# Approximating dynamics of a singularly perturbed stochastic wave equation with a random dynamical boundary condition * 

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#### Abstract

This work is concerned with a singularly perturbed stochastic nonlinear wave equation with a random dynamical boundary condition. A splitting skill is used to derive the approximating equation of the system in the sense of probability distribution, when the singular perturbation parameter is sufficiently small. The approximating equation is a stochastic parabolic equation when the power exponent of singular perturbation parameter is in $[1 / 2,1)$, but a deterministic hyperbolic (wave) equation when the power exponent is in $(1,+\infty)$. Key words: Stochastic wave equation; random dynamical boundary condition; singular limit, convergence in probability distribution, weak convergence. AMS Subject Classifications (2010): 60H15, 37L55, 37D10, 37L25, 37H05.


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## 1 Introduction

Stochastic nonlinear wave equations play an important role in describing the propagation of waves in certain systems or media, such as atmosphere, oceans, sonic booms, traffic flows, optic devices and quantum fields, when random fluctuations are taken into account ( $5, ~ 10,13,27$, 31, 34]). They have been studied recently by a number of authors(see [6, 7, 11, 14, 20, 25, 35]). For some wave systems on bounded domains, noise may affect the system evolution through the boundary in terms of random boundary conditions. Dirichlet, Neumann and Robin boundary conditions are static boundary conditions, because they are not involved with time derivatives of the system state variables. On the contrary, dynamical boundary conditions contain time derivatives of the system state variables and arise in many physical problems ([16, [17, 28]).

In this paper, we investigate a singularly perturbed stochastic nonlinear wave equation with a random dynamical boundary condition

$$
\begin{cases}\varepsilon u_{t t}^{\varepsilon}+u_{t}^{\varepsilon}-\triangle u^{\varepsilon}+u^{\varepsilon}-f\left(u^{\varepsilon}\right)=\varepsilon^{\alpha} \dot{W}_{1} & \text { in } D,  \tag{1.1}\\ \varepsilon \delta_{t t}^{\varepsilon}+\delta_{t}^{\varepsilon}+\delta^{\varepsilon}=-u_{t}^{\varepsilon}+\varepsilon^{\alpha} \dot{W}_{2} & \text { on } \partial D, \\ \delta_{t}^{\varepsilon}=\frac{\partial u^{\varepsilon}}{\partial \mathbf{n}} & \text { on } \partial D, \\ u^{\varepsilon}(0)=u_{0}, u_{t}^{\varepsilon}(0)=u_{1}, \delta^{\varepsilon}(0)=\delta_{0}, \delta_{t}^{\varepsilon}(0)=\delta_{1} . & \end{cases}
$$

Here $u^{\varepsilon}(x, t)$ is the unknown wave amplitude, $\varepsilon$ is a small positive singular perturbation parameter $(0<\varepsilon \ll 1)$, and the power exponent $\alpha$ is in $[1 / 2,1)$ or $(1,+\infty)$. Moreover, $W_{1}$ and $W_{2}$ are two independent Wiener processes, which will be specified in details in the next section. The symbol $\frac{\partial}{\partial \mathbf{n}}$ denotes the unit outer normal derivative on the boundary $\partial D$ of a bounded domain $D$ in $\mathbb{R}^{3}$. Note that $\delta_{t}^{\varepsilon}$ is the outer normal derivative of $u^{\varepsilon}$ on the boundary. We often write $u^{\varepsilon}(x, t)$ as $u^{\varepsilon}(t)$. In particular, in this paper we will only concern with the case of the nonlinear term $f\left(u^{\varepsilon}\right)=\sin u^{\varepsilon}$ (the Sine-Gordon equation).

The system (1.1) arises in the modeling of gas dynamics in an open bounded domain $D$, with points on boundary acting like a spring reacting to the excess pressure of the gas (see [24]). Chen and Zhang [6] studied the long time behavior of the solutions of the system (1.1) without the singular perturbation parameter. Also Chen, Duan and Zhang [7] derived the effective dynamics of the system (1.1) on a bounded domain perforated with small holes. For the deterministic case of the system (1.1), Beale [1, 2] and Mugnolo [26] established the well-posedness in some special cases. Cousin, Frota and Larkin [12] studied the global solvability and asymptotic behavior. Frigeri [15] considered large time dynamical behavior.

The singular perturbation issues of wave equations have been studied extensively. On the one hand, for a deterministic wave equation with a static boundary condition, Hale and Raugel
[18] and Mora [23] studied the approximation as the perturbation parameter $\varepsilon$ goes to zero. For deterministic wave equations with dynamical boundary conditions, Rodriguez-Bernal and Zuazua [29, 30] and Popsescu and Rodriguez-Bernal [28] considered the singular limiting equations. On the other hand, for stochastic wave equations with homogeneous boundary conditions, Cerrai and Freidlin [3, 4, Lv, Roberts and Wang [21, 22, 32, 33] investigated the approximation as the perturbation parameter $\varepsilon$ goes to zero.

In the present paper, we investigate the singular perturbations of the stochastic wave equation with random dynamical boundary conditions. Our goal is to derive the approximating equation of the system (1.1) for sufficiently small parameter $\varepsilon$. There are two key points to achieve this goal: The first is to establish the tightness of the solutions, and the second is to construct the approximating equation of the system (1.1).

For the first key point, the tightness of solutions for the system (1.1) heavily depends on the almost sure boundedness of the solutions, independent of the parameter $\varepsilon$. However, since the parameter $\varepsilon$ disturbs the system (1.1), it is difficult to derive the almost sure boundedness independent of $\varepsilon$. As showing in Chen and Zhang [6], the classic energy relation of this stochastic system (1.1) does not directly imply the a priori estimate of the solutions. Meanwhile, as we will see, the pseudo energy argument especially proposed in Chow [11] and Chen and Zhang [6] for stochastic wave equations also does not lead to the a priori estimate of the solutions. Therefore, for the system (1.1), we will explore a new way to establish the a priori estimate of solutions. By applying the a priori estimate, we could then obtain the global well-posedness and the almost sure boundedness independent of $\varepsilon$, which further implies the tightness of the solutions.

For the second key point, we use a splitting skill to construct the approximating equation. Firstly, we split the solution of the system (1.1) into three parts: the solution of a linear random ordinary differential equation (RODE), the solution of a random partial differential equation (RPDE), and the solution of a linear stochastic ordinary differential equation (SODE). Then we analyze their respective approximations for sufficiently small $\varepsilon$. Finally, we derive the approximating equation of the system (1.1) for the sufficiently small $\varepsilon$ in the sense of probability distribution, which is a stochastic parabolic equation with a dynamical boundary condition for $\alpha \in[1 / 2,1)$, and a deterministic wave equation with a dynamical boundary condition for $\alpha \in(1,+\infty)$.

We especially remark that the power exponent (of the singular perturbation parameter), $\alpha$, is in the set $[1 / 2,1) \cup(1,+\infty)$. The case of $0<\alpha<1 / 2$ is not covered in our results, as the condition of $\alpha \geq 1 / 2$ plays two crucial roles in our work: One is in deriving the a priori estimate
of solutions for the system (1.1) (see Proposition 3.2), and the other is in deriving the almost sure boundedness of solutions for the split linear stochastic ordinary differential equation (see the proof of Theorem 4.1). In addition, for the case of $\alpha=1$, in analyzing the approximation of the decomposition of the system (1.1) (see the proof of Theorem 4.1), there is no difference of convergence velocity between $O(\varepsilon)$ and $O\left(\varepsilon^{\alpha}\right)$ as $\varepsilon$ tends to zero, which means that the final approximating equation of the system (1.1) is just itself.

This paper is organized as follows. In the next section, we present preliminary results including the local well-posedness of the system (1.1) (Proposition 2.1). In section 3, we first derive the pseudo energy relation of the system (1.1) (Proposition 3.1), which implies certain estimates, independent of $\varepsilon$, for one part of the solution of the system (1.1) (Remark 3.2). With the help of these estimates, we establish the estimates, independent of $\varepsilon$, for the other parts of the solution of the system (1.1) (Proposition 3.4). All the a priori estimates for the solution play an important role in proving the global well-posedness (Proposition 3.5), the almost sure boundedness (Remark 3.3) and the tightness (Proposition 3.8). In section 4, we examine the solution as decomposed into three parts (Proposition 4.1), and further derive the approximating equation of the system (1.1), in the sense of probability distribution (Theorem 4.1).

## 2 Preliminaries

Consider the Wiener processes $W_{1}(t)$ and $W_{2}(t)$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\left\{\mathcal{F}_{t}\right\}_{t \in \mathbb{R}}$, and two-sided in time with values in $L^{2}(D)$ and $L^{2}(\partial D)$, respectively. Further assume that $W_{1}(t)$ and $W_{2}(t)$ are independent and that their covariance operators, $Q_{1}$ and $Q_{2}$, are symmetric nonnegative operators, satisfying $\operatorname{Tr} Q_{1}<+\infty$ and $\operatorname{Tr} Q_{2}<$ $+\infty$. Their expansions are given as follows

$$
\begin{aligned}
& W_{1}(t)=\sum_{i=1}^{+\infty} \sqrt{\alpha_{1 i}} \beta_{1 i} e_{i}, \quad \text { with } \quad Q_{1} e_{i}=\alpha_{1 i} e_{i}, \\
& W_{2}(t)=\sum_{i=1}^{+\infty} \sqrt{\alpha_{2 i}} \beta_{2 i} \gamma\left(e_{i}\right), \quad \text { with } \quad Q_{2} e_{i}=\alpha_{2 i} \gamma\left(e_{i}\right),
\end{aligned}
$$

where $\left\{e_{i}\right\}_{i \in \mathbb{N}}$ is an orthonormal basis of $L^{2}(D)$, and $\gamma$ is the trace operator from $D$ to $\partial D$. Moreover, $\left\{\beta_{1 i}\right\}_{i \in \mathbb{N}}$ and $\left\{\beta_{2 i}\right\}_{i \in \mathbb{N}}$ are two sequences of mutually independent (two-sided in time) standard scalar Wiener processes in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Then the system (1.1) can be written in the Itô form as follows

$$
\begin{cases}d u^{\varepsilon}=v^{\varepsilon} d t & \text { in } D,  \tag{2.1}\\ d v^{\varepsilon}=\left(\frac{1}{\varepsilon} \triangle u^{\varepsilon}-\frac{1}{\varepsilon} u^{\varepsilon}-\frac{1}{\varepsilon} v^{\varepsilon}+\frac{1}{\varepsilon} \sin u^{\varepsilon}\right) d t+\varepsilon^{\alpha-1} d W_{1} & \text { in } D, \\ d \delta^{\varepsilon}=\theta^{\varepsilon} d t & \text { on } \partial D, \\ d \theta^{\varepsilon}=\left(-\frac{1}{\varepsilon} \theta^{\varepsilon}-\frac{1}{\varepsilon} \delta^{\varepsilon}-\frac{1}{\varepsilon} v^{\varepsilon}\right) d t+\varepsilon^{\alpha-1} d W_{2} & \text { on } \partial D, \\ \delta_{t}^{\varepsilon}=\frac{\partial u^{\varepsilon}}{\partial \mathbf{n}} & \text { on } \partial D, \\ u^{\varepsilon}(0)=u_{0}, v^{\varepsilon}(0)=v_{0}=u_{1}, \delta^{\varepsilon}(0)=\delta_{0}, \theta^{\varepsilon}(0)=\theta_{0}=\delta_{1} . & \end{cases}
$$

Now we define

$$
A^{\varepsilon}=\left(\begin{array}{cccc}
0 & I & 0 & 0 \\
\frac{1}{\varepsilon}(\triangle-I) & -\frac{1}{\varepsilon} I & 0 & 0 \\
0 & 0 & 0 & I \\
0 & -\frac{1}{\varepsilon} I & -\frac{1}{\varepsilon} I & -\frac{1}{\varepsilon} I
\end{array}\right), F^{\varepsilon}\left(U^{\varepsilon}\right)=\left(\begin{array}{c}
0 \\
\frac{1}{\varepsilon} \sin u^{\varepsilon} \\
0 \\
0
\end{array}\right), W=\left(\begin{array}{c}
0 \\
W_{1} \\
0 \\
W_{2}
\end{array}\right) .
$$

Let $U^{\varepsilon}:=\left(u^{\varepsilon}, v^{\varepsilon}, \delta^{\varepsilon}, \theta^{\varepsilon}\right)^{T}$ be in the Hilbert space

$$
\mathcal{H}:=\left\{U^{\varepsilon} \in H^{1}(D) \times L^{2}(D) \times L^{2}(\partial D) \times L^{2}(\partial D) \left\lvert\, \frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}=\theta^{\varepsilon}\right.\right\},
$$

with norm

$$
\left\|U^{\varepsilon}\right\|_{\mathcal{H}}^{2}=\left\|u^{\varepsilon}\right\|_{H^{1}(D)}^{2}+\left\|v^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+\left\|\theta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} .
$$

Here and hereafter, the superscript " $T$ " denotes the transpose for a matrix.
Thus the system (2.1) is further rewritten as

$$
\left\{\begin{array}{l}
d U^{\varepsilon}=A^{\varepsilon} U^{\varepsilon} d t+F^{\varepsilon}\left(U^{\varepsilon}\right) d t+\varepsilon^{\alpha-1} d W(t)  \tag{2.2}\\
U^{\varepsilon}(0)=U_{0}^{\varepsilon}=\left(u_{0}, v_{0}, \delta_{0}, \theta_{0}\right)^{T}
\end{array}\right.
$$

For the Cauchy problem (2.2), it follows from Frigeri [15] that the operator $A^{\varepsilon}$ generates a strongly continuous semigroup $S(t)=e^{A^{\varepsilon} t}$ for $t \geq 0$ on $\mathcal{H}$. Then Equation (2.2) can be formulated in the mild sense

$$
\begin{equation*}
U^{\varepsilon}(t)=S(t) U^{\varepsilon}(0)+\int_{0}^{t} S(t-s) F^{\varepsilon}\left(U^{\varepsilon}(s)\right) d s+\int_{0}^{t} S(t-s) \varepsilon^{\alpha-1} d W(s) \tag{2.3}
\end{equation*}
$$

Proposition 2.1 (Local well-posedness) Let the initial datum $U^{\varepsilon}(0)$ be a $\mathcal{F}_{0}$-measurable random variable with value in $\mathcal{H}$. Then the Cauchy problem (2.2) has a unique local mild solution $U^{\varepsilon}(t)$ in $C\left(\left[0, \tau^{*}\right), \mathcal{H}\right)$, where $\tau^{*}$ is a stopping time depending on $U^{\varepsilon}(0)$ and $\omega$. Moreover, the mild solution $U^{\varepsilon}(t)$ is also a weak solution in the following sense
$\left\langle U^{\varepsilon}(t), \phi\right\rangle_{\mathcal{H}}=\left\langle U^{\varepsilon}(0), \phi\right\rangle_{\mathcal{H}}+\int_{0}^{t}\left\langle A^{\varepsilon} U^{\varepsilon}(s), \phi\right\rangle_{\mathcal{H}} d s+\int_{0}^{t}\left\langle F^{\varepsilon}\left(U^{\varepsilon}(s)\right), \phi\right\rangle_{\mathcal{H}} d s+\int_{0}^{t}\left\langle\varepsilon^{\alpha-1} d W(s), \phi\right\rangle_{\mathcal{H}}$
for any $t \in\left[0, \tau^{*}\right)$ and $\phi \in \mathcal{H}$.
Using the cut-off function method and combining with Theorem 7.4 and the stochastic Fubini theorem in [13], we can prove Proposition 2.1. Please refer to Chen and Zhang [6].

