## A RELAXATION OF STEINBERG'S CONJECTURE

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ABSTRACT. A graph is  $(c_1, c_2, \dots, c_k)$ -colorable if the vertex set can be partitioned into k sets  $V_1, V_2, \dots, V_k$ , such that for every  $i : 1 \le i \le k$  the subgraph  $G[V_i]$  has maximum degree at most  $c_i$ . We show that every planar graph without 4- and 5-cycles is (1, 1, 0)-colorable and (3, 0, 0)-colorable. This is a relaxation of the Steinberg Conjecture that every planar graph without 4- and 5-cycles are properly 3-colorable (i.e., (0, 0, 0)-colorable).

#### 1. INTRODUCTION

It is well-known that the problem of deciding whether a planar graph is properly 3-colorable is NP-complete. Grötzsch in 1959 [5] showed the famous theorem that every triangle-free planar graph is 3-colorable. A lot of research was devoted to find sufficient conditions for a planar graph to be 3-colorable, by allowing a triangle together with some other conditions. One of such efforts is the following famous conjecture made by Steinberg in 1976.

Conjecture 1 (Steinberg, [7]). All planar graphs without 4-cycles and 5-cycles are 3-colorable.

Not much progress in this direction was made until Erdös proposed to find a constant C such that a planar graph without cycles of length from 4 to C is 3-colorable. Borodin, Glebov, Raspaud, and Salavatipour [2] showed that  $C \leq 7$ . For more results, see the recent nice survey by Borodin [1].

Yet another direction of relaxation of the Conjecture is to allow some defects in the color classes. A graph is  $(c_1, c_2, \dots, c_k)$ -colorable if the vertex set can be partitioned into k sets  $V_1, V_2, \dots, V_k$ , such that for every  $i : 1 \leq i \leq k$  the subgraph  $G[V_i]$  has maximum degree at most  $c_i$ . Thus a (0, 0, 0)-colorable graph is properly 3-colorable.

Eaton and Hull [4] and independently Skrekovski [6] showed that every planar graph is (2, 2, 2)-colorable (actually choosable). Xu [8] proved that all planar graphs with no adjacent triangles or 5-cycles are (1, 1, 1)-colorable. Chang, Havet, Montassier, and Raspaud [3] proved that all planar graphs without 4-cycles or 5-cycles are (2, 1, 0)-colorable and (4, 0, 0)-colorable. In this paper, we further prove the following relaxation of the Steinberg Conjecture.

**Theorem 1.** All planar graphs without 4-cycles and 5-cycles are (1, 1, 0)-colorable.

**Theorem 2.** All planar graphs without 4-cycles and 5-cycles are (3,0,0)-colorable.

We will use a discharging argument in the proofs. We let the initial charge of vertex  $u \in G$  be  $\mu(u) = 2d(u) - 6$ , and the initial charge of face f be  $\mu(f) = d(f) - 6$ . Then by Euler's formula, we have

(1) 
$$\sum_{v \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = -12$$

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Our goal is to show that we may re-distribute the charges among vertices and faces so the final charges of the vertices and faces are non-negative, which would be a contradiction. In the process of discharging, we will see that some configurations prevent us from showing some vertices or faces to have non-negative charges. Those configurations will be shown to be reducible configurations, that is, a valid coloring outside of the configurations can be extended to the whole graph. It is worth to note that in the proof of Theorem 1, we prove a somewhat global structure, a special chain of triangles, to be reducible.

The following are some simple observations about the minimal counterexamples to the above theorems.

**Proposition 1.** Among all planar graphs without 4-cycles and 5-cycles that are not (1, 1, 0)colorable or (3, 0, 0)-colorable, let G be one with minimum number of vertices. Then
(a) G contains no  $2^-$  vertices.

(b) a k-vertex in G can have  $\alpha \leq \lfloor \frac{k}{2} \rfloor$  incident 3-faces, and at most  $k - 2\alpha$  pendant 3-faces.

We will use the following notations in the proofs. A k-vertex  $(k^+$ -vertex,  $k^-$ -vertex) is a vertex of degree k (at least k, at most k resp.). The same notation will apply to faces. An  $(\ell_1, \ell_2, \ldots, \ell_k)$ -face is a k-face with incident vertices of degree  $\ell_1, \ell_2, \ldots, \ell_k$ . A bad 3-vertex is a 3-vertex on a 3-face. A face f is a pendant 3-face to vertex v if v is adjacent to some bad 3-vertex on f. The pendant neighbor of a 3-vertex v on a 3-face is the neighbor of v not on the 3-face. A vertex v is properly colored if all neighbors of v have different colors from v. A vertex v is nicely colored if it shares colors with at most max $\{s_i - 1, 0\}$  neighbors, thus if a vertex v is nicely colored by a color c which allows deficiency  $s_i > 0$ , then an uncolored neighbor of v can be colored by c.

In the next section, we will give a proof to Theorem 1; and in the last section, we will give a proof to Theorem 2.

### 2. (1, 1, 0)-coloring of planar graphs

We will use a discharging argument in our proof. First we will prove some reducible configurations.

Let G be a minimum counterexample to Theorem 1, that is, G is a planar graph without 4-cycles and 5-cycles, and G is not (1,1,0)-colorable, but any proper subgraph of G is (1,1,0)-colorable.

The following is a very useful tool in the proofs.

**Lemma 1.** Let H be a proper subgraph of G so that there is a (1,1,0)-coloring of G - H. If vertex  $v \in H$  satisfies either (i) 3 neighbors of v are colored, with at least two properly colored, or (ii) 4 neighbors of v are colored, all properly, then the coloring of G - H can be extended to G - (H - v).

*Proof.* (i) Let  $v \in H$  be a vertex with 3 colored neighbors, two of which are properly colored, such that the coloring of G - H can not be extended to v. Since v is not (1, 1, 0)-colorable, the three neighbors of v must have different colors, and furthermore, two of the colored neighbors cannot be properly colored, a contradiction to the assumption that two of the colored neighbors of v are properly colored.

(ii) Let  $v \in H$  be a vertex of degree 4 with all neighbors properly colored such that the coloring of G - H can not be extended to v. Then due to the coloring deficiencies, v must have at least 2 neighbors colored by 1, at least 2 neighbors colored by 2, and at least 1 neighbor colored by 1. Then v has at least five colored neighbors, a contradiction.

**Lemma 2.** There is no  $(3, 3, 4^{-})$ -face in G.

*Proof.* Let uvw be a  $(3,3,4^-)$ -face in G with d(u) = d(v) = 3 and  $d(w) \le 4$ . Then  $G \setminus \{u, v, w\}$  is (1,1,0)-colorable. Color w and v properly, then u is colorable by Lemma 1, thus G is (1,1,0)-colorable, a contradiction.

**Lemma 3.** There is no 5-vertex that is incident to two  $(3, 4^-, 5)$ -faces and adjacent to a 3-vertex in G.

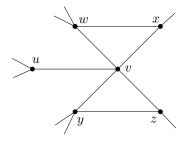


FIGURE 1. Figure for Lemma 3

Proof. Let v be a 5-vertex with neighbors u, w, x, y, z so that  $wx, yz \in E(G)$  and d(u) = d(x) = d(z) = 3 and  $d(w), d(y) \leq 4$  (See Figure 1). By the minimality of  $G, G \setminus \{u, v, w, x, y, z\}$  is (1, 1, 0)-colorable. Properly color u, w, and y, then properly color x and z. For v to not be colorable, v must have two neighbors colored by 1, two neighbors colored by 2 and one neighbor colored by 3. Since the w, x and y, z vertex pairs must be colored differently, one of them must have the colors 1 and 2. W.l.o.g. we can assume that w is colored by 1 and x by 2. Then since w is properly colored, we can either recolor x by 1 or 3, and color v by 2 obtaining a coloring of G, a contradiction.

**Lemma 4.** No 3-vertex in G can be adjacent to two other 3-vertices. In particular, the 3-vertices on a  $(3,3,5^+)$ -face must have another neighbor with degree four or higher.

*Proof.* Let v be a 3-vertex with x and y being two neighbors of degree 3. By the minimality of G,  $G \setminus \{v, x, y\}$  is (1, 1, 0)-colorable. Then we can first properly color x and y, and then by Lemma 1 color v to get a coloring of G, a contradiction.

