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Abstract

The global in time existence of strong solutions to the compressible Navier-Stokes equation around time-periodic parallel flows in $\mathbb{R}^n$, $n \geq 2$, is established under smallness conditions on Reynolds number, Mach number and initial perturbations. Furthermore, it is proved for $n = 2$ that the asymptotic leading part of solutions is given by a solution of one-dimensional viscous Burgers equation multiplied by time-periodic function. In the case $n \geq 3$ the asymptotic leading part of solutions is given by a solution of $n-1$-dimensional heat equation with convective term multiplied by time-periodic function.

Mathematics Subject Classification

Keywords. Compressible Navier-Stokes equation, global existence, asymptotic behavior, time-periodic, viscous Burgers equation.

1 Introduction

In this paper we study the stability of solutions around a time-periodic parallel flow to the compressible Navier-Stokes equation with time-periodic external force and time-periodic boundary conditions.

We consider the system of equations

$$\partial_{\tilde{t}}\tilde{\rho} + \operatorname{div}(\tilde{\rho}\tilde{v}) = 0, \tag{1.1}$$

$$\tilde{\rho}(\partial_{\tilde{t}}\tilde{v} + \tilde{v}\cdot\nabla\tilde{v}) - \mu\Delta\tilde{v} - (\mu+\mu')\nabla\operatorname{div}\tilde{v} + \nabla\tilde{P}(\tilde{\rho}) = \tilde{\rho}\tilde{\boldsymbol{g}}, \tag{1.2}$$

in an n dimensional infinite layer $\Omega_\ell = \mathbb{R}^{n-1}\times(0,\ell)$:

$$\begin{aligned}\Omega_\ell \quad = \quad & \{\tilde{x} = {}^T(\tilde{x}', \tilde{x}_n)\,; \\ & \quad \tilde{x}' = {}^T(\tilde{x}_1,\ldots,\tilde{x}_{n-1}) \in \mathbb{R}^{n-1},\ 0 < \tilde{x}_n < \ell\}.\end{aligned}$$

Here, $n \geq 2$; $\tilde{\rho} = \tilde{\rho}(\tilde{x},\tilde{t})$ and $\tilde{v} = {}^T(\tilde{v}^1(\tilde{x},\tilde{t}),\ldots,\tilde{v}^n(\tilde{x},\tilde{t}))$ denote the unknown density and velocity at time $\tilde{t} \geq 0$ and position $\tilde{x} \in \Omega_\ell$, respectively; $\tilde{P}$ is the pressure, smooth function of $\tilde{\rho}$, where for given $\rho_* > 0$ we assume

$$\tilde{P}'(\rho_*) > 0;$$

μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu > 0$, $\frac{2}{n}\mu + \mu' \geq 0$; div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to $\tilde{x}$. Here and in what follows ${}^T\cdot$ denotes the transposition.

In (1.2) $\tilde{\boldsymbol{g}}$ is assumed to have the form

$$\tilde{\boldsymbol{g}} = {}^T(\tilde{g}^1(\tilde{x}_n,\tilde{t}), 0, \ldots, 0, \tilde{g}^n(\tilde{x}_n)),$$

with $\tilde{g}^1$ being a τ-periodic function in time, where $\tau > 0$.

The system (1.1)–(1.2) is considered under boundary condition

$$\tilde{v}|_{\tilde{x}_n=0} = \tilde{V}^1(t)\boldsymbol{e}_1, \quad \tilde{v}|_{\tilde{x}_n=\ell} = 0, \tag{1.3}$$

and initial condition

$$(\tilde{\rho},\tilde{v})|_{\tilde{t}=0} = (\tilde{\rho}_0,\tilde{v}_0), \tag{1.4}$$

where $\tilde{V}^1$ is a τ-periodic function of time and $\boldsymbol{e}_1 = {}^T(1,0,\ldots,0) \in \mathbb{R}^n$.

Under suitable conditions on $\tilde{\boldsymbol{g}}$ and $\tilde{V}^1$, problem (1.1)–(1.3) has smooth time-periodic solution $\bar{u}_p = {}^T(\bar{\rho}_p, \bar{v}_p)$ satisfying

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$$\overline{\rho}_p = \overline{\rho}_p(\widetilde{x}_n) \geq \widetilde{\rho}_1, \quad \frac{1}{\ell}\int_0^{\ell} \overline{\rho}_p(\widetilde{x}_n)\, d\widetilde{x}_n = \rho_*,$$

$$\overline{v}_p = {}^T(\overline{v}_p^1(\widetilde{x}_n, \widetilde{t}), 0, \ldots, 0), \quad \overline{v}_p^1(\widetilde{x}_n, \widetilde{t}+\tau) = \overline{v}_p^1(\widetilde{x}_n, \widetilde{t}),$$

for a positive constant $\widetilde{\rho}_1$.

The aim of this paper is to give an asymptotic description of large time behavior of perturbations from $\overline{u}_p$ when Reynolds and Mach numbers are sufficiently small.

To formulate the problem for perturbations, we introduce the following dimensionless variables:

$$\widetilde{x} = \ell x, \quad \widetilde{t} = \frac{\ell}{V} t, \quad \widetilde{v} = V v, \quad \widetilde{\rho} = \rho_* \rho, \quad \widetilde{P} = \rho_* V^2 P,$$

with

$$\widetilde{w} = V w, \quad \widetilde{\phi} = \rho_* \gamma^{-2} \phi, \quad \widetilde{V}^1 = V V^1, \quad \widetilde{\boldsymbol{g}} = \frac{\mu V}{\rho_* \ell^2} \boldsymbol{g},$$

where

$$\gamma = \frac{\sqrt{\widetilde{P}'(\rho_*)}}{V}, \quad V = \frac{\rho_* \ell^2}{\mu}\left\{ |\partial_{\widetilde{t}} \widetilde{V}^1|_{C^0(\mathbb{R})} + |\widetilde{g}^1|_{C^0(\mathbb{R}\times[0,\ell])} \right\} + |\widetilde{V}^1|_{C^0(\mathbb{R})} > 0.$$

In this paper we assume $V > 0$. Under this change of variables the domain Ω_ℓ is transformed into $\Omega = \mathbb{R}^{n-1} \times (0,1)$; and $g^1(x_n, t)$, $V^1(t)$ are periodic in t with period $T > 0$ defined by

$$T = \frac{V}{\ell}\tau.$$

The time-periodic solution $\overline{u}_p$ is transformed into $u_p = {}^T(\rho_p, v_p)$ satisfying

$$\rho_p = \rho_p(x_n) > 0, \quad \int_0^1 \rho_p(x_n)\, dx_n = 1,$$

$$v_p = {}^T(v_p^1(x_n, t), 0, \ldots, 0), \quad v_p^1(x_n, t+T) = v_p^1(x_n, t).$$

It then follows that the perturbation $u(t) = {}^T(\phi(t), w(t)) \equiv {}^T(\gamma^2(\rho(t) - \rho_p), v(t) - v_p(t))$ is governed by the following system of equations

$$\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \mathrm{div}\,(\rho_p w) = f^0, \tag{1.5}$$

$$\begin{aligned} \partial_t w - \tfrac{\nu}{\rho_p}\Delta w - \tfrac{\widetilde{\nu}}{\rho_p}\nabla \mathrm{div}\, w + v_p^1 \partial_{x_1} w + (\partial_{x_n} v_p^1) w^n \boldsymbol{e}_1 & \\ + \tfrac{\nu}{\gamma^2 \rho_p^2}(\partial_{x_n}^2 v_p^1)\phi\, \boldsymbol{e}_1 + \nabla\left(\tfrac{P'(\rho_p)}{\gamma^2 \rho_p}\phi\right) &= \boldsymbol{f}, \end{aligned} \tag{1.6}$$

$$w|_{\partial\Omega} = 0, \tag{1.7}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0), \tag{1.8}$$

where f^0 and $\boldsymbol{f} = {}^T(f^1, \cdots, f^n)$ denote nonlinearities, i.e.,

$$f^0 = -\mathrm{div}\,(\phi w),$$

$$\begin{aligned} \boldsymbol{f} = -w\cdot\nabla w + \frac{\nu\phi}{\gamma^2\rho_p^2}\left(-\Delta w + \frac{\partial_{x_n}^2 v_p^1}{\rho_p\gamma^2}\phi\boldsymbol{e}_1\right) - \frac{\nu\phi^2}{\gamma^2\rho_p^2(\gamma^2\rho_p+\phi)}\left(-\Delta w + \frac{\partial_{x_n}^2 v_p^1}{\rho_p\gamma^2}\phi\boldsymbol{e}_1\right) \\ - \frac{\widetilde{\nu}\phi}{\rho_p(\gamma^2\rho_p+\phi)}\nabla\mathrm{div}\, w + \frac{\phi}{\gamma^2\rho_p}\nabla\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\phi\right) - \frac{1}{2\gamma^4\rho_p}\nabla(P''(\rho_p)\phi^2) + \widetilde{P}_3(\rho_p, \phi, \partial_x\phi), \end{aligned}$$

$$\begin{aligned} \widetilde{P}_3(\rho_p, \phi, \partial_x\phi) &= \tfrac{\phi^3}{\gamma^4(\gamma^2\rho_p+\phi)\rho_p^3}\nabla P(\rho_p) + \tfrac{\phi\nabla(P''(\rho_p)\phi^2)}{2\gamma^4\rho_p(\gamma^2\rho_p+\phi)} \\ &\quad - \tfrac{\phi^2\nabla(P'(\rho_p)\phi)}{\gamma^4\rho_p^2(\gamma^2\rho_p+\phi)} - \tfrac{1}{2\gamma^4(\gamma^2\rho_p+\phi)}\nabla(\phi^3 P_3(\rho_p, \phi)), \end{aligned}$$

with

$$P_3(\rho_p, \phi) = \int_0^1 (1-\theta)^2 P'''(\theta\gamma^{-2}\phi + \rho_p)d\theta.$$

Here, $\mathrm{div}\,, \nabla$ and Δ denote the usual divergence, gradient and Laplacian with respect to x; ν, ν' and $\widetilde{\nu}$ are the non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}, \quad \widetilde{\nu} = \nu + \nu'.$$

We note that the Reynolds number Re and Mach number Ma are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively. See [1] for the derivation of (1.5)–(1.8).

In the case g^1 and V^1 do not depend on t, problem (1.1)–(1.3) has a stationary parallel flow. The stability of stationary parallel flows were studied in [5, 6, 7, 11]. It was shown in [6] and [7] that the stationary parallel flow is asymptotically stable under sufficiently small initial perturbations in $H^m(\Omega) \cap L^1(\Omega)$ with $m \geq [n/2] + 1$, provided that $Re \ll 1$, $Ma \ll 1$ and density of the parallel flow is sufficiently close to a positive constant. Furthermore, the asymptotic behavior of perturbations from the stationary parallel flow is described by $n-1$ dimensional linear heat equation in the case $n \geq 3$ ([6]) and by one-dimensional viscous Burgers equation in the case $n = 2$ ([7]).

The case of time-periodic parallel flows was considered in [1, 2] for $Re \ll 1$ and $Ma \ll 1$. In [1, 2] the authors investigated the linearized problem, i.e., (1.5)–(1.8) with $(f^0, \boldsymbol{f}) = (0, 0)$, which is written as

$$\partial_t u + L(t)u = 0, \quad w|_{x_n=0,1} = 0, \quad u|_{t=s} = u_0. \tag{1.9}$$

Here, $u = {}^T(\phi, w)$ and $L(t)$ is operator of the form

$$\begin{aligned} L(t) &= \begin{pmatrix} v_p^1(t)\partial_{x_1} & \gamma^2 \mathrm{div}\,(\rho_p \,\cdot\,) \\ \nabla \left(\frac{P'(\rho_p)}{\gamma^2 \rho_p} \,\cdot \right) & -\frac{\nu}{\rho_p}\Delta I_n - \frac{\widetilde{\nu}}{\rho_p}\nabla \mathrm{div} \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ \frac{\nu}{\gamma^2 \rho_p^2}\partial_{x_n}^2 v_p^1(t)\boldsymbol{e}_1 & v_p^1(t)\partial_{x_1} I_n + (\partial_{x_n} v_p^1(t))\boldsymbol{e}_1{}^T\boldsymbol{e}_n \end{pmatrix}. \end{aligned} \tag{1.10}$$

Note that $L(t)$ satisfies $L(t) = L(t+T)$.

In [1, 2] spectral properties of the solution operator $U(t, s)$ for (1.9) were studied by using Fourier transform with respect to $x' \in \mathbb{R}^{n-1}$. The Fourier transform of (1.9) can be written in the form:

$$\frac{d}{dt}\widehat{u} + \widehat{L}_{\xi'}(t)\widehat{u} = 0, \ t > s, \quad \widehat{u}|_{t=s} = \widehat{u}_0, \tag{1.11}$$

where $\widehat{u}$ denotes the Fourier transform of u in x'; and ξ' is dual variable to x'. For each $\xi' \in \mathbb{R}^{n-1}$ and for all $t \geq s$ there exists a unique evolution operator $\widehat{U}_{\xi'}(t, s)$ for (1.11).

Since $\widehat{L}_{\xi'}(t)$ is T-time periodic, the spectrum of $\widehat{U}_{\xi'}(T, 0)$ plays an important role in the study of large time behavior. It was shown in [1] that the spectrum of $\widehat{U}_{\xi'}(T, 0)$ satisfies the following inclusion

$$\sigma(\widehat{U}_{\xi'}(T, 0)) \subseteq \begin{cases} \{e^{\lambda_{\xi'} T}\} \cup \{|\lambda| < q_1\} & (|\xi'| < r), \\ \{|\lambda| < q_1\} & (|\xi'| \geq r), \end{cases}$$

for a constant $0 < q_1 < 1$ and $0 < r \ll 1$. Here, $e^{\lambda_{\xi'} T}$ is the simple eigenvalue of $\widehat{U}_{\xi'}(T, 0)$ and $\lambda_{\xi'} = -i\kappa_0 \xi_1 - \kappa_1 \xi_1^2 - \kappa'' |\xi''|^2 + O(|\xi'|^3)$ with $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$, $\kappa'' > 0$ and $\xi' = {}^T(\xi_1, \xi'')$, $\xi'' = (\xi_2, \ldots, \xi_{n-1})$.

In [2] spectral properties of $\widehat{U}_{\xi'}(t, s)$ were investigated for $|\xi'| < r$ by using the Floquet theory. A family $\{\mathbb{P}(t)\}_{t \in \mathbb{R}}$ of bounded projections on $L^2(\Omega)$ was constructed to represent $\mathbb{P}(t)U(t, s)$ as

$$\mathbb{P}(t)U(t, s) = \mathscr{Q}(t)e^{(t-s)\Lambda}\mathscr{P}(s). \tag{1.12}$$

Here, $e^{t\Lambda} = \mathscr{F}^{-1}\widehat{\chi}_1 e^{\lambda_{\xi'} t}\mathscr{F}$ with frequency cut off function $\widehat{\chi}_1 : \widehat{\chi}_1(\xi') = 1$ $(|\xi'| < r)$, $\widehat{\chi}_1(\xi') = 0$ $(|\xi'| \geq r)$, and $\mathscr{Q}(t) = \mathscr{F}^{-1}\widehat{\chi}_1 \widehat{\mathscr{Q}}_{\xi'}(t)\mathscr{F}$ and $\mathscr{P}(t) = \mathscr{F}^{-1}\widehat{\chi}_1 \widehat{\mathscr{P}}_{\xi'}(t)\mathscr{F}$ with

$$\widehat{\mathscr{Q}}_{\xi'}(t) : \mathbb{C} \to L^2(0, 1) \text{ and } \widehat{\mathscr{P}}_{\xi'}(t) : L^2(0, 1) \to \mathbb{C},$$

expanded as

$$\begin{aligned} \widehat{\mathscr{Q}}_{\xi'}(t) &= \mathscr{Q}^{(0)}(t) + i\xi' \cdot \mathscr{Q}^{(1)}(t) + O(|\xi'|^2), \\ \widehat{\mathscr{P}}_{\xi'}(t) &= \mathscr{P}^{(0)} + i\xi' \cdot \mathscr{P}^{(1)}(t) + O(|\xi'|^2), \end{aligned}$$

for $|\xi'| \le r$, where $\mathscr{Q}^{(0)}(t)\sigma = \sigma u^{(0)}(\cdot,t)$ $(\sigma \in \mathbb{C})$, $u^{(0)}(\cdot,t) = u^{(0)}(x_n,t)$ is a function T-periodic in t and $\mathscr{P}^{(0)}u = [\phi]$ $(u = {}^T(\phi,w) \in L^2(0,1))$. One consequence of (1.12) is that

$$\|\partial_{x'}^k \partial_{x_n}^l \mathbb{P}(t)U(t,s)u_0\|_{L^2(\Omega)} \le C(1+t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_{L^1(\Omega)},$$

$$\|(I-\mathbb{P}(t))U(t,s)u_0\|_{H^1(\Omega)} \le e^{-d(t-s)}(\|u_0\|_{H^1\times L^2} + \|\partial_{x'}w_0\|_{L^2}),$$

$$\|\partial_{x'}^k \partial_{x_n}^l (\mathbb{P}(t)U(t,s)u_0 - \sigma_{t,s}[u_0]u^{(0)}(t))\|_{L^2(\Omega)} \le C(t-s)^{-\frac{n-1}{4}-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{L^1(\Omega)},$$

for $t-s \ge T$, $s \ge 0$; $k = 0,1,\dots,$ $l = 0,\dots,m$ for $m \ge 2$. Here, $\sigma_{t,s}[u_0] = \sigma_{t,s}(x')[u_0]$ is a function whose Fourier transform in x' is given by

$$\mathscr{F}(\sigma_{t,s}[u_0]) = e^{-(i\kappa_0\xi_1 + \kappa_1\xi_1^2 + \kappa''|\xi''|^2)(t-s)}[\widehat{\phi}_0(\xi')],$$

where $[\widehat{\phi}_0(\xi')]$ is a quantity given by

$$[\widehat{\phi}_0(\xi')] = \int_0^1 \widehat{\phi}_0(\xi', x_n)\, dx_n,$$

with $\widehat{\phi}_0$ being the Fourier transform of ϕ_0 in x' and $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$, $\kappa'' > 0$ are positive constants depending on ρ_*, l, V, μ, μ' and $\widetilde{P}'(\rho_*)$.

Another consequence of (1.12) is that if $u(t)$ is a solution of

$$\partial_t u + L(t)u = f,\ u|_{t=0} = u_0,$$

then $\mathbb{P}(t)u(t)$ is represented as

$$\mathbb{P}(t)u(t) = \mathscr{Q}(t)\left(e^{t\Lambda}\mathscr{P}(0)u_0 + \int_0^t e^{(t-z)\Lambda}\mathscr{P}(z)f(z)dz\right). \tag{1.13}$$

In this paper we show the following results. Let u_0 be sufficiently small in $H^m(\Omega) \cap L^1(\Omega)$ for a given $m \ge [n/2]+1$; and let u_0 satisfy a suitable compatibility condition, then there exists unique solution $u(t)$ of (1.5)–(1.8) in $C([0,\infty); H^m(\Omega))$, provided that $Re \ll 1$, $Ma \ll 1$ and $|1-\rho_p|_{C^{m+1}([0,1])} \ll 1$. Furthermore, $u(t)$ satisfies

$$\|\partial_{x'}^k u(t)\|_{L^2(\Omega)} \le O(t^{-\frac{n-1}{4}-\frac{k}{2}}),\ k = 0,1,$$

as $t \to \infty$.

In the case $n = 2$, we show that the asymptotic leading term of perturbation $u(t)$ is described by a solution of one-dimensional viscous Burgers equation, i.e.,

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{3}{4}+\delta}),\ \forall\delta > 0,$$

as $t \to \infty$. Here, $u^{(0)} = u^{(0)}(x_2,t)$ is a given time-periodic function; and $\sigma = \sigma(x_1,t)$ is function satisfying

$$\partial_t\sigma - \kappa_1\partial_{x_1}^2\sigma + \kappa_0\partial_{x_1}\sigma + \omega_0\partial_{x_1}(\sigma^2) = 0,\ \sigma|_{t=0} = \int_0^1 \phi_0(x_1,x_2)\ dx_2, \tag{1.14}$$

with constants $\kappa_0 \in \mathbb{R}$, $\kappa_1 > 0$ being the same ones as those in λ_{ξ_1} and $\omega_0 \in \mathbb{R}$ determined by the nonlinearity $\boldsymbol{F}$.

In the case $n \ge 3$, we show that the asymptotic leading term of $u(t)$ is the same one as for the linearized problem and thus it is given by $n-1$-dimensional heat equation with convective term, i.e.,

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t)),$$

as $t \to \infty$. Here, $\sigma = \sigma(x',t)$ is function satisfying

$$\partial_t\sigma - \kappa_1\partial_{x_1}^2\sigma - \kappa''\Delta''\sigma + \kappa_0\partial_{x_1}\sigma = 0,\ \sigma|_{t=0} = \int_0^1 \phi_0(x',x_n)\ dx_n,$$

with constants $\kappa_0 \in \mathbb{R}$, $\kappa_1, \kappa'' > 0$ being the same ones as those in $\lambda_{\xi'}$; where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$; and $\eta_n(t) = \log(1+t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \ge 4$.

The proof of the main results is given by a combination of various estimates for $\mathbb{P}(t)U(t,s)$ mentioned above and a variant of Matsumura-Nishida energy method ([6, 7], cf. [12]). We decompose the solution $u(t)$

of (1.5)–(1.8) into the $\mathbb{P}(t)$-part and $(I-\mathbb{P}(t))$-part. Considering the $\mathbb{P}(t)$-part, we represent $\mathbb{P}(t)u(t)$ as in (1.13) with $f = {}^T(f^0, \boldsymbol{f})$ being the nonlinearity given in (1.5) and (1.6). We then combine various estimates on $\mathbb{P}(t)$ and $\mathbb{P}(t)U(t,s)$ to obtain the necessary estimates on $\mathbb{P}(t)u(t)$. On the other hand, $(I-\mathbb{P}(t))u(t)$ can be estimated by a variant of Matsumura-Nishida energy method as in the case of the stationary parallel flow ([7]). However, in contrast to [7], the linearized operator has time-dependent coefficients. Therefore a modification of the argument in [7] is needed for the time-periodic case to aquire the necessary energy estimate. It is worth mentioning that in the case $n=2$ the asymptotic leading part of $u(t)$ is not described by the linearized problem due to the quadratic nonlinearities $-\mathrm{div}\,(\phi w)$, $\frac{\nu\phi}{\gamma^2\rho_p^2}\left(-\partial_{x_n}^2 w^1 + \frac{\partial_{x_n}^2 v_p^1}{\rho_p\gamma^2}\phi\right)$ and $-\frac{1}{2\gamma^4\rho_p}\partial_{x_n}(P''(\rho_p)\phi^2)$. This leads to the 1-dimensional Burgers equation (1.14).

Our result is an extension of previous results on the stationary case [5, 6, 7, 11] to the case of time-periodic external force and time-periodic boundary conditions.

Structure of this paper is the following. In Section 2 we introduce basic notations that are used throughout the paper. In Section 3 we state the main results. In Section 4 we present the results on spectral properties of the linearized problem obtained in [2]. In Section 5 we introduce decomposition of solution $u(t)$ to (1.5)–(1.8) based on the spectral properties of $L(t)$ introduced in Section 4. Moreover, we prove the a priori estimate using the estimates on $\mathbb{P}(t)u(t)$ and $(I-\mathbb{P}(t))u(t)$ from subsequent sections 6, 7 and 8. In Section 6 we show estimate for $\mathbb{P}(t)u(t)$ using properties of $\mathbb{P}(t)$ and $\mathbb{P}(t)U(t,s)$. In Section 7 we obtain estimate on $(I-\mathbb{P}(t))u(t)$ using energy method. In Section 8 we estimate the nonlinearities f^0 and $\boldsymbol{f}$. Finally, in Section 9 we prove the asymptotic behavior of solutions.

