On Near-controllability, Nearly-controllable Subspaces, and Near-controllability Index of a Class of Discrete-time Bilinear Systems: A Root Locus Approach

Lin Tie

Abstract

This paper studies near-controllability of a class of discrete-time bilinear systems via a root locus approach. A necessary and sufficient criterion for the systems to be nearly controllable is given. In particular, by using the root locus approach, the control inputs which achieve the state transition for the nearly controllable systems can be computed. Furthermore, for the non-nearly controllable systems, nearly-controllable subspaces are derived and near-controllability index is defined. Accordingly, the controllability properties of such class of discrete-time bilinear systems are fully characterized. Finally, examples are provided to demonstrate the results of the paper.

Index Terms

discrete-time bilinear systems, near-controllability, nearly-controllable subspaces, near-controllability index, root locus approach.

I. Introduction

Given the capability of modeling a large number of processes in the real world, bilinear systems have received considerable attention [1-4]. Furthermore, bilinear systems are thought to be simpler and better understood than most other nonlinear systems. Such systems have particular advantages on structural properties, optimization, identification, and control. The relaxed version of a linear switched system is actually a bilinear system [5]. Owing to both the practical and theoretical importance, bilinear systems have been a hot topic in the literature of nonlinear systems over decades.

Controllability is one fundamental concept in mathematical control theory. It was identified in the early 1960s, and then the theory of controllability for linear systems based on the state space description was systematically established [6,7]. During nearly the same period, controllability of nonlinear systems also was considered. Since the 1970s, Lie algebra methods and other powerful tools of differentiable manifold theory have been developed to study controllability of nonlinear systems [8-11]. Today, controllability has played an essential role in the development and application of mathematical control theory. There are many different kinds of definitions on controllability, such as local controllability, global controllability, approximate controllability, positive controllability, and null controllability. Roughly speaking, controllability is defined as the ability of a system that the system can be steered from an arbitrary initial state to an arbitrary terminal state under the action of admissible controls. It is an important as well as fundamental property of control systems and is of great practical relevance.

In bilinear system theory, controllability is one of the main research topics. This is particularly true for the continuous-time case. More specifically, controllability of continuous-time bilinear systems has

This work was supported by the China Postdoctoral Science Foundation funded project and the National Natural Science Foundation of China.

The author is with the School of Automation Science and Electrical Engineering, Beihang University (Beijing University of Aeronautics and Astronautics), 100191, Beijing, P. R. China. *E-mail address*: tielinllc@gmail.com.

been extensively investigated profiting from the Lie algebra methods. Various Lie-algebraic criteria on controllability of continuous-time bilinear systems were obtained in the literature, which have been summarized and updated in a recent monograph [4] on bilinear systems. However, for discrete-time bilinear systems, the controllability results are rather sparse compared with their continuous-time counterparts. Most of the works on controllability of discrete-time bilinear systems were done in the 1970s [12-15], dealing with systems of the form

$$x\left(k+1\right)=\left(A+u\left(k\right)B\right)x\left(k\right) \tag{1}$$

where $x(k)\in\mathbb{R}^n$, $A,B\in\mathbb{R}^{n\times n}$, and $u(k)\in\mathbb{R}$. *System (1) is said to be controllable if, for any ξ,η in $\mathbb{R}^n_*$ ($\mathbb{R}^n_*:=\mathbb{R}^n\setminus\{0\}$), there exist a positive integer l and a finite control sequence $u(k)$ ($k=0,1,\ldots,l-1$) such that the system can be steered from ξ to η at $k=l$.* In particular, [12] gave a sufficient condition for controllability of system (1), which requires, at least, A is similar to an orthogonal matrix. [13] studied controllability of system (1) under the assumption of rank$B=1$ and presented necessary as well as sufficient conditions; based on the work in [13], [15] improved these conditions by raising necessary and sufficient ones. It can be seen that, for controllability of discrete-time bilinear systems, only specific subclasses have been considered, while most cases remain unsolved. Since the middle 1980s, few work has been reported until the 2000s. One of the reasons leading to such a few results on controllability of discrete-time bilinear systems is that the controllability problems are quite difficult to deal with due to the systems' nonlinearity. Another one is that, for discrete-time nonlinear systems, semigroups tend to appear so that less algebraic structure of the systems is available [16]. Recently, [17] continued the study on controllability of system (1) and obtained necessary and sufficient conditions for the case of $AB=BA$, where it was shown that the system can be controllable only if its dimension is no greater than two. Nevertheless, an uncontrollable system can still have a large *controllable region*[1]. This has first been proved for system (1) with $A=I$, i.e.

$$x\left(k+1\right)=\left(I+u\left(k\right)B\right)x\left(k\right) \tag{2}$$

which is uncontrollable with dimension greater than two [17]. Indeed, if B has only real eigenvalues that are nonzero and pairwise distinct, then the system (2) has a large controllable region, which nearly covers the whole space, and is nearly controllable.

If we only use "uncontrollable" to describe a system which is not controllable according to the general controllability definition, we may miss some valuable properties of it. Near-controllability is thus introduced to describe those systems that are uncontrollable but have a very large controllable region. Nearly controllable systems exist widely in nonlinear systems. This property was first defined and was demonstrated on system (2) in [18], and it was then generalized to system (1) with $B=I$ in [19] and to both continuous-time and discrete-time nonlinear systems that are not necessarily bilinear in [20]. The definition of near-controllability, which was first given in [18], has now been updated in [20]. *A continuous-time system $\dot{x}\left(t\right)=f\left(x\left(t\right),u\left(t\right)\right)$ (discrete-time system $x\left(k+1\right)=f\left(x\left(k\right),u\left(k\right)\right)$) is said to be nearly controllable if, for any $\xi\in\mathbb{R}^n\setminus\mathcal{E}$ and any $\eta\in\mathbb{R}^n\setminus\mathcal{F}$, there exist a piecewise continuous control $u\left(t\right)$ and $T>0$ (a finite control sequence $u\left(k\right)$, $k=0,1,\ldots,l-1$, where l is a positive integer) such that the system can be steered from ξ to η at some $t\in(0,T)$ ($k=l$), where $\mathcal{E}$ and $\mathcal{F}$ are two sets of zero Lebesgue measure in $\mathbb{R}^n$.* If we let $\mathcal{E},\mathcal{F}=\varnothing$, then the near-controllability definition degenerates to the general controllability definition. Thus, near-controllability includes the notion of controllability and may better characterize the controllability properties of nonlinear systems. However, most of the existing works on near-controllability are reported for discrete-time bilinear systems and the study on this topic is just at the beginning.

[1]A controllable region is a region in $\mathbb{R}^n$ on which the system is controllable. Namely, for any ξ,η in this region, there exist control inputs that steer the system from ξ to η.

In this paper, we continue the research on near-controllability of system (2) with B having real eigenvalues only. We improve the sufficient condition for near-controllability of the system (2) in [18] by giving a necessary and sufficient one. In particular, we apply a new approach this time to prove near-controllability. That is, a root locus approach is proposed in this paper. Compared with the technique used in [18,19] which is based on the implicit function theorem, by the root locus approach we can not only improve the obtained result on near-controllability, but also compute the required control inputs that achieve the state transition[2]. We thus present a useful algorithm. Furthermore, inspired by the state space description for linear systems, we also consider the non-nearly controllable systems. It is well known that if a linear time-invariant system is uncontrollable, then the state space can be decomposed as a direct sum of a controllable subspace and an uncontrollable subspace. For the non-nearly controllable systems, we derive nearly-controllable subspaces and define near-controllability index by using the improved near-controllability result, which shows that the non-nearly controllable systems can be controllable on the nearly-controllable subspaces. The near-controllability index is used to determine the largest nearly-controllable subspaces that a non-nearly controllable system has. In summary, the controllability properties of the system (2) are fully characterized. Finally, we provide examples to illustrate the conceptions and the results of this paper.

II. A Root Locus Approach to Near-controllability

In this section, we propose a root locus approach and apply it to prove near-controllability of the system (2). The idea of the root locus approach is to use the root locus theory to achieve the state transition, including transferring any initial state to itself and to a state close to it. More importantly, the control inputs are computable in these steps. Then, we use the matrix theory by establishing a transition matrix and some algebraic techniques to prove near-controllability. The proof steps are similar to those in [18,19], but the factor that plays the key role has changed. That is, the implicit function theorem has been replaced by the root locus approach.

Lemma 1: If $B \in \mathbb{R}^{n\times n}$ has only nonzero and real eigenvalues, then there exist nonzero real numbers $u(0), u(1), \ldots, u(L-1), u(L)$ such that

$$(I + u(L)B)(I + u(L-1)B)\cdots(I + u(1)B)(I + u(0)B) = I \tag{3}$$

if and only if the dimension of the largest Jordan block in the Jordan canonical form of B is no greater than two. Further, if B not only satisfies the necessary and sufficient condition but also is cyclic[3], then there exist $(2m+3)$ nonzero real and pairwise distinct numbers $u(0), u(1), \ldots, u(2m+1), u(2m+2)$ such that (3) holds, where m is the number of the distinct eigenvalues of B.

Proof: We put the proof of necessity in Appendix and only prove sufficiency here. Assume first that B has the following cyclic form

$$\begin{bmatrix} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_r & 1 & & & \\ & & & & \lambda_r & & & \\ & & & & & \lambda_{r+1} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_m \end{bmatrix} \tag{4}$$

[2]For nonlinear systems, it is, in general, hard to compute the control inputs to achieve state transition even if controllability has been proved. In this paper, we obtain both the controllability and "control computability" (the ability of computing the required control inputs for the transition of any given initial and terminal states).