**Lemma 5.** The pendant neighbor of the 3-vertex on a (3, 4, 4)-face must have degree 4 or higher.

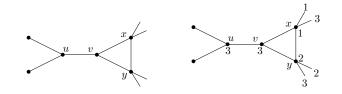


FIGURE 2. Figure for Lemma 5

*Proof.* Let vxy be a (3, 4, 4)-face in G such that the pendant neighbor u of the 3-vertex v has degree 3 (See Figure 2). By the minimality of G,  $G \setminus \{u, v\}$  is (1, 1, 0)-colorable. We properly color u and then color v differently from both x and y. If u and v are not both colored by 3, then we get a coloring for G, a contradiction, so we may assume both u and v are colored by 3. This means that both u and v have two remaining neighbors colored by 1 and 2. Let x and y be colored by 1 and 2 respectively. The neighbors of x must be colored by 1 and 3 or else we could recolor v by 1 and x by 3 if necessary to obtain a coloring of G. Likewise, the neighbors of y must be colored by 2 and 3. In this case we switch the colors of x and y and color v by 1 to obtain a coloring of G, a contradiction again.

Let a  $(T_0, T_1, \ldots, T_n)$ -chain be a sequence of triangles,  $T_0, T_1, \ldots, T_n$ , such that (i)  $T_0$  is a (3, 4, 4)-face and  $T_n$  is a  $(3^+, 4, 4^+)$ -face, and all other triangles are (4, 4, 4)-faces, and (ii) for  $0 \le i \le n - 1$ ,  $T_i$  and  $T_{i+1}$  share a 4-vertex  $t_i$ . In a  $(T_0, T_1, \ldots, T_n)$ -chain, let  $x_i \in T_i$  for  $0 \le i \le n$  be a non-connecting 4<sup>+</sup>-vertex.

Let a *special 4-vertex* be a 4-vertex that is incident to one 3-face and has two pendant 3-faces, and let a 3-face be a *special 3-face* if it has at least one special 4-vertex. Let a *good* 4-vertex be a 4-vertex with only one incident 3-face and at most one pendant 3-face.

We will prove in the following lemmas that a (3, 4, 4)-face  $T_0$  may get help in discharging from a  $(3^+, 4^+, 5^+)$ -face or special 3-face  $T_n$  through a  $(T_0, T_1, \ldots, T_n)$ -chain.

**Lemma 6.** There are no special (3, 4, 4)-faces in G.

Proof. Let uvw be a special (3, 4, 4)-face in G such that d(v) = d(w) = 4. W.l.o.g. we can assume that v is a special 4-vertex with pendant neighbors  $v_1$  and  $v_2$ . By the minimality of  $G, G \setminus \{u, v, v_1, v_2, w\}$  is (1, 1, 0)-colorable. We can properly color w and u in that order then properly color  $v_1$  and  $v_2$ . Then by Lemma 1, we can color u, obtaining a coloring of G, a contradiction.

The following is a very useful tool in extending a coloring to a chain.

**Lemma 7.** Consider a  $(T_0, T_1, \dots, T_n)$ -chain with  $n \ge 1$  and  $T_n$  being a  $(4, 4^-, k)$ -face. If  $G \setminus \{T_0, T_1, \dots, T_{n-1}\}$  has a coloring such that the k-vertex of  $T_n$  is properly colored, or it shares the same color with the 4<sup>-</sup>-vertex, then the coloring can be extended to G.

*Proof.* We assume that the  $(4, 4^-, k)$ -face  $T_n$  has k-vertex  $x_n$  and  $4^-$ -vertex  $t_n$ . Also let  $G \setminus \{T_0, T_1, \dots, T_{i-1}\}$  has a coloring such that  $x_n$  is properly colored or shares the same color with  $t_n$  and G does not have a (1, 1, 0)-coloring. Finally let u be the 3-vertex of  $T_0$  and let w be the pendant neighbor of u.

We consider two cases. First let n = 1. If  $x_1$  and  $t_1$  have the same color, then we can properly color  $x_0$  and  $t_0$  in that order, thus by Lemma 1 we can color u so G has a (1, 1, 0)coloring, a contradiction. So we know that  $x_1$  and  $t_1$  must be colored differently, and further  $x_1$  is colored properly. We can properly color  $x_0$ . If  $x_0$  and w share the same color then we can color  $t_0$  by Lemma 1 and properly color u, a contradiction. So we may assume that  $x_0$  and w are colored differently. If any two of  $x_0, x_1$ , and  $t_1$  are colored the same then we could color  $t_0$  properly and color u by Lemma 1, a contradiction. Since  $x_0, x_1$ , and  $t_1$  are colored differently, if  $x_0$  is not colored by 3 then we could color  $t_0$  by the same color as  $x_0$  and properly color u, a contradiction. So  $x_0$  must be colored by 3 and w.l.o.g. we can assume that w is colored by 1. Since  $x_1$  is properly colored, it must be colored by 2, or we could color  $t_0$  by 1 and properly color u, a contradiction. It follows that  $t_1$  is colored by 1. If  $t_1$  is colored properly, then we could color  $t_0$  by 1 and properly color u, a contradiction, so we may assume that  $t_1$  is not colored properly. Further, neither z nor z' (the two other neighbors of  $t_1$ ) can be colored by 2, or we could recolor  $t_1$  properly, then color  $t_0$  by 1 and u properly, a contradiction. So we color  $t_1$  by 2 and  $t_0$  by 1, and properly color u, a contradiction.

Now we assume that  $n \ge 2$ . For all  $j: 1 \le j \le n$ , properly color  $x_{n-j}$  and color  $t_{n-j}$  by Lemma 1, or properly if possible. Then since  $x_1$  was properly colored, and  $t_1$  was colored after  $x_1$ , either  $x_1$  remains properly colored, or  $t_1$  has the same color as  $x_1$ . Also, we know that  $T_1$  must be a (4, 4, 4)-face, so by the previous case, we can extend the coloring to  $T_0$ and get a coloring of G, a contradiction.

**Lemma 8.** There is no  $(T_0, \ldots, T_n)$ -chain so that (i)  $n \ge 1$  and  $T_n$  is a special (4, 4, 4)-face or (ii)  $n \ge 2$  and  $T_n$  is a (3, 4, k)-face or (iii) n = 1 and  $T_n$  is a  $(3, 4, 4^-)$ -face.

*Proof.* Let  $T_0 = ux_0t_0$  be a (3, 4, 4)-face with d(u) = 3.

(i) Let v be a special 4-vertex of  $T_n$  and let y and z be the neighbors of v other than  $t_n$  and  $x_n$ . Let  $S = \{t_i, x_i : 0 \le i \le n-1\}$ . By the minimality of  $G, G \setminus (S \cup \{u, v, x_n, y, z\})$  has a (1, 1, 0)-coloring. Properly color  $x_n, y$  and z, then by Lemma 1 color v. Then, either  $x_n$  remains properly colored or v shares the same color, so by Lemma 7 we can extend the coloring to  $\{T_0, T_1, \dots, T_{n-1}\}$  to obtain a coloring of G.

(ii) Let v be the 3-vertex of  $T_n$  and let  $S = \{t_i, x_i : 0 \le i \le n-1\}$ . By the minimality of  $G, G \setminus (S \cup \{u, v\})$  has a (1, 1, 0) coloring. Properly color v and  $x_{n-1}$ . Then by Lemma 1, we can color  $t_{n-1}$ . Either  $x_{n-1}$  remains properly colored or  $t_{n-1}$  shares the same color, so by Lemma 7 we can extend the coloring to  $\{T_0, T_1, \cdots, T_{n-2}\}$  to obtain a coloring of G.

(iii) Assume that n = 1 and  $T_n$  is a (3, 4, 4)-face with 3-vertex v. By the minimality of  $G, G \setminus \{t_0, u, v, x_0, x_1\}$  has a (1, 1, 0)-coloring. Properly color  $x_0$  and u in that order and properly color  $x_1$  and v in that order. Then  $t_0$  has four neighbors colored, all properly, so by Lemma 1 we can color  $t_0$  to get a coloring for G.

**Remark:** By above lemma, a  $(T_0, T_1)$ -chain with  $T_1$  being a  $(3, 4, 5^+)$ -face is not necessarily reducible. Let a *bad*  $(3, 4, 5^+)$ -*face* be a  $(3, 4, 5^+)$ -face that shares a 4-vertex with a (3, 4, 4)-face.

