

ON NEW SUM-PRODUCT TYPE ESTIMATES

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ABSTRACT. New lower bounds involving sum, difference, product, and ratio sets for $A \subset \mathbb{C}$ are given.

1. INTRODUCTION

Erdős and Szemerédi ([3]) conjectured that if A is a set of integers, then

$$|A + A| + |A \cdot A| \gg |A|^{2-o(1)},$$

where

$$A + A = \{a_1 + a_2 : a_{1,2} \in A\}$$

is called the sum set of A , the product $A \cdot A$, difference $A - A$, and ratio $A : A$ sets being similarly defined. (In the latter case one should not divide by zero.) The notations \ll , \gg , \approx are being used throughout to suppress absolute constants in inequalities, the symbol $o(1)$ in exponents absorbs logarithmic factors in the asymptotic parameter $|A|$, the cardinality of A .

Variations of the Erdős-Szemerédi question consider the set A living in other rings or fields, as well as replacing, e.g., the sum set with the difference set $A - A$. The conjecture is far from being settled, and therefore partial current “word records” on it vary with such variations of the input.

The best result for reals, for instance, is due to Solymosi ([14]), claiming

$$(1) \quad |A + A| + |A \cdot A| \gg |A|^{1+\frac{1}{3}-o(1)},$$

and would include the endpoint exponent $\frac{4}{3}$ if $A \cdot A$ were replaced by $A : A$.

However, the construction is specific for reals, and does not appear either to extend easily to the case $A \subset \mathbb{C}$, or to allow for replacing the sum set $A + A$ with the difference set $A - A$. So, if $A \in \mathbb{C}$ or if $A + A$ for reals gets replaced by $A - A$, the best known result comes from an older paper of Solymosi ([13]), claiming

$$(2) \quad |A \pm A| + |A \cdot A| \gg |A|^{1+\frac{3}{11}-o(1)},$$

also without the $o(1)$ term if $A \cdot A$ gets replaced by $A : A$.

It may be worth mentioning that the result (2) is based on the – rather sophisticated – Szemerédi-Trotter theorem, while (1) is not, using just elementary order properties of positive reals, expressed in the fact that if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a+c}{b+d}$ falls in between. This note is almost entirely based on the Szemerédi-Trotter theorem and succeeds in slightly improving on (2), yet not enough to beat (1).

Theorem 1. *For any $A \subset \mathbb{C}$, with two or more elements one has*

$$(3) \quad \begin{aligned} |A - A| + |A : A| &\gg |A|^{1+\frac{9}{31}-o(1)}, \\ |A + A| + |A : A| &\gg |A|^{1+\frac{15}{53}-o(1)}, \\ |A - A| + |A \cdot A| &\gg |A|^{1+\frac{11}{39}-o(1)}, \\ |A + A| + |A \cdot A| &\gg |A|^{1+\frac{19}{69}-o(1)}. \end{aligned}$$

2. LEMMATA

The main tool behind the above estimates is the Szemerédi-Trotter incidence theorem. For a set P of points and a set of L straight lines in a plane let

$$I(P, L) = \{(p, l) \in P \times L : p \in l\}$$

be the set of incidences.

Theorem 2 (Szemerédi and Trotter [16]). *The maximum number of incidences in \mathbb{R}^2 is bounded as follows:*

$$(4) \quad |I(P, L)| \ll (|P||L|)^{\frac{2}{3}} + |P| + |L|.$$

In particular, if P_t (or L_t) denote the sets of points (or lines) incident to at least $t \geq 1$ lines (or points) of L (or P), then

$$(5) \quad \begin{aligned} |P_t| &\ll \frac{|L|^2}{t^3} + \frac{|L|}{t}, \\ |L_t| &\ll \frac{|P|^2}{t^3} + \frac{|P|}{t}. \end{aligned}$$

Let us note that the linear in $|P|, |L|$ terms in the estimates (4, 5) are essentially trivial and usually of no interest in the sense of being dominated by the non-linear ones, whenever these estimates are being used. This is also the case in this paper.