[3]A matrix is said to be cyclic if its characteristic polynomial is equal to its minimal polynomial. Namely, only one Jordan block exists for each eigenvalue in the Jordan canonical form of the matrix.

where B is already in Jordan canonical form since a nonsingular transformation to B does not affect the proof and $\lambda_1, \ldots, \lambda_m$ are nonzero real and pairwise distinct. In addition, $m + r = n$. We now show that the equation

$$(I + u(2m+2)B)(I + u(2m+1)B)\cdots(I + u(1)B)(I + u(0)B) = I \tag{5}$$

admits a solution of nonzero real $u(0), u(1), \ldots, u(2m+1), u(2m+2)$. To this end, it will be proved that the reciprocals of $u(0), u(1), \ldots, u(2m+1), u(2m+2)$ are on the root loci of the characteristic equation of a closed transfer function related to B's eigenvalues. Multiplying both sides of equation (5) by

$$\prod_{k=0}^{2m+2} \frac{1}{u(k)}$$

we have

$$\left(\frac{1}{u(2m+2)}I + B\right)\left(\frac{1}{u(2m+1)}I + B\right)\cdots\left(\frac{1}{u(1)}I + B\right)\left(\frac{1}{u(0)}I + B\right) = \prod_{k=0}^{2m+2}\frac{1}{u(k)}I, \tag{6}$$

from which[4]

$$B^{2m+3} + \sum_{k=0}^{2m+2}\frac{1}{u(k)}B^{2m+2} + \cdots + \sum_{k=0}^{2m+2}\frac{u(k)}{\prod_{j=0}^{2m+2}u(j)}B = 0. \tag{7}$$

Writing (7) through two groups of equations yields

$$\begin{cases} \lambda_1^{2m+3} + \sum_{k=0}^{2m+2}\frac{1}{u(k)}\lambda_1^{2m+2} + \cdots + \sum_{k=0}^{2m+2}\frac{u(k)}{\prod_{j=0}^{2m+2}u(j)}\lambda_1 = 0 \\ (2m+3)\lambda_1^{2m+2} + (2m+2)\sum_{k=0}^{2m+2}\frac{1}{u(k)}\lambda_1^{2m+1} + \cdots + \sum_{k=0}^{2m+2}\frac{u(k)}{\prod_{j=0}^{2m+2}u(j)} = 0 \\ \vdots \\ \lambda_r^{2m+3} + \sum_{k=0}^{2m+2}\frac{1}{u(k)}\lambda_r^{2m+2} + \cdots + \sum_{k=0}^{2m+2}\frac{u(k)}{\prod_{j=0}^{2m+2}u(j)}\lambda_r = 0 \\ (2m+3)\lambda_r^{2m+2} + (2m+2)\sum_{k=0}^{2m+2}\frac{1}{u(k)}\lambda_r^{2m+1} + \cdots + \sum_{k=0}^{2m+2}\frac{u(k)}{\prod_{j=0}^{2m+2}u(j)} = 0 \end{cases},$$

$$\begin{cases} \lambda_{r+1}^{2m+3} + \sum_{k=0}^{2m+2}\frac{1}{u(k)}\lambda_{r+1}^{2m+2} + \cdots + \sum_{k=0}^{2m+2}\frac{u(k)}{\prod_{j=0}^{2m+2}u(j)}\lambda_{r+1} = 0 \\ \vdots \\ \lambda_m^{2m+3} + \sum_{k=0}^{2m+2}\frac{1}{u(k)}\lambda_m^{2m+2} + \cdots + \sum_{k=0}^{2m+2}\frac{u(k)}{\prod_{j=0}^{2m+2}u(j)}\lambda_m = 0 \end{cases}. \tag{8}$$

[4]Throughout this paper, such kind of expression $\sum_{k=0}^{2m+2}\frac{u(k)}{\prod_{j=0}^{2m+2}u(j)}$ is used to represent the summation

$$\sum \frac{1}{u(k_0)}\frac{1}{u(k_1)}\cdots\frac{1}{u(k_{2m+1})} = \frac{1}{u(0)}\frac{1}{u(1)}\cdots\frac{1}{u(2m+1)} + \frac{1}{u(0)}\cdots\frac{1}{u(2m)}\frac{1}{u(2m+2)} + \cdots + \frac{1}{u(1)}\cdots\frac{1}{u(2m+1)}\frac{1}{u(2m+2)}$$

for simplicity. There is no meaning of division in the expression.

By adding the following $(m-r)$ constraints

$$(2m+3)\,\lambda_{r+1}^{2m+2}+(2m+2)\sum_{k=0}^{2m+2}\frac{1}{u\,(k)}\lambda_{r+1}^{2m+1}+\cdots+\sum_{k=0}^{2m+2}\frac{u\,(k)}{\prod_{j=0}^{2m+2}u\,(j)}=0,$$

$$\vdots$$

$$(2m+3)\,\lambda_{m}^{2m+2}+(2m+2)\sum_{k=0}^{2m+2}\frac{1}{u\,(k)}\lambda_{m}^{2m+1}+\cdots+\sum_{k=0}^{2m+2}\frac{u\,(k)}{\prod_{j=0}^{2m+2}u\,(j)}=0$$

to the second group of equations in (8) and the two constraints

$$\begin{aligned}\lambda_{m+1}^{2m+3}+\sum_{k=0}^{2m+2}\frac{1}{u\,(k)}\lambda_{m+1}^{2m+2}+\cdots+\sum_{k=0}^{2m+2}\frac{u\,(k)}{\prod_{j=0}^{2m+2}u\,(j)}\lambda_{m+1} &= 0,\\ \lambda_{m+2}^{2m+3}+\sum_{k=0}^{2m+2}\frac{1}{u\,(k)}\lambda_{m+2}^{2m+2}+\cdots+\sum_{k=0}^{2m+2}\frac{u\,(k)}{\prod_{j=0}^{2m+2}u\,(j)}\lambda_{m+2} &= 0\end{aligned}$$

to all equations in (8), where $\lambda_{m+1},\lambda_{m+2}$ are chosen such that

$$0<|\lambda_{m+1}|<\min\left\{|\lambda_1|,\ldots,|\lambda_m|\right\}<\max\left\{|\lambda_1|,\ldots,|\lambda_m|\right\}<-\lambda_{m+2}, \tag{9}$$

we can put the above $(m+r)+(m-r)+2=2m+2$ equations into the matrix form

$$\begin{bmatrix}\lambda_1^{2m+2} & \cdots & \lambda_1\\ (2m+2)\,\lambda_1^{2m+1} & \cdots & 1\\ \vdots & \vdots & \vdots\\ \lambda_m^{2m+2} & \cdots & \lambda_m\\ (2m+2)\,\lambda_m^{2m+1} & \cdots & 1\\ \lambda_{m+1}^{2m+2} & \cdots & \lambda_{m+1}\\ \lambda_{m+2}^{2m+2} & \cdots & \lambda_{m+2}\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{2m+2}\frac{1}{u(k)}\\ \vdots\\ \sum_{k=0}^{2m+2}\frac{u(k)}{\prod_{j=0}^{2m+2}u(j)}\end{bmatrix}=\begin{bmatrix}-\lambda_1^{2m+3}\\ -(2m+3)\,\lambda_1^{2m+2}\\ \vdots\\ -\lambda_m^{2m+3}\\ -(2m+3)\,\lambda_m^{2m+2}\\ -\lambda_{m+1}^{2m+3}\\ -\lambda_{m+2}^{2m+3}\end{bmatrix}.$$

From Lemma 15 in Appendix, we obtain

$$\begin{aligned}\sum_{k=0}^{2m+2}\frac{1}{u\,(k)} &= (-1)\left(2\sum_{i=1}^{m}\lambda_i+\lambda_{m+1}+\lambda_{m+2}\right),\\ &\vdots\\ \sum_{k=0}^{2m+2}\frac{u\,(k)}{\prod_{j=0}^{2m+2}u\,(j)} &= (-1)^{2m+2}\,\lambda_{m+1}\lambda_{m+2}\prod_{i=1}^{m}\lambda_i^2.\end{aligned}$$

If we denote $\prod_{k=0}^{2m+2}\frac{1}{u(k)}$ by

$$(-1)^{2m+3}\,K,$$

then, from the Viète's formulas, $\frac{1}{u(0)}, \frac{1}{u(1)}, \ldots, \frac{1}{u(2m+1)}, \frac{1}{u(2m+2)}$ are the roots of the following $(2m+3)$th-degree equation

$$s^{2m+3} + \left(2\sum_{i=1}^{m}\lambda_i + \lambda_{m+1} + \lambda_{m+2}\right) s^{2m+2} + \cdots + \left(\lambda_{m+1}\lambda_{m+2}\prod_{i=1}^{m}\lambda_i^2\right) s + K = 0$$

and are thus on the root loci of the characteristic equation of the closed loop transfer function

$$\begin{aligned} 1 + KG(s) \quad &: \quad = 1 + \frac{K}{s^{2m+3} + \left(2\sum_{i=1}^{m}\lambda_i + \lambda_{m+1} + \lambda_{m+2}\right) s^{2m+2} + \cdots + \left(\lambda_{m+1}\lambda_{m+2}\prod_{i=1}^{m}\lambda_i^2\right) s} \\ &= \quad 1 + \frac{K}{s(s+\lambda_1)^2 \cdots (s+\lambda_m)^2 (s+\lambda_{m+1})(s+\lambda_{m+2})}. \end{aligned}$$

$G(s)$ has $(2m+3)$ real poles only and has no zero. By condition (9), in the $(2m+3)$ real poles, $-\lambda_{m+2}$ is the largest pole of $G(s)$ and is a single pole; $-\lambda_1, \ldots, -\lambda_m$ are double poles of $G(s)$; 0 and $-\lambda_{m+1}$ are two single poles of $G(s)$ between two double poles in $-\lambda_1, \ldots, -\lambda_m$. Therefore, we can see from the root locus theory [21] that, as K increases from 0 to $+\infty$, all root loci of $1+KG(s)=0$ that start at the $(2m+3)$ real poles will first move on the real axis. That is to say, we can always choose a K such that $1+KG(s)=0$ has $(2m+3)$ *nonzero real* and *pairwise distinct* roots. Indeed, by making K small enough, $\frac{1}{u(0)}, \frac{1}{u(1)}, \ldots, \frac{1}{u(2m+1)}, \frac{1}{u(2m+2)}$ are perturbed away from $-\lambda_1, -\lambda_1, \ldots, -\lambda_m, -\lambda_m, -\lambda_{m+1}, -\lambda_{m+2}, 0$ and hence are not only nonzero real but also pairwise distinct. Then, the reciprocals of the nonzero real and pairwise distinct roots are the real numbers that satisfy equation (5) since $\frac{1}{u(0)}, \frac{1}{u(1)}, \ldots, \frac{1}{u(2m+1)}, \frac{1}{u(2m+2)}$ satisfy the equations in (8), (7), and (6).

