Bounding the fractional chromatic number of K_{Δ} -free graphs

Katherine Edwards*

Department of Computer Science Princeton University, Princeton, NJ

Andrew D. King[†]

Departments of Mathematics and Computing Science Simon Fraser University, Burnaby, BC

October 29, 2018

Abstract

King, Lu, and Peng recently proved that for $\Delta \geq 4$, any K_{Δ} -free graph with maximum degree Δ has fractional chromatic number at most $\Delta - \frac{2}{67}$ unless it is isomorphic to $C_5 \boxtimes K_2$ or C_8^2 . Using a different approach we give improved bounds for $\Delta \geq 6$ and pose several related conjectures. Our proof relies on a weighted local generalization of the fractional relaxation of Reed's ω , Δ , χ conjecture.

1 Introduction

In this paper we consider simple, undirected graphs, and refer the reader to [21] for unspecified terminology and notation. We also work completely within the rational numbers.

The idea of bounding the chromatic number χ based on the clique number ω and maximum degree Δ goes all the way back to Brooks' Theorem, which states that for $\Delta \geq 3$, any $K_{\Delta+1}$ -free graph with maximum degree Δ has chromatic number at most Δ . More recently, Borodin and Kostochka conjectured that if $\Delta \geq 9$, then any K_{Δ} -free graph with maximum degree Δ has chromatic number at most $\Delta - 1$ [4]. The example of $C_5 \boxtimes K_3$ (see Figure 2) tells us that we cannot improve the condition that $\Delta \geq 9$. Reed [19] proved a weaker result that had been conjectured independently by Beutelspacher and Hering [3]:

Theorem 1. For graph with $\Delta \geq 10^{14}$, if $\omega \leq \Delta - 1$ then $\chi \leq \Delta - 1$.

In the paper, Reed claims that a more careful analysis could replace 10^{14} with 10^3 .

^{*}Email: ke@princeton.edu. Supported by an NSERC PGS Fellowship and a Gordon Wu Fellowship.

[†]Email: adk7@sfu.ca. Supported by an EBCO/Ebbich Postdoctoral Scholarship and the NSERC Discovery Grants of Pavol Hell and Bojan Mohar.

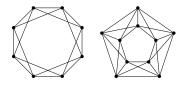


Figure 1: C_8^2 (left) and $C_5 \boxtimes K_2$ (right).

This is the state of the art on the chromatic number of K_{Δ} -free graphs, but what about the *fractional* chromatic number χ_f (we will define it soon) of K_{Δ} -free graphs? Albertson, Bollobás, and Tucker noted in the 1970s that even when $\Delta \geq 3$, there are at least two K_{Δ} -free graphs with $\chi_f = \Delta$, namely C_8^2 and $C_5 \boxtimes K_2$ [2] (see Figure 1). It turns out that these are the only such graphs. For $\Delta \geq 3$ we define $f(\Delta)$ as:

$$f(\Delta) = \min_{G} \left\{ \Delta - \chi_f(G) \mid \Delta(G) \leq \Delta; \ \omega(G) < \Delta; \ G \notin \{C_8^2, C_5 \boxtimes K_2\} \right\}.$$

From Brooks' Theorem we know that $f(\Delta)$ is always nonnegative. Considering Theorem 1, one may be inclined to believe that $f(\Delta)$ increases with Δ . As proven by King, Lu, and Peng, this is indeed the case for $\Delta \geq 4$ [14]¹. In Table 1 we show the known and conjectured bounds for various values of Δ . Figure 2 shows graphs demonstrating the best known (and conjectured) upper bounds on $f(\Delta)$ for $3 \leq \Delta \leq 8$.

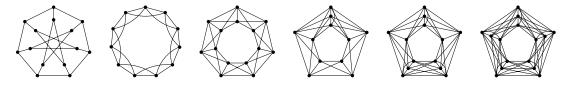


Figure 2: From left to right, the graphs P(7,2), C_{11}^2 , $C_7 \boxtimes K_2$, $(C_5 \boxtimes K_3) - 4v$, $(C_5 \boxtimes K_3) - 2v$, and $C_5 \boxtimes K_3$.

In this paper we give improved bounds on $f(\Delta)$ for $\Delta \geq 6$ up until whenever Theorem 1 takes effect, which we assume to be $\Delta = 1000$. We also conjecture that the upper bound of $f(\Delta) \leq \frac{1}{2}$ is tight for $\Delta \in \{6,7,8\}$:

Conjecture 1. For $\Delta \in \{6,7,8\}$, let G be a graph with maximum degree Δ and clique number at most $\Delta - 1$. Then the fractional chromatic number of G is at most $\Delta - \frac{1}{2}$.

One of the major questions in this area, as is evident from Table 1, is the following: Conjecture 2. For $\Delta \geq 3$, $f(\Delta) \leq f(\Delta + 1)$.

2 Fractionally colouring weighted and unweighted graphs

In this paper we must consider fractional colourings of both vertex-weighted and unweighted graphs, because we will begin to fractionally colour an unweighted graph G in one way that

¹For $\Delta \geq 6$, this is a consequence of the fact that when $\omega > \frac{2}{3}(\Delta + 1)$, there is a stable set hitting every maximum clique [13]. For $\Delta \in \{4,5\}$ more work is required.

	$f(\Delta)$			$f(\Delta)$			conjectured	
Δ	lower bounds			upper bound			value	
3	3/64	0.0468	[9]					
3	3/43	0.0697	[16]					
3	1/11	0.0909	[8]					
3	2/15	0.1333	[15]	1/5	P(7, 2)	[7]	1/5	[10]
4	2/67	0.0298	[14]	1/3	C_{11}^{2}		1/3	[14]
5	2/67	0.0298	[14]	1/3	$C_7 \boxtimes K_2$		1/3	[14]
6	1/22.5	0.0445		1/2	$(C_5 \boxtimes K_3) - 4v$		1/2	
7	1/11.2	0.0899		1/2	$(C_5 \boxtimes K_3) - 2v$		1/2	
8	1/8.9	0.1135		1/2	$C_5 \boxtimes K_3$	[5]	1/2	
9	1/7.7	0.1307		1	K_8		1	[4]
10	1/7.1	0.1423		1	K_9		1	[4]
1000	1	1	[19]	1	K_{999}		1	[3]

Table 1: The state of the art. New results and conjectures are in boldface. For $\Delta \leq 5$, the fractional bound is the proven bound. For $\Delta \geq 6$, the decimal bound approximates the proven bound, and the fractional expression approximates the decimal bound for ease of comparison.

does very well on particularly tricky vertices, then finish the colouring in another way that does fairly well on all vertices. The second step requires a weighted generalization of a known result; the weight on a vertex reflects how much colour we have yet to assign to the vertex.

Let G = (V, E) be a graph, let S = S(G) be the set of stable sets of G, and let k be a nonnegative rational. Now let $\kappa : S \to \mathcal{P}([0, k))$ be a function assigning each stable set S of G a subset of [0, k) such that for every $S \in S$, $\kappa(S)$ is the union of disjoint half-open intervals² with rational endpoints between 0 and k, and for any distinct S, S' in S, $\kappa(S) \cap \kappa(S') = \emptyset$. For a set $S' \subseteq S$ of stable sets, define $\kappa(S')$ as $\bigcup_{S \in S'} \kappa(S)$. For each $v \in V$, define $\kappa[v]$ as $\bigcup_{S \ni v} \kappa(S)$. For a set $X \subseteq V$, define $\kappa[X]$ as $\bigcup_{S \cap X \neq \emptyset} \kappa(S) = \bigcup_{v \in X} \kappa[v]$.

Now consider a nonnegative vertex weight function $w:V\to [0,\infty)$; in this case we say that G is a w-weighted graph. (Recall that w, like all numbers considered in this paper, is rational.) If for every vertex $v\in V$ we have $|\kappa[v]|\geq w(v)$, then κ is a fractional ℓw -colouring of G with weight k; in other words it is a fractional $k \ell w$ -colouring of G. The minimum weight of a fractional ℓw -colouring G is denoted $\chi_f^w(G)$, or simply χ_f^w when the context is clear. If w=1 (i.e. the weight function uniformly equal to 1), then we may omit it from the notation, i.e. we define fractional colourings and the fractional chromatic number of unweighted graphs. If some vertex v has $|\kappa[v]| < w(v)$, we say that we have a partial fractional $k \ell w$ -colouring of G.

In both settings, $\kappa[v]$ is the *colour set* assigned to v. We denote the colours *available to* v (i.e. not appearing on the neighbourhood of v) by $\alpha(v)$, that is, $\alpha(v) = [0, k) \setminus \kappa[N(v)]$.

This is just one of several ways to think about fractional colourings; we hold the following

²(containing their lower endpoint but not their upper)

proposition to be self-evident³:

Proposition 2. Let G be a w-weighted graph. The following are equivalent:

- (1) G has a fractional $k \wr w$ -colouring.
- (2) There is an integer c and a multiset of ck stable sets of G such that every vertex v is contained in at least $c \cdot w(v)$ of them.
- (3) There is a probability distribution on S such that for each $v \in V$, given a stable set S drawn from the distribution, $\Pr(v \in S) \ge w(v)/k$.

For more background on fractional colourings we refer the reader to [20]. At this point it is convenient to prove a useful consequence of Hall's Theorem that we will use repeatedly in Section 7:

Lemma 3. Let κ be a partial fractional $k \wr w$ -colouring of G, and let X be the set of vertices v with $|\kappa[v]| < w(v)$. Suppose for every $X' \subseteq X$ we have

$$\left| \bigcup_{v \in X'} \alpha(v) \right| \ge \sum_{v \in X'} w(v). \tag{1}$$

Then there is a fractional $k \wr w$ -colouring of G.

Proof. We may assume (by uncolouring X) that for every $v \in X$, $\kappa[v] = \emptyset$. Thus we have a fractional $k \wr w$ -colouring of G - X. By Proposition 2 there is an integer c and a multiset of ck stable sets S_1, \ldots, S_{ck} of G - X such that every vertex $v \notin X$ is in at least $c \cdot w(v)$ of them.

We now set up Hall's Theorem by constructing a bipartite graph H with vertex set $A \cup B$. Let A consist of, for every $v \in X$, $c \cdot w(v)$ copies of v. Let B consist of vertices $b_1, \ldots b_{ck}$. For every vertex a of A, let a be adjacent to b_i if and only if the vertex v in X corresponding to a has no neighbour in S_i . Equation (1) guarantees that for every $A' \subseteq A$, $|N(A')| \geq |A'|$, so by Hall's Theorem we have a matching in H saturating A. This matching corresponds to a partial mapping $m : [ck] \to X$ such that

- for every $i \in [ck]$ in the domain of $m, S_i \cup m(i)$ is a stable set, and
- for every $v \in X$, at least $c \cdot w(v)$ elements of [ck] map to v.

Thus we can extend the stable sets S_i appropriately; by Proposition 2, this gives the desired fractional $k \wr w$ -colouring of G.

We remark that this lemma is most sensibly applied when X is a clique.

³The unweighted version is described as folklore in [8] and was used earlier in [11], and probably elsewhere.

2.1 Reed's Conjecture and fractional colourings

Our approach to fractionally colouring K_{Δ} -free graphs is inspired by the following result of Reed ([17], §21.3):

Theorem 4. Every graph G satisfies $\chi_f(G) \leq \frac{1}{2}(\Delta(G) + 1 + \omega(G))$.

This is the fractional relaxation of Reed's ω , Δ , χ conjecture [18], which proposes that every graph satisfies $\chi \leq \lceil \frac{1}{2}(\Delta+1+\omega) \rceil$. However, we do not consider the conjecture, or even the fractional relaxation, but rather a weighted version of a local strengthening observed by McDiarmid ([17], p.246). For a vertex v let $\omega(v)$ be the size of the largest clique containing v. Then:

Theorem 5. Every graph G satisfies $\chi_f(G) \leq \max_v \frac{1}{2}(d(v) + 1 + \omega(v))$.

The proof of this theorem was never published, but appears in Section 2.2 of [12] and is almost identical to the proof of Theorem 4. What we need is a new weighted version of this theorem, which we prove here. First we need some notation. For a vertex v let $\tilde{N}(v)$ denote the closed neighbourhood of v. Given a w-weighted graph G and a vertex $v \in V(G)$, we define:

- The degree weight $w_d(v)$ of v, defined as $\sum_{u \in \tilde{N}(v)} w(u)$.
- The clique weight $w_c(v)$ of v, defined as the maximum over all cliques C containing v of $\sum_{u \in C} w(u)$.
- The Reed weight $\rho_w(v)$ of v, defined as $\frac{1}{2}(w_d(v) + w_c(v))$ (we sometimes denote ρ_1 by ρ). For a graph G, we define $\rho_w(G)$ as $\max_{v \in V(G)} \rho_w(v)$.

Our result is a natural generalization of McDiarmid's:

Theorem 6. Every graph G satisfies $\chi_f^w(G) \leq \rho_w(G)$.

Proof. Let c be a positive integer such that for every v, cw(v) is an integer; c exists since the weights are rational. Let G_w be the graph constructed from G by replicating each vertex v into a clique C_v of size cw(v). ⁴ Applying Theorem 5 to G_w tells us that there is a fractional $c\rho_w(G)$ -colouring κ_w of G_w . From this we construct a cw-fractional $c\rho_w(G)$ -colouring κ of G by setting, for each $v \in V(G)$,

$$\kappa[v] = \kappa_w[C_v].$$

The result follows from Proposition 2 (3).