**Lemma 9.** There is no  $(T_0, \ldots, T_n)$ -chain with  $T_i = T_n$  for some  $i \neq n$ .

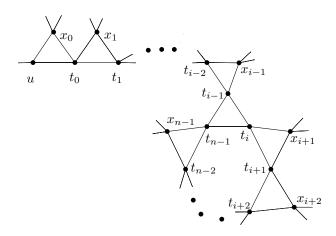


FIGURE 3. Figure for Lemma 9

Proof. Let  $(T_0, \ldots, T_n)$ -chain be a chain with  $T_i = T_n$  for some i < n. Let u be the 3-vertex of  $T_0$  and let  $S = \{t_j, x_j : 0 \le j \le n-1\}$ . Since  $T_i = T_n$ , the vertex that would have been labelled  $x_i$  is instead labelled  $t_{n-1}$  (See Figure 3). By the minimality of G,  $G \setminus (S \cup \{u\})$  is (1, 1, 0)-colorable. Start by properly coloring  $x_{i+1}, x_{i+2}$ , and  $t_{i+1}$ . Then for all  $j : i + 2 \le j \le n - 2$ , properly color  $x_{j+1}$  and color  $t_j$  by Lemma 1. Next, properly color  $t_{n-1}$ , and we have two cases:

**Case 1:** i = 0. We can properly color u, then color  $t_i$  by Lemma 1 to get a coloring of G, a contradiction.

**Case 2:** i > 0. We can then color  $t_i$  by Lemma 1 and then either  $t_{n-1}$  is properly colored, or  $t_i$  shares the same color, so by Lemma 7 we can extend the coloring to  $\{T_0, T_1, \dots, T_{i-1}\}$  to obtain a coloring of G, a contradiction.

**Lemma 10.** For each (3, 4, 4)-face  $T_0$  without good 4-vertices, there exist two chains,  $(T_0, \ldots, T_n)$ chain and  $(T_0, \ldots, T'_m)$ -chain, such that  $T_n$  and  $T'_m$  are either bad  $(3, 4, 5^+)$ -faces,  $(4, 4^+, 5^+)$ faces, or (4, 4, 4)-faces with a good 4-vertex. Furthermore,  $T_n \neq T'_m$ .

*Proof.* As G is finite, any chain of triangles in G must be finite. By Lemma 8 and 9, no chain of triangles in G can end with a special 3-face or a non-bad  $(3, 4, 5^+)$ -face, thus it must end with a bad  $(3, 4, 5^+)$ -face or a  $(4, 4^+, 4^+)$ -face. Since a (4, 4, 4)-face in a chain can not be a special 3-face, any chain of triangles in G must end with a bad  $(3, 4, 5^+)$ -face, a  $(4, 4^+, 5^+)$ -face or a (4, 4, 4)-face with a good 4-vertex.

Now we assume that  $T_n = T_m$ . Then by Lemma 7,  $T_n$  must be a  $(4, 4, 5^+)$ -face, and since G has no 4- and 5-cycles,  $n + m \ge 6$ . Assume that  $n \le m$ . Let  $S = \{t_i, x_i : 1 \le i \le n - 1\}$ , where  $S = \emptyset$  if n = 1, and  $S' = \{t'_j, x'_j : 0 \le j \le m - 1\}$  and let u be the 3-vertex of  $T_0$ . By the minimality of  $G, G \setminus S \cup S' \cup \{u\}$  has a (1, 1, 0)-coloring. We have two cases:

If n = 1, properly color  $x'_{m-1}$  and  $t'_{m-1}$ . Then, by Lemma 7 we can extend the coloring to  $\{T_0, T'_1, \cdots, T'_{m-2}\}$  to obtain a coloring of G, a contradiction.

If  $n \ge 2$ , then properly color  $x_{n-1}$ ,  $t_{n-1}$  and  $x'_{m-1}$  in that order, then by Lemma 1 we can color  $t'_{m-1}$ . If  $n \ge 3$ , for all  $i: 2 \le i \le n-1$ , properly color  $x_{n-i}$  and by Lemma 1 we can color  $t_{n-1}$ . Then since either  $x'_{m-1}$  is still properly colored or shares the same color as  $t'_{m-1}$ , by Lemma 7 we can extend the coloring to  $\{T_0, T'_1, \dots, T'_{m-2}\}$  to obtain a coloring of G, a contradiction.

We will now prove some lemmas which will ensure that bad  $(3, 4, 5^+)$ -faces will have extra charge to help (3, 4, 4)-faces.

**Lemma 11.** A 5-vertex incident to a bad (3,4,5)-face cannot be incident to another bad (3,4,5)-face or a (3,3,5)-face.

*Proof.* We only show the case when a 5-vertex v is incident to two bad (3, 4, 5)-faces, and it is very similar (and easier!) to show the case when it is incident to a bad (3, 4, 5)-face and a (3, 3, 5)-face.

Let v be a 5-vertex that is incident two bad (3, 4, 5)-faces,  $f_1$  and  $f_2$ , and let u be a k-vertex adjacent v (see Figure 4). Let  $f_3$  be the (3, 4, 4)-face sharing a 4-vertex with  $f_1$  and let  $f_4$  be the (3, 4, 4)-face sharing a 4-vertex with  $f_2$ . Let  $f_3$  and  $f_4$  have outer 4-vertices of x and x' respectively and 3-vertices of y and y' respectively. Also, let  $f_1$  and  $f_2$  have 4-vertices z and z'. Then, by the minimality of G,  $G \setminus \{f_1, f_2, f_3, f_4\}$  has a (1, 1, 0)-coloring.

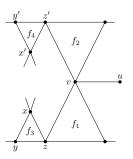


FIGURE 4. Figure for Lemma 11

If u is colored by 1 or 2, then we can color v by 3 and color the 3-vertices of  $f_1$  and  $f_2$  properly. Since v is properly colored, by Lemma 7 we can extend the coloring to  $f_1$  and  $f_3$ . Then, since v is colored by 3, it would remain properly colored, so again by Lemma 7 we can extend the coloring to  $f_2$  and  $f_4$  to get a coloring of G.

If u is colored by 3, then we properly color x and x' then properly color y and y'. We then properly color z and z'. If either z or z' is colored by 3, then we can properly color the 3-vertices of  $f_1$  and  $f_2$  and color v by either 1 or 2 getting a coloring for G. So we can assume neither is colored by 3, and w.l.o.g. we can assume that z is colored by 1. Then since z and z' are properly colored, we can color the 3-vertices of  $f_1$  and  $f_2$  by either 1 or 3. Then since v will have at most one neighbor colored by 2, and that neighbor colored properly, we can color v by 2 to obtain a coloring for G.

**Lemma 12.** A (3,5,k)-face in G that is incident a 5-vertex that is also incident to a bad (3,4,5)-face and a pendant  $(3,4^-,4^-)$ -face will have a pendant neighbor that is a 4<sup>+</sup>-vertex.

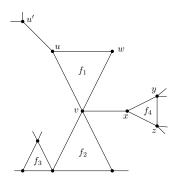


FIGURE 5. Figure for Lemma 12

Proof. Let  $f_1$  be a (3, 5, k)-face in G with a 5-vertex v, a 3-vertex u, and a pendant neighbor u' that is a 3-vertex. Let the k-vertex of  $f_1$  be w. Let v be incident a bad (3, 4, 5)-face  $f_2$  with neighbor (3, 4, 4)-face  $f_3$ , and let v have a pendant (3, 4, 4)-face  $f_4$ . Let the 3-vertex of  $f_4$  be x and the 4-vertices of  $f_4$  be y and z (See Figure 5). By the minimality of G,  $G \setminus \{f_2, f_3, u, u', x\}$  has a (1, 1, 0)-coloring. Properly color x. If w and x share the same color, then we can properly color u' and u, then properly color v and the 3-vertex of  $f_2$ . Then the coloring can be extended to  $f_3$  by Lemma 7, obtaining a coloring of G. So we can assume that w and x are colored differently. If x is colored by 1 or 2 (w.l.o.g. we may assume that

x is colored by 1), then we can color u' properly and color u by 1. Then we can properly color v and properly color the 3-vertex of  $f_2$ . Finally we can apply Lemma 7 to extend the coloring to  $f_3$ , obtaining a coloring of G. So we can assume that x is colored by 3.

Since x is colored by 3, we may assume that w is colored by 1. Properly color u' and color u by 2. Since x is properly colored, y and z must be colored by 1 and 2. W.l.o.g. let y be colored by 1. Then to avoid being able to re-color x by 1, the two other neighbors of y must be colored 1 and 3. For similar reasons the other two neighbors of z must be colored 2 and 3. Then switch the colors of y and z and color x by 1 or 2 and color v by 3, we can color the 3-vertex of  $f_2$  properly and by Lemma 7, extend the coloring to  $f_3$ , obtaining a coloring of G.

**Lemma 13.** A (3, 5, 5)-face in G can not have both 5-vertices also be incident to bad (3, 4, 5)-faces and have pendant (3, 4, 4)-faces.

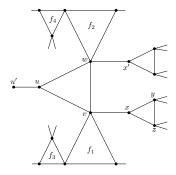


FIGURE 6. Figure for Lemma 13

Proof. Let uvw be a (3, 5, 5)-face in G where d(v) = d(w) = 5 and u has pendant neighbor u'. Also let v and w both be incident bad (3, 4, 5)-faces,  $f_1$  and  $f_2$  with neighbor (3, 4, 4)-faces  $f_3$  and  $f_4$  respectively and let v and w have pendant (3, 4, 4)-faces. Let the pendant (3, 4, 4)-faces to v and w have 3-vertices x and x' respectively (See Figure 6). By the minimality of  $G, G \setminus \{f_1, f_2, f_3, f_4, u, x, x'\}$  has a (1, 1, 0)-coloring.

