof of Propositions 8.1 and 8.2, we need to rule out the possibility that solutions to the homogeneous system (8.26) decay at a rate that is faster than any exponential. For $\gamma > 0$ we can proceed exactly as in [25], but for $\gamma = 0$ we need to exploit the special properties of the restated system (8.30).

LEMMA 8.15 (cf. [25, Lem. A.1]). *Consider the homogeneous system* (8.26) *and suppose that* (HA) *and* (hb) *are satisfied. Fix $\gamma > 0$, $c \in \mathbb{R}$, and $\tau \in \mathbb{R}$. There exists a constant $\vartheta > 0$ such that any function*

$$v \in W^{2,\infty}([\tau - r_{\min}, \infty), \mathbb{R}^n) \cap L^\infty([\tau, \infty), \mathbb{R}^n) \tag{8.76}$$

that satisfies (8.26) *for all $\xi \ge \tau - r_{\min}$ and has $v(\xi) \ge \mathbf{0}$ for all $\xi \ge \tau$ must have*

$$\frac{d}{d\xi} |v(\xi)| \ge -\vartheta |v(\xi)|, \qquad \xi \ge \tau - r_{\min}. \tag{8.77}$$

Proof. Upon writing

$$w(\xi) = e^{\frac{c}{2\gamma}\xi} v(\xi), \tag{8.78}$$

a short computation shows that for all $\xi \ge \tau - r_{\min}$ we have

$$-\gamma w''(\xi) + \frac{c^2}{4\gamma} w(\xi) = \sum_{j=0}^{N} A_j e^{-\frac{c}{2\gamma} r_j} w(\xi + r_j) + B(\xi) w(\xi). \tag{8.79}$$

Recalling the constant $\kappa > 0$ appearing in (hb), we can use this to estimate

$$\begin{aligned} w''(\xi) &= \frac{c^2}{4\gamma^2} w(\xi) - \frac{1}{\gamma} \sum_{j=0}^{N} A_j e^{-\frac{c}{2\gamma} r_j} w(\xi + r_j) - \frac{1}{\gamma} B(\xi) w(\xi) \\ &\le \left[\frac{c^2}{4\gamma^2} + \frac{1}{\gamma}\kappa \right] w(\xi). \end{aligned} \tag{8.80}$$

We now write $K = [\frac{c^2}{4\gamma^2} + \frac{1}{\gamma}\kappa]^{1/2}$ and fix an arbitrary $\xi_0 \ge \tau - r_{\min}$. A standard differential inequality shows that for every integer $1 \le i \le n$, we have

$$w_i(\xi) \le C_{1,i} e^{K(\xi - \xi_0)} + C_{2,i} e^{-K(\xi - \xi_0)}, \qquad \xi \ge \xi_0, \tag{8.81}$$

in which

$$C_{1,i} = \frac{1}{2K}[K w_i(\xi_0) + w_i'(\xi_0)], \qquad C_{2,i} = \frac{1}{2K}[K w_i(\xi_0) - w_i'(\xi_0)]. \tag{8.82}$$

Since $w \ge \mathbf{0}$, we must have $C_{1,i} \ge 0$ for all $1 \le i \le n$, which implies $w'(\xi_0) \ge -K w(\xi_0)$. The bound (8.77) follows directly from this. ∎

LEMMA 8.16 (cf. [32, Prop. 4.5]). *Consider the homogeneous system* (8.26) *and suppose that* (HA) *and* (hb) *are satisfied. Fix $\gamma = 0$, $c \ne 0$, and $\tau \in \mathbb{R}$. There exist constants $R > 0$, $\vartheta > 0$, and $\sigma \in \mathbb{R}^n$ such that any function*

$$v \in W^{1,\infty}([\tau - r_{\min}, \infty), \mathbb{R}^n) \cap L^\infty([\tau, \infty), \mathbb{R}^n) \tag{8.83}$$

that satisfies (8.26) *for all $\xi \ge \tau - r_{\min}$ and has $v(\xi) \ge \mathbf{0}$ for all $\xi \ge \tau$ must have*

$$\frac{d}{d\xi} |v^\sigma(\xi)| \ge -\vartheta |v^\sigma(\xi)|, \qquad \xi \ge \tau + R. \tag{8.84}$$

Proof. We restrict ourselves to the case $c > 0$, noting that the case $c < 0$ can be treated similarly. We recall the constant σ_* appearing in Lemma 8.6 and pick $\sigma = \sigma_*$. Choosing $R = -r_{\min} + 2|\sigma|$, an initial estimate shows that for all $\xi \geq \tau + R$ we have

$$D_\xi v^\sigma(\xi) = -c^{-1}[J^\sigma v^\sigma](\xi) - c^{-1}B^\sigma_{\mathrm{diag}}(\xi)v^\sigma(\xi) \leq c^{-1}\kappa v^\sigma(\xi). \tag{8.85}$$

In particular, upon writing $\nu = c^{-1}\kappa$ and $w(\xi) = e^{-\nu\xi}v^\sigma(\xi)$, we have $w'(\xi) \leq \mathbf{0}$ for $\xi \geq \tau + R$.