## 3 Boundedness and tightness

In this section, we will establish the almost sure boundedness independent of the parameter $\varepsilon$ and the tightness of solutions for the system (1.1). Due to the singular perturbation in the system (1.1), the classic energy method and the pseudo energy method does not directly imply the almost sure boundedness, independent of the parameter $\varepsilon$, of solutions. We will explore a new way to do it.

For a real parameter $r$ in $(0,1)$, we define

$$
\begin{equation*}
v_{r}^{\varepsilon}=v^{\varepsilon}+r u^{\varepsilon} \quad \text { and } \quad \theta_{r}^{\varepsilon}=\theta^{\varepsilon}+r \delta^{\varepsilon} \tag{3.1}
\end{equation*}
$$

with $\left(u^{\varepsilon}, v^{\varepsilon}, \delta^{\varepsilon}, \theta^{\varepsilon}\right)^{T}$ being the solution of the Cauchy problem (2.1). Then $U_{r}^{\varepsilon}=\left(u^{\varepsilon}, v_{r}^{\varepsilon}, \delta^{\varepsilon}, \theta_{r}^{\varepsilon}\right)^{T} \in$ $\mathcal{H}$ satisfies the following equation

$$
\begin{cases}d u^{\varepsilon}=\left(v_{r}^{\varepsilon}-r u^{\varepsilon}\right) d t & \text { in } D \times\left[0, \tau^{*}\right)  \tag{3.2}\\ d v_{r}^{\varepsilon}=\left(\frac{1}{\varepsilon} \triangle u^{\varepsilon}-\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right) u^{\varepsilon}-\frac{1}{\varepsilon}(1-\varepsilon r) v_{r}^{\varepsilon}+\frac{1}{\varepsilon} \sin u^{\varepsilon}\right) d t+\varepsilon^{\alpha-1} d W_{1}(t) & \text { in } D \times\left[0, \tau^{*}\right) \\ d \delta^{\varepsilon}=\left(\theta_{r}^{\varepsilon}-r \delta^{\varepsilon}\right) d t & \text { on } \partial D \times\left[0, \tau^{*}\right) \\ d \theta_{r}^{\varepsilon}=\left(-\frac{1}{\varepsilon}(1-\varepsilon r) \theta_{r}^{\varepsilon}-\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right) \delta^{\varepsilon}-\frac{1}{\varepsilon} v_{r}^{\varepsilon}+\frac{r}{\varepsilon} u^{\varepsilon}\right) d t+\varepsilon^{\alpha-1} d W_{2}(t) & \text { on } \partial D \times\left[0, \tau^{*}\right) \\ \delta_{t}^{\varepsilon}=\frac{\partial u^{\varepsilon}}{\partial \mathbf{n}} & \text { on } \partial D \times\left[0, \tau^{*}\right) \\ u^{\varepsilon}(0)=u_{0}, v_{r}^{\varepsilon}(0)=v_{0}+r u_{0}:=v_{r 0} \\ \delta^{\varepsilon}(0)=\delta_{0}, \theta_{r}^{\varepsilon}(0)=\theta_{0}+r \delta_{0}:=\theta_{r 0} & \end{cases}
$$

Define a pseudo energy functional $\mathcal{E}_{r}^{\varepsilon}(t)$ for the Cauchy problem (2.2) as follows

$$
\begin{aligned}
\mathcal{E}_{r}^{\varepsilon}(t):= & \varepsilon\left\|v_{r}^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\left\|\nabla u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\left(1-r+\varepsilon r^{2}\right)\left\|u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2} \\
& +\varepsilon\left\|\theta_{r}^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}+\left(1-r+\varepsilon r^{2}\right)\left\|\delta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}+4\left\|\cos \frac{u^{\varepsilon}(t)}{2}\right\|_{L^{2}(D)}^{2} \\
& +2 r\left\langle u^{\varepsilon}(t), \delta^{\varepsilon}(t)\right\rangle_{L^{2}(\partial D)}
\end{aligned}
$$

Proposition 3.1 (Pseudo energy equation) Let the initial datum $U_{r}^{\varepsilon}(0)$ be a $\mathcal{F}_{0}$ measurable random variable in $L^{2}(\Omega, \mathcal{H})$. Then for any time $t \in\left[0, \tau^{*}\right)$, we have

$$
\begin{aligned}
\mathcal{E}_{r}^{\varepsilon}(t)= & \mathcal{E}_{r}^{\varepsilon}(0)-\int_{0}^{t}\left[2(1-\varepsilon r)\left\|v_{r}^{\varepsilon}\right\|_{L^{2}(D)}^{2}+2 r\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+2\left(1-r+\varepsilon r^{2}\right) r\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}\right. \\
& \left.+2(1-\varepsilon r)\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+2\left(1-r+\varepsilon r^{2}\right) r\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}\right] d s \\
& +2 r \int_{0}^{t}\left\langle u^{\varepsilon}, \sin u^{\varepsilon}\right\rangle_{L^{2}(D)} d s+4 r \int_{0}^{t}\left\langle u^{\varepsilon}, \theta_{r}^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s-4 r^{2} \int_{0}^{t}\left\langle u^{\varepsilon}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s \\
& +\int_{0}^{t}\left\langle 2 v_{r}^{\varepsilon}, \varepsilon^{\alpha} d W_{1}(s)\right\rangle_{L^{2}(D)}+\int_{0}^{t}\left\langle 2 \theta_{r}^{\varepsilon}, \varepsilon^{\alpha} d W_{2}(s)\right\rangle_{L^{2}(\partial D)} \\
& +\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{1} \cdot t+\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{2} \cdot t .
\end{aligned}
$$

Moreover,

$$
\begin{align*}
\mathbb{E} \mathcal{E}_{r}^{\varepsilon}(t)= & \mathbb{E} \mathcal{E}_{r}^{\varepsilon}(0)-\int_{0}^{t}\left[2(1-\varepsilon r) \mathbb{E}\left\|v_{r}^{\varepsilon}\right\|_{L^{2}(D)}^{2}+2 r \mathbb{E}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}\right. \\
& +2\left(1-r+\varepsilon r^{2}\right) r \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+2(1-\varepsilon r) \mathbb{E}\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} \\
& \left.+2\left(1-r+\varepsilon r^{2}\right) r \mathbb{E}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}\right] d s+2 r \int_{0}^{t} \mathbb{E}\left\langle u^{\varepsilon}, \sin u^{\varepsilon}\right\rangle_{L^{2}(D)} d s  \tag{3.4}\\
& +4 r \int_{0}^{t} \mathbb{E}\left\langle u^{\varepsilon}, \theta_{r}^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s-4 r^{2} \int_{0}^{t} \mathbb{E}\left\langle u^{\varepsilon}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s \\
& +\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{1} \cdot t+\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{2} \cdot t .
\end{align*}
$$

Proof. Firstly, we examine the second equation of (3.2). Set $M\left(v_{r}^{\varepsilon}\right):=\int_{D}\left|v_{r}^{\varepsilon}\right|^{2} d x$. Then from the Itô formula, we deduce that

$$
\begin{align*}
M\left(v_{r}^{\varepsilon}(t)\right)= & M\left(v_{r}^{\varepsilon}(0)\right)+\int_{0}^{t}\left\langle M^{\prime}\left(v_{r}^{\varepsilon}\right), \varepsilon^{\alpha-1} d W_{1}(s)\right\rangle_{L^{2}(D)} \\
& +\int_{0}^{t} \frac{1}{2} T r\left[M^{\prime \prime}\left(v_{r}^{\varepsilon}\right)\left(\varepsilon^{\alpha-1} Q_{1}\right)^{\frac{1}{2}}\left(\varepsilon^{\alpha-1} Q_{1}^{\frac{1}{2}}\right)^{*}\right] d s \\
& +\int_{0}^{t}\left\langle M^{\prime}\left(v_{r}^{\varepsilon}\right),\left(\frac{1}{\varepsilon} \triangle u^{\varepsilon}-\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right) u^{\varepsilon}-\frac{1}{\varepsilon}(1-\varepsilon r) v_{r}^{\varepsilon}+\frac{1}{\varepsilon} \sin u^{\varepsilon}\right)\right\rangle_{L^{2}(D)} d s, \tag{3.5}
\end{align*}
$$

with $M^{\prime}\left(v_{r}^{\varepsilon}\right)=2 v_{r}^{\varepsilon}$ and $M^{\prime \prime}\left(v_{r}^{\varepsilon}\right)=2 \varphi$ for any $\varphi$ in $L^{2}(D)$. After some calculations, we conclude that

$$
\begin{aligned}
& \left\langle M^{\prime}\left(v_{r}^{\varepsilon}\right)\left(\frac{1}{\varepsilon} \triangle u^{\varepsilon}-\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right) u^{\varepsilon}-\frac{1}{\varepsilon}(1-\varepsilon r) v_{r}^{\varepsilon}+\frac{1}{\varepsilon} \sin u^{\varepsilon}\right)\right\rangle_{L^{2}(D)} \\
= & -\frac{d}{d S}\left[\frac{1}{\varepsilon}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right)\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{4}{\varepsilon}\left\|\cos \frac{u^{\varepsilon}}{2}\right\|_{L^{2}(D)}^{2}\right] \\
& -\left[\frac{2 r}{\varepsilon}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{2}{\varepsilon} \cdot\left(1-r+\varepsilon r^{2}\right) \cdot r\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{2}{\varepsilon}(1-\varepsilon r)\left\|v_{r}^{\varepsilon}\right\|_{L^{2}(D)}^{2}\right] \\
& +\frac{2}{\varepsilon}\left\langle v_{r}^{\varepsilon}, \frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}\right\rangle_{L^{2}(\partial D)}^{2}+\frac{2 r}{\varepsilon}\left\langle u^{\varepsilon}, \sin u^{\varepsilon}\right\rangle_{L^{2}(D) .}
\end{aligned}
$$

It further follows from (3.5) and (3.6) that

$$
\begin{align*}
& \left\|v_{r}^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left\|\nabla u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon} \cdot\left(1-r+\varepsilon r^{2}\right)\left\|u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\frac{4}{\varepsilon}\left\|\cos \frac{u^{\varepsilon}(t)}{2}\right\|_{L^{2}(D)}^{2} \\
= & \left\|v_{r}^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left\|\nabla u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon} \cdot\left(1-r+\varepsilon r^{2}\right)\left\|u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{4}{\varepsilon}\left\|\cos \frac{u^{\varepsilon}(0)}{2}\right\|_{L^{2}(D)}^{2} \\
& -\int_{0}^{t}\left[\frac{2}{\varepsilon}(1-\varepsilon r)\left\|v_{r}^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{2 r}{\varepsilon}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{2}{\varepsilon}\left(1-r+\varepsilon r^{2}\right) r\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}\right] d s \\
& +\frac{2}{\varepsilon} \int_{0}^{t}\left\langle v_{r}^{\varepsilon}, \frac{\left.\partial u^{\varepsilon}\right\rangle^{2}}{\partial \mathbf{n}}\right\rangle_{L^{2}(\partial D) d} d s+\frac{2 r}{\varepsilon} \int_{0}^{t}\left\langle u^{\varepsilon}, \sin u^{\varepsilon}\right\rangle_{L^{2}(D)} d s \\
& +\int_{0}^{t}\left\langle 2 v_{r}^{\varepsilon}, \varepsilon^{\alpha-1} d W_{1}(s)\right\rangle_{L^{2}(D)}+\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{1} \cdot t . \tag{3.7}
\end{align*}
$$