The Szemerédi-Trotter theorem is also true in the plane over \mathbb{C} . This was proved by Tóth ([17]). Recently there has been a new proof by Solymosi and Tao ([15]) which had to sacrifice the endpoint in the exponent for the sake of elegance of the method.

One can easily develop a weighted version of the Szemerédi-Trotter theorem, see Iosevich et al. ([5]). Suppose each line $l \in L$ has been assigned a weight $m(l) \geq 1$. The number of weighted incidences $i_m(P, L)$ is obtained by summing over the set $I(P, L)$, with each pair (p, l) being counted $m(l)$ times. Suppose, the total weight of all lines is W and the maximum weight per line is \bar{m} . It is argued in [5] that the worst possible case for the weighted incidence estimate is the uniform one, when there are $\frac{W}{\bar{m}}$ lines of equal weight \bar{m} , hence the following theorem.

Theorem 3. *The maximum number of weighted incidences between a point set P and a set of lines with the total weight W and maximum weight per line \bar{m} is*

$$(6) \quad i_m(P, L) \ll \bar{m}^{\frac{1}{3}}(|P|W)^{\frac{2}{3}} + \bar{m}|P| + W.$$

This paper uses in its core the same geometric construction as Solymosi ([13]) did, which yielded the exponent $\frac{14}{11}$, with some more detailed analysis of the incidences involved by dealing with the weighted case. Yet the improvements over (2) are due to combining this construction with a recent purely additive-combinatorial observation by Shkredov and Schoen ([9], Lemma 3.1) which allowed for a series of the latest state-of-the art improvements in incremental progress towards a number of open questions in field combinatorics in [9], [12], [10].

The above-mentioned additive-combinatorial observation is the content of the following Lemma 1, quoting which requires some notation used in the sequel. Throughout the rest of this section A, B denote any sets in an Abelian group. The following energy notations E , when applied in a field will bifurcate into E and E_* , respectively, relative to the addition and multiplication operations.

For any $d \in A - A$, set

$$(7) \quad A_d = \{a \in A : a + d \in A\}.$$

Denote

$$E(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 - a_2 = b_1 - b_2\}|,$$

referred to as the “additive energy” of A, B . By the Cauchy-Schwarz inequality, rearranging the terms in the above definition of $E(A, B)$, one has

$$(8) \quad E(A, B)|A \pm B| \geq |A|^2|B|^2.$$

Indeed, if d is an element of $A - B$ or s is an element of $A + B$, and $n(d)$, $n(s)$ are the number of realisations of d and s , respectively as a difference or sum of a pair of elements from $A \times B$, (8) follows from the fact that

$$(9) \quad E(A, B) = \sum_{d \in A-B} n^2(d) = \sum_{s \in A+B} n^2(s).$$

Also, $E(A, A) = E(A)$ is referred to as the “energy of A ”. In this case note that according to the notation (7), $n(d) = |A_d|$.

Moreover, the following inequality will be useful. If

$$(10) \quad D^+ = \left\{ d \in A - A : n(d) \geq \frac{1}{2} \frac{|A|^2}{|A + A|} \right\},$$

then “the energy supported on D^+ ”, namely the left-hand side of the next inequality, satisfies

$$(11) \quad \sum_{d \in D^+} n^2(d) \gg \frac{|A|^4}{|A + A|}.$$

Indeed, by (8) with $A = B$, the energy supported on the complement of D^+ is trivially bounded from above by the right-hand side of (11).

Also useful will be the “cubic energy” of A , which is

$$(12) \quad E_3(A) = |\{(a_1, \dots, a_6) \in A \times \dots \times A : a_1 - a_2 = a_3 - a_4 = a_5 - a_6\}|.$$

This definition implies that ([10], Lemma 2)

$$(13) \quad E_3(A) = \sum_{d \in A-A} E(A, A_d).$$

To see this, fix any $d = a_1 - a_2$ in (12) and observe that if one is to count every representation $d = a_3 - a_4$ as many times as $a_3, a_4 \in A_d$ for some d , this will happen exactly $n(d)$ times, for different $d = a_5 - a_3 = a_6 - a_4$ in (12).