2 Notation

In this section we introduce some notations which are used throughout the paper. For a domain E we denote by $L^p(E)$ the usual Lebesgue space on E and its norm is denoted by $\|\cdot\|_{L^p(E)}$ for $1 \le p \le \infty$. Let k be a nonnegative integer. $H^k(E)$ denotes the k-th order L^2 Sobolev space on E with norm $\|\cdot\|_{H^k(E)}$. $C_0^k(E)$ stands for the set of all C^k functions which have compact support in E. We denote by $H_0^1(E)$ the completion of $C_0^1(E)$ in $H^1(E)$.

We simply denote by $L^p(E)$ (resp., $H^k(E)$) the set of all vector fields $w = {}^T(w^1,\dots,w^n)$ on E with $w^j \in L^p(E)$ (resp., $H^k(E)$), $j=1,\dots,n$, and its norm is also denoted by $\|\cdot\|_{L^p(E)}$ (resp., $\|\cdot\|_{H^k(E)}$). For $u = {}^T(\phi, w)$ with $\phi \in H^k(E)$ and $w = {}^T(w^1,\dots,w^n) \in H^l(E)$, we define $\|u\|_{H^k(E)\times H^l(E)}$ by $\|u\|_{H^k(E)\times H^l(E)} = \|\phi\|_{H^k(E)} + \|w\|_{H^l(E)}$. When $k=l$, we simply write $\|u\|_{H^k(E)\times H^k(E)} = \|u\|_{H^k(E)}$.

In the case $E=\Omega$ we abbreviate $L^p(\Omega)$ (resp., $H^k(\Omega)$) as L^p (resp., H^k). In particular, we denote the norm of L^p (resp., H^k) by $\|\cdot\|_p$ (resp., $\|\cdot\|_{H^k}$).

In the case $E=(0,1)$ we denote the norm $|\cdot|_{L^2(0,1)}$ (resp., $|\cdot|_{H^k(0,1)}$) by $|\cdot|_2$ (resp., $|\cdot|_{H^k}$).

The inner product of L^2 is denoted by

$$(f,g) = \int_\Omega f(x)g(x)\,dx, \quad f,g \in L^2.$$

Furthermore, we introduce a weighted inner product $\langle\cdot,\cdot\rangle_\Omega$ defined by

$$\langle u_1, u_2\rangle_\Omega = \int_\Omega \phi_1\phi_2 \frac{P'(\rho_p)}{\gamma^4\rho_p}\,dx + \int_\Omega w_1 w_2 \rho_p\,dx,$$

for $u_j = {}^T(\phi_j, w_j) \in L^2$, $j=1,2$; and for $u_j = {}^T(\phi_j, w_j) \in L^2(0,1)$, $j=1,2$, we also define $\langle u_1, u_2\rangle$ by

$$\langle u_1, u_2\rangle = \int_0^1 \phi_1\overline{\phi}_2 \frac{P'(\rho_p)}{\gamma^4\rho_p}\,dx_n + \int_0^1 w_1\overline{w}_2\rho_p\,dx_n.$$

Here, $\overline{g}$ denotes the complex conjugate of g.

Furthermore, for $f \in L^1(0,1)$ we denote the mean value of f in $(0,1)$ by $[f]$:

$$[f] = (f,1) = \int_0^1 f(x_n)\,dx_n.$$

For $u = {}^T(\phi, w) \in L^1(0,1)$ with $w = {}^T(w^1,\dots,w^n)$ we define $[u]$ by

$$[u] = [\phi] + [w^1] + \cdots + [w^n].$$

We often write $x \in \Omega$ as

$$x = {}^T(x', x_n),\ x' = {}^T(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

Partial derivatives of function u in x, x', x_n and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$; by $\Delta' = \sum_{i=1}^{n-1} \partial_{x_i}^2$, $\nabla' = (\partial_{x_1}, \dots, \partial_{x_{n-1}})$ and $\mathrm{div}' = \nabla' \cdot$ we denote the Lapacian, gradient and divergence with respect to x', respectively.

We denote $k \times k$ identity matrix by I_k. In particular, when $k = n+1$, we simply write I for I_{n+1}. We define $(n+1) \times (n+1)$ diagonal matrices Q_j, Q' and $\widetilde{Q}$ by

$$Q_j = \mathrm{diag}\,(0, \dots, 0, \underbrace{1}_{j\text{-}th}, 0, \dots, 0), \quad j = 0, 1, \dots, n,$$

and

$$Q' = \mathrm{diag}\,(0, 1, \dots, 1, 0),\ \widetilde{Q} = \mathrm{diag}\,(0, 1, \dots, 1).$$

We then have for $u = {}^T(\phi, w) \in \mathbb{R}^{n+1}$, $w = {}^T(w^1, \dots, w^n) = {}^T(w', w^n)$,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad Q_j u = \begin{pmatrix} 0 \\ w^j \\ 0 \end{pmatrix}, \quad Q_n u = \begin{pmatrix} 0 \\ 0 \\ w^n \end{pmatrix}, \quad Q' u = \begin{pmatrix} 0 \\ w' \\ 0 \end{pmatrix}, \quad \widetilde{Q} u = \begin{pmatrix} 0 \\ w \end{pmatrix}.$$

We denote $\boldsymbol{e}_1' = {}^T(1, 0, \dots, 0) \in \mathbb{R}^{n-1}$. We note that

$$[Q_0 u] = [\phi] \ \text{ for } u = {}^T(\phi, w).$$

For a function $f = f(x')$ $(x' \in \mathbb{R}^{n-1})$, we denote its Fourier transform by $\widehat{f}$ or $\mathscr{F} f$:

$$\widehat{f}(\xi') = (\mathscr{F} f)(\xi') = \int_{\mathbb{R}^{n-1}} f(x') e^{-i\xi' \cdot x'}\, dx'.$$

The inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$(\mathscr{F}^{-1} f)(x') = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'}\, d\xi'.$$

For closed linear operator A in X we denote the spectrum of A by $\sigma(A)$. We denote the set of all bounded linear operators from X_0 into itself by $L(X_0)$ and denote the norm by $|\cdot|_{L(X_0)}$. For operators A, B we denote $[A, B]$ the commutator, i.e., $[A, B] = AB - BA$. For time interval $[a, b] \subset \mathbb{R}$, we denote the usual Bochner spaces by $L^2(a, b; X)$, $H^m(a, b; X)$, etc., where X denotes a Banach space.

Definition 2.1 *For a domain E we define the following function spaces:*

$$X_0 = H^1(0,1) \times L^2(0,1), \quad H_*^j(E) = \begin{cases} H^{-1}(E) = (H_0^1)^*(E) & \textit{for } j = -1, \\ L^2(E) & \textit{for } j = 0, \\ H^j(E) \cap H_0^1(E) & \textit{for } j \geq 1. \end{cases}$$

Definition 2.2 *We introduce the following norms:*

$$[\![f(t)]\!]_k = \left(\sum_{j=0}^{[\frac{k}{2}]} \| \partial_t^j f(t) \|_{H^{k-2j}} \right)^{\frac{1}{2}},$$

$$|||Df(t)|||_k = \begin{cases} \|\partial_x f(t)\|_2 & \textit{for } k = 0, \\ ([\![\partial_x f(t)]\!]_k^2 + [\![\partial_t f(t)]\!]_{k-1}^2)^{\frac{1}{2}} & \textit{for } k \geq 1. \end{cases}$$

Remark 2.3 *Let us note that*

$$|||Dv|||_{m-1} \leq 2[\![v]\!]_m \ \textit{and}\ [\![v]\!]_m \leq \|v\|_2 + |||Dv|||_{m-1},$$

for $[\![v]\!]_m < \infty$.

Definition 2.4 *Let $m \geq [n/2]+1$. For $\tau > 0$ we define a function space $Z^m(\tau)$ by*

$$Z^m(\tau) = \{u \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\tau]; H^{m-2j}),\ \|u\|_{Z^m(\tau)} < \infty\},$$

where

$$\|u\|_{Z^m(\tau)} = \sup_{0\leq z\leq \tau} [\![u(z)]\!]_m + \left(\int_0^\tau |\!|\!|Dw(z)|\!|\!|_m^2 dz\right)^{\frac{1}{2}}.$$

3 Main results

In this section we state the main results of this paper.

In the whole article we assume the following regularity for $\widetilde{\boldsymbol{g}}$ and $\widetilde{V}^1$.

Assumptions 3.1 *For a given integer $m \geq [n/2]+1$ assume that $\widetilde{\boldsymbol{g}} = {}^T(\widetilde{g}^1(\widetilde{x}_n, \widetilde{t}), 0, \ldots, 0, \widetilde{g}^n(\widetilde{x}_n))$ and $\widetilde{V}^1(\widetilde{t})$ belong to the following spaces:*

$$\widetilde{g}^n \in C^{m+1}[0,\ell],$$

and

$$\widetilde{g}^1 \in \bigcap_{j=0}^{[\frac{m+1}{2}]} C^j_{per}([0,\tau]; H^{m+1-2j}(0,\ell)),$$

$$\widetilde{V}^1 \in C_{per}^{[\frac{m+2}{2}]}([0,\tau]).$$

Furthermore, we assume

$$\widetilde{P}(\cdot) \in C^{m+2}(\mathbb{R}).$$

It is straightforward that $\boldsymbol{g}$ and V^1 belong to similar spaces as $\widetilde{\boldsymbol{g}}$ and $\widetilde{V}^1$.

Under Assumptions 3.1 one can see that flow u_p has the following properties (see [1]).

Proposition 3.2 *There exists $\delta_0 > 0$ such that if*

$$\nu |g^n|_{C^{m+1}([0,1])} \leq \delta_0,$$

then the following assertions hold true(see [1]). The flow $u_p = {}^T(\rho_p(x_n), v_p(x_n,t))$ exists and under Assumptions 3.1, it satisfies

$$v_p \in \bigcap_{j=0}^{[\frac{m+3}{2}]} C^j_{per}(J_T; H^{m+3-2j}(0,1)), \quad \rho_p \in C^{m+2}[0,1],$$

and

$$0 < \rho_1 \leq \rho_p(x_n) \leq \rho_2,\ \int_0^1 \rho_p(x_n)dx_n = 1,\ v_p(x_n,t) = {}^T(v_p^1(x_n,t), 0),$$

with

$$P'(\rho) > 0 \text{ for } \rho_1 \leq \rho \leq \rho_2,$$

$$|1-\rho_p|_{C^{k+1}([0,1])} \leq \frac{C}{\gamma^2}\nu(|P''|_{C^{k-1}(\rho_1,\rho_2)} + |g^n|_{C^k([0,1])}), \quad k = 1,\ldots,m+1, \tag{3.1}$$

$$|P'(\rho_p) - \gamma^2|_{C^0([0,1])} \leq \frac{C}{\gamma^2}\nu|g^n|_{C^0([0,1])},$$

for some constants $0 < \rho_1 < 1 < \rho_2$.

First, let us introduce the local existence result. To do so, we rewrite (1.5)–(1.8) in the form

$$\partial_t\phi + v\cdot\nabla\phi = -\gamma^2 w\cdot\nabla\rho_p - \rho\,\mathrm{div}\, w, \tag{3.2}$$

$$\rho\partial_t w - \nu\Delta w - \widetilde{\nu}\nabla\mathrm{div}\, w = -\frac{\nu}{\gamma^2\rho_p}\partial_{x_n}^2 v_p\phi - \nabla(P(\rho)-P(\rho_p)) - \rho(v\cdot\nabla v), \tag{3.3}$$

$$w|_{\partial\Omega} = 0, \tag{3.4}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0), \tag{3.5}$$

where $\rho = \rho_p + \gamma^{-2}\phi$ and $v = v_p + w$.

Next, let us mention the compatibility condition for $u_0 = {}^T(\phi_0, w_0)$. We look for a solution $u = {}^T(\phi, w)$ of (3.2)–(3.5) in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j})$ satisfying $\int_0^t \|\partial_x w(z)\|_{H^m}^2 dz < \infty$ for all $t \ge 0$ with $m \ge [n/2]+1$. Therefore, we need to require the compatibility condition for the initial value $u_0 = {}^T(\phi_0, w_0)$, which is formulated as follows.

Let $u = {}^T(\phi, w)$ be a smooth solution of (3.2)–(3.5). Then $\partial_t^j u = {}^T(\partial_t^j\phi, \partial_t^j w)$, $j \ge 1$ is inductively determined by

$$\partial_t^j\phi = -v\cdot\nabla\partial_t^{j-1}\phi - \rho\,\mathrm{div}\,\partial_t^{j-1}w - \gamma^2\partial_t^{j-1}w\cdot\nabla\rho_p - \{[\partial_t^{j-1}, v\cdot\nabla]\phi + [\partial_t^{j-1}, \rho]\mathrm{div}\, w\},$$

and

$$\begin{aligned}\partial_t^j w = -\rho^{-1}\{-\nu\Delta\partial_t^{j-1}w - \widetilde{\nu}\nabla\mathrm{div}\,\partial_t^{j-1}w + P'(\rho)\nabla\partial_t^{j-1}\rho\} - \rho^{-1}\{\gamma^{-2}[\partial_t^{j-1}, \phi]\partial_t w + [\partial_t^{j-1}, P'(\rho)]\nabla\rho\}\\ -\rho^{-1}\{\tfrac{\nu}{\gamma^2\rho_p}\partial_t^{j-1}(\partial_{x_n}^2 v_p\phi) - \partial_t^{j-1}\nabla P(\rho_p)\} - \rho^{-1}\partial_t^{j-1}(\rho(v\cdot\nabla v)).\end{aligned}$$

From these relations we see that $\partial_t^j u|_{t=0} = {}^T(\partial_t^j\phi, \partial_t^j w)|_{t=0}$ is inductively given by $u_0 = {}^T(\phi_0, w_0)$ in the following way:

$$\partial_t^j u|_{t=0} = {}^T(\partial_t^j\phi, \partial_t^j w)|_{t=0} = {}^T(\phi_j, w_j) = u_j,$$

where

$$\phi_j = -v_0\cdot\nabla\phi_{j-1} - \rho_0\mathrm{div}\, w_{j-1} - \gamma^2 w_{j-1}\cdot\nabla\rho_p - \sum_{l=1}^{j-1}\begin{pmatrix} j-1 \\ l\end{pmatrix}\{v_l\cdot\nabla\phi_{j-1-l} + \gamma^{-2}\phi_l\mathrm{div}\, w_{j-1}\},$$

and

$$\begin{aligned}w_j = -\rho_0^{-1}\{-\nu\Delta w_{j-1} - \widetilde{\nu}\nabla\mathrm{div}\, w_{j-1} + P'(\rho_0)\nabla\rho_{j-1}\} - \rho_0^{-1}\sum_{l=1}^{j-1}\begin{pmatrix} j-1 \\ l\end{pmatrix}\{\gamma^{-2}\phi_l w_{j-l}\\ +a_l(\phi_0;\phi_1,\ldots,\phi_l)\nabla\rho_{j-1-l}\} - \rho_0^{-1}\frac{\nu}{\gamma^2\rho_p}\sum_{l=0}^{j-1}\begin{pmatrix} j-1 \\ l\end{pmatrix}\partial_t^{j-1-l}\partial_{x_n}^2 v_p(0)\phi_l + \delta_{1j}\rho_0^{-1}\nabla P(\rho_p)\\ -\rho_0^{-1}G_{j-1}(\phi_0, w_0, \partial_x w_0; \phi_1,\ldots,\phi_{j-1}, w_1,\ldots,w_{j-1}, \partial_x w_1,\ldots,\partial_x w_{j-1}),\end{aligned}$$

with $v_l = \partial_t^l v_p(0) + w_l$, $\rho_l = \delta_{1l}\rho_p + \gamma^{-2}\phi_l$; and $a_l(\phi_0;\phi_1,\ldots,\phi_l)$ is certain polynomial in $\phi_1,\ldots,\phi_l$; and analogously. Here, δ_{1j} denotes the Kronecker's delta.

By the boundary condition $w|_{\partial\Omega} = 0$ in (3.4), we necessarily have $\partial_t^j w|_{\partial\Omega} = 0$, and hence,

$$w_j|_{\partial\Omega} = 0.$$

Assume that $u = {}^T(\phi, w)$ is a solution of (3.2) –(3.5) in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\tau_0]; H^{m-2j})$ for some $\tau_0 > 0$. Then from above observation, we need the regularity $u_j = {}^T(\phi_j, w_j) \in H^{m-2j}\times H^{m-2j}$ for $j = 1,\ldots,[m/2]$, which, indeed follows from the fact that $u_0 = {}^T(\phi_0, w_0) \in H^m$ with $m \ge [n/2]+1$. Furthermore, it is necessary to require that $u_0 = {}^T(\phi_0, w_0)$ satisfies the $\widehat{m}$*-th order compatibility condition*:

$$w_j \in H_0^1 \text{ for } j = 0,\ldots,\widehat{m} = \left[\frac{m-1}{2}\right].$$

Now, we can apply local solvability result obtained in [8] to show the following assertion.

Proposition 3.3 *Let $n \geq 2$, m be an integer satisfying $m \geq [n/2]+1$ and $M > 0$. Assume that $u_0 = {}^T(\phi_0, w_0) \in H^m$ satisfies the following conditions:*

(a) $\|u_0\|_{H^m} \leq M$ and u_0 satisfies the $\widehat{m}$-th compatibility condition,

(b) $-\frac{\gamma^2}{4}\rho_1 \leq \phi_0$.

Then there exists a positive number τ_0 depending on M and ρ_1 such that problem (3.2)–(3.5) has a unique solution $u(t)$ on $[0, \tau_0]$ satisfying $u(t) \in Z^m(\tau_0)$.

Remark 3.4 *It is straightforward to see that solution $u(t)$ of (3.2)–(3.5) is a solution of (1.5)–(1.8). Condition (b) in previous proposition assures that $\gamma^{-2}\phi_0 + \rho_p > \frac{3}{4}\rho_1$.*

We are in a position to state our main results of this paper.

Theorem 3.5 *Suppose that $n = 2$ and let m be an integer satisfying $m \geq 2$. There are positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_0^2$ then the following assertions hold true.*

There is a positive number ε_0 such that if $u_0 = {}^T(\phi_0, w_0) \in H^m \cap L^1$ satisfies the $\widehat{m}$-th compatibility condition and $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_0$, then there exists a unique global solution $u(t) = {}^T(\phi(t), w(t))$ of (1.5)–(1.8) with $n = 2$ in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j})$ which satisfies

$$\|\partial_{x'}^k u(t)\|_2 = O(t^{-\frac{1}{4}-\frac{k}{2}}), \ k = 0, 1, \tag{3.6}$$

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{3}{4}+\delta}), \ \forall \delta > 0, \tag{3.7}$$

as $t \to \infty$. Here, $u^{(0)} = u^{(0)}(x_2, t)$ is function given in Lemma 4.9 below; $\sigma = \sigma(x_1, t)$ is function satisfying

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma + \kappa_0 \partial_{x_1} \sigma + \omega_0 \partial_{x_1}(\sigma^2) = 0, \ \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2) \, dx_2,$$

with given constants $\kappa_0, \omega_0 \in \mathbb{R}$, $\kappa_1 > 0$.

Theorem 3.6 *Suppose that $n \geq 3$ and let m be an integer satisfying $m \geq [n/2]+1$. There are positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_0^2$ then the following assertions hold true.*

There is a positive number ε_0 such that if $u_0 = {}^T(\phi_0, w_0) \in H^m \cap L^1$ satisfies the $\widehat{m}$-th compatibility condition and $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_0$, then there exists a unique global solution $u(t) = {}^T(\phi(t), w(t))$ of (1.5)–(1.8) in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j})$ which satisfies

$$\|\partial_{x'}^k u(t)\|_2 = O(t^{-\frac{n-1}{4}-\frac{k}{2}}), \ k = 0, 1, \tag{3.8}$$

$$\|u(t) - (\sigma u^{(0)})(t)\|_2 = O(t^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t)), \tag{3.9}$$

as $t \to \infty$. Here, $\sigma = \sigma(x', t)$ is function satisfying

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + \kappa_0 \partial_{x_1} \sigma = 0, \ \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n) \, dx_n,$$

with given constants $\kappa_0 \in \mathbb{R}$, $\kappa_1, \kappa'' > 0$; where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$; and $\eta_n(t) = \log(1+t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \geq 4$.

As in [8, 12], the global existence result in Theorem 3.5 and Theorem 3.6 is proved by combining the local existence and the a priori estimate. Next we introduce the a priori estimate.

Proposition 3.7 *Let $n \geq 2$ and m be an integer satisfying $m \geq [n/2]+1$. There are positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_0^2$, then the following assertion holds true.*

There exists number $\varepsilon_1 > 0$ such that if solution $u(t)$ of (1.5)–(1.8) is in $Z^m(\tau)$ and $u(t)$ satisfies $\|u_0\|_{H^m \cap L^1} \leq \varepsilon_1$, then there holds the estimate

$$[\![u(t)]\!]_m^2 \leq C_1 \|u_0\|_{H^m \cap L^1}^2,$$

for a constant $C_1 > 0$ independent of τ.

Remark 3.8 *In the proof of Proposition 3.7 we use the estimate* (3.1) *with* $k = m$ *only, i.e.,*

$$|1-\rho_p|_{C^{m+1}([0,1])} \le \frac{C}{\gamma^2}\nu(|P''|_{C^{m-1}(\rho_1,\rho_2)} + |g^n|_{C^m([0,1])}).$$

Moreover, we require the boundedness of ρ_p *in* $C^{m+1}([0,1])$ *only.*

The global existence of the solution $u(t)$ follows from Proposition 3.3 and the a priori estimate in Proposition 3.7 in standard manner as follows.

Proof of global existence. Let $n \ge 2$ and let us fix $m \ge [n/2]+1$ and $\nu \ge \nu_0$, $\gamma^2/(\nu+\widetilde{\nu}) \ge \gamma_0^2$ such that Proposition 3.7 holds true.

Since $m \ge [n/2]+1$ we have the Sobolev inequality

$$\|f\|_\infty \le C_S\|f\|_{H^m}, \text{ for any } f \in H^m(\Omega). \tag{3.10}$$

Let us define $\varepsilon_0 > 0$ as

$$\varepsilon_0 = \min\{\varepsilon_1, \frac{\gamma^2}{4C_S}\rho_1, \frac{\varepsilon_1}{\sqrt{C_1}}, \frac{\gamma^2}{4C_S\sqrt{C_1}}\rho_1\}.$$

Here, ε_1 and C_1 are given by Proposition 3.7.

Let $\|u_0\|_{H^m\cap L^1} \le \varepsilon_0$ satisfies $\widehat{m}$-th compatibility condition. It is easy to see that such u_0 satisfies conditions (a), (b) of Proposition 3.3 and therefore, there exits $\tau_0 > 0$, which is determined by ε_1, such that the problem (1.5)–(1.8) has a unique solution $u(\cdot) \in Z^m(\tau_0)$.

Since $\varepsilon_0 \le \varepsilon_1$ we see from Proposition 3.7 that $u(t)$ satisfies

$$[\![u(\tau_0)]\!]_m^2 \le C_1\|u_0\|^2_{H^m\cap L^1} \le \min\{\varepsilon_1^2, \left(\frac{\gamma^2}{4C_S}\rho_1\right)^2\}. \tag{3.11}$$

Thus, $\|u(\tau_0)\|_{H^m} \le \varepsilon_1$ and $u(\tau_0)$ satisfies conditions (a) and (b) of Proposition 3.3. Hence, there exists unique extension of solution $u(t)$ of (1.5)–(1.8) on $[\tau_0, 2\tau_0]$ and we get

$$u(\cdot) \in Z^m(2\tau_0).$$

It is straightforward to see that we can use Proposition 3.7 again, to obtain estimate (3.11) for $u(2\tau_0)$, which enables us to extend solution $u(t)$ on $[0, 3\tau_0]$. By repeating this procedure the existence on $[0,\infty)$ is showed. This concludes the proof.