Secondly, we examine the fourth equation of (3.2). Set $M\left(\theta_{r}^{\varepsilon}\right):=\int_{\partial D}\left|\theta_{r}^{\varepsilon}\right|^{2} d x$. Note that

$$
\begin{align*}
M\left(\theta_{r}^{\varepsilon}(t)\right)= & M\left(\theta_{r}^{\varepsilon}(0)\right)+\int_{0}^{t}\left\langle M^{\prime}\left(\theta_{r}^{\varepsilon}\right), \varepsilon^{\alpha-1} d W_{2}(s)\right\rangle_{L^{2}}(\partial D) \\
& +\int_{0}^{t} \frac{1}{2} T r\left[M^{\prime \prime}\left(\theta_{r}^{\varepsilon}\right)\left(\varepsilon^{\alpha-1} Q_{2}\right)^{\frac{1}{2}}\left(\varepsilon^{\alpha-1} Q_{2}^{\frac{1}{2}}\right)^{*}\right] d s  \tag{3.8}\\
& +\int_{0}^{t}\left\langle M^{\prime}\left(\theta_{r}^{\varepsilon}\right),\left(-\frac{1}{\varepsilon}(1-\varepsilon r) \theta_{r}^{\varepsilon}-\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right) \delta^{\varepsilon}-\frac{1}{\varepsilon} v_{r}^{\varepsilon}+\frac{r}{\varepsilon} u^{\varepsilon}\right)\right\rangle_{L^{2}(\partial D)} d s,
\end{align*}
$$

with $M^{\prime}\left(\theta_{r}^{\varepsilon}\right)=2 \theta_{r}^{\varepsilon}$ and $M^{\prime \prime}\left(\theta_{r}^{\varepsilon}\right)=2 \phi$ for any $\phi$ in $L^{2}(\partial D)$. After some calculations, we obtain that

$$
\begin{align*}
& \left\langle M^{\prime}\left(\theta_{r}^{\varepsilon}\right),\left(-\frac{1}{\varepsilon}(1-\varepsilon r) \theta_{r}^{\varepsilon}-\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right) \delta^{\varepsilon}-\frac{1}{\varepsilon} v_{r}^{\varepsilon}+\frac{r}{\varepsilon} u^{\varepsilon}\right)\right\rangle_{L^{2}(\partial D)} \\
= & -\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right) \frac{d}{d s}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}-\frac{2}{\varepsilon} \cdot\left(1-r+\varepsilon r^{2}\right) r\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}-\frac{2}{\varepsilon}(1-\varepsilon r)\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}  \tag{3.9}\\
& -\frac{2}{\varepsilon}\left\langle\frac{\partial \varepsilon^{\varepsilon}}{\partial \mathbf{n}}, v_{r}^{\varepsilon}\right\rangle_{L^{2}}(\partial D)-\frac{2 r}{\varepsilon}\left\langle\delta^{\varepsilon}, v_{r}^{\varepsilon}\right\rangle_{L^{2}(\partial D)}+\frac{2 r}{\varepsilon}\left\langle\theta_{r}^{\varepsilon}, u^{\varepsilon}\right\rangle_{L^{2}(\partial D) .}
\end{align*}
$$

Then it follows from (3.8) and (3.9) that

$$
\begin{align*}
& \left\|\theta_{r}^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}+\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right)\left\|\delta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2} \\
= & \left\|\theta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}+\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right)\left\|\delta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2} \\
& -\int_{0}^{t}\left[\frac{2}{\varepsilon}(1-\varepsilon r)\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+\frac{2}{\varepsilon} \cdot\left(1-r+\varepsilon r^{2}\right) r\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}\right] d s  \tag{3.10}\\
& \left.-\frac{2}{\varepsilon} \int_{0}^{t}\left\langle\frac{\partial u^{\varepsilon}}{\partial \mathrm{n}}, v_{r}^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s-\frac{2 r}{\varepsilon} \int_{0}^{t}\left\langle\delta^{\varepsilon}, v_{r}^{\varepsilon}\right\rangle\right\rangle_{L^{2}(\partial D)} d s+\frac{2 r}{\varepsilon} \int_{0}^{t}\left\langle\theta_{r}^{\varepsilon}, u^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s \\
& +\int_{0}^{t}\left\langle 2 \theta_{r}^{\varepsilon}, \varepsilon^{\alpha-1} d W_{2}(s)\right\rangle_{L^{2}(\partial D)}+\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{2} \cdot t .
\end{align*}
$$

Thus, from (3.7) and (3.10), we have

$$
\begin{align*}
& \left\|v_{r}^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left\|\nabla u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right)\left\|u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\left\|\theta_{r}^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2} \\
& +\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right)\left\|\delta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}+\frac{4}{\varepsilon}\left\|\cos \frac{u^{\varepsilon}(t)}{2}\right\|_{L^{2}(D)}^{2} \\
= & \left\|v_{r}^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left\|\nabla u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right)\left\|u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\left\|\theta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2} \\
& +\frac{1}{\varepsilon}\left(1-r+\varepsilon r^{2}\right)\left\|\delta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}+\frac{4}{\varepsilon}\left\|\cos \frac{u^{\varepsilon}(0)}{2}\right\|_{L^{2}(D)}^{2} \\
& -\int_{0}^{t}\left[\frac{2}{\varepsilon}(1-\varepsilon r)\left\|v_{r}^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{2 r}{\varepsilon}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{2}{\varepsilon} \cdot\left(1-r+\varepsilon r^{2}\right) r\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}\right.  \tag{3.11}\\
& \left.+\frac{2}{\varepsilon}(1-\varepsilon r)\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+\frac{2}{\varepsilon} \cdot\left(1-r+\varepsilon r^{2}\right) r\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}\right] d s \\
& +\frac{2 r}{\varepsilon} \int_{0}^{t}\left\langle u^{\varepsilon}, \sin u^{\varepsilon}\right\rangle_{L^{2}(D)} d s-\frac{2 r}{\varepsilon} \int_{0}^{t}\left\langle\delta^{\varepsilon}, v_{r}^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s+\frac{2 r}{\varepsilon} \int_{0}^{t}\left\langle\theta_{r}^{\varepsilon}, u^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s \\
& +\int_{0}^{t}\left\langle 2 v_{r}^{\varepsilon}, \varepsilon^{\alpha-1} d W_{1}(s)\right\rangle_{L^{2}(D)}+\int_{0}^{t}\left\langle 2 \theta_{r}^{\varepsilon}, \varepsilon^{\alpha-1} d W_{2}(s)\right\rangle_{L^{2}(\partial D)} \\
& +\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{1} \cdot t+\varepsilon^{2 \alpha-2} T r Q_{2} \cdot t .
\end{align*}
$$

Meanwhile, we observe that

$$
\begin{aligned}
& \left\langle u^{\varepsilon}(t), \delta^{\varepsilon}(t)\right\rangle_{L^{2}(\partial D)} \\
= & \left\langle u^{\varepsilon}(0), \delta^{\varepsilon}(0)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle\left(u^{\varepsilon}\right)_{s}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle u^{\varepsilon},\left(\delta^{\varepsilon}\right)_{s}\right\rangle_{L^{2}(\partial D)} d s \\
= & \left\langle u^{\varepsilon}(0), \delta^{\varepsilon}(0)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle v^{\varepsilon}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle u^{\varepsilon}, \theta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s \\
= & \left\langle u^{\varepsilon}(0), \delta^{\varepsilon}(0)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle v_{r}^{\varepsilon}-r u^{\varepsilon}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle u^{\varepsilon}, \theta_{r}^{\varepsilon}-r \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s \\
= & \left\langle u^{\varepsilon}(0), \delta^{\varepsilon}(0)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle v_{r}^{\varepsilon}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle u^{\varepsilon}, \theta_{r}^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s \\
& -2 r \int_{0}^{t}\left\langle u^{\varepsilon}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s,
\end{aligned}
$$

which implies that

$$
\begin{align*}
& -\frac{2 r}{\varepsilon} \int_{0}^{t}\left\langle v_{r}^{\varepsilon}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s \\
= & -\frac{2 r}{\varepsilon}\left\langle u^{\varepsilon}(t), \delta^{\varepsilon}(t)\right\rangle_{L^{2}(\partial D)}+\frac{2 r}{\varepsilon}\left\langle u^{\varepsilon}(0), \delta^{\varepsilon}(0)\right\rangle_{L^{2}(\partial D)}+\frac{2 r}{\varepsilon} \int_{0}^{t}\left\langle u^{\varepsilon}, \theta_{r}^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s  \tag{3.12}\\
& -\frac{4 r^{2}}{\varepsilon} \int_{0}^{t}\left\langle u^{\varepsilon}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s .
\end{align*}
$$

Hence, it follows from (3.11) and (3.12) that (3.3) and (3.4) hold.
Proposition 3.2 Let $\alpha \in[1 / 2,1) \cup(1,+\infty)$. Assume that the initial datum $U_{r}^{\varepsilon}(0)$ is a $\mathcal{F}_{0}$-measurable random variable in $L^{2}(\Omega, \mathcal{H})$. Then for any time $t \in\left[0, \tau^{*}\right), \varepsilon \in(0,1 / 2)$ and a sufficiently small $r \in(0,1 / 2)$, there exists a positive constant $C$, independent of the parameter $\varepsilon$, such that

$$
\left.\begin{array}{l}
\frac{d}{d t}\left[\varepsilon \mathbb{E}\left\|v_{r}^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\varepsilon \mathbb{E}\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+\mathbb{E}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}\right] \\
\leq  \tag{3.13}\\
-C\left[\varepsilon \mathbb{E}\left\|v_{r}^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\varepsilon \mathbb{E}\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+\mathbb{E}\left\|\delta^{\delta}\right\|_{L^{2}(\partial D)}^{2}\right] \\
\\
\end{array}\right]\left[\operatorname{Tr} Q_{1}+\operatorname{Tr} Q_{2}+1\right] . \quad \text {. }
$$

Proof. On the one hand, it follows from the Cauchy inequality and the trace inequality that there exists a positive constant $C_{T I}>0$ (here and hereafter $C_{T I}$ denotes the positive constant in the trace inequality) such that

$$
\begin{aligned}
0 & \leq r \mathbb{E}\left\|u^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}+2 r \mathbb{E}\left\langle u^{\varepsilon}(t), \delta^{\varepsilon}(t)\right\rangle_{L^{2}(\partial D)}+r \mathbb{E}\left\|\delta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2} \\
& \leq r C_{T I}^{2} \mathbb{E}\left\|u^{\varepsilon}(t)\right\|_{H^{1}(D)}^{2}+2 r \mathbb{E}\left\langle u^{\varepsilon}(t), \delta^{\varepsilon}(t)\right\rangle_{L^{2}(\partial D)}+r \mathbb{E}\left\|\delta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2},
\end{aligned}
$$

which implies that

$$
\begin{align*}
\mathbb{E} \mathcal{E}_{r}^{\varepsilon}(t) \geq & \varepsilon \mathbb{E}\left\|v_{r}^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\left(1-r C_{T I}^{2}\right) \mathbb{E}\left\|\nabla u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2} \\
& +\left(1-r-r C_{T I}^{2}+\varepsilon r^{2}\right) \mathbb{E}\left\|u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\varepsilon \mathbb{E}\left\|\theta_{r}^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}  \tag{3.14}\\
& +\left(1-2 r+\varepsilon r^{2}\right) \mathbb{E}\|\delta(t)\|_{L^{2}(\partial D)}^{2}
\end{align*}
$$

On the other hand, by the Hölder inequality, the Young inequality and the trace inequality, we obtain that

$$
\begin{aligned}
\mathbb{E}\left\langle u^{\varepsilon}, \theta_{r}^{\varepsilon}\right\rangle_{L^{2}(\partial D)} & \leq \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(\partial D)} \cdot \mathbb{E}\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)} \\
& \leq r \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+\frac{1}{4 r} \mathbb{E}\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} \\
& \leq r C_{T I}^{2} \mathbb{E}\left\|u^{\varepsilon}\right\|_{H^{1}(D)}^{2}+\frac{1}{4 r} \mathbb{E}\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}
\end{aligned}
$$

which implies that

$$
\begin{equation*}
4 r \mathbb{E}\left\langle u^{\varepsilon}, \theta_{r}^{\varepsilon}\right\rangle_{L^{2}(\partial D)} \leq 4 r^{2} C_{T I}^{2} \mathbb{E}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+4 r^{2} C_{T I}^{2} \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} \tag{3.15}
\end{equation*}
$$

At the same time, it follows from the Cauchy inequality and the trace inequality that

$$
\begin{align*}
-4 r^{2} \mathbb{E}\left\langle u^{\varepsilon}, \delta^{\varepsilon}\right\rangle_{L^{2}(\partial D)} & \leq 2 r^{2} \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+2 r^{2} \mathbb{E}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} \\
& \leq 2 r^{2} C_{T I}^{2} \mathbb{E}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+2 r^{2} C_{T I}^{2} \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+2 r^{2} \mathbb{E}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} \tag{3.16}
\end{align*}
$$

Also the Cauchy inequality leads to

$$
\begin{align*}
2 r \mathbb{E}\left\langle u^{\varepsilon}, \sin u^{\varepsilon}\right\rangle_{L^{2}(D)} & \leq r \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+r \mathbb{E}\left\|\sin u^{\varepsilon}\right\|_{L^{2}(D)}^{2}  \tag{3.17}\\
& \leq r \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+C .
\end{align*}
$$

Then it follows from Proposition 3.1 and (3.15)-(3.17) that

$$
\begin{align*}
\mathbb{E}_{r}^{\varepsilon}(t) \leq & \mathbb{E} \mathcal{E}_{r}^{\varepsilon}(0)-\int_{0}^{t}\left[2(1-\varepsilon r) \mathbb{E}\left\|v^{\varepsilon}\right\|_{L^{2}(D)}^{2}+2 r\left(1-3 r C_{T I}^{2}\right) \mathbb{E}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}\right. \\
& +r\left[1-2 r-6 r C_{T I}^{2}+2 \varepsilon r^{2}\right] \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2} \\
& \left.+(1-2 \varepsilon r) \mathbb{E}\left\|\theta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+2 r\left(1-2 r+\varepsilon r^{2}\right) \mathbb{E}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}\right] d s  \tag{3.18}\\
& +\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{1} \cdot t+\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{2} \cdot t+C t
\end{align*}
$$

For $\varepsilon \in(0,1 / 2)$, choose $r$ in $(0,1 / 2)$ sufficiently small such that

$$
\begin{equation*}
\min \left\{1-3 r C_{T I}^{2}, 1-r-r C_{T I}^{2}+\varepsilon r^{2}, 1-2 r-6 r C_{T I}^{2}+2 \varepsilon r^{2}, 1-2 r+\varepsilon r^{2}\right\}>0 \tag{3.19}
\end{equation*}
$$

Then

$$
\begin{equation*}
2(1-\varepsilon r)>\varepsilon, \quad \text { and } \quad 1-2 \varepsilon r>\varepsilon \tag{3.20}
\end{equation*}
$$

Furthermore, noticing that $\alpha \in[1 / 2,1) \bigcup(1,+\infty)$, we have

$$
\begin{equation*}
0<\varepsilon^{2 \alpha-1} \leq 1 \tag{3.21}
\end{equation*}
$$

Therefore, from (3.14), (3.18)-(3.21), there exists a positive constant $C$ independent of the parameter $\varepsilon$ such that (3.13) holds.