Finally, if B is noncyclic, we can choose the largest cyclic main submatrix of B, named B_{sub}, and there exist real numbers $u(0), u(1), \ldots, u(L-1), u(L)$ such that

$$(I + u(L) B_{sub})(I + u(L-1) B_{sub}) \cdots (I + u(1) B_{sub})(I + u(0) B_{sub}) = I \tag{10}$$

from the above analysis. By noting the fact that

$$\left(I + u(L)\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right) \cdots \left(I + u(0)\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right) = I$$

implies

$$\left(I + u(L)\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \cdots \left(I + u(0)\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = I,$$

one can easily verify that eq. (10) still works if B_{sub} is replaced by B. ■

By using Lemma 1, we can transfer an arbitrary state to itself and improve Theorem 2 obtained in [18]. Moreover, by the root locus approach, we can compute the required control inputs to achieve the state transition.

Theorem 2: Consider that B in (2) has only real eigenvalues. Then, the system (2) is nearly controllable if and only if B is nonsingular, cyclic, and does not have a Jordan block with dimension greater than two in its Jordan canonical form.

Proof: The proof of necessity is put in Appendix. For sufficiency, we still assume that B is of the Jordan canonical form given in (4) without loss of generality. According to Lemma 1, there exist $(2m+3)$ nonzero real and pairwise distinct values $u_0, u_1, \ldots, u_{2m+1}, u_{2m+2}$ such that

$$(I + u_{2m+2}B)(I + u_{2m+1}B) \cdots (I + u_1 B)(I + u_0 B) = I. \tag{11}$$

We next prove that, for almost any ξ in $\mathbb{R}^n$, we can construct control inputs such that ξ can be transferred to an arbitrary state which is close to ξ. We first prove the existence of such control inputs, which is similar to what we have done in [18]. We then show how to compute such control inputs by applying the root locus approach, which is concluded in the final step of Algorithm 5 (to shorten the proof we do not write it here).

Consider the following function

$$\begin{aligned} F\left(t_0, t_1, \ldots, t_{n-2}, t_{n-1}, y\right) := & \left(I + t_{n-1}B\right)\left(I + t_{n-2}B\right)\cdots\left(I + t_1 B\right)\left(I + t_0 B\right)\xi - \\ & \left(I + u_{n-1}B\right)\left(I + u_{n-2}B\right)\cdots\left(I + u_1 B\right)\left(I + u_0 B\right)\left(\xi + y\right) \end{aligned}$$

where $t_0, t_1, \ldots, t_{n-2}, t_{n-1} \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Apparently, $(u_0, u_1, \ldots, u_{n-2}, u_{n-1}, \mathbf{0})$ is a zero of F. Using Lemma 1 in [18] yields

$$\left|\frac{\partial\left(F\left(t_0, t_1, \ldots, t_{n-2}, t_{n-1}, y\right)\right)}{\partial\left(t_0, t_1, \ldots, t_{n-2}, t_{n-1}\right)}\right| = |B|\, V\left(t_{n-1}, t_{n-2}, \ldots, t_1, t_0\right)\left|\begin{array}{ccccc} \xi & B\xi & \cdots & B^{n-2}\xi & B^{n-1}\xi \end{array}\right| \quad (12)$$

where V is the Vandermonde determinant and $|\cdot|$ denotes the determinant of a matrix. Thus, for any

$$\xi \notin \left\{\xi \left| \left|\begin{array}{ccccc} \xi & B\xi & \cdots & B^{n-2}\xi & B^{n-1}\xi \end{array}\right| = 0\right.\right\},$$

the Jacobian determinant (12) does not vanish at $(u_0, u_1, \ldots, u_{n-2}, u_{n-1}, \mathbf{0})$. According to the implicit function theorem, there exist two open neighborhoods, named

$$O\left(\left[\begin{array}{ccccc} u_0 & u_1 & \cdots & u_{n-2} & u_{n-1} \end{array}\right]^T, \rho_1\right),\ O\left(\mathbf{0}, \rho_2\right)$$

respectively, such that

$$F\left(t_0\left(y\right), t_1\left(y\right), \ldots, t_{n-2}\left(y\right), t_{n-1}\left(y\right), y\right) = \mathbf{0}$$

where $\left[\begin{array}{ccccc} t_0(y) & t_1(y) & \cdots & t_{n-2}(y) & t_{n-1}(y) \end{array}\right]^T \in O\left(\left[\begin{array}{ccccc} u_0 & u_1 & \cdots & u_{n-2} & u_{n-1} \end{array}\right]^T, \rho_1\right)$ and $y \in O\left(\mathbf{0}, \rho_2\right)$, i.e.

$$\begin{aligned} & \left(I + t_{n-1}\left(y\right)B\right)\left(I + t_{n-2}\left(y\right)B\right)\cdots\left(I + t_1\left(y\right)B\right)\left(I + t_0\left(y\right)B\right)\xi \\ = \ & \left(I + u_{n-1}B\right)\left(I + u_{n-2}B\right)\cdots\left(I + u_1 B\right)\left(I + u_0 B\right)\left(\xi + y\right). \end{aligned}$$

Then, by (11) we have

$$\begin{aligned} & \left(I + u_{2m+2}B\right)\cdots\left(I + u_n B\right)\left(I + t_{n-1}\left(y\right)B\right)\cdots\left(I + t_0\left(y\right)B\right)\xi \\ = \ & \left(I + u_{2m+2}B\right)\cdots\left(I + u_n B\right)\left(I + u_{n-1}B\right)\cdots\left(I + u_0 B\right)\left(\xi + y\right) = \xi + y. \end{aligned} \quad (13)$$

This means ξ can be transferred to any state that is close enough to ξ. From B's structure and PBH test [22],

$$\left\{\xi \left| \left|\begin{array}{ccccc} \xi & B\xi & \cdots & B^{n-2}\xi & B^{n-1}\xi \end{array}\right| = 0\right.\right\} = \left\{\xi = \left[\begin{array}{cccc} \xi_1 & \xi_2 & \cdots & \xi_n \end{array}\right]^T \mid \xi_2\xi_4\cdots\xi_{2r}\xi_{2r+1}\cdots\xi_n = 0\right\}$$

that is a hypersurface in $\mathbb{R}^n$ and separates $\mathbb{R}^n$ into 2^m open orthants (which are the same as those in (12) in [19]). We now prove that the system (2) is controllable on each of the 2^m open orthants. For any two

states ξ, η in one orthant, we establish the *transition matrix*

$$T_{\xi\to\eta} = \begin{bmatrix} \frac{\eta_2}{\xi_2} & \frac{\eta_1}{\xi_2} - \frac{\xi_1\eta_2}{\xi_2^2} & & & & & & & \\ & \frac{\eta_2}{\xi_2} & & & & & & & \\ & & \ddots & & & & & & \\ & & & \frac{\eta_{2r}}{\xi_{2r}} & \frac{\eta_{2r-1}}{\xi_{2r}} - \frac{\xi_{2r-1}\eta_{2r}}{\xi_{2r}^2} & & & & \\ & & & & \frac{\eta_{2r}}{\xi_{2r}} & & & & \\ & & & & & \frac{\eta_{2r+1}}{\xi_{2r+1}} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \frac{\eta_n}{\xi_n} \end{bmatrix}. \tag{14}$$

It can be seen that $\eta = T_{\xi\to\eta}\xi$ and all the eigenvalues of $T_{\xi\to\eta}$ are positive since ξ, η belong to the same orthant. Furthermore,

$$\begin{aligned} \lim_{q\to+\infty} T_{\xi\to\eta}^{\frac{1}{q}} &= \lim_{q\to+\infty} \begin{bmatrix} \left(\frac{\eta_2}{\xi_2}\right)^{\frac{1}{q}} & \frac{\frac{\eta_1}{\xi_2} - \frac{\xi_1\eta_2}{\xi_2^2}}{q\left(\frac{\eta_2}{\xi_2}\right)^{\frac{q-1}{q}}} & & & & & & & \\ & \left(\frac{\eta_2}{\xi_2}\right)^{\frac{1}{q}} & & & & & & & \\ & & \ddots & & & & & & \\ & & & \left(\frac{\eta_{2r}}{\xi_{2r}}\right)^{\frac{1}{q}} & \frac{\frac{\eta_{2r-1}}{\xi_{2r}} - \frac{\xi_{2r-1}\eta_{2r}}{\xi_{2r}^2}}{q\left(\frac{\eta_{2r}}{\xi_{2r}}\right)^{\frac{q-1}{q}}} & & & & \\ & & & & \left(\frac{\eta_{2r}}{\xi_{2r}}\right)^{\frac{1}{q}} & & & & \\ & & & & & \left(\frac{\eta_{2r+1}}{\xi_{2r+1}}\right)^{\frac{1}{q}} & & & \\ & & & & & & \ddots & & \\ & & & & & & & \left(\frac{\eta_n}{\xi_n}\right)^{\frac{1}{q}} \end{bmatrix} \\ &= I. \end{aligned}$$

Therefore, we can choose a positive integer q such that $T_{\xi\to\eta}^{\frac{1}{q}}\xi$ is sufficiently close to ξ and hence can be reached from ξ according to (13). That is, there exist control inputs $\bar{u}_0, \ldots, \bar{u}_{n-1}, u_n, \ldots, u_{2m+2}$ such that

$$(I + u_{2m+2}B)\cdots(I + u_nB)(I + \bar{u}_{n-1}B)\cdots(I + \bar{u}_0B)\xi = T_{\xi\to\eta}^{\frac{1}{q}}\xi,$$

where $\begin{bmatrix} \bar{u}_0 & \bar{u}_1 & \cdots & \bar{u}_{n-2} & \bar{u}_{n-1} \end{bmatrix}^T \in O\left(\begin{bmatrix} u_0 & u_1 & \cdots & u_{n-2} & u_{n-1} \end{bmatrix}^T, \rho_1\right)$. Note that $T_{\xi\to\eta}^{\frac{1}{q}}$ and B commute with each other. Applying q groups of $\bar{u}_0, \ldots, \bar{u}_{n-1}, u_n, \ldots, u_{2m+2}$ yields

$$\begin{aligned} & [(I + u_{2m+2}B)\cdots(I + u_nB)(I + \bar{u}_{n-1}B)\cdots(I + \bar{u}_0B)]^q\xi \\ =\ & [(I + u_{2m+2}B)\cdots(I + u_nB)(I + \bar{u}_{n-1}B)\cdots(I + \bar{u}_0B)]^{q-1}T_{\xi\to\eta}^{\frac{1}{q}}\xi \\ =\ & T_{\xi\to\eta}^{\frac{1}{q}}[(I + u_{2m+2}B)\cdots(I + u_nB)(I + \bar{u}_{n-1}B)\cdots(I + \bar{u}_0B)]^{q-1}\xi \\ =\ & \cdots = \left(T_{\xi\to\eta}^{\frac{1}{q}}\right)^q\xi = T_{\xi\to\eta}\xi = \eta. \end{aligned}$$