Properly color x and x'. If either x or x' has a coloring different from u', w.l.o.g. we can assume x, then we color u the same as x. We can properly color w and v in that order, then properly color the 3-vertices of  $f_1$  and  $f_2$ . Then by Lemma 7 we can extend the coloring to  $f_3$  and  $f_4$  to obtain a coloring of G. So we can assume that x, x', and u' are colored the same. If x is colored by 3, since x is properly colored, y and z must be colored by 1 and 2. Then to avoid being able to re-color x by 1, the other two neighbors of y must be colored 1 and 3. For similar reasons the other two neighbors of z must be colored 2 and 3. Then we can switch the colors of y and z and color x differently from u'. Then we follow the above procedure to obtain a coloring for G.

So we may assume that w.l.o.g. x, x', and u' are all colored by 1. Then we color u by 2 and w by 3. Color the 3-vertex of  $f_2$  properly and by Lemma 7, extend the coloring to  $f_4$ . We now have v adjacent to 3 differently and properly colored vertices. Properly color the outer 4-vertex and the 3-vertex of  $f_3$  in that order, then properly color the 4-vertex of  $f_1$ . If it is colored by 3, then properly color the 3-vertex of  $f_1$  and color v by either 1 or 2 to obtain a coloring of G. If it is not colored by 3, then w.l.o.g. we can assume that it is colored by 1. Then since it is properly colored, we can color the 3-vertex of  $f_1$  by either 1 or 3 and color v by 2, obtaining a coloring of G.

**Lemma 14.** A 5-vertex in G that is incident a bad (3, 4, 5)-face and has a pendant (3, 4, 4)-face cannot also be incident a  $(4, 4^+, 5)$ -face  $T_n$  that is in a  $(T_0, \ldots, T_n)$ -chain.

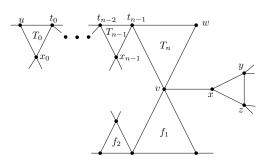


FIGURE 7. Figure for Lemma 14

Proof. Let v be a 5-vertex in G that is incident a bad (3, 4, 5)-face  $f_1$  with neighbor (3, 4, 4)-face  $f_2$ . Let v have a pendant (3, 4, 4)-face with 3-vertex w and 4-vertices y and z. Also let v be incident a  $(4, 4^+, 5)$ -face  $T_n$  such that there exists a chain of triangles from  $T_0$  to  $T_n$ . Let the 4<sup>+</sup>-vertex of  $T_n$  be w. Let  $S = \{t_i, x_i : 0 \le i \le n-1\}$  and let u be the 3-vertex of  $T_0$  (See Figure 7). By the minimality of  $G, G \setminus (S \cup \{f_1, f_2, u, x\})$  has a (1, 1, 0)-coloring.

Properly color x. If x and w are colored the same then we can properly color  $x_{n-1}$ ,  $t_{n-1}$ , and v. If n = 1, then by Lemma 1, we can color u. If  $n \ge 2$ , then by Lemma 7 we can extend the coloring to  $\{T_0, T_1, \dots, T_{n-1}\}$ . Then we can properly color the 3-vertex of  $f_1$  and by Lemma 7 we can extend the coloring to  $f_2$  obtaining a coloring for G. So we can assume that x and w are colored differently.

Let x be colored 1 or 2 and w.l.o.g. we can assume that x is colored by 1. Then we can properly color  $x_{n-1}$  and color  $t_{n-1}$  by 1. Since w and x are colored differently, either  $x_{n-1}$ and  $t_{n-1}$  are both colored properly or share the same color. If n = 1, then either we can color u properly or we can color u by Lemma 1. If  $n \ge 2$ , then by Lemma 7 we can extend the coloring to  $\{T_0, T_1, \dots, T_{n-1}\}$ . Then since  $t_{n-1}$  and x are colored the same we can properly color v and the 3-vertex of  $f_1$ . By Lemma 7 we can extend the coloring to  $f_2$  to obtain a coloring of G.

So let x be colored by 3 (then w is colored 1 or 2). Then y and z must be colored by 1 and 2, respectively. To avoid being able to re-color x by 1 or 2, the two other neighbors of y must be colored 1 and 3 and the two other neighbors of z must be colored 2 and 3. Then we switch the colors of y and z and re-color x to be the same as w, and proceed as above to get a coloring for G.

**Lemma 15.** Every 6-vertex in G that is incident a bad (3, 4, 6)-face can be incident at most two  $(3, 4^-, 6)$ -faces.

*Proof.* Let v be a 6-vertex in G. Let vwx be a bad (3, 4, 6)-face with d(w) = 3 and neighbor (3, 4, 4)-face xyz with 3-vertex y. Let v also be incident non-bad (3, 4, 6)-faces  $t_1t_2v$  and  $u_1u_2v$ 

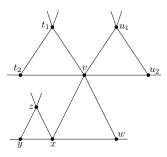


FIGURE 8. Figure for Lemma 15

where  $d(t_1) = d(u_1) = 4$  (See Figure 8). By the minimality of G,  $G \setminus \{t_1, t_2, u_1, u_2, v, w, x, y, z\}$  has a (1, 1, 0)-coloring. Properly color  $t_1, t_2, u_1$ , and  $u_2$ . If the color set of  $\{t_1, t_2, u_1, u_2\}$  is not  $\{1, 2, 3\}$ , then we can properly color v and w. Then by Lemma 7, we can extend the coloring to x, y, and z, obtaining a coloring of G. So we can assume that the color set of  $\{t_1, t_2, u_1, u_2\}$  includes 1, 2, and 3.

If two of  $\{t_1, t_2, u_1, u_2\}$  are colored by 3, then we can color z, y, and x properly. If x is colored by 3, then we can color w properly and color v by 1 or 2 to get a coloring of G. If x is colored by 1 or 2, then since x is properly colored we can color w by 3 or the same as x. Then we can color v differently from 3 and x to obtain a coloring of G.

So we can assume that exactly one of vertices in the set  $\{t_1, t_2, u_1, u_2\}$  is colored by 3. Then w.l.o.g. we may assume that the color set of  $\{t_1, t_2\}$  is  $\{1, 3\}$  and the color set of  $\{u_1, u_2\}$  is  $\{1, 2\}$ . Since  $u_1$  and  $u_2$  were colored properly, the outside neighbor of  $u_2$  must be 3. Let  $u_1$  be colored by 1, then since it is colored properly we can recolor  $u_2$  by 1. Then we can color v and w properly, and extend to x, y, and z to obtain a coloring of G. So we can assume that  $u_1$  is colored by 2.

Now color z, y, and x properly in that order. If x is colored by 3 then color w properly. If w is colored by 1, then color v by 2 to get a coloring for G. If w is colored by 2, then since  $u_1$  is colored properly recolor  $u_2$  by 2 and color v by 1 to get a coloring for G. So we can assume that x is colored by 1 or 2. Since x is properly colored we can color w by 3 or the same as x. Then either 1 or 2 but not both is in the color set of  $\{x, w\}$ . If 1 is in the color set, then v will have only one neighbor colored by 2 so we can color v by 2 and obtain a coloring of G. If 2 is in the color set, then v will have two neighbors colored by 1, but we can recolor  $u_2$  by 2 and color v by 1 to obtain a coloring of G.

The following lemma says that a 3-face with k vertices of degree 4 can have at most k chains of triangles ending at it.

**Lemma 16.** If a  $(T_0, T_1, \ldots, T_n)$ -chain and a  $(T'_0, T'_1, \ldots, T'_m)$ -chain with  $T'_m = T_n$  satisfy  $T_{n-1} \cap T_n = \{t_n\} = T'_{m-1} \cap T'_m$ , then  $T_0 = T'_0$ .

*Proof.* For otherwise, the two chains have a common (4, 4, 4)-face T so that  $T = T_a$  and  $T = T'_b$ . Then we would have a  $(T_0, T_1, T_{a-1}, T, T'_{b-1}, \ldots, T'_1, T'_0)$ -chain. But by Lemma 8, this chain cannot exist in G.

# **Discharging Procedure**

As we mentioned in the introduction, we set the initial charge of a vertex v to be  $\mu(v) = 2d(v) - 6$  and the initial charge of a face f to be  $\mu(f) = d(f) - 6$ . For the discharging procedure we must introduce the notion of a bank, which serves as a temporary placeholder for charges. We set the bank with initial charge zero and will show it has a non-negative final charge.