The following statement is the content of Corollary 3 in [10], with trivial variations.

Lemma 1. *One has the following identities, for any $D' \in A - A$:*

$$(14) \quad \sum_{d \in D'} |A_d| |A \pm A_d| \geq \frac{|A|^2 \left(\sum_{d \in D'} |A_d|^{\frac{3}{2}} \right)^2}{E_3(A)},$$

$$\sum_{d \in D'} |A_d|^2 |A \pm A_d| \geq \frac{|A|^2 \left(\sum_{d \in D'} |A_d|^2 \right)^2}{E_3(A)}.$$

Proof. To verify the first inequality of (16) observe that by the Cauchy-Schwarz inequality, for each d :

$$\sqrt{|A_d| |A \pm A_d|} \sqrt{E(A, A_d)} \geq |A| |A_d|^{\frac{3}{2}}.$$

Summing over $d \in D'$ and applying once again the Cauchy-Schwartz inequality to the left-hand side yields

$$\sqrt{\sum_{d \in D'} |A_d| |A \pm A_d|} \sqrt{\sum_{d \in D'} E(A, A_d)} \geq |A| \sum_{d \in D'} |A_d|^{\frac{3}{2}}.$$

squaring both sides and using (13) does the job. Verifying the second inequality in (14) requires merely a straightforward modification of the above. \square

The identities (14) suggest that the cubic energy estimate from above can be quite useful, since $A - A_d \subseteq (A - A) \cap (A - A - d)$, as well as $A + A_d \subseteq (A + A) \cap (A + A + d)$. The latter observation, which [10] credits to Katz and Koester ([6]), allows for the following interpretation of (14), producing lower bounds for $E(A, A - A)$ and $E(A, A + A)$. Indeed, the left-hand side in (1) itself is a lower bound for both $E(A, A \pm A)$. However, in order to be able to express it in terms of the sum set, via (9), one needs to deal with the sum of $|A_d|^2$ in the right-hand side, rather than of $|A_d|^{\frac{3}{2}}$.

In the first formula of (14) assume that D' is a popular subset of $A - A$, namely

$$(15) \quad D' = \left\{ d \in A - A : |A_d| \geq \frac{1}{2} \frac{|A|^2}{|A - A|} \right\}.$$

Then

$$\sum_{d \in D'} |A_d|^{\frac{3}{2}} \gg \left(\frac{|A|^2}{|A - A|} \right)^{\frac{1}{2}} \sum_{d \in D'} |A_d| \gg \left(\frac{|A|^2}{|A - A|} \right)^{\frac{1}{2}} |A|^2.$$

In the second formula of (14) replace D' with D^+ , defined by (10). Then

$$(16) \quad \begin{aligned} E(A, A - A) &\geq \frac{|A|^8}{|A - A| E_3(A)}, \\ E(A, A + A) &\geq \frac{|A|^{10}}{|A + A|^2 E_3(A) \max_{d \in D^+} |A_d|}. \end{aligned}$$

2.1. Some applications of Lemma 1. The estimates (16) enabled Schoen and Shkredov ([10]) to achieve progress on the sum set of a convex set problem. If $A = f([1, \dots, N])$, where f is a strictly convex real-valued function, then $|A - A| \gg |A|^{\frac{5}{3} - o(1)}$, with a slightly worse estimate $|A + A| \gg |A|^{\frac{14}{9} - o(1)}$ for the sum-set. The reason for the two estimates being different is that dealing with the sum set necessitated bootstrapping the earlier established exponent $\frac{3}{2}$, which had been obtained over the past ten years or so in various guises, with or without using the Szemerédi-Trotter theorem. (See e.g. [2], [4], [5]. The conjectured exponent in the convex set sum set problem is $2 - o(1)$.)

Li ([7]) – see also his recent work with Roche-Newton ([8]) – has adapted the approach of [10] to the sum-product problem, using a specific convex function, the exponential. This was also observed in another paper of Schoen and Shkredov ([11], Corollary 25). The result was the exponent $\frac{14}{11} - o(1)$. A closer look at this adaptation – see the Appendix in this paper – reveals that the exponential function has as much to do with it as basically replacing a with $\exp(\log a)$ in a variant of the old sum-product construction by Elekes ([1]): applying the Schoen-Shkredov trick, expressed in (14) to the estimate (41) in the Appendix would improve the Elekes exponent $\frac{5}{4}$ to $\frac{14}{11} - o(1)$.