Proposition 3.3 Let $\alpha \in[1 / 2,1) \cup(1,+\infty)$ and $\varepsilon \in(0,1 / 2)$. Assume that the initial datum $U_{r}^{\varepsilon}(0)$ is a $\mathcal{F}_{0}$-measurable random variable in $L^{2}(\Omega, \mathcal{H})$. Then there exists a positive constant $C$ independent of the parameter $\varepsilon$ such that

$$
\begin{equation*}
\varepsilon \mathbb{E}\left\|v_{r}^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\varepsilon \mathbb{E}\left\|\theta_{r}^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}+\mathbb{E}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} \leq C, \forall t \in\left[0, \tau^{*}\right) . \tag{3.22}
\end{equation*}
$$

Proposition 3.3 is easily deduced from the Gronwall inequality and Proposition 3.2.
Remark 3.1 From Frigeri [15], for $r \in(0,1 / 2), \mathbb{E}\left\|U_{r}^{\varepsilon}\right\|_{\mathcal{H}}^{2} \geq \frac{1}{2} \mathbb{E}\left\|U^{\varepsilon}\right\|_{\mathcal{H}}^{2}$. Therefore, if we obtain the almost sure boundedness of $U_{r}^{\varepsilon}$ in $\mathcal{H}$, we naturally derive the almost sure boundedness of $U^{\varepsilon}$ in $\mathcal{H}$. But from Proposition 3.3, since the parameter $\varepsilon$ disturbs the system (1.1), we can not use the pseudo energy method to directly derive the almost sure boundedness, while this method is effective for wave equations without the singular parameter $\varepsilon$ (see Chen and Zhang [6]).

Remark 3.2 Although Proposition 3.3 does not implies the almost sure boundedness of $U^{\varepsilon}$ in $\mathcal{H}$, we can obtain that under the condition of Proposition 3.3, there exists a positive constant $C$ independent of the parameter $\varepsilon$ such that

$$
\begin{equation*}
\mathbb{E}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2} \leq C, \quad \mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2} \leq C, \quad \mathbb{E}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} \leq C, \quad \forall t \in\left[0, \tau^{*}\right) \tag{3.23}
\end{equation*}
$$

In the following, we will continue to derive the almost sure boundedness of the solution for the Cauchy problem (2.2).

Proposition 3.4 Let $\alpha \in[1 / 2,1) \cup(1,+\infty)$ and $\varepsilon \in(0,1 / 2)$. Assume that the initial datum $U^{\varepsilon}(0)$ is a $\mathcal{F}_{0}$-measurable random variable in $L^{2}(\Omega, \mathcal{H})$. Then there exists a positive constant $C$ independent of the parameter $\varepsilon$ such that

$$
\begin{equation*}
\mathbb{E}\left\|v^{\varepsilon}\right\|_{L^{2}(D)}^{2} \leq C, \quad \mathbb{E}\left\|\theta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} \leq C, \quad \forall t \in\left[0, \tau^{*}\right) \tag{3.24}
\end{equation*}
$$

Proof. Set $M\left(v^{\varepsilon}(t)\right)=\int_{D}\left|v^{\varepsilon}(t)\right|^{2} d x$. For the second equation of (2.1), from the Itô formula, we get

$$
\begin{align*}
M\left(v^{\varepsilon}(t)\right)= & M\left(v^{\varepsilon}(0)\right)+\int_{0}^{t}\left\langle M^{\prime}\left(v^{\varepsilon}\right), \varepsilon^{\alpha-1} d W_{1}(s)\right\rangle_{L^{2}(D)} \\
& +\int_{0}^{t}\left\langle M^{\prime}\left(v^{\varepsilon}\right),\left(\frac{1}{\varepsilon} \triangle u^{\varepsilon}-\frac{1}{\varepsilon} u^{\varepsilon}-\frac{1}{\varepsilon} v^{\varepsilon}+\frac{1}{\varepsilon} \sin u^{\varepsilon}\right)\right\rangle_{L^{2}(D)} d s  \tag{3.25}\\
& +\int_{0}^{t} \frac{1}{2} T r\left[M^{\prime \prime}\left(v^{\varepsilon}\right)\left(\varepsilon^{\alpha-1} Q_{1}\right)^{\frac{1}{2}}\left(\varepsilon^{\alpha-1} Q_{1}^{\frac{1}{2}}\right)^{*}\right] d s,
\end{align*}
$$

with $M^{\prime}\left(v^{\varepsilon}\right)=2 v^{\varepsilon}$ and $M^{\prime \prime}\left(v^{\varepsilon}\right)=2 \varphi$ for any $\varphi$ in $L^{2}(D)$. Thus we deduce that

$$
\begin{align*}
& \left\langle M^{\prime}\left(v^{\varepsilon}\right),\left(\frac{1}{\varepsilon} \triangle u^{\varepsilon}-\frac{1}{\varepsilon} u^{\varepsilon}-\frac{1}{\varepsilon} v^{\varepsilon}+\frac{1}{\varepsilon} \sin u^{\varepsilon}\right)\right\rangle_{L^{2}(D)} \\
= & -\frac{d}{d s}\left[\frac{1}{\varepsilon}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left\|u^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\frac{4}{\varepsilon}\left\|\cos \frac{u^{\varepsilon}}{2}\right\|_{L^{2}(D)}^{2}\right]  \tag{3.26}\\
& -\frac{2}{\varepsilon}\left\|v^{\varepsilon}\right\|_{L^{2}(D)}^{2}+2\left\langle v^{\varepsilon}, \frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}\right\rangle_{L^{2}(\partial D) .} .
\end{align*}
$$

It immediately follows from (3.25) and (3.26) that

$$
\begin{align*}
& \left\|v^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left\|\nabla u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left\|u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\frac{4}{\varepsilon}\left\|\cos \frac{u^{\varepsilon}(t)}{2}\right\|_{L^{2}(D)}^{2} \\
= & \left\|v^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left\|\nabla u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon}\left\|u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{4}{\varepsilon}\left\|\cos \frac{u^{\varepsilon}(0)}{2}\right\|_{L^{2}(D)}^{2}  \tag{3.27}\\
& -\frac{2}{\varepsilon} \int_{0}^{t}\left\|v^{\varepsilon}\right\|_{L^{2}(D)}^{2} d s+\frac{2}{\varepsilon} \int_{0}^{t}\left\langle v^{\varepsilon}, \frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}\right\rangle_{L^{2}(\partial D)} d s \\
& +\int_{0}^{t}\left(2 v^{\varepsilon}, \varepsilon^{\alpha-1} d W_{1}(s)\right\rangle_{L^{2}(D)}+\varepsilon^{2 \alpha-2} T r Q_{1} \cdot t .
\end{align*}
$$

Secondly, noticing that the fourth equation of (2.1) and putting $M\left(\theta^{\varepsilon}\right)=\int_{\partial D}\left|\theta^{\varepsilon}\right|^{2} d x$, using the Itô formula, we have

$$
\begin{align*}
M\left(\theta^{\varepsilon}(t)\right)= & M\left(\theta_{0}^{\varepsilon}\right)+\int_{0}^{t}\left\langle M^{\prime}\left(\theta^{\varepsilon}\right), \varepsilon^{\alpha-1} d W_{2}(s)\right\rangle_{L^{2}(\partial D)} \\
& +\int_{0}^{t}\left\langle M^{\prime}\left(\theta^{\varepsilon}\right),\left(-\frac{1}{\varepsilon} \theta^{\varepsilon}-\frac{1}{\varepsilon} \delta^{\varepsilon}-\frac{1}{\varepsilon} v^{\varepsilon}\right)\right\rangle_{L^{2}(\partial D)} d s  \tag{3.28}\\
& +\int_{0}^{t} \frac{1}{2} \operatorname{Tr}\left[M^{\prime \prime}\left(\theta^{\varepsilon}\right)\left(\varepsilon^{\alpha-1} Q_{2}^{\frac{1}{2}}\right)\left(\varepsilon^{\alpha-1} Q_{2}^{\frac{1}{2}}\right)^{*}\right] d s,
\end{align*}
$$

with $M^{\prime}\left(\theta^{\varepsilon}\right)=2 \theta^{\varepsilon}$ and $M^{\prime \prime}\left(\theta^{\varepsilon}\right)=2 \phi$ for any $\phi$ in $L^{2}(\partial D)$. After some further calculation, we conclude that

$$
\begin{align*}
& \left\langle M^{\prime}\left(\theta^{\varepsilon}\right),\left(-\frac{1}{\varepsilon} \theta^{\varepsilon}-\frac{1}{\varepsilon} \delta^{\varepsilon}-\frac{1}{\varepsilon} \varepsilon^{\varepsilon}\right)\right\rangle_{L^{2}(\partial D)} \\
= & -\frac{1}{\varepsilon} \frac{d}{d s}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}-\frac{2}{\varepsilon}\left\|\theta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}-\frac{2}{\varepsilon}\left\langle\frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}, v^{\varepsilon}\right\rangle_{L^{2}(\partial D)} . \tag{3.29}
\end{align*}
$$

Thus, by (3.28) and (3.29),

$$
\begin{align*}
& \left\|\theta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}+\frac{1}{\varepsilon}\left\|\delta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2} \\
= & \left\|\theta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}+\frac{1}{\varepsilon}\left\|\delta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}-\frac{2}{\varepsilon} \int_{0}^{t}\left\|\theta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} d s-\frac{2}{\varepsilon} \int_{0}^{t}\left\langle\frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}, v^{\varepsilon}\right\rangle_{L^{2}(\partial D)} d s  \tag{3.30}\\
& +\int_{0}^{t}\left\langle 2 \theta^{\varepsilon}, \varepsilon^{\alpha-1} d W_{2}(s)\right\rangle_{L^{2}(\partial D)}+\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{2} \cdot t .
\end{align*}
$$

Then it follows from (3.27) and (3.30) that

$$
\begin{aligned}
& \mathbb{E}\left\|v^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}+\frac{1}{\varepsilon} \mathbb{E}\left\|\nabla u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon} \mathbb{E}\left\|u^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2} \\
& +\frac{1}{\varepsilon} \mathbb{E}\left\|\delta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}+\frac{4}{\varepsilon} \mathbb{E}\left\|\cos \frac{u^{\varepsilon}(t)}{2}\right\|_{L^{2}(D)}^{2} \\
= & \mathbb{E}\left\|v^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}+\frac{1}{\varepsilon} \mathbb{E}\left\|\nabla u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon} \mathbb{E}\left\|u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2} \\
& +\frac{1}{\varepsilon} \mathbb{E}\left\|\delta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}+\frac{4}{\varepsilon} \mathbb{E}\left\|\cos \frac{u^{\varepsilon}(0)}{2}\right\|_{L^{2}(D)}^{2} \\
& \left.-\frac{2}{\varepsilon} \int_{0}^{t} \mathbb{E}\left\|v^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}\right] d s \\
& +\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{1} \cdot t+\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{2} \cdot t,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \mathbb{E}\left\|v^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2} \\
\leq & \mathbb{E}\left\|v^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}+\frac{1}{\varepsilon} \mathbb{E}\left\|\nabla u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\frac{1}{\varepsilon} \mathbb{E}\left\|u^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2} \\
& +\frac{1}{\varepsilon} \mathbb{E}\left\|\delta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}+\frac{4}{\varepsilon} \mathbb{E}\left\|\cos \frac{u^{\varepsilon}(0)}{2}\right\|_{L^{2}(D)}^{2} \\
& -\frac{2}{\varepsilon} \int_{0}^{t}\left[\mathbb{E}\left\|v^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}\right] d s \\
& +\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{1} \cdot t+\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{2} \cdot t .
\end{aligned}
$$

Therefore, we have

$$
\begin{align*}
& \frac{d}{d t}\left[\mathbb{E}\left\|v^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2}\right] \\
\leq & -\frac{2}{\varepsilon}\left[\mathbb{E}\left\|v^{\varepsilon}\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2}\right]+\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{1}+\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{2} . \tag{3.31}
\end{align*}
$$

By the Gronwall inequality, and noticing that $\alpha \in[1 / 2,1) \cup(1,+\infty)$ and $\varepsilon \in(0,1 / 2)$, it follows from (3.31) that for arbitrary $t \in\left[0, \tau^{*}\right)$,

$$
\begin{align*}
& \mathbb{E}\left\|v^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}(t)\right\|_{L^{2}(\partial D)}^{2} \\
\leq & {\left[\mathbb{E}\left\|v^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}\right] e^{-\frac{2 t}{\varepsilon}}+\frac{1}{2}\left(\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{1}+\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{2}\right) \cdot\left(1-e^{-\frac{2 t}{\varepsilon}}\right) }  \tag{3.32}\\
\leq & {\left[\mathbb{E}\left\|v^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}\right]+\frac{1}{2}\left(\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{1}+\varepsilon^{2 \alpha-1} \operatorname{Tr} Q_{2}\right) } \\
\leq & {\left[\mathbb{E}\left\|v^{\varepsilon}(0)\right\|_{L^{2}(D)}^{2}+\mathbb{E}\left\|\theta^{\varepsilon}(0)\right\|_{L^{2}(\partial D)}^{2}\right]+\frac{1}{2}\left(\operatorname{Tr} Q_{1}+\operatorname{Tr} Q_{2}\right) . }
\end{align*}
$$

This completes the proof of Proposition 3.4.
Proposition 3.5 Let $\alpha \in[1 / 2,1) \cup(1,+\infty)$ and $\varepsilon \in(0,1 / 2)$. Assume that the initial datum $U^{\varepsilon}(0)$ is a $\mathcal{F}_{0}$-measurable random variable in $L^{2}(\Omega, \mathcal{H})$. Then the solution $U^{\varepsilon}(t)$ of the Cauchy problem (2.2) globally exists in $\mathcal{H}$, i.e. $\tau^{*}=+\infty$ almost surely.