For any $1 \leq i \leq n$, we write $e_i \in \mathbb{R}^n$ for the standard unit vector $(e_i)_j = \delta_{ij}$. Using these vectors, we construct the matrix

$$\mathcal{A}^\sigma_- = \sum_{(j,k,l)\in\mathcal{I}^\sigma_-} \alpha^-_{jkl} e^{\nu(r_j+\sigma_k-\sigma_l)} e_k e_l^\dagger. \tag{8.86}$$

We now pick $\epsilon > 0$ to be so small that $r_j + \sigma_k - \sigma_l \leq -2\epsilon$ holds for all $(j,k,l) \in \mathcal{I}^\sigma_-$. In addition, we pick any $\xi_1 \geq \tau + R - r_{\min} + \epsilon$. For any $\tau + R - r_{\min} \leq \xi \leq \xi_1$, we have the inequality

$$\begin{aligned} w'(\xi) = -c^{-1} &\sum_{(j,k,l)\in\mathcal{I}} e_k \beta^\sigma_{jkl}(\xi) e^{\nu(r_j+\sigma_k-\sigma_l)} w_l(\xi + r_j + \sigma_k - \sigma_l) \\ &- c^{-1}[B^\sigma_{\mathrm{diag}}(\xi) + \kappa]w(\xi) \\ \leq{}& -c^{-1}\mathcal{A}^\sigma_- w(\xi - 2\epsilon). \end{aligned} \tag{8.87}$$

Integrating (8.87) from $\xi_1 - \epsilon$ to ξ_1, we obtain

$$w(\xi_1) - w(\xi_1 - \epsilon) \leq -\epsilon c^{-1}\mathcal{A}^\sigma_- w(\xi_1 - 2\epsilon). \tag{8.88}$$

Discarding the term $w(\xi_1) \geq \mathbf{0}$, this gives

$$\epsilon c^{-1}\mathcal{A}^\sigma_- w(\xi_1 - 2\epsilon) \leq w(\xi_1 - \epsilon). \tag{8.89}$$

Let us now consider any $v \geq \mathbf{0}$ that has $\mathcal{A}^\sigma_- v = 0$. Since $\mathcal{A}^\sigma_- \geq \mathbf{0}$, it is not hard to see that $(I - \pi^\sigma_-)v = 0$. In particular, upon writing $\mathcal{K} = \mathrm{Ker}(\mathcal{A}^\sigma_-)$, we can pick $\mathcal{K}_{\Sigma^\perp} \subset \mathbb{R}^n$ and $\mathcal{K}_\perp \subset \mathbb{R}^n$ in such a way that we have the decompositions

$$\mathbb{R}^n = \mathcal{K}_\perp \oplus \mathcal{K}, \qquad \mathcal{K} = \mathcal{K}_{\Sigma^\perp} \oplus \mathrm{span}_{i\in\Sigma^\sigma_-}\{e_i\}, \qquad \pi^\sigma_-(\mathcal{K}_{\Sigma^\perp}) = \{0\}. \tag{8.90}$$

There now exists a bounded operator $\mathcal{Q} : \mathrm{Range}(\mathcal{A}^\sigma_-) \to \mathcal{K}_\perp$ such that the identity $\mathcal{A}^\sigma_- v = w$ implies that

$$v = \mathcal{Q}w + q_{\Sigma^\perp} + q_\Sigma \tag{8.91}$$

for some $q_{\Sigma^\perp} \in \mathcal{K}_{\Sigma^\perp}$ and $q_\Sigma \in \mathrm{span}_{i\in\Sigma^\sigma_-}\{e_i\}$. By compactness, there exists $\epsilon_2 > 0$ such that for all $q \in \mathcal{K}_{\Sigma^\perp}$ with $|q| = 1$ we have

$$\min_{1\leq i\leq n} q_i < -\epsilon_2, \qquad \max_{1\leq i\leq n} q_i > \epsilon_2. \tag{8.92}$$

If we require $v \geq \mathbf{0}$ in (8.91), we may hence estimate

$$|q_{\Sigma^\perp}| \leq C_1 |\mathcal{Q}w| \tag{8.93}$$

for some $C_1 > 0$. In addition, since our special choice of σ implies that $\pi_-^\sigma \mathcal{A}_-^\sigma = 0$, there exists a constant $C_2 > 0$ such that the inequality

$$\left|(I - \pi_-^\sigma)v\right| \leq C_2 \left|(I - \pi_-^\sigma)w\right| \tag{8.94}$$

holds whenever $\mathcal{A}_-^\sigma v = w$ for some $v \geq \mathbf{0}$.