The same exponent $\frac{14}{11} - o(1)$ had been coincidentally obtained in Solymosi's work [14], as stated in (2) above.

This note uses the construction of [14] and combines it with an estimate of the type (14), thereby getting an improvement (3) over (2). The estimates involving the sum set end up being worse, because they require bootstrapping the estimates which themselves would yield the exponent $\frac{14}{11} - o(1)$ only. It will also use the Elekes construction in order to channel the estimates involving the ratio set into ones using the product set. These will have to bootstrap the Elekes exponent $\frac{5}{4}$, thus making the estimates involving the product set still worse.

3. PROOF OF THEOREM 1

The condition $|A| \geq 2$ is tantamount to assuming that $0 \notin A$. Consider a point set $A \times A$ in the coordinate plane.

The multiplicative energy $E_*(A)$ of A is defined as the number of solutions of the following equation

$$E_*(A) = |\{(a_1, \dots, a_4) \in A \times \dots \times A : a_1/a_2 = a_3/a_4\}|.$$

By the Cauchy-Schwarz inequality

$$(17) \quad E_*(A) \geq \max \left(\frac{|A|^4}{|A \cdot A|}, \frac{|A|^4}{|A : A|} \right).$$

Geometrically, $E_*(A)$ is the number of ordered pairs of points of $A \times A$ in the plane on straight lines through the origin, whose slopes r are members of the ratio set $A : A$. Hence, elements of $A : A$ can be identified with lines through the origin supporting points of $A \times A$.

The proof will deal with an in some sense “popular” subset L of these lines, P denoting a subset of $A \times A$ supported on the lines in L . There will be two cases to consider.

Ratio set case. In order to establish the first two estimates of (3) the notation L will stand for the set of popular lines through the origin, namely those supporting at least $\frac{1}{2}|A|^2|A : A|$ points of $A \times A$. The subset of $A \times A$ supported on these lines L will be denoted as P . One has $|P| \gg |A|^2$, as well as $|A| \leq |L| \leq |A : A|$. Let us also use the notation N for the maximum number of points per line, clearly $N \leq |A|$. It follows from (17) that if $n(l)$ denotes the number of elements of P supported on a line $l \in L$,

$$(18) \quad \sum_{l \in L} n^2(l) \gg \frac{|A|^4}{|A : A|}.$$

Product set case. In order to establish the last two estimates of (3), the same notations P, L will apply to slightly differently defined sets, as well as the quantity N , as follows. The set P will be a “popular multiplicative energy” subset of $A \times A$, constructed by the standard dyadic pigeonholing procedure by popularity, in terms of supporting points of $A \times A$, of all lines through the origin. The elements of P are those points in $A \times A$ which lie on a set L of straight lines passing through the origin, supporting between $N/2$ and N points each, and such that

$$(19) \quad |L|N^2 \gg \frac{|A|^4}{|A \cdot A| \log |A|}.$$

Such sets L, P always exist, by the pigeonhole principle and (17).

Important in the product set case will be the following bounds on N .

Lemma 2. *There exists L, N satisfying (19) and such that*

$$(20) \quad \frac{1}{2} \frac{|A|^2}{|A \cdot A|} \leq N \ll \frac{|A \pm A|^2 |A \cdot A|}{|A|^3}.$$

The lower bound in (20) follows right away from (17) and a popularity argument. The upper bound comes from a variant of the Elekes construction ([1]) apropos of sum-products. A variant of Lemma 2 can be found in the above-mentioned works [8], [11]. A short proof is given in the Appendix.

In both the ratio and product set cases, it can be assumed that $|L| \gg N$. It is clear in the ratio set case, where $N \leq |A|$ and thus, by (18), $|L| \gg \frac{|A|^3}{|A : A|} \geq |A|$.