$\square$

Proposition 3.7 together with L^2-decay estimates (3.6) and (3.8) are proved in Sections 4-8. The asymptotic behavior, i.e., (3.7) and (3.9), is proved in Section 9.

4 Spectral properties of the linearized operator

Let us write (1.5)–(1.8) in the form

$$\begin{gathered}\partial_t u + L(t)u = \boldsymbol{F},\\ w|_{\delta\Omega} = 0,\ u|_{t=0} = u_0.\end{gathered} \tag{4.1}$$

Here, $u = {}^T(\phi, w)$; $\boldsymbol{F} = {}^T(f^0, \boldsymbol{f})$ with $\boldsymbol{f} = {}^T(f^1, \cdots, f^n)$ is the nonlinearity; and $L(t)$ is the operator given in (1.10)

In this section we introduce the spectral properties of the linearized problem, i.e., (4.1) with $\boldsymbol{F} = 0$. These results were established in [2]. At the end of this section we show regularity improvements for ϕ.

Now, let us consider the linearized problem

$$\partial_t u + L(t)u = 0,\ t > s,\ w|_{x_n=0,1} = 0,\ u|_{t=s} = u_0. \tag{4.2}$$

We introduce space Z_s defined by

$$Z_s = \{u = {}^T(\phi, w); \phi \in C_{loc}([s,\infty); H^1),\ \partial_{x'}^{\alpha'} w \in C_{loc}([s,\infty); L^2)\cap L^2_{loc}([s,\infty); H^1_0)\ (|\alpha'| \le 1),\ w \in C_{loc}((s,\infty); H^1_0)\}.$$

In [1] we showed that for any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in H^1 \cap L^2$ with $\partial_{x'} w_0 \in L^2$ there exists a unique solution $u(t)$ of linear problem (4.2) in Z_s. We denote $U(t,s)$ the evolution operator for (4.2) given by

$$u(t) = U(t,s)u_0.$$

To investigate problem (4.2) we consider the Fourier transform of (4.2). We thus obtain

$$\frac{d}{dt}\widehat{u} + \widehat{L}_{\xi'}(t)\widehat{u} = 0,\ t > s,\quad \widehat{u}|_{t=s} = \widehat{u}_0. \tag{4.3}$$

Here $\widehat{\phi} = \widehat{\phi}(\xi', x_n, t)$ and $\widehat{w} = \widehat{w}(\xi', x_n, t)$ are the Fourier transforms of $\phi = \phi(x', x_n, t)$ and $w = w(x', x_n, t)$ in $x' \in \mathbb{R}^{n-1}$ with $\xi' \in \mathbb{R}^{n-1}$ being the dual variable; $\widehat{L}_{\xi'}(t)$ is an operator on X_0 with domain $D(\widehat{L}_{\xi'}(t)) = H^1(0,1) \times H^2_*(0,1)$, which takes the form

$$\widehat{L}_{\xi'}(t) = \begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2 {\rho_p}^T\xi' & \gamma^2\partial_{x_n}(\rho_p\ \cdot\ ) \\ i\xi'\frac{P'(\rho_p)}{\gamma^2\rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \frac{\tilde{\nu}}{\rho_p}\xi'^T\xi' & -i\frac{\tilde{\nu}}{\rho_p}\xi'\partial_{x_n} \\ \partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\ \cdot\right) & -i\frac{\tilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1(t))\boldsymbol{e}_1' & i\xi_1 v_p^1(t) I_{n-1} & \partial_{x_n}(v_p^1(t))\boldsymbol{e}_1' \\ 0 & 0 & i\xi_1 v_p^1(t) \end{pmatrix}.$$

Let us note that $\widehat{L}_{\xi'}(t)$ is sectorial uniformly with respect to $t \in \mathbb{R}$ for each $\xi' \in \mathbb{R}^{n-1}$. As for the evolution operator $\widehat{U}_{\xi'}(t,s)$ for (4.3) we have the following results.

Lemma 4.1 *For each $\xi' \in \mathbb{R}^{n-1}$ and for all $t \geq s$ there exists unique evolution operator $\widehat{U}_{\xi'}(t,s)$ for (4.3) that satisfies*

$$|\widehat{L}_{\xi'}(t)\widehat{U}_{\xi'}(t,s)|_{L(X_0)} \leq C_{t_1 t_2},\ t_1 \leq s < t \leq t_2.$$

Furthermore, for $u_0 \in X_0$, $f \in C^\alpha([s,\infty); X_0), \alpha \in (0,1]$ there exists unique classical solution u of inhomogeneous problem

$$\frac{d}{dt}u + \widehat{L}_{\xi'}(t)u = f,\ t > s,\quad u|_{t=s} = u_0, \tag{4.4}$$

satisfying $u \in C_{loc}([s,\infty); X_0) \cap C^1(s,\infty; X_0) \cap C(s,\infty; H^1(0,1) \times H^2_*(0,1))$; and the solution u is given by

$$u(t) = (\phi(t), w(t)) = \widehat{U}_{\xi'}(t,s)u_0 + \int_s^t \widehat{U}_{\xi'}(t,z)f(z)dz.$$

Next, let us introduce *adjoint problem* to

$$\partial_t u + \widehat{L}_{\xi'}(t)u = 0,\ t > s,\quad u|_{t=s} = u_0.$$

Lemma 4.2 *For each $\xi' \in \mathbb{R}^{n-1}$ and for all $s \leq t$ there exists unique evolution operator $\widehat{U}^*_{\xi'}(s,t)$ for adjoint problem*

$$-\partial_s u + \widehat{L}^*_{\xi'}(s)u = 0,\ s < t,\ u|_{s=t} = u_0,$$

on X_0. Here, $\widehat{L}^*_{\xi'}(s)$ is an operator on X_0 with domain $D(\widehat{L}^*_{\xi'}(s)) = H^1(0,1) \times H^2_*(0,1)$, which takes the form

$$\widehat{L}^*_{\xi'}(s) = \begin{pmatrix} -i\xi_1 v_p^1(s) & -i\gamma^2 {\rho_p}^T\xi' & -\gamma^2\partial_{x_n}(\rho_p\ \cdot\ ) \\ -i\xi'\frac{P'(\rho_p)}{\gamma^2\rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \frac{\tilde{\nu}}{\rho_p}\xi'^T\xi' & -i\frac{\tilde{\nu}}{\rho_p}\xi'\partial_{x_n} \\ -\partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\ \cdot\right) & -i\frac{\tilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \frac{\nu\gamma^2}{P'(\rho_p)}(\partial_{x_n}^2 v_p^1(s))^T\boldsymbol{e}_1' & 0 \\ 0 & -i\xi_1 v_p^1(s) I_{n-1} & 0 \\ 0 & \partial_{x_n}(v_p^1(s))^T\boldsymbol{e}_1' & -i\xi_1 v_p^1(s) \end{pmatrix}.$$

Moreover, $\widehat{L}^*_{\xi'}(s)$ satisfies $\langle \widehat{L}_{\xi'}(s)u, v\rangle = \langle u, \widehat{L}^*_{\xi'}(s)v\rangle$ for $s \in \mathbb{R}$ and $u, v \in H^1 \times H^2_*$ and

$$|\widehat{L}^*_{\xi'}(s)\widehat{U}^*_{\xi'}(s,t)|_{L(X_0)} \leq C_{t_1t_2},\ t_1 \leq s < t \leq t_2.$$

Furthermore, for $u_0 \in X_0$, $f \in C^\alpha((-\infty,t];X_0), \alpha \in (0,1]$ *there exists unique classical solution* u *of inhomogeneous problem*

$$-\frac{d}{ds}u + \widehat{L}^*_{\xi'}(s)u = f,\ s < t, \quad u|_{s=t} = u_0, \tag{4.5}$$

satisfying $u \in C_{loc}((-\infty,t];X_0) \cap C^1(-\infty,t;X_0) \cap C(-\infty,t;H^1(0,1) \times H^2_*(0,1))$*; and the solution* u *is given by*

$$u(s) = (\phi(s), w(s)) = \widehat{U}^*_{\xi'}(s,t)u_0 + \int_s^t \widehat{U}^*_{\xi'}(s,z)f(z)dz.$$

Note that $\widehat{U}_{\xi'}(t,s)$ and $\widehat{U}^*_{\xi'}(s,t)$ are defined for all $t \geq s$ and

$$\widehat{U}_{\xi'}(t+T,s+T) = \widehat{U}_{\xi'}(t,s),\ \widehat{U}^*_{\xi'}(s+T,t+T) = \widehat{U}^*_{\xi'}(s,t).$$

Lemma 4.3 *There exist positive numbers* ν_1 *and* γ_1 *such that if* $\nu \geq \nu_1$ *and* $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_1^2$ *then there exists* $r_0 > 0$ *such that for each* ξ' *with* $|\xi'| \leq r_0$ *there hold the following statements.*

(i) The spectrum of operator $\widehat{U}_{\xi'}(T,0)$ *on* $H^1(0,1) \times H^1_0(0,1)$ *satisfies*

$$\sigma(\widehat{U}_{\xi'}(T,0)) \subset \{\mu_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\},$$

with constant $q_0 < \mathrm{Re}\,\mu_{\xi'} < 1$*. Here,* $\mu_{\xi'} = e^{\lambda_{\xi'}T}$ *is simple eigenvalue of* $\widehat{U}_{\xi'}(T,0)$ *and* $\lambda_{\xi'}$ *has an expansion*

$$\lambda_{\xi'} = -i\kappa_0\xi_1 - \kappa_1\xi_1^2 - \kappa''|\xi''|^2 + O(|\xi'|^3), \tag{4.6}$$

where $\kappa_0 \in \mathbb{R}$ *and* $\kappa_1 > 0$, $\kappa'' > 0$*.*

Moreover, let $\widehat{\Pi}_{\xi'}$ *denote the eigenprojection associated with* $\mu_{\xi'}$*. There holds*

$$|\widehat{U}_{\xi'}(t,s)(I-\widehat{\Pi}_{\xi'})u|_{H^1} \leq Ce^{-d(t-s)}|(I-\widehat{\Pi}_{\xi'})u|_{X_0},$$

for $u \in X_0$ *and* $T \leq t-s$*. Here,* d *is a positive constant depending on* r_0*.*

(ii) The spectrum of operator $\widehat{U}^*_{\xi'}(0,T)$ *on* $H^1(0,1) \times H^1_0(0,1)$ *satisfies*

$$\sigma(\widehat{U}^*_{\xi'}(0,T)) \subset \{\overline{\mu}_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\}.$$

Here, $\overline{\mu}_{\xi'}$ *is simple eigenvalue of* $\widehat{U}^*_{\xi'}(0,T)$*.*

Furthermore, let $\widehat{\Pi}^*_{\xi'}$ *denote the eigenprojection associated with* $\overline{\mu}_{\xi'}$*. There holds*

$$\langle \widehat{\Pi}_{\xi'}u, v\rangle = \langle u, \widehat{\Pi}^*_{\xi'}v\rangle,$$

for $u, v \in X_0$*.*

Next, we introduce Floquet theory.

Definition 4.4 *Let* $k = 1, 2, \ldots$*. Let us define spaces* Y^k_{per} *as*

$$Y^1_{per} = L^2_{per}([0,T];X_0),$$

$$Y^k_{per} = \bigcap_{j=0}^{[\frac{k}{2}]} H^j_{per}([0,T];H^{k-2j}(0,1) \times H^{k-1-2j}(0,1)),\ \text{for } k \geq 2.$$

Here, for Banach space X *and* $j = 0, \ldots$ *spaces* $L^2_{per}([0,T];X)$ *and* $H^j_{per}([0,T];X)$ *consist of functions from* $L^2([0,T];X)$ *and* $H^j([0,T];X)$*, respectively, that are restrictions of* T*-periodic functions.*

Definition 4.5 *We define operator $B_{\xi'}$ on space Y^1_{per} with domain*

$$D(B_{\xi'}) = H^1_{per}([0,T]; X_0) \cap L^2_{per}([0,T]; H^1(0,1) \times H^2_*(0,1)),$$

in the following way

$$B_{\xi'} v = \partial_t v + \widehat{L}_{\xi'}(\cdot) v,$$

for $v \in D(B_{\xi'})$. Moreover, we define formal adjoint operator $B^*_{\xi'}$ with respect to inner product $\frac{1}{T}\int_0^T \langle \cdot, \cdot \rangle dt$ as

$$B^*_{\xi'} v = -\partial_t v + \widehat{L}^*_{\xi'}(\cdot) v,$$

for $v \in D(B^*_{\xi'}) = D(B_{\xi'})$.

Remark 4.6 *Operators $B_{\xi'}$ and $B^*_{\xi'}$ are closed, densely defined on Y^1_{per} for each fixed $\xi' \in \mathbb{R}^{n-1}$.*

Definition 4.7 *Let $k \geq 1$. We say that $u = {}^T(\phi, w)$ is k-regular function on time interval $[a,b]$ whenever*

$$u \in \bigcap_{j=0}^{\left[\frac{k}{2}\right]} C^j([a,b]; (H^{k-2j} \times H^{k-2j}_*)(\Omega)),$$

$$\phi \in \bigcap_{j=0}^{\left[\frac{k}{2}\right]} H^{j+1}(a,b; H^{k-2j}(\Omega)), \;\; w \in \bigcap_{j=0}^{\left[\frac{k+1}{2}\right]} H^j(a,b; H^{k+1-2j}_*(\Omega)).$$

Lemma 4.8 *There exist positive numbers $\nu_2 \geq \nu_1$ and $\gamma_2 \geq \gamma_1$ such that if $\nu \geq \nu_2$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_2^2$ then there exists $0 < r_1 \leq 1$ such that for each $|\xi'| \leq r_1$ there hold the following statements.*

(i) Let $1 \leq k \leq m+1$. There exists $q_1 > 0$ such that spectrum of operator $B_{\xi'}$ on Y^k_{per} satisfies

$$\sigma(B_{\xi'}) \subset \{-\lambda_{\xi'}\} \cup \{\lambda : \mathrm{Re}\,\lambda \geq q_1\},$$

with $0 \leq -\mathrm{Re}\,\lambda_{\xi'} \leq \frac{1}{2} q_1$ uniform for all k. Here, $-\lambda_{\xi'}$ is simple eigenvalue of $B_{\xi'}$.

(ii) Let $1 \leq k \leq m+1$. Spectrum of operator $B^*_{\xi'}$ on Y^k_{per} satisfies

$$\sigma(B^*_{\xi'}) \subset \{-\overline{\lambda}_{\xi'}\} \cup \{\lambda : \mathrm{Re}\,\lambda \geq q_1\}.$$

Here, $-\overline{\lambda}_{\xi'}$ is simple eigenvalue of $B^*_{\xi'}$.

(iii) There exist $u_{\xi'}$ and $u^*_{\xi'}$ eigenfunctions associated with $-\lambda_{\xi'}$ and $-\overline{\lambda}_{\xi'}$, respectively, with the following properties:

$$\langle u_{\xi'}(t), u^*_{\xi'}(t) \rangle = 1,$$

$$u_{\xi'}(t) = u^{(0)}(t) + i\xi' \cdot u^{(1)}(t) + |\xi'|^2 u^{(2)}(\xi', t),$$

$$u^*_{\xi'}(t) = u^{*(0)} + i\xi' \cdot u^{*(1)}(t) + |\xi'|^2 u^{*(2)}(\xi', t),$$

for $t \in \mathbb{R}$. Here, all functions

$$u_{\xi'}, u^*_{\xi'}, u^{(0)}, u^{(0)*}, u^{(1)}, u^{(1)*}, u^{(2)}(\xi'), u^{(2)*}(\xi'),$$

are T-periodic in t, $m+1$-regular on $[0,T]$ and we have estimate

$$\sup_{z \in J_T} \sum_{j=0}^{\left[\frac{m+1}{2}\right]} |\partial_z^j u(z)|^2_{H^{m+1-2j}} + \int_0^T \sum_{j=0}^{\left[\frac{m}{2}\right]} |\partial_z^{j+1} u|^2_{H^{m+1-2j} \times H^{m-2j}} + |\partial_z^{\left[\frac{m+3}{2}\right]} Q_0 u|_2^2 + |u|^2_{H^{m+1} \times H^{m+2}} dz \leq C,$$

for $u \in \{u_{\xi'}, u^*_{\xi'}, u^{(2)}(\xi'), u^{(2)*}(\xi')\}$ and a constant $C > 0$ depending on r_1.

Let us introduce more properties of $u^{(0)}$.

Lemma 4.9 *Function $u^{(0)}(t)$ satisfies $\partial_t u^{(0)} + \widehat{L}_0(t)u^{(0)} = 0$ and $u^{(0)}(t) = u^{(0)}(t+T)$ for all $t \in \mathbb{R}$. Function $u^{(0)}(t)$ is given as*

$$u^{(0)}(x_n,t) = {}^T(\phi^{(0)}(x_n), w^{(0),1}(x_n,t), 0).$$

Here,

$$\phi^{(0)}(x_n) = \alpha_0 \frac{\gamma^2 \rho_p(x_n)}{P'(\rho_p(x_n))}, \qquad \alpha_0 = \left[\frac{\gamma^2\rho_p}{P'(\rho_p)}\right]^{-1},$$

$$w^{(0),1}(x_n,t) = -\frac{1}{\gamma^2}\int_{-\infty}^{t} e^{-(t-s)\nu A}\nu \frac{\alpha_0\gamma^2}{P'(\rho_p)\rho_p}(\partial_{x_n}^2 v_p^1(s))\,ds,$$

where A denotes the uniformly elliptic operator on $L^2(0,1)$ with domain $D(A) = (H^2 \cap H_0^1)(0,1)$ and

$$Av = -\frac{1}{\rho_p(x_n)}\partial_{x_n}^2 v,$$

for $v \in D(A)$. Moreover, function $w^{(0),1}$ satisfies

$$\partial_t w^{(0),1}(t) - \frac{\nu}{\rho_p(x_n)}\partial_{x_n}^2 w^{(0),1}(t) = -\frac{\nu}{\gamma^2}\frac{\alpha_0\gamma^2}{P'(\rho_p)\rho_p}(\partial_{x_n}^2 v_p^1(t)), \tag{4.7}$$

for all $t \in \mathbb{R}$ and

$$\|w^{(0),1}(t)\|_{C^{m+1}(\Omega)} = O(\frac{1}{\gamma^2}).$$

In the rest of this section we assume that $\nu \geq \nu_2$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_2^2$.

Definition 4.10 *We define $\widehat{\chi}_1$ by*

$$\widehat{\chi}_1(\xi') = \begin{cases} 1, & |\xi'| < r_1, \\ 0, & |\xi'| \geq r_1, \end{cases}$$

for $\xi' \in \mathbb{R}^{n-1}$.

Now, we introduce time-periodic operators based on $u_{\xi'}$ and $u^*_{\xi'}$.

Definition 4.11 *We define operators $\mathscr{P}(t) : L^2(\Omega) \to L^2(\mathbb{R}^{n-1})$ by*

$$\mathscr{P}(t)u = \mathscr{F}^{-1}\{\widehat{\mathscr{P}}_{\xi'}(t)\widehat{u}\},$$

$$\widehat{\mathscr{P}}_{\xi'}(t)\widehat{u} = \widehat{\chi}_1\langle \widehat{u}, u^*_{\xi'}(t)\rangle,$$

for $u \in L^2$ and $t \in [0,\infty)$.

We define operators $\mathscr{Q}(t) : L^2(\mathbb{R}^{n-1}) \to L^2(\Omega)$ by

$$\mathscr{Q}(t)\sigma = \mathscr{F}^{-1}\{\widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\sigma}\},$$

$$\widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\sigma} = \widehat{\chi}_1 u_{\xi'}(\cdot,t)\widehat{\sigma},$$

for $t \in [0,\infty)$ and multiplier $\Lambda : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ by

$$\Lambda\sigma = \mathscr{F}^{-1}\{\widehat{\chi}_1\lambda_{\xi'}\widehat{\sigma}\},$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$.

Moreover, we define projections $\mathbb{P}(t)$ and $\mathbb{P}^*(t)$ on $L^2(\Omega)$ as

$$\mathbb{P}(t)u = \mathscr{F}^{-1}\{\widehat{\chi}_1\langle u, u^*_{\xi'}(t)\rangle u_{\xi'}(\cdot,t)\} = \mathscr{Q}(t)\mathscr{P}(t)u,$$

$$\mathbb{P}^*(t)u = \mathscr{F}^{-1}\{\widehat{\chi}_1\langle u, u_{\xi'}(t)\rangle u^*_{\xi'}(\cdot,t)\},$$

for $t \in [0,\infty)$ and $u \in L^2$.

We define projection $\mathit{\Pi}^{(0)}(t)$ on $L^2(\Omega)$ as

$$\mathit{\Pi}^{(0)}(t)u = [Q_0 u]u^{(0)}(t),$$

for $t \in [0,\infty)$ and $u \in L^2$.

In terms of $P(t)$ we have the following decomposition of $U(t,s)$.

Lemma 4.12 *$\mathbb{P}(t)$ and $\mathbb{P}^*(t)$ satisfies the following:*

(i)

$$\mathbb{P}(t)(\partial_t + L(t))u(t) = (\partial_t + L(t))\mathbb{P}(t)u(t) = \mathscr{Q}(t)[(\partial_t - \Lambda)\mathscr{P}(t)u(t)],$$

for $u \in L^2([0,T]; H^1 \times H^2_*) \cap H^1([0,T]; L^2)$.

(ii)

$$\mathbb{P}(t)U(t,s) = U(t,s)\mathbb{P}(s) = \mathscr{Q}(t)e^{(t-s)\Lambda}\mathscr{P}(s).$$

If $u \in L^1$, then

$$\|\partial_t^j \partial_{x'}^k \partial_{x_n}^l \mathbb{P}(t)U(t,s)u\|_2 \leq C(1+t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u\|_1,$$

for $0 \leq 2j + l \leq m+1$, $k = 0, \ldots$.

(iii) For $u, v \in L^2$ there holds

$$\langle \mathbb{P}(t)u, v\rangle = \langle u, \mathbb{P}^*(t)u\rangle.$$

If $u \in L^2$, then

$$\|\partial_t^j \partial_{x'}^k \partial_{x_n}^l (\mathbb{P}^*(t)u)\|_2 \leq C\|u\|_2,$$

for $0 \leq 2j + l \leq m+1$, $k = 0, 1, \ldots$.

(iv) $(I - \mathbb{P}(t))U(t,s) = U(t,s)(I - \mathbb{P}(s))$ satisfies

$$\|(I - \mathbb{P}(t))U(t,s)u\|_{H^1} \leq Ce^{-d(t-s)}(\|u\|_{H^1 \times L^2} + \|\partial_{x'}w\|_2),$$

for $t - s \geq T$. Here d is a positive constant.

Next, let us show the asymptotic properties of $U(t,s)$. First, let us define a semigroup $\mathscr{H}(t)$ on $L^2(\mathbb{R}^{n-1})$ associated with a linear heat equation with a convective term:

$$\partial_t \sigma - \kappa_1 \partial_{x_1}^2 \sigma - \kappa'' \Delta'' \sigma + \kappa_0 \partial_{x_1} \sigma = 0.$$

Definition 4.13 *We define operator $\mathscr{H}(t)$ as*

$$\mathscr{H}(t)\sigma = \mathscr{F}^{-1}[e^{-(i\kappa_0\xi_1 + \kappa_1\xi_1^2 + \kappa''|\xi''|^2)t}\widehat{\sigma}],$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$. Here, κ_0, κ_1 and κ'' are given by (4.6).