3 The general approach

Fix some $\Delta \geq 6$ and $0 < \epsilon \leq \frac{1}{2}$, and suppose we wish to prove that $f(\Delta) \geq \epsilon$. Let G be a graph with maximum degree Δ and clique number $\omega \leq \Delta - 1$; by Theorem 4 we know that $\chi_f(G) \leq \Delta - \frac{1}{2}$ if $\omega \leq \Delta - 2$, so we assume G has clique number $\omega = \Delta - 1$. We define V_{ω} as the set of vertices in ω -cliques, and V'_{ω} as the set of vertices in V_{ω} with degree Δ . Let

⁴That is, $x \in C_u$ and $y \in C_v$ are adjacent precisely if u, v are adjacent or if u = v and x, y are distinct.

 G_{ω} and G'_{ω} denote the subgraphs of G induced on V_{ω} and V'_{ω} respectively. Notice that a vertex v will have $\rho_{\mathbf{1}}(v) > \Delta - \frac{1}{2}$ if and only if v is in V'_{ω} . In plain language, our approach is:

- 1. Prove that in a minimum counterexample, G_{ω} has a nice structure.
- 2. Spend a little bit of weight on a fractional colouring that lowers the Reed weight for vertices in V'_{ω} at a rate of $(1 + \epsilon')$ per weight spent, i.e. we spend y weight and $(1 + \epsilon')y = y + \epsilon$. If y is sufficiently small, this lowers the maximum Reed weight over all vertices of G by $y + \epsilon$.
- 3. Having already "won" by ϵ , i.e. having lowered $\rho(G)$ by $y + \epsilon$ using only y colour weight, we can finish the colouring using Theorem 6.

More specifically, we find a vertex weighting w such that we have a fractional $y \wr w$ -colouring of G, and such that $\rho_{(1-w)}(G) \leq \Delta - y - \epsilon$. We then apply Theorem 6 to find a fractional $(\Delta - y - \epsilon) \wr (1 - w)$ -colouring of G. Combining these two partial fractional colourings gives us a fractional $(\Delta - \epsilon)$ -colouring of G.

Since any $v \notin V'_{\omega}$ satisfies $\rho_{\mathbf{1}}(v) \leq \Delta - \frac{1}{2}$, if $(1 + \epsilon')y \leq \frac{1}{2}$ we only need to ensure that ρ drops by $(1 + \epsilon')y$ for vertices with $\rho_{\mathbf{1}}(v) = \Delta$. Actually we can ensure that while we do this, ρ also drops at a decent rate (easily at least $\frac{1}{2}y$) for vertices with $\rho < \Delta$. This means that we can spend more weight (i.e. increase y), thereby improving ϵ . It is in our interests to first worry about maximizing ϵ' , then worry about maximizing y.

This method depends heavily on properly understanding the structure of vertices with $\rho_1(v) = \Delta$. We simplify this structure through reductions, or if you prefer, the structural characterization of a minimum counterexample:

Lemma 7. Fix some $\Delta \geq 5$ and some $\epsilon \leq \frac{1}{2}$, with the further restriction that $\epsilon \leq \frac{1}{3}$ if $\Delta = 5$. Let G be a graph with maximum degree Δ and clique number at most $\Delta - 1$ such that

- if $\Delta = 5$, no component of G is isomorphic to $C_5 \boxtimes K_2$,
- G has fractional chromatic number greater than $\Delta \epsilon$, and
- no graph on fewer vertices has these properties.

Then

- (i) the maximum cliques of G are pairwise disjoint, and
- (ii) there is no vertex v outside a maximum clique C such that $|N(v) \cap C| > 1$.

Together, these properties allow us to apply the following result of Aharoni, Berger, and Ziv [1]:

Theorem 8. Let k be a positive integer and let G be a graph whose vertices are partitioned into cliques of size $\omega \geq 2k$. If G has maximum degree at most $\omega + k - 1$, then $\chi_f(G) = \omega$.

Applying this theorem to an induced subgraph of G_{ω} is the key to proving that we can lower ρ quickly for any vertex v with $\rho_{\mathbf{1}}(v) = \Delta$. The proof of Lemma 7 is technical,

independent of the main proof, and does not give insight to our approach, so we defer it to Section 7. We now consider the probability distribution on stable sets that, via Proposition 2, characterizes our initial colouring phase.

From now until Section 7, we consider G to be a graph with maximum degree $\Delta \geq 6$, clique number $\omega = \Delta - 1$, and satisfying properties (i) and (ii) of Lemma 7. We remark that Lemma 7 gives a characterization of minimum counterexamples with $\Delta = 5$; although we do not make use of the characterization in this paper, it is likely to be useful in the future.

4 A probability distribution

For every vertex v of G, let $N_{\omega}(v)$ denote $N(v) \cap V_{\omega}$ and let $d_{\omega}(v)$ denote $|N_{\omega}(v)|$. The initial phase of our colouring involves choosing a random stable set S_w of G_w , then extending it randomly to a stable set S of G such that S_w and S have the following desirable properties:

1. For every
$$v \in V_{\omega}$$
,
$$\Pr(v \in S_{\omega}) = \frac{1}{\omega}. \tag{2}$$

2. For every $v \notin V_{\omega}$,

$$\Pr(N_{\omega}(v) \cap S_{\omega} = \emptyset) \geq \sum_{i=0}^{3} \frac{1}{4} \Pr\left(\operatorname{Bin}(d_{\omega}(v), \frac{4}{\omega}) \leq i\right)$$

$$= \sum_{i=0}^{3} \frac{4-i}{4} \Pr\left(\operatorname{Bin}(d_{\omega}(v), \frac{4}{\omega}) = i\right).$$
(3)

3. For every $v \notin V_{\omega}$,

$$\Pr(v \in S) \ge \frac{\Pr(N_{\omega}(v) \cap S_{\omega} = \emptyset)}{(d(v) - d_{\omega}(v)) + 1} \ge \frac{\sum_{i=0}^{3} \frac{4-i}{4} \Pr\left(\operatorname{Bin}(d_{\omega}(v), \frac{4}{\omega}) = i\right)\right)}{(d(v) - d_{\omega}(v)) + 1}.$$
 (4)

4. S is maximal.

We will put weight on stable sets according to this distribution until we can no longer guarantee that ρ is dropping quickly. We discuss this stopping condition in Section 5.1.

4.1 Choosing S_{ω}

Denote the maximum cliques of G by B_1, \ldots, B_ℓ , bearing in mind that they are vertexdisjoint. To choose S_ω we first select, for each $1 \leq i \leq \ell$, a subset B_i' of B_i of size 4, uniformly at random and independently for each i. Setting \tilde{G}_ω to be the subgraph of Ginduced on $\bigcup_i B_i'$, note that every vertex in B_i has at most two neighbours outside B_i and therefore $\Delta(\tilde{G}_\omega) \leq 5$. Thus Theorem 8 tells us that \tilde{G}_ω is fractionally 4-colourable. It follows from Proposition 2 that there is a probability distribution on the stable sets of \tilde{G}_ω such that given a stable set \tilde{S} chosen from this distribution, for any $v \in \tilde{G}_\omega$, $\Pr(v \in \tilde{S}) = \frac{1}{4}$. We therefore choose S_{ω} from this distribution, subject to our random choice of \tilde{G}_{ω} . Since every $v \in G_{\omega}$ satisfies $\Pr(v \in \tilde{G}_{\omega}) = \frac{4}{\omega}$, for any $v \in G_{\omega}$ we clearly have $\Pr(v \in S_{\omega}) = \frac{1}{\omega}$, i.e. (2) holds. We must now prove that (3) holds (the reader may have noticed that any old fractional ω -colouring of G_{ω} would have given us S_{ω} satisfying (2)).

The first step is to observe that for $v \notin G_{\omega}$ and $0 \le i \le 3$,

$$\Pr\left(\left(N_{\omega}(v)\cap S_{\omega}=\emptyset\right)\mid \left(|N_{\omega}(v)\cap \tilde{G}_{\omega}|=i\right)\right)\geq \frac{4-i}{4}.\tag{5}$$

This is because every neighbour of v in \tilde{G}_{ω} is in S_w with probability $\frac{1}{4}$, and in the worst case these events may be disjoint for all i such neighbours (we later conjecture that it is possible to avoid this worst case; this would improve our bounds substantially for $\Delta \in \{5,6\}$).

The second step is to observe that for $v \notin G_{\omega}$ and $0 \le i \le d_{\omega}(v)$,

$$\Pr\left(|N_{\omega}(v)\cap \tilde{G}_{\omega}|=i\right) = \Pr\left(\operatorname{Bin}(d_{\omega}(v), \frac{4}{\omega})=i)\right). \tag{6}$$

To see this, note that Lemma 7 tells us that any two neighbours $x, y \in G_{\omega}$ of v are in different blocks B_i , and therefore the events of x being in \tilde{G}_{ω} and y being in \tilde{G}_{ω} are independent. Equation (3) follows immediately from Equations (5) and (6).

4.2 Choosing S

Given a choice of S_{ω} , we randomly extend to S as follows:

- 1. Choose an ordering π of $V(G) \setminus V_{\omega}$ uniformly at random, and label the vertices of $V(G) \setminus V_{\omega}$ as v_1, \ldots, v_r in the order in which they appear in π .
- 2. Set $S = S_{\omega}$.
- 3. For each of i = 1, ..., r in order, put v_i in S if and only if it currently has no neighbour in S.

Since every vertex in V_{ω} is in S_{ω} or has a neighbour in S_{ω} , and every vertex not in V_{ω} is in S or has a neighbour in S, we can see that S is always a maximal stable set. A vertex $v_i \in V(G) \setminus V_{\omega}$ is in S if it has no neighbours in S_{ω} , and it is not adjacent to any $v_j \in V(G) \setminus V_{\omega}$ for j < i. Since we choose π uniformly at random, any vertex $v \in V(G) \setminus V_{\omega}$ satisfies

$$\Pr\left(\left(v \in S\right) \mid \left(N_{\omega}(v) \cap S_{\omega} = \emptyset\right)\right) \ge \frac{1}{|N(v) \setminus V_{\omega}| + 1}.\tag{7}$$

Equation (4) follows immediately from Equation (7).

4.3 Bounding the rate at which ρ initially decreases

Suppose we spend weight y to colour G according to the probability distribution on S that we just described. That is, for $S' \in \mathcal{S}(G)$, we place weight q(S') on S', where

$$q(S') = y \cdot \Pr(S = S').$$

Δ	$\mu(\Delta)$	$\mu(\Delta)(\Delta+1)$	d for which $\mu(\Delta) = p(\Delta, d)$	$\tilde{y}(\Delta)$	$\tilde{y}(\Delta)\mu(\Delta)$
6	.029376	.205	6	1.518	0.04459
7	.054869	.439	6	1.640	0.08999
8	.062947	.567	7	1.804	0.11353
9	.066406	.664	7	1.969	0.13077
10	.066328	.730	8	2.146	0.14234
100	.009843	.994	29	20.003	0.19691
1000	.000998	.999	135	199.979	0.19973

Table 2: Some values of $\mu(\Delta)$, where they are achieved, and corresponding values of \tilde{y} , which we discuss later. Note that $p(\Delta, 0) = 1/(\Delta + 1)$ is an upper bound for $\mu(\Delta)$. These values are calculated in [6].

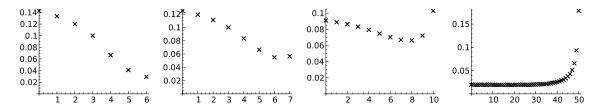


Figure 3: Values of d versus $p(\Delta, d)$ for $\Delta \in \{6, 7, 10, 50\}$.

Then we wish to argue that $\rho(G)$ drops by $(1+\epsilon')y$ for some positive ϵ' . For now, to avoid consideration of stopping conditions⁵, suppose that y is very small $(y=\frac{1}{10} \text{ will do for now})$. For a fixed Δ and $0 \le d \le \Delta$ we define $p(\Delta, d)$ as

$$p(\Delta, d) = \frac{\sum_{i=0}^{3} \frac{1}{4} \Pr\left(\text{Bin}(d, \frac{4}{\omega}) \le i)\right)}{(\Delta - d) + 1},$$
(8)

noting that a vertex $v \notin G_{\omega}$ with $d_{\omega}(v) = d$ is in S with probability at least $p(\Delta, d)$. Following this, we define

$$\mu_k(\Delta) = \min_{0 \le d \le k} p(\Delta, d)$$
 and $\mu(\Delta) = \mu_{\Delta}(\Delta) = \min_{0 \le d \le \Delta} p(\Delta, d),$

noting that any vertex $v \notin G_{\omega}$ is in S with probability at least $\mu(\Delta)$.

Lemma 9. For every vertex $v \in V(G)$, $\Pr(v \in S) \ge \mu(\Delta)$.

Proof. To see this we only need to prove that $v \in G_{\omega}$ is in S with probability at least $\mu(\Delta)$. This is clearly the case since v is in S with probability $\frac{1}{\omega} > \frac{1}{\Delta+1} = p(\Delta, 0) \ge \mu(\Delta)$.

We now set ϵ' to be $\mu(\Delta)$. Table 2 gives some computed values of $\mu(\Delta)$, and Figure 3 shows some values of $p(\Delta, d)$. (We will define and consider $\tilde{y}(\Delta)$ in the next section.) These numbers were computed using Sage; the code is available at [6].