The following are the rules for discharging:

- (R1) Each 4-vertex gives  $\frac{1}{2}$  to each pendant 3-face and the rest to the incident 3-faces evenly.
- (R2) Every 6-vertex gives  $\frac{9}{4}$  to incident bad (3, 4, 6)-faces, 2 to other incident (3, 4<sup>-</sup>, 6)-faces and  $\frac{3}{2}$  to all other incident 3-faces; every 7<sup>+</sup>-vertex gives  $\frac{9}{4}$  to all incident 3-faces.
- (R3) Every 6<sup>+</sup>-vertex gives  $\frac{1}{2}$  to all pendant 3-faces.
- (R4) Every  $(4^+, 4^+, 5^+)$ -face and every (4, 4, 4)-face with a good 4-vertex give  $\frac{1}{2}$  to the bank and every bad  $(3, 4, 5^+)$ -face gives  $\frac{1}{4}$  to the bank.
- (R5) The bank gives  $\frac{1}{2}$  to each (3, 4, 4)-face without good 4-vertices.
- (R6) Every 5-vertex gives
  - (a) 2 to each incident (3, 3, 5)-face and 9/4 to each incident bad (3, 4, 5)-face.
  - (b) 7/4 to incident non-bad (3, 4, 5)-faces when also incident a bad (3, 4, 5)-face, and gives 2 to incident non-bad (3, 4, 5)-faces otherwise.
  - (c) 5/4 to incident  $(3, 5^+, 5^+)$ -faces when also incident to a bad (3, 4, 5)-face and a pendant  $(3, 4^-, 4^-)$ -face, and gives 3/2 to incident  $(3, 5^+, 5^+)$ -faces otherwise.
  - (d) 3/2 to all  $(4, 4^+, 5)$ -faces with a chain of triangles to a (3, 4, 4)-face and gives 1 to  $(4, 4^+, 5)$ -faces otherwise.
  - (e) 1/2 to each pendant  $(3, 4^-, 4^-)$ -face and (3, 3, k)-face and 1/4 to all other pendant 3-faces.

Let v be a k-vertex. By Proposition 1,  $k \ge 3$ .

For k = 3, the final charge  $\mu^*(v)$  of v is  $\mu^*(v) = \mu(v) = 0$ .

For k = 4, by (R1), the final charge of v is 0. We note that v gives at least 1 to each incident 3-face, and gives at least 3/2 to 3-faces when v is a good 4-vertex.

For k = 5, if v has at most one incident 3-face, then by (R6a) and (R6e),  $\mu^*(v) \ge \mu(v) - \frac{9}{4} \cdot 1 - \frac{1}{2} \cdot 3 = 1/4 > 0$ . Let v have two incident 3-faces  $f_1$  and  $f_2$  and a pendant 3-face  $f_3$ .

Let  $f_3$  be a  $(3, 4^-, 4^-)$ -face. When  $f_1$  is a bad (3, 4, 5)-face, by Lemma 3  $f_2$  cannot be a  $(3, 4^-, 5)$ -face. By Lemma 14, if  $f_2$  is a  $(4, 4^+, 5)$ -face, then there is no chain of triangles from some (3, 4, 4)-face to f, so by (R6a), (R6c), (R6d), and (R6e),  $\mu^*(v) \ge \mu(v) - \frac{1}{2} \cdot 1 - \frac{9}{4} \cdot 1 - \frac{5}{4} \cdot 1 = 0$ . When  $f_1$  is a non-bad (3, 4, 5)-face, then by Lemma 3,  $f_2$  cannot be a  $(3, 4^-, 5)$ -face, so by (R6b), (R6c), (R6d), and (R6e),  $\mu^*(v) \ge \mu(v) - \frac{1}{2} \cdot 1 - \frac{3}{2} \cdot 1 = 0$ . When neither  $f_1$  nor  $f_2$  are  $(3, 4^-, 5)$ -faces, by (R6c), (R6d), and (R6e),  $\mu^*(v) \ge \mu(v) - \frac{1}{2} \cdot 1 - \frac{3}{2} \cdot 2 = \frac{1}{2} > 0$ .

Now let  $f_3$  be a (3, 4, 5)-face. When  $f_1$  or  $f_2$  is  $(3, 4^-, 5)$ -face, by Lemma 3, the other one cannot be a  $(3, 4^-, 5)$ -face, so by (R6b), (R6c), (R6d), and (R6e),  $\mu^*(v) \ge \mu(v) - \frac{1}{4} \cdot 1 - \frac{9}{4} \cdot 1 - \frac{3}{2} \cdot 1 = 0$ . When neither  $f_1$  nor  $f_2$  are  $(3, 4^-, 5)$ -faces, by rules (R6c), (R6d), and (R6e),  $\mu^*(v) \ge \mu(v) - \frac{1}{4} \cdot 1 - \frac{3}{2} \cdot 2 = \frac{3}{4} > 0$ .

Finally, let v have two incident 3-faces  $f_1$  and  $f_2$ , and no pendant 3-face. If  $f_1$  is a bad (3, 4, 5)-face, then by Lemma 11,  $f_2$  cannot also be a bad (3, 4, 5)-face or a (3, 3, 5)-face.

Then by (R6),  $\mu^*(v) \ge \mu(v) - \frac{9}{4} \cdot 1 - \frac{7}{4} \cdot 1 = 0$ . If neither  $f_1$  nor  $f_2$  is a bad (3, 4, 5)-face, then by (R6b), (R6c), and (R6d),  $\mu^*(v) \ge \mu(v) - 2 \cdot 2 = 0$ .

For k = 6, if v is incident to at most two 3-faces, then by (R2) and (R3),  $\mu^*(v) \ge \mu(v) - \frac{9}{4} \cdot 2 - \frac{1}{2} \cdot 2 = \frac{1}{2}$ . So we can assume that v is incident to three 3-faces. If v is incident a bad (3, 4, 6)-face then by Lemma 15 only one other incident 3-face can be a  $(3, 4^-, 6)$ -face. So by (R2),  $\mu^*(v) \ge \mu(v) - \frac{9}{4} \cdot 2 - \frac{3}{2} \cdot 1 = 0$ . If v is not incident a bad (3, 4, 6)-face, then by (R2),  $\mu^*(v) \ge \mu(v) - 2 \cdot 3 = 0$ .

For  $k \ge 7$ , if k is odd, then  $\mu^*(v) \ge \mu(v) - \frac{k-1}{2} \cdot \frac{9}{4} - \frac{1}{2} \cdot 1 = 2k - 6 - \frac{9k-9}{8} - \frac{4}{8} = \frac{7k-43}{8} \ge \frac{3}{4}$ . If k is even, then  $\mu^*(v) \ge \mu(v) - \frac{k}{2} \cdot \frac{9}{4} = 2k - 6 - \frac{9k}{8} = \frac{7k-43}{8} \ge 1$ .

Now let f be a k-face. Since G is a simple graph,  $k \ge 3$ . By the condition that there is no 4-cycle and 5-cycle, k = 3 or  $k \ge 6$ . Since no faces above degree 3 are involved in the discharging procedure, the final charge of 6<sup>+</sup>-face f is  $\mu^*(f) = \mu(f) = d(f) - 6 \ge 0$ .

For k = 3, by Lemma 2, we have no  $(3, 3, 4^{-})$ -faces, but we still have a few different cases:

**Case 1:** Face f is a  $(3,3,5^+)$ -face. By Lemma 4, f will have two pendant neighbors of degree 4 or higher. So by (R1), (R2), (R4), and (R7),  $\mu^*(f) \ge (3-6) + 2 \cdot 1 + \frac{1}{2} \cdot 2 = 0$ .

**Case 2:** Face f is a (3, 4, 4)-face. By Lemma 5, f will have a pendant neighbor of degree 4 or higher. If f has a good 4-vertex, then by (R1),  $\mu^*(f) \ge \mu(f) + \frac{3}{2} \cdot 1 + 1 \cdot 1 + \frac{1}{2} \cdot 1 = 0$ . If f has no good 4-vertices, then by (R5), f receives 1/2 from the bank, so  $\mu^*(f) = \mu(f) + 1 \cdot 2 + \frac{1}{2} \cdot 1 + \frac{1}{2} = 0$ .

**Case 3:** Face f is a bad (3,4,5)-face. By (R1), (R4) and (R6a),  $\mu^*(f) = \mu(f) + 1 \cdot 1 + \frac{9}{4} \cdot 1 - \frac{1}{4} \cdot 1 = 0$ .