The estimate (8.89) now implies

$$\left|(I - \pi_-^\sigma)w(\xi_1 - 2\epsilon)\right| \leq c\epsilon^{-1} C_2 \left|(I - \pi_-^\sigma)w(\xi_1 - \epsilon)\right|. \tag{8.95}$$

In particular, for all $\xi \geq \tau + R - r_{\min}$ and $0 \leq \delta \leq \epsilon$ we have

$$\left|(I - \pi_-^\sigma)w(\xi - \delta)\right| \leq \left|(I - \pi_-^\sigma)w(\xi - \epsilon)\right| \leq c\epsilon^{-1} C_2 \left|(I - \pi_-^\sigma)w(\xi)\right|. \tag{8.96}$$

Repeating this estimate a sufficient number of times, we see that there exists a constant $C_3 > 0$ such that for all $\xi \geq \tau + R - 2r_{\min} + 2\left|\sigma\right|$ we have

$$\left|(I - \pi_-^\sigma)w(\xi + r_{\min} - 2\left|\sigma\right|)\right| \leq C_3 \left|(I - \pi_-^\sigma)w(\xi)\right|. \tag{8.97}$$

Using the fact that $\alpha_{jkl}^+ = 0$ whenever $\alpha_{jkl}^- = 0$, this allows us to compute

$$\begin{aligned} w'(\xi) \geq{}& -c^{-1} \sum_{(j,k,l)\in\mathcal{I}_-^\sigma} e_k \alpha_{jkl}^+ e^{\nu(r_j + \sigma_k - \sigma_l)} [(I - \pi_-^\sigma)w(\xi + r_j + \sigma_k - \sigma_l)]_l \\ & - c^{-1} \sum_{(j,k,l)\in\mathcal{I}_0^\sigma \cup \mathcal{I}_+^\sigma} e_k \alpha_{jkl}^+ e^{\nu(r_j + \sigma_k - \sigma_l)} [w(\xi + r_j + \sigma_k - \sigma_l)]_l \\ & - c^{-1}[B_{\mathrm{diag}}^\sigma(\xi) + \kappa]w(\xi) \\ \geq{}& -C_4 \left|w(\xi)\right| \mathbf{1} \end{aligned} \tag{8.98}$$

for some constant $C_4 > 0$. Since $w \geq \mathbf{0}$, this yields

$$\frac{d}{d\xi}\left|w(\xi)\right|^2 = 2\langle w(\xi), w'(\xi)\rangle \geq -2C_4 \langle w(\xi), \mathbf{1}\rangle \left|w(\xi)\right| \geq -2C_4 \left|\mathbf{1}\right| \left|w(\xi)\right|^2 \tag{8.99}$$

for all $\xi \geq \tau + R - 2r_{\min} + 2\left|\sigma\right|$. Upon increasing the constant R appropriately, this estimate is sufficiently strong to complete the proof. ◻

LEMMA 8.17. *Consider the homogeneous system* (8.26) *and suppose that* (HA), (hb)*, and* (hs) *are satisfied. Fix $\gamma = 0$, $c = 0$, and $\tau \in \mathbb{R}$. There exist constants $K > 0$, $b > 0$, $R > 0$, and $\sigma \in \mathbb{R}^n$ such that any function*

$$v \in L^\infty([\tau, \infty), \mathbb{R}^n) \tag{8.100}$$

that satisfies (8.26) *for all $\xi \geq \tau - r_{\min}$ and has $\mathbf{0} < v(\xi_2) \leq v(\xi_1)$ whenever $\tau \leq \xi_1 \leq \xi_2$ must have*

$$\left|v^\sigma(\xi_1)\right| \leq K e^{b(\xi_2 - \xi_1)} \left|v^\sigma(\xi_2)\right| \tag{8.101}$$

for all $\tau + R \leq \xi_1 \leq \xi_2$.

Proof. We recall the constant σ_{**} appearing in Lemma 8.6 and pick $\sigma = \sigma_{**}$. As above, we pick $\epsilon > 0$ to be so small that $r_j + \sigma_k - \sigma_l \leq -2\epsilon$ holds for all $(j,k,l) \in \mathcal{I}_-^\sigma$. Upon writing

$$\mathcal{A}_-^\sigma = \sum_{(j,k,l)\in\mathcal{I}_-^\sigma} \alpha_{jkl}^- e_k e_l^\dagger, \tag{8.102}$$

we can pick $R = -r_{\min} + 2|\sigma|$ and obtain the estimate

$$\begin{aligned} \mathcal{A}_-^\sigma v^\sigma(\xi - 2\epsilon) &\leq \sum_{(j,k,l)\in\mathcal{I}_-^\sigma} e_k \beta_{jkl}^\sigma(\xi) v_l^\sigma(\xi + r_j + \sigma_k - \sigma_l) \\ &= -\sum_{(j,k,l)\in\mathcal{I}_0^\sigma\cup\mathcal{I}_+^\sigma} e_k \beta_{jkl}^\sigma(\xi) v_l^\sigma(\xi + r_j + \sigma_k - \sigma_l) \\ &\quad - B_{\mathrm{diag}}^\sigma(\xi) v^\sigma(\xi) \\ &\leq \kappa v^\sigma(\xi) \end{aligned} \tag{8.103}$$

for all $\xi \geq \tau + R$. Arguing as in the proof of Lemma 8.16 and remembering that $\Sigma_-^\sigma = \emptyset$, we see that there exists $C_2 > 0$ such that

$$|v^\sigma(\xi - 2\epsilon)| \leq C_2 |v^\sigma(\xi)| \tag{8.104}$$

holds for all $\xi \geq \tau + R$. Repeating this estimate and exploiting the fact that v is nonincreasing yields the desired bound (8.101). ☐

Proof of Proposition 8.1. It suffices to show that the existence of u_- implies that $\Delta_{c,\gamma,q_*}(\lambda_-)v_- = 0$ for some $\lambda_- < 0$ and nonzero $v_- \in \mathbb{R}^n_{\geq \mathbf{0}}$, while the existence of u_+ implies that $\Delta_{c,\gamma,q_*}(\lambda_+)v_+ = 0$ for some $\lambda_+ > 0$ and nonzero $v_+ \in \mathbb{R}^n_{\geq \mathbf{0}}$. Indeed, Lemma 8.4 precludes these two consequences from occurring simultaneously.