In the product set cases $|P| \approx |L|N$. Assume that $N \gg |L|$. Since N satisfies (20) and LN^2 is bounded from below by (19), it follows that

$$|A \pm A|^6 |A \cdot A|^4 \gg \frac{|A|^{13}}{\log |A|},$$

which is far better than the last two claims in (3).

Beginning now the proof proper, in either of the two cases above consider the sum set of P with some other set Q in the plane, with $|Q| \geq |P|$. (In the sequel $Q = \pm P$ or $P \pm P$). To obtain the vector sums, one translates the lines from L to each point of Q , getting thereby some set \mathcal{L} of lines with $|\mathcal{L}| \leq |L||Q|$.

The Szemerédi-Trotter theorem, namely (4), enables one to estimate $|\mathcal{L}|$ from below:

$$|L||Q| \ll |\mathcal{L}|^{\frac{2}{3}} |Q|^{\frac{2}{3}}.$$

(It is easy to see that the fact that since $|L| \leq |Q| \leq |\mathcal{L}| \leq |L||Q|$, trivial terms in this application of (4) can be ignored.) Thus

$$(21) \quad |\mathcal{L}| \gg |L|^{\frac{3}{2}} |Q|^{\frac{1}{2}}.$$

Let us call the number of points of Q on a particular line $l \in \mathcal{L}$, the weight $m(l)$ of l . The total weight W of all lines in the collection \mathcal{L} is by construction equal to $|L||Q|$.

The lines in \mathcal{L} have been given weights, because the same line $l \in \mathcal{L}$ can contribute to the same vector sum in $P + Q$ at most $\max(N, m(l))$ times. Hence, let us lower the weights of lines, which are “too heavy”: whenever $m(l) \geq N$, redefine it as N .

Therefore, for the total weight W and the mean weight \bar{m} per line, from (21) one has

$$(22) \quad W \leq |L||Q|, \quad \bar{m} = \frac{W}{|\mathcal{L}|} \ll \sqrt{\frac{|Q|}{|L|}}.$$

The Szemerédi-Trotter theorem, namely (5), tells one that the weight distribution over \mathcal{L} obeys the inverse cube law. I.e., for $t \leq N$, one has

$$(23) \quad |\mathcal{L}_t| = |\{l \in \mathcal{L} : m(l) \geq t\}| \ll \frac{|Q|^2}{t^3} + \frac{|Q|}{t} \ll \frac{|Q|^2}{t^3},$$

as since $N \ll \sqrt{|Q|}$, the trivial term $\frac{|Q|}{t}$ gets dominated by the first term.

Now, let us look at the set $\mathcal{P}(\mathcal{L})$ of all pair-wise intersections of lines from \mathcal{L} , and for an intersection point $p \in \mathcal{P}(\mathcal{L})$ of some $k \geq 2$ lines l_1, \dots, l_k look at the sum of the weights of the lines that intersect there:

$$m(p) = \sum_{i=1}^k m(l_i).$$

For any point set $\mathcal{P} \subseteq \mathcal{P}(\mathcal{L})$, the number of weighted incidences between \mathcal{P} and \mathcal{L} is the sum over all pairs $(p, l) \in I(\mathcal{P}, \mathcal{L})$, counting each pair (p, l) with the weight $m(p)$.

The inverse cube weight distribution over the set of lines \mathcal{L} enables one to use the Szemerédi-Trotter theorem rather efficiently for counting weighted incidences, similar to how it was done in the paper of Iosevich et al. (See [5], Lemma 6).