From Proposition 2.1, Remark 3.2 and Proposition 3.4, using the Borel-Cantelli lemma, we easily obtain Proposition 3.5.

Remark 3.3 (Almost sure boundedness) From Remark 3.2, Proposition 3.4 and Proposition 3.5, we know that the global solution $U^{\varepsilon}(t)$ of the Cauchy problem (2.2) is bounded in $\mathcal{H}$ almost surely.

Introduce another space

$$
\Sigma:=\left\{U^{\varepsilon} \in H^{2}(D) \times H^{1}(D) \times H^{1 / 2}(\partial D) \times H^{1 / 2}(\partial D) \left\lvert\, \frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}=\theta^{\varepsilon}\right. \text { on } \partial D\right\}
$$

Proposition 3.7 Let $\alpha \in[1 / 2,1) \bigcup(1,+\infty)$ and $\varepsilon \in(0,1 / 2)$. Assume that the initial datum $U^{\varepsilon}(0)$ is a $\mathcal{F}_{0}$-measurable random variable in $L^{2}(\Omega, \Sigma)$. Then the global solution $U^{\varepsilon}(t)$ of the Cauchy problem (2.2) is also bounded in $\Sigma$ almost surely.

Using a similar process for proving the almost sure boundedness of the solution for the Cauchy problem (2.2) in $\mathcal{H}$, we can prove Proposition 3.7. It is omitted here.

We now establish the tightness of solutions for the system (1.1). To begin with it, we recall some related results. Let $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$ be three reflective Banach spaces and $\mathcal{X} \hookrightarrow \mathcal{Y}$ being a compact and dense embedding. Define a new Banach space

$$
\mathcal{G}=\left\{\varphi: \varphi \in L^{2}(0, T ; \mathcal{X}), \frac{d \varphi}{d t} \in L^{2}(0, T ; \mathcal{Z})\right\}
$$

with norm

$$
\|\varphi\|_{\mathcal{G}}^{2}=\int_{0}^{T}\|\varphi(s)\|_{\mathcal{X}}^{2} d s+\int_{0}^{T}\left\|\frac{d \varphi(s)}{d s}\right\|_{\mathcal{Z}}^{2} d s
$$

Lemma 3.1 19 If $K$ is bounded in $\mathcal{G}$, then $K$ is precompact in $L^{2}(0, T ; \mathcal{Y})$.
Proposition 3.8 (Tightness) Let $\alpha \in[1 / 2,1) \cup(1,+\infty)$ and $\varepsilon \in(0,1 / 2)$. Assume that the initial datum $U^{\varepsilon}(0)$ is a $\mathcal{F}_{0}$-measurable random variable in $L^{2}(\Omega, \mathcal{H})$. Then for a given positive $T$, the solution $u^{\varepsilon}(t)$ and $\delta^{\varepsilon}(t)$ of the system (1.1) is tight in $L^{2}\left(0, T ; L^{2}(D)\right)$ and $L^{2}\left(0, T ; L^{2}(\partial D)\right)$, respectively.

Proof. First, for the solution $u^{\varepsilon}$ of the system (1.1), let $\mathcal{X}=H^{1}(D)$ and $\mathcal{Y}=\mathcal{Z}=L^{2}(D)$. Then from Remark 3.3 and the Chebyshev inequality, for any $\rho>0$, there exists a bounded ball of radius $\rho$ centered at zero, $K_{\rho} \subset \mathcal{G}$, such that $\mathbb{P}\left\{u^{\varepsilon} \in K_{\rho}\right\}>1-\rho$. By Lemma 3.1, $K_{\rho}$ is precompact in $L^{2}(0, T ; \mathcal{Y})$. Then the solution $u^{\varepsilon}$ is tight in $L^{2}\left(0, T ; L^{2}(D)\right)$.

Similarly, for $\delta^{\varepsilon}$, let $\mathcal{X}=\mathcal{Y}=\mathcal{Z}=L^{2}(\partial D)$. Using the same process of $u^{\varepsilon}$, we see that $\delta^{\varepsilon}$ is tight in $L^{2}\left(0, T ; L^{2}(\partial D)\right)$.

## 4 Approximating equation

In this section, we use a splitting method [33] to derive the approximating equation of the system (1.1) for $\varepsilon$ sufficiently small, in the sense of probability distribution. We consider the solutions of the system (1.1) in the weak sense. The main result is as follows.

Theorem 4.1 (Approximating equation) For the system (1.1), let the initial datum $\left(u_{0}, u_{1}, \delta_{0}, \delta_{1}\right)^{T}$ be a $\mathcal{F}_{0}$-measurable random variable in $L^{2}(\Omega, \mathcal{H})$. Let $T$ be a given positive number. Then we have the following conclusions:
(i) If $\alpha \in[1 / 2,1)$, then for sufficiently small $\varepsilon$,

$$
\begin{align*}
& \left\|u^{\varepsilon}-\bar{u}^{\varepsilon}\right\|_{L^{2}\left(0, T ; L^{2}(D)\right)}=O\left(\varepsilon^{\alpha}\right), \\
& \left\|\delta^{\varepsilon}-\bar{\delta}^{\varepsilon}\right\|_{L^{2}\left(0, T ; L^{2}(\partial D)\right)}=O\left(\varepsilon^{\alpha}\right), \tag{4.1}
\end{align*}
$$

where $\bar{u}^{\varepsilon}$ and $\bar{\delta}^{\varepsilon}$ are the solutions of the following stochastic parabolic equation with a dynamical boundary condition

$$
\begin{cases}\bar{u}_{t}^{\varepsilon}-\triangle \bar{u}^{\varepsilon}+\bar{u}^{\varepsilon}-\sin \bar{u}^{\varepsilon}=\varepsilon^{\alpha} \dot{\bar{W}}_{1}, & \text { in } D,  \tag{4.2}\\ \bar{\delta}_{t}^{\varepsilon}+\bar{\delta}^{\varepsilon}=-\bar{u}_{t}^{\varepsilon}+\varepsilon^{\alpha} \overline{\bar{W}}_{2}, & \text { on } \partial D, \\ \bar{\delta}_{t}^{\varepsilon}=\frac{\partial \bar{u}^{\varepsilon}}{\partial \mathbf{n}}, & \text { on } \partial D, \\ \bar{u}^{\varepsilon}(0)=u_{0}, \bar{\delta}^{\varepsilon}(0)=\delta_{0} . & \end{cases}
$$

(ii) If $\alpha \in(1,+\infty)$, then for sufficiently small $\varepsilon$,

$$
\begin{align*}
& \left\|u^{\varepsilon}-\bar{u}^{\varepsilon}\right\|_{L^{2}\left(0, T ; L^{2}(D)\right)}=O(\varepsilon),  \tag{4.3}\\
& \left\|\delta^{\varepsilon}-\bar{\delta}^{\varepsilon}\right\|_{L^{2}\left(0, T ; L^{2}(\partial D)\right)}=O(\varepsilon),
\end{align*}
$$

where $\bar{u}^{\varepsilon}$ and $\bar{\delta}^{\varepsilon}$ are the solutions of the following deterministic wave equation with a dynamical boundary condition

$$
\begin{cases}\varepsilon \bar{u}_{t t}^{\varepsilon}+\bar{u}_{t}^{\varepsilon}-\triangle \bar{u}^{\varepsilon}+\bar{u}^{\varepsilon}-\sin \bar{u}^{\varepsilon}=0, & \text { in } D  \tag{4.4}\\ \varepsilon \bar{\delta}_{t t}^{\varepsilon}+\bar{\delta}_{t}^{\varepsilon}+\bar{\delta}^{\varepsilon}=-\bar{u}_{t}^{\varepsilon}, & \text { on } \partial D \\ \bar{\delta}_{t}^{\varepsilon}=\frac{\partial \bar{u}^{\varepsilon}}{\partial \mathbf{n}}, & \text { on } \partial D \\ \bar{u}^{\varepsilon}(0)=u_{0}, \bar{u}_{t}^{\varepsilon}(0)=u_{1}, \bar{\delta}^{\varepsilon}(0)=\delta_{0}, \bar{\delta}_{t}^{\varepsilon}(0)=\delta_{1} & \end{cases}
$$

Remark 4.1 By the method of Chueshov and Schmalfuss [8, 9], we can show that Equation (4.2) is well-posed. In addition, Equation (4.4) is also well-posed (see [15]).

In the following, we will prove Theorem 4.1. We state some preliminary results.
For the system (2.1), we give the decomposition as follows. Firstly,

$$
\begin{equation*}
v^{\varepsilon}=\bar{v}_{1}^{\varepsilon}+\bar{v}_{2}^{\varepsilon}+\bar{v}_{3}^{\varepsilon} \tag{4.5}
\end{equation*}
$$

where

$$
\begin{gather*}
\left\{\begin{array}{l}
\frac{d \bar{v}_{1}^{\varepsilon}}{d t}=-\frac{1}{\varepsilon} \bar{v}_{1}^{\varepsilon}, \quad \text { in } D, \\
\bar{v}_{1}^{\varepsilon}(0)=v_{0},
\end{array}\right.  \tag{4.6}\\
\left\{\begin{array}{l}
\frac{d \bar{v}_{2}^{\varepsilon}}{d t}=-\frac{1}{\varepsilon} \bar{v}_{2}^{\varepsilon}+\frac{1}{\varepsilon} \triangle u^{\varepsilon}-\frac{1}{\varepsilon} u^{\varepsilon}+\frac{1}{\varepsilon} \sin u^{\varepsilon}, \quad \text { in } D, \\
\bar{v}_{2}^{\varepsilon}(0)=0,
\end{array}\right. \tag{4.7}
\end{gather*}
$$

and

$$
\left\{\begin{array}{l}
\frac{d \bar{v}_{3}^{\varepsilon}}{d t}=-\frac{1}{\varepsilon} \bar{v}_{3}^{\varepsilon}+\varepsilon^{\alpha-1} \dot{W}_{1}, \quad \text { in } \quad D  \tag{4.8}\\
\bar{v}_{3}^{\varepsilon}(0)=0
\end{array}\right.
$$

Secondly,

$$
\begin{equation*}
\theta^{\varepsilon}=\bar{\theta}_{1}^{\varepsilon}+\bar{\theta}_{2}^{\varepsilon}+\bar{\theta}_{3}^{\varepsilon}, \tag{4.9}
\end{equation*}
$$

where

$$
\begin{gather*}
\left\{\begin{array}{l}
\frac{d \bar{\theta}_{1}^{\varepsilon}}{d t}=-\frac{1}{\varepsilon} \bar{\theta}_{1}^{\varepsilon}, \quad \text { on } \quad \partial D, \\
\bar{\theta}_{1}^{\varepsilon}(0)=\theta_{0},
\end{array}\right.  \tag{4.10}\\
\left\{\begin{array}{l}
\frac{d \bar{\theta}_{2}^{\varepsilon}}{d t}=-\frac{1}{\varepsilon} \bar{\theta}_{2}^{\varepsilon}-\frac{1}{\varepsilon} \delta^{\varepsilon}+\frac{1}{\varepsilon} v^{\varepsilon}, \quad \text { on } \quad \partial D, \\
\bar{\theta}_{2}^{\varepsilon}(0)=0,
\end{array}\right. \tag{4.11}
\end{gather*}
$$

and

$$
\left\{\begin{array}{l}
\frac{d \bar{\theta}_{3}^{\varepsilon}}{d t}=-\frac{1}{\varepsilon} \bar{\theta}_{3}^{\varepsilon}+\varepsilon^{\alpha-1} \dot{W}_{2}, \quad \text { on } \quad \partial D  \tag{4.12}\\
\bar{\theta}_{3}^{\varepsilon}(0)=0
\end{array}\right.
$$

Proposition 4.1 Let $\alpha \in[1 / 2,1) \cup(1,+\infty)$ and $\varepsilon \in(0,1 / 2)$. Assume that the initial data $v_{0}$ and $\theta_{0}$ are $\mathcal{F}_{0}$-measurable random variables in $L^{2}\left(\Omega, L^{2}(D)\right)$ and $L^{2}\left(\Omega, L^{2}(\partial D)\right.$ ), respectively. Then we have that
(i) For Equation (4.6) and Equation (4.19)

$$
\begin{equation*}
\bar{v}_{1}^{\varepsilon}(t)=v_{0} e^{-\frac{t}{\varepsilon}}, \quad \bar{\theta}_{1}^{\varepsilon}(t)=\theta_{0} e^{-\frac{t}{\varepsilon}}, \quad \forall t \geq 0 . \tag{4.13}
\end{equation*}
$$

(ii) For Equation (4.7) and Equation (4.11), there is a positive constant $C$ independent of the parameter $\varepsilon$ such that

$$
\begin{equation*}
\mathbb{E}\left\|\bar{v}_{2}^{\varepsilon}(t)\right\|_{H^{-1}(D)} \leq C, \quad \mathbb{E}\left\|\bar{\theta}_{2}^{\varepsilon}(t)\right\|_{H^{-1 / 2}(\partial D)} \leq C, \quad \forall t \geq 0 \tag{4.14}
\end{equation*}
$$

(iii) For Equation (4.8) and Equation (4.12),

$$
\begin{align*}
& \mathbb{E}\left\|\bar{v}_{3}^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}=-\frac{2}{\varepsilon} \int_{0}^{t} \mathbb{E}\left\|\bar{v}_{3}^{\varepsilon}(s)\right\|_{L^{2}(D)}^{2} d s+\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{1} \cdot t, \quad \forall t \geq 0,  \tag{4.15}\\
& \mathbb{E}\left\|\theta_{3}^{\varepsilon}(t)\right\|_{L^{2}(D)}^{2}=-\frac{2}{\varepsilon} \int_{0}^{t} \mathbb{E}\left\|\bar{\theta}_{3}^{\varepsilon}(s)\right\|_{L^{2}(\partial D)}^{2} d s+\varepsilon^{2 \alpha-2} \operatorname{Tr} Q_{2} \cdot t, \quad \forall t \geq 0 .
\end{align*}
$$

Proof. For Equation (4.6) and Equation (4.10), we can directly solve them to obtain (4.13). And for Equation (4.8) and Equation (4.12), applying the Itô formula, we immediately obtain (4.15).