Assuming the existence of u_-, we write $y(\xi) = q_* - u_-(\xi)$ and observe that $y(\xi) \geq \mathbf{0}$ for all $\xi \in \mathbb{R}$ because u_- is nondecreasing. Either Proposition 8.8 or 8.9 imply that in fact $y(\xi) > \mathbf{0}$ holds for all $\xi \in \mathbb{R}$. Pick a sequence $\xi_n \to \infty$ and define the functions $z_n(\xi) = y(\xi + \xi_n)/|y(\xi_n)|$, which all satisfy $|z_n(0)| = 1$. After passing to a subsequence, we have the pointwise convergence $z_n \to z$ for some nonincreasing function $z \in L^\infty(\mathbb{R}, \mathbb{R}^n)$. We claim that z satisfies the autonomous system (8.9) with $q = q_*$ and does not decay faster than exponentially as $\xi \to \infty$.

To see this, we will assume without loss of generality that $\sigma_* = \sigma_{**} = \mathbf{0}$ holds for the constants appearing in Lemmas 8.16 and 8.17. If $\gamma + |c| > 0$, we can use either Lemma 8.15 or Lemma 8.16 to conclude that

$$0 \geq \frac{d}{d\xi} |z_n(\xi)| \geq -\vartheta |z_n(\xi)| \tag{8.105}$$

for all $\xi \in \mathbb{R}$. In particular, since $|z_n(0)| = 1$, the sequences z_n and $\gamma z_n'$ are uniformly bounded and equicontinuous on each compact interval, which implies that the convergence $z_n \to z$ is in fact uniform on such intervals. To see that z satisfies (8.9), it now suffices to look at an integrated version of (8.9), as in [29, proof of Thm. 3.1]. In addition, the estimate (8.105) carries over to z, which together with $|z(0)| = 1$ shows that z does not decay faster than exponentially as $\xi \to \infty$. On the other hand, if $\gamma = c = 0$, then the fact that z solves (8.9) is immediate and we can use Lemma 8.17 to rule out the faster than exponential decay of z.

Applying either [31, Prop. 7.2] or an argument similar to the proof of [32, Lem. 5.3], we now obtain the asymptotic expansion

$$z(\xi) = \sum_{i=1}^{\ell} K_i(\xi) p_i(\xi) e^{-b\xi} e^{i\nu_i \xi} + O(e^{-(b+\epsilon)\xi}), \qquad \xi \to \infty, \tag{8.106}$$

for some $b \geq 0$ and $\ell \geq 1$, in which each K_i is a scalar function that never vanishes. Furthermore, K_i is periodic if $\gamma = c = 0$ and the shifts $\{r_j\}$ are rationally related but constant otherwise. In addition, each p_i is a $\mathbb{C}^n$-valued nonzero polynomial for which $\xi \mapsto p_i(\xi)e^{-b\xi}e^{i\nu_i\xi}$ is an eigensolution to (8.9). Since $z(\xi) \geq \mathbf{0}$, we must have $\nu_i = 0$ for each $1 \leq i \leq \ell$, together with

$$v_i := \lim_{\xi \to \infty} \xi^{-\deg(p_i)} p_i(\xi) \in \mathbb{R}^n_{\geq \mathbf{0}}. \tag{8.107}$$

In particular, we must have $\Delta_{c,\gamma,q_*}(-b)v_i = 0$, as desired. ◻

Proof of Proposition 8.2. We first use the spectral flow theorem [31, Thm. C] to show that $\mathrm{ind}(\Lambda_{c,\gamma}) = 0$. In particular, let us write

$$M(\vartheta) = (1-\vartheta)Df(\mathbf{0}) + \vartheta Df(\mathbf{1}) + \nu(\vartheta)I, \tag{8.108}$$

where the scalar function ν satisfies $\nu(0) = \nu(1) = 0$ and is further determined below. In addition, we write

$$\Delta_\vartheta(z) = -\gamma z^2 - cz - \sum_{j=0}^{N} A_j(e^{zr_j} - 1) - M(\vartheta). \tag{8.109}$$

Since off-diagonal elements of $M(\vartheta)$ are nonnegative, we can introduce the functions $\lambda_l(\vartheta)$ and $\lambda_r(\vartheta)$ that track the roots λ^- and λ^+ featured in Lemma 8.5, where we use the function ν to ensure that these two roots never collide. Since the characteristic equation

$$\det \Delta_\vartheta(z) = 0 \tag{8.110}$$

has no solutions with $\lambda_l(\vartheta) < \mathrm{Re}\, z < \lambda_r(\vartheta)$, we can use the inequalities

$$\lambda_l(0) < 0 < \lambda_r(0), \qquad \lambda_l(1) < 0 < \lambda_r(1) \tag{8.111}$$

to conclude that every root of (8.110) that crosses the imaginary axis as ϑ is increased from zero to one must also cross back. In particular, the crossing number for this transition is zero, as desired.