Lemma 3. *Suppose, $|Q| \geq |P|$, and the weights of lines in \mathcal{L} have been capped by N . For $x \in P + Q$, let $n(x)$ be the number of realisations of x as a sum. Then for $t : N \ll t \leq |P|$,*

$$(24) \quad |\{x \in P + Q : n(x) \geq t\}| \ll \frac{|L|^{\frac{3}{2}}|Q|^{\frac{5}{2}}}{t^3},$$

Proof. The condition $|Q| \geq |P| \gg N^2$ (which holds in both the ratio and product set cases) ensures that (23) is valid, since for all $l \in \mathcal{L}$, $m(l) \leq N$. Observe that for any point set \mathcal{P} , the number of weighted incidences of \mathcal{L} with \mathcal{P} can be bounded from above using dyadic decomposition of \mathcal{L} by weight in excess of \bar{m} , via

$$(25) \quad \mathcal{I} \ll i_m(\mathcal{P}, \mathcal{L}_{\bar{m}}) + \sum_{j=1}^{\log_2 N - \log_2 \bar{m}} (2^j \bar{m}) |I(\mathcal{P}, \mathcal{L}_{2^j \bar{m}})|,$$

where $\mathcal{L}_{\bar{m}}$ denotes the set of lines from \mathcal{L} whose weight is at most \bar{m} , $\mathcal{L}_{2^j \bar{m}}$ denotes the dyadic group of lines whose weights are approximately $2^j \bar{m}$, and the notation i_m has been introduced in the context of Theorem 3.

One uses the Szemerédi-Trotter theorem to estimate each number of incidences $|I(\mathcal{P}, \mathcal{L}_{2^j \bar{m}})|$, for $j > 0$ involved and its weighted version for the first term. The condition (23) ensures that the weighted Szemerédi-Trotter estimate (6) for the first term in (25) dominates as follows. For $j \geq 1$, one has

$$(26) \quad |I(\mathcal{P}, \mathcal{L}_{2^j \bar{m}})| \ll (|\mathcal{P}| |\mathcal{L}_{2^j \bar{m}}|)^{\frac{2}{3}} + |\mathcal{P}| + |\mathcal{L}_{2^j \bar{m}}|.$$

For the case $j = 0$, Theorem 3 is used.

$$(27) \quad i_m(\mathcal{P}, \mathcal{L}_{\bar{m}}) \ll \bar{m}^{\frac{1}{3}} (|\mathcal{P}| W)^{\frac{2}{3}} + \bar{m} |\mathcal{P}| + W.$$

Together with the rest of the bounds for each $|\mathcal{L}_{2^j \bar{m}}|$ coming from (23) this yields

$$(28) \quad \mathcal{I} \ll \bar{m}^{\frac{1}{3}} (|\mathcal{P}| W)^{\frac{2}{3}} + N |\mathcal{P}| + W.$$

(See (31) below where the bounds in (26) for the terms for $j > 1$ are spelled out explicitly.)

Observe now that for $x \in P + Q \subseteq \mathcal{P}(\mathcal{L})$, with $n(x)$ denoting the number of realisations of a sum set element x , one always has $n(x) \leq m(x)$, where $m(x)$ is the weight of x as a member of $\mathcal{P}(\mathcal{L})$.

Applying (28) to the point set

$$(29) \quad \mathcal{P}_t = \{x \in P + Q : n(x) \geq t\},$$

with the lower bound $t|\mathcal{P}_t|$ for \mathcal{I} , one sees that for $t \gg N$ the term $N|\mathcal{P}|$ in the right-hand side of (28) cannot possibly dominate the remaining terms. Hence for $t \gg N$ one has

$$(30) \quad |\mathcal{P}_t| \ll \frac{|L|^{\frac{3}{2}}|Q|^{\frac{5}{2}}}{t^3} + \frac{|L||Q|}{t},$$

and has to be slightly more careful with the trivial term $\frac{|L||Q|}{t}$ in the bound (30) and refine it, so that it becomes absorbed into the first term to get (24). This is clearly the case for $t \leq \sqrt[4]{|L||Q|^3}$, but not yet for higher t .