Now we prove (4.14).
Noticing that

$$
\begin{aligned}
& \mathbb{E}\left\|\bar{v}_{2}^{\varepsilon}(t)\right\|_{H^{-1}(D)}=\sup _{\phi \in H^{1}(D)} \frac{\mathbb{E}\left|\left\langle\bar{v}_{\tilde{c}}^{\varepsilon}, \phi\right\rangle_{L^{2}(D)}\right|}{\|\phi\|_{H^{1}(D)}}, \\
& \mathbb{E}\left\|\bar{\theta}_{2}^{\varepsilon}(t)\right\|_{H^{-1 / 2}(\partial D)}=\sup _{\psi \in H^{1 / 2}(\partial D)} \frac{\mathbb{E}\left|\left\langle\bar{\theta}_{2}^{\varepsilon}, \psi\right\rangle_{L^{2}(\partial D)}\right|}{\|\psi\|_{H^{1 / 2}(\partial D)}},
\end{aligned}
$$

we only need to prove

$$
\begin{equation*}
\mathbb{E}\left|\left\langle\bar{v}_{2}^{\varepsilon}, \phi\right\rangle_{L^{2}(D)}\right| \leq C\|\phi\|_{H^{1}(D)}, \quad \forall \phi \in H^{1}(D) \tag{4.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{E}\left|\left\langle\bar{\theta}_{2}^{\varepsilon}, \psi\right\rangle_{L^{2}(\partial D)}\right| \leq C\|\psi\|_{H^{1 / 2}(D)}, \quad \forall \psi \in H^{1 / 2}(D) \tag{4.17}
\end{equation*}
$$

Firstly, for arbitrary $\phi \in H^{1}(D)$, it follows from (4.7) that

$$
\frac{d}{d t}\left\langle\bar{v}_{2}^{\varepsilon}, \phi\right\rangle_{L^{2}(D)}=-\frac{1}{\varepsilon}\left\langle\bar{v}_{2}^{\varepsilon}, \phi\right\rangle_{L^{2}(D)}+\frac{1}{\varepsilon}\left\langle\Delta u^{\varepsilon}, \phi\right\rangle_{L^{2}(D)}-\frac{1}{\varepsilon}\left\langle u^{\varepsilon}, \phi\right\rangle_{L^{2}(D)}+\frac{1}{\varepsilon}\left\langle\sin u^{\varepsilon}, \phi\right\rangle_{L^{2}(D)},
$$

which implies, from Remark 3.3, that

$$
\begin{aligned}
\mathbb{E}\left\langle\bar{v}_{2}^{\varepsilon}, \phi\right\rangle_{L^{2}(D)}= & \frac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \int_{0}^{t} e^{\frac{s}{\varepsilon}}\left[-\mathbb{E}\left\langle\nabla u^{\varepsilon}, \nabla \phi\right\rangle_{L^{2}(D)}+\mathbb{E}\left\langle\frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}, \phi\right\rangle_{L^{2}(\partial D)}-\mathbb{E}\left\langle u^{\varepsilon}, \phi\right\rangle_{L^{2}(D)}\right. \\
& \left.+\mathbb{E}\left\langle\sin u^{\varepsilon}, \phi\right\rangle_{L^{2}(D)}\right] d s \\
\leq & \frac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \int_{0}^{t} e^{\frac{s}{\varepsilon}}\left[\mathbb{E}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(D)} \cdot\|\phi\|_{H^{1}(D)}+\mathbb{E}\left\|\theta^{\varepsilon}\right\|_{L^{2}(\partial D)} \cdot\|\phi\|_{H^{1}(D)}\right. \\
& \left.+\mathbb{E}\left\|u^{\varepsilon}\right\|_{L^{2}(D)} \cdot\|\phi\|_{H^{1}(D)}+\mathbb{E}\left\|\sin u^{\varepsilon}\right\|_{L^{2}(D)} \cdot\|\phi\|_{H^{1}(D)}\right] d s \\
\leq & \frac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \int_{0_{0}^{t}} e^{\frac{s}{\varepsilon}} d s \cdot C\|\phi\|_{H^{1}(D)} \\
= & {\left[1-e^{-\frac{t}{\varepsilon}}\right] \cdot C\|\phi\|_{H^{1}(D)} } \\
\leq & C\|\phi\|_{H^{1}(D)} .
\end{aligned}
$$

which arrives at (4.16).
Secondly, for arbitrary $\psi \in H^{1 / 2}(\partial D)$, it follows from (4.11) that

$$
\frac{d}{d t}\left\langle\bar{\theta}_{2}^{\varepsilon}, \psi\right\rangle_{L^{2}(\partial D)}=-\frac{1}{\varepsilon}\left\langle\bar{\theta}_{2}^{\varepsilon}, \psi\right\rangle_{L^{2}(\partial D)}-\frac{1}{\varepsilon}\left\langle\delta^{\varepsilon}, \psi\right\rangle_{L^{2}(\partial D)}+\frac{1}{\varepsilon}\left\langle v^{\varepsilon}, \psi\right\rangle_{L^{2}(\partial D)}
$$

which implies, from the trace inequality, Remark 3.3 and Proposition 3.7, that

$$
\begin{aligned}
\mathbb{E}\left\langle\bar{\theta}_{2}^{\varepsilon}, \psi\right\rangle_{L^{2}(\partial D)} & =\frac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \int_{0}^{t} e^{\frac{s}{\varepsilon}}\left[-\mathbb{E}\left\langle\delta^{\varepsilon}, \psi\right\rangle_{L^{2}(\partial D)}+\mathbb{E}\left\langle v^{\varepsilon}, \psi\right\rangle_{L^{2}(\partial D)}\right] d s \\
& \leq \frac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \int_{0}^{t} e^{\frac{s}{\varepsilon}}\left[\mathbb{E}\left\|\delta^{\varepsilon}\right\|_{L^{2}(\partial D)}^{2} \cdot\|\psi\|_{H^{1 / 2}(\partial D)}+C_{T I} \mathbb{E}\left\|v^{\varepsilon}\right\|_{H^{1}(D)} \cdot\|\psi\|_{H^{1 / 2}(\partial D)}\right] d s \\
& \leq \frac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \int_{0}^{t} e^{\frac{s}{\varepsilon}} d s \cdot C\|\psi\|_{H^{1 / 2}(\partial D)} \\
& =\left[1-e^{-\frac{t}{\varepsilon}}\right] \cdot C\|\psi\|_{H^{1 / 2}(\partial D)} \\
& \leq C\|\psi\|_{H^{1 / 2}(\partial D)} .
\end{aligned}
$$

which leads to (4.17).
Lemma 4.1 (Prohorov Theorem) 13 Assume that $\mathcal{M}$ is a separable Banach space. The set of probability measures $\left\{\mathcal{L}\left(X_{n}\right)\right\}_{n}$ on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ is relatively compact if and only if $\left\{X_{n}\right\}$ is tight.

Lemma 4.2 (Skorohod Theorem) [13] For an arbitrary sequence of probability measures $\left\{\mu_{n}\right\}$ on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ weakly converges to probability measures $\mu$, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables, $X, X_{1}, X_{2}, \cdots, X_{n}, \cdots$ such that $X_{n}$ distributes as $\mu_{n}$ and $X$ distributes as $\mu$, and $\lim _{n \rightarrow \infty} X_{n}=X, \mathbb{P}$-a.s.

## Proof of Theorem 4.1

From Proposition 3.8, for $t \in[0, T]$, the solution $u^{\varepsilon}(t)$ and $\delta^{\varepsilon}(t)$ of the system (1.1), are tight in $L^{2}\left(0, T ; L^{2}(D)\right)$ and $L^{2}\left(0, T ; L^{2}(\partial D)\right)$, respectively. Therefore, for arbitrary $\rho>0$, there exist two bounded balls of radius $\rho$ centered at zero, $K_{\rho} \subset H^{1}(D)$ and $B_{\rho} \subset L^{2}(\partial D)$, which are compact in $L^{2}(D)$ and $L^{2}(\partial D)$, such that

$$
\mathbb{P}\left\{u^{\varepsilon} \in K_{\rho}\right\}>1-\rho, \quad \text { and } \quad \mathbb{P}\left\{\delta^{\varepsilon} \in B_{\rho}\right\}>1-\rho .
$$

According to Lemma 4.1 and Lemma 4.2, we know that for every sequence $\left\{\varepsilon_{j}\right\}_{j=1}^{j=\infty}$ with $\varepsilon_{j} \rightarrow 0$ as $j \rightarrow \infty$, there exists a subsequence $\left\{\varepsilon_{j(k)}\right\}_{k=1}^{k=\infty}$, random variables $u^{* \varepsilon_{j(k)}} \subset L^{2}\left(0, T ; L^{2}(D)\right)$ and $\delta^{* \varepsilon_{j(k)}} \subset L^{2}\left(0, T ; L^{2}(\partial D)\right)$, and $u^{*} \in L^{2}\left(0, T ; L^{2}(D)\right)$ and $\delta^{*} \in L^{2}\left(0, T ; L^{2}(\partial D)\right)$ defined on a new probability space $\left(\Omega^{*}, \mathcal{F}^{*}, \mathbb{P}^{*}\right)$, such that for almost all $\omega \in \Omega^{*}$,

$$
\begin{aligned}
& \mathcal{L}\left(u^{* \varepsilon_{j(k)}}\right)=\mathcal{L}\left(u^{\varepsilon_{j}(k)}\right), \\
& \mathcal{L}\left(\delta^{* \varepsilon_{j(k)}}\right)=\mathcal{L}\left(\delta^{\varepsilon_{j(k)}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& u^{* \varepsilon_{j(k)}} \longrightarrow u^{*}, \quad \text { in } \quad L^{2}\left(0, T ; L^{2}(D)\right) \quad \text { as } \quad k \rightarrow \infty \\
& \delta^{* \varepsilon_{j(k)}} \longrightarrow \delta^{*}, \quad \text { in } \quad L^{2}\left(0, T ; L^{2}(\partial D)\right) \quad \text { as } \quad k \rightarrow \infty
\end{aligned}
$$

In the meantime, $u^{* \varepsilon_{j(k)}}$ and $\delta^{* \varepsilon_{j(k)}}$ solve the system (1.1) with $W_{1}$ and $W_{2}$ being replaced by the Wiener processes $W_{1}^{*}$ and $W_{2}^{*}$, defined on the probability space $\left(\Omega^{*}, \mathcal{F}^{*}, \mathbb{P}^{*}\right)$ but with the same distributions as $W_{1}$ and $W_{2}$, respectively. In the following, we will derive the approximating equation for $u^{*}$ and $\delta^{*}$ and present the error estimates between the approximating equation and the original system (1.1) as in Theorem 4.1.

Now, for the above $\rho$, it follows from (4.14) and the Chebyshev inequality that there exists a positive constant $C_{\rho}$ independent of the parameter $\varepsilon$ such that

$$
\mathbb{P}\left\{\left\|\bar{v}_{2}^{\varepsilon}\right\|_{H^{-1}(D)} \leq C_{\rho}\right\}>1-\rho, \quad \text { and } \quad \mathbb{P}\left\{\left\|\bar{\theta}_{2}^{\varepsilon}\right\|_{H^{-1 / 2}(\partial D)} \leq C_{\rho}\right\}>1-\rho .
$$

Define

$$
\begin{gathered}
\Omega_{\rho}=\left\{\omega \in \Omega: u^{\varepsilon}(\omega) \in K_{\rho}, \delta^{\varepsilon}(\omega) \in B_{\rho},\left\|\bar{v}_{2}^{\varepsilon}(\omega)\right\|_{H^{-1}(D)} \leq C_{\rho},\left\|\bar{\theta}_{2}^{\varepsilon}(\omega)\right\|_{H^{-1 / 2}(\partial D)} \leq C_{\rho}\right\}, \\
\mathcal{F}_{\rho}=\left\{F \bigcap \Omega_{\rho}: F \in \mathcal{F}\right\},
\end{gathered}
$$

and

$$
\mathbb{P}_{\rho}(F)=\frac{\mathbb{P}\left(F \bigcap \Omega_{\rho}\right)}{\mathbb{P}\left(\Omega_{\rho}\right)}, \quad \text { for } \quad F \in \mathcal{F}_{\rho} .
$$

Then $\left(\Omega_{\rho}, \mathcal{F}_{\rho}, \mathbb{P}_{\rho}\right)$ is a new probability space, whose expectation operator is denoted by $\mathbb{E}_{\rho}$. We will work in the probability space $\left(\Omega_{\rho}, \mathcal{F}_{\rho}, \mathbb{P}_{\rho}\right)$ in stead of $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity, we will omit the subscript $\rho$ unless we specifically stated otherwise.