Proposition 8.8 immediately implies that $p > \mathbf{0}$. Either Lemma 8.15 or Lemma 8.16 imply that $p(\xi)$ does not decay faster than exponentially as $\xi \to \pm\infty$. In particular, we can use Lemma 8.3 and [31, Prop. 7.2] together with the inequality $p \geq \mathbf{0}$ to obtain the asymptotic expressions

$$p(\xi) = \begin{cases} C^p_- v_- e^{-\lambda_-|\xi|} + O(e^{-(\lambda_- + \epsilon)|\xi|}), & \xi \to -\infty, \\ C^p_+ v_+ e^{-\lambda_+|\xi|} + O(e^{-(\lambda_+ + \epsilon)|\xi|}), & \xi \to \infty \end{cases} \tag{8.112}$$

for some $\epsilon > 0$ with

$$\lambda_- > 0, \qquad \lambda_+ > 0, \qquad v_- > \mathbf{0}, \qquad v_+ > \mathbf{0} \tag{8.113}$$

and positive constants $C^p_\pm > 0$.

Suppose that there exists some $x \in \mathrm{Ker}(\Lambda_{c,\gamma})$ that is linearly independent of p. By adding some multiple of p and replacing x by $-x$ if necessary, we may assume that x satisfies a similar asymptotic expansion (8.112) with the same quantities (8.113) but with $C^x_- \leq 0$ and $C^x_+ = 0$. We claim that there exists an integer $1 \leq i_0 \leq n$ and $\xi_0 \in \mathbb{R}$ for which $x_{i_0}(\xi_0) > 0$. Indeed, assuming to the contrary that $x(\xi) \leq \mathbf{0}$ for all $\xi \in \mathbb{R}$, we may argue as above to conclude that $x(\xi) < \mathbf{0}$ for all $\xi \in \mathbb{R}$ and hence also

$C_+^x < 0$, in contrast to our assumption. By choosing a sufficiently large $\mu_0 \gg 1$, we hence see that

$$p_{i_0}(\xi_0) - \mu_0 x_{i_0}(\xi_0) < 0. \tag{8.114}$$

We now consider the family $p - \mu x \in \mathrm{Ker}(\Lambda_{c,\gamma})$ for $0 \le \mu \le \mu_0$. The asymptotic expressions for p and x ensure that there exist $\tau,\ K,\ \lambda \in \mathbb{R}$ such that

$$p(\xi) - \mu x(\xi) \ge K e^{-\lambda|\xi|}\mathbf{1} > \mathbf{0}, \qquad |\xi| > \tau, \qquad 0 \le \mu \le \mu_0. \tag{8.115}$$

This allows us to define the quantity

$$\mu_* = \sup\left\{\mu \in [0, \mu_0] \mid p(\xi) - \mu x(\xi) \ge \mathbf{0} \text{ for all } \xi \in \mathbb{R}\right\}. \tag{8.116}$$

In view of the asymptotics (8.115), we must have $p_{i_*}(\xi_*) - \mu_* x_{i_*}(\xi_*) = 0$ for some integer $1 \le i_* \le n$ and $\xi_* \in \mathbb{R}$. As above, this however immediately implies that $p(\xi) = \mu_* x(\xi)$ for all $\in \mathbb{R}$, which establishes $\dim \mathrm{Ker}(\Lambda_{c,\gamma}) = 1$.

It now suffices to show that there exists a nontrivial $p_* \in \mathrm{Ker}(\Lambda_{c,\gamma}^*)$ that satisfies $p_* \ge 0$, since the strict inequality $p_* > \mathbf{0}$ can then be obtained by repeating the arguments used above for p. Assuming to the contrary that $(p_*)_{i_+}(\xi_+) > 0 > (p_*)_{i_-}(\xi_-)$ for two pairs $1 \le i_\pm \le n$ and $\xi_\pm \in \mathbb{R}$, we remark that Lemma 8.12 implies that we can pick a compactly supported continuous function $h : \mathbb{R} \to \mathbb{R}^n_{\ge \mathbf{0}}$ for which

$$\int_{-\infty}^{\infty} \langle p_*(\xi), h(\xi)\rangle \, d\xi = 0. \tag{8.117}$$

In particular, we have $h = \Lambda_{c,\gamma} x$ for some bounded function $x : \mathbb{R} \to \mathbb{R}^n$.

