

Minimizing the number of carries in addition

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Abstract

When numbers are added in base b in the usual way, carries occur. If two random, independent 1-digit numbers are added, then the probability of a carry is $\frac{b-1}{2b}$. Other choices of digits lead to less carries. In particular, if for odd b we use the digits $\{-(b-1)/2, -(b-3)/2, \dots, (b-1)/2\}$ then the probability of carry is only $\frac{b^2-1}{4b^2}$. Diaconis, Shao and Soundararajan conjectured that this is the best choice of digits, and proved that this is asymptotically the case when $b = p$ is a large prime. In this note we prove this conjecture for all odd primes p .

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1 The problem and result

When numbers are added in base b in the usual way, carries occur. If two added one-digit numbers are random and independent, then the probability of a carry is $\frac{b-1}{2b}$. Other choices of digits lead to less carries. In particular, if for odd b we use *balanced digits*, that is, the digits

$$\{-(b-1)/2, -(b-3)/2, \dots, 0, 1, 2, \dots, (b-1)/2\}$$

then the probability of carry is only $\frac{b^2-1}{4b^2}$. Diaconis, Shao and Soundararajan [4] conjectured that this is the best choice of digits, and proved that this conjecture is asymptotically correct when $b = p$ is a large prime. More precisely, they proved the following.

Theorem 1.1 ([4]) *For every $\epsilon > 0$ there exists a number $p_0 = p_0(\epsilon)$ so that for any prime $p > p_0$ the probability of carry when adding two random independent one-digit numbers using any fixed set of digits in base p is at least $\frac{1}{4} - \epsilon$.*

The estimate given in [4] to $p_0 = p_0(\epsilon)$ is a tower function of $1/\epsilon$.

Here we establish a tight result for any prime p , proving the conjecture for any prime.

Theorem 1.2 *For any odd prime p , the probability of carry when adding two random independent one-digit numbers using any fixed set of digits in base p is at least $\frac{p^2-1}{4p^2}$.*

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The proof is very short, and is in fact mostly an observation that the above result follows from a theorem of J. M. Pollard proved in the 70s. The conjecture for non-prime values of p remains open.

The proof can be extended to show that balanced digits minimize the probability of carry while adding k numbers, for any $k \geq 2$.

The rest of this short note is organized as follows. The next section contains a brief description of the digit systems considered. In Section 3 we present the short derivation of Theorem 1.2 from Pollard's Theorem, and in Section 4 we describe the proof of the extension to the addition of more than two summands.

2 Addition and choices of digits

A simple example illustrating the advantage of using digits that minimize the probability of carry is that of adding numbers in the finite cyclic group Z_{b^2} . Here the basis used is b . Since Z_{b^2} is a finite group, one can choose random members of it g_1, g_2, \dots, g_n uniformly and independently and consider their sum in Z_{b^2} . Following the discussion in Section 6 of [1], consider the normal subgroup $Z_b \triangleleft Z_{b^2}$, where Z_b is the subgroup of Z_{b^2} consisting of the elements $\{0, b, 2b, \dots, (b-1)b\}$.

Let $A \subset Z_{b^2}$ be a set of representatives of the cosets of Z_b in Z_{b^2} . Therefore $|A| = b$ and no two elements of A are equal modulo b . These are the digits we use. Any element $g \in Z_{b^2}$ now has a unique representation of the form $g = x + y$, where $x \in A$ and $y \in Z_b$. Indeed, x is the member of A representing the coset of Z_b that contains g , and $y \in Z_b$ is determined by the equality $g = x + y$.

Suppose, now that $g_i = x_i + y_i$ with $x_i \in A$ and $y_i \in Z_b$ is the representation of n elements g_i of Z_{b^2} that we wish to sum. We start by computing $g_1 + g_2$. To do so one first adds $x_1 + x_2$ (in Z_{b^2}). If their sum, call it z_2 , is a member of A , then there is no carry in this stage. We can then compute the sum $y_1 + y_2$ in Z_b , and get w_2 (in this stage there is no carry, as the addition here is modulo b). Therefore, in this case the representation of $g_1 + g_2$ using our digits is $g_1 + g_2 = z_2 + w_2$ with $z_2 \in A$ and $w_2 \in Z_b$, and we can now proceed by induction and compute the sum of this element with $g_3 + g_4 + \dots + g_n$. If, on the other hand, $x_1 + x_2 \notin A$ then there is a carry. In this case we let z_2 be the unique member of A so that $x_1 + x_2$ lies in the coset of Z_b containing z_2 . This determines the element $t_2 \in Z_b$ so that $x_1 + x_2 = z_2 + t_2$. The carry here is t_2 , and we can now proceed and compute the sum $t_2 + y_1 + y_2$ in Z_b getting an element w_2 . Therefore in this case too the unique representation of $g_1 + g_2$ using our digits is $z_2 + w_2$, but since the process of computing them involved the carry t_2 the number of additions performed during the computation in the group Z_b was 2 and not 1 as in the case that involved no carry.