The system (2.1), combining with (4.5) and (4.9), can be rewritten as follows

$$
\begin{cases}u_{t}^{\varepsilon}=v^{\varepsilon}=\bar{\sigma}_{\varepsilon}^{\varepsilon}+\bar{v}_{\varepsilon}^{\varepsilon}+\overline{\bar{v}}_{\varepsilon}^{\varepsilon}, & u^{\varepsilon}(0)=u_{0}, \\ \delta_{t}^{\varepsilon} \theta^{\varepsilon}=\bar{\theta}_{1}^{\varepsilon}+\bar{\theta}_{2}^{\varepsilon}+\bar{\theta}_{3}^{\varepsilon}, & \delta^{\varepsilon}(0)=\delta_{0}, \\ \delta_{t}^{\varepsilon}=\frac{\partial u^{\varepsilon}}{\partial \mathbf{n}}, & \text { on } \quad \partial D, \\ & \text { on } \partial D,\end{cases}
$$

whose weak sense formulation is

$$
\begin{align*}
& \left\langle u^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}+\left\langle\delta^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)} \\
= & \left\langle u^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(D)}+\int_{0}^{t}\left\langle u^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
& +\left\langle\delta^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle\delta^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s  \tag{4.18}\\
& +\int_{0}^{t}\left\langle\bar{v}_{1}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(D)} d s+\int_{0}^{t}\left\langle\bar{v}_{2}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(D)} d s+\int_{0}^{t}\left\langle\bar{v}_{3}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(D)} d s \\
& +\int_{0}^{t}\left\langle\bar{\theta}_{1}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle\bar{\theta}_{2}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle\bar{\theta}_{3}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(\partial D)} d s,
\end{align*}
$$

for every $\varphi \in C_{0}^{\infty}([0,+\infty) \times D)$.
We consider the case of $\alpha \in[1 / 2,1)$.

From (4.13), it follows that for every $\varphi \in C_{0}^{\infty}([0,+\infty) \times D)$,

$$
\begin{align*}
\int_{0}^{t}\left\langle\bar{v}_{1}^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s & =\int_{0}^{t}\left\langle v_{0} e^{-\frac{s}{\varepsilon}}, \varphi(s)\right\rangle_{L^{2}(D)} d s \\
& =\varepsilon \int_{0}^{\frac{t}{\varepsilon}}\left\langle v_{0}, \varphi(\varepsilon \tau)\right\rangle_{L^{2}(D)} e^{-\tau} d \tau  \tag{4.19}\\
& =O(\varepsilon),
\end{align*}
$$

for sufficiently small $\varepsilon$.
In addition, it follows from (4.7) that

$$
\begin{align*}
\int_{0}^{t}\left\langle\frac{d \bar{v}_{2}^{\varepsilon}}{d s}, \varphi\right\rangle_{L^{2}(D)} d s= & -\frac{1}{\varepsilon} \int_{0}^{t}\left\langle\bar{v}_{2}^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s+\frac{1}{\varepsilon} \int_{0}^{t}\left\langle\Delta u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s  \tag{4.20}\\
& -\frac{1}{\varepsilon} \int_{0}^{t}\left\langle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s+\frac{1}{\varepsilon} \int_{0}^{t}\left\langle\sin u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s .
\end{align*}
$$

Meanwhile, noticing that $\bar{v}_{2}^{\varepsilon}(0)=0$ and that

$$
\int_{0}^{t}\left\langle\frac{d \bar{v}_{2}^{\varepsilon}}{d s}, \varphi\right\rangle_{L^{2}(D)} d s=\left\langle\bar{v}_{2}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}-\left\langle\bar{v}_{2}^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(D)}-\int_{0}^{t}\left\langle\bar{v}_{2}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s,
$$

we infer from (4.20) that

$$
\begin{align*}
& =\int_{0}^{t}\left\langle\bar{v}_{2}^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s \\
& =-\varepsilon\left\langle\bar{v}_{2}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}+\varepsilon \int_{0}^{t}\left\langle\bar{v}_{2}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s  \tag{4.21}\\
& \\
& \quad+\int_{0}^{t}\left\langle\triangle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s-\int_{0}^{t}\left\langle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s+\int_{0}^{t}\left\langle\sin u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s,
\end{align*}
$$

which implies from (4.14) that

$$
\begin{align*}
& \int_{0}^{t}\left\langle\bar{v}_{2}^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s  \tag{4.22}\\
= & O(\varepsilon)+O(\varepsilon)+\int_{0}^{t}\left\langle\triangle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s-\int_{0}^{t}\left\langle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s+\int_{0}^{t}\left\langle\sin u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s,
\end{align*}
$$

for sufficiently small $\varepsilon$.
Also, it follows from (4.8) that

$$
\begin{equation*}
\int_{0}^{t}\left\langle\frac{d \bar{v}_{3}^{\varepsilon}}{d s}, \varphi\right\rangle_{L^{2}(D)} d s=-\frac{1}{\varepsilon} \int_{0}^{t}\left\langle\bar{v}_{3}^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s+\varepsilon^{\alpha-1} \int_{0}^{t}\left\langle d W_{1}(s), \varphi\right\rangle_{L^{2}(D)} . \tag{4.23}
\end{equation*}
$$

Noticing that $\bar{v}_{3}^{\varepsilon}(0)=0$ and that

$$
\int_{0}^{t}\left\langle\frac{d \bar{v}_{3}^{\varepsilon}}{d s}, \varphi\right\rangle_{L^{2}(D)} d s=\left\langle\bar{v}_{3}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}-\left\langle\bar{v}_{3}^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(D)}-\int_{0}^{t}\left\langle\bar{v}_{3}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s,
$$

we deduce from (4.23) that

$$
\begin{align*}
\int_{0}^{t}\left\langle\bar{v}_{3}^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s= & -\varepsilon\left\langle\bar{v}_{3}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)} d s+\varepsilon \int_{0}^{t}\left\langle\bar{v}_{3}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s  \tag{4.24}\\
& +\varepsilon^{\alpha} \int_{0}^{t}\left\langle d W_{1}(s), \varphi\right\rangle_{L^{2}(D)} .
\end{align*}
$$

Combining with (4.15) and $\alpha \in[1 / 2,1) \cup(1,+\infty)$, and using the Gronwall inequality, we further obtain that

$$
\begin{equation*}
\mathbb{E}\left\|\bar{v}_{3}^{\varepsilon}(t)\right\|_{L^{2}(D)} \leq \operatorname{Tr} Q_{1}, \quad \forall t \geq 0 \tag{4.25}
\end{equation*}
$$

which immediately implies from (4.24) and $\alpha \in[1 / 2,1)$ that,

$$
\begin{equation*}
\int_{0}^{t}\left\langle\bar{v}_{3}^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s=O(\varepsilon)+O(\varepsilon)+\varepsilon^{\alpha} \int_{0}^{t}\left\langle d W_{1}(s), \varphi\right\rangle_{L^{2}(D)} \tag{4.26}
\end{equation*}
$$

for sufficiently small $\varepsilon$.
Similarly, for every $\varphi \in C_{0}^{\infty}([0,+\infty) \times D)$ and for sufficiently small $\varepsilon$,

$$
\begin{gather*}
\int_{0}^{t}\left\langle\bar{\theta}_{1}^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s=O(\varepsilon),  \tag{4.27}\\
\int_{0}^{t}\left\langle\bar{\theta}_{2}^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s=\begin{array}{c}
-\varepsilon\left\langle\bar{\theta}_{2}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}+\varepsilon \int_{0}^{t}\left\langle\bar{\theta}_{2}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s \\
-\int_{0}^{t}\left\langle\delta \delta^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle v^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s,
\end{array} \tag{4.28}
\end{gather*}
$$

which implies from (4.14) that

$$
\begin{align*}
\int_{0}^{t}\left\langle\bar{\theta}_{2}^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s= & O(\varepsilon)+O(\varepsilon) \\
& -\int_{0}^{t}\left\langle\delta^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle v^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s, \tag{4.29}
\end{align*}
$$

and

$$
\begin{align*}
\int_{0}^{t}\left\langle\bar{\theta}_{3}^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s= & -\varepsilon\left\langle\bar{\theta}_{3}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)} d s+\varepsilon \int_{0}^{t}\left\langle\bar{\theta}_{3}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s  \tag{4.30}\\
& +\varepsilon^{\alpha} \int_{0}^{t}\left\langle d W_{2}(s), \varphi\right\rangle_{L^{2}(\partial D)},
\end{align*}
$$

which leads to

$$
\begin{equation*}
\left.\int_{0}^{t} \bar{\theta}_{3}^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s=O(\varepsilon)+O(\varepsilon)+\varepsilon^{\alpha} \int_{0}^{t}\left\langle d W_{2}(s), \varphi\right\rangle_{L^{2}(\partial D)} . \tag{4.31}
\end{equation*}
$$

Thus, by the Gronwall inequality, (4.15) and the condition $\alpha \in[1 / 2,1) \cup(1,+\infty)$, we have $\mathbb{E}\left\|\bar{\theta}_{3}^{\varepsilon}(t)\right\|_{L^{2}(\partial D)} \leq \operatorname{Tr} Q_{2}$ for $t \in[0,+\infty)$.

Therefore, substituting (4.19), (4.22), (4.26), (4.27), (4.29) and (4.31) into (4.18), for every $\varphi \in C_{0}^{\infty}([0,+\infty) \times D)$, we conclude that for sufficiently small $\varepsilon$,

$$
\begin{align*}
& \left\langle u^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}+\left\langle\delta^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)} \\
= & \left\langle u^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(D)}+\int_{0}^{t}\left\langle u^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
& +\left\langle\delta^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle\delta^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s \\
& +\int_{0}^{t}\left\langle\triangle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s-\int_{0}^{t}\left\langle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s+\int_{0}^{t}\left\langle\sin u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s  \tag{4.32}\\
& -\int_{0}^{t}\left\langle\delta^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle v^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s \\
& +\varepsilon^{\alpha} \int_{0}^{t}\left\langle d W_{1}(s), \varphi\right\rangle_{L^{2}(D)}+\varepsilon^{\alpha} \int_{0}^{t}\left\langle d W_{2}(s), \varphi\right\rangle_{L^{2}(\partial D)}+O(\varepsilon) .
\end{align*}
$$

Projecting (4.32) onto $L^{2}(D)$ and $L^{2}(\partial D)$, respectively, we derive that for sufficiently small $\varepsilon$, as $\alpha \in[1 / 2,1)$, the approximating equation for $u^{\varepsilon}$ is

$$
\begin{cases}\bar{u}_{t}^{\varepsilon}=\triangle \bar{u}^{\varepsilon}-\bar{u}^{\varepsilon}+\sin \bar{u}^{\varepsilon}+\varepsilon^{\alpha} \dot{\bar{W}}_{1}, \quad \bar{u}^{\varepsilon}(0)=u_{0}, & \text { in } D,  \tag{4.33}\\ \bar{\delta}_{t}^{\varepsilon}=-\bar{\delta}^{\varepsilon}+\bar{u}_{t}^{\varepsilon}+\varepsilon^{\alpha} \overline{\bar{W}}_{2}, \quad \bar{\delta}^{\varepsilon}(0)=\delta_{0}, & \text { on } \partial D, \\ \bar{\delta}_{t}^{\varepsilon}=\frac{\bar{u}^{\varepsilon}}{\partial \mathbf{n}}, & \text { on } \partial D .\end{cases}
$$

Then it follows from (4.32) and (4.33) that the result under the condition $[1 / 2,1)$ holds.
It remains to consider the case of $\alpha \in(1,+\infty)$.
It follows from (4.18), (4.21) and (4.28) that

$$
\begin{align*}
& \left\langle u^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}+\left\langle\delta^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)} \\
= & \left\langle u^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(D)}+\int_{0}^{t}\left\langle u^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
& +\left\langle\delta^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle\delta^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s \\
& +\int_{0}^{t}\left\langle\bar{v}_{1}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(D)} d s+\int_{0}^{t}\left\langle\bar{v}_{3}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(D)} d s \\
& \left.+\int_{0}^{t}\left\langle\bar{\theta}_{1}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t} \bar{\theta}_{3}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(\partial D)} d s  \tag{4.34}\\
& -\varepsilon\left\langle\bar{v}_{2}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}+\varepsilon \int_{0}^{t}\left\langle\hat{v}_{2}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
& +\int_{0}^{t}\left\langle\triangle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s-\int_{0}^{t}\left\langle u^{\varepsilon}\langle, \varphi\rangle_{L^{2}(D)} d s+\int_{0}^{t}\left\langle\sin u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s\right. \\
& -\varepsilon\left\langle\bar{\theta}_{2}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}+\varepsilon \int_{0}^{t}\left\langle\bar{\theta}_{2}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s \\
& -\int_{0}^{t}\left\langle\delta^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s+\int_{0}^{t}\left\langle v^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s .
\end{align*}
$$