Since x satisfies the homogeneous system (8.26) for all sufficiently large $|\xi|$, we see that x enjoys the asymptotic expressions (8.112). In particular, the quantity

$$\mu_* = \inf\left\{\mu \in \mathbb{R} \mid x(\xi) + \mu p(\xi) \ge 0 \text{ for all } \xi \in \mathbb{R}\right\} \tag{8.118}$$

is finite and we may write $y = x + \mu_* p$. Obviously, we have $y(\xi) \ge \mathbf{0}$ for all $\xi \in \mathbb{R}$, but y may not vanish identically since $\Lambda_{c,\gamma} y = h$. Proposition 8.8 now implies that in fact $y(\xi) > \mathbf{0}$ for all $\xi \in \mathbb{R}$. In particular, y also enjoys the asymptotic expression (8.112) with constants $C_\pm^y > 0$. This however means that for all sufficiently small $\epsilon > 0$ we have $y - \epsilon p \ge \mathbf{0}$, which is a direct violation of the definition of μ_*. ∎

9. Proof of main results. In this final section we prove the main results formulated in section 2 for the family of nonlocal systems

$$\partial_t u(x,t) = [\mathcal{D}u](x,t) + f\big(u(x,t);\rho\big). \tag{9.1}$$

In order to accomplish this, we study how solutions to the traveling wave MFDE

$$-\gamma u''(\xi) - c u'(\xi) = \sum_{j=0}^{N} A_j[u(\xi + r_j) - u(\xi)] + f\big(u(\xi);\rho\big) \tag{9.2}$$

behave as the parameter ρ is varied, paying special attention to the singular limit $\gamma \to 0$. In particular, we establish the following key result, which is stronger than Theorem 2.2 and instrumental in the proof of the remaining theorems.

PROPOSITION 9.1. *Suppose that* (HA) *and* (Hf1)–(Hf3) *are satisfied and consider a sequence*

$$(\gamma_n, c_n, \rho_n, P_n)_{n\in\mathbb{N}} \in [0,\infty)\times\mathbb{R}\times V\times W^{2,\infty}(\mathbb{R},\mathbb{R}^n) \tag{9.3}$$

for which $\gamma_n + |c_n| > 0$ *for all* $n \in \mathbb{N}$ *and for which we have the limits* $\gamma_n \to \gamma_* \geq 0$ *and* $\rho_n \to \rho_* \in V$ *as* $n \to \infty$. *Suppose furthermore that for every* $n \in \mathbb{N}$, *the function* P_n *has* $P_n'(\xi) > 0$ *for all* $\xi \in \mathbb{R}$, *solves the traveling wave MFDE* (9.2) *with* $c = c_n$, $\gamma = \gamma_n$, *and* $\rho = \rho_n$, *and satisfies the limits*

$$\lim_{\xi\to-\infty} P_n(\xi) = \mathbf{0}, \qquad \lim_{\xi\to\infty} P_n(\xi) = \mathbf{1}. \tag{9.4}$$

Then, possibly after passing to a subsequence, we have $c_n \to c_* \in \mathbb{R}$ *and the limit*

$$P_*(\xi) := \lim_{n\to\infty} P_n(\xi) \tag{9.5}$$

exists pointwise. The function P_* *is nondecreasing and satisfies the limits*

$$\lim_{\xi\to-\infty} P_*(\xi) = \mathbf{0}, \qquad \lim_{\xi\to+\infty} P_*(\xi) = \mathbf{1}. \tag{9.6}$$

In addition, for almost all $\xi \in \mathbb{R}$ *the function* P_* *satisfies the MFDE* (9.2) *with* $\gamma = \gamma_*$, $c = c_*$, *and* $\rho = \rho_*$.

Our proof of the above result is largely based on ideas developed in [32, Thm. 2.3] and [25, Thm. 3.10]. However, we borrow a technique from [9] in order to establish that the wave speeds $\{c_n\}$ are bounded.

LEMMA 9.2 (cf. [9, Thm. 3.5]). *Consider the setting of Proposition* 9.1. *We have the uniform bound*

$$\sup_{n\in\mathbb{N}} |c_n| < \infty. \tag{9.7}$$

Proof. Pick any $n \in \mathbb{N}$ and write $f_n = f(\cdot;\rho_n)$, together with

$$[\mathcal{D}_n u](x,t) = \gamma_n \partial_{xx} u(x,t) + [J * u](x,t). \tag{9.8}$$

In addition, write $v_n^l > \mathbf{0}$ and $v_n^r > \mathbf{0}$ for the eigenvectors described in (4.27) for Df_n.

Pick $\delta > 0$, $\epsilon > 0$, and $C \gg 1$ and consider the function

$$w_n^-(x,t) = -\delta v_n^l H_-\big(\epsilon(x - Ct)\big) + (\mathbf{1} - \delta v_n^r) H_+\big(\epsilon(x - Ct)\big). \tag{9.9}$$