**Case 4:** Face f is a non-bad (3, 4, 5)-face. If the 5-vertex of f is not incident a bad (3, 4, 5)-face, then by (R1) and (R6b),  $\mu^*(f) = \mu(f) + 1 \cdot 1 + 2 \cdot 1 = 0$ . If the 5-vertex of f is incident a bad (3, 4, 5)-face, then by Lemma 12, f has a pendant neighbor of degree 4 or higher. So by (R1), (R6b), and (R6e),  $\mu^*(f) \ge \mu(f) + 1 \cdot 1 + \frac{7}{4} \cdot 1 + \frac{1}{4} \cdot 1 = 0$ .

**Case 5:** Face f is a (3, 4, 6)-face. If f is a bad (3, 4, 6)-face, then by (R1), (R2), and (R4),  $\mu^*(f) = \mu(f) + 1 \cdot 1 + \frac{9}{4} \cdot 1 - \frac{1}{4} \cdot 1 = 0$ . If f is a non-bad (3, 4, 6)-face then by (R1) and (R2),  $\mu^*(f) = \mu(f) + 1 \cdot 1 + 2 \cdot 1 = 0$ .

**Case 6:** Face f is a  $(3, 4, 7^+)$ -face. By (R1) and (R2),  $\mu^*(f) = \mu(f) + 1 \cdot 1 + \frac{9}{4} \cdot 1 = \frac{1}{4}$ .

**Case 7:** Face f is a (3, 5, 5)-face. If neither 5-vertex of f is also incident to a bad (3, 4, 5)-face and a pendant  $(3, 4^-, 4^-)$ -face, then by (R6c),  $\mu^*(f) = \mu(f) + \frac{3}{2} \cdot 2 = 0$ . If one of the 5-vertices of f is also incident to a bad (3, 4, 5)-face and a pendant  $(3, 4^-, 4^-)$ -face then by Lemma 12, f must have a pendant neighbor of degree 4 or higher. In addition, by Lemma 13 the other 5-vertex of f cannot have both an incident bad (3, 4, 5)-face and a pendant  $(3, 4^-, 4^-)$ -face. So by (R6c) and (R6e),  $\mu^*(f) = \mu(f) + \frac{5}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{3}{2} \cdot 1 = 0$ .

**Case 8:** Face f is a  $(3, 5, 6^+)$ -face. If the 5-vertex of f is not incident to a bad (3, 4, 5)-face and a pendant  $(3, 4^-, 4^-)$ -face then by (R2) and (R6c),  $\mu^*(f) \ge \mu(f) + \frac{3}{2} \cdot 2 = 0$ . If the 5-vertex of f has both an incident bad (3, 4, 5)-face and a pendant  $(3, 4^-, 4^-)$ -face, then by Lemma 12 f must have a pendant neighbor of degree 4 or higher. So by (R2), (R6c), and (R6e),  $\mu^*(f) \ge \mu(f) + \frac{5}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{3}{2} \cdot 1 = 0$ .

**Case 9:** Face f is a  $(3, 6^+, 6^+)$ -face. By (R2),  $\mu^*(f) \ge \mu(f) + \frac{3}{2} \cdot 2 = 0$ .

**Case 10:** Face f is a (4, 4, 4)-face. If f has no good 4-vertices then by (R1),  $\mu^*(f) = \mu(f) + 1 \cdot 3 = 0$ . If f has a good 4-vertex then by (R1) and (R4),  $\mu^*(f) \ge \mu(f) + 1 \cdot 2 + \frac{3}{2} \cdot 1 - \frac{1}{2} \cdot 1 = 0$ .

**Case 11:** Face f is a  $(4^+, 4^+, 5^+)$ -face. If f has no chains of triangles to a (3, 4, 4)-face, then each incident vertex gives at least 1 to f, so  $\mu^*(f) \ge \mu(f) + 1 \cdot 3 = 0$ . If f has a chain of triangles to a (3, 4, 4)-face then by (R6d), at least one vertex must give  $\frac{3}{2}$  to f, so combined with (R4),  $\mu^*(v) \ge \mu(v) + 1 \cdot 2 + \frac{3}{2} \cdot 1 - \frac{1}{2} \cdot 1 = 0$ .

Finally, we show that the bank has a non-negative charge. By Lemma 10, for each (3, 4, 4)-face without good 4-vertices in G, there exist at least two chains of triangles from the (3, 4, 4)-face to a bad  $(3, 4, 5^+)$ -face, a (4, 4, 4)-face with a good 4-vertex, or a  $(4^+, 4^+, 5^+)$ -face. Then by Lemma 16, there exist at most two chains of triangles to  $(4^+, 4^+, 5^+)$ -face from (3, 4, 4)-faces and at most one chain of triangles to a  $(3, 4, 5^+)$ -face from (3, 4, 4)-faces. So we can see the transfer of charge from triangles with extra charge to the bank and back to (3, 4, 4)-faces is a transfer of  $\frac{1}{4}$  charge over each chain of triangles. Each (4, 4, 4)-face with a good 4-vertex and  $(4^+, 4^+, 5^+)$ -face gives  $\frac{1}{2}$  to the bank, and the bank will give at most  $\frac{1}{4} \cdot 2$  to (3, 4, 4)-faces for each (4, 4, 4)-face with a good 4-vertex or  $(4^+, 4^+, 5^+)$ -face. Also, each bad  $(3, 4, 5^+)$ -face gives  $\frac{1}{4}$  to the bank, and the bank will give at most  $\frac{1}{4} \cdot 1$  to (3, 4, 4)-faces for each bank will always have a non-negative charge.

This completes the discharging, showing that the final charges of all faces, vertices, and the bank are non-negative, a contradiction to (1). This completes the proof of Theorem 1.1.

## 3. (3,0,0)-coloring of planar graphs

In this section, we give a proof for Theorem 2. Our proof will again use a discharging method. Let G be a minimum counterexample to Theorem 2, that is, G is a planar graph without 4-cycles and 5-cycles and is not (3, 0, 0)-colorable, but any proper subgraph of G is properly (3, 0, 0)-colorable. We may assume that vertices colored by 1 may have up to three neighbors colored by 1.

The following is a very useful tool to extend a coloring on a subgraph of G to include more vertices.

**Lemma 17.** Let H be a proper subgraph of G. Given a (3,0,0)-coloring of G - H, if two neighbors of  $v \in H$  are colored so that one is a 5<sup>-</sup>-vertex and the other is nicely colored, then the coloring can be extended to G - (H - v) such that v is nicely colored by 1.

Proof. Let H be a subgraph of G such that G - H has a (3, 0, 0)-coloring. Let  $v \in H$  have neighbors u and w that are colored. Let  $d(u) \leq 5$  and let w be nicely colored. Color vby 1. Since w is nicely colored, if this coloring is invalid, then u must be colored by 1. In addition, u must have at least 3 neighbors colored by 1. To avoid recoloring u by 2 or 3, umust have at least one neighbor of color 2 and at least one neighbor of color 3. This implies that  $d(u) \geq 6 > 5$ , a contradiction. So v is colorable by 1. In addition, since the deficiency of color 1 is 3 and v only has 2 neighbors, it follows that v is nicely colored.

**Lemma 18.** Every 3-vertex in G has a  $6^+$ -vertex as a neighbor.

*Proof.* Let v be a vertex in G such that each neighbor vertex of v has degree 5. By the minimality of G, G - v is (3, 0, 0)-colorable. If two vertices in N(v) share the same color, then v can be properly colored, so we can assume all the neighbors of v are colored differently. Let u be the neighbor of v that is colored by 1. Then u must have 3 neighbors colored by 1 to forbid v to be colored by 1. In addition, u must have neighbors colored by 2 and 3 to forbid v to be colored by 2 or 3. Then, u has at least 6 neighbors, a contradiction.

Let a  $(3, 3, 3^+)$ -face to be *poor* if the pendant neighbors of the two 3-vertices have degrees at most 5. A  $(3, 3^+, 3^+)$ -face is *semi-poor* if exactly one of the pendant neighbors of the 3-vertices has degree 5 or less. A 3-face is *non-poor* if each 3-vertex on it has the pendant neighbor being a 6<sup>+</sup>-vertex. Finally, a *poor 3-vertex* is a 3-vertex on a poor or semi-poor 3-face that has a 5<sup>-</sup>-vertex as its pendant neighbor.

# **Lemma 19.** All $(3,3,6^-)$ -faces in G are non-poor.

*Proof.* For all  $(3, 3, 5^-)$ -faces in G, the proof is trivial by Lemma 18. Let uvw be a (3, 3, 6)-face in G with d(u) = d(v) = 3 such that the pendant neighbor v' of v has degree at most 5. By the minimality of G,  $G \setminus \{u, v\}$  is (3, 0, 0)-colorable. Properly color u and color v differently than both w and v'. Then u and v are both colored by 2 or 3, w.l.o.g. assume 2. This means that u' and v' share the same color (where u' is the pendant neighbor of u), different from the color of w.