Lemma 4.14 *There hold the following estimates for $1 \leq p \leq 2$ and $k = 0, 1, \ldots$.*

(i)

$$\|\partial_{x'}^k(\mathscr{H}(t)\sigma)\|_{L^2(\mathbb{R}^{n-1})} \leq Ct^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})}, \tag{4.8}$$

for $\sigma \in L^p(\mathbb{R}^{n-1})$.

(ii) Λ generates uniformly continuous group $\{e^{t\Lambda}\}_{t\in\mathbb{R}}$ and

$$\|\partial_{x'}^k e^{t\Lambda}\sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})}, \tag{4.9}$$

for $\sigma \in L^p(\mathbb{R}^{n-1})$.

(iii) It holds the relation,

$$\mathscr{P}(t)U(t,s) = e^{(t-s)\Lambda}\mathscr{P}(s).$$

Set $\sigma = [Q_0 u]$. Then

$$\|\partial_{x'}^k(e^{(t-s)\Lambda}\mathscr{P}(s)u - \mathscr{H}(t-s)\sigma)\|_{L^2(\mathbb{R}^{n-1})} \leq C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|u\|_p, \tag{4.10}$$

for $u \in L^p$. Furhermore, for any $\sigma \in L^p(\mathbb{R}^{n-1})$ there holds

$$\|(e^{(t-s)\Lambda} - \mathscr{H}(t-s))\partial_{x'}^k\sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})}. \tag{4.11}$$

Next, we introduce the properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$.

Proposition 4.15 *$\mathscr{Q}(t)$ has the following properties:*

(i)
$$\mathscr{Q}(t+T)=\mathscr{Q}(t),\ \partial_{x'}^{k}\mathscr{Q}(t)=\mathscr{Q}(t)\partial_{x'}^{k}.$$

(ii)
$$\mathscr{Q}(t)\sigma\in\bigcap_{j=0}^{\left[\frac{m+1}{2}\right]}C_{per}^{j}(J_T;H^{m+1-2j}\times H_{*}^{m+1-2j}),$$
$$\widetilde{Q}\mathscr{Q}(t)\sigma\in\bigcap_{j=0}^{\left[\frac{m+2}{2}\right]}H_{per}^{j}(J_T;H_{*}^{m+2-2j}),$$

and

$$\|\partial_t^j\partial_{x'}^k\partial_{x_n}^l(\mathscr{Q}(t)\sigma)\|_2\le C\|\sigma\|_{L^2(\mathbb{R}^{n-1})},\ 0\le 2j+l\le m+1,\ k=0,1,\dots,$$

for $\sigma\in L^2(\mathbb{R}^{n-1})$.

(iii)
$$(\partial_t+L(t))(\mathscr{Q}(t)\sigma(t))=\mathscr{Q}(t)(\partial_t-\Lambda)\sigma(t),$$

for $\sigma\in H_{loc}^1([0,\infty);L^2(\mathbb{R}^{n-1}))$.

(iv) $\mathscr{Q}(t)$ is decomposed as

$$\mathscr{Q}(t)=\mathscr{Q}^{(0)}(t)+\mathrm{div}'\mathscr{Q}^{(1)}(t)+\Delta'\mathscr{Q}^{(2)}(t).$$

Here, $\mathscr{Q}^{(0)}(t)\sigma=(\mathscr{F}^{-1}\{\widehat{\chi}_1\widehat{\sigma}\})u^{(0)}(\cdot,t)$, $\mathscr{Q}^{(1)}(t)$ and $\mathscr{Q}^{(2)}(t)$ share the same properties given in (i) and (ii) for $\mathscr{Q}(t)$.

Proposition 4.16 *$\mathscr{P}(t)$ has the following properties:*

(i)
$$\mathscr{P}(t+T)=\mathscr{P}(t),\ \partial_{x'}^{k}\mathscr{P}(t)=\mathscr{P}(t)\partial_{x'}^{k},\ \partial_{x_n}\mathscr{P}(t)=0.$$

(ii)
$$\mathscr{P}(t)u\in\bigcap_{j=0}^{\left[\frac{m+1}{2}\right]}C_{per}^{j}(J_T;H^{k}(\mathbb{R}^{n-1})),\ \textit{for all}\ k=0,1,\dots,$$

and
$$\|\partial_t^j\partial_{x'}^k(\mathscr{P}(t)u)\|_{L^2(\mathbb{R}^{n-1})}\le C\|u\|_2,\ 0\le 2j\le m+1,\ k=0,1,\dots,$$

for $u\in L^2$.

Moreover,
$$\|\mathscr{P}(t)u\|_{L^2(\mathbb{R}^{n-1})}\le C\|u\|_p,$$

for $u\in L^p$ and $1\le p\le 2$.

(iii)
$$\mathscr{P}(t)(\partial_t+L(t))u(t)=(\partial_t-\Lambda)(\mathscr{P}(t)u(t)),\tag{4.12}$$

for $u\in L_{loc}^2([0,\infty);H^1\times H_*^2)\cap H_{loc}^1([0,\infty);L^2)$.

(iv) $\mathscr{P}(t)$ is decomposed as

$$\mathscr{P}(t)=\mathscr{P}^{(0)}+\mathrm{div}'\mathscr{P}^{(1)}(t)+\Delta'\mathscr{P}^{(2)}(t).$$

Here,

$$\mathscr{P}^{(0)}u=\mathscr{F}^{-1}\{\widehat{\chi}_1\langle\widehat{u},u^{*(0)}\rangle\}=\mathscr{F}^{-1}\{\widehat{\chi}_1[Q_0\widehat{u}]\},$$

$$\mathscr{P}^{(1)}(t)u = \mathscr{F}^{-1}\{\widehat{\chi}_1\langle\widehat{u}, u^{*(1)}(t)\rangle\},$$

$$\mathscr{P}^{(2)}(t)u = \mathscr{F}^{-1}\{-\widehat{\chi}_1\langle\widehat{u}, u^{*(2)}(\xi', t)\rangle\}.$$

$\mathscr{P}^{(p)}(t)$, $p = 0, 1, 2$, share the same properties given in (i) and (ii) for $\mathscr{P}(t)$.

(v) There holds

$$\|\partial_{x'}^k e^{(t-s)\Lambda}\mathscr{P}(s)u\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|u\|_p, \tag{4.13}$$

$$\|\partial_{x'}^k e^{(t-s)\Lambda}\mathscr{P}^{(q)}(s)u\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|u\|_p,\ q = 0, 1, 2, \tag{4.14}$$

for $u \in L^p$, $1 \leq p \leq 2$ and $k = 0, 1, \ldots$.

Remark 4.17 *Note that $\Pi^{(0)}(t)$ and $\mathscr{Q}^{(0)}(t)\mathscr{P}^{(0)}$ are not identical operators.*

To close this section, we show improvements of regularity for ϕ.

Proposition 4.18 *Let $u = {}^T(\phi, w) \in Z^m(\tau)$ for $m \geq [n/2] + 1$ be a solution of (4.1). There holds*

$$\phi \in \bigcap_{j=1}^{[\frac{m+1}{2}]} C^j([0, \tau]; H^{m+1-2j}). \tag{4.15}$$

Proof. From definition of $Z^m(\tau)$ we have

$$u \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j([0, \tau]; H^{m-2j}) \ \text{ and } \ \sup_{0\leq z\leq\tau} [\![u(z)]\!]_m < \infty.$$

We write (1.5) as

$$\partial_t\phi = -v_p^1\partial_{x_1}\phi - \gamma^2\text{div}\,(\rho_p w) - \text{div}\,(\phi w).$$

Taking $[\![\cdot]\!]_{m-1}$-norm we obtain

$$[\![\partial_t\phi]\!]_{m-1} \leq [\![v_p^1\partial_{x_1}\phi]\!]_{m-1} + \gamma^2[\![\rho_p w]\!]_m + [\![\phi w]\!]_m.$$

Since $m \geq [n/2] + 1$ we get using Lemma 8.3 (ii) that

$$[\![\partial_t\phi]\!]_{m-1} \leq C([\![\phi]\!]_m)\{[\![v_p^1]\!]_m[\![\phi]\!]_m + \|\rho_p\|_{H^m}[\![w]\!]_m + [\![\phi]\!]_m[\![w]\!]_m\}.$$

This concludes the proof. □

5 Decomposition of the solution

In this section we decompose solution $u(t)$ of (4.1) and we prove the a priori estimate in Proposition 3.7. We decompose $u(t)$ into several parts based on the spectral properties of $L(t)$. In this section we assume that $\nu \geq \nu_2$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_2^2$ unless further restricted.

Let us first introduce some notation and projection operators. Let $\widehat{\chi}_2$ and $\widehat{\chi}_3$ be defined by

$$\widehat{\chi}_2(\xi') = \mathbf{1}_{[r_1,1)}(|\xi'|) \ \text{ and } \ \widehat{\chi}_3(\xi') = \mathbf{1}_{[1,\infty)}(|\xi'|).$$

We then define $[f]_j$, $j = 1, 2, \infty$ by

$$[f]_j = \mathscr{F}^{-1}(\widehat{\chi}_j[\widehat{f}]), \quad j = 1, 2,$$

$$[f]_\infty = [f]_1 + [f]_2 = \mathscr{F}^{-1}((\widehat{\chi}_1 + \widehat{\chi}_2)[\widehat{f}]).$$

Next, we define $P_{\infty,j}$, $j = 1, 2, 3$ as

$$P_{\infty,1}(t)u = \mathscr{F}^{-1}(\widehat{P}_{\infty,1}(t)\widehat{u}), \quad \widehat{P}_{\infty,1}(t)\widehat{u} = \widehat{\chi}_1(I - \widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\mathscr{P}}_{\xi'}(t))\widehat{u},$$

$$P_{\infty,j}u = \mathscr{F}^{-1}(\widehat{P}_{\infty,j}\widehat{u}), \quad \widehat{P}_{\infty,j}\widehat{u} = \widehat{\chi}_j\widehat{u} \quad (j = 2, 3).$$

By setting

$$\widetilde{P}_\infty(t) = I - \mathbb{P}(t), \quad P_\infty^{(0)}(t) = P_{\infty,1}(t) + P_{\infty,2},$$

we get

$$I = \mathbb{P}(t) + \widetilde{P}_\infty(t), \quad \widetilde{P}_\infty(t) = P_\infty^{(0)}(t) + P_{\infty,3}.$$

Using above operators we decompose solution $u(t)$ into

$$u(t) = \mathbb{P}(t)u(t) + \widetilde{P}_\infty(t)u(t),$$

with

$$\mathbb{P}(t)u(t) = \sigma_1(t)u^{(0)}(t) + u_1(t),$$

$$\widetilde{P}_\infty(t)u(t) = \sigma_\infty(t)u^{(0)}(t) + u_\infty(t),$$

where

$$\sigma_1(t) = \mathscr{P}(t)u(t), \quad u_1(t) = (\mathscr{Q}(t) - \mathscr{Q}^{(0)}(t))\mathscr{P}(t)u(t),$$

$$\sigma_\infty(t) = [Q_0 P_\infty^{(0)}(t)u(t)] = [Q_0 P_\infty^{(0)}(t)u(t)]_\infty, \quad u_\infty(t) = P_\infty(t)u(t).$$

By P_∞ we denote the operator defined as

$$P_\infty(t) = (I - \Pi^{(0)}(t))P_\infty^{(0)}(t) + P_{\infty,3}.$$

Remark 5.1 *Notice that $\sigma_1(t)$, $\sigma_\infty(t)$ and $u^{(0)}(t)$ are separate functions. Furthermore, notice that*

$$\|u_1(t)\|_2 \le C\|\partial_{x'}\sigma_1(t)\|_2.$$

Next, we derive the equations for σ_1, σ_∞ and u_∞. We define $\mathscr{M}(t)$ by

$$\mathscr{M}(t) = \widetilde{A} + \widetilde{B}(t),$$

with

$$\widetilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\nu}{\rho_p}\Delta' I_{n-1} - \frac{\widetilde{\nu}}{\rho_p}\nabla'\mathrm{div} \\ 0 & (-\frac{\widetilde{\nu}}{\rho_p}\partial_{x_n}\mathrm{div}', -\frac{\nu}{\rho_p}\Delta') \end{pmatrix}, \quad \widetilde{B}(t) = \begin{pmatrix} v_p^1(t)\partial_{x_1} & \gamma^2\rho_p\mathrm{div}' & 0 \\ \frac{P'(\rho_p)}{\gamma^2\rho_p}\nabla' & v_p^1(t)\partial_{x_1}I_{n-1} & 0 \\ 0 & 0 & v_p^1(t)\partial_{x_1} \end{pmatrix}.$$

Proposition 5.2 *Let $\tau > 0$ and $u(t)$ be a solution of (4.1) in $Z^m(\tau)$. Then there hold*

$$\sigma_k \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\tau]; H^l(\mathbb{R}^{n-1})), \quad \int_0^\tau |||D\sigma_k(z)|||_m dz < \infty, \ k = 1, \infty, \ l = 0, 1, \dots,$$

$$u_1 \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\tau]; H^{m+1-2j}), \quad \int_0^\tau |||Dw_1(z)|||_m dz < \infty,$$

$$u_\infty \in Z^m(\tau), \quad \int_0^\tau [\![\partial_t\phi_\infty(z)]\!]_{m-1} dz < \infty.$$

Moreover, σ_1, σ_∞ and u_∞ satisfy

$$\sigma_1(t) = e^{(t-s)\Lambda}\mathscr{P}(s)u_0 + \int_s^t e^{(t-z)\Lambda}\mathscr{P}(z)\boldsymbol{F}(z)dz, \tag{5.1}$$

$$\partial_t \sigma_\infty + [Q_0 \widetilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty = [Q_0 P_\infty^{(0)} \boldsymbol{F}]_\infty, \tag{5.2}$$

$$\partial_t u_\infty + L(t) u_\infty + \mathscr{M}(t)(\sigma_\infty u^{(0)}) - [Q_0 \widetilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty u^{(0)} = P_\infty \boldsymbol{F}. \tag{5.3}$$

$$w_\infty|_{x_n=0,1} = 0,$$

$$\sigma_\infty|_{t=0} = \sigma_{\infty,0}, \quad u_\infty|_{t=0} = u_{\infty,0}.$$

Here, $\sigma_{\infty,0} = [Q_0 P_\infty^{(0)}(0) u_0]_\infty$, $u_{\infty,0} = P_\infty u_0$.

Proof. Since $u \in Z^m(\tau)$, the regularity assertions on σ_k $(k = 1, \infty)$ and u_1 follow from properties of $\mathscr{P}(t)$, $\mathscr{Q}(t)$ and (4.15). As for u_∞, we already know that $\mathbb{P} Z^m(\tau) \subset Z^m(\tau)$ and therefore $\widetilde{P}_\infty Z^m(\tau) \subset Z^m(\tau)$. Since it is straightforward to see that $P_{\infty,3} Z^m(\tau) \subset Z^m(\tau)$ we have $P_\infty^{(0)} Z^m(\tau) \subset Z^m(\tau)$. Finally, from properties of $u^{(0)}(t)$ we obtain $\Pi^{(0)} P_\infty^{(0)} Z^m(\tau) \subset Z^m(\tau)$. Therefore, $u_\infty \in Z^m(\tau)$. $\int_0^\tau [\![\partial_t \phi_\infty(z)]\!]_{m-1} dz < \infty$ follows in analogous way using (4.15).

As for (5.1), it follows from (4.1) and (4.12) that

$$(\partial_t - \Lambda)\sigma_1(t) = \mathscr{P}(t) \boldsymbol{F}(t).$$

The rest is standard.

As for (5.2) and (5.3), we first apply $\widetilde{P}_\infty(t)$ to (4.1) to get

$$\partial_t(\widetilde{P}_\infty u) + L(t) \widetilde{P}_\infty u = \widetilde{P}_\infty \boldsymbol{F}. \tag{5.4}$$

Next, we apply $P_\infty^{(0)}(t)$ and $P_{\infty,3}$ to (5.4) to obtain

$$\partial_t(P_\infty^{(0)} u) + L(t) P_\infty^{(0)} u = P_\infty^{(0)} \boldsymbol{F}, \tag{5.5}$$

$$\partial_t(P_{\infty,3} u) + L(t) P_{\infty,3} u = P_{\infty,3} \boldsymbol{F}. \tag{5.6}$$

Since $[Q_0 L(t) v] = [Q_0 \mathscr{M}(t) v]$ for $\widetilde{Q} v|_{x_2=0,1} = 0$ and $[Q_0 \mathscr{M}(t) v] = [Q_0 \widetilde{B}(t) v]$ for any v, we get by applying $[Q_0 \cdot]$ to (5.5)

$$\partial_t \sigma_\infty + [Q_0 \widetilde{B}(t) P_\infty^{(0)} u] = [Q_0 P_\infty^{(0)} \boldsymbol{F}]. \tag{5.7}$$

There holds

$$[Q_0 \widetilde{B}(t) P_\infty^{(0)} u] = [Q_0 \widetilde{B}(\sigma_\infty u^{(0)}(t) + (I - \Pi^{(0)}(t)) P_\infty^{(0)} u)]_\infty = [Q_0 \widetilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty,$$

and thus (5.2) follows from (5.7).

To obtain (5.3) we use the fact that $\partial_t \Pi^{(0)}(t) + L(t) \Pi^{(0)}(t) = \mathscr{M}(t) \Pi^{(0)}(t)$. Applying $I - \Pi^{(0)}(t)$ to (5.5) gives us

$$\partial_t((I - \Pi^{(0)}(t)) P_\infty^{(0)} u) + L(t)(I - \Pi^{(0)}(t)) P_\infty^{(0)} u + \mathscr{M}(t)(\sigma_\infty u^{(0)}(t)) - \Pi^{(0)}(t) \widetilde{B}(t)(P_\infty^{(0)} u) = (I - \Pi^{(0)}(t)) P_\infty^{(0)} \boldsymbol{F}. \tag{5.8}$$

(5.3) now follows by adding (5.6) and (5.8). This completes the proof.

$\square$

Let us state some properties of σ_1, σ_∞ and u_∞ parts.

Lemma 5.3 *There hold the following inequalities.*

(i)

$$\|\partial_{x'}^k [Q_0 P_\infty^{(0)} u]_\infty\|_2 \le \|[Q_0 P_\infty^{(0)} u]_\infty\|_2, \ k = 0, 1, \dots,$$

(ii)

$$\|P_\infty u\|_2 \le C \|\partial_x P_\infty u\|_2 \quad \textit{if } \widetilde{Q} u|_{x_n=0,1} = 0.$$

(iii) Let $\tau > 0$ and $u(t)$ be a solution of (4.1) *in $Z^m(\tau)$. Then there hold*

$$\|\partial_{x'}^k \sigma_1\|_2 \le C\|\partial_{x'}\sigma_1\|_2,\ \|\partial_{x'}^k \sigma_\infty\|_2 \le C\|\partial_{x'}\sigma_\infty\|_2,\ k = 1, 2, \dots,$$

$$\|\phi_\infty\|_2 \le C\|\partial_x \phi_\infty\|_2,$$

$$\|w_\infty\|_2 \le C\|\partial_x w_\infty\|_2,$$

$$\|\Lambda\sigma_1\|_2 \le C\|\partial_{x'}\sigma_1\|_2.$$

Proof. Inequality (i) is obvious since $\mathrm{supp}\,(\widehat{\chi}_1 + \widehat{\chi}_2) \subset \{|\xi'| \le 1\}$. As for (ii), since $\mathrm{supp}\,\widehat{\chi}_3 \subset \{|\xi'| \ge 1\}$, we see that

$$\|P_{\infty,3}u\|_2 \le \|\partial_{x'}P_{\infty,3}u\|_2.$$

Since $\widetilde{Q}u|_{x_n=0,1} = 0$, we have $\widetilde{Q}P_\infty^{(0)}u|_{x_n=0,1} = 0$, and hence, $\widetilde{Q}(I - \Pi^{(0)}(t))P_\infty^{(0)}u|_{x_n=0,1} = 0$. By the Poincaré ineguality we obtain

$$\|\widetilde{Q}(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2 \le \|\partial_{x_n}\widetilde{Q}(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2.$$

Furthermore, since $[Q_0(I - \Pi^{(0)}(t))P_\infty^{(0)}u] = 0$, we see from the Poincaré inequality that

$$\|Q_0(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2 \le \|\partial_{x_n}Q_0(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2.$$

It then follows that

$$\|P_\infty u\|_2 \le C\{\|\partial_x(I - \Pi^{(0)}(t))P_\infty^{(0)}u\|_2 + \|\partial_x P_{\infty,3}u\|_2\} \le C\|\partial_x P_\infty u\|_2.$$

Here, we used $(\partial_x(I - \Pi^{(0)}(t))P_\infty^{(0)}u, \partial_x P_{\infty,3}u) = 0$, which follows from the fact $\widehat{\chi}_1\widehat{\chi}_3 = \widehat{\chi}_2\widehat{\chi}_3 = 0$ and the Plancherel theorem.

As for (iii), it follows from the proof of (i) and (ii).

□

We prove the a priori estimate in Proposition 3.7 by estimating the following quantities.

Let $u(t)$ be solution of (4.1) in $Z^m(\tau)$ and let $u(t)$ be decomposed as above, i.e.,

$$u(t) = \sigma_1(t)u^{(0)}(t) + u_1(t) + \sigma_\infty(t)u^{(0)}(t) + u_\infty(t).$$

We define $M(t) \ge 0$ by

$$M(t)^2 = M_1(t)^2 + \sup_{0\le z\le t}(1 + z)^{\frac{n+1}{2}}E_\infty(z), \quad t \in [0, \tau].$$

Here, $M_1(t)$ and $E_\infty(t)$ are defined as

$$M_1(t) = \sup_{0\le z\le t}(1 + z)^{\frac{n-1}{4}}\|\sigma_1(z)\|_2 + \sup_{0\le z\le t}(1 + z)^{\frac{n+1}{4}}\{\|\partial_{x'}\sigma_1(z)\|_2 + \sum_{j=1}^{[\frac{m}{2}]}\|\partial_z^j\sigma_1(z)\|_2\},$$

and

$$E_\infty(t) = [\![u_\infty(t)]\!]_m^2 + [\![\sigma_\infty(t)]\!]_m^2.$$

Finally, we introduce quantity $D_\infty(t)$ for $u_\infty(t) = {}^T(\phi_\infty(t), w_\infty(t))$:

$$D_\infty(t) = [\![\partial_x\phi_\infty(t)]\!]_{m-1}^2 + [\![\partial_t\phi_\infty]\!]_{m-1}^2 + |||Dw_\infty(t)|||_m^2 + |||D\sigma_\infty(t)|||_m^2.$$

Remark 5.4 *From properties of* $\mathscr{Q}^{(p)}(t)$, $p=1,2$, *we see that*

$$\|\partial_{x'}^k\partial_{x_n}^l\partial_t^j u_1(t)\|_2 \le C\||D\sigma_1(t)\||_m,\ 0\le 2j+l\le m+1,\ k=0,1,\ldots,$$

and there holds

$$\sup_{0\le z\le t}(1+z)^{\frac{n+1}{4}}\{[\![u_1(z)]\!]_m + [\![\partial_x u_1(z)]\!]_m\} \le CM_1(t).$$

Therefore, we do not need special estimates for $u_1(t)$.

We show the following estimates for $M_1(t)$ and $E_\infty(t)$.