⁵i.e. when y is large enough to make our model fail

Lemma 10. For any vertex v in V'_{ω} , $E(|S \cap \tilde{N}(v)|) \geq 1 + 2\epsilon'$.

Proof. Since v is in some B_i and has degree $\Delta = 1 + \omega$, v has exactly two neighbours outside B_i . Each is in S with probability at least ϵ' , and S contains a vertex in B_i with probability 1. Therefore the lemma follows from linearity of expectation.

Let v be a vertex in $V'_{\omega} \cap B_i$. Since $E(|S \cap B_i|) = 1$, and B_i is the unique maximum clique containing v, we know that at the outset, when we spend weight y, $\rho(v)$ will drop by $\frac{1}{2}(1+1+2\epsilon')y = (1+\epsilon')y$.

For $k \leq \omega$, let V_k be the set of vertices in a clique of size k but not a clique of size k+1, noting that these vertex sets partition V(G). We note the following.

Lemma 11. If $4 \le k \le \omega - 1$ and v is a vertex in V_k , then v has at most $\Delta + 1 - k$ neighbours in V_{ω} .

Proof. It suffices to prove that if X is a k-clique containing v, then X does not intersect an ω -clique. Suppose it does intersect some B_i , and note that it may only intersect B_i once by Lemma 7. Since any vertex in B_i has at most two neighbours outside B_i , |X| must be at most 3, a contradiction.

Corollary 12. If $v \in V_k$ for some $4 \le k \le \omega - 1$, then $\Pr(v \in S) \ge \mu_{\Delta+1-k}(\Delta)$.

5 The initial colouring

The probability distribution described in the previous section tells us what to do in the initial colouring phase: we choose colour classes according to the distribution. The only thing we need to worry about is giving a vertex more than colour weight 1. To avoid this, when a vertex is full we simply delete it and continue as though it never existed. This is the same approach taken in the proof of Theorems 4 and 5. Vertices in V_{ω} will never be full before the end of our process.

Lemma 13. For any $y \in [0, \omega]$ there exists a vertex weighting w and a fractional $y \wr w$ colouring of G such that w satisfies the following conditions:

- (a) Every vertex v in V_{ω} has $w(v) = y/\omega$.
- (b) For $0 \le \ell \le \Delta$, every vertex $v \notin V_{\omega}$ with exactly ℓ neighbours in V_{ω} has $w(v) \ge \min\{p(\Delta, \ell)y, 1\}$.
- (c) For $1 \le k < \omega$, every clique X of size k has $w(X) \ge k \min\{\mu(\Delta)y, 1\}$.
- (d) For $4 \le k < \omega$, every clique X of size k has $w(X) \ge k \min\{\mu_{\Delta+1-k}(\Delta)y, 1\}$.
- (e) Every vertex v with w(v) < 1 has $w(\tilde{N}(v)) \ge y$.

Note that $\mu(\Delta)y$ and $\mu_{\Delta+1-k}(\Delta)y$ are less than 1.

Proof. We proceed using the following algorithm.

Initially, set $H_0 = G$, set $leftover_0 = y$, and set $capacity_0(v) = 1$ for every vertex in H_0 . For i = 0, 1, ... do the following.

- 1. Let R_i be a random stable set drawn from the distribution giving S described in Section 4. For every vertex v we set $prob_i(v)$ as $Pr(v \in R_i)$.
- 2. Set y_i' to be $\min_{v \in V(H_i)}(capacity_i(v)/prob_i(v))$, and set y_i to be $\min\{leftover_i, y_i'\}$.
- 3. For every $v \in V(H_i)$, set $w_i(v)$ to be $prob_i(v)y_i$.
- 4. For every $v \in V(H_i)$, set $capacity_{i+1}(v)$ to be $capacity_i(v) w_i(v)$.
- 5. Set $leftover_{i+1}$ to be $leftover_i y_i$.
- 6. If $leftover_{i+1} = 0$, we terminate the process. Otherwise, let U_i be the vertex set $\{v \in V(H_i) \mid capacity_{i+1}(v) = 0\}$, and set H_{i+1} to be $H_i U_i$.

Let ν denote the value of i for which $leftover_{i+1} = 0$. For every vertex v, let $w(v) = \sum_{i=0}^{\nu} w_i(v)$. Observe that $y = \sum_{i=0}^{\nu} y_i$.

We first prove that this process must terminate. Our choice of each y_i implies that either $leftover_{i+1} = 0$, or $|U_{i+1}| < |U_i|$. Thus we terminate after at most |V(G)| iterations. Now observe that every vertex $v \in G_{\omega}$ has $prob_i(v) = 1/\omega$ throughout the process, and therefore $capacity_{\nu}(v) > 0$ since $leftover_0 = y \le \omega$ (this can easily be proved by induction on i). Note that (a) also follows from this observation. As a further consequence, we can see that G_{ω} is a subgraph of every H_i .

We claim that we actually have a collection of fractional $y_i \wr w_i$ -colourings for $0 \le i \le \nu$. To see this we simply appeal to Proposition 2 (3), noting that $\Pr(v \in R_i) = w_i(v)/y_i$. Since $w = \sum_{i=1}^{\nu} w_i$ and $y = \sum_{i=0}^{\nu} y_i$, it follows immediately that these colourings together give us a fractional $y \wr w$ -colouring of G.

To prove (b), we take $v \notin V_{\omega}$ with ℓ neighbours in V_{ω} , and assume that w(v) < 1, otherwise we are done. Since every H_i contains G_{ω} , we can see that

$$\Pr(v \in R_i) \ge \frac{\sum_{i=0}^3 \frac{4-i}{4} \Pr\left(\text{Bin}(d_{\omega}(v), \frac{4}{\omega}) = i\right)}{|N(v) \cap V(H_i)| - d_{\omega}(v) + 1} \ge p(\Delta, \ell). \tag{9}$$

Consequently $prob_i(v) \ge p(\Delta, \ell)$ for all i, and (b) follows. Note that (c) follows immediately from (b). Similarly, (d) follows from (b) and Lemma 11.

To see that (e) holds, simply note that R_i is always a maximal stable set in H_i . Therefore if w(v) < 1, then $capacity_{\nu}(v) > 0$, thus $v \in H_i$ for every i, meaning that R_i intersects $\tilde{N}(v)$ with probability 1.

5.1 Maximizing the expenditure

Here we consider the best possible choice of y in Lemma 13. The optimal value of y will be the largest possible such that the upper bound on $\rho_{1-w}(G)$ is still achieved by some vertex in G_{ω} . If we increase y beyond this point, we will find that $\rho_{1-w}(G)$ is no longer guaranteed to drop as fast as y increases.

In light of this goal, for $1 \leq k \leq 3$ we let $\tilde{y}_k(\Delta)$ denote the maximum value of y such that

$$(1 + \mu(\Delta))y \le \frac{1}{2}(\Delta - 1 - k) + (\frac{1}{2} + \frac{1}{2}k\mu(\Delta))y. \tag{10}$$

For $4 \le k \le \Delta - 2$ we let $\tilde{y}_k(\Delta)$ denote the maximum value of y such that

$$(1 + \mu(\Delta))y \le \frac{1}{2}(\Delta - 1 - k) + \left(\frac{1}{2} + \frac{1}{2}k\mu_{\Delta + 1 - k}(\Delta)\right)y. \tag{11}$$

Now let $\tilde{y}(\Delta)$ denote $\min\{\min_k \tilde{y}_k(\Delta), \omega, \frac{\omega-3}{1-3\mu(\Delta)}\}$ (the latter two bounds are for convenience of proof, and do not affect our results). Our initial colouring phase culminates in the following consequence of Lemma 13.

Theorem 14. For any $0 \le y \le \tilde{y}(\Delta)$, there is a vertex weighting w and fractional $y \wr w$ colouring of G such that $\rho_{1-w}(G) \le \Delta - (1 + \mu(\Delta))y$.

Proof. Let v be any vertex in G; it suffices to prove that $\rho_{1-w}(v) \leq \Delta - (1 + \mu(\Delta))y$. We take the fractional $y \wr w$ -colouring guaranteed by Lemma 13.

First suppose $v \in G_{\omega}$, and assume without loss of generality that $v \in B_1$. We know that $w(B_1) = y$ by Lemma 13(a), and that for any $u \in \tilde{N}(v) \setminus B_1$, $w(u) \geq y\mu(\Delta)$ (by Lemma 13(b)). Therefore $|\tilde{N}(v)| - w(\tilde{N}(v)) \leq \omega - y + 2(1 - y\mu(\Delta)) = \Delta + 1 - y - 2y\mu(\Delta)$. We now claim that for any clique C containing $v, |C| - w(C) \leq \omega - y$. Clearly $w(B_1) = y$. For C not equal to B_1 , Lemma 7 tells us that $|C| \leq 3$. Therefore $|C| - w(C) \leq 3 - 3y\mu(\Delta)$. If $\omega - y < 3 - 3y\mu(\Delta)$, then $\omega - 3 < y(1 - 3\mu(\Delta))$, contradicting the fact that $y \leq \tilde{y}(\Delta) \leq \frac{\omega - 3}{1 - 3\mu(\Delta)}$. Therefore $|C| - w(C) \leq \omega - y = \Delta - 1 - y$. Thus

$$\rho_{1-w}(v) \le \frac{1}{2}(\Delta - 1 - y) + \frac{1}{2}(\Delta + 1 - y - 2y\mu(\Delta)) = \Delta - (1 + \mu(\Delta))y. \tag{12}$$

Now suppose that v is not in V_{ω} , and let C be a clique containing v such that |C| - w(C) is maximum. Denote the size of C by k. By Lemma 13(e), we know that $w(\tilde{N}(v)) \geq y$, so

$$|\tilde{N}(v)| - w(\tilde{N}(v)) \le \Delta + 1 - y. \tag{13}$$

Therefore to prove that $\rho_{1-w}(v) \leq \Delta - (1+\mu(\Delta))y$, it is sufficient to prove that

$$k - w(C) \le \Delta - 1 - y - 2y\mu(\Delta),\tag{14}$$

i.e.

$$(\mu(\Delta) + \frac{1}{2})y \le \frac{1}{2}(\Delta - 1 - k) + \frac{1}{2}w(C). \tag{15}$$

By Lemma 13(c) we know that $w(C) \ge k\mu(\Delta)y$. If $k \ge 4$, by Lemma 13(d) we know that $w(C) \ge k\mu_{\Delta+1-k}(\Delta)y$. We also know that $y \le \tilde{y}(\Delta) \le \tilde{y}_k(\Delta)$, so if $k \le 3$ then

$$(1 + \mu(\Delta))y \le \frac{1}{2}(\Delta - 1 - k) + (\frac{1}{2} + \frac{1}{2}k\mu(\Delta))y, \tag{16}$$

and if $k \geq 4$ then

$$(1 + \mu(\Delta))y \le \frac{1}{2}(\Delta - 1 - k) + \left(\frac{1}{2} + \frac{1}{2}k\mu_{\Delta + 1 - k}(\Delta)\right)y. \tag{17}$$

In either case,

$$(1 + \mu(\Delta))y \le \frac{1}{2}(\Delta - 1 - k) + \left(\frac{1}{2}y + \frac{1}{2}w(C)\right),\tag{18}$$

so

$$(\mu(\Delta) + \frac{1}{2})y \le \frac{1}{2}(\Delta - 1 - k) + \frac{1}{2}w(C),\tag{19}$$

as desired. Thus $\rho_{1-w}(v) \leq \Delta - (1 + \mu(\Delta))y$.

Since equations 10 and 11 are linear, we can easily find the optimal values of $\tilde{y}_k(\Delta)$ by solving for

$$\tilde{y}_k(\Delta) = \frac{\frac{1}{2}(\Delta - 1 - k)}{\frac{1}{2} + \mu(\Delta) - \frac{1}{2}k\mu(\Delta)}$$

$$\tag{20}$$

when $k \leq 3$ and for

$$\tilde{y}_k(\Delta) = \frac{\frac{1}{2}(\Delta - 1 - k)}{\frac{1}{2} + \mu(\Delta) - \frac{1}{2}k\mu_{\Delta + 1 - k}(\Delta)}$$
(21)

when $\Delta - 2 \ge k \ge 4$. See [6] and Table 2 for numerical values.

6 Proving the main result

We now have enough results in hand to prove the main result easily.

Theorem 15. For $\Delta \geq 6$, let G be a graph with maximum degree Δ and clique number at most $\Delta - 1$. Then G has fractional chromatic number at most $\Delta - \min\{\frac{1}{2}, \tilde{y}(\Delta)\mu(\Delta)\}$.

Proof. Let G be a minimum counterexample; Theorem 5 tells us that G has maximum degree Δ and clique number $\omega = \Delta - 1$. Lemma 7 tells us that all ω -cliques of G are disjoint, and that no vertex v has two neighbours in an ω -clique not containing v.

We may therefore set $y = \tilde{y}(\Delta)$ and apply Theorem 14. This gives us a vertex weighting w and fractional $y \wr w$ -colouring of G such that $\rho_{1-w}(G) \leq \Delta - (1+\mu(\Delta))y$. By Theorem 6, $\chi_f^{1-w} \leq \rho_{1-w}(G) \leq \Delta - (1+\mu(\Delta))y$. That is, there is a fractional $(\Delta - (1+\mu(\Delta))y)\wr (1-w)$ -colouring of G. Combining this colouring with the initial fractional $y \wr w$ -colouring gives us a fractional $(\Delta - \tilde{y}(\Delta)\mu(\Delta))$ -colouring, which tells us that $\chi_f(G) \leq \Delta - \tilde{y}(\Delta)\mu(\Delta)$.