That is, controllability on each of the 2^m open orthants has been proved. The rest is to "connect" the 2^m open orthants, i.e. to prove that the system is controllable on the union of the 2^m open orthants. One can readily finish this by using Lemma 5 in [18] and following the arguments used in [18] (pp. 2856-2857, from eq. (33) to eq. (37)), where the only explanation we should make here is that, although $\lambda_1, \ldots, \lambda_r$ are double eigenvalues of B, the fact does affect connecting the 2^m open orthants. Indeed, just by replacing the subscript n by m in equations from (33) to (37) in [18], we can complete the proof.

So far, we have proved that the system (2) is controllable on

$$\mathbb{R}^n \setminus \left\{ \xi \,\middle|\, \left| \begin{array}{cccc} \xi & B\xi & \cdots & B^{n-2}\xi \quad B^{n-1}\xi \end{array} \right| = 0 \right\}. \tag{15}$$

Recalling the near-controllability definition, we have that

$$\mathcal{E} = \mathcal{F} = \left\{ \xi \,\middle|\, \left| \begin{array}{cccc} \xi & B\xi & \cdots & B^{n-2}\xi \quad B^{n-1}\xi \end{array} \right| = 0 \right\}$$

and the system (2) is nearly controllable since $\mathcal{E}, \mathcal{F}$ are two sets of zero Lebesgue measure in $\mathbb{R}^n$. ■

Remark 3: Lemma 1 is important to obtain the stronger result on near-controllability of the system (2). One can see that Lemma 2 in [18] is a special case of Lemma 1 and Theorem 2 in [18] is a special case of Theorem 2 also. In fact, [18] was focusing on the case of B being diagonalizable, while the case dealt with in this paper is more general. Moreover, the *transition* matrix $T_{\xi \to \eta}$ with the root locus approach will make it possible to compute the required control inputs that achieve the state transition.

Remark 4: One can further prove that $\mathcal{F}$ in the proof of Theorem 2 can be $\varnothing$. That is, for any ξ in (15) and any $\eta \in \mathbb{R}^n$, ξ can be transferred to η. See the corresponding part in the proof of Theorem 1 in [19] (from the last equation in p. 653 to the end of the proof) for reference.

By using the root locus approach, an algorithm is given to compute the required control inputs that steer the nearly controllable system (2) from one state to another, which both belong to (15).

Algorithm 5: Steps on computing control inputs for given initial and terminal states:

- 1. Transform B into the Jordan canonical form as given in (4) by a nonsingular matrix P. The initial and terminal states ξ, η are thus transformed into $P\xi, P\eta$, respectively.
- 2. Find the control inputs that transfer $P\xi$ to a state ζ which belongs to the same orthant as $P\eta$ belongs to (Lemma 5 in [18] will be helpful and its proof includes the details on how to find such control inputs).
- 3. Get the transition matrix $T_{\zeta \to P\eta}$ for $\zeta, P\eta$ from (14).
- 4. Choose $\lambda_{m+1}, \lambda_{m+2}$ that satisfy (9).
- 5. Choose a positive integer q and compute $T_{\zeta \to P\eta}^{\frac{1}{q}}$.
- 6. Obtain the root loci of the characteristic equation of the following closed loop transfer function

$$1 + KG(s) := 1 + \frac{K\left((-1)^{2m+2}\mu_1 s^{2m+2} + \cdots + (-1)\mu_{2m}s + 1\right)}{s(s+\lambda_1)^2 \cdots (s+\lambda_m)^2 (s+\lambda_{m+1})(s+\lambda_{m+2})} \tag{16}$$

where

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{2m} \end{bmatrix} := \begin{bmatrix} \lambda_1^{2m+2} & \cdots & \lambda_1 \\ (2m+2)\lambda_1^{2m+1} & \cdots & 1 \\ \vdots & \vdots & \vdots \\ \lambda_m^{2m+2} & \cdots & \lambda_m \\ (2m+2)\lambda_m^{2m+1} & \cdots & 1 \\ \lambda_{m+1}^{2m+2} & \cdots & \lambda_{m+1} \\ \lambda_{m+2}^{2m+2} & \cdots & \lambda_{m+2} \end{bmatrix}^{-1} \begin{bmatrix} T_{\zeta\to P\eta}^{\frac{1}{q}}(2,2)-1 \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(1,2) \\ \vdots \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(2r,2r)-1 \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(2r-1,2r) \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(2r+1,2r+1)-1 \\ 0 \\ \vdots \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(n,n)-1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

with $T_{\zeta\to P\eta}^{\frac{1}{q}}(i,j)$ denoting the (i,j)th entry of $T_{\zeta\to P\eta}^{\frac{1}{q}}$ and K increases from 0 to $+\infty$. If any of the root loci leaves the real axis directly at the pole, return to the former step and choose another integer q greater than the previous one. Otherwise, choose a suitable K such that the roots of $1+KG(s)=0$ are all real. Then, the reciprocals of the real roots are the control inputs that transfer ζ to $T_{\zeta\to P\eta}^{\frac{1}{q}}\zeta$. q groups of such control inputs together with the control inputs that transfer $P\xi$ to ζ are the desired ones which steer the nearly controllable system (2) from ξ to η.

An Explanation to Step 6 of Algorithm 5. Consider the equation

$$(I+v_{2m+2}B)(I+v_{2m+1}B)\cdots(I+v_1B)(I+v_0B)\zeta = T_{\zeta\to P\eta}^{\frac{1}{q}}\zeta. \tag{18}$$

From the proof of Theorem 2, we know that if q is large enough, then eq. (18) admits a real solution of $v_0, v_1, \ldots, v_{2m+1}, v_{2m+2}$. Eq. (18) is equivalent to the matrix equation

$$(I+v_{2m+2}B)(I+v_{2m+1}B)\cdots(I+v_1B)(I+v_0B) = T_{\zeta\to P\eta}^{\frac{1}{q}}.$$

Let

$$\prod_{k=0}^{2m+2}\frac{1}{v_k} = (-1)^{2m+3}K. \tag{19}$$

As shown for deriving eq. (7) in the proof of Lemma 1, we can obtain

$$B^{2m+3} + \sum_{k=0}^{2m+2}\frac{1}{v_k}B^{2m+2} + \cdots + \sum_{k=0}^{2m+2}\frac{v_k}{\prod_{j=0}^{2m+2}v_j}B = (-1)^{2m+3}K\left(T_{\zeta\to P\eta}^{\frac{1}{q}} - I\right)$$

which can be written via equations as those in (8). Using $\lambda_{m+1}, \lambda_{m+2}$ chosen in Step 4 and introducing

the same constraints as those for (8), we can deduce

$$\begin{bmatrix} \lambda_1^{2m+2} & \cdots & \lambda_1 \\ (2m+2)\lambda_1^{2m+1} & \cdots & 1 \\ \vdots & \vdots & \vdots \\ \lambda_m^{2m+2} & \cdots & \lambda_m \\ (2m+2)\lambda_m^{2m+1} & \cdots & 1 \\ \lambda_{m+1}^{2m+2} & \cdots & \lambda_{m+1} \\ \lambda_{m+2}^{2m+2} & \cdots & \lambda_{m+2} \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{2m+2} \frac{1}{v_k} \\ \vdots \\ \sum_{k=0}^{2m+2} \frac{v_k}{\prod_{j=0}^{2m+2} v_j} \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda_1^{2m+3} \\ -(2m+3)\lambda_1^{2m+2} \\ \vdots \\ -\lambda_m^{2m+3} \\ -(2m+3)\lambda_m^{2m+2} \\ -\lambda_{m+1}^{2m+3} \\ -\lambda_{m+2}^{2m+3} \end{bmatrix} + (-1)^{2m+3} K \begin{bmatrix} T_{\zeta\to P\eta}^{\frac{1}{q}}(2,2) - 1 \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(1,2) \\ \vdots \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(2r,2r) - 1 \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(2r-1,2r) \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(2r+1,2r+1) - 1 \\ 0 \\ \vdots \\ T_{\zeta\to P\eta}^{\frac{1}{q}}(n,n) - 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

By Lemma 15,

$$\begin{bmatrix} \sum_{k=0}^{2m+2} \frac{1}{v_k} \\ \vdots \\ \sum_{k=0}^{2m+2} \frac{v_k}{\prod_{j=0}^{2m+2} v_j} \end{bmatrix} = \begin{bmatrix} (-1)\left(2\sum_{i=1}^{m}\lambda_i + \lambda_{m+1} + \lambda_{m+2}\right) \\ \vdots \\ (-1)^{2m+2}\lambda_{m+1}\lambda_{m+2}\prod_{i=1}^{m}\lambda_i^2 \end{bmatrix} + (-1)^{2m+3} K \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{2m} \end{bmatrix}$$

where $\mu_1, \ldots, \mu_{2m}$ are given in (17). Then, with (19) we have that $\frac{1}{v_0}, \frac{1}{v_1}, \ldots, \frac{1}{v_{2m+1}}, \frac{1}{v_{2m+2}}$ are the roots of the following $(2m+3)$th-degree equation

$$s^{2m+3} + \left(2\sum_{i=1}^{m}\lambda_i + \lambda_{m+1} + \lambda_{m+2} + (-1)^{2m+2} K\mu_1\right) s^{2m+2} + \cdots + \left(\lambda_{m+1}\lambda_{m+2}\prod_{i=1}^{m}\lambda_i^2 + (-1) K\mu_{2m}\right) s + K = 0$$

which is equivalent to

$$s(s+\lambda_1)^2 \cdots (s+\lambda_m)^2 (s+\lambda_{m+1})(s+\lambda_{m+2}) + K\left((-1)^{2m+2}\mu_1 s^{2m+2} + \cdots + (-1)\mu_{2m} s + 1\right) = 0.$$

Thus, $\frac{1}{v_0}, \frac{1}{v_1}, \ldots, \frac{1}{v_{2m+1}}, \frac{1}{v_{2m+2}}$ are on the root loci of the characteristic equation of the closed loop transfer function given in (16).