Proposition 5.5 *There exist positive constants* ν_0 *and* γ_0 *such that if* $\nu\ge\nu_0$ *and* $\gamma^2/(\nu+\widetilde{\nu})\ge\gamma_0^2$, *then the following assertions hold true.*

There exists $\varepsilon_2>0$ *such that if solution* $u(t)$ *of (4.1) in* $Z^m(\tau)$ *satisfies* $\sup_{0\le z\le t}[\![u(z)]\!]_m\le\varepsilon_2$ *and* $M(t)\le 1$ *for all* $t\in[0,\tau]$, *then the following estimates hold uniformly for* $t\in[0,\tau]$ *with* $C>0$ *independent of* τ.

$$M_1(t)\le C\{\|u_0\|_1 + M(t)^2\}, \tag{5.9}$$

$$E_\infty(t)+\int_0^t e^{-a(t-z)}D_\infty(z)dz \le C\{e^{-at}E_\infty(0)+(1+t)^{-\frac{n+1}{2}}M(t)^4+\int_0^t e^{-a(t-z)}\widetilde{R}(z)dz\}. \tag{5.10}$$

Here, $a=a(\nu,\widetilde{\nu},\gamma)$ *is a positive constant; and* $\widetilde{R}(t)$ *is quantity that satisfies*

$$\widetilde{R}(t)\le C\{(1+t)^{-\frac{n+1}{2}}M(t)^3+M(t)D_\infty(t)\}, \tag{5.11}$$

whenever $\sup_{0\le z\le t}[\![u(z)]\!]_m\le\varepsilon_2$ *and* $M(t)\le 1$.

The proof of Proposition 5.5 is given in Sections 6-8. We prove (5.9), (5.10) and (5.11) in Sections 6, 7 and 8, respectively.

Assuming that Proposition 5.5 holds true, we can show the following estimate.

Proposition 5.6 *If* $\nu\ge\nu_0$ *and* $\gamma^2/(\nu+\widetilde{\nu})\ge\gamma_0^2$, *then the following assertion holds true. There exists number* $\varepsilon_3>0$ *such that if solution* $u(t)$ *of (4.1) in* $Z^m(\tau)$ *satisfies* $\|u_0\|_{H^m\cap L^1}\le\varepsilon_3$, *then there holds the estimate*

$$M(t)\le C\|u_0\|_{H^m\cap L^1}, \tag{5.12}$$

for a constant $C>0$ *independent of* τ.

As an immediate consequence of (5.12) we see that the a priory estimate in Proposition 3.7 holds true. Moreover, (5.12) provides us with the following decay estimates:

$$[\![u(t)]\!]_m\le C(1+t)^{-\frac{n-1}{4}}\|u_0\|_{H^m\cap L^1},$$

$$\|\partial_{x'}^k u(t)\|_2\le C(1+t)^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_{H^m\cap L^1},\ k=0,1,$$

and

$$\|u(t)-\sigma_1(t)u^{(0)}(t)\|_2\le C(1+t)^{-\frac{n+1}{4}}\|u_0\|_{H^m\cap L^1}, \tag{5.13}$$

for $t\in[0,\tau]$. This proves (3.6) and (3.8).

Proof of Proposition 5.6 If $\sup_{0\le z\le t}[\![u(z)]\!]_m\le\varepsilon_2$ and $M(t)\le 1$, then we see from (5.10) and (5.11) that

$$\begin{aligned}E_\infty(t)+\int_0^t e^{-a(t-z)}D_\infty(z)dz &\le C\{e^{-at}E_\infty(0)+(1+t)^{-\frac{n+1}{2}}M(t)^4\\ &+\int_0^t e^{-a(t-z)}\{(1+z)^{-\frac{n+1}{2}}M(z)^3+M(z)D_\infty(z)\}dz\}\end{aligned}$$

$$\leq C\{e^{-at}E_\infty(0) + (1+t)^{-\frac{n+1}{2}}M(t)^3 + M(t)\int_0^t e^{-a(t-z)}D_\infty(z)dz\}.$$

Therefore, using continuity of $E_\infty(t)$ and compatibility conditions we obtain

$$(1+t)^{\frac{n+1}{2}}E_\infty(t) + \mathscr{D}(t) + \leq C\{\|u_0\|^2_{H^m} + M(t)^3 + M(t)\mathscr{D}(t)\}, \tag{5.14}$$

with

$$\mathscr{D}(t) = (1+t)^{\frac{n+1}{2}}\int_0^t e^{-a(t-z)}D_\infty(z)dz.$$

It follows from (5.9) and (5.14) that

$$M(t)^2 + \sup_{0\leq z\leq t}\mathscr{D}(z) \leq C_1\{\|u_0\|^2_{H^m\cap L^1} + M(t)^3 + M(t)\sup_{0\leq z\leq t}\mathscr{D}(t)\}, \tag{5.15}$$

whenever $\sup_{0\leq z\leq t}[\![u(z)]\!]_m \leq \varepsilon_2$ and $M(t)\leq 1$.

In the same way as in [7, Proof of Proposition 5.4] using (5.15) one can show that there exists $\varepsilon_3 > 0$ such that if $\|u_0\|_{H^m\cap L^1} < \varepsilon_3$, then

$$M(t) < 2C_2\|u_0\|_{H^m\cap L^1},$$

for all $t\in[0,\tau]$ with $C_2>0$ independent of τ. This concludes the proof.

□

6 Estimates on $\sigma_1(t)$

In this section we estimate the $\mathbb{P}(t)$-part of $u(t)$. Since

$$\mathbb{P}(t)u(t) = \sigma_1(t)u^{(0)}(t) + u_1(t),$$

where $\sigma_1(t) = \mathscr{P}(t)u(t)$ and $u_1(t) = (\mathscr{Q}(t) - \mathscr{Q}^{(0)}(t))\mathscr{P}(t)u(t)$, it is enough to obtain estimates for σ_1 (see Remark 5.4). In this section we assume that $\nu\geq\nu_2$ and $\gamma^2/(\nu+\widetilde{\nu})\geq\gamma_2^2$.

Let us first make an observation. Regarding the spectral properties of linearized operator, we expect $\sigma_1(t)$ to be the most slowly decaying part of $u(t)$. Therefore, the most slowly decaying part of the nonlinearity $\boldsymbol{F}(t,u(t))$ would be given by the terms containing only $\sigma_1(t)^2$. There are two such terms in $\boldsymbol{F}(t,u(t))$,

$$\frac{\nu\phi}{\gamma^2\rho_p^2}\left(-\partial_{x_n}^2 w^1 + \frac{\partial_{x_n}^2 v_p^1}{\gamma^2\rho_p}\phi\right)\boldsymbol{e}_1 \quad\text{and}\quad -\frac{1}{2\gamma^4\rho_p}\partial_{x_n}\left(P''(\rho_p)\phi^2\right)\boldsymbol{e}_n.$$

Since $w^{(0),1}$ satisfies (4.7), we can define $\sigma_1^2\boldsymbol{F}_1$ with $\boldsymbol{F}_1 = \boldsymbol{F}_1(x_n,t)$ as

$$\boldsymbol{F}_1 = {}^T\left(0,\ \frac{\phi^{(0)}(x_n)}{\gamma^2\rho_p}\partial_t w^{(0),1}(x_n,t),\ \frac{1}{2\gamma^4\rho_p(x_n)}\partial_{x_n}\left(P''(\rho_p(x_2))\{\phi^{(0)}(x_n)\}^2\right)\right).$$

We thus write

$$\boldsymbol{F} = \sigma_1^2\boldsymbol{F}_1 + \boldsymbol{F}_2, \tag{6.1}$$

where $\boldsymbol{F}_2 = \boldsymbol{F} - \sigma_1^2\boldsymbol{F}_1$ contains terms involving u_∞, its derivatives and terms of order $O(\sigma_1\partial_{x'}\sigma_1)$ like $\sigma_1 u_1$, and $O(\sigma_1^3)$, but not just $O(\sigma_1^2)$. In particular, we have that $Q_0\boldsymbol{F} = Q_0\boldsymbol{F}_2$.

First we introduce two lemmas.

Lemma 6.1 *There hold the following relations.*

(i)

$$[Q_0\boldsymbol{F}] = -\mathrm{div}'[\phi w'],$$

(ii)

$$\mathscr{P}(t)\boldsymbol{F}(t) = -\mathrm{div}'[\phi(t)w'(t)]_1 + \mathrm{div}'\mathscr{P}^{(1)}(t)\boldsymbol{F}(t) + \Delta'\mathscr{P}^{(2)}(t)\boldsymbol{F}(t).$$

Proof. Since $w|_{x_n=0,1} = 0$, by integration by parts, we have

$$[Q_0\boldsymbol{F}] = -\mathrm{div}'[\phi w'].$$

This shows (i). As for (ii), it is straightforward from definition of $\mathscr{P}^{(0)}$ and (i).

$\square$

Remark 6.2 *Let $\varepsilon_5 > 0$ be number such that*

$$C_s \varepsilon_5 \leq \frac{\gamma^2 \rho_1}{4}.$$

Here, $C_S > 0$ comes from Sobolev inequality (3.10). Then whenever $[\![u(t)]\!]_m \leq \varepsilon_5$, we have

$$\|\phi(t)\|_\infty \leq C_S [\![u(t)]\!]_m \leq C_S \varepsilon_5 \leq \frac{\gamma^2 \rho_1}{4},$$

and hence,

$$\rho(x,t) = \rho_p(x_2) + \gamma^{-2}\phi(x,t) \geq \rho_1 - \gamma^{-2}\|\phi(t)\|_\infty \geq \frac{3\rho_1}{4} > 0.$$

Therefore, we see that $\widetilde{Q}\boldsymbol{F}(t)$ is smooth whenever $[\![u(t)]\!]_m \leq \varepsilon_5$.

Using inequality $\|\sigma_1\|_\infty \leq C\|\sigma_1\|_2^{1/2}\|\partial_{x'}\sigma_1\|_2^{1/2}$ (see Lemmas 8.2 (iii) and 5.3 (iii)), it is not difficult to verify the following estimates on nonlinearities. We omit the proof.

Lemma 6.3 *Let solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\sup_{0\leq z\leq t}[\![u(z)]\!]_m \leq \varepsilon_5$ and $M(t) \leq 1$ for $t \in [0,\tau]$, then there hold the following estimates for $t \in [0,\tau]$ with $C > 0$ independent of τ.*

(i)
$$\|\partial_{x'}(\sigma_1^2(t))\|_1 \leq C(1+t)^{-\frac{n}{2}} M(t)^2,$$

(ii)
$$\|\mathrm{div}'[\phi w'](t)\|_1 \leq C(1+t)^{-\frac{n}{2}} M(t)^2,$$

(iii)
$$\|[\phi w'](t)\|_1 \leq C(1+t)^{-\frac{n-1}{2}} M(t)^2,$$

(iv)
$$\|\boldsymbol{F}(t)\|_1 \leq C(1+t)^{-\frac{n-1}{2}} M(t)^2,$$

(v)
$$\|\boldsymbol{F}_2(t)\|_1 \leq C(1+t)^{-\frac{n}{2}} M(t)^2,$$

(vi)
$$\|\boldsymbol{F}(t)\|_2 \leq C(1+t)^{-\frac{2n-1}{4}} M(t)^2,$$

(vii)
$$\|\partial_{x'}(\sigma_1^2(t))\|_2 \leq C(1+t)^{-\frac{2n+1}{4}} M(t)^2.$$

Finally, we prove (5.9).

Proposition 6.4 *There exists number $\varepsilon_4 > 0$ such that if a solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\sup_{0\leq z\leq t}[\![u(z)]\!]_m \leq \varepsilon_4$ and $M(t) \leq 1$ for all $t \in [0,\tau]$, then the estimate*

$$M_1(t) \leq C\{\|u_0\|_1 + M(t)^2\},$$

holds uniformly for $t \in [0,\tau]$ with $C > 0$ independent of τ.

Proof. We write (5.1) for $s = 0$ as

$$\sigma_1(t) = e^{t\Lambda}\mathscr{P}(0)u_0 + I(t),$$

where

$$I(t) = \int_0^t e^{(t-z)\Lambda}\mathscr{P}(z)F(z)dz.$$

(4.13) yields

$$\|\partial_{x'}^k e^{t\Lambda}\mathscr{P}(0)u_0\|_2 \le C(1+t)^{-\frac{n-1}{4}-\frac{k}{2}}\|u_0\|_1,$$

for $k = 0, 1$. Next, we estimate $I(t)$ which we write it as

$$I(t) = I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_0^{\frac{t}{2}} e^{(t-z)\Lambda}\mathscr{P}(z)F(z)dz,$$

$$I_2(t) = \int_{\frac{t}{2}}^t e^{(t-z)\Lambda}\mathscr{P}(z)F(z)dz.$$

By Lemma 6.1 (ii), we have

$$e^{(t-z)\Lambda}\mathscr{P}(z)\boldsymbol{F}(z) = \mathrm{div}'e^{(t-z)\Lambda}\{-[\phi w']_1 + \mathscr{P}^{(1)}\boldsymbol{F} + \nabla'\mathscr{P}^{(2)}\boldsymbol{F}\}(z).$$

It then follows from (4.14) and Lemma 6.3 that

$$\begin{aligned}\|\partial_{x'}^k I_1(t)\|_2 &\le C\int_0^{\frac{t}{2}}(1+t-z)^{-\frac{n+1}{4}-\frac{k}{2}}\{\|[\phi w'](z)\|_1 + \|\boldsymbol{F}(z)\|_1\}\,dz \\ &\le CM(t)^2\int_0^{\frac{t}{2}}(1+t-z)^{-\frac{n+1}{4}-\frac{k}{2}}(1+z)^{-\frac{n-1}{2}}dz \le C(1+t)^{-\frac{n-1}{4}-\frac{k}{2}}M(t)^2,\end{aligned}$$

for $k = 0, 1$.

As for $I_2(t)$, using (6.1) and Lemma 6.1 (ii) we write $\mathscr{P}(z)\boldsymbol{F}(z)$ as

$$\mathscr{P}\boldsymbol{F} = -\mathrm{div}'[\phi w']_1 + (\mathscr{P}^{(1)} + \nabla'\mathscr{P}^{(2)})\cdot\nabla'(\sigma_1)^2\boldsymbol{F}_1 + (\mathrm{div}'\mathscr{P}^{(1)} + \Delta'\mathscr{P}^{(2)})\boldsymbol{F}_2.$$

It then follows from (4.14) and Lemma 6.3 that

$$\begin{aligned}\|\partial_{x'}^k I_2(t)\|_2 &\le C\int_{\frac{t}{2}}^t(1+t-z)^{-\frac{n-1}{4}-\frac{k}{2}}\{\|\mathrm{div}'[\phi w'](z)\|_1 + \|\nabla'(\sigma_1(z))^2\|_1 + \|\boldsymbol{F}_2(z)\|_1\}dz \\ &\le CM(t)^2\int_{\frac{t}{2}}^t(1+t-z)^{-\frac{n-1}{4}-\frac{k}{2}}(1+z)^{-\frac{n}{2}}dz \le C(1+t)^{-\frac{n-1}{4}-\frac{k}{2}}M(t)^2,\end{aligned}$$

for $k = 0, 1$. We thus obtain

$$\sum_{k=0}^1(1+t)^{\frac{n-1}{4}+\frac{k}{2}}\|\partial_{x'}^k\sigma_1(t)\|_2 \le C\{\|u_0\|_1 + M(t)^2\}. \tag{6.2}$$

It remains to estimate time derivatives. From (5.1) we see that

$$\partial_t\sigma_1(t) = \Lambda\sigma_1(t) + \mathscr{P}(t)\boldsymbol{F}(t). \tag{6.3}$$

It then follows from Lemma 6.3 (vi) and previous result that

$$\|\partial_t\sigma_1(t)\|_2 \le C\{\|\partial_{x'}\sigma_1(t)\|_2 + \|\mathscr{P}(t)\boldsymbol{F}(t)\|_2\} \le C(1+t)^{-\frac{n+1}{4}}\{\|u_0\|_1 + M(t)^2\}.$$

Concerning $\|\partial_t^{j+1}\sigma_1(t)\|_2$ for $j = 1, \ldots, [\frac{m}{2}]-1$, we obtain from (6.3)

$$\|\partial_t^{j+1}\sigma_1(t)\|_2 \le C\{\|\partial_t^j\sigma_1(t)\|_2 + \|\partial_t^j(\mathscr{P}(t)\boldsymbol{F}(t))\|_2\}.$$

Since

$$\|\partial_t^j \boldsymbol{F}(t)\|_2 \le C(1+t)^{-\frac{n+1}{4}} M(t)^2,$$

for $0 \le 2j \le m-2$ as we see in Propositions 8.5 (i)–(iii) and 8.6 (i), we find by induction on j, that estimate

$$\|\partial_t^{j+1}\sigma_1(t)\|_2 \le C_j(1+t)^{-\frac{n+1}{4}}\{\|u_0\|_1 + M(t)^2\}, \tag{6.4}$$

holds for $j = 0, 1, \dots, [\frac{m}{2}] - 1$.

The desired result now follows from (6.2) and (6.4). This completes the proof.

$\square$

7 Estimates on $\widetilde{P}_\infty u(t)$

In this section we prove estimate (5.10) for σ_∞ and u_∞ by a variant of Matsumura-Nishida energy method as in the case of stationary parallel flow ([7]). Since coefficients of the linearized operator depend on time some extra terms arise in contrast to [7]. We omit the proofs that can be obtained as modification of those in [7]. In this section we assume that $\nu \ge \nu_2$ and $\gamma^2/(\nu+\widetilde{\nu}) \ge \gamma_2^2$ unless further restricted.

First, let us show the following inequality.

Proposition 7.1 *There exists $\nu_0 \ge \nu_2$ and $\gamma_0 \ge \gamma_2$ such that if $\nu \ge \nu_0$ and $\gamma^2/(\nu+\widetilde{\nu}) \ge \gamma_0^2$ the solution $u(t)$ of* (4.1) *in $Z^m(\tau)$ satisfies*

$$\frac{d}{dt}\widetilde{E}(t) + 2D(t) \le \widetilde{R}(t). \tag{7.1}$$

Here, $\widetilde{E}(t)$, $D(t)$ and $\widetilde{R}(t)$ are quantities such that

(i) $\widetilde{E}(t) + [\![\partial_{x_n}^2 w_\infty(t)]\!]_{m-2}^2$ is equivalent to $E_\infty(t)$,

(ii) $D(t)$ is equivalent to $D_\infty(t)$,

(iii) $\widetilde{R}(t)$ satisfies estimate (5.11)*.*

We introduce some quantities. Let $E^{(0)}[\widetilde{P}_\infty u]$ and $D^{(0)}[w]$ be defined by

$$E^{(0)}[\widetilde{P}_\infty u] = \frac{\alpha_0}{\gamma^2}\|\sigma_\infty\|_2^2 + \frac{1}{\gamma^2}\|\sqrt{\frac{P'(\rho_p)}{\gamma^2\rho_p}}\phi_\infty\|_2^2 + \|\sqrt{\rho_p}w_\infty\|_2^2,$$

for $\widetilde{P}_\infty(t) = \sigma_\infty u^{(0)} + u_\infty$ with $u_\infty = {}^T(\phi_\infty, w_\infty)$; and

$$D^{(0)}[w_\infty] = \nu\|\nabla w_\infty\|_2^2 + \widetilde{\nu}\|\mathrm{div}\, w_\infty\|_2^2.$$

Note that,

$$\langle Au(t), u(t)\rangle_\Omega = D^{(0)}[w(t)],$$

for $u = {}^T(\phi, w) \in Z^m(\tau)$, and

$$\langle B(t)u(t), v(t)\rangle_\Omega = -\langle u(t), B(t)v(t)\rangle_\Omega,$$

for $u, v \in Z^m(\tau)$, $\widetilde{Q}u|_{x_n=0,1} = \widetilde{Q}v|_{x_n=0,1} = 0$. In particular,

$$\langle B(t)u(t), u(t)\rangle_\Omega = 0,$$

for $u \in Z^m(\tau)$, $\widetilde{Q}u|_{x_n=0,1} = 0$.

We denote the tangential derivatives $\partial_t^j\partial_{x'}^k$ by $T_{j,k}$:

$$T_{j,k}u = \partial_t^j\partial_{x'}^k u.$$

In this section we often use $\|w^{(0),1}(t)\|_{C^{m+1}(\Omega)} = O(\frac{1}{\gamma^2})$ in calculations (see Lemma 4.9). It is straightforward to see that following lemma holds true.

Lemma 7.2 *There hold the following assertions.*

(i)
$$\|T_{j,k+1}\sigma_\infty\|_2^2 \le \|T_{j,1}\sigma_\infty\|_2^2,\ k \ge 0,\ 2j \le m,$$

$$\|T_{j,k}\phi_\infty\|_2^2 \le C\|\partial_x T_{j,k}\phi_\infty\|_2^2,\ 2j+k \le m-1,$$

$$\|T_{j,k}w_\infty\|_2^2 \le C\|\partial_x T_{j,k}w_\infty\|_2^2,\ 2j+k \le m-1.$$

(ii)
$$\|[Q_0\widetilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty\|_2^2 \le C(\|\partial_{x'}\sigma_\infty\|_2^2 + \|\partial_{x'}\phi_\infty\|_2^2 + \gamma^4\|\partial_{x'}w_\infty\|_2^2).$$

(iii) If $w_\infty^2|_{x_n=0,1} = 0$, then $[Q_0\widetilde{B}u_\infty]_\infty = [Q_0 B u_\infty]_\infty = [v_p^1\partial_{x_1}\phi_\infty + \gamma^2 \mathrm{div}\,(\rho_p w_\infty)]_\infty$.

(iv) If $w_\infty^2|_{x_n=0,1} = 0$ and $2j+k \le m$, then

$$\|\partial_{x'}^k\partial_t^j[Q_0\widetilde{B}(\sigma_\infty u^{(0)} + u_\infty)]_\infty\|_2^2$$

$$\le C\sum_{i=0}^{j}(\|\partial_{x'}^p\partial_t^i\sigma_\infty\|_2^2 + \|\partial_{x'}^q\partial_t^i\phi_\infty\|_2^2) + \gamma^4\|\mathrm{div}\,(\partial_{x'}^r\partial_t^j w_\infty)\|_2^2 + \gamma^4|\partial_{x_n}\rho_p|_\infty^2\|\partial_{x'}^s\partial_t^j w_\infty\|_2^2),$$

for $0 \le p,q \le k+1$, $0 \le r,s \le k$.

We begin with L^2-energy estimates for tangential derivatives. We set

$$\sigma_* = \sigma_1 + \sigma_\infty,\quad \phi_* = \phi_1 + \phi_\infty,\quad w_* = w_1 + w_\infty,$$

$$u_* = {}^T(\phi_*, w_*) = u_1 + u_\infty.$$

We write $\widetilde{Q}\boldsymbol{F} = {}^T(0, \boldsymbol{f})$ in the form

$$\widetilde{Q}\boldsymbol{F} = \widetilde{\boldsymbol{F}}_0 + \widetilde{\boldsymbol{F}}_1 + \widetilde{\boldsymbol{F}}_2 + \widetilde{\boldsymbol{F}}_3.$$

Here, $\widetilde{\boldsymbol{F}}_l = {}^T(0, \boldsymbol{f}_l)$, $l = 0,1,2,3$, with

$$\boldsymbol{f}_0 = -w\cdot\nabla w - f_1(\rho_p,\phi)\Delta'\sigma_* w^{(0),1}\boldsymbol{e}_1 - f_2(\rho_p,\phi)\nabla(\partial_{x_1}\sigma_* w^{(0),1})$$

$$+\boldsymbol{f}_{01}(x_n,t,\phi)\phi\sigma_* + \boldsymbol{f}_{02}(x_n,\phi)\phi\nabla'\sigma_* + \boldsymbol{f}_{03}(x_n,t,\phi)\phi\phi_*,$$

$$\boldsymbol{f}_1 = -f_1(\rho_p,\phi)\Delta w_* = -\mathrm{div}\,(f_1(\rho_p,\phi)\nabla w_*) + {}^T(\nabla w_*)\nabla(f_1(\rho_p,\phi)),$$

$$\boldsymbol{f}_2 = -f_2(\rho_p,\phi)\nabla\mathrm{div}\, w_* = -\nabla(f_2(\rho_p,\phi)\mathrm{div}\, w_*) + (\mathrm{div}\, w_*)\nabla(f_2(\rho_p,\phi)),$$

$$\boldsymbol{f}_3 = -f_3(x_n,\phi)\phi\nabla\phi_*.$$

Here, ∇w_* denotes the $n\times n$ matrix $(\partial_{x_i}w_*^j)$; $f_1 = \frac{\nu\phi}{\rho_p(\gamma^2\rho_p+\phi)}$; $f_2 = \frac{\widetilde{\nu}\phi}{\rho_p(\gamma^2\rho_p+\phi)}$; and $\boldsymbol{f}_{0l}(x_n,t,\phi)$, $l=1,2,3$ and $f_3(x_n,\phi)$ are smooth functions of x_n, t and ϕ.