Let w be colored by 1, then to avoid being able to recolor u or v by 1, w must have 3 outer neighbors colored by 1. Then w can be recolored by 2 or 3 depending on the color of its fourth colored neighbor. We recolor w by 2 or 3 and recolor u and v by 1 to get a coloring of G, a contradiction.

So we may assume that w is colored by 3, and that u' and v' are colored by 1. To avoid recoloring v by 1, v' must have at least 3 neighbors colored by 1. In addition, to avoid recoloring v' by 2 or 3 and coloring v by 1, v' must have neighbors colored by both 2 and 3. This contradicts that v' has degree less than 6.

# **Lemma 20.** No vertex $v \in V(G)$ can have $\lfloor \frac{d(v)}{2} \rfloor$ incident poor 3-faces.

*Proof.* Let v be a k-vertex in G with  $\lfloor \frac{k}{2} \rfloor$  incident poor (3, 3, k)-faces. Let  $u_1, u_2, \dots, u_k$  be the neighbors of v, and let  $u'_i$  be the pendant neighbor if  $u_i$  is in a poor 3-face. Note that  $d(u'_i) \leq 5$  and we know that all except possibly  $u_k$  are in poor 3-faces.

By the minimality of G,  $G \setminus \{v, u_1, u_2, \dots, u_{k-1}\}$  is (3, 0, 0)-colorable. If d(v) is odd, then by Lemma 17, for all i with  $1 \leq i \leq k-1$ , we can color  $u_i$  by 1. Then we can properly color v to get a coloring of G, so we can assume that d(v) is even. If d(v) is even, then by Lemma 17, for all i with  $1 \leq i \leq k-2$ , we can color  $u_i$  by 2. Then if  $u_k$  is colored by 1 we can color  $u_{k-1}$  properly and v properly to get a coloring of G. If  $u_k$  is colored by 2 or 3, then it is colored properly and by Lemma 17 we can color  $u_{k-1}$  by 1. Then we can properly color v to get a coloring of G, a contradiction.

**Lemma 21.** If an 8-vertex v is incident to three incident poor (3,3,8)-faces, then it cannot be incident to a semi-poor face, nor two pendant 3-faces.

*Proof.* Let v be an 8-vertex in G with 3 incident poor (3,3,8)-faces. Let  $u_1, u_2, \dots, u_6$  be the 3-vertices in the poor (3,3,8)-face and let  $u'_1, u'_2, \dots, u'_6$  be the corresponding pendant neighbors, respectively. We know that for all i with  $1 \le i \le 6$ ,  $d(u'_i) \le 5$ .

(i) Let  $vu_7u_8$  be the incident semi-poor face with  $u_7$  being the poor 3-vertex. Then by the minimality of  $G, G \setminus \{v, u_1, u_2, \dots, u_7\}$  is (3, 0, 0)-colorable. By Lemma 17,  $u_1, u_2, \dots, u_6$ can be colored by 1. Then if  $u_8$  is colored by 1, we can properly color  $u_7$  and then v to get a coloring of G. So we may assume that  $u_8$  is not colored by 1, in which case it is nicely colored and we may color  $u_7$  with 1 by Lemma 17, and then properly color v to get a coloring of G, a contradiction.

(ii) Let  $u_7$  and  $u_8$  be the bad 3-vertices adjacent to v. Then  $G \setminus \{v, u_1, u_2, \dots, u_7, u_8\}$  is (3,0,0)-colorable, by the minimality of G. Properly color both  $u_7$  and  $u_8$ . If either  $u_7$  or  $u_8$  is colored by 1 or both have the same color, then by Lemma 17, we may color  $u_1, u_2, \dots, u_6$  by 1 and then properly color v. So we may assume that  $u_7$  is colored by 2 and  $u_8$  is colored by 3. Then we properly color  $u_1, u_2, \dots, u_6$ , and it follows that for each i with  $1 \le i \le 3$ ,  $u_{2i-1}$  and  $u_{2i}$  must be colored differently. Then v can have at most 3 neighbors colored by 1, all properly colored, so v can be colored by 1, a contradiction.

**Lemma 22.** If a 7-vertex v is incident to two poor (3,3,7)-faces, then it cannot be (i) incident to a semi-poor  $(3,6^-,7)$ -face and adjacent to a pendant 3-face, or (ii) adjacent to three pendant 3-faces.

*Proof.* Let v be a 7-vertex in G with 2 incident poor (3, 3, 7)-faces. Let  $u_1, u_2, u_3$ , and  $u_4$  be the 3-vertices on the poor (3, 3, 7)-faces and let  $u'_1, u'_2, u'_3$ , and  $u'_4$  be their corresponding pendant neighbors, respectively. We know that for all i with  $1 \le i \le 4$ ,  $d(u'_i) \le 5$ .

(i) Let  $vu_5u_6$  be a semi-poor face with  $u_5$  being a poor 3-vertex and  $d(u_6) \leq 6$  and let  $u_7$  be a bad 3-vertex adjacent to v. By the minimality of G,  $G \setminus \{v, u_1, u_2, u_3, u_4, u_5, u_7\}$  is (3, 0, 0)-colorable. Since at this point  $u_6$  has only 4 colored neighbors, if  $u_6$  is colored by 1 then either it is nicely colored or it can be recolored properly. If  $u_6$  is not nicely colored, then recolor  $u_6$  properly.

Color  $u_7$  properly. If  $u_7$  is colored by 1, then by Lemma 17, we can color  $u_1, u_2, \dots, u_5$  by 1 and then color v properly, a contradiction. So we may assume w.l.o.g. that  $u_7$  is colored by 2. Color  $u_1, u_2, \dots, u_5$  properly. Then, for each i with  $1 \leq i \leq 3$ ,  $u_{2i}$  and  $u_{2i-1}$  are colored differently and nicely. This leaves v with at most 3 neighbors colored by 1, all nicely, so we may color v by 1 to get a coloring of G, a contradiction.

(ii) Let  $u_5$ ,  $u_6$ , and  $u_7$  be the bad 3-vertices adjacent to v. By the minimality of G,  $G \setminus \{v, u_1, \ldots, u_7\}$  is (3, 0, 0)-colorable. Properly color  $u_5$ ,  $u_6$ , and  $u_7$ . If the set  $\{u_5, u_6, u_7\}$  does not contain both colors 2 and 3, then by Lemma 17, we can color  $u_1$ ,  $u_2$ ,  $u_3$ , and  $u_4$  by 1 and color v properly. So we can assume that  $\{u_5, u_6, u_7\}$  contains both colors 2 and 3. This implies that at most one vertex is colored by 1. So we properly color  $u_1$ ,  $u_2$ ,  $u_3$ , and  $u_4$ . Then v has at most 3 neighbors colored by 1, all nicely, so we can color v by 1 to get a coloring of G, a contradiction.

**Lemma 23.** Let uvw be a semi-poor (3,7,7)-face in G such that d(v) = d(w) = 7. Then vertices v and w cannot both be 7-vertices that are incident to two poor 3-faces, one semi-poor (3,7,7)-face, and adjacent to one pendant 3-face.

*Proof.* Let uvw be a semi-poor (3, 7, 7)-face in G such that d(v) = d(w) = 7 and both v and w are incident to two poor 3-faces, one (3, 7, 7)-face, and adjacent to one pendant 3-face. Let the neighbors of v and w be  $t_1, t_2, \cdots, t_5$  and  $z_1, z_2, \cdots, z_5$ , respectively such that  $t_5$  and  $z_5$  are bad 3-vertices (See Figure 9).

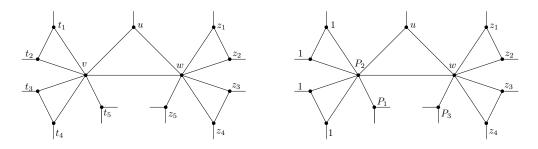


FIGURE 9. Figure for Lemma 23

By the minimality of G,  $G \setminus \{u, v, w, t_1, t_2, \dots, t_5, z_1, z_2, \dots, z_5\}$  is (3, 0, 0)-colorable. By Lemma 17, we can color  $t_1, t_2, t_3$ , and  $t_4$  by 1. Then properly color  $t_5$ , v, and  $z_5$  in that order. Vertex v will not be colored by 1, so w.l.o.g. lets assume that v is properly colored by 2. If  $z_5$  is colored by 1, then by Lemma 17, we can color  $z_1, z_2, z_3, z_4$ , and u by 1 and then properly color w, to get a coloring of G, a contradiction. So we can assume that  $z_5$  is not colored by 1. Then we properly color  $z_1, z_2, z_3, z_4$  and u, so w can have at most 3 neighbors colored by 1, all properly. We can color v by 1 to get a coloring of G, a contradiction.  $\Box$ 

# **Discharging Procedure:**

We start the discharging process now. Recall that the initial charge for a vertex v is  $\mu(v) = 2d(v) - 6$  and the initial charge for a face f is  $\mu(f) = d(f) - 6$ .