An example will be provided in Section 4 to show the effectiveness of Algorithm 5.

Remark 6: By using the similar idea, one can try to derive an algorithm for computing the control inputs to achieve the state transition for the nearly controllable bilinear systems studied in [19]. Furthermore, although the result obtained in [19], namely Theorem 1 in [19], seems similar to Theorem 2, it cannot yield Theorem 2 (vice versa). To see this, for ξ, η of the system (2) that belong to (15), from Theorem 1 in [19] we can have $u(0), u(1), \ldots, u(L_1 - 1), u(L_1)$ such that

$$\left(B^{-1} + u(L_1) I\right)\left(B^{-1} + u(L_1 - 1) I\right) \cdots \left(B^{-1} + u(1) I\right)\left(B^{-1} + u(0) I\right)\xi = B^{-L_2-1}\eta$$

where B satisfies the conditions in Theorem 2 and ξ, $B^{-L_2-1}\eta$ are the corresponding initial and terminal states, respectively. This implies

$$(I + u(L_1) B)(I + u(L_1 - 1) B) \cdots (I + u(1) B)(I + u(0) B)\xi = B^{L_1-L_2}\eta.$$

Unfortunately, we can transfer ξ to $B^{L_1-L_2}\eta$ but not to η since we do not have $L_1 = L_2$ either from Theorem 1 in [19] or from Theorem 2.

Remark 7: Note that the opposite numbers of B's eigenvalues are poles of $G(s)$ given in (16), which are all real so that it is possible for the root loci of $1 + KG(s) = 0$ to first move on the real axis. However, if B has complex eigenvalues, then $G(s)$ may have complex poles and some of the root loci of $1 + KG(s) = 0$ will not start at the real axis. In such a case, it is rather difficult to ensure that all the root loci can have the same moment moving on the real axis, then the real control inputs that achieve the state transition cannot be obtained. Therefore, for near-controllability of system (2) with B having complex eigenvalues, we need to develop the proposed root locus approach or to find a new method.

III. Nearly-controllable subspaces and Near-controllability index

Consider the system

$$x(k+1) = (I + u(k) B) x(k) = \left(I + u(k) \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \right) x(k) \tag{20}$$

where $x(k) \in \mathbb{R}^3$, $u(k) \in \mathbb{R}$, and $\lambda \neq 0$. Since B has a Jordan block with dimension greater than two, system (20) is non-nearly controllable according to Theorem 2. Nevertheless, consider system (20) on region

$$\left\{ \xi = \begin{bmatrix} \xi_1 & \xi_2 & 0 \end{bmatrix}^T \right\}. \tag{21}$$

It can be seen that the system is invariant on (21), i.e.

$$\begin{aligned} x_{1,2}(k+1) &= (I + u(k) B_{1,2}) x_{1,2}(k) = \left(I + u(k) \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \right) x_{1,2}(k), \\ x_3(k+1) &= 0 \end{aligned} \tag{22}$$

where $x_{1,2}(k) = \begin{bmatrix} x_1(k) & x_2(k) \end{bmatrix}^T$ and $B_{1,2}$ is the main submatrix of B by taking the entries of B in both rows $1, 2$ and columns $1, 2$. From Theorem 2, subsystem (22) is nearly controllable. More specifically, it is controllable on

$$\mathbb{R}^2 \setminus \left\{ \xi_{1,2} \,\middle|\, \begin{vmatrix} \xi_{1,2} & B_{1,2}\xi_{1,2} \end{vmatrix} = 0 \right\} = \left\{ \xi_{1,2} = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^T \mid \xi_2 \neq 0 \right\}. \tag{23}$$

Therefore, system (20) is controllable on

$$\left\{\xi_{1,2}=\left[\begin{array}{cc}\xi_1 & \xi_2\end{array}\right]^T \mid \xi_2\neq 0\right\}\otimes\{\xi_3=0\}=\left\{\xi=\left[\begin{array}{ccc}\xi_1 & \xi_2 & 0\end{array}\right]^T \mid \xi_2\neq 0\right\} \tag{24}$$

which is a two-dimensional region in $\mathbb{R}^3$. Similarly, one can deduce that the one-dimensional region

$$\left\{\xi=\left[\begin{array}{ccc}\xi_1 & 0 & 0\end{array}\right]^T \mid \xi_1\neq 0\right\}$$

is also a region on which system (20) is controllable.

In this section, we study the non-nearly controllable systems. It is well known that if a linear time-invariant system is uncontrollable, then the state space can be decomposed as a direct sum of a controllable subspace and an uncontrollable subspace. On the controllable subspace the linear system is controllable. Furthermore, the controllable subspace corresponds to a controllable linear subsystem. Inspired by these facts, we will derive nearly-controllable subspaces of the system (2) by using Theorem 2. Actually, the regions in (23) and (24) are nearly-controllable subspaces of system (20). We do not use "controllable subspace" here since a subspace is in general a linear space, while a nearly-controllable subspace does not contain the zero element (the origin is an isolated state of system (1) that once it is reached, then the system cannot be steered away).

Definition 8: A nearly-controllable subspace of the system (2) is a controllable region of the system (2) that is derived from the corresponding nearly controllable subsystem of the system (2) on the region.

By the following Lemma, we will show how to obtain a nearly-controllable subspace.

Lemma 9: Consider the system

$$x(k+1)=(I+u(k)B)x(k)=\left(I+u(k)\left[\begin{array}{cccc}\lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1\\ & & & \lambda\end{array}\right]\right)x(k) \tag{25}$$

where $x(k)\in\mathbb{R}^n$, $n\geq 3$, $u(k)\in\mathbb{R}$, and $\lambda\neq 0$. Then, any state in a controllable region of system (25), which contains more than one state, has the property that only the first two entries can be nonzero.

The proof of this lemma can be found in Appendix.

Theorem 10: Consider the system (2) with B in Jordan canonical form

$$\left[\begin{array}{ccc}J(\lambda_1) & & \\ & \ddots & \\ & & J(\lambda_m)\end{array}\right]$$

without loss of generality, where $\lambda_i\neq\lambda_j$ if $i\neq j$ and $J(\lambda_i)$ is the Jordan matrix associated with eigenvalue λ_i for $i=1,\ldots,m$. Let h be the dimension of B's largest main submatrix that is nonsingular, cyclic, and does not have a Jordan block with dimension greater than two. Then, the system (2) has an e-dimensional nearly-controllable subspace for $e=1,\ldots,h$.

Proof: Choose any main submatrix $B_{i_1,\ldots,i_e}$ of B that is nonsingular, cyclic, and does not have a Jordan block with dimension greater than two, where i_j is required to correspond to the first or the second row (and column) of a Jordan block in B in view of Lemma 9 for $j=1,\ldots,e$ and if some i_j

corresponds to the second row (and column) of a Jordan block, then i_{j-1} corresponds to the first row (and column) of the same one. Consider the system (2) on

$$\left\{\xi = \left[\begin{array}{ccccccccccc} 0 & \cdots & 0 & \xi_{i_1} & 0 & \cdots & 0 & \xi_{i_e} & 0 & \cdots & 0 \end{array}\right]^T\right\}.$$

Then, the system can be rewritten as

$$\begin{aligned} x_{i_1,\ldots,i_e}(k+1) &= \left(I + u\left(k\right) B_{i_1,\ldots,i_e}\right) x_{i_1,\ldots,i_e}(k), \\ x_{1,\ldots,i_1-1,i_1+1,\ldots,i_e-1,i_e+1,\ldots,n}(k+1) &= \mathbf{0} \end{aligned} \tag{26}$$

where

$$x_{i_1,\ldots,i_e}(k) = \left[\begin{array}{cccc} x_{i_1}(k) & x_{i_2}(k) & \cdots & x_{i_e}(k) \end{array}\right]^T$$

and

$$\begin{aligned} &x_{1,\ldots,i_1-1,i_1+1,\ldots,i_e-1,i_e+1,\ldots,n}(k) \\ &\qquad = \left[\begin{array}{ccccccccccc} x_1(k) & \cdots & x_{i_1-1}(k) & x_{i_1+1}(k) & \cdots & x_{i_e-1}(k) & x_{i_e+1}(k) & \cdots & x_n(k) \end{array}\right]^T. \end{aligned}$$

Due to the chosen $B_{i_1,\ldots,i_e}$, subsystem (26) is nearly controllable by Theorem 2. Thus, the system (2) is controllable on the e-dimensional region

$$\left\{\xi = \left[\begin{array}{ccccccccccc} 0 & \cdots & 0 & \xi_{i_1} & 0 & \cdots & 0 & \xi_{i_e} & 0 & \cdots & 0 \end{array}\right]^T\right\} \setminus H \tag{27}$$

where

$$\begin{aligned} H = \Big\{ \xi = \Big[\begin{array}{ccccccccccc} 0 & \cdots & 0 & \xi_{i_1} & 0 & \cdots & 0 & \xi_{i_e} & 0 & \cdots & 0 \end{array}\Big]^T \Big| \\ \Big| \begin{array}{cccc} \xi_{i_1,\ldots,i_e} & B_{i_1,\ldots,i_e}\xi_{i_1,\ldots,i_e} & \cdots & B_{i_1,\ldots,i_e}^{e-1}\xi_{i_1,\ldots,i_e} \end{array} \Big| = 0,\ \xi_{i_1,\ldots,i_e} = \Big[\begin{array}{cccc} \xi_{i_1} & \cdots & \xi_{i_e} \end{array}\Big]^T \Big\}. \end{aligned}$$

That is, region (27) is an e-dimensional nearly-controllable subspace of the system (2). ■

Remark 11: From the proof of Theorem 10, in order to obtain a nearly-controllable subspace of the system (2), the main submatrix $B_{i_1,\ldots,i_e}$ of B that the nearly-controllable subspace corresponds to must be chosen to be nonsingular, cyclic, and have no Jordan block with dimension greater than two, where i_j is required to correspond to the first or the second row (and column) of a Jordan block in B for $j = 1, \ldots, e$ and if some i_j corresponds to the second row (and column) of a Jordan block, then i_{j-1} corresponds to the first row (and column) of the same one.