Upon writing

$$\mathcal{J}_n^-(x,t) = \partial_t w_n^-(x,t) - [\mathcal{D}_n w_n^-](x,t) - f_n\big(w_n^-(x,t)\big) \tag{9.10}$$

and introducing the shorthand $y = \epsilon(x - Ct)$, we may compute

$$\begin{aligned}
\mathcal{J}_n^-(x,t) &= C\epsilon\delta v_n^l H_-'(y) - C\epsilon(\mathbf{1} - \delta v_n^r) H_+'(y) \\
&\quad + \delta[\mathcal{D}_n v_n^l H_-](y) - [\mathcal{D}_n(\mathbf{1} - \delta v_n^r) H_+](y) \\
&\quad - f_n\Big(-\delta v_n^l H_-(y) + (\mathbf{1} - \delta v_n^r) H_+(y)\Big) \\
&= -C\epsilon(\mathbf{1} - \delta v_n^r + \delta v_n^l) H_+'(y) \\
&\quad + \delta[\mathcal{D}_n v_n^l H_-](y) - [\mathcal{D}_n(\mathbf{1} - \delta v_n^r) H_+](y) \\
&\quad - f_n\Big(-\delta v_n^l H_-(y) + (\mathbf{1} - \delta v_n^r) H_+(y)\Big),
\end{aligned} \tag{9.11}$$

in which we have used $H'_-(y) = -H'_+(y)$. We now pick $\delta > 0$ to be sufficiently small to ensure that there exist constants $\kappa > 0$ and $\vartheta > 0$ such that for all $n \in \mathbb{N}$ the inequality

$$f_n\Big(-\delta v_n^l H_-(y) + (\mathbf{1} - \delta v_n^r) H_+(y)\Big) > \vartheta \mathbf{1} \tag{9.12}$$

holds whenever $|H_-(y)| \le \kappa$ or $|H_+(y)| \le \kappa$. This is possible because of (Hf1) and the convergence $\rho_n \to \rho_* \in V$. In addition, we pick $\epsilon > 0$ to be so small that

$$\big|[\mathcal{D}_n \delta v_n^l H_-](y) - [\mathcal{D}_n(\mathbf{1} - \delta v_n^r)H_+](y)\big| < \frac{\vartheta}{2} \tag{9.13}$$

holds for all $y \in \mathbb{R}$ and $n \in \mathbb{N}$. This is possible because we have a uniform bound on γ_n. Finally, we pick $C \gg 1$ to be so large that for all $n \in \mathbb{N}$ we have

$$C\epsilon(\mathbf{1} - \delta v_n^r + \delta v_n^l)H'_+(y) \quad > \quad \tfrac{\vartheta}{2}\mathbf{1} + \Big| f_n\Big(-\delta v_n^l H_-(y) + (\mathbf{1} - \delta v_n^r)H_+(y)\Big)\Big|\mathbf{1} \tag{9.14}$$

whenever $\kappa < H_+(y) < 1 - \kappa$. This is possible because $H'_+(y)$ is bounded away from zero on this region. Note that these choices ensure that $\mathcal{J}_n^-(x,t) \le 0$ for all $x \in \mathbb{R}$ and $t \ge 0$.

For each $n \in \mathbb{N}$ there exists a constant $\theta_n \gg 1$ such that

$$P_n(x + \theta_n) \ge w_n^-(x, 0) \tag{9.15}$$

holds for all $x \in \mathbb{R}$, while also

$$P_n(x_* - \theta_n) < w_n^-(x_*, 0) \tag{9.16}$$

for some $x_* \in \mathbb{R}$. The comparison principle stated in Proposition 4.1 now implies that

$$P_n(x + \theta_n - c_n t) \ge w_n^-(x,t) = w_n^-(x - Ct, 0). \tag{9.17}$$

We claim that this implies that $c_n \le C$. Indeed, if this is not the case, a contradiction can be obtained by choosing $t = 2\theta_n(c_n - C)^{-1}$ and $x = x_* + Ct$. Since the constant $C \gg 1$ does not depend on n, we have obtained a uniform upper bound for the wave speed. A similar argument can be used to obtain a uniform lower bound. ∎

Proof of Proposition 9.1. The existence of the limiting function P_* follows from the fact that $P_n' > 0$, while the existence of $c_* \in \mathbb{R}$ follows from Lemma 9.2. Arguing as in the proof of [25, Thm. 3.10], we can conclude that P_* satisfies the MFDE (9.2) with $\gamma = \gamma_*$, $c = c_*$, and $\rho = \rho_*$ for almost all $\xi \in \mathbb{R}$. In addition, if either $\gamma_* > 0$ or $c_* \neq 0$, then (9.2) is in fact satisfied for all $\xi \in \mathbb{R}$. Finally, both limits

$$v_- = \lim_{\xi \to -\infty} P_*(\xi) \ge \mathbf{0}, \qquad v_+ = \lim_{\xi \to +\infty} P_*(\xi) \le \mathbf{1} \tag{9.18}$$

exist and satisfy $f(v_\pm; \rho_*) = 0$.