Proposition 7.3 *There exists $\nu_3 > \nu_2$ such that for $\nu \ge \nu_3$ the following estimate holds for $0 \le 2j+k \le m$:*

$$\frac{1}{2}\frac{d}{dt}E^{(0)}[T_{j,k}\widetilde{P}_\infty u] + \frac{1}{2}D^{(0)}[T_{j,k}w_\infty] \tag{7.2}$$

$$\le R_{j,k}^{(1)} + C\{(\frac{\nu}{\gamma^4} + \frac{1}{\gamma^4\nu} + \frac{1}{\gamma^2})\sum_{i=0}^{j}\|T_{i,k}\phi_\infty\|_2^2 + (\frac{\nu+\widetilde{\nu}}{\gamma^4} + \frac{1}{\nu\gamma^4} + \frac{1}{\gamma^2})\sum_{i=0}^{j}\|T_{i,k+1}\sigma_\infty\|_2^2$$

$$+\frac{1}{\gamma^2}\sum_{i=0}^{j-1}\|T_{i,k+1}\phi_\infty\|_2^2 + \frac{1}{\gamma^2}(1-\delta_{j0})\|\partial_t T_{j-1,k}\sigma_\infty\|_2^2 + \frac{1}{\nu^2}\sum_{i=0}^{j-1}D^{(0)}[T_{i,k}w_\infty]\},$$

where δ_{j0} denotes Kronecker's delta and $R_{j,k}^{(1)}$ is given by

$$R_{j,k}^{(1)} = \frac{\alpha_0}{\gamma^2}([Q_0 T_{j,k}\boldsymbol{F}]_\infty, T_{j,k}\sigma_\infty) - \frac{\alpha_0}{\gamma^2}([Q_0 T_{j,k}(\mathbb{P}\boldsymbol{F})]_\infty, T_{j,k}\sigma_\infty) + \widetilde{R}_{j,k}^{(1)} - \langle T_{j,k}([Q_0\boldsymbol{F}]_\infty u^{(0)}), T_{j,k}u_\infty\rangle_\Omega$$

$$-\langle T_{j,k}(\mathbb{P}\boldsymbol{F}), T_{j,k}u_\infty\rangle_\Omega + \langle T_{j,k}([Q_0(\mathbb{P}\boldsymbol{F})]_\infty u^{(0)}), T_{j,k}u_\infty\rangle_\Omega.$$

Here,

$$\widetilde{R}_{j,k}^{(1)} = \langle T_{j,k}\boldsymbol{F}, T_{j,k}u_\infty\rangle_\Omega,$$

when $2j+k \le m-1$, *and*

$$\widetilde{R}_{j,k}^{(1)} = -(T_{j,k}(\phi \operatorname{div} w), T_{j,k}\phi_\infty \frac{P'(\rho_p)}{\gamma^4\rho_p}) + \frac{1}{2}(\operatorname{div}(\frac{P'(\rho_p)}{\gamma^4\rho_p}w), |T_{j,k}\phi_\infty|^2)$$

$$-(w\nabla T_{jk}(\sigma_*\phi^{(0)} + \phi_1), T_{jk}\phi_\infty \frac{P'(\rho_p)}{\gamma^4\rho_p}) - ([T_{j,k}, w]\nabla\phi, T_{j,k}\phi_\infty \frac{P'(\rho_p)}{\gamma^4\rho_p})$$

$$+(T_{j,k}\boldsymbol{f}_0, T_{j,k}w_\infty\rho_p) + \sum_{l=1}^{3}\langle T_{j,k}\boldsymbol{f}_l, T_{j,k}w_\infty\rho_p\rangle_{-1},$$

when $2j+k=m$. *Here and in what follows, for* $G = g + \partial_{x_j}\widetilde{g}$ *with* $g, \widetilde{g} \in L^2$ *and* $v \in H_0^1$, $\langle G, v\rangle_{-1}$ *denotes*

$$\langle G, v\rangle_{-1} = (g, v) - (\widetilde{g}, \partial_{x_j}v).$$

Proof. We apply $T_{j,k}$ to (5.2) and (5.3). We then take the inner products of the resulting equations with $T_{j,k}\sigma_\infty$ and $T_{j,k}u_\infty$, respectively. Integration by parts together with symmetric properties of A and B gives us the desired result in the same manner as in [7, Proposition 7.4]. □

We next derive the H^1-parabolic estimates for w_∞. We define $J[\widetilde{P}_\infty u]$ by

$$J[\widetilde{P}_\infty u] = -2\langle \sigma_\infty u^{(0)} + u_\infty, B\widetilde{Q}u_\infty\rangle_\Omega \text{ for } \widetilde{P}_\infty u = \sigma_\infty u^{(0)} + u_\infty.$$

A direct computation shows that if $\gamma^2 \ge 1$ then

$$|J[\widetilde{P}_\infty u]| \le \frac{b_0\gamma^2}{\nu}E^{(0)}[\widetilde{P}_\infty] + \frac{1}{2}D^{(0)}[w_\infty],$$

for some constant $b_0 > 0$.

Let b_1 be a positive constant (to be determined later) and define $E^{(1)}[\widetilde{P}_\infty u]$ by

$$E^{(1)}[\widetilde{P}_\infty u] = \frac{2b_1\gamma^2}{\nu}E^{(0)}[\widetilde{P}_\infty u] + D^{(0)}[w_\infty] + J[\widetilde{P}_\infty u].$$

Note that if $b_1 \ge b_0$ then $E^{(1)}[\widetilde{P}_\infty u]$ is equivalent to $E^{(0)}[\widetilde{P}_\infty u] + D^{(0)}[w_\infty]$.

Proposition 7.4 *There exists* $b_1 \ge \max\{b_0, 8C_0\}$ *such that if* $\nu \ge \nu_3$, $\gamma^2 \ge 1$ *and* $\frac{\gamma^2}{\nu+\widetilde{\nu}} \ge \max\{1, \gamma_2^2\}$ *then the following estimate holds for* $0 \le 2j+k \le m-1$,

$$\frac{1}{2}\frac{d}{dt}E^{(1)}[T_{j,k}\widetilde{P}_\infty u] + \frac{b_1\gamma^2}{\nu}\frac{3}{4}D^{(0)}[T_{j,k}w_\infty] + \frac{1}{2}\|\sqrt{\rho_p}T_{j,k}\partial_t w_\infty\|_2^2 \le R_{j,k}^{(2)} \tag{7.3}$$

$$+C\sum_{i=0}^{j}\{(\frac{\nu^2}{\gamma^4} + \frac{1}{\nu})\|T_{i,k}\phi_\infty\|_2^2 + \left(\frac{\nu+\widetilde{\nu}}{\gamma^2} + \frac{1}{\nu}\right)\|T_{i,k+1}\sigma_\infty\|_2^2 + \frac{1}{\nu}\|T_{i,k+1}\phi_\infty\|_2^2$$

$$+\frac{1}{\nu}(1-\delta_{j0})\|\partial_t T_{j-1,k}\sigma_\infty\|_2^2\} + C_0\frac{\gamma^2}{\nu}(1+\frac{C}{\nu^2})\sum_{i=0}^{j-1}D^{(0)}[T_{i,k}w_\infty].$$

where

$$R_{j,k}^{(2)} = \frac{2b_1\gamma^2}{\nu}R_{j,k}^{(1)} + C\|T_{j,k}\boldsymbol{F}\|_2^2.$$

Proof. We apply $T_{j,k}$ to (5.3) and take inner product with $\partial_t T_{j,k}\widetilde{Q}u_\infty$ to obtain the desired result in the same manner as in [7, Proposition 7.5]. □

As for the disipative estimates for x_n-derivatives of ϕ_∞, we have the following inequality.

Proposition 7.5 *The following estimate holds for* $0 \le 2j+k+l \le m-1$*:*

$$\frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2}\|\sqrt{\frac{P'(\rho_p)}{\gamma^2\rho_p}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\|_2^2 + \frac{1}{2(\nu+\widetilde{\nu})}\|\frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}^{l+1}T_{j,k}\phi_\infty\|_2^2 \le R_{j,k,l}^{(3)} + C\frac{\nu+\widetilde{\nu}}{\gamma^4}\|K_{j,k,l}\|_2^2, \tag{7.4}$$

where

$$R_{j,k,l}^{(3)} = \left|\frac{1}{2}(\mathrm{div}\left(\frac{P'(\rho_p)}{\gamma^4\rho_p}w\right), |T_{j,k}\partial_{x_n}^{l+1}\phi_\infty|^2)\right| + C\frac{\nu+\widetilde{\nu}}{\gamma^4}\|H_{j,k,l}\|_2^2,$$

with

$$\|H_{j,k,l}\|_2^2 \le C\{\|[T_{j,k}\partial_{x_n}^{l+1}, w]\cdot\nabla\phi_\infty\|_2^2 + \|T_{j,k}\partial_{x_n}^{l+1}(\widetilde{Q_0P_\infty\boldsymbol{F}})\|_2^2 + \|\frac{\gamma^2\rho_p^2}{\nu+\widetilde{\nu}}T_{j,k}\partial_{x_n}^{l}(Q_nP_\infty\boldsymbol{F})\|_2^2\},$$

and

$$\widetilde{Q_0P_\infty\boldsymbol{F}} = -\phi\,\mathrm{div}\,w - w\cdot\nabla(\sigma_*\phi^{(0)}+\phi_1) - \{Q_0\mathbb{P}\boldsymbol{F} + [Q_0P_\infty^{(0)}\boldsymbol{F}]_\infty\phi^{(0)}\}$$

Here, $K_{j,k,l}$ *is estimated as*

$$\begin{aligned}\frac{\nu+\widetilde{\nu}}{\gamma^4}\|K_{j,k,l}\|_2^2 \le C\frac{\nu+\widetilde{\nu}}{\gamma^2}\{&\frac{\nu^2}{\nu+\widetilde{\nu}}\|T_{j,k+1}\partial_{x_n}^{l}\partial_x w_\infty\|_2^2 + \frac{1}{\nu+\widetilde{\nu}}\|\sqrt{\rho_p}T_{j,k}\partial_{x_n}^{l}\partial_t w_\infty\|_2^2\\ &+\frac{\nu^2}{\gamma^2}(\sum_{q=0}^{l-1}\|T_{j,k+1}\partial_{x_n}^{q}\partial_x w_\infty\|_2^2 + \sum_{q=0}^{l}\|T_{j,k}\partial_{x_n}^{q}\partial_x w_\infty\|_2^2 + \sum_{i=0}^{j}\|T_{i,k+1}w_\infty\|_2^2)\\ &+\frac{1}{\nu+\widetilde{\nu}}\sum_{i=0}^{j}\sum_{q=0}^{l}\|T_{i,k+1}\partial_{x_n}^{q}w_\infty\|_2^2\\ &+\frac{1}{\gamma^2}\left(\sum_{i=0}^{j}\sum_{q=0}^{l+1}{}_{(i,q)\ne(j,l+1)}\|T_{i,k+1}\partial_{x_n}^{q}\phi_\infty\|_2^2 + \sum_{i=0}^{j}\|T_{i,k+1}\sigma_\infty\|_2^2 + \sum_{q=0}^{l}\|\partial_{x_n}^{q}T_{j,k}\phi_\infty\|_2^2\right)\}.\end{aligned}$$

Proof. We obtain the desired result in the same manner as in [7, Proposition 7.6]. □

The following estimate for the material derivative of ϕ_∞ plays an important role to obtain the dissipative estimate for higher order x_2-derivatives of w_∞. We denote the material derivative of ϕ_∞ by $\dot{\phi}_\infty$:

$$\dot{\phi}_\infty = \partial_t\phi_\infty + (v_p+w)\cdot\nabla\phi_\infty.$$

Proposition 7.6 *The following estimates hold for* $0 \le 2j+k+l \le m-1$*:*

(i)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2}\|\sqrt{\frac{P'(\rho_p)}{\gamma^2\rho_p}}T_{j,k}\partial_{x_n}^{l+1}\phi_\infty\|_2^2 + \frac{1}{4(\nu+\widetilde{\nu})}\|\frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}^{l+1}T_{j,k}\phi_\infty\|_2^2 \\ +c_0\frac{\nu+\widetilde{\nu}}{\gamma^4}\|T_{j,k}\partial_{x_n}^{l+1}\dot{\phi}_\infty\|_2^2 \le R_{j,k,l}^{(3)} + C\frac{\nu+\widetilde{\nu}}{\gamma^4}\|K_{j,k,l}\|_2^2,\end{aligned} \tag{7.5}$$

where c_0 *is a positive constant and* $R_{j,k,l}^{(3)}$ *and* $K_{j,k,l}$ *satisfy the same estimates as in Proposition 7.5.*

(ii) Let $0 \le q \le k$ *and* $0 \le 2j+k \le m$*. Then*

$$\begin{aligned}\frac{\nu+\widetilde{\nu}}{\gamma^4}\|T_{j,k}\dot{\phi}_\infty\|_2^2 \le C\{R_{j,k}^{(4)} + D^{(0)}[T_{j,k}w_\infty] + \frac{\nu^2(\nu+\widetilde{\nu})}{\gamma^4}\|T_{j,k}w_\infty\|_2^2 \\ +\frac{\nu+\widetilde{\nu}}{\gamma^4}\sum_{i=0}^{j}\|T_{i,k+1}\sigma_\infty\|_2^2 + \frac{\nu+\widetilde{\nu}}{\gamma^4}\sum_{i=0}^{j}\|T_{i,q}\phi_\infty\|_2^2\},\end{aligned} \tag{7.6}$$

where $R_{j,k}^{(4)} = \frac{\nu+\widetilde{\nu}}{\gamma^4}\|T_{j,k}\widetilde{Q_0P_\infty\boldsymbol{F}}\|_2^2$.

Proof. The desired result is obtained in the same manner as in [7, Proposition 7.7]. $\square$

Let us derive the dissipative estimates for σ_∞.

Proposition 7.7 *Let $\gamma^2/(\nu+\widetilde{\nu}) \geq \max\{1, \gamma_2^2\}$, then there holds the following estimate for $0 \leq 2j+k \leq m-1$:*

$$\frac{1}{2}\frac{d}{dt}\frac{\nu}{\gamma^2(\nu+\widetilde{\nu})}\|T_{j,k}\sigma_\infty\|_2^2 + \frac{\alpha_1}{2(\nu+\widetilde{\nu})}\|\nabla' T_{j,k}\sigma_\infty\|_2^2 \leq R_{j,k}^{(5)} \tag{7.7}$$

$$+C\frac{\nu^2}{\gamma^4(\nu+\widetilde{\nu})}(1-\delta_{j0})\|\partial_t T_{j-1,k}\sigma_\infty\|_2^2 + \frac{1}{\nu+\widetilde{\nu}}(\frac{\alpha_1}{16} + C\frac{\nu+\widetilde{\nu}}{\gamma^2})\sum_{i=0}^{j-1}\|T_{i,k+1}\sigma_\infty\|_2^2$$

$$+C\{\frac{1}{\nu+\widetilde{\nu}}\|\sqrt{\rho_p}T_{j,k}\partial_t w_\infty\|_2^2 + \sum_{i=0}^{j} D^{(0)}[T_{i,k}w_\infty] + \frac{1}{\nu+\widetilde{\nu}}\sum_{i=0}^{j}\|\frac{P'(\rho_p)}{\gamma^2}T_{i,p}\partial_{x_n}\phi_\infty\|_2^2\},$$

where $\alpha_1 > 0$ is a constant, p is any integer satisfying $0 \leq p \leq k$, and

$$R_{j,k}^{(5)} = \frac{\nu}{\gamma^2(\nu+\widetilde{\nu})}(Q_0 T_{j,k}(P_\infty^{(0)}\boldsymbol{F}), T_{j,k}\sigma_\infty) - \frac{1}{\nu+\widetilde{\nu}}(\mathrm{div}'[\rho_p(-\Delta)^{-1}(\rho_p T_{j,k}Q'(P_\infty\boldsymbol{F}))]_\infty, T_{j,k}\sigma_\infty).$$

Here, $(-\Delta)^{-1}$ is the inverse of $-\Delta$ on $L^2(\Omega)$ with domain $D(-\Delta) = H^2(\Omega)\cap H_0^1(\Omega)$.

Proof. The desired result is obtained in the same manner as in [7, Proposition 7.8]. $\square$

Next, we estimate the higher order derivatives.

Proposition 7.8 *If $\nu \geq 1$ then there holds the following estimate for $0 \leq 2j+k+l \leq m-1$:*

$$\frac{\nu^2}{\nu+\widetilde{\nu}}\|\partial_x^{l+2}T_{j,k}w_\infty\|_2^2 + \frac{1}{\nu+\widetilde{\nu}}\|\partial_x^{l+1}T_{j,k}\phi_\infty\|_2^2 \leq CR_{j,k,l}^{(6)} + C\{(\frac{1}{\nu+\widetilde{\nu}} + \frac{\nu+\widetilde{\nu}}{\gamma^4})\sum_{i=0}^{j}\|T_{i,k+1}\sigma_\infty\|_2^2 \tag{7.8}$$

$$+\frac{\nu+\widetilde{\nu}}{\gamma^4}\sum_{i=0}^{j}\|T_{i,k}\phi_\infty\|_{H^l}^2 + \frac{\nu+\widetilde{\nu}}{\gamma^4}\|T_{j,k}\dot{\phi}_\infty\|_{H^{l+1}}^2 + \frac{1}{\nu+\widetilde{\nu}}\|\partial_t T_{j,k}w_\infty\|_{H^l}^2$$

$$+(\frac{1}{\nu+\widetilde{\nu}} + \frac{\nu^2(\nu+\widetilde{\nu})}{\gamma^4})\sum_{i=0}^{j}\|T_{i,k}w_\infty\|_{H^{l+1}}^2 + D^{(0)}[T_{j,k}w_\infty]\},$$

where

$$R_{j,k,l}^{(6)} = \frac{\nu+\widetilde{\nu}}{\gamma^4}\|T_{j,k}\widetilde{Q_0 P_\infty}\boldsymbol{F}\|_{H^{l+1}}^2 + \frac{1}{\nu+\widetilde{\nu}}\|T_{j,k}(\widetilde{Q}P_\infty\boldsymbol{F})\|_{H^l}^2.$$

Proof. We use the estimates for the Stokes system. Let ${}^T(\widetilde{\phi}, \widetilde{w})$ be the solution of the Stokes system

$$\mathrm{div}\,\widetilde{w} = F \quad \text{in } \Omega,$$

$$-\Delta\widetilde{w} - \nabla\widetilde{\phi} = G \quad \text{in } \Omega,$$

$$\widetilde{w}|_{\delta\Omega} = 0.$$

Then for any $l \in \mathbb{Z}$, $l \geq 0$, there exists a constant $C > 0$ such that

$$\|\partial_x^{l+2}\widetilde{w}\|_2^2 + \|\partial_x^{l+1}\widetilde{\phi}\|_2^2 \leq C\{\|F\|_{H^{l+1}}^2 + \|G\|_{H^l}^2 + \|\partial_x\widetilde{w}\|_2^2\}, \tag{7.9}$$

(see, e.g., [3][4, Appendix]).

We rewrite (5.3) in the form of Stokes system with $\widetilde{w} = T_{j,k}w_\infty$ and $\widetilde{\phi} = \frac{P'(\rho_p)}{\nu\gamma^2}T_{j,k}\phi_\infty$. The desired result is then obtained by using (7.9) (cf. [7, Proposition 7.9]). $\square$

At last we estimate the time derivatives of σ_∞ and ϕ_∞.

Proposition 7.9 *(i) If $0 \leq 2j+k \leq m-1$, then there holds the following estimate:*

$$\|\partial_t T_{j,k}\sigma_\infty\|_2^2 \leq C\{R_{j,k}^{(7)} + \sum_{i=0}^{j}(\|T_{i,k+1}\sigma_\infty\|_2^2 + \|T_{i,k+1}\phi_\infty\|_2^2) + \gamma^4\|T_{j,k+1}w_\infty\|_2^2\}, \tag{7.10}$$

Here, $R_{j,k}^{(7)} = \|[Q_0 T_{j,k}(P_\infty^{(0)}\boldsymbol{F})]_\infty\|_2^2$.

(ii) If $0 \leq k+2j \leq m-1$ then there holds the following estimate:

$$\|\partial_t^{j+1}\phi_\infty\|_{H^k}^2 \leq C\{R_j^{(8)} + \sum_{i=0}^{j}(\|\partial_{x'}\partial_t^i\phi_\infty\|_{H^k}^2 + \|\partial_{x'}\partial_t^i\sigma_\infty\|_2^2) + \gamma^4\|\partial_x\partial_t^j w_\infty\|_{H^k}^2\} \tag{7.11}$$

Here, $R_{j,k}^{(8)} = \|\partial_t^j(Q_0 P_\infty \boldsymbol{F})\|_{H^k}^2$.

Proof. The estimates (7.10) and (7.11) follow from (5.2) and the first line of (5.3). $\square$

Proposition 7.1 now follows from combination of results in Propositions 7.3–7.9.