Definition 12: h in Theorem 10 is called the near-controllability index of the system (2).

If $h = n$, then the system (2) is nearly controllable. Otherwise, we can have the nearly-controllable subspaces with dimension from 1 to h. In particular, the state transition on every nearly-controllable subspace can also be achieved through Algorithm 5. Additionally, even for the nearly controllable system (2), it has nearly-controllable subspaces in the removed region of (15) (i.e. $\mathcal{E}$) and (15) can be regarded as an n-dimensional nearly-controllable subspace. An example is given in the next section.

IV. EXAMPLES

Example 13: Consider the system

$$x(k+1) = (I + u(k) B) x(k) = \left(I + u(k) \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & -4 \\ 0 & 0 & -2 \end{bmatrix} \right) x(k) \tag{28}$$

where $x(k) \in \mathbb{R}^3$ and $u(k) \in \mathbb{R}$. Given $\xi = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T$, $\eta = \begin{bmatrix} 27 & 21 & 12 \end{bmatrix}^T$. Find the control inputs such that ξ is transferred to η.

We now apply Algorithm 5 to compute the required control inputs. **Step 1**: let $\bar{x}(k) = Px(k)$ where

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$\bar{x}(k+1) = \left(I + u(k) PBP^{-1}\right) \bar{x}(k) = \left(I + u(k) \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \right) \bar{x}(k) \tag{29}$$

and $P\xi = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$, $P\eta = \begin{bmatrix} 6 & 9 & 12 \end{bmatrix}^T$. From Theorem 2, system (29) is nearly controllable (so is system (28)) and is controllable on

$$\begin{aligned} & \mathbb{R}^3 \setminus \left\{ \xi \,\middle|\, \begin{array}{cccc} \xi & PBP^{-1}\xi & \left(PBP^{-1}\right)^2 \xi \end{array} \right| = 0 \} \\ = \; & \mathbb{R}^3 \setminus \left\{ \xi = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix}^T \mid \xi_2 \xi_3 = 0 \right\} \\ = \; & \{ \xi \mid \xi_2 > 0, \xi_3 > 0 \} \cup \{ \xi \mid \xi_2 < 0, \xi_3 > 0 \} \cup \{ \xi \mid \xi_2 < 0, \xi_3 < 0 \} \cup \{ \xi \mid \xi_2 > 0, \xi_3 < 0 \} \end{aligned}$$

that consists of four open orthants. **Step 2**: since $P\xi, P\eta$ belong to different orthants, let $u(0) = 1$. It follows

$$\left(I + u(0) PBP^{-1}\right) P\xi = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}^T \triangleq \zeta$$

which is in the orthant that $P\eta$ belongs to. **Step 3**: obtain from (14) the transition matrix

$$T_{\zeta \to P\eta} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 12 \end{bmatrix}.$$

Step 4: choose $\lambda_3 = 1, \lambda_4 = -4$ in view of (9). **Step 5**: choose $q = 4$. Then,

$$T_{\zeta \to P\eta}^{\frac{1}{4}} = \begin{bmatrix} 3^{\frac{1}{4}} & \frac{1}{4 \times 3^{\frac{3}{4}}} & 0 \\ 0 & 3^{\frac{1}{4}} & 0 \\ 0 & 0 & 12^{\frac{1}{4}} \end{bmatrix}.$$

Step 6: by (17)

$$\mu_1 \approx 0.000742, \; \mu_2 \approx -0.00511, \; \mu_3 \approx -0.0393, \; \mu_4 \approx -0.0648, \; \mu_5 \approx 0.293, \; \mu_6 \approx 0.314.$$

Consider

$$1 + KG(s) = 1 + \frac{K\left((-1)^6 \mu_1 s^6 + \cdots + (-1)\mu_6 s + 1\right)}{s(s+2)^2 (s-2)^2 (s+1)(s-4)}.$$

By Matlab the root loci of $1 + KG(s) = 0$ are shown in the following figure, where "$\times$" and "$\circ$" respectively denote the poles and zeros of $G(s)$ and the colored curves are the root loci of $1 + KG(s) = 0$ starting at the poles and ending at the zeros. From Fig. 1, the root loci of $1 + KG(s) = 0$ first move on

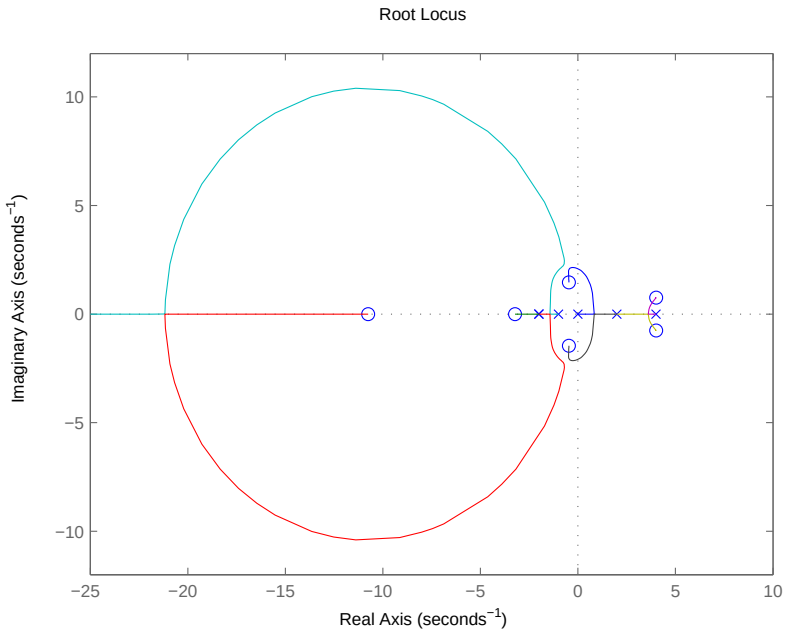

Fig. 1. Root loci of $1 + KG(s) = 0$

the real axis. Choose $K = 10$ and obtain the roots of $1 + KG(s) = 0$. Computing the reciprocals of the roots yields

$$v_1 \approx -0.770,\ v_2 \approx -0.643,\ v_3 \approx -0.452,\ v_4 \approx 0.250,\ v_5 \approx 0.439,\ v_6 \approx 0.612,\ v_7 \approx 6.650.$$

One can now verify that, by 4 groups of the above control inputs with $u(0) = 1$, system (28) is steered from ξ to η.

Example 14: Consider the system

$$x(k+1) = (I + u(k)B)\,x(k) = \left(I + u(k) \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \right) x(k) \tag{30}$$

where $x(k) \in \mathbb{R}^4$, $u(k) \in \mathbb{R}$, and λ_1, λ_2 are nonzero real and distinct. Find its nearly-controllable subspaces.

According to Theorem 2, system (30) is non-nearly controllable. The dimension of B's largest main submatrix, which is nonsingular, cyclic, and does not have a Jordan block with dimension greater than two, is three, so the near-controllability index of system (30) $h = 3$. More specifically, $B_{1,2,4}, B_{2,3,4}$ are such main submatrices of B. From the proof of Theorem 10 $B_{1,2,4}$ corresponds to the following three-dimensional nearly-controllable subspace of system (30)

$$\begin{aligned} & \left\{ \xi = \begin{bmatrix} \xi_1 & \xi_2 & 0 & \xi_4 \end{bmatrix}^T \right\} \setminus \left\{ \xi \,\middle|\, \left| \begin{array}{ccc} \xi_{1,2,4} & B_{1,2,4}\xi_{1,2,4} & B_{1,2,4}^2\xi_{1,2,4} \end{array} \right| = 0 \right\} \\ = & \left\{ \xi = \begin{bmatrix} \xi_1 & \xi_2 & 0 & \xi_4 \end{bmatrix}^T \mid \xi_2\xi_4 \neq 0 \right\} \end{aligned} \tag{31}$$

where $\xi_{1,2,4} = \begin{bmatrix} \xi_1 & \xi_2 & \xi_4 \end{bmatrix}^T$. However, the region

$$\left\{ \xi = \begin{bmatrix} 0 & \xi_2 & \xi_3 & \xi_4 \end{bmatrix}^T \mid \xi_3\xi_4 \neq 0 \right\}$$

that $B_{2,3,4}$ corresponds to is not a nearly-controllable subspace in view of Lemma 9. Besides (31), system (30) has the two-dimensional nearly-controllable subspaces

$$\left\{\xi = \left[\begin{array}{cccc} \xi_1 & \xi_2 & 0 & 0 \end{array}\right]^T \mid \xi_2 \neq 0\right\}, \ \left\{\xi = \left[\begin{array}{cccc} \xi_1 & 0 & 0 & \xi_4 \end{array}\right]^T \mid \xi_1\xi_4 \neq 0\right\}$$

that $B_{1,2}, B_{1,4}$ respectively correspond to and the one-dimensional nearly-controllable subspaces

$$\left\{\xi = \left[\begin{array}{cccc} \xi_1 & 0 & 0 & 0 \end{array}\right]^T \mid \xi_1 \neq 0\right\}, \ \left\{\xi = \left[\begin{array}{cccc} 0 & 0 & 0 & \xi_4 \end{array}\right]^T \mid \xi_4 \neq 0\right\}$$

that B_1, B_4 respectively corresponds to. In all, system (30) has five nearly-controllable subspaces from Theorem 10.