For all values of Δ we have investigated, $\tilde{y}(\Delta)\mu(\Delta) < \frac{1}{5}$. We believe that this is always the case.

7 The structural reduction

In this section we prove Lemma 7, which tells us that we need only consider graphs whose maximum cliques behave nicely. First observe that every proper induced subgraph of G is fractionally $(\Delta - \epsilon)$ -colourable, since deleting vertices from a graph with $\Delta = 5$ cannot create a copy of $C_5 \boxtimes K_2$. We prove the lemma in two parts:

Lemma 16. Part (i) of Lemma 7 holds.

Lemma 17. Part (ii) of Lemma 7 holds.

7.1 Part (i)

We actually split the proof of Lemma 16 into three parts. Suppose C and C' are two intersecting ω -cliques. Since $\omega = \Delta - 1$, we can immediately observe that $|C \cap C'| \ge \omega - 2$. Therefore Lemma 16 follows as an easy corollary of the next three Lemmas 18, 19, 20. Throughout this section we will make implicit use of the fact that every vertex in G has at least $\Delta - 1$ neighbours, as is trivially implied by the minimality of G. Furthermore

note that whenever we reduce G to a graph G', no component of which is 5-regular, no component of G' can be isomorphic to $C_5 \boxtimes K_2$.

Lemma 18. G does not contain three ω -cliques mutually intersecting in $\omega-1$ vertices.

Proof. Suppose that G contains an $(\omega - 1)$ -clique X and vertices x_1, x_2, x_3 each of which is complete to X. Because there is no $(\omega + 1)$ -clique, $\{x_1, x_2, x_3\}$ is a stable set. Let $G' = G \setminus (X \cup \{x_1, x_2, x_3\})$; as previously observed, since G' is a proper induced subgraph of G, there is a fractional $(\Delta - \epsilon)$ -colouring κ of G'. We extend κ to a fractional $(\Delta - \epsilon)$ -colouring of G to obtain a contradiction, beginning by colouring $\{x_1, x_2, x_3\}$ using weight at most $2 - \epsilon$.

First suppose $\Delta = 5$, so $\epsilon \leq \frac{1}{3}$. Since each x_i has at most two neighbours in G', we have $|\alpha(x_i)| \geq \Delta - \epsilon - 2$. Note that $|\alpha(x_i) \cup \alpha(x_j)| \leq \Delta - \epsilon$, so for any $\{i, j\} \subseteq \{1, 2, 3\}$ we have $|\alpha(x_i) \cap \alpha(x_j)| \geq \Delta - \epsilon - 4 \geq 1 - \epsilon \geq \frac{2}{3}$. We extend κ to $\{x_1, x_2, x_3\}$ such that

- $|\kappa[x_1] \cap \kappa[x_2]| \geq \frac{2}{3}$, and
- There exist disjoint subsets s_1 and s_2 of $\kappa[x_3]$, each of size $\frac{1}{3}$, such that $s_1 \subset \kappa[x_1]$ and $s_2 \subset \kappa[x_2]$.

To do this, we first give x_1 and x_2 weight $\frac{2}{3}$ of colour in common, then give x_1 and x_3 weight $\frac{1}{3}$ of colour each such that all the colour on x_3 is in $\kappa[x_1]$, then give x_2 and x_3 weight $\frac{1}{3}$ of colour each such that all the new colour on x_3 is in $\kappa[x_2]$. Finally we complete the colouring of x_3 arbitrarily. Confirming that this is possible is straightforward given the pairwise intersections of $\alpha(x_i)$. Furthermore since $|\kappa[\{x_1, x_2\}]| \leq \frac{4}{3}$ and at least $\frac{2}{3}$ of the colour in $\kappa[x_3]$ is in $\kappa[\{x_1, x_2\}]$, we use weight at most $2 - \epsilon$ on $\{x_1, x_2, x_3\}$.

Now suppose $\Delta \geq 6$, so $\epsilon \leq \frac{1}{2}$. Our approach is the same as before, except now for any $\{i,j\} \subseteq \{1,2,3\}$ we have $|\alpha(x_i) \cap \alpha(x_j)| \geq \Delta - \epsilon - 4 \geq \frac{3}{2}$. Thus we can proceed by giving x_1 and x_2 weight $\frac{1}{2}$ of colour in common, then assign s_1 and s_2 as before, but with size $\frac{1}{2}$ each. Again we use weight at most $2 - \epsilon$ on $\{x_1, x_2, x_3\}$.

We now have $\{v \in V(G) : |\kappa(v)| < 1\} = V(X)$. For every $v \in V(X)$, we have $|\alpha(v)| \ge \Delta - \epsilon - (2 - \epsilon) = \omega - 1 = |V(X)|$. We may therefore apply Lemma 3 and extend κ to a fractional $(\Delta - \epsilon)$ -colouring of G.

Lemma 19. G does not contain two ω -cliques intersecting in $\omega-1$ vertices.

Proof. Suppose C and C' are two ω -cliques intersecting in $\omega-1$ vertices. Let $v_1,\ldots,v_{\omega-1}$ be the vertices in $C\cap C'$, let x be the vertex in $C\setminus C'$, and let y be the vertex in $C'\setminus C$, noting that x and y are nonadjacent. For $1\leq i\leq \omega-1$, if v_i has a neighbour outside $C\cup C'$ call it u_i .

Claim 1. There exists a fractional $(\Delta - \epsilon)$ -colouring κ of $G \setminus (C \cap C')$ satisfying the following:

- (1) If $\Delta = 5$, then $|\kappa[\{x,y\}]| \le 1 + \epsilon$.
- (2) If $\Delta \geq 6$, then $|\kappa[\{x,y\}]| = 1$.
- (3) $|\bigcap_{i<\omega-1}\kappa[u_i]|\leq \epsilon.$

We first show how the claim implies the lemma. For each $v_i \in C \cap C'$, $|\alpha(v_i)| \ge \Delta - \epsilon - |\kappa[\{x,y,u_i\}]| \ge \Delta - \epsilon - 1 - |\kappa[\{x,y\}]| \ge \omega - 2$. Thus to apply Lemma 3 and extend κ to G it is enough to show that $|\bigcup_{i \le \omega - 1} \alpha(v_i)| \ge \omega - 1$. Indeed, the set of colours available to at least some of the vertices in $C \cap C'$ are those which are not forbidden to all of them: If $\Delta \ge 6$, then

$$\left| \bigcup_{i \le \omega - 1} \alpha(v_i) \right| \ge \Delta - \epsilon - |\kappa[\{x, y\}]| - \left| \bigcap_{i \le \omega - 1} \kappa[u_i] \right| \ge \omega - 2\epsilon \ge \omega - 1$$

and if $\Delta = 5$,

$$\left| \bigcup_{i \le \omega - 1} \alpha(v_i) \right| \ge \omega - 3\epsilon \ge \omega - 1.$$

Lemma 3 then guarantees a fractional $(\Delta - \epsilon)$ -colouring of G, a contradiction.

Proof of Claim 1. There are two cases. Note that by Lemma 18, if u_i exists for each i then $|\{u_i: 1 \leq i \leq \omega - 1\}| \geq 2$.

Case 1: $2 \le |\{u_i : 1 \le i \le \omega - 1\}| < \omega - 1 \text{ and } u_i \text{ exists for each } i.$

Without loss of generality suppose that $u_1 = u_2$ and consider $G' = G \setminus (C \cup C' \cup \{u_1\})$. Again, since G' is a proper induced subgraph of G, there exists a fractional $(\Delta - \epsilon)$ -colouring κ of G'. We extend κ to a fractional colouring of $G \setminus (C \cap C')$, first colouring x and y, then u_1 .

Each of x and y has at most two neighbours in G' so we have $|\alpha(x)|, |\alpha(y)| \ge \Delta - \epsilon - 2$. Since $|\alpha(x) \cup \alpha(y)| \le \Delta - \epsilon$ it follows that $|\alpha(x) \cap \alpha(y)| \ge \Delta - \epsilon - 4 \ge 1$ when $\Delta \ge 6$, and $|\alpha(x) \cap \alpha(y)| \ge 1 - \epsilon$ when $\Delta = 5$. We extend κ in the obvious way so that if $\Delta \ge 6$ then $\kappa[x] = \kappa[y]$, and if $\Delta = 5$ then $|\kappa[x] \cap \kappa[y]| \ge 1 - \epsilon$, satisfying (1) and (2). It remains to colour u_1 . Note that u_1 has degree at most $\omega - 1$ in $G \setminus (C \cap C')$ so $|\alpha(u_1)| \ge 2 - \epsilon$. Because $|\bigcap_{3 \le i \le \omega - 1} \kappa[u_i]| \le 1$, we can choose $\kappa[u_1]$ from $\alpha(u_1)$ in such a way that $|\kappa[u_1] \cap \bigcap_{3 \le i \le \omega - 1} \kappa[u_i]| \le \epsilon$, satisfying (3).

Case 2: $|\{u_i : 1 \le i \le \omega - 1\}| = \omega - 1$ or u_i does not exist for some i.

If there exists an edge u_iu_j in G for some $i \neq j$, then let $G' = G \setminus (C \cup C')$. Otherwise choose $i \neq j$ such that adding the edge u_iu_j to $G \setminus (C \cup C')$ yields a graph with $\omega < \Delta$ and let $G' = G \setminus (C \cup C') \cup u_iu_j$. To see that such i and j exist, consider u_1, u_2 and u_3 and suppose that each pair of these has an $(\omega - 1)$ -clique in the common neighbourhood. Because $\Delta = \omega + 1$ there must be a vertex contained in each of these three cliques, but Lemma 18 forbids the existence of three pairwise intersecting ω -cliques.

By the minimality of G, there exists a fractional $(\Delta - \epsilon)$ -colouring κ of G'. We need to extend κ to x and y. Because each of x and y has at most two neighbours in G' we have $|\alpha(x)|, |\alpha(y)| \ge \Delta - \epsilon - 2$. It follows that $|\alpha(x) \cap \alpha(y)| \ge 1$ if $\Delta \ge 6$ and $|\alpha(x) \cap \alpha(y)| \ge 1 - \epsilon$ if $\Delta = 5$ so we can extend κ in the obvious way to satisfy (1) and (2). Requirement (3) is guaranteed by the existence of the edge $u_i u_j$. This proves the claim.

As we have shown, the claim implies the lemma.

Lemma 20. G does not contain two ω -cliques intersecting in $\omega-2$ vertices.

Proof. Suppose C and C' are two ω -cliques intersecting in $\omega - 2$ vertices. Let x, x' be the vertices in $C \setminus C'$ and let y, y' be those in $C' \setminus C$. Suppose that x is adjacent to y. Then C and $(C \setminus \{x'\}) \cup \{y\})$ are two ω -cliques intersecting in $\omega - 1$ vertices, contradicting Lemma 19. By symmetry we may therefore assume there is no edge between $\{x, x'\}$ and $\{y, y'\}$. The case $\Delta = 5$ gives us the most difficulty by far, so we deal with it separately.

Case 1: $\Delta \geq 6$.

We construct the graph G' from G by identifying x, y and x', y' into two new vertices z and z', respectively, and deleting $C \cap C'$. Clearly $\Delta(G') \leq \Delta(G)$. If G' contains a Δ -clique, then since z and z' have degree at most 5, we have $\Delta = 6$, and furthermore the Δ -clique must contain both z and z'. Thus there is a set of four vertices C'' forming a 6-clique with z and z'. This means there must be eight edges between $\{x, x', y, y'\}$ and C'' in G.

If any vertex in C'' has a neighbour outside of $\{x, x', y, y'\}$ then C'' is a clique cutset in G, contradicting the fact that every proper induced subgraph of G is fractionally $(\Delta - \epsilon)$ -colourable. Thus $V(G) = V(C) \cup V(C') \cup V(C'')$. Further, $(N(x) \cup N(y)) \cap V(C'') = V(C'')$ and $(N(x') \cup N(y')) \cap V(C'') = V(C'')$. If x and x' have the same two neighbours in C'' then G is the graph $(C_5 \boxtimes K_3) - 4v$ shown in Figure 2, contradicting the assumption that $\chi_f(G) > \Delta - \frac{1}{2}$. Thus x and y' have a common neighbour in C''. We may safely switch the roles of y and y' in this case to ensure that $\omega(G') < \omega(G)$.

It now follows from the minimality of G that there exists a fractional $(\Delta - \epsilon)$ -colouring κ of G'. By unidentifying x, y and x', y', we may think of κ as a fractional colouring of $G \setminus (C \cap C')$ where $\kappa[x] = \kappa[y]$ and $\kappa[x'] = \kappa[y']$. We now extend κ to a $(\Delta - \epsilon)$ -colouring of G. We have $\{v \in V(G) : |\kappa(v)| < 1\} = V(C \cap C')$. Further, for each $v \in V(C \cap C')$, $|\alpha(v)| \geq \Delta - \epsilon - 2 \geq \omega - 2$. Thus applying Lemma 3 gives the extension of κ to G, a contradiction.

Case 2: $\Delta = 5$.