The key issue is to show that

$$v_- = \mathbf{0}, \qquad v_+ = \mathbf{1}. \tag{9.19}$$

To see this, let us pick $\delta > 0$ in such a way that $f(v; \rho_*) = 0$ has no solutions $v \in [0,1]^n \setminus \{\mathbf{0}, \mathbf{1}\}$ that have either $|v| \le \delta$ or $|v - \mathbf{1}| \le \delta$. We then consider the two sequences $\{\zeta_n^-\}, \{\zeta_n^+\} \subset \mathbb{R}$ that are uniquely determined by the identities

$$\big|P_n(\zeta_n^-)\big| = \delta, \qquad \big|P_n(\zeta_n^+) - \mathbf{1}\big| = \delta. \tag{9.20}$$

By shifting the functions P_n appropriately, we may assume that $\zeta_n^- < 0 < \zeta_n^+$ holds for all $n \in \mathbb{N}$. Note that it suffices to show that $\zeta_n^+ - \zeta_n^-$ is bounded. Indeed, this means that the sequences $\zeta_n^\pm$ are both bounded separately, which in view of our choice of $\delta > 0$ directly implies the limits (9.19).

Arguing by contradiction, let us assume that $\zeta_n^+ - \zeta_n^- \to \infty$ and define the functions

$$x_n^-(\xi) = P_n(\xi + \zeta_n^-), \qquad x_n^+(\xi) = P_n(\xi + \zeta_n^+). \tag{9.21}$$

Arguing as above, we may pass to a subsequence for which we have the pointwise limits $x_n^- \to x_*^-$ and $x_{n+} \to x_*^+$, where both $x_*^\pm$ solve the MFDE (9.2). In addition, using the fact that there do not exist $\mathbf{0} < q_1 < q_2 < \mathbf{1}$ for which $f(q_1) = f(q_2) = 0$, we have the identical limits

$$\lim_{\xi \to +\infty} x_*^- = q = \lim_{\xi \to -\infty} x_*^+ \tag{9.22}$$

for some $\mathbf{0} < q < \mathbf{1}$ that has $f(q) = 0$. Proposition 8.1 now gives the desired contradiction. ☐

Proof of Theorem 2.1. For each $\rho \in V$, the existence of $P_\gamma(\rho)$ and $c_\gamma(\rho)$ can be obtained by approximating the nonlinearity f with a sequence of nonlinearities f_n that satisfy the assumption (H$\mathcal{W}$) and using Proposition 9.1 to show that the traveling waves obtained in Proposition 7.1 converge to a traveling wave for (9.1) with the desired nonlinearity f.

In view of the preparatory results obtained in section 8, the uniqueness of this pair $P_\gamma(\rho)$, $c_\gamma(\rho)$ can be obtained by following the proof of [32, Prop. 6.5]. In addition, the smooth dependence of P_γ and c_γ on the parameter ρ can be obtained by following the proof of [25, Prop 3.2] and invoking Proposition 9.1. ☐

Proof of Theorem 2.2. The statements follow directly from Proposition 9.1. ☐

Proof of Theorem 2.3. For each $\rho \in V$, the existence of the wave speed c_0 and the profile P described in (ii) and (iii) follows upon using Proposition 9.1 to write $c_0 = \lim_{\gamma \to 0} c_\gamma(\rho)$ and $P(\xi) = \lim_{\gamma \to 0} P_\gamma(\rho)(\xi)$, where the limits are taken after passing to an appropriate subsequence. In view of the preparatory results obtained in section 8, the smoothness properties in (i) and (ii) can be obtained by following the proof of [32, Prop. 6.4], while the uniqueness claims in (iv) and (v) can be established as in the proof of [32, Prop. 6.5]. ☐

10. Discussion. In this paper we constructed traveling wave solutions to a class of high dimensional LDEs that includes bistable reaction diffusion problems with spatially periodic diffusion coefficients. This was achieved by adding a small continuous diffusion term to the system and using parameter continuation techniques together with the comparison principle.

A key ingredient in our approach was the analysis of the operator $\Lambda_{c,\gamma}$ given in (1.13), which arises when considering the linearization of (1.1) around a traveling wave. We established a Krein–Rutman-type result for this Fredholm operator, allowing the use of the implicit function theorem to construct solutions to parameter-dependent families of reaction diffusion systems.

Let us emphasize here that we expect the results for $\Lambda_{c,\gamma}$ to be useful in further applications. Indeed, when considering LDEs posed on one-dimensional lattices, the Fredholm properties of similar operators have been used to study the stability of waves [24], glue waves together [26], and analyze singular perturbations [23]. In addition, a recent result [20] provides a set of spectral conditions on $\Lambda_{c,0}$ that are sufficient to guarantee the nonlinear stability of the waves constructed in this paper for LDEs posed on high dimensional lattices. Our results here are sufficiently strong to verify an important subset of these conditions. Let us also mention the recent work [40], where our results on $\Lambda_{c,0}$ are used to construct traveling waves for reaction diffusion systems that do not admit a comparison principle, even after variable transformations of the type used in section 3.2.

Models involving infinite range nonlocal interactions are attracting increasing attention, motivated for example by the desire to discretize fractional Laplacians. At present however, our techniques require us to limit our attention chiefly to finite range interactions. We do wish to point out that small infinite range tails can be incorporated into our framework from a bifurcational point of view. The chief obstacle toward removing this restriction is that the Fredholm results developed in [31] and [29] are stated only for finite-range operators. We believe that a careful study of the proofs should allow most of the results to be extended to infinite range interactions. Technical complications will however undoubtedly arise, similar to those encountered when studying delay differential equations with infinite delays.

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