V. Conclusions

In this paper, near-controllability of a class of discrete-time bilinear systems is studied by proposing a root locus approach. A necessary and sufficient criterion for the systems to be nearly controllable is obtained. In particular, the control inputs which achieve the state transition for the nearly controllable systems are computed and an algorithm is presented. Furthermore, the nearly-controllable subspaces are derived and the near-controllability index is defined. Two examples are provided to demonstrate the results of the paper. Future work should consider the near-controllability and controllability problems of general bilinear systems and develop the proposed root locus approach.

References

[1] C. Bruni, G.D. Pillo, and G. Koch, *Bilinear systems: an appealing class of "nearly linear" systems in theory and applications*, IEEE Transactions on Automatic Control, 19 (1974), pp. 334-348.

[2] R.R. Mohler and W.J. Kolodziej, *An overview of bilinear system theory and applications*, IEEE Transactions on System, Man, and Cybernetics, 10 (1980), pp. 683-688.

[3] R.R. Mohler, *Nonlinear Systems: Vol. II, Applications to Bilinear Control*, Englewood Cliffs, Prentice-Hall, NJ, 1991.

[4] D.L. Elliott, *Bilinear Control Systems: Matrices in Action*, Springer, Dordrecht, 2009.

[5] T. Monovich and M. Margaliot, Analysis of discrete-time linear switched systems: a variational approach, SIAM Journal on Control and Optimization, 49 (2011), pp. 808-829.

[6] R.E. Kalman, *Mathematical description of linear dynamical systems*, SIAM Journal on Control and Optimization, 1 (1963), pp. 152-192.

[7] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach, 3rd ed*, Springer-Verlag, New York, 1985.

[8] H.J. Sussmann and V. Jurdjevic, *Controllability of nonlinear systems*, Journal of Differential Equations, 12 (1972), pp. 95-116.

[9] R. Hermann and A.J. Krener, *Nonlinear controllability and observability*, IEEE Transactions on Automatic Control, 22 (1977), pp. 728-740.

[10] A. Isidori, *Nonlinear Control Systems, 3rd ed*, Springer-Verlag, New York, 1995.

[11] V. Jurdjevic, *Geometric Control Theory*, Cambridge University Press, New York, 1997.

[12] T.J. Tarn, D.L. Elliott, and T. Goka, *Controllability of discrete bilinear systems with bounded control*, IEEE Transactions on Automatic Control, 18 (1973), pp. 298-301.

[13] T. Goka, T.J. Tarn, and J. Zaborszky, *On the controllability of a class of discrete bilinear systems*, Automatica, 9 (1973), pp. 615-622.

[14] G.S.J. Cheng, T.J. Tarn, and D.L. Elliott, *Controllability of bilinear systems*, in Variable Structure Systems with Application to Economics and Biology, A. Ruberti and R.R. Mohler eds. Springer-Verlag, New York, pp. 83-100, 1975.

[15] M.E. Evans and D.N.P. Murthy, *Controllability of a class of discrete time bilinear systems*, IEEE Transactions on Automatic Control, 22 (1977), pp. 78-83.

[16] B. Jakubczyk and E.D. Sontag, *Controllability of nonlinear discrete-time systems: a Lie-algebraic approach*, SIAM Journal on Control and Optimization, 28 (1990), pp. 1-33.

[17] L. Tie and K.-Y. Cai, *On controllability of a class of discrete-time homogeneous bilinear systems with solvable controls*, International Journal of Robust and Nonlinear Control, 22 (2012), pp. 591-603.

[18] L. Tie, K.-Y. Cai, and Y. Lin, *On uncontrollable discrete-time bilinear systems which are "nearly" controllable*, IEEE Transactions on Automatic Control, 55 (2010), pp. 2853-2858.

[19] L. Tie and K.-Y. Cai, *On near-controllability and stabilizability of a class of discrete-time bilinear systems*, Systems & Control Letters, 60 (2011), pp. 650-657.

[20] L. Tie and K.-Y. Cai, *On near-controllability of nonlinear control systems*, the 30th Chinese Control Conference, Yantai, Shandong, (2011), pp. 131-136.

[21] R.C. Dorf and R.H. Bishop, *Modern Control Systems*, Englewood Cliffs, Prentice-Hall, NJ, 2000.

[22] T. Kailath, *Linear systems*, Englewood Cliffs, Prentice-Hall, NJ, 1980.

VI. APPENDIX

Lemma 15: Let

$$C=\begin{bmatrix} \lambda_1^{2m+2} & \cdots & \lambda_1 \\ (2m+2)\,\lambda_1^{2m+1} & \cdots & 1 \\ \vdots & \vdots & \vdots \\ \lambda_m^{2m+2} & \cdots & \lambda_m \\ (2m+2)\,\lambda_m^{2m+1} & \cdots & 1 \\ \lambda_{m+1}^{2m+2} & \cdots & \lambda_{m+1} \\ \lambda_{m+2}^{2m+2} & \cdots & \lambda_{m+2} \end{bmatrix},\ d=\begin{bmatrix} -\lambda_1^{2m+3} \\ -(2m+3)\,\lambda_1^{2m+2} \\ \vdots \\ -\lambda_m^{2m+3} \\ -(2m+3)\,\lambda_m^{2m+2} \\ -\lambda_{m+1}^{2m+3} \\ -\lambda_{m+2}^{2m+3} \end{bmatrix}$$

where $\lambda_1,\ldots,\lambda_{m+2}$ are nonzero real and pairwise distinct. Then, the linear equation

$$Cz=d \tag{32}$$

has a unique solution

$$z=C^{-1}d=\begin{bmatrix} (-1)\left(2\sum_{i=1}^{m}\lambda_i+\lambda_{m+1}+\lambda_{m+2}\right) \\ \vdots \\ (-1)^{2m+2}\,\lambda_{m+1}\lambda_{m+2}\prod_{i=1}^{m}\lambda_i^2 \end{bmatrix}.$$

Proof: Note that the transpose of C is a general Vandermonde matrix. C is nonsingular since $\lambda_1,\ldots,\lambda_{m+2}$ are nonzero and pairwise distinct. Therefore, linear equation (32) has a unique solution. Assume that $z=\begin{bmatrix} z_1 & z_2 & \cdots & z_{2m+2} \end{bmatrix}^T$ is the solution. From (32) we have

$$\lambda_i^{2m+3}+z_1\lambda_i^{2m+2}+\cdots+z_{2m+2}\lambda_i=0$$

for $i=1,\ldots,m+2$ and

$$(2m+3)\,\lambda_i^{2m+2}+z_1\,(2m+2)\,\lambda_i^{2m+1}+\cdots+z_{2m+2}=0$$

for $i=1,\ldots,m$. Since λ_i are nonzero, it follows

$$\lambda_i^{2m+2}+z_1\lambda_i^{2m+1}+\cdots+z_{2m+2}=0$$

for $i=1,\ldots,m+2$ and hence

$$\begin{aligned} & (2m+3)\,\lambda_i^{2m+2}+z_1\,(2m+2)\,\lambda_i^{2m+1}+\cdots+z_{2m+2}-\left(\lambda_i^{2m+2}+z_1\lambda_i^{2m+1}+\cdots+z_{2m+2}\right) \\ = \ & (2m+2)\,\lambda_i^{2m+2}+z_1\,(2m+1)\,\lambda_i^{2m+1}+\cdots+z_{2m+1}\lambda_i=0 \\ \Rightarrow \ & (2m+2)\,\lambda_i^{2m+1}+z_1\,(2m+1)\,\lambda_i^{2m}+\cdots+z_{2m+1}=0 \end{aligned}$$

for $i=1,\ldots,m$. We can thus conclude that $\lambda_1,\ldots,\lambda_{m+2}$ are all the roots of the following $(2m+2)$th-degree equation

$$s^{2m+2}+z_1s^{2m+1}+\cdots+z_{2m+2}=0,$$

in which $\lambda_1,\ldots,\lambda_m$ are double roots. By the Viète's formulas,

$$\begin{aligned} z_1 &= (-1)\left(2\sum_{i=1}^{m}\lambda_i+\lambda_{m+1}+\lambda_{m+2}\right), \\ &\vdots \\ z_{2m+2} &= (-1)^{2m+2}\,\lambda_{m+1}\lambda_{m+2}\prod_{i=1}^{m}\lambda_i^2. \end{aligned}$$

■

Proof of Necessity of Lemma 1: We only need to show that there do not exist nonzero real numbers for (3) when

$$B=\left[\begin{array}{cccc}\lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda\end{array}\right]\in\mathbb{R}^{N\times N}$$

where $N\geq 3$ and $\lambda\neq 0$. Reduction to absurdity: assume that such nonzero real numbers exist for (3) with the above given B. From (3),

$$\left(\frac{1}{u\left(L\right)}I+B\right)\left(\frac{1}{u\left(L-1\right)}I+B\right)\cdots\left(\frac{1}{u\left(1\right)}I+B\right)\left(\frac{1}{u\left(0\right)}I+B\right)=\prod_{k=0}^{L}\frac{1}{u\left(k\right)}I. \tag{33}$$

It follows

$$\begin{aligned}B^{L+1}+\sum_{k=0}^{L}\frac{1}{u\left(k\right)}B^{L}+\cdots+\sum_{k=0}^{L}\frac{u\left(k\right)}{\prod_{j=0}^{2m+2}u\left(j\right)}B &= 0\Rightarrow\\ B^{L}+\sum_{k=0}^{L}\frac{1}{u\left(k\right)}B^{L-1}+\cdots+\sum_{k=0}^{L}\frac{u\left(k\right)}{\prod_{j=0}^{2m+2}u\left(j\right)}I &= 0,\end{aligned} \tag{34}$$

which implies $L\geq N$. Otherwise the $(1,L+1)$th entry in the sum of the left side of matrix equation (34), which is determined by B^{L}, does not vanish. Next, let $w\left(k\right)=\frac{1}{u(k)}+\lambda$ for $k=0,1,\ldots,L$. From (33),

$$\left(w\left(L\right)I+\bar{B}\right)\left(w\left(L-1\right)I+\bar{B}\right)\cdots\left(w\left(1\right)I+\bar{B}\right)\left(w\left(0\right)I+\bar{B}\right)=\prod_{k=0}^{L}\left(w\left(k\right)-\lambda\right)I$$

where

$$\bar{B}=\left[\begin{array}{cccc}0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0\end{array}\right].$$