Proof of Proposition 7.1. Let us define

$$\widetilde{E}^{(0)}(t) = \sum_{\substack{2j+k\leq m\\ 2j\neq m}} E^{(0)}[T_{j,k}\widetilde{P}_\infty u(t)],\quad \widetilde{E}^{(1)}(t) = \sum_{2j+k\leq m-1} E^{(1)}[T_{j,k}\widetilde{P}_\infty u(t)],$$

$$E^{(2)}(t) = \sum_{2j+k\leq m-1}\frac{1}{\gamma^2}\|\sqrt{\frac{P'(\rho_p)}{\gamma^2\rho_p}}\partial_{x_n}T_{j,k}\phi_\infty(t)\|_2^2,\quad E^{(3)}(t) = \sum_{2j+k\leq m-1}\frac{\nu}{\gamma^2(\nu+\widetilde{\nu})}\|T_{j,k}\sigma_\infty(t)\|_2^2,$$

and

$$\widetilde{D}^{(0)}(t) = \sum_{\substack{2j+k\leq m\\ 2j\neq m}} D^{(0)}[T_{j,k}w_\infty(t)],$$

$$D^{(1)}(t) = \sum_{2j+k\leq m-1}\left(\frac{3b_1\gamma^2}{2\nu(\nu+\widetilde{\nu})}D^{(0)}[T_{j,k}w_\infty(t)] + \frac{1}{\nu+\widetilde{\nu}}\|\sqrt{\rho_p}T_{j,k}\partial_t w_\infty(t)\|_2^2\right),$$

$$D^{(2)}(t) = \sum_{2j+k\leq m-1}\left(\frac{1}{2(\nu+\widetilde{\nu})}\|\frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}T_{j,k}\phi_\infty(t)\|_2^2 + \min\{1,2c_0\}\frac{\nu+\widetilde{\nu}}{\gamma^4}\|T_{j,k}\dot{\phi}_\infty(t)\|_{H^1}^2\right),$$

$$D^{(3)}(t) = \sum_{2j+k\leq m-1}\frac{\alpha_1}{\nu+\widetilde{\nu}}\|\nabla' T_{j,k}\sigma_\infty(t)\|_2^2.$$

Let b_l, $l = 2,\ldots,5$, be positive numbers and let us consider

$$\sum_{2j+k\leq m|2j\neq m}\{2\times(7.2) + 2b_2\times(7.6)\}$$

$$+\sum_{2j+k\leq m-1}\{\frac{2}{\nu+\widetilde{\nu}}\times(7.3) + 2b_2\times(7.5)_{l=0} + 2b_3\times(7.7) + b_4\times(7.8)_{l=0}\}$$

$$+\sum_{2j+k\leq m-1}\frac{1}{(\nu+\widetilde{\nu})\gamma^2}b_5\times((7.10)+(7.11)) + 2\frac{b_6}{\nu+\widetilde{\nu}}\times(7.2)_{2j=m}.$$

Then we obtain

$$\frac{d}{dt}E^{(4)}(t) + D^{(4)}(t) + \frac{1}{(\nu+\widetilde{\nu})\gamma^2}b_5\sum_{2j+k\leq m-1}(\|\partial_t T_{j,k}\sigma_\infty\|_2^2 + \|\partial_t^{j+1}\phi_\infty(t)\|_k^2) \tag{7.12}$$

$$\leq \sum_{j=1}^{8} R^{(j)}(t) + RHS.$$

Here,

$$E^{(4)}(t) = \widetilde{E}^{(0)} + \frac{1}{\nu+\widetilde{\nu}}\widetilde{E}^{(1)}(t) + b_2 E^{(2)}(t) + b_3 E^{(3)}(t) + \frac{b_6}{\nu+\widetilde{\nu}} E^{(0)}[\partial_t^{[\frac{m}{2}]}\widetilde{P}_\infty u(t)],$$

$$D^{(4)} = \widetilde{D}^{(0)}(t) + D^{(1)}(t) + b_2 D^{(2)}(t) + b_3 D^{(3)}(t) + b_4 \sum_{2j+k\leq m-1} (\frac{\nu^2}{\nu+\widetilde{\nu}}\|\partial_x^2 T_{j,k} w_\infty\|_2^2 + \frac{1}{\nu+\widetilde{\nu}}\|\partial_x T_{j,k}\phi_\infty\|_2^2)$$

$$+\frac{b_6}{\nu+\widetilde{\nu}} D^{(0)}[\partial_t^{[\frac{m}{2}]} w_\infty],$$

and

$$R^{(1)} = \sum_{2j+k\leq m} R^{(1)}_{j,k},\ R^{(p)} = \sum_{2j+k\leq m-1} R^{(p)}_{j,k},\ p = 2,5,7,8,\ R^{(4)} = \sum_{\substack{2j+k\leq m\\ 2j\neq m}} R^{(4)}_{j,k},$$

$$R^{(p)} = \sum_{2j+k\leq m-1} R^{(p)}_{j,k,0},\ p = 3,6,$$

with

$$RHS = C(\frac{\nu+\widetilde{\nu}}{\gamma^2} + \frac{1}{\nu})\frac{1}{\nu+\widetilde{\nu}} \sum_{2j+k\leq m-1} \|\partial_x T_{j,k}\phi_\infty\|_2^2 + C(\frac{\nu+\widetilde{\nu}}{\gamma^2} + \frac{1}{\nu} + b_4)\frac{1}{\nu+\widetilde{\nu}} \sum_{2j+k\leq m-1} \|\partial_{x'} T_{j,k}\sigma_\infty\|_2^2$$

$$+C(\frac{1}{\gamma^2} + \frac{1}{\nu(\nu+\widetilde{\nu})}) \sum_{2j+k\leq m-2} \|\partial_t T_{j,k}\sigma_\infty\|_2^2 + C(\frac{1}{\nu^2} + b_2 + b_3 + b_4) \sum_{2j+k\leq m|2j\neq m} D^{(0)}[T_{j,k} w_\infty]$$

$$+(\frac{b_1}{4} + b_5 C)\frac{\gamma^2}{\nu(\nu+\widetilde{\nu})} \sum_{2j+k\leq m-1} D^{(0)}[T_{j,k} w_\infty] + C(b_2 + b_3 + b_4) \sum_{2j+k\leq m-1} \frac{1}{\nu+\widetilde{\nu}}\|\sqrt{\rho_p} T_{j,k}\partial_t w_\infty\|_2^2$$

$$+\frac{b_3\alpha_1}{4(\nu+\widetilde{\nu})} \sum_{2j+k\leq m-3} \|T_{j,k+1}\sigma_\infty\|_2^2 + C b_3 \frac{1}{\nu+\widetilde{\nu}} \sum_{2j+k\leq m-1} \|\frac{P'(\rho_p)}{\gamma^2}\partial_{x_n} T_{j,k}\phi_\infty\|_2^2$$

$$+C b_4 \frac{\nu+\widetilde{\nu}}{\gamma^4} \sum_{2j+k\leq m-1} \|T_{j,k}\dot{\phi}_\infty\|_{H^1}^2 + C b_6 \frac{1}{(\nu+\widetilde{\nu})\gamma^2} (\|\partial_t^{[\frac{m}{2}]}\phi_\infty\|_2^2 + \|\partial_t^{[\frac{m}{2}]}\sigma_\infty\|_2^2).$$

There exists $\nu_0 \geq \max\{1, \nu_3\}$, $\gamma_0 \geq \max\{1, \gamma_2\}$ and $1 > b > 0$ such that if $b_4 < b_3 < b_2$ and $b_6 \leq b_5 \leq b_1$ appropriately with $b_l \leq b$ for $l = 2, \ldots, 4$, and $\nu \geq \nu_0$ and $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_0^2$ we can absorb most of the terms from RHS in the left-hand side of (7.12) to get

$$\frac{d}{dt}E^{(4)}(t) + \frac{1}{2}D^{(4)}(t) + \frac{1}{2}\frac{1}{(\nu+\widetilde{\nu})\gamma^2} b_5 \sum_{2j+k\leq m-1} \|\partial_t T_{j,k}\sigma_\infty\|_2^2 + \frac{1}{2}\frac{1}{(\nu+\widetilde{\nu})\gamma^2} b_5 \sum_{2j\leq m-2} \|\partial_t^{j+1}\phi_\infty(t)\|_2^2 \quad (7.13)$$

$$\leq C\sum_{j=1}^{8} R^{(j)}(t) + C \sum_{2j+k\leq m-2} \|\partial_t T_{j,k}\sigma_\infty\|_2^2 \frac{1}{\nu+\widetilde{\nu}}.$$

Next, we estimate higher order derivatives in x_n. For $1 \leq l \leq m-1$, we set

$$E_l^{(4)}(t) = \sum_{2j+k\leq m-1-l} \frac{1}{\gamma^2}\|\sqrt{\frac{P'(\rho_p)}{\gamma^2}} T_{j,k}\partial_{x_n}^{l+1}\phi_\infty(t)\|_2^2,$$

and

$$D_l^{(4)}(t)=\sum_{2j+k\le m-1-l}\left(\frac{1}{2(\nu+\widetilde{\nu})}\|\frac{P'(\rho_p)}{\gamma^2}\partial_{x_n}^{l+1}T_{j,k}\phi_\infty(t)\|_2^2+\frac{2c_0(\nu+\widetilde{\nu})}{\gamma^4}\|T_{j,k}\partial_{x_n}^{l+1}\dot{\phi}_\infty(t)\|_2^2\right)$$

$$+b_7\sum_{2j+k\le m-1-l}(\frac{\nu^2}{\nu+\widetilde{\nu}}\|\partial_x^{l+2}T_{j,k}w_\infty(t)\|_2^2+\frac{1}{\nu+\widetilde{\nu}}\|\partial_x^{l+1}T_{j,k}\phi_\infty(t)\|_2^2).$$

We add 2×(7.5) to b_7×(7.8) and sum over $2j+k\le m-1-l$ to obtain

$$\frac{d}{dt}E_l^{(4)}(t)+D_l^{(4)}(t)\le CR_l^{(9)}+b_7C\frac{\nu+\widetilde{\nu}}{\gamma^4}\sum_{2j+k\le m-1-l}\|T_{j,k}\dot{\phi}_\infty\|_{H^{l+1}}^2$$

$$+C(\frac{\nu+\widetilde{\nu}}{\gamma^2}+b_7)\frac{1}{\nu+\widetilde{\nu}}\sum_{2j+k\le m-1-l}\|T_{j,k}\partial_t w_\infty\|_{H^l}^2+C(\frac{1}{\nu^2}+\frac{\nu+\widetilde{\nu}}{\gamma^2})\frac{\nu^2}{\nu+\widetilde{\nu}}\sum_{2j+k\le m-1-l}\|T_{j,k}w_\infty\|_{H^{l+2}}^2$$

$$+C(b_7+\frac{\nu+\widetilde{\nu}}{\gamma^2})\frac{1}{\nu+\widetilde{\nu}}\sum_{2j+k\le m-1-l}\|T_{j,k+1}\sigma_\infty\|_2^2+C\frac{\nu+\widetilde{\nu}}{\gamma^2}\frac{1}{\nu+\widetilde{\nu}}\sum_{2j+k\le m-1-l}\|T_{j,k}\phi_\infty\|_{H^{l+1}}^2.$$

Here, $R_l^{(9)}=\sum_{2j+k\le m-1-l}(R_{j,k,l}^{(3)}+R_{j,k,l}^{(6)})$.

Let us sum up to l to get

$$\frac{d}{dt}\sum_{p=1}^{l}E_p^{(4)}(t)+\sum_{p=1}^{l}D_p^{(4)}(t)\le C\sum_{p=1}^{l}R_p^{(9)}$$

$$+b_7C\frac{\nu+\widetilde{\nu}}{\gamma^4}\sum_{p=1}^{l}\sum_{2j+k\le m-1-p}\|T_{j,k}\dot{\phi}_\infty\|_{H^{p+1}}^2+C(\frac{\nu+\widetilde{\nu}}{\gamma^2}+b_7)\frac{1}{\nu+\widetilde{\nu}}\sum_{p=1}^{l}\sum_{2j+k\le m-1-p}\|T_{j,k}\partial_t w_\infty\|_{H^p}^2$$

$$+C(\frac{1}{\nu^2}+\frac{\nu+\widetilde{\nu}}{\gamma^2})\frac{\nu^2}{\nu+\widetilde{\nu}}\sum_{p=1}^{l}\sum_{2j+k\le m-1-p}\|T_{j,k}w_\infty\|_{H^{p+2}}^2+C(b_7+\frac{\nu+\widetilde{\nu}}{\gamma^2})\frac{1}{\nu+\widetilde{\nu}}\sum_{p=1}^{l}\sum_{2j+k\le m-1-p}\|T_{j,k+1}\sigma_\infty\|_2^2$$

$$+C\frac{\nu+\widetilde{\nu}}{\gamma^2}\frac{1}{\nu+\widetilde{\nu}}\sum_{p=1}^{l}\sum_{2j+k\le m-1-p}\|T_{j,k}\phi_\infty\|_{H^{p+1}}^2.$$

Taking b_7 appropriately small, ν_0 and γ_0 possibly larger (based on l) we obtain

$$\frac{d}{dt}\sum_{p=1}^{l}E_p^{(4)}(t)+\frac{1}{2}\sum_{p=1}^{l}D_p^{(4)}(t)\le C\sum_{p=1}^{l}R_p^{(9)}+b_7C\frac{\nu+\widetilde{\nu}}{\gamma^4}\sum_{2j+k\le m-2}\|T_{j,k}\dot{\phi}_\infty\|_{H^1}^2 \tag{7.14}$$

$$+C\frac{1}{\nu^2}\frac{\nu^2}{\nu+\widetilde{\nu}}\sum_{2j+k\le m-1}\|T_{j,k}\partial_x^2 w_\infty\|_2^2+C\frac{1}{\nu+\widetilde{\nu}}\sum_{2j+k\le m|2j\ne m}\|T_{j,k}w_\infty\|_{H^1}^2+C(b_7+\frac{\nu+\widetilde{\nu}}{\gamma^2})\frac{1}{\nu+\widetilde{\nu}}D^{(0)}[\partial_t^{[\frac{m}{2}]}w_\infty]$$

$$+C(\frac{1}{\nu^2}+\frac{\nu+\widetilde{\nu}}{\gamma^2})\frac{\nu^2}{\nu+\widetilde{\nu}}\sum_{2j+k\le m-2}\|\partial_x^2T_{j,k}w_\infty\|_2^2+C(\frac{1}{\nu^2}+\frac{\nu+\widetilde{\nu}}{\gamma^2})\sum_{2j+k\le m-2}D^{(0)}[T_{j,k}w_\infty]$$

$$+C(b_7+\frac{\nu+\widetilde{\nu}}{\gamma^2})\frac{1}{\nu+\widetilde{\nu}}\sum_{2j+k\le m-1}\|T_{j,k+1}\sigma_\infty\|_2^2+C\frac{\nu+\widetilde{\nu}}{\gamma^2}\frac{1}{\nu+\widetilde{\nu}}\sum_{2j+k\le m-2}\|\partial_xT_{j,k}\phi_\infty\|_2^2.$$

Now adding 2×(7.13) together with (7.14) and taking possibly b_7 smaller, ν_0 and γ_0 larger we obtain

$$\frac{d}{dt}(2E^{(4)}(t)+\sum_{p=1}^{l}E_p^{(4)}(t))+(D^{(4)}(t)+\sum_{p=1}^{l}D_p^{(4)}(t)) \tag{7.15}$$

$$+\frac{1}{(\nu+\widetilde{\nu})\gamma^2}b_5\sum_{2j+k\le m-1}\|\partial_tT_{j,k}\sigma_\infty\|_2^2+\frac{1}{(\nu+\widetilde{\nu})\gamma^2}b_5\sum_{2j\le m-2}\|\partial_t^{j+1}\phi_\infty(t)\|_2^2$$

$$\leq C(\sum_{j=1}^{8} R^{(j)}(t) + \sum_{p=1}^{l} R_p^{(9)}) + C \sum_{2j+k\leq m-2} \|\partial_t T_{j,k}\sigma_\infty\|_2^2 \frac{1}{\nu+\widetilde{\nu}}.$$

To absorb the last term on the right-hand side we use induction on m.

Let $m = 1$ then we have from (7.15) that

$$\frac{d}{dt}(2E_1^{(4)}(t) + \sum_{p=1}^{l} E_{p,1}^{(4)}(t)) + (D_1^{(4)}(t) + \sum_{p=1}^{l} D_{p,1}^{(4)}(t)) \tag{7.16}$$

$$+\frac{1}{(\nu+\widetilde{\nu})\gamma^2} b_5 \|\partial_t \sigma_\infty\|_2^2 \leq C(\sum_{j=1}^{8} R^{(j)}(t) + \sum_{p=1}^{l} R_p^{(9)}).$$

Let $m = 2$ then

$$\frac{d}{dt}(2E^{(4)}(t) + \sum_{p=1}^{l} E_p^{(4)}(t)) + (D^{(4)}(t) + \sum_{p=1}^{l} D_p^{(4)}(t)) \tag{7.17}$$

$$+\frac{1}{(\nu+\widetilde{\nu})\gamma^2} b_5 \sum_{k\leq 1} \|\partial_t T_{0,k}\sigma_\infty\|_2^2 + \frac{1}{(\nu+\widetilde{\nu})\gamma^2} b_5 \|\partial_t^1 \phi_\infty(t)\|_2^2 \leq C(\sum_{j=1}^{8} R^{(j)}(t) + \sum_{p=1}^{l} R_p^{(9)})$$

$$+C\|\partial_t \sigma_\infty\|_2^2 \frac{1}{\nu+\widetilde{\nu}}.$$

By adding $b_8\gamma^2\times$(7.16) to (7.17) with appropriately large $b_8 > 0$ we can absorb $\|\partial_t\sigma_\infty\|_2^2 \frac{1}{\nu+\widetilde{\nu}}$ to the left-hand side. It is straightforward to see that this can be done from m to $m+1$. Therefore, we have

$$\begin{aligned} C_1 \frac{d}{dt}(2E^{(4)}(t) + \sum_{p=1}^{l} E_p^{(4)}(t)) + (D^{(4)}(t) + \sum_{p=1}^{l} D_p^{(4)}(t)) + \frac{1}{(\nu+\widetilde{\nu})\gamma^2} b_5 \sum_{2j+k\leq m-1} \|\partial_t T_{j,k}\sigma_\infty\|_2^2 \\ + \frac{1}{(\nu+\widetilde{\nu})\gamma^2} b_5 \sum_{2j\leq m-2} \|\partial_t^{j+1}\phi_\infty(t)\|_2^2 \leq C_2(\sum_{j=1}^{8} R^{(j)}(t) + \sum_{p=1}^{l} R_p^{(9)}), \end{aligned} \tag{7.18}$$

with $C_1, C_2 > 0$. The desired estimate (7.1) now follows from (7.18) with $l = m-1$.

Estimate (5.11) for $\widetilde{R}(t)$ is given in Proposition 8.1 (ii) below. This concludes the proof. $\square$

To prove (5.10) we employ the following lemma.

Lemma 7.10 *There exists $\widetilde{r}_0 = \widetilde{r}_0(\nu, \widetilde{\nu}, \gamma)$ such that if $r_1 \leq \widetilde{r}_0$, then there holds the estimate*

$$\|[Q_0 P_{\infty,1}(t)u]\|_2 \leq C\|\partial_{x'}(I - \Pi^{(0)}(t))P_{\infty,1}(t)u\|_2.$$

Proof. We set

$$\mathscr{R}(\xi',t) = \widehat{\mathscr{Q}}^{(0)}(t)(i\xi'\widehat{\mathscr{P}}^{(1)}(t) - |\xi'|^2\widehat{\mathscr{P}}^{(2)}(\xi',t)) + (i\xi'\widehat{\mathscr{Q}}^{(1)}(t) - |\xi'|^2\widehat{\mathscr{Q}}^{(2)}(\xi',t))\widehat{\mathscr{P}}_{\xi'}(t).$$

Since

$$[Q_0\widehat{\chi}_1(\xi')(I - \widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\mathscr{P}}_{\xi'}(t))] = [Q_0\widehat{\chi}_1(\xi')(I - \widehat{\mathscr{Q}}^{(0)}(t)\widehat{\mathscr{P}}^{(0)} - \mathscr{R}(\xi',t))]$$

$$= -[Q_0\widehat{\chi}_1(\xi')\mathscr{R}(\xi',t)],$$

we see that

$$[Q_0\widehat{P_{\infty,1}u}(t)] = [Q_0\widehat{\chi}_1(\xi')(I - \widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\mathscr{P}}_{\xi'}(t))\widehat{u}].$$

It then follows that

$$|[Q_0\widehat{P_{\infty,1}(t)}u]|_2 \leq C|\xi'||\widehat{\chi}_1\widehat{u}|_2$$

$$\leq C|\xi'|(\widehat{\chi}_1|(I-\Pi^{(0)}(t))\widehat{u}|_2+\widehat{\chi}_1|\Pi^{(0)}(t)\widehat{u}|_2).$$

Since $(P_{\infty,1}(t))^2 = P_{\infty,1}$, we see that

$$|[Q_0\widehat{P_{\infty,1}(t)u}]|_2 \leq C|\xi'|(|(I-\Pi^{(0)}(t))\widehat{P_{\infty,1}(t)u}|_2+|[Q_0\widehat{P_{\infty,1}(t)u}]|_2),$$

for $|\xi'|\leq r_1$. Therefore, there exists a positive number $\widetilde{r}_0$ such that if $r_1\leq\widetilde{r}_0$ then

$$|[Q_0\widehat{P_{\infty,1}(t)u}]|_2 \leq C|\xi'||(I-\Pi^{(0)}(t))\widehat{P_{\infty,1}u}|_2,$$

for $|\xi'|\leq r_1$, from which we obtain

$$\|[Q_0P_{\infty,1}(t)u]\|_2 \leq C\|\partial_{x'}(I-\Pi^{(0)}(t))P_{\infty,1}(t)u\|_2.$$

This completes the proof.

□

Finally, we prove (5.10).

Proof of (5.10)**.** We fix $\nu,\widetilde{\nu},\gamma$ so that inequality (7.1) in Proposition 7.1 holds true and set $r_1=\min\{r_0,\widetilde{r}_0,1\}$. Then we proceed as in [7, Proof of (5.15)] to obtain

$$\widetilde{E}(t)+\frac{\nu^2}{\nu+\widetilde{\nu}}[\![\partial_{x_n}^2 w_\infty(t)]\!]_{m-2}^2+\int_0^t e^{-\widetilde{a}(t-z)}D(z)dz \leq C\{e^{-\widetilde{a}t}\widetilde{E}(0)+R^{(10)}(t)+\int_0^t e^{-\widetilde{a}(t-z)}\widetilde{R}(z)dz\}. \quad (7.19)$$

Since

$$R^{(10)}(t)\leq C(1+t)^{-\frac{n+1}{2}}M(t)^4, \quad (7.20)$$

as we show in Proposition 8.1 (i) below, we deduce (5.10) from (7.19), Proposition 7.1 (i) and (7.20). This completes the proof.

□

8 Estimates on the nonlinearities

In this section we estimates the nonlinearities, e.g., we prove (5.11) and (7.20). In this section we assume that $\nu\geq\nu_2$ and $\gamma^2/(\nu+\widetilde{\nu})\geq\gamma_2^2$.

Proposition 8.1 *There exists number $\varepsilon_6>0$ such that if solution $u(t)$ of (4.1) in $Z^m(\tau)$ satisfies $\sup_{0\leq z\leq t}[\![u(z)]\!]_m\leq\varepsilon_6$ and $M(t)\leq 1$ for all $t\in[0,\tau]$, then the following estimates hold for all $t\in[0,\tau]$ with $C>0$ independent of τ.*

(i)

$$[\![\widetilde{Q}P_\infty\boldsymbol{F}(t)]\!]_{m-2}\leq C(1+t)^{-\frac{2n-1}{4}}M(t)^2,$$

(ii)

$$\widetilde{R}(t)\leq C\{(1+t)^{-\frac{3n}{4}}M(t)^3+(1+t)^{-\frac{n-1}{4}}M(t)D_\infty(t)\}.$$

To show the estimates in Proposition 8.1 we use the following inequalities.

Lemma 8.2 *(i) Let $2\leq p\leq\infty$ and let j and k be integers satisfying*

$$0\leq j<k,\quad k>j+n\left(\frac{1}{2}-\frac{1}{p}\right).$$

Then there exists a constant $C>0$ such that

$$\|\partial_x^j f\|_{L^p(\mathbb{R}^n)}\leq C\|f\|_{L^2(\mathbb{R}^n)}^{1-\theta}\|\partial_x^k f\|_{L^2(\mathbb{R}^n)}^{\theta},$$

where $\theta=\frac{1}{k}(j+\frac{n}{2}-\frac{n}{p})$.