Let us now address the issue of large t . The lines in \mathcal{L} come in $|L|$ possible directions, and therefore no more than $|L|$ lines can be incident to a single point. Hence, lines from a dyadic group $\mathcal{L}_{2^j\bar{m}}$ do not contribute to sets \mathcal{P}_t , whenever $t \gg |L| \cdot (2^j\bar{m})$. This means that one needs, with (23) in view, only a “tail” estimate for the right-hand side of (25), with $t \approx |L|(2^j\bar{m})$:

$$(31) \quad t|\mathcal{P}_t| \ll \sum_{i=j}^{\log_2 N - \log_2 \bar{m}} (2^i\bar{m})|I(\mathcal{P}_t, \mathcal{L}_{2^i\bar{m}})| \ll \frac{|Q|^{\frac{4}{3}}|\mathcal{P}_t|^{\frac{2}{3}}}{2^j\bar{m}} + N|\mathcal{P}_t| + \frac{|Q|^2}{(2^j\bar{m})^2}.$$

It follows that, since $N \ll t \leq |P|$,

$$|\mathcal{P}_t| \ll \frac{|L|^{\frac{3}{2}}|Q|^{\frac{5}{2}}}{2^{3j}t^3} + \frac{|Q|^2}{(2^j\bar{m})^2t} \ll \frac{|L|^{\frac{3}{2}}|Q|^{\frac{5}{2}}}{t^3} + \frac{|Q|^2|L|^2}{t^3}.$$

Since $|Q| \geq |P| \geq |L|$, the first term in the last estimate dominates the second term, thus proving (24). \square

Let us now apply the estimate (24) with $Q = \pm P$; of these two the case $Q = -P$ will serve to estimate $E_3(P)$.

The assumptions of Lemma 3 are satisfied, and therefore

$$(32) \quad E_3(P) = \sum_{x \in P-P} n^3(x) \ll N^2|P|^2 + |L|^{\frac{3}{2}}|P|^{\frac{5}{2}} \sum_{j=1}^{2\log_2 |A|} \frac{2^{3j}}{2^{3j}} \ll |L|^{\frac{3}{2}}|P|^{\frac{5}{2}} \log |A|,$$

since in both the ratio and product set cases $N^2 \ll \sqrt{|P||L|^3}$.

Using this, the first formula in (16), with A replaced by P , yields:

$$(33) \quad E(P, P-P) \gg \frac{|P|^{\frac{11}{2}}}{|L|^{\frac{3}{2}}|P-P| \log |A|}.$$

To get a similar estimate for the sum set $P+P$, let us use the second formula in (16), with A replaced by P and $D = P-P$. To estimate the quantity $t = \max_{d \in D^+} |P_d|$ it suffices to observe that part of the energy $E(P)$ supported on D^+ is bounded from below by (11), where A gets replaced by P , and from above, by Lemma 3, as

$$O\left(N|P|^2 + \frac{|L|^{\frac{3}{2}}|P|^{\frac{5}{2}}}{t}\right).$$

It follows that unless $|P+P| \gg \frac{|P|^2}{N}$, which is much better than (3), one has

$$(34) \quad \max_{d \in D^+} |P_d| \ll \frac{|L|^{\frac{3}{2}}|P+P|}{|P|^{\frac{3}{2}}}.$$

Then, using the second estimate in (16) and (32), one gets

$$(35) \quad E(P, P+P) \gg \frac{|P|^9}{|L|^3 |P+P|^3 \log |A|}.$$

On the other hand, one can use Lemma 3 with $Q = P \pm P$ and estimate the quantity $E(P, P \pm P)$ from above. Then for any $t \gg N$:

$$(36) \quad E(P, P \pm P) \ll |P| |P \pm P| t + \frac{|L|^{\frac{3}{2}} |P \pm P|^{\frac{5}{2}}}{t},$$

and choosing

$$t = \frac{|P \pm P|^{\frac{3}{4}} |L|^{\frac{3}{4}}}{\sqrt{|P|}} \gg N,$$

to match the two terms in (36) yields

$$(37) \quad E(P, P \pm P) \ll \sqrt{|P|} |P \pm P|^{\frac{7}{4}} |L|^{\frac{3}{4}}.$$

Combining this with (33) results in

$$(38) \quad |P - P|^{\frac{11}{4}} |L|^{\frac{9}{4}} \gg \frac{|P|^5}{\log |A|}.$$

To obtain the first estimate of (3) it suffices to note that one has (with what the notations P, L stand for in the ratio set case) $|P| \gg |A|^2$, $|L| \leq |A : A|$, plus $|P - P| \leq |A - A|^2$.