We introduce the following discharging rules:

- (R1) Every 4-vertex gives 1 to each incident 3-face.
- (R2) Every 5 and 6-vertex gives 2 to each incident 3-face.
- (R3) every  $6^+$ -vertex gives 1 to each adjacent pendant 3-face.
- (R4) Each *d*-vertex with  $7 \le d \le 10$  gives 3 to each incident poor (3, 3, \*)-face, 2 to each incident semi-poor 3-face, except 7-vertices give 1 to special semi-poor 3-face, where a special semi-poor (3, 7, 7+)-face is a semi-poor 3-face incident to a 7-vertex which is also incident to two poor 3-faces and adjacent to one pendant 3-face. Each *d*-vertex with  $7 \le d \le 10$  gives 1 to all other incident 3-faces.
- (R5) Every  $11^+$ -vertex gives 3 to all incident 3-faces.

Now let v be a k-vertex. By Proposition 1,  $k \geq 3$ .

When k = 3, v is not involved in the discharging process, so  $\mu^*(v) = \mu(v) = 0$ .

When k = 4, by Proposition 1, v can have at most 2 incident 3-faces. By (R1),  $\mu^*(v) \ge \mu(v) - 1 \cdot 2 = 0$ .

When k = 5, by Proposition 1, v can have at most 2 incident 3-faces. By (R2),  $\mu^*(v) \ge \mu(v) - 2 \cdot 2 = 0$ .

When k = 6, by Proposition 1, v can have  $\alpha \leq 3$  incident 3-faces, and at most  $(k - 2\alpha)$  pendant 3-faces. By (R2) and (R3),  $\mu^*(v) \geq \mu(v) - 2 \cdot \alpha - 1 \cdot (k - 2\alpha) = k - 6 = 0$ .

When k = 7, v has an initial charge  $\mu(v) = 7 \cdot 2 - 6 = 8$ . By Lemma 20, v has at most two poor 3-faces. If v has less than two incident poor 3-faces, then by (R3) and (R4),  $\mu^*(v) \ge \mu(v) - 3 \cdot 1 - 1 \cdot 5 = 0$  since v gives at most one charge per vertex excluding vertices in poor 3-faces. So assume that v has exactly 2 incident poor 3-faces. By Lemma 22, v is

adjacent to at most two pendant 3-faces, and if it is incident to a semi-poor  $(3, 6^-, 7)$ -face, then v is not adjacent to a pendant 3-face. So if v is not incident to a semi-poor  $(3, 7^+, 7)$ face, then by (R3) and (R4),  $\mu^*(v) \ge \mu(v) - 3 \cdot 2 - 2 \cdot 1 = 0$ ; If v is incident to a semi-poor  $(3, 7^+, 7)$ -face, then by rules (R3) and (R4),  $\mu^*(v) \ge \mu(v) - 3 \cdot 2 - 1 \cdot 1 - 1 \cdot 1 = 0$ .

When k = 8, v has an initial charge  $\mu(v) = 8 \cdot 2 - 6 = 10$ . By Lemma 20, v has at most three poor 3-faces. If v has less than 3 incident poor 3-faces, then by (R3) and (R4),  $\mu^*(v) \ge \mu(v) - 3 \cdot 2 - 1 \cdot 4 = 10 - 6 - 4 = 0$  since v gives at most one charge per vertex excluding vertices in poor 3-faces. So let v is incident to exactly 3 poor 3-faces. By Lemma 21, v cannot be incident to a semi-poor 3-face or adjacent to two pendant 3-faces, then  $\mu^*(v) \ge \mu(v) - 3 \cdot 3 - 1 \cdot 1 = 0$ .

When k = 9, by Lemma 20, v is incident to at most three poor 3-faces. The worst case occurs when v is incident 3 poor (3, 3, 9)-faces, incident one semi-poor (3, 3, 9)-face, and pendant one 3-face. So by (R3) and (R4),  $\mu^*(v) \ge \mu(v) - 1 \cdot 1 - 3 \cdot 3 - 2 \cdot 1 = 12 - 1 - 9 - 2 = 0$ .

When k = 10, by Lemma 20, v is incident to at most four poor (3, 3, 10)-faces. So by (R3) and (R4),  $\mu^*(v) \ge \mu(v) - 3 \cdot 4 - 2 \cdot 1 = 14 - 3 \cdot 4 - 2 \cdot 1 = 0$ .

When  $k \ge 11$ , we assume that v is incident to  $\alpha$  3-faces, then by Proposition 1,  $\alpha \le \lfloor k/2 \rfloor$ . Thus the final charge of v is  $\mu^* \ge 2k - 6 - 3\alpha - 1 \cdot (k - 2\alpha) = k - \alpha - 6 \ge 0$ .

Now let f be a k-face in G. By the conditions on G, k = 3 or  $k \ge 6$ . When  $k \ge 6$ , f is not involved in the discharging procedure, so  $\mu * (f) = \mu(f) = k - 6 \ge 0$ . So in the following we only consider 3-faces.

**Case 1:** f is a  $(4^+, 4^+, 4^+)$ -face. By the rules, each  $4^+$ -vertex on f gives at least 1 to f, so  $\mu * (f) \ge \mu(f) + 1 \cdot 3 = 0$ .

**Case 2:** f is a  $(3, 4^+, 4^+)$ -face with vertices u, v, w such that d(u) = 3. If u is not a poor 3-vertex, then by (R2), f gains 1 from the pendant neighbor of u and by the other rules, f gains at least 2 from vertices on f, thus  $\mu^*(f) \ge \mu(f) + 1 \cdot 3 = 0$ . If u is a poor vertex (it follows that f is a semi-poor 3-face), then by Lemma 18, f is a  $(3, 4^+, 6^+)$ -face. Since vor w is a 6<sup>+</sup>-vertex, it gives at least 2 to f unless f is a special semi-poor  $(3, 7, 7^+)$ -face, and as the other is a 4<sup>+</sup>-vertex, it gives at least 1 to f. Therefore, if f is not a special semi-poor 3-face, then  $\mu^*(f) \ge \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$ ; if f is a special semi-poor  $(3, 7, 8^+)$ -face, then f receives at least 2 from the 8<sup>+</sup>-vertex, so  $\mu^*(v) \ge \mu(v) + 2 \cdot 1 + 1 \cdot 1 = 0$ . If f is a special semi-poor (3, 7, 7)-face so that both v and w are incident to two poor 3-faces, one semi-poor (3, 7, 7)-face and adjacent to one pendant 3-face, then by Lemma 23, is impossible.

**Case 3:** f is a  $(3,3,4^+)$ -face with  $4^+$ -vertex v. If  $d(v) \ge 11$ , then by (R5),  $\mu^*(f) \ge \mu(f) + 3 = 0$ . So assume  $d(v) \le 10$ . By Lemma 18, if  $4 \le d(v) \le 6$ , then each 3-vertex has the pendant neighbor of degree 6 or higher. So by (R1) and (R3) (when d(v) = 4),  $\mu^*(f) \ge \mu(f) + 1 \cdot 3 = 0$ , or by (R1) and (R2) (when d(v) > 4),  $\mu^*(f) = \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$ . Let  $7 \le d(v) \le 10$ . If f is poor, then by (R4),  $\mu^*(f) = \mu(f) + 3 \cdot 1 = 0$ . If f is semi-poor, then one 3-vertex on f is adjacent to a 6<sup>+</sup>-vertex and thus by (R3) f gains 1 from it, together

the 2 that f gains from v by (R4), we have  $\mu^*(f) = \mu(f) + 2 \cdot 1 + 1 \cdot 1 = 0$ . If f is non-poor, then both 3-vertices on f are adjacent to the pendant neighbors of degrees more than 5, thus by (R3) and (R4),  $\mu^*(f) = \mu(f) + 1 \cdot 2 + 1 \cdot 1 = 0$ .

**Case 4:** f is a (3,3,3)-face. By Lemma 18, each 3-vertex will have the pendant neighbor of degree 6 or higher, so by (R3),  $\mu^*(f) = \mu(f) + 1 \cdot 3 = 0$ .

Since for all  $x \in V \cup F$ ,  $\mu^*(x) \ge 0$ ,  $\sum_{v \in V} \mu^*(v) + \sum_{f \in F} \mu^*(f) \ge 0$ , a contradiction. This completes the proof of Theorem 1.2.

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