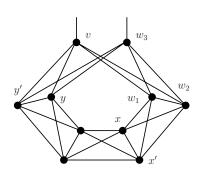
We construct G' as in the previous case. If G' has a fractional $(\Delta - \epsilon)$ -colouring, we reach a contradiction as before. Otherwise, it must be the case that G' contains a Δ -clique or $C_5 \boxtimes K_2$. To deal with these cases we prove four claims.

Our first claim is that no vertex in $G \setminus (C \cup C')$ has a neighbour in both $\{x, x'\}$ and $\{y, y'\}$. To prove this, assume for a contradiction that x and y have a common neighbour $w \notin C \cup C'$. Let $G'' = G \setminus (C \cup C')$. By the minimality of G there exists a fractional $(\Delta - \epsilon)$ -colouring κ of G'' that we now extend to a fractional colouring of G. We do so in two steps, first colouring $\{x, y, x', y'\}$.

Since x and y have a common neighbour plus at most one other coloured neighbour each, we have $|\alpha(x) \cap \alpha(y)| \geq \Delta - \epsilon - 3$. On the other hand, each of x' and y' has at most two coloured neighbours, so $|\kappa[N(x') \cup N(y')]| \leq 4$. We choose $\kappa[x] = \kappa[y]$ from $\alpha(x) \cap \alpha(y)$ maximizing its intersection with $\kappa[N(x') \cup N(y')]$, so that after colouring x and y we still have $|\kappa[N(x') \cup N(y')]| \leq 4$. This means that $|\alpha(x') \cap \alpha(y')| \geq 1 - \epsilon$ so we may choose colours for x' and y' so that $|\kappa[x'] \cap \kappa[y']| \geq 1 - \epsilon$. This ensures that $|\kappa[\{x, y, x', y'\}]| \leq 2 + \epsilon$.

It remains to extend the colouring to the vertices in $C \cap C'$. For each vertex $v \in V(C \cap C')$, $|\alpha(v)| \geq \Delta - \epsilon - (2 + \epsilon) \geq \omega - 2$. Applying Lemma 3, we find a fractional $(\Delta - \epsilon)$ -colouring of G, a contradiction. This proves the first claim, so we may henceforth assume no vertex in $G \setminus (C \cup C')$ has a neighbour in both $\{x, x'\}$ and $\{y, y'\}$.

Our second claim is that G does not contain an edge cut of size at most two. For if it



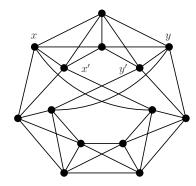


Figure 4: Left: If G contains $(C_5 \boxtimes K_2) - e$, we can easily reduce. Right: Reducing on the six top vertices renders G' isomorphic to $C_5 \boxtimes K_2$.

does, we can take a fractional $(\Delta - \epsilon)$ -colouring of either side of this cut. The edges of the cut have colour weight at most four on their endpoints, and since $\Delta - \epsilon > 2 \cdot 2$, we can safely merge the $(\Delta - \epsilon)$ -colouring of either side of the cut into a fractional $(\Delta - \epsilon)$ -colouring of G, a contradiction. This proves the second claim.

Our third claim is that G' does not contain a Δ -clique. Suppose it does; we now investigate the structure of G. In $G \setminus (C \cup C')$ there is an $\omega - 1$ clique C'', each vertex of which is complete (in G) to either $\{x, x'\}$ or $\{y, y'\}$, since no vertex has neighbours in both $\{x, x'\}$ and $\{y, y'\}$ (by the first claim). Since |C''| = 3, we may assume that x and x' have two common neighbours w_1 and w_2 in C'', and y and y' have a common neighbour w_3 in $C'' \setminus \{w_1, w_2\}$. Call the neighbours of y and y' in $G \setminus (C' \cup C'')$ v and v' respectively, if these vertices exist. We assume v and v' exist, as adding them as pendant vertices does not affect the proof adversely. Let G'' be the graph obtained from $G \setminus (C \cup C' \cup C'')$ by adding the edge vv' if possible (v and v' may not be two distinct vertices, or may already be adjacent). This construction does not create a Δ -clique in G'' since no pair of cliques in G intersects in $\omega - 1$ vertices by Lemma 19. Bearing in mind that $\Delta = 5$, G'' cannot contain a copy of $C_5 \boxtimes K_2$, since the existence of $(C_5 \boxtimes K_2) - e$ in G would violate the second claim. Therefore the minimality of G guarantees that G'' has a fractional $(\Delta - \epsilon)$ -colouring κ . We extend in two cases based on whether or not $|\{v, v'\}| = 2$.

Note that if $|\{v,v'\}| = 1$, we may assume one of w_1, w_2 is nonadjacent to v, say w_1 is nonadjacent to v, otherwise G contains a copy of $(C_5 \boxtimes K_2) - e$, violating the second claim (see Figure 4 (left)). Now assume $|\{v,v'\}| \leq 1$. We recolour v (if it exists) such that $|\kappa[v] \cap (\alpha(w_1) \cup \alpha(w_2))| \geq 1 - \epsilon$. This is possible because $|\alpha(v)| \geq 2 - \epsilon$ and $|\alpha(w_1)| \geq 4 - \epsilon$, so the intersection of these two sets is at least $(6 - 2\epsilon) - (5 - \epsilon) = 1 - \epsilon$. Now we may easily extend κ by colouring w_1 such that $|\kappa[v] \cap \kappa[w_1]| \geq 1 - \epsilon$. Next we extend κ by colouring w_2 and w_3 , which we can do greedily since each of these vertices has at most three neighbours in $G \setminus (C \cup C')$. Now it remains to colour $C \cup C'$. Since $|\kappa[\{v, w_1, w_2, w_3\}]| \leq 4 - (1 - \epsilon)$, there is weight $\frac{4}{3}$ of colour we can use on both $\{x, x'\}$ and $\{y, y'\}$. Since each vertex in $\{x, x', y, y'\}$ has only three neighbours in $G \setminus (C \cap C')$, we can extend κ to a colouring of $G \setminus (C \cap C')$ such that $|\kappa[\{x, x'\}] \cap \kappa[\{y, y'\}]| \geq \frac{4}{3}$. After doing this we can easily extend κ to a fractional $(5 - \epsilon)$ -colouring of G by applying Lemma 3, a contradiction.

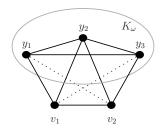


Figure 5: A bump.

Now we handle the case $|\{v,v'\}| = 2$, starting with a fractional $(5 - \epsilon)$ -colouring of G'' which we take as a partial coloring of G. We begin by extending κ by colouring w_3 such that $\kappa[w_3] \subset \kappa[\{v,v'\}]$, which is possible because $\kappa[v]$ and $\kappa[v']$ are disjoint (and w_3 is adjacent to at most one of v and v', since it is adjacent to w_1, w_2, y and y'). We now extend κ by colouring w_1 and w_2 in any way, which we can do greedily. At this point, we have $|\alpha(y)| \geq \frac{8}{3}$, $|\alpha(y')| \geq \frac{8}{3}$, and $|\alpha(y) \cup \alpha(y')| \geq \frac{11}{3}$. Therefore $|\alpha(y) \setminus \kappa[\{w_1, w_2\}]| \geq \frac{2}{3}$, $|\alpha(y') \setminus \kappa[\{w_1, w_2\}]| \geq \frac{2}{3}$, and $|(\alpha(y) \cup \alpha(y')) \setminus \kappa[\{w_1, w_2\}]| \geq \frac{5}{3}$. We may therefore give y weight $\frac{2}{3}$ of colour not in $\kappa[\{w_1, w_2\}]$, and give y' weight $\frac{2}{3}$ of colour in $\kappa[\{w_1, w_2\}]$, then finish colouring y and y' greedily, since each has at most three neighbours in $G \setminus C$. It follows that $|\kappa[\{w_1, w_2\}] \cap \kappa[\{y, y'\}]| \leq \frac{2}{3}$, so we can extend κ by colouring $\{x, x'\}$ such that $|\kappa[\{w_1, w_2\}] \cap \kappa[\{x, x'\}]| \geq \frac{4}{3}$. We can now extend κ to a fractional $(\Delta - \epsilon)$ -colouring of G by applying Lemma 3 as in the previous case. This contradiction proves the third claim.

Our fourth claim, which is sufficient to complete the proof, is that G' does not contain $C_5 \boxtimes K_2$. If it does, there must be four vertices w, w', v, and v' such that in G', $\{w, w', z, z'\}$ and $\{v, v', z, z'\}$ are cliques. Each of w, w', v, and v' therefore has two neighbours in $\{x, x', y, y'\}$. By the first claim, there are two cases, by symmetry: w and w' are adjacent to both x and x', or w and v are adjacent to both x and x'. In the first case, the component of G containing G is isomorphic to $G_7 \boxtimes G_2 = \frac{14}{3}$. In the second case, the component of G containing G is isomorphic to the graph shown in Figure 4 (right). Observe that the outer seven vertices induce G_7 , as do the inner seven vertices. Therefore $\chi_f(G) \leq 2\chi_f(G_7) = 5 - \frac{1}{3}$, a contradiction. This completes the proof of the lemma.

7.2 Part (ii)

Our approach to proving Lemma 17 involves reducing G to a smaller graph G'. Either G' is fractionally $(\Delta - \epsilon)$ -colourable by minimality, in which case we finish easily, or G' contains a K_{Δ} or $C_5 \boxtimes K_2$ (when $\Delta = 5$), in which case we proceed on a case-by-case basis.

To simplify things, we first need to prove a couple of lemmas that exclude induced subgraphs of G.

Definition 1. Suppose we have a set $Y = \{y_1, y_2, y_3\}$ of vertices in a maximum clique C, and two adjacent vertices v_1 and v_2 such that $N(v_1) \cap Y = \{y_1, y_2\}$ and $N(v_2) \cap Y = \{y_2, y_3\}$. Then we say that the set $X = C \cup \{v_1, v_2\}$ is a bump (see Figure 5).

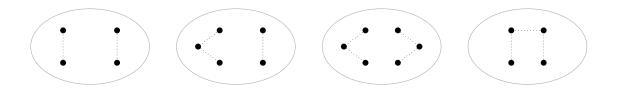


Figure 6: Configurations of edges missing from a K_{Δ} that are forbidden for, respectively, $\Delta \geq 5$ (Lemma 22), $\Delta \geq 6$ (Lemma 23), $\Delta \geq 7$ (Lemma 24), and $\Delta \geq 7$ (Lemma 25).

Lemma 21. G does not contain a bump.

Proof. Suppose to the contrary that G contains a bump X. To reach a contradiction we take a fractional $(\Delta - \epsilon)$ -colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$ -colouring of G as follows.

First, extend κ by colouring v_1 and y_3 with the same set of colours. This is possible because v_1 has at most $\Delta - 3$ neighbours in G', and y_3 has at most one neighbour in G', so $|\alpha(v_1) \cap \alpha(y_3)| \ge \Delta - \epsilon - (\Delta - 3) - 1 > 1$.

Next we extend κ by giving v_2 and y_1 common colour of total weight $\frac{1}{2}$, and leaving them only partially coloured. This is possible because at this point, v_2 has at most $\Delta - 1$ coloured neighbours, and y_1 has at most 3 coloured neighbours, but both are adjacent to v_1 and y_3 . Therefore $|\kappa[N(y_1) \cup N(v_2)]| \leq \Delta - 2 + 1 = \Delta - 1$, and so $|\alpha(y_1) \cap \alpha(v_2)| \geq 1 - \epsilon \geq \frac{1}{2}$.

At this point observe that $|\kappa[Y]| = \frac{3}{2} \leq (\Delta - \epsilon) - 2 - (\Delta - 4)$, so we may now greedily extend κ by colouring the $\Delta - 4$ vertices in $C \setminus Y$, since each of these has at most two coloured neighbours in G'. All that remains is to complete the colouring of v_2 , y_1 , and y_2 . First we finish colouring y_1 ; we can do this greedily because at this point $|\kappa[N(y_1)]| \leq \Delta - 2$, since y_2 is uncoloured and $\kappa[v_1] = \kappa[y_3]$. Next we greedily finish colouring v_2 , which again we can do because at this point $|\kappa[N(v_2)]| \leq \Delta - 2$, since y_2 is uncoloured and $\kappa[v_1] = \kappa[y_3]$.

Finally we must extend to y_2 , which we can do greedily: since $\kappa[v_1] = \kappa[y_3]$ and $|\kappa[v_2] \cap \kappa[y_1]| \geq \frac{1}{2}$, $|\kappa[N(y_2)]| \leq \Delta - \frac{3}{2}$, so $|\alpha(y_2)| \geq \frac{3}{2} - \epsilon \geq 1$. Thus G is fractionally $(\Delta - \epsilon)$ -colourable, a contradiction.

We already know, thanks to Lemma 16, that K_{Δ} minus an edge cannot appear in G. But given restrictions on Δ , we can forbid other subgraphs arising as K_{Δ} minus a small number of edges. We use variations of the approach for bumps: we extend a partial fractional colouring of the graph by leaving a set of vertices to the end, then finishing greedily, having already given their neighbourhoods lots of repeated colour.

Lemma 22. G cannot contain K_{Δ} minus a matching of size two.

Proof. Suppose to the contrary that G contains a subgraph X on Δ vertices, with vertices $v_1, v_2, v_3, v_4 \in V(X)$ such that the non-edges of G[X] are exactly $\{v_1v_2, v_3v_4\}$. We first consider the case where $\Delta \geq 6$. We begin with a fractional $(\Delta - \epsilon)$ -colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$ -colouring of G as follows.