(ii) Let $2 \le p \le \infty$ and let j and k be integers satisfying

$$0 \le j < k, \quad k > j + n\left(\frac{1}{2} - \frac{1}{p}\right).$$

Then there exists a constant $C > 0$ such that

$$\|\partial_x^j f\|_{L^p(\Omega)} \le C\|f\|_{H^k(\Omega)}.$$

(iii) If $f \in H^{n-1}(\Omega)$ and $f = f(x')$ is independent of x_n, then

$$\|f\|_{L^\infty(\Omega)} \le C\|f\|^{\frac{1}{2}}_{L^2(\Omega)}\|\partial_{x'}^{n-1} f\|^{\frac{1}{2}}_{L^2(\Omega)}.$$

Proof. The inequality in (i) is a special case of the Galiardo-Nirenberg-Sobolev inequality which can be proved using Fourier transform. Inequality in (ii) can be obtained by (i) and the standard extension argument. As for (iii), since

$$\|f\|_{L^p(\Omega)} = \|f\|_{L^p(\mathbb{R}^{n-1})}, \ 1 \le p \le \infty, \ \|\partial_{x'} f\|_{L^2(\Omega)} = \|\partial_{x'} f\|_{L^2(\mathbb{R}^{n-1})},$$

the inequality is a consequence of (i) with $n = n-1$, $p = \infty$, $j = 0$ and $k = n-1$. $\square$

Lemma 8.3 *(i) Let m and m_k, $k = 1, \dots, l$ be nonnegative integers and let α_k $k = 1, \dots, l$ be multi-indeces. Suppose that*

$$m \ge \left[\frac{n}{2}\right] + 1, \ 0 \le |\alpha_k| \le m_k \le m + |\alpha_k|, \ k = 1, \dots, l,$$

and

$$m_1 + \cdots + m_l \ge (l-1)m + |\alpha_1| + \cdots + |\alpha_l|.$$

Then there exists a constant $C > 0$ such that

$$\|\partial_x^{\alpha_1} f_1 \cdots \partial_x^{\alpha_l} f_l\|_2 \le C \prod_{1 \le k \le l} \|f_k\|_{H^{m_k}}.$$

(ii) Let $1 \le k \le m$. Suppose that $F(x,t,y)$ is a smooth function on $\Omega \times [0,\infty) \times I$, where I is a compact interval in $\mathbb{R}$. Then for $|\alpha| + 2j = k$ there hold

$$\|[\partial_x^\alpha \partial_t^j, F(x,t,f_1)]f_2\|_2 \le \begin{cases} C_0(t,f_1(t))[\![f_2]\!]_{k-1} + C_1(t,f_1(t))\{1 + |||Df_1|||_{m-1}^{|\alpha|+j-1}\}|||Df_1|||_{m-1}[\![f_2]\!]_k, \\ C_0(t,f_1(t))[\![f_2]\!]_{k-1} + C_1(t,f_1(t))\{1 + |||Df_1|||_{m-1}^{|\alpha|+j-1}\}|||Df_1|||_m[\![f_2]\!]_{k-1}. \end{cases}$$

Here

$$C_0(t,f_1(t)) = \sum_{\substack{(\beta,l) \le (\alpha,j) \\ (\beta,l) \ne (0,0)}} \sup_x |(\partial_x^\beta \partial_t^l F)(x,t,f_1(x,t))|,$$

and

$$C_1(t,f_1(t)) = \sum_{\substack{(\beta,l) \le (\alpha,j) \\ 1 \le p \le j + |\alpha|}} \sup_x |(\partial_x^\beta \partial_t^l \partial_y^p F)(x,t,f_1(x,t))|.$$

(iii) Let $m \ge [n/2] + 1$ then there exist constants $C, C' > 0$ such that

$$\|f_1 \cdot f_2\|_{H^m} \le C\|f_1\|_{H^m}\|f_2\|_{H^m},$$

and when $[\![f_1]\!]_m \le 1$,

$$[\![f_1 \cdot f_2]\!]_m \le C'[\![f_1]\!]_m[\![f_2]\!]_m.$$

Proof of previous lemma can be found in [9, 10].

We recall that $u(t)$ is decomposed into

$$u = \sigma_1 u^{(0)} + u_1 + \sigma_\infty u^{(0)} + u_\infty,$$

and we write

$$\sigma_* = \sigma_1 + \sigma_\infty, \quad \phi_* = \phi_1 + \phi_\infty, \quad w_* = w_1 + w_\infty,$$

$$u_* = {}^T(\phi_*, w_*) = u_1 + u_\infty.$$

Before investigating the nonlinearities we present some basic estimates.

Lemma 8.4 *Let $u(t) = {}^T(\phi(t), w(t)) = (\sigma_* u^{(0)})(t) + u_*(t)$ be solution of (4.1) in $Z^m(\tau)$. The following estimates hold for all $t \in [0, \tau]$ with $C > 0$ independent of τ.*

(i)
$$\|\sigma_*(t)\|_2 \le C(1+t)^{-\frac{n-1}{4}} M(t),$$

(ii)
$$|||D\sigma_*(t)|||_{m-1} + [\![u_*(t)]\!]_m \le C(1+t)^{-\frac{n+1}{4}} M(t),$$

(iii)
$$[\![\phi(t)]\!]_m + [\![w(t)]\!]_m \le C(1+t)^{-\frac{n-1}{4}} M(t),$$

(iv)
$$\|\sigma_*(t)\|_\infty \le C(1+t)^{-\frac{n}{4}} M(t),$$

(v)
$$\|u_*(t)\|_\infty \le C(1+t)^{-\frac{n+1}{4}} M(t),$$

(vi)
$$\|\phi(t)\|_\infty + \|w(t)\|_\infty \le C(1+t)^{-\frac{n}{4}} M(t).$$

Proof. Estimates (i), (ii) and (iii) immediately follow from definition of $M(t)$. As for (iv), we see from Lemma 8.2 (iii) and Lemma 5.3 (iii) that

$$\|\sigma_*(t)\|_\infty \le C\|\sigma_*(t)\|_2^{\frac{1}{2}} \|\partial_{x'}^{n-1}\sigma_*(t)\|_2^{\frac{1}{2}} \le C\|\sigma_*(t)\|_2^{\frac{1}{2}} \|\partial_{x'}\sigma_*(t)\|_2^{\frac{1}{2}} \le C(1+t)^{-\frac{n}{4}} M(t).$$

This shows (iv). Since $\|u_*(t)\|_\infty \le C\|u_*(t)\|_{H^m}$ by Lemma 8.2 (ii) we get get (v) from (ii). Estimate (vi) now follows from (iv) and (v). This completes the proof.

□

First, we consider the estimates on $Q_0\boldsymbol{F}$.

Proposition 8.5 *Let $u(t)$ be a solution of (4.1) in $Z^m(\tau)$ such that $M(t) \le 1$ for all $t \in [0, \tau]$. There hold the following estimates for all $t \in [0, \tau]$ with $C > 0$ independent of τ.*

(i)
$$[\![\phi \mathrm{div}\, w]\!]_l \le C \begin{cases} (1+t)^{-\frac{2n+1}{4}} M(t)^2 + (1+t)^{-\frac{n}{4}} M(t) |||Dw_\infty(t)|||_m, & l = m, \\ (1+t)^{-\frac{2n+1}{4}} M(t)^2, & l = m-1, \end{cases}$$

(ii)
$$[\![w \cdot \nabla(\sigma_* \phi^{(0)} + \phi_1)]\!]_m \le C(1+t)^{-\frac{2n+1}{4}} M(t)^2,$$

(iii)
$$[\![w \cdot \nabla \phi_\infty]\!]_{m-1} \le C(1+t)^{-\frac{2n+1}{4}} M(t)^2,$$

(iv)
$$|(\mathrm{div} \left(\frac{P'(\rho_p)}{\gamma^4 \rho_p} w\right), |\partial_t^j \partial_x^k \phi_\infty|^2)| \le C(1+t)^{-\frac{n+1}{4}} M(t) D_\infty(t),$$

for $2j + k \le m$,

(v)

$$\|[\partial_t^j\partial_x^k, w]\cdot\nabla\phi\|_2 \le C(1+t)^{-\frac{n}{2}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)\sqrt{D_\infty(t)},$$

for $2j+k\le m$,

(vi)

$$\|T_{j,k}(\phi w)\|_2 \le (1+t)^{-\frac{2n-1}{4}}M(t)^2,$$

for $2j+k\le m$.

Proof. By Lemma 5.3 (iii) we have

$$[\![\partial_{x'}\sigma_*(t)]\!]_m \le [\![\partial_{x'}\sigma_*(t)]\!]_{m-1} + [\![\partial_t\sigma_*(t)]\!]_{m-2} \le |||D\sigma_*(t)|||_{m-1}.$$

We use this estimate and others that come from properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$, e.g.,

$$\|\partial_{x'}^k\sigma_1\|_2 \le \|\partial_{x'}\sigma_1\|_2,\ k=1,\dots,$$

and

$$[\![\nabla u_1]\!]_m \le C[\![u_1]\!]_m,$$

together with Lemma 8.3 and Lemma 8.4 to obtain estimates (i)–(vi).

In the case of estimates (i)–(iii), we first use the following expansions and then apply above estimates:

$$\phi\mathrm{div}\, w = \sigma_*\phi^{(0)}w^{(0),1}\partial_{x_1}\sigma_* + \sigma_*\phi^{(0)}\mathrm{div}\, w_* + \phi_* w^{(0),1}\partial_{x_1}\sigma_* + \phi_*\mathrm{div}\, w_*,$$

$$\begin{aligned} w\cdot\nabla(\sigma_*\phi^{(0)}+\phi_1) &= \sigma_*\phi^{(0)}w^{(0),1}\partial_{x_1}\sigma_* + w'_*\cdot\nabla'\sigma_*\phi^{(0)} + w_*^n\sigma_*\partial_{x_n}\phi^{(0)} \\ &\quad +\sigma_* w^{(0),1}\partial_{x_1}\phi_1 + w_*\cdot\nabla\phi_1, \end{aligned}$$

$$w\cdot\nabla\phi_\infty = \sigma_* w^{(0),1}\partial_{x_1}\phi_\infty + w_*\cdot\nabla\phi_\infty.$$

This concludes the proof. □

Second, we consider $\widetilde{Q}\boldsymbol{F} = {}^T(0,\boldsymbol{f})$. Recall that $\widetilde{Q}\boldsymbol{F}$ is written in the form

$$\widetilde{Q}\boldsymbol{F} = \widetilde{\boldsymbol{F}}_0 + \widetilde{\boldsymbol{F}}_1 + \widetilde{\boldsymbol{F}}_2 + \widetilde{\boldsymbol{F}}_3.$$

Here, $\widetilde{\boldsymbol{F}}_l = {}^T(0,\boldsymbol{f}_l)$, $l=0,1,2,3$, with

$$\boldsymbol{f}_0 = -w\cdot\nabla w - f_1(\rho_p,\phi)\Delta'\sigma_* w^{(0),1}\boldsymbol{e}_1 - f_2(\rho_p,\phi)\nabla(\partial_{x_1}\sigma_* w^{(0),1})$$

$$+\boldsymbol{f}_{01}(x_n,t,\phi)\phi\sigma_* + \boldsymbol{f}_{02}(x_n,\phi)\phi\nabla'\sigma_* + \boldsymbol{f}_{03}(x_n,t,\phi)\phi\phi_*,$$

$$\boldsymbol{f}_1 = -f_1(\rho_p,\phi)\Delta w_* = -\mathrm{div}\,(f_1(\rho_p,\phi)\nabla w_*) + {}^T(\nabla w_*)\nabla(f_1(\rho_p,\phi)),$$

$$\boldsymbol{f}_2 = -f_2(\rho_p,\phi)\nabla\mathrm{div}\, w_* = -\nabla(f_2(\rho_p,\phi)\mathrm{div}\, w_*) + (\mathrm{div}\, w_*)\nabla(f_2(\rho_p,\phi)),$$

$$\boldsymbol{f}_3 = -f_3(x_n,\phi)\phi\nabla\phi_*.$$

Here, ∇w_* denotes the $n\times n$ matrix $(\partial_{x_i}w_*^j)$; $f_1 = \frac{\nu\phi}{\rho_p(\gamma^2\rho_p+\phi)}$; $f_2 = \frac{\widetilde{\nu}\phi}{\rho_p(\gamma^2\rho_p+\phi)}$; and $\boldsymbol{f}_{0l}(x_n,t,\phi)$, $l=1,2,3$ and $f_3(x_n,\phi)$ are smooth functions of x_n, t and ϕ.

Proposition 8.6 *Let $u(t)$ be solution of (4.1) in $Z^m(\tau)$ and assume that $\sup_{0\le z\le t}[\![u(z)]\!]_m \le \varepsilon_5$ and $M(t)\le 1$ for all $t\in[0,\tau]$. The following estimates hold for all $t\in[0,\tau]$ with $C>0$ independent of τ.*

(i)

$$[\![\widetilde{Q}\boldsymbol{F}(t)]\!]_{m-2} \le C(1+t)^{-\frac{2n-1}{4}}M(t)^2,$$

(ii)

$$[\![\boldsymbol{f}_0(t)]\!]_m \le C\{(1+t)^{-\frac{2n-1}{4}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)|||Dw_\infty(t)|||_m\},$$

(iii)

$$\sum_{l=1}^{3} [\![\boldsymbol{f}_l(t)]\!]_{m-1} \le C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) |\!|\!| D w_\infty(t) |\!|\!|_m\},$$

(iv)

$$\sum_{l=1}^{3} \|T_{j,k}\boldsymbol{f}_l\|_{H^{-1}} \le C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) |\!|\!| D w_\infty(t) |\!|\!|_m\},$$

for $2j+k=m$. Here, we regard $T_{j,k}\boldsymbol{f}_l$ with $2j+k=m$ as an element in H^{-1} by $(T_{j,k}\boldsymbol{f}_l)[v] = \langle T_{j,k}\boldsymbol{f}_l, v\rangle_{-1}$ for $v \in H_0^1$,

Proof. Since $[\![u(t)]\!]_m \le \varepsilon_5$ we see that $\widetilde{Q}\boldsymbol{F}(t)$ is smooth. Estimates (i)–(iii) can be obtained in similar manner to the proof of Proposition 8.5 and we omit the proof.

Let us prove estimate (iv). Let $2j+k=m$ and let $v \in H_0^1$. If $k \ge 1$ then we see from (iii) that

$$|\langle T_{j,k}\boldsymbol{f}_l, v\rangle_{-1}| = |-(T_{j,k-1}\boldsymbol{f}_l, \partial_{x'} v)| \le \|T_{j,k-1}\boldsymbol{f}_l\|_2 \|\partial_{x'} v\|_v$$

$$\le C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) |\!|\!| D w_\infty(t) |\!|\!|_m\} \|v\|_{H_0^1}.$$

We thus conclude that

$$\|T_{j,k}\boldsymbol{f}_l\|_{H^{-1}} \le C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) |\!|\!| D w_\infty(t) |\!|\!|_m\},$$

in the case $2j+k=m$, $k \ge 1$ and $l = 1,2,3$.

If $k=0$, i.e., $m=2j$, we write $\langle \partial_t^j \boldsymbol{f}_1, v\rangle_{-1}$ as

$$\langle \partial_t^j \boldsymbol{f}_1, v\rangle_{-1} = (\partial_t^j (f_1(\rho_p,\phi)\nabla w_*), \nabla v) + (\partial_t^j ({}^T(\nabla w_*)\partial_{\rho_p} f_1(\rho_p,\phi)\nabla\rho_p), v)$$

$$+([\partial_t^j, {}^T(\nabla w_*)\partial_\phi f_1(\rho_p,\phi)]\nabla\phi, v) - ({}^T(\nabla w_*)\partial_\phi f_1(\rho_p,\phi)\partial_t^j\phi, \operatorname{div} v)$$

$$-({}^T(\nabla^2 w_*)\partial_\phi f_1(\rho_p,\phi)\partial_t^j\phi, v) - ({}^T(\nabla w_*)\nabla_{\rho_p,\phi}\partial_\phi f_1(\rho_p,\phi)(\nabla\rho_p + \nabla\phi)\partial_t^j\phi, v) \equiv \sum_{i=1}^{6} I_i.$$

As for I_1, we have

$$|I_1| \le \|\partial_t^j (f_1(\rho_p,\phi)\nabla w_*)\|_2 \|\nabla v\|_2.$$

As in the proof of Proposition 8.5 (i) one can estimate $\|\partial_t^j (f_1(\rho_p,\phi)\nabla w_*)\|_2$ to obtain

$$|I_1| \le C\{(1+t)^{-\frac{2n+1}{4}} M(t)^2 + (1+t)^{-\frac{n}{4}} M(t) |\!|\!| D w_\infty(t) |\!|\!|_m\} \|v\|_{H_0^1}.$$

Similarly, we have

$$|I_2| \le C\{(1+t)^{-\frac{2n+1}{4}} M(t)^2 + (1+t)^{-\frac{n}{4}} M(t) |\!|\!| D w_\infty(t) |\!|\!|_m\} \|v\|_{H_0^1}.$$

Next, we consider I_3 which we write as follows:

$$I_3 = (\partial_\phi f_1(\rho_p,\phi)[\partial_t^j, {}^T(\nabla w_*)]\nabla\phi, v) + ([\partial_t^j, \partial_\phi f_1(\rho_p,\phi)]({}^T(\nabla w_*)\nabla\phi), v) \equiv J_1 + J_2.$$

First, we treat J_1, we have

$$|[\partial_t^j, {}^T\nabla w_*]\nabla\phi| \le C \sum_{l=0}^{j-1} |\partial_t^l \nabla\phi||\partial_t^{j-l}\partial_x w_*|.$$

Since

$$\frac{1}{2} - \frac{m-1-2l}{n} + \frac{1}{2} - \frac{m-2(j-l)}{n} = 1 - \frac{m-1}{n} < 1,$$

we can find p_{1l}, $p_{2l} \ge 2$ satisfying

$$\frac{1}{p_{1l}} > \frac{1}{2} - \frac{m-1-2l}{n}, \ \frac{1}{p_{2l}} > \frac{1}{2} - \frac{m-2(j-l)}{n}, \ \frac{1}{2} \le \frac{1}{p_{1l}} + \frac{1}{p_{2l}} < 1.$$

Now, we take number $p_{3l} \geq 2$ satisfying $\frac{1}{p_{3l}} = 1 - (\frac{1}{p_{1l}} + \frac{1}{p_{2l}}) > 0$. It then follows from Lemma 8.2 (ii) that

$$|(\partial_\phi f_1(\rho_p, \phi)[\partial_t^j, {}^T(\nabla w_*)]\nabla\phi, v)| \leq C \sum_{l=0}^{j-1} \|\partial_t^l \partial_x \phi\|_{p_{1l}} \|\partial_t^{j-l} \partial_x w_*\|_{p_{2l}} \|v\|_{p_{3l}}$$

$$\leq C \sum_{l=0}^{j-1} \|\partial_t^l \partial_x \phi\|_{H^{m-1-2l}} \|\partial_t^{j-l} \partial_x w_*\|_{H^{m-2(j-l)}} \|v\|_{H_0^1} \leq C[\![\phi]\!]_m \{[\![\partial_x w_1]\!]_m + |||Dw_\infty|||_m\} \|v\|_{H_0^1}.$$

Using Lemma 8.4 we conclude that

$$|J_1| \leq C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) |||Dw_\infty(t)|||_m\} \|v\|_{H_0^1}.$$

Second, we estimate J_2. By Lemma 8.3 (i) we have for $m_k = m - 1 - 2k$ and $m_l = m - 1 - 2l$ that

$$\|[\partial_t^j, \partial_\phi f_1(\rho_p, \phi)]({}^T(\nabla w_*)\nabla\phi)\|_2 \leq C \sum_{k+1+l=j} \|\partial_t^k(\partial_\phi^2 f_1(\rho_p, \phi)\partial_t \phi)\|_{m_k} \|\partial_t^l({}^T(\nabla w_*)\nabla\phi)\|_{m_l}$$

$$\leq C[\![\partial_t \phi]\!]_{m-1} [\![{}^T(\nabla w_*)\nabla\phi]\!]_{m-1}.$$

Therefore, we obtain

$$|J_2| \leq C[\![\partial_t \phi]\!]_{m-1} [\![{}^T(\nabla w_*)\nabla\phi]\!]_{m-1} \|v\|_2,$$

By Lemma 8.3 (ii) we get

$$[\![{}^T(\nabla w_*)\nabla\phi]\!]_{m-1} \leq C\{\|\nabla w_*\|_\infty [\![\partial_x \phi]\!]_{m-1} + |||D\nabla w_*|||_{m-2} [\![\nabla\phi]\!]_{m-1}\}$$

$$\leq C\{\|\partial_x w_1\|_{H^m} + |||Dw_\infty|||_m\} [\![\partial_x \phi]\!]_{m-1} \leq C\{(1+t)^{-\frac{n}{2}} M(t)^2 + (1+t)^{-\frac{n-1}{4}} M(t) |||Dw_\infty(t)|||_m\}.$$

As for $[\![\partial_t \phi]\!]_{m-1}$, we see from (6.3) and Proposition 7.9 that

$$[\![\partial_t \phi]\!]_{m-1} \leq C\{[\![\partial_t \sigma_*]\!]_{m-1} + [\![\partial_t \phi_*]\!]_{m-1}\} \leq C\{[\![\Lambda \sigma_1(t)]\!]_{m-1} + [\![\mathscr{P}(t)\boldsymbol{F}(t)]\!]_{m-1}$$

$$+ [\![\partial_{x'} \phi_\infty]\!]_{m-1} + [\![\partial_x w_\infty]\!]_{m-1} + [\![\partial_{x'} \sigma_\infty]\!]_{m-1} + [\![[Q_0(P_\infty^{(0)} \boldsymbol{F})]_\infty]\!]_{m-1} + [\![Q_0 P_\infty \boldsymbol{F}]\!]_{m-1}\}.$$

Using

$$P_\infty \boldsymbol{F} = \boldsymbol{F} - [Q_0 \boldsymbol{F}]_\infty u^{(0)} - \{\mathbb{P}\boldsymbol{F} - [Q_0 \mathbb{P}\boldsymbol{F}]_\infty u^{(0)}\},$$

and

$$[Q_0 P_\infty^{(0)} \boldsymbol{F}]_\infty = [Q_0 \boldsymbol{F}]_\infty - [Q_0 \mathbb{P}\boldsymbol{F}]_\infty,$$

together with Lemma 8.4 we get

$$[\![\partial_t \phi]\!]_{m-1} \leq C\{(1+t)^{-\frac{n+1}{4}} M(t) + [\![\mathscr{P}(t)\boldsymbol{F}(t)]\!]_{m-1} + [\![Q_0 \boldsymbol{F}]\!]_{m-1}\}.$$

Since $2j = m$, we have $\left[\frac{m-1}{2}\right] = \left[\frac{m-2}{2}\right]$, and hence, by properties of $\mathscr{P}(t)$,

$$[\![\mathscr{P}(t)\boldsymbol{F}(t)]\!]_{m-1} \leq C[\![\boldsymbol{F}(t)]\!]_{m-2}.$$

It then follows from Propositions 8.5 (i)–(iii) and 8.6 (i) that

$$[\![\boldsymbol{F}(t)]\!]_{m-2} + [\![Q_0 \boldsymbol{F}(t)]\!]_{m-1} \leq 2[\![Q_0 \boldsymbol{F}(t)]\!]_{m-1} + [\![\widetilde{Q}\boldsymbol{F}(t)]\!]_{m-2} \leq C(1+t)^{-\frac{2n-1}{4}} M(t)^2,$$

which implies

$$[\![\partial_t \phi]\!]_{m-1} \leq C(1+t)^{-\frac{n+1}{4}} M(t).$$

We thus conclude

$$|J_2| \leq C\{(1+t)^{-\frac{3n+1}{4}} M(t)^3 + (1+t)^{-\frac{n}{2}} M(t)^2 |||Dw_\infty(t)|||_m\} \|v\|_2.$$

Consequently,

$$|I_3| \le C\{(1+t)^{-\frac{n}{2}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H^1_0}.$$

Since

$$|I_4| \le |({}^T(\nabla w_*)\partial_\phi f_1(\rho_p,\phi)\partial_t^j\phi, \operatorname{div} v)| \le C\|\nabla w_*\|_\infty \|\partial_t^j\phi\|_2\|\nabla v\|_2,$$

and by Lemma 8.3 (i),

$$|I_5| \le |({}^T