As g_1 and g_2 are random independent elements chosen uniformly in Z_{b^2} , their sum is also uniform in this group, implying that the element $z_2 + w_2$ is also uniform. Therefore, when we now proceed and compute the sum of this element with g_3 the probability of carry is again exactly as it has been before (though the conditional probability of a carry given that there has been one in the previous step may be different). We conclude that the expected number of carries during the whole process of adding $g_1 + g_2 + \dots + g_n$ that consist of $n - 1$ additions is exactly $(n - 1)$ times the probability of getting a

carry in one addition of two independent uniform random digits in the set A .

Therefore, the problem of minimizing the expected number of these carries is that of selecting the set of coset representatives A so that the probability that the addition of two random members of A in Z_{b^2} does not lie in A is minimized.

This leads to the following equivalent formulation of Theorem 1.2.

Theorem 2.1 *Let p be an odd prime. For any subset A of the group Z_{p^2} of integers modulo p^2 so that $|A| = p$ and the members of A are pairwise distinct modulo p , the number of ordered pairs $(a, b) \in A \times A$ so that $a + b \pmod{p^2} \notin A$ is at least $\frac{p^2-1}{4}$.*

3 Adding two numbers

The result of Pollard needed here is the following.

Theorem 3.1 (Pollard [7]) *For an integer m and two sets A and B of residues modulo m , and for any positive integer r , let $N_r = N_r(A, B)$ denote the number of all residues modulo m that have a representation as a sum $a + b$ with $a \in A$ and $b \in B$ in at least r ways. (The representations are counted as ordered pairs, that is, if a and b differ and both belong to $A \cap B$, then $a + b = b + a$ are two distinct representations of the sum). If $(x - y, m) = 1$ for any two distinct elements $x, y \in B$ then for any $1 \leq r \leq \min\{|A|, |B|\}$:*

$$N_1 + N_2 + \cdots + N_r \geq r \cdot \min\{m, |A| + |B| - r\}.$$

Note that the case $r = 1$ is the classical theorem of Cauchy and Davenport (see [2], [3]). The proof is short and clever, following the approach in the original papers of Cauchy and Davenport. It proceeds by induction on $|B|$, where in the induction step one first replaces B by a shifted copy $B' = B - g$ so that $I = A \cap B'$ satisfies $1 \leq |I| < |B'| = |B|$, and then applies the induction hypothesis to the pair $(A \cup B', I)$ and to the pair $(A - I, B' - I)$. The details can be found in [7] (see also [6]).

Proof of Theorem 1.2: As mentioned above, the statement of Theorem 1.2 is equivalent to that of Theorem 2.1: for any subset A of the group Z_{p^2} of integers modulo p^2 so that $|A| = p$ and the members of A are pairwise distinct modulo p , the number of ordered pairs $(a, b) \in A \times A$ so that $a + b \pmod{p^2} \notin A$ is at least $\frac{p^2-1}{4}$.

Given such a set A , note that $(x - y, p^2) = 1$ for every two distinct elements $x, y \in A$. We can thus apply Pollard's Theorem stated above with $m = p^2$, $A = B$ and $r = (p - 1)/2$ to conclude that

$$N_1 + N_2 + \cdots + N_r \geq r \cdot \min\{p^2, |A| + |B| - r\} = r \cdot (2p - r).$$

The sum $N_1 + N_2 + \cdots + N_r$ counts every element $x \in Z_{p^2}$ exactly $\min\{r, n(x)\}$ times, where $n(x)$ is the number of representations of x as an ordered sum $a + b$ with $a, b \in A$. The total contribution to this sum arising from elements $x \in A$ is at most $r|A| = rp$. Therefore, there are at least $r(p - r) = \frac{p-1}{2} \frac{p+1}{2} = \frac{p^2-1}{4}$ ordered pairs $(a, b) \in A \times A$ so that $a + b \notin A$, completing the proof. \square

Remark: The above proof shows that even if we use one set of digits for one summand, a possibly different set of digits for the second summand, and a third set of digits for the sum, then still the probability of carry must be at least $\frac{b^2-1}{4b^2}$.