First, we extend κ by colouring v_1 and v_2 with the same set of colours. Each of v_1, v_2 has at most two coloured neighbours in G', and so $|\alpha(v_1) \cap \alpha(v_2)| \geq (\Delta - \epsilon) - 4 \geq 1$. Thus it is possible to choose $\kappa[v_1] = \kappa[v_2]$.

Next, we extend κ by colouring v_3 and v_4 in such a way that $\kappa[v_3] \cap \kappa[v_4] \geq \frac{1}{2}$. Each of v_3, v_4 has at most two coloured neighbours in G' as well as neighbours v_1 and v_2 which have the same set of colours, and so $|\alpha(v_3) \cap \alpha(v_4)| \geq (\Delta - \epsilon) - 5 \geq 1 - \epsilon \geq \epsilon \geq \frac{1}{2}$. Thus we may choose $\kappa[v_3]$ and $\kappa[v_4]$ as claimed. We now have $|\kappa[v_1, v_2, v_3, v_4]| \leq \frac{5}{2}$.

It remains to colour the $\Delta-4$ vertices in $X\setminus\{v_1,v_2,v_3,v_4\}$. We can do this easily because for each such vertex, the total weight of colours appearing twice in its neighbourhood is at least $1+\epsilon$. Therefore as we colour greedily, the weight on the closed neighbourhood will never exceed $\Delta-\epsilon$. Thus G is fractionally $(\Delta-\epsilon)$ -colourable, a contradiction.

Now we consider the case where $\Delta = 5$. Let u denote the neighbour of v_5 outside X; if u does not exist, we can add a pendant vertex to v_5 and call it u, for the sake of our argument. We begin with a fractional $(\Delta - \epsilon)$ -colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$ -colouring of G as follows, considering three subcases based on $f = |\kappa[N(v_3)] \cap \kappa[N(v_4)]|$.

If $f < \frac{1}{3}$, we give v_1 and v_2 common colour of weight $\frac{2}{3}$, leaving them only partially coloured. We then put weight $\frac{2}{3} - |\kappa[u] \cap \kappa[v_1]|$ of colour from $\kappa[u] \setminus \kappa[v_1]$ onto $\{v_3, v_4\}$ (putting none on both), which is possible because there is at least $\frac{2}{3}$ colour in $\kappa[u] \cap (\alpha(v_3) \cup \alpha(v_4))$. We now extend κ to completely colour v_1 and v_2 , which is possible because at this point $|\kappa[\{v_3, v_4\}]| \leq \frac{2}{3}$. Next we extend κ to completely colour v_3 and v_4 , which is possible because at this point v_5 is uncoloured and $|\kappa[v_1] \cap \kappa[v_2]| \geq \frac{2}{3}$. Finally we extend the colouring to include v_5 , which is possible because $|\kappa[v_1] \cap \kappa[v_2]| \geq \frac{2}{3}$ and $|\kappa[u] \cap \kappa[\{v_1, v_2, v_3, v_4\}]| \geq \frac{2}{3}$. So we may assume $f \geq \frac{1}{3}$.

If $f < \frac{2}{3}$, we give v_1 and v_2 common colour of weight $\frac{2}{3}$, leaving them only partially coloured. We then give v_3 and v_4 common colour of weight $\frac{1}{3}$, so at this point the total colour appearing on $N(v_3) \cup N(v_4)$ is at most $4 - \frac{1}{3} + \frac{2}{3} \leq 5 - \epsilon - \frac{1}{3}$ (because $f \geq \frac{1}{3}$). We then give $\{v_3, v_4\}$ enough colour from $\kappa[u]$ so that $|\kappa[u] \cap \kappa[\{v_1, v_2, v_3, v_4\}]| \geq \frac{1}{3}$; this is possible because $f < \frac{2}{3}$, and so $(\alpha(v_3) \cup \alpha(v_4) \cup \kappa[v_1, v_2]) \geq 4$. We may extend to finish colouring v_1, v_2, v_3, v_4 greedily, since v_5 is uncoloured and both $\kappa[v_1] \cap \kappa[v_2]$ and $\kappa[v_3] \cap \kappa[v_4]$ have size at least $\frac{1}{3}$. Finally we can extend the colouring to v_5 , since the weight of colours appearing at least twice on $N(v_5)$ is at least $\frac{4}{3} \geq 1 + \epsilon$. So we may assume $f \geq \frac{2}{3}$.

This final case is easiest: we give v_1 and v_2 common colour of weight $\frac{2}{3}$, then give v_3 and v_4 common colour of weight $\frac{2}{3}$, then extend to completely colour $\{v_1, v_2, v_3, v_4\}$ greedily, then extend to v_5 greedily. The details are as in the previous cases, but easier. Thus G is fractionally $(\Delta - \epsilon)$ -colourable, a contradiction.

Lemma 23. If $\Delta \geq 6$, G cannot contain K_{Δ} minus the edges of vertex disjoint paths, one of length one and one of length two.

Proof. Suppose to the contrary that $\Delta \geq 6$ and G contains a subgraph X on Δ vertices, with vertices $v_1, v_2, v_3, v_4, v_5 \in V(X)$ such that the non-edges of G[X] are exactly $\{v_1v_2, v_1v_3, v_4v_5\}$. We begin with a fractional $(\Delta - \epsilon)$ -colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$ -colouring of G as follows.

First we give v_1 and v_2 weight $\frac{1}{2}$ of common colour, leaving them only partially coloured. This is possible because v_1 and v_2 have, in total, at most $5 \le \Delta - \epsilon - \frac{1}{2}$ coloured neighbours in $G \setminus X$. Next we give v_4 and v_5 the same colour, which is possible because at this point the weight of colour on their neighbourhoods totals at most $2 + 2 + \frac{1}{2} \le \Delta - \epsilon - 1$, since they are both adjacent to v_1 and v_2 . Next we extend κ to complete the colouring of v_1 , v_2 , v_3 , v_4 , and v_5 greedily, which we can do since each of these vertices has at least $\frac{1}{2}$ weight of

repeated colour in its neighbourhood, and at least one uncoloured neighbour in X. Finally we extend greedily to the remaining vertices of X, which we can do since each such vertex is adjacent to v_1 , v_2 , v_4 , and v_5 , and therefore has repeated colour of weight at least $\frac{3}{2}$ in its neighbourhood. Thus G is fractionally $(\Delta - \epsilon)$ -colourable, a contradiction.

Lemma 24. If $\Delta \geq 7$, G cannot contain K_{Δ} minus the edges of two vertex-disjoint paths of length two.

Proof. Suppose to the contrary that $\Delta \geq 7$ and G contains a subgraph X on Δ vertices, with vertices $v_1, \ldots, v_6 \in V(X)$ such that the non-edges of G[X] are exactly $\{v_1v_2, v_2v_3, v_4v_5, v_5v_6\}$. We begin with a fractional $(\Delta - \epsilon)$ -colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$ -colouring of G as follows.

First we give v_1 and v_2 the same colour. Next we give v_4 and v_5 weight $\frac{1}{2}$ of common colour. We then extend greedily to complete the colouring of v_3 , v_4 , v_5 , and v_6 , then extend greedily to complete the colouring of G. We can do this because, similar to Lemma 22, v_1 and v_2 together have at most 5 neighbours in $G \setminus X$, as do v_4 and v_5 .

Lemma 25. If $\Delta \geq 7$, G cannot contain K_{Δ} minus the edges of a three-edge path.

Proof. Suppose to the contrary that $\Delta \geq 7$ and G contains a subgraph X on Δ vertices, with vertices $v_1, v_2, v_3, v_4 \in V(X)$ such that the non-edges of G[X] are exactly $\{v_1v_2, v_2v_3, v_3v_4\}$. We begin with a fractional $(\Delta - \epsilon)$ -colouring κ of $G' = G \setminus X$ and extend it to a $(\Delta - \epsilon)$ -colouring of G as follows.

We first extend κ by colouring v_1 and v_2 with the same set of colours. Since v_1 has at most two coloured neighbours in G' and v_2 has at most three coloured neighbours, we have $|\alpha(v_1) \cap \alpha(v_2)| \geq (\Delta - \epsilon - 5) \geq 2 - \epsilon \geq 1$, and so from this set we choose $\kappa[v_1] = \kappa[v_2]$.

We next extend κ by giving v_3 and v_4 weight $\frac{1}{2}$ of common colour, which is possible because v_3 and v_4 together have at most 5 neighbours in $G \setminus X$, and weight 1 of colour appearing in their neighbourhood in X. We may then extend greedily to complete the colouring of v_3 and v_4 . Now since the weight of colours appearing twice in X is at least $\frac{3}{2}$, we may extend the colouring to the rest of X greedily. Thus G is fractionally $(\Delta - \epsilon)$ -colourable, a contradiction.

We are now ready to prove Lemma 17.

Proof of Lemma 17. Suppose G contains a clique C of size $\Delta - 1$ and a vertex w outside C with at least two neighbours in C. Call the vertices in C v_1, \ldots, v_{ω} , and suppose w is adjacent to v_1 and v_2 . Let the neighbours of v_1 and v_2 outside $C \cup \{w\}$ be denoted y and z, if they exist. We may actually assume they exist, since adding them as pendant vertices does not affect our proof adversely.

We choose w, v_1 , and v_2 such that if possible, w is in a K_{ω} , and subject to that, if possible, v_1 and v_2 do not have a common neighbour outside $C \cup \{w\}$, i.e. $y \neq z$. We construct one of two reduced graphs from G, depending on whether or not y and z are distinct.

Case 1: $y \neq z$.

Let p and p' be the neighbours of v_3 outside C. Subject to whether or not we can choose w to be in a K_{ω} and whether or not we can choose v_1 and v_2 such that $y \neq z$, we choose w,

 v_1 , v_2 , and v_3 such that w and v_3 are nonadjacent and $|\{p,p'\}\cap\{y,z\}|$ is minimum. Choose v_4 nonadjacent to w as well, noting that this is possible since by Lemma 16, w has at least two non-neighbours in C. Construct the graph G_1 from G-C by making w adjacent to w and making w adjacent to w and w. Clearly w w w w definition.

We claim that G_1 is not fractionally $(\Delta - \epsilon)$ -colourable; if it is then we extend a $(\Delta - \epsilon)$ -colouring κ of G_1 to a colouring of G as follows. First, we extend κ by giving v_3 the same colours as w. Since all the coloured neighbours of v_3 are adjacent to w in G_1 , we have $\kappa[w] \subseteq \alpha(v_3)$, and so we may choose $\kappa[v_3] = \kappa[w]$. We now greedily extend to the vertices v_4, \ldots, v_{ω} , which is possible because v_1 and v_2 remain uncoloured; it now remains to colour v_1 and v_2 . Since $\kappa[v_3] = \kappa[w]$, it follows that $|\alpha(v_1)| \ge (\Delta - \epsilon) - (\Delta - 3) - 1 \ge 2 - \epsilon$ and $|\alpha(v_2)| \ge 2 - \epsilon$. Further, since $|\kappa[y] \cap \kappa[z]| = 0$ we have $|\kappa[N(v_1)] \cap \kappa[N(v_2)]| \le \Delta - 3$, and so $|\alpha(v_1) \cup \alpha(v_2)| \ge 2$. Thus we may apply Lemma 3 to extend κ to v_1 and v_2 . It follows that G is fractionally $(\Delta - \epsilon)$ -colourable, a contradiction. This proves the claim.

Therefore by the minimality of G we may assume that either G_1 contains a Δ -clique, or $\Delta = 5$ and G_1 contains a copy of $C_5 \boxtimes K_2$.

We claim that if $\Delta = 5$, G_1 does not contain a copy X of $C_5 \boxtimes K_2$. Suppose to the contrary that adding the edges wp, wp', yz to G yields a copy of $C_5 \boxtimes K_2$. Since G does not contain two intersecting copies of K_4 , X contains two disjoint edges that are not edges of G. It follows that $w, y, z \in V(X)$. Further, since $C_5 \boxtimes K_2$ is 5-regular, wp and wp' both belong to E(X) and further no vertex in V(X) has a neighbour in $G \setminus (X \cup \{v_1, v_2, v_3\})$. Therefore $\{v_1, v_2, v_3\}$ is a clique cutset of size three, contradicting the fact that every proper induced subgraph of G is fractionally $(\Delta - \epsilon)$ -colourable. This proves the claim.

We may now move on to the more complicated task of proving that $\omega(G_1) = \omega$. Suppose G_1 contains a Δ -clique C'.

Our first claim is that $\{w, y, z\} \in C'$ and $yz \notin E(G)$. By Lemma 16, adding a single edge to G cannot create a Δ -clique. It follows that $w \in V(C')$. Suppose that $|\{y, z\} \cap C'| \leq 1$ or that $yz \in E(G)$. Again by Lemma 16, p, p' must be distinct and belong to C'. Now, in G, w has $\omega - 2$ neighbours in the ω -clique C' - w, and so w does not belong to an ω -clique by Lemma 16. On the other hand, v_3 has two neighbours in a ω -clique (namely p and p') and does belong to a maximum clique, contradicting our choice of w. This proves the first claim.