From (4.5) and (4.9), we have

$$
\begin{align*}
& \bar{v}_{2}^{\varepsilon}=v^{\varepsilon}-\bar{v}_{1}^{\varepsilon}-\bar{v}_{3}^{\varepsilon}, \\
& \bar{\theta}_{2}^{\varepsilon}=\theta^{\varepsilon}-\bar{\theta}_{1}^{\varepsilon}-\bar{\theta}_{3}^{\varepsilon} . \tag{4.35}
\end{align*}
$$

Then, from (4.34) and 4.35), we infer that

$$
\begin{align*}
& \left\langle u^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}-\left\langle u^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(D)}-\int_{0}^{t}\left\langle u^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
& \quad+\left\langle\delta^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}-\left\langle\delta^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}}(\partial D)-\int_{0}^{t}\left\langle\delta^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s \\
& \quad-\int_{0}^{t}\left\langle\triangle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s+\int_{0}^{t}\left\langle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s-\int_{0}^{t}\left\langle\sin u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s \\
& \quad+\varepsilon\left\langle v^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}-\varepsilon \int_{0}^{t}\left\langle v^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
& \quad+\int_{0}^{t}\left\langle\delta^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s-\int_{0}^{t}\left\langle v^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s  \tag{4.36}\\
& \quad+\varepsilon\left\langle\theta^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}-\varepsilon \int_{\langle }^{t}\left\langle\theta^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} \\
& = \\
& \varepsilon\left\langle\bar{v}_{1}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}+\int_{0}^{t}\left\langle\bar{v}_{1}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(D)} d s-\varepsilon \int_{0}^{t}\left\langle\bar{v}_{1}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
& \\
& \quad+\varepsilon\left\langle\bar{v}_{3}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}+\int_{0}^{t}\left\langle v_{3}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(D)} d s-\varepsilon \int_{0}^{t}\left\langle\bar{v}_{3}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
& \\
& \quad+\varepsilon\left\langle\bar{\theta}_{1}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle\bar{\theta}_{1}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(\partial D)} d s-\varepsilon \int_{0}^{t}\left\langle\bar{\theta}_{1}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s \\
& \\
& \quad+\varepsilon\left\langle\bar{\theta}_{3}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle\bar{\theta}_{3}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(\partial D)} d s-\varepsilon \int_{0}^{t}\left\langle\bar{\theta}_{3}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s .
\end{align*}
$$

For (4.6), we see that for $\varphi \in C_{0}^{\infty}([0,+\infty) \times D)$,

$$
\begin{equation*}
\int_{0}^{t}\left\langle\frac{d \bar{v}_{1}^{\varepsilon}}{d s}, \varphi\right\rangle_{L^{2}(D)} d s=-\frac{1}{\varepsilon} \int_{0}^{t}\left\langle\bar{v}_{1}^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s \tag{4.37}
\end{equation*}
$$

Noticing that $\bar{v}_{1}^{\varepsilon}(0)=v_{0}$ and that

$$
\int_{0}^{t}\left\langle\frac{d \bar{v}_{1}^{\varepsilon}}{d s}, \varphi\right\rangle_{L^{2}(D)} d s=\left\langle\bar{v}_{1}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}-\left\langle\bar{v}_{1}^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(D)}-\int_{0}^{t}\left\langle\bar{v}_{1}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s
$$

we deduce from (4.37) that

$$
\begin{align*}
& \varepsilon\left\langle\bar{v}_{1}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}+\int_{0}^{t}\left\langle\bar{v}_{1}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(D)} d s-\varepsilon \int_{0}^{t}\left\langle\bar{v}_{1}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s  \tag{4.38}\\
= & \varepsilon\left\langle v_{0}, \varphi(0)\right\rangle_{L^{2}(D)} .
\end{align*}
$$

Also, from (4.24), (4.25) and $\alpha \in(1,+\infty)$, we have that for sufficiently small $\varepsilon$,

$$
\begin{align*}
& \varepsilon\left\langle\bar{v}_{3}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}+\int_{0}^{t}\left\langle\bar{v}_{3}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(D)} d s-\varepsilon \int_{0}^{t}\left\langle\bar{u}_{3}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
= & \varepsilon^{\alpha} \int_{0}^{t}\left\langle\varphi, d W_{1}(s)\right\rangle_{L^{2}(D)}  \tag{4.39}\\
= & O\left(\varepsilon^{\alpha}\right) .
\end{align*}
$$

Similarly, we derive that

$$
\begin{align*}
& \varepsilon\left\langle\bar{\theta}_{1}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle\bar{\theta}_{1}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(\partial D)} d s-\varepsilon \int_{0}^{t}\left\langle\bar{\theta}_{1}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s  \tag{4.40}\\
= & \varepsilon\left\langle\theta_{0}, \varphi(0)\right\rangle_{L^{2}(\partial D)},
\end{align*}
$$

and

$$
\begin{align*}
& \varepsilon\left\langle\bar{\theta}_{3}^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}+\int_{0}^{t}\left\langle\bar{\theta}_{3}^{\varepsilon}(s), \varphi(s)\right\rangle_{L^{2}(\partial D)} d s-\varepsilon \int_{0}^{t}\left\langle\bar{\theta}_{3}^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s  \tag{4.41}\\
= & O\left(\varepsilon^{\alpha}\right),
\end{align*}
$$

for sufficiently small $\varepsilon$.
Substituting (4.38)-(4.41) into (4.36), for every $\varphi \in C_{0}^{\infty}([0,+\infty) \times D)$, we have that for sufficiently small $\varepsilon$,

$$
\begin{align*}
& \left\langle u^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}-\left\langle u^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(D)}-\int_{0}^{t}\left\langle u^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s \\
& \quad+\left\langle\delta^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}-\left\langle\delta^{\varepsilon}(0), \varphi(0)\right\rangle_{L^{2}(\partial D)}-\int_{0}^{t}\left\langle\delta^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} d s \\
& \quad-\int_{0}^{t}\left\langle\Delta u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s+\int_{0}^{t}\left\langle u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s-\int_{0}^{t}\left\langle\sin u^{\varepsilon}, \varphi\right\rangle_{L^{2}(D)} d s \\
& \quad+\varepsilon\left\langle v^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(D)}-\varepsilon \int_{0}^{t}\left\langle v^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(D)} d s  \tag{4.42}\\
& \quad+\int_{0}^{t}\left\langle\delta^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s-\int_{0}^{t}\left\langle v^{\varepsilon}, \varphi\right\rangle_{L^{2}(\partial D)} d s \\
& \quad+\varepsilon\left\langle\theta^{\varepsilon}(t), \varphi(t)\right\rangle_{L^{2}(\partial D)}-\varepsilon \int_{0}^{t}\left\langle\theta^{\varepsilon}(s), \varphi_{s}(s)\right\rangle_{L^{2}(\partial D)} \\
& =\varepsilon\left\langle v_{0}, \varphi(0)\right\rangle_{L^{2}(D)}+\varepsilon\left\langle\theta_{0}, \varphi(0)\right\rangle_{L^{2}(\partial D)}+O\left(\varepsilon^{\alpha}\right) .
\end{align*}
$$

Projecting (4.42) onto $L^{2}(D)$ and $L^{2}(\partial D)$, respectively, we obtain that for sufficiently small $\varepsilon$ and for $\alpha \in(1,+\infty)$, the approximating equation of $u^{\varepsilon}$ is

$$
\begin{cases}\varepsilon \bar{u}_{t t}^{\varepsilon}+\bar{u}_{t}^{\varepsilon}-\triangle \bar{u}^{\varepsilon}+\bar{u}^{\varepsilon}-\sin \bar{u}^{\varepsilon}=0, & \text { in } D,  \tag{4.43}\\ \varepsilon \delta_{t t}^{\varepsilon}+\bar{\delta}_{t}^{\varepsilon}+\bar{\delta}^{\varepsilon}=-\bar{u}_{t}^{\varepsilon}, & \text { on } \partial D, \\ \bar{\delta}_{t}^{\varepsilon}=\frac{\partial_{\varepsilon^{\varepsilon}}}{\partial \mathbf{n}}, & \text { on } \partial D, \\ \bar{u}^{\varepsilon}(0)=u_{0}, \bar{u}_{t}^{\varepsilon}(0)=u_{1}, \bar{\delta}^{\varepsilon}(0)=\delta_{0}, \bar{\delta}_{t}^{\varepsilon}(0)=\delta_{1} . & \end{cases}
$$

Therefore, it follows from (4.42) and (4.43) that the result under the condition $(1,+\infty)$ holds. This completes the proof of Theorem 4.1.

## References

[1] J. T. Beale, Spectral properties of an acoustic boundary condition, Indiana Univ. Math. J., 25(1976): 895-917.
[2] J. T. Beale, Acoustic scattering from locally reacting surfaces, Indiana Univ. Math. J., 26(1977): 199-222.
[3] S. Cerrai and M. I. Freidlin, On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom, Prob. Th. and Relat. Fields, 135(2006): 363-394.
[4] S. Cerrai and M. I. Freidlin, Smoluchowski-Kramers approximation for a general class of SPDEs, J. Evol. Equ., 6(2006): 657-689.
[5] G. Chen, J. Duan, and J. Zhang, Geometric shape of invariant manifolds for a class of stochastic partial differential equations, Journal of Mathematical Physics, 52,(2011), 072702.
[6] G. Chen and J. Zhang, Asymptotic behavior for a stochastic wave equation with dynamical boundary conditions, Discrete and Continuous Dynamical Systems Series B, 17(2012): 1441-1453.
[7] G. Chen, J. Duan, and J. Zhang, Effective dynamics of stochastic wave equation with a random dynamical boundary condition, http://arxiv.org/abs/1205.5948, 2012.
[8] I. Chueshov and B. Schmalfuss, Parabolic stochastic partial differential equations with dynamical boundary conditions, Differntial and Integral Equation, 17(2008): 751-780.
[9] I. Chueshov and B. Schmalfuss, Qualitative behavior of a class of stochastic parabolic PDEs with dynamical boundary conditions, Discrte and Continuous dynamcial Systems, 18 (2007): 315-338.
[10] P. L. Chow, Stochastic wave equations with polynomial nonlinearity, Ann. Appl. Probab., 12(2002): 361-381.
[11] P. L. Chow, Asymptotics of solutions to semilinear stochastic wave equations, Ann. Appl. Probab., 16(2006): 757-780.
[12] A. T. Cousin, C. L. Frota and N. A. Larkin, Global solvability and asymptotic behavior of a hyperbolic problem with acoustic boundary condition, Funkcialaj Ekvacioj, 44(2001): 471-484.
[13] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, 1992.
[14] X. M. Fan and Y. G. Wang, Fractal dimensional of attractors for a stochastic wave equation with nonlinear damping and white noise, Stoch. Anal. Appl., 25(2007): 381-396.
[15] S. Frigeri, Attractors for semilinear damped wave equations with an acoustic boundary condition, J. Evol. Equ., 10(2010): 29-58.
[16] C. L. Frota and J. A. Goldstein, Some Nonlinear Wave Equations with Acoustic Boundary Conditions, Journal of Differential Equations, 164(2000): 92-109.
[17] S. Gerbi and B. Said-Houari, Local existence and exponential growth for a semilinear damped wave equation with dynamic boundary conditions, Advances in Differential Equations, 13(2008): 1051-1074.
[18] J. Hale and G. Raugel, Upper semicontinuity of the attractor for a singularly perturbed hyperbolic equation, J. Diff.Eqs., 73(1988): 197-214.
[19] J. L. Lions, Quelques méthodes de résolution des problèmes non linéaires, Dunod, Paris, 1969.
[20] K. Lu and B. Schmalfuss, Invariant manifolds for stochastic wave equations, J. Differential Equations, 236(2007): 460-492.
[21] Y. Lv and A. J. Roberts, Averaging approximation to singularly perturbed nonlinear stochastic wave equations, http://arxiv.org/abs/1107.4184, 2011.
[22] Y. Lv and W. Wang, Limiting dynamics for stochastic wave equations, J. Differential Equations, 244(2008): 1-23.
[23] X. Mora, Finite-dimensional attracting invariant manifolds for damped semilinear wave equations, Res. Notes in Math., 155(1987): 172-183.
[24] P. M. Morse and K. U. Ingard, Theoretical acoustics, McGraw-Hill, New York, 1968.
[25] C. Mueller, Long time existence for the wave equation with a noise term, Ann. Probab., 25(1997): 133-151.
[26] D. Mugnolo, Abstract wave equations with acoustic boundary conditions, Math. Nachr., 279(2006): 299-318.
[27] E. Pardoux and A. Piatnitski, Homogenization of a singular random one-dimensional PDE with time-varying coefficients, The Annals of Probability, 40(2012): 1316-1356.
[28] L. Popescu and A. Rodriguez-Bernal, On a singularly perturbed wave equation with dynamic boundary conditions, Proceedings of the Royal Society of Edinburgh, 134A(2004): 389-413.
[29] A. Rodriguez-Bernal and E. Zuazua, Parabolic singular limit of a wave equation with localized boundary damping, Discrete and Continuous Dynamical Systems, 1(1995): 303-346.
[30] A. Rodriguez-Bernal and E. Zuazua, Parabolic singular limit of wave equation with localized interior damping, Commun. Conemp. Math., 3(2001): 215-257.
[31] M. Reed and B. Simon, Methods of Modern Mathematical Physics II, Academic Press, New York, 1975.
[32] W. Wang and Y. Lv, Limit behavior of nonlinear stochastic wave equations with singular perturbation, Disc. and Cont. Dyna. Syst. B, 13(2010): 175-193 .
[33] W. Wang, Y. Lv and A. J. Roberts, On the approximation for singularly perturbed stochastic wave equations, http://arxiv.org/abs/1109.3000, 2011.
[34] G. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.
[35] S. F. Zhou, F. Q. Yin and Z. G. Ouyang, Random attractor for damped nonlinear wave equations with white noise, SIAM J. Appl. Dyn. Syst., 4(2005): 883-903.


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