To prove the second estimate in (3) the estimates (37) and (35) get put together. It follows that

$$(39) \quad |P + P|^{\frac{19}{4}} |L|^{\frac{15}{4}} \gg \frac{|P|^{\frac{34}{4}}}{\log |A|}.$$

Using $|L| \leq |A : A|$, $|P| \gg |A|^2$ and $|P + P| \leq |A + A|^2$ results in the second estimate in (3).

In the product set case, where $|P| = |L|N$, the estimates (38, 39) become

$$(40) \quad \begin{aligned} |A - A|^{\frac{11}{2}} &\gg \frac{(|L|N^2)^{\frac{11}{4}}}{\sqrt{N} \log |A|}, \\ |A + A|^{\frac{19}{2}} &\gg \frac{(|L|N^2)^{\frac{19}{4}}}{N \log |A|}. \end{aligned}$$

Lemma 2 now supplies a non-trivial upper bound on N . Substituting the bounds from (19) and (20) into (40) then yields the last two estimates of (3) and completes the proof of Theorem 1. \square

3.1. Appendix. Proof of Lemma 2. The lower bound of (20) is merely the multiplicative version of the popularity argument behind (10), resulting in (11). Indeed, the multiplicative energy of A supported on those lines through the origin, which correspond to ratios $r \in A : A$, whose number of realisations $n(r) \geq \frac{1}{2} \frac{|A|^2}{|A \cdot A|}$, is $\gg \frac{|A|^4}{|A \cdot A|}$.

On the other hand, it was proven in [8], [11] that the multiplicative energy coming from the lines through the origin, supporting at least N points of $A \times A$ (that from the set of all ratios $r \in A : A$ such that $n(r) \geq N$), is $O\left(\frac{|A \pm A|^2 |A|}{N}\right)$. This, together with (19) settles the upper bound in (20) and proves Lemma 2.

For completeness sake, a simple version of the proof of the upper bound in (20) for N is given below. This bound was derived in [8], [11] via the Szemerédi-Trotter type estimates for convex functions, using a particular example of a convex function, the exponential. Let us show that the bound in question, in fact, represents a variant of the well known construction of Elekes ([1]) apropos of sum-products, which gave the exponent $\frac{5}{4}$ implicit in the bounds (20). The notation in the forthcoming argument is somewhat independent from the rest of the paper.

Consider a set A , not containing zero and a set of lines $L = \{y = \frac{d+x}{a}\}$, where d is an element of the difference set $A - A$ (or the sum set $A + A$, the modification required being trivial) and $a \in A$. Clearly there are $|A - A||A|$ lines. Therefore, the number of points in a set P_t , where at least $t \leq |A|$ lines intersect is, by (5), bounded as

$$|P_t| \ll \frac{|A - A|^2 |A|^2}{t^3}.$$

Given a point $(x, y) \in P_t$, one has an intersections of at least t lines of L there, namely $y = \frac{d_i+x}{a_i}$ for at least t different pairs (d_i, a_i) . For each d_i take some fixed representation $d_i = u_i - v_i$, let $a \in A$ be a variable. Clearly $d_i + x = (u_i - a) + (a - v_i + x) = d'_i + x'$. I.e., P_t is such that if it contains one point (x, y) , then it contains at least $|A|$ distinct points with the same ordinate.

Conversely, if R_t is the subset of ratios from $A : A$ which have at least t realisations, it is clearly a subset of the set of ordinates of the points from P_t . Indeed, if $r = \frac{a'_1}{a_1} = \dots = \frac{a'_k}{a_k}$, then it equals to the ordinate of a horizontal family of intersection points of at least k lines, identified by the pairs $(d_i = a'_i - a, a_i)$, these intersection points having the abscissae $x = a$.

It follows that

$$(41) \quad |R_t| \ll \frac{|A - A|^2 |A|}{t^3},$$

and hence the multiplicative energy supported on R_t is $O\left(\frac{|A - A|^2 |A|}{t}\right)$. Comparing this with the lower bound in terms of $\frac{|A|^4}{|A \cdot A|}$ gives the upper bound in (20), where t has been replaced by N . \square

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