Suppose $|\{p,p'\}\cap\{y,z\}|=2$. We may assume p=y and p'=z. Recall that we have

chosen v_3 so as to minimize $|\{p,p'\}\cap\{y,z\}|$. Each of w,y,z has at most three neighbours in C, since $\{w,y,z\}\in C'$ by the first claim. Furthermore, if w belongs to a K_{ω} in G, then it has only two neighbours in C. If $\Delta=5$, by Lemma 16, v_4 sees neither y nor z (since v_3 sees y and z), contradicting our choice of v_3 . If $\Delta\geq 6$ and w is in a K_{ω} in G, then there is a vertex in $C\setminus N(w)$ that is adjacent to at most one of y,z and nonadjacent to w, contradicting our choice of v_3 . If $\Delta\geq 6$ and w is not in a K_{ω} in G, then either there is a vertex in $C\setminus N(w)$ adjacent to at most one of y,z, contradicting our choice of v_3 , or else every vertex in $C\setminus N(w)$ sees both of y,z. In this latter case we can relabel: relabel y to w', v_1 to v'_1 , v_3 to v'_2 , w to y', z to z', and v_4 to v'_3 . Since v_4 was chosen to be nonadjacent to w, we have a labelling that contradicts the minimality of $|\{p,p'\}\cap \{y,z\}|$. This proves the second claim. We may now assume that y=p and that $|\{y,z,p'\}|=3$.

Our third claim is that $p' \in C'$. Suppose to the contrary that $p' \notin C'$. Then in G, w has $\omega - 1$ neighbours in V(C'). Thus w belongs to an ω -clique in G - C, and therefore has exactly two neighbours in C. Also, since $wz \in E(G)$ and G does not contain a bump by Lemma 21, v_2 is the only neighbour of z in C. Further, y belongs to an $(\omega - 1)$ -clique in G - C and has at most three neighbours in C, and at most two if $\Delta = 5$. Therefore, there is a vertex in C with no neighbour in $\{w, y, z\}$, contradicting our choice of v_3 . This proves the third claim.

We now know that y = p and $\{w, y, p', z\} \subseteq V(C')$. Since G does not contain a bump and since $wz \in E(G)$, we know that z has only one neighbour in C. Therefore by our choice of v_3 minimizing $|\{p, p'\} \cap \{y, z\}|$, every vertex in C is adjacent to w or y. Thus $\Delta = 6$ and each of w and y has three neighbours in C.

To complete the proof, we now fractionally colour G directly, beginning with a fractional $(\Delta - \epsilon)$ -colouring κ of $G - C - \{w, y\}$. We first extend κ by colouring w and v_3 with the same set of colours. Since v_3 and w together have at most four coloured neighbours, we have $|\alpha(v_3) \cap \alpha(w)| \ge (6 - \epsilon) - 4 \ge 1$, and so we may choose $\kappa[v_3] = \kappa[w]$.

Next we extend κ by colouring y and v_2 so that $|\kappa[y] \cap \kappa[v_2]| \geq \frac{1}{2}$, which is possible because at this point, $|\kappa[N(y) \cup N(v_2)]| = 5$, since the only coloured vertices in $N(y) \cup N(v_2)$ are C' - y and v_3 (which has the same colour as w). We now have $\kappa[\{v_2, v_3, w, y\}] \leq \frac{5}{2}$.

Next we extend κ by colouring v_4 and v_5 . Since each of v_4, v_5 is adjacent to either w or y, we have $|\kappa[N(v_4)]| \leq \frac{7}{2}$ and $|\kappa[N(v_5)]| \leq \frac{7}{2}$. Thus $|\alpha(v_4)|, |\alpha(v_5)| \geq 2$ and so we may apply Lemma 3 to choose $\kappa[v_4]$ and $\kappa[v_5]$ greedily.

Finally we greedily extend κ to v_1 . We have $\kappa[N(v_1)] \leq \frac{9}{2}$ since v_1 is adjacent to v_2 , v_3 , w, and y. Applying Lemma 3, we may choose $\kappa[v_1]$ from $\alpha(v_1)$. Thus G is fractionally $(\Delta - \epsilon)$ -colourable, a contradiction.

This completes the proof of Case 1.

Case 2: y = z and w is in a K_{ω} in G.

In this case, we know that we can choose w to be in a maximum clique, but we cannot make such a choice of w, v_1, v_2 for which $y \neq z$. Since w is in a maximum clique, it has only two neighbours in C. Therefore we may choose v_3 and v_4 to be nonadjacent to both w and y, since Lemma 16 implies that y has at least two non-neighbours in C. But we need further conditions on our vertex labelling. Denote by p, p' and q, q' the neighbours of v_3 and v_4 outside C, respectively. We choose a labelling of the vertices satisfying the following conditions:

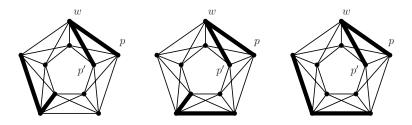


Figure 7: Three ways to form $C_5 \boxtimes K_2$ in Case 2.

 $L1 \ w$ is in a maximum clique. Subject to this condition,

 $L2 \ y$ is in a maximum clique if possible. Subject to this condition,

L3 v_3 and v_4 are not adjacent to w nor to y. Subject to satisfying the previous conditions,

L4 v_3 is chosen so that $|N(p) \cap N(p') \cap N(y)|$ is maximized.

Construct the graph G_2 from G - C by making w adjacent to p and p' and making y adjacent to q and q'. Clearly $\Delta(G_1) \leq \Delta$.

We claim that G_2 is not fractionally $(\Delta - \epsilon)$ -colourable; if it is then we extend a $(\Delta - \epsilon)$ -colouring κ of G_2 to a colouring of G as follows. We begin by extending κ to colour v_3 with the same colour as w. Since v_3 's only coloured neighbours are p and p', which are adjacent to w in G_2 , we may choose $\kappa[v_3] = \kappa[w]$. We now extend κ to the remaining vertices in C. By the choice of $\kappa[v_3]$, we have $|\alpha(v_1)|, |\alpha(v_2)| \geq \Delta - \epsilon - 2$. Since each of the $\Delta - 4$ other uncoloured vertices has at most three coloured neighbours we find $|\alpha(v_i)| \geq \Delta - \epsilon - 3$ for $4 \leq i \leq \omega$. Third, the edges yq, yq' in G_2 ensure that $|\alpha(v_1) \cup \alpha(v_4)|, |\alpha(v_2) \cup \alpha(v_4)| \geq \Delta - \epsilon - 1$. Applying Lemma 3 to $C \setminus \{v_3\}$ (which has size $\Delta - 2$), we find a $(\Delta - \epsilon)$ -colouring of G, a contradiction. This proves the claim.

Therefore by the minimality of G we may assume that either G_2 contains a Δ -clique, or $\Delta = 5$ and G_2 contains a copy of $C_5 \boxtimes K_2$. Let $F = E(G_2) \setminus E(G) \subseteq \{wp, wp', yq, yq'\}$. Let F_w and F_y denote the edges incident to w and y in G_2 , respectively.

We claim that if $\Delta = 5$, G_2 does not contain a copy X of $C_5 \boxtimes K_2$. Suppose to the contrary that adding the edges wp, wp', yq, yq' to G creates a copy of $C_5 \boxtimes K_2$. Since G does not contain two intersecting copies of K_4 , X contains at least two vertex-disjoint edges that are not edges of G. It follows that $w, y \in V(X)$. Further, since $C_5 \boxtimes K_2$ is 5-regular, $\{p, p', q, q'\} \subseteq V(X)$ and F contains all four edges wp, wp', yq, yq'. Since w belongs to a K_4 in G, p and p' must form the intersection of two K_4 s in X. Since G does not contain a pair of intersecting K_4 s, q and q' do not form the intersection of two K_4 s in X, and moreover, y cannot be in $N(w) \cup N(p) \cup N(p')$ in X. Hence y does not belong to a 4-clique in G. See Figure 7, where y is the bottom left vertex. Observe that v_3 belongs to a maximum clique in G, and its neighbours p and p' belong to another maximum clique. Further, p and p' have a common neighbour in a third maximum clique. Since y is not in a maximum clique, this contradicts L2 in our choice of w and y, and proves the claim.

We now move on to the task of proving that $\omega(G_2) = \omega$. Suppose G_2 contains a Δ -clique C'.

Our first claim is that $|E(C') \cap F_y| \ge 1$ and $|E(C') \cap F_w| \ge 1$. We can see that

 $|E(C') \cap F_y| \ge 1$, otherwise $C' \setminus w$ is a maximum clique in G intersecting a maximum clique containing w, contradicting Lemma 16.

Suppose now that $|E(C') \cap F_w| = 0$. By the same argument, y cannot belong to a maximum clique in G. We know that C' must contain at least two edges in F, so $yq, yq' \in F \cap E(C')$. Therefore $|N(q) \cap N(q') \cap N(y)| \ge \omega - 2$ and these vertices, along with y form an $(\omega - 1)$ -clique. Further $qq' \in E(G)$, and so q, q' belong to an ω -clique in G.

If $|\{p,p'\} \cap \{q,q'\}| = 1$, then we can relabel v_4 as w'; since v_4 is in a K_{ω} in G and has two neighbours in $C' \setminus \{y\}$, one but not both of which are adjacent to v_3 , contradicting the fact that we are not in Case 1. If $|\{p,p'\} \cap \{q,q'\}| = 2$, this contradicts condition L2 in our choice of labelling, since G contains two vertices in the K_{ω} C having two neighbours in common in a disjoint K_{ω} $C' \setminus \{y\}$. Therefore $|\{p,p'\} \cap \{q,q'\}| = 0$.

Note that by L3 we have chosen v_3 , v_4 nonadjacent to both w and y. In particular this means that $\{p,p'\}$ and $\{q,q',y\}$ are disjoint. By L4, we know $|N(p) \cap N(p') \cap N(y)| \ge |N(q) \cap N(q') \cap N(y)| \ge \omega - 2$. In particular this set must intersect $N(q) \cap N(q') \cap N(y)$. But since $\{p,p'\}$ and $\{q,q',y\}$ are disjoint, if $\{p,p'\} \cap C' = \emptyset$, there is a vertex of degree $\Delta + 1$, a contradiction. Therefore we may assume without loss of generality that $p \in C' \setminus \{q,q',y\}$. But then in G, p is adjacent to every other vertex in C', so its only other neighbour is v_3 . Since y is nonadjacent to v_3 , q, and q', $N(p) \cap N(p') \cap N(y) \subseteq C' \setminus \{q,q',y,p\}$, contradicting the fact that its size is at least $\omega - 2$. This proves the first claim.

Our second claim is that w and y belong to an ω -clique W in G. As a consequence, since this makes $\{w, y, v_1, v_2\}$ a clique, Lemma 16 tells us that $\Delta \geq 6$. To prove this, let W be the maximum clique in G containing w, and note that W is the closed neighbourhood of w in G - C. By the first claim, $w \in V(C')$ and $y \in V(C')$. By the choice of $v_3, y \notin \{p, p'\}$ and $w \notin \{q, q'\}$. It follows that $wy \in E(G)$, and so $y \in W$. This proves the second claim.

Our third claim is that the only edges between C and W are between $\{v_1, v_2\}$ and $\{w, y\}$. To see this assume otherwise, and denote the vertices of W $\{w, y, w_3, \ldots, w_{\omega}\}$. By the maximum degree, there must exist $3 \le i, j \le \omega$ such that v_i and w_j are adjacent.

To reach a contradiction we extend a fractional $(\Delta - \epsilon)$ -colouring κ of G - W - C as follows. First assign w and v_i the same colour, which is possible because together these vertices have at most weight 1 of colour on (the union of) their neighbourhoods. Then for some $i' \notin \{1,2,i\}$, give y and $v_{i'}$ colour $\frac{1}{2}$ in common, leaving them only partially coloured, noting that this is possible because at this point y and $v_{i'}$ have colour at most 1+2=3 on their neighbourhoods (since w and v_i have the same colour). Next we greedily extend to all vertices of $W \setminus \{w, y, w_j\}$, noting that this is possible because all these vertices are adjacent to y and w_j , which together have only $\frac{1}{2}$ colour on them at this point. We then greedily extend to w_j , which is possible because w_j is adjacent to w_j , and v_i , which together have weight $\frac{3}{2}$ colour on them. Next we greedily extend to complete the colouring of all vertices of $(C \cup \{y\}) \setminus \{v_1, v_2\}$, which is clearly possible because v_1 and v_2 are still uncoloured. Finally we extend to v_1 and v_2 , which is possible because both are complete to $\{w, y, v_i, v_{i'}\}$, a set of four vertices with at most $\frac{5}{2}$ colour on them. This contradicts the fact that G is not fractionally $(\Delta - \epsilon)$ -colourable, and proves the third claim.

Our fourth claim is that $\{p, p'\} \cap \{q, q'\} \neq \emptyset$. By the second claim, neither w nor y has any neighbours outside of W in G - C. By the first claim, C' contains an edge in F_w and an edge in F_y ; we may assume without loss of generality that $p \in V(C')$. By the third claim $\{p, p', q, q'\} \cap W = \emptyset$, so G contains no edges between $\{w, y\}$ and $\{p, p', q, q'\}$.

Therefore since C' is a clique in G_2 , yp must be in F, so $p \in \{q, q'\}$. This proves the fourth claim.

Without loss of generality, for the remainder of Case 2 we assume $p \in V(C')$ and p = q. Thus we can also assume that p is adjacent to w_3 and w_4 in W. By the third claim p does not belong to W.

We now complete the proof of Case 2. To do so we fractionally colour G by extending a $(\Delta - \epsilon)$ -colouring κ of $G - W - C - \{p\}$ as follows. We begin to extend κ by assigning $\kappa[w] = \kappa[v_5]$, noting that v_5 may or may not be adjacent to p. This is possible since together these vertices have at most two coloured neighbours. Next we give v_1 and w_5 colour $\frac{1}{2}$ in common, leaving them partially uncoloured. This is possible since at this point $|\kappa[N(v_1) \cup N(w_5)]| \leq 3$. Next, we extend κ by giving w_4 and v_4 the same set of colours, noting that since both are adjacent to p, at this point at most $\frac{7}{2} \leq \Delta - \epsilon - 1$ colour appears on their neighbourhoods, so this is possible. Next, we give w_3 and v_3 common colour $\frac{1}{2}$, noting that both are adjacent to p. Since $\kappa[N(v_3) \cap C] = \kappa[(N(w_3) \cap W) \cup \{v_3\}]$ and $|\kappa[N(v_3)\cap C]|=\frac{5}{2}$, we have $|\alpha(v_3)\cap\alpha(w_3)|\geq (\Delta-\epsilon)-\frac{5}{2}-2\geq 1$. We now greedily extend κ to colour $W - \{w, y, w_3, w_4\}$, which is possible since y and w_3 together have weight $\frac{3}{2}$ not yet coloured. Next we give y and v_3 weight $\frac{1}{2}$ of colour in common and leave them partially uncoloured, which is possible because at this point $|\alpha(y)| \geq \frac{3}{2}$, and $|\kappa[\tilde{N}(v_3)] \setminus \kappa[\tilde{N}(y)]| \leq 1$. We can now greedily extend to $C - \{v_1, \ldots, v_5\}$, since v_1 and v_2 together have weight $\frac{3}{2}$ not yet coloured. Next we can extend to complete the colouring of w_3 , since y and p together have weight $\frac{3}{2}$ not yet coloured. Next we can extend to complete the colouring of y, since v_1 and v_2 together have weight $\frac{3}{2}$ not yet coloured.

Finally we can complete the colouring by extending greedily to complete the colouring of v_1 and v_2 , since each has weight at least $\frac{3}{2}$ of colour appearing twice on its neighbourhood. This completes the proof of Case 2.

This completes the proof of Case 2.

Case 3: y = z and w is not in a K_{ω} in G.

In this case, by the choice of w, there exists no vertex in G belonging to a maximum clique that has two neighbours in a different maximum clique. Also, we know that every pair of vertices in C has either zero or two common neighbours outside of C, for otherwise with a better choice of w, v_1, v_2 we would be in Case 1. Thus $N(w) \cap V(C) = N(y) \cap V(C)$. By Lemma 16, $|V(C) \setminus N(w)| \geq 2$. Again denote by p, p' and q, q' the neighbours of v_3 and v_4 outside C, respectively. We choose v_3 and v_4 from $V(C) \setminus N(w)$ to maximize $|\{p, p', q, q'\}|$. Subject to this, v_3 and v_4 are chosen to maximize $|\{wp, wp', yq, yq'\} \cap E(G)|$. Note that $|\{p, p'\} \cap \{q, q'\}| \in \{0, 2\}$, that $\{w, y\} \cap \{p, p', q, q'\} = \emptyset$, and that in particular, y is nonadjacent to v_4 .

Noting that $w, y \notin \{p, p', q, q'\}$, we construct the graph G_2 from G - C as in Case 2 by making w adjacent to p and p' and making p adjacent to p and p'. As in Case 2, we may assume G_2 is not fractionally $(\Delta - \epsilon)$ -colourable; if it is then we extend a $(\Delta - \epsilon)$ -colouring κ of G_2 to a colouring of G. (Observe that the colouring argument given in Case 2 does not make use of the fact that p belongs to a maximum clique in that case.)

Therefore we may assume that either G_2 contains a Δ -clique, or $\Delta = 5$ and G_2 contains a copy of $C_5 \boxtimes K_2$. As in the previous case, let $F = E(G_2) \setminus E(G) \subseteq \{wp, wp', yq, yq'\}$. Let F_w and F_y denote the edges of F incident to w and y in G_2 , respectively.

We claim that if $\Delta = 5$, G_2 does not contain a copy X of $C_5 \boxtimes K_2$. Suppose to the contrary that adding the edges wp, wp', yq, yq' to G creates a copy of $C_5 \boxtimes K_2$. Since G does not contain two intersecting copies of K_4 , X contains two vertex-disjoint edges of F. It follows that $w, y \in V(X)$, and since $\Delta = 5$, Lemma 16 tells us that w and y are not adjacent. Further, since $C_5 \boxtimes K_2$ is 5-regular, $\{p, p', q, q'\} \subseteq V(X)$ and F contains all four edges wp, wp', yq, yq'. Since w does not belong to a K_4 in G, p and p' do not form the intersection of two K_4 s in X. Likewise, neither do q and q'. Also, if $\{p, p'\} \cap \{q, q'\} \neq \emptyset$ then $|\{p, p'\} \cap \{q, q'\}| = 2$ (since we are not in Case 1), which is impossible because intersection of the neighbourhoods of two nonadjacent vertices in $C_5 \boxtimes K_2$ is the intersection of two K_4 s, a contradiction. Therefore w, y, p, p', q, q' are six distinct vertices.

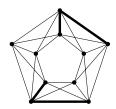


Figure 8: The only way to form $C_5 \boxtimes K_2$ in Case 3. If w is the top vertex, we may instead choose w as the vertex immediately below it to put us in Case 1.

Since exchanging the roles of v_3 and v_4 cannot reduce |F|, G contains no edges from $\{wq, wq', yp, yp'\}$. It follows that $pp' \in E(G)$ and $qq' \in E(G)$. Therefore by symmetry, bearing in mind that w and y are nonadjacent in both G and G_2 , the only possible case is shown in Figure 8. Note here that there is a different choice of w that would put us in Case 1, a contradiction.

We now proceed to prove that $\omega(G_2) < \Delta$. Suppose G_2 contains a Δ -clique C'.

Our first claim is that $|E(C') \cap F_w| \ge 1$ and $|E(C') \cap F_y| \ge 1$. Suppose that $|E(C') \cap F_w| = 0$. Then clearly $y \in V(C')$, and by Lemma 16, both edges yq, yq' belong to E(C'), and $qq' \in E(G)$. But then C' - y is an ω -clique containing two neighbours of v_4 , which also belongs to an ω -clique. This contradicts our choice of w. By a symmetric argument, $|E(C') \cap F_y| \ge 1$. This proves the first claim.

Our second claim is that $wy \in E(G)$ and $\Delta \geq 6$. By the first claim, w and y belong to V(C'). By the choice of v_3 and v_4 , $w, y \notin \{p, p', q, q'\}$. Thus $wy \in E(G)$, and so w, y, v_1, v_2 form a K_4 . If $\Delta = 5$ this contradicts Lemma 16. This proves the second claim.

Our third claim is that $|E(C') \cap F| \ge 3$. Suppose that $|E(C') \cap F| = 2$. By Lemma 22, the two edges in $E(C') \cap F$ do not form a matching, and so they form a two-edge path. By the first claim, one of the edges must be between w and y, contradicting the second claim. This proves the third claim.

Our fourth claim is that $|E(C') \cap F| = 4$. Suppose that $|E(C') \cap F| = 3$. By Lemma 23, at least two pairs of the edges in $E(C') \cap F$ intersect. Since $w, y \notin \{p, p', q, q'\}$ the edges $E(C') \cap F$ do not form a triangle, so they form a three-edge path. By Lemma 25 and the second claim, $\Delta = 6$.

Since $wy \in E(G)$ and by symmetry between w and y and between p and p', we may assume p = q and the edges of the path are p'w, wp, py. Since $|\{p, p'\} \cap \{q, q'\}| \neq 1$, p' = q'

and $pp' \in E(G)$. By the choice of v_3, v_4 maximizing $|\{p, p', q, q'\}|$, v_5 must be complete to $\{p, p'\}$ or to $\{w, y\}$. But then v_5 belongs to two 5-cliques in G, contradicting Lemma 16. This proves the fourth claim.

We now know that $|E(C') \cap F| = 4$. Suppose that the edges in $E(C') \cap F$ form two vertex-disjoint two-edge paths. Then by Lemma 24, $\Delta = 6$. Now $|\{p, p', q, q'\}| = 4$ and so $wq, wq', yp, yp' \in E(G)$. This contradicts the choice of v_3 and v_4 , for reversing their roles would yield |F| = 0.

Since we are in Case 3, the edges in $E(C') \cap F$ therefore form a cycle of length four. It follows that $\{p,p'\}=\{q,q'\}$ and $wy,pp'\in E(G)$. By the choice of v_3 and v_4 maximizing $|\{p,p',q,q'\}|$, each of v_5,\ldots,v_{ω} is complete to either $\{w,y\}$ or $\{p,p'\}$. Therefore by Lemma 16, $\Delta \geq 7$. Since each of w,y,p,p' is adjacent to $\Delta - 3$ vertices of C' in G, each has at most three neighbours in C. Therefore $\Delta = 7$, and G is isomorphic to the graph $(C_5 \boxtimes K_3) - 2v$ pictured in Figure 2. Thus G is indeed fractionally $\frac{13}{2}$ -colourable and thus fractionally $(\Delta - \epsilon)$ -colourable, a contradiction.

This completes the proof of Case 3, and the proof of the lemma.

8 Future directions

We have already given several open problems that are worthy of consideration, namely Conjectures 1 and 2, which propose, respectively, that $f(6) = f(7) = f(8) = \frac{1}{2}$ and that $f(4) \ge f(3)$. We conclude the paper with one more conjecture:

Conjecture 3. Let G be a graph with maximum degree 5 and clique number 4 such that no two 4-cliques intersect and such that no vertex outside any maximum clique C has more than one neighbour in C. Then there is a fractional 4-colouring of the vertices in 4-cliques such that for any vertex v not in a 4-clique, $|\alpha(v)| \geq 1$.

If Conjecture 3 were to hold, our fractional colouring method could be applied to greater effect. In particular, we could easily prove that $f(5) \ge 1/11$ and $f(6) \ge 1/8$. The improvements would be smaller for larger values of Δ .

9 Acknowledgements

The authors are very grateful to the two referees for their thorough, helpful, and speedy reviews.

References

- [1] R. Aharoni, E. Berger, and R. Ziv. Independent systems of representatives in weighted graphs. *Combinatorica*, 27(3):253–267, 2007.
- [2] M. Albertson, B. Bollobas, and S. Tucker. The independence ratio and maximum degree of a graph. In *Proceedings of the 7th Southeastern Conference on Combinatorics*, Graph Theory and Computing, pages 43–50, 1976.

- [3] A. Beutelspacher and P. Hering. Minimal graphs for which the chromatic number equals the maximal degree. *Ars Combinatorica*, 18:201–216, 1984.
- [4] O. Borodin and A. Kostochka. On an upper bound on a graph's chromatic number, depending on the graph's degree and density. *J. Comb. Theory Ser. B*, 23:247–250, 1977.
- [5] P. A. Catlin. Hajós' graph-coloring conjecture: Variations and counterexamples. J. Comb. Theory Ser. B, 26(2):268–274, 1979.
- [6] K. Edwards and A. D. King. Supplemental material: Sage worksheet http://www.sagenb.org/home/pub/4712/, 2012.
- [7] S. Fajtlowicz. The independence ratio for cubic graphs. In *Proceedings of the 8th Southeastern Conference on Combinatorics, Graph Theory and Computing*, pages 273–277, 1977.
- [8] D. Ferguson, T. Kaiser, and D. Král'. The fractional chromatic number of triangle-free subcubic graphs. arXiv preprint 1203.1308, 2012.
- [9] H. Hatami and X. Zhu. The fractional chromatic number of graphs of maximum degree at most three. SIAM J. Discrete Math., 23(4):1762–1775, 2009.
- [10] C. C. Heckman and R. Thomas. A new proof of the independence ratio of triangle-free cubic graphs. *Discrete Math.*, 233:233–237, April 2001.
- [11] T. Kaiser, A. D. King, and D. Král'. Fractional total colourings of graphs of high girth. J. Comb. Theory Ser. B, 2010. Accepted.
- [12] A. D. King. Claw-free graphs and two conjectures on ω , Δ , and χ . PhD thesis, McGill University, October 2009.
- [13] A. D. King. Hitting all maximum cliques with a stable set using lopsided independent transversals. *J. Graph Theory*, 67(4):300–305, 2011.
- [14] A. D. King, L. Lu, and X. Peng. A fractional analogue of Brooks' theorem. SIAM J. Discrete Math., 26(2):452–471, 2012.
- [15] C.-H. Liu. Personal communication, 2012.
- [16] L. Lu and X. Peng. The fractional chromatic number of triangle-free graphs with $\Delta \leq 3$. arXiv preprint 1011.2500, 2010.
- [17] M. Molloy and B. Reed. Graph Colouring and the Probabilistic Method. Springer, 2000.
- [18] B. A. Reed. ω , Δ , and χ . J. Graph Theory, 27:177–212, 1998.
- [19] B. A. Reed. A strengthening of Brooks' Theorem. J. Comb. Theory Ser. B, 76:136–149, 1999.

- [20] E. R. Scheinerman and D. H. Ullman. Fractional Graph Theory. John Wiley & Sons, Inc., New York, 1997.
- [21] D. B. West. *Introduction to Graph theory, 2nd ed.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 2000.