4 Adding more numbers

The theorem of Pollard is more general than the statement above and deals with addition of $k > 2$ sets as well. This can be used in determining the minimum possible probability of carry in the addition of k random 1-digit numbers in a prime base p with the best choice of the p digits. In fact, for every k and every odd prime p , the minimum probability is obtained by using the balanced digits $\{-(p-1)/2, -(p-3)/2, \dots, (p-1)/2\}$. Therefore, the minimum possible probability of carry in adding k 1-digit numbers in a prime base $p > 2$ is exactly the probability that the sum of k independent random variables, each distributed uniformly on the set

$$\{-(p-1)/2, -(p-3)/2, \dots, (p-1)/2\},$$

is of absolute value exceeding $(p-1)/2$.

Here are the details. We need the following.

Theorem 4.1 (Pollard [7]) *Let m be a positive integer and let A_1, A_2, \dots, A_k be subsets of Z_m . Assume, further, that for every $2 \leq i \leq k$ every two distinct elements x, y of A_i satisfy $(x - y, m) = 1$. Let A'_1, A'_2, \dots, A'_k be another collection of subsets of Z_m , in which each A'_i consists of consecutive elements and satisfies $|A'_i| = |A_i|$. For an $x \in Z_m$ let $n(x)$ denote the number of representations of x as an ordered sum of the form $x = a_1 + a_2 + \dots + a_k$ with $a_i \in A_i$, and let $n'(x)$ denote the number of representations of x as an ordered sum of the form $x = a'_1 + a'_2 + \dots + a'_k$ with $a'_i \in A'_i$ for all i . Then, for any integer $r \geq 1$*

$$\sum_{x \in Z_m} \min\{r, n(x)\} \geq \sum_{x \in Z_m} \min\{r, n'(x)\}.$$

Corollary 4.2 *Let p be an odd prime, and let A be a subset of cardinality p of Z_{p^2} . Assume, further, that the members of A are pairwise distinct modulo p . Put $A' = \{-(p-1)/2, -(p-3)/2, \dots, (p-1)/2\}$. Then, for any positive integer k , the number of ordered sums modulo p^2 of k elements of A that do not belong to A is at least as large as the number of ordered sums modulo p^2 of k elements of A' that do not belong to A' .*

Proof: Let r be the number of ordered sums modulo p^2 of k elements of A' whose value is precisely $(p-1)/2$. It is not difficult to check that for any other member g of A' there are at least r ordered sums modulo p^2 of k elements of A' whose value is precisely g . Similarly, for any $x \notin A'$, the number of ordered sums of k elements of A' whose value is precisely x is at most r . Indeed, the number of times an element is obtained as an ordered sum modulo p^2 of k elements of A' is a monotone non-increasing

function of its distance from 0. (This can be easily proved by induction on k). Therefore,

$$\sum_{x \in Z_{p^2}} \min\{r, n'(x)\} = rp + \sum_{x \in Z_{p^2} - A'} n'(x).$$

By Theorem 4.1 with $m = p^2$, $A_1 = A_2 = \dots = A_k = A$ and the value of r above

$$\sum_{x \in Z_{p^2}} \min\{r, n(x)\} \geq \sum_{x \in Z_{p^2}} \min\{r, n'(x)\}.$$

However, clearly,

$$rp + \sum_{x \in Z_{p^2} - A} n(x) \geq \sum_{x \in Z_{p^2}} \min\{r, n(x)\},$$

and therefore

$$rp + \sum_{x \in Z_{p^2} - A} n(x) \geq \sum_{x \in Z_{p^2}} \min\{r, n'(x)\} = rp + \sum_{x \in Z_{p^2} - A'} n'(x).$$

This implies that

$$\sum_{x \in Z_{p^2} - A} n(x) \geq \sum_{x \in Z_{p^2} - A'} n'(x),$$

as needed. \square

The corollary clearly implies that the minimum possible probability of carry in adding k 1-digit numbers in a prime base $p > 2$ is exactly the probability that the sum of k independent random variables, each distributed uniformly on the set

$$\{-(p-1)/2, -(p-3)/2, \dots, (p-1)/2\},$$

is of absolute value exceeding $(p-1)/2$.

Remarks:

- As in the case of two summands, the proof implies that the assertion of the last paragraph holds even if we are allowed to choose a different set of digits for each summand and for the result.
- After the completion of this note I learned from the authors of [4] that closely related results (for addition in Z_p , not in Z_{p^2}) appear in the paper of Lev [5].

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