We have

$$\begin{aligned}&\bar{B}^{L+1}+\sum_{k=0}^{L}w\left(k\right)\bar{B}^{L}+\cdots+\sum_{k=0}^{L}\frac{\prod_{j=0}^{L}w\left(j\right)}{w\left(k\right)}\bar{B}+\prod_{k=0}^{L}w\left(k\right)I\\ &=\sum w\left(k_{0}\right)w\left(k_{1}\right)\cdots w\left(k_{L-N+1}\right)\bar{B}^{N-1}+\sum w\left(k_{0}\right)w\left(k_{1}\right)\cdots w\left(k_{L-N+2}\right)\bar{B}^{N-2}+\\ &\cdots+\sum_{k=0}^{L}\frac{\prod_{j=0}^{L}w\left(j\right)}{w\left(k\right)}\bar{B}+\prod_{k=0}^{L}w\left(k\right)I\\ &=\prod_{k=0}^{L}\left(w\left(k\right)-\lambda\right)I\end{aligned} \tag{35}$$

since $\bar{B}^j = 0$ for $j \geq N$. Therefore, from (35)

$$\begin{aligned} \sum w(k_0)\, w(k_1) \cdots w(k_{L-N+1}) &= 0, \\ &\vdots \\ \sum_{k=0}^{L} \frac{\prod_{j=0}^{L} w(j)}{w(k)} &= 0, \\ \prod_{k=0}^{L} w(k) &= \prod_{k=0}^{L} (w(k) - \lambda). \end{aligned} \tag{36}$$

Let

$$\begin{aligned} \sum_{k=0}^{L} w(k) &= (-1)\, c_1, \\ &\vdots \\ \sum w(k_0)\, w(k_1) \cdots w(k_{L-N}) &= (-1)^{L-N+1} c_{L-N+1}. \end{aligned} \tag{37}$$

For the last equation in (36), by noting the other equations in (36) and the equations in (37) we have

$$\begin{aligned} \prod_{k=0}^{L} (w(k) - \lambda) &= \prod_{k=0}^{L} w(k) + \sum_{k=0}^{L} \frac{\prod_{j=0}^{L} w(j)}{w(k)} (-\lambda) + \cdots + \sum_{k=0}^{L} w(k) (-\lambda)^L + (-\lambda)^{L+1} \\ &= \prod_{k=0}^{L} w(k) + (-1)^{L-N+1} c_{L-N+1} (-\lambda)^N + \cdots + (-1)\, c_1 (-\lambda)^L + (-\lambda)^{L+1} \\ &= \prod_{k=0}^{L} w(k) + (-1)^{L+1} \left(c_{L-N+1} \lambda^N + \cdots + c_1 \lambda^L + \lambda^{L+1} \right) = \prod_{k=0}^{L} w(k). \end{aligned}$$

Hence,

$$\begin{aligned} c_{L-N+1} \lambda^N + \cdots + c_1 \lambda^L + \lambda^{L+1} &= 0 \Rightarrow \\ c_{L-N+1} &= -\left(\lambda^{L-N+1} + c_1 \lambda^{L-N} + \cdots + c_{L-N} \lambda \right). \end{aligned} \tag{38}$$

Now, let

$$\prod_{k=0}^{L} w(k) = (-1)^{L+1} c. \tag{39}$$

Combining (37), (36), (39) and noting (38), one can see from the Viète's formulas that $w(0), w(1), \ldots, w(L-1), w(L)$ are all the roots of the following $(L+1)$th-degree equation

$$s^{L+1} + c_1 s^L + \cdots + c_{L-N} s^{N+1} - \left(\lambda^{L-N+1} + c_1 \lambda^{L-N} + \cdots + c_{L-N} \lambda \right) s^N + c = 0. \tag{40}$$

However, according to Lemma 5 in [19], eq. (40) must have complex roots. This contradicts the fact that $u(0), u(1), \ldots, u(L-1), u(L)$ are all real. Therefore, if there exist nonzero real numbers to satisfy eq. (3), then B does not have a Jordan block with dimension greater than two in its Jordan canonical form. ■

Proof of Necessity of Theorem 2: For necessity, we show that if B is singular, or noncyclic, or has a Jordan block with dimension greater than two in its Jordan canonical form, then the system (2) is non-nearly controllable. Firstly, if B is singular, we can write

$$B = \begin{bmatrix} * & * \\ \mathbf{0} & 0 \end{bmatrix}$$

without loss of generality. Then, $u(k)$ loses the ability in controlling $x_n(k)$ and the system is non-nearly controllable.

Secondly, if B is not cyclic, it has an eigenvalue that corresponds to at least two Jordan blocks in its Jordan canonical form and can be simply written as

$$\begin{bmatrix} * & * & & & \\ & \lambda & & & \\ & & * & * & \\ & & & \lambda & \\ & & & & * \end{bmatrix}$$

without loss of generality. Then, for any initial state $\xi = \begin{bmatrix} \cdots & \xi_i & \cdots & \xi_j & \cdots \end{bmatrix}^T$, it follows

$$\begin{aligned} & \prod_{k=0}^{L} (I + u(k) B) \begin{bmatrix} \cdots & \xi_i & \cdots & \xi_j & \cdots \end{bmatrix}^T \\ = & \begin{bmatrix} * & & * & & & \\ & \prod_{k=0}^{L} (1 + u(k)\lambda) & & & & \\ & & * & & * & \\ & & & \prod_{k=0}^{L} (1 + u(k)\lambda) & & \\ & & & & & * \end{bmatrix} \begin{bmatrix} \vdots \\ \xi_i \\ \vdots \\ \xi_j \\ \vdots \end{bmatrix} \\ = & \begin{bmatrix} \cdots & \prod_{k=0}^{L} (1 + u(k)\lambda)\xi_i & \cdots & \prod_{k=0}^{L} (1 + u(k)\lambda)\xi_j & \cdots \end{bmatrix}^T \triangleq \begin{bmatrix} \cdots & \eta_i & \cdots & \eta_j & \cdots \end{bmatrix}^T, \end{aligned}$$

from which we can see $\xi_i \eta_j = \xi_j \eta_i$. This implies that some of the state variables in the terminal states are linearly dependent. Thus, the system does not have a large controllable region and is non-nearly controllable.

Finally, if B has a Jordan block with dimension greater than two in its Jordan canonical form, then the system is non-nearly controllable by Lemma 9. ■

Proof of Lemma 9: From the analysis on system (20), we can see that system (25) is controllable on

$$\left\{ \xi = \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \end{bmatrix}^T \mid \xi_1 \neq 0 \right\}, \ \left\{ \xi = \begin{bmatrix} \xi_1 & \xi_2 & 0 & \cdots & 0 \end{bmatrix}^T \mid \xi_2 \neq 0 \right\}$$

respectively. Now, if system (25) has a controllable region that contains more than one state and contains such a state

$$\bar{\xi} = \begin{bmatrix} \xi_1 & \cdots & \xi_j & 0 & \cdots & 0 \end{bmatrix}^T$$

where $\xi_j \neq 0$ and $j \geq 3$, then, for another state η ($\eta \neq \bar{\xi}$) in this region, there exist nonzero real control inputs $u(0), u(1), \ldots, u(L_1 - 1), u(L_1)$ such that $\bar{\xi}$ is transferred to η and nonzero real control inputs $v(0), v(1), \ldots, v(L_2 - 1), v(L_2)$ such that η is transferred to $\bar{\xi}$. As a result, we have

$$\prod_{k=0}^{L_1+L_2+1} (I + u(k) B) \bar{\xi} = \prod_{k=L_1+1}^{L_1+L_2+1} (I + u(k) B) \eta = \prod_{k=0}^{L_2} (I + v(k) B) \eta = \bar{\xi}$$

where $u(L_1+1)=v(0), u(L_1+2)=v(1), \ldots, u(L_1+L_2)=v(L_2-1), u(L_1+L_2+1)=v(L_2)$, i.e.

$$\prod_{k=0}^{L_1+L_2+1}(I+u(k)B)\begin{bmatrix}\xi_1 & \cdots & \xi_j & 0 & \cdots & 0\end{bmatrix}^T=\begin{bmatrix}\xi_1 & \cdots & \xi_j & 0 & \cdots & 0\end{bmatrix}^T.$$

This implies

$$\prod_{k=0}^{L_1+L_2+1}(I+u(k)B_{1,\ldots,j})\begin{bmatrix}\xi_1 & \cdots & \xi_j\end{bmatrix}^T=\begin{bmatrix}\Pi & \Sigma_1 & \cdots & \Sigma_{j-1}\\ & \ddots & \ddots & \vdots\\ & & \Pi & \Sigma_1\\ & & & \Pi\end{bmatrix}\begin{bmatrix}\xi_1\\ \vdots\\ \xi_j\end{bmatrix}=\begin{bmatrix}\xi_1\\ \vdots\\ \xi_j\end{bmatrix} \tag{41}$$

where

$$\Pi=\prod_{k=0}^{L_1+L_2+1}(1+u(k)\lambda),\ \Sigma_i=\frac{1}{i!}\frac{d^i\prod_{k=0}^{L_1+L_2+1}(1+u(k)y)}{dy^i}\bigg|_{y=\lambda}\quad \text{for } i=1,\ldots,j-1.$$

From (41)

$$\begin{aligned}\Pi\xi_1+\Sigma_1\xi_2+\cdots+\Sigma_{j-1}\xi_j &= \xi_1,\\ &\vdots\\ \Pi\xi_{j-1}+\Sigma_1\xi_j &= \xi_{j-1},\\ \Pi\xi_j &= \xi_j.\end{aligned}$$

Using the last equation in the above equations, we can deduce

$$\Pi=1,\ \Sigma_1=0,\ldots,\Sigma_{j-1}=0.$$

Then, it results in

$$\prod_{k=0}^{L_1+L_2+1}(I+u(k)B_{1,\ldots,j})=I. \tag{42}$$

However, since $j\geq 3$, there do not exist such nonzero control inputs to satisfy (42) according to Lemma 1. Therefore, system (25) can only have the states of which the latter $(n-2)$ entries are all zero in its controllable regions that contain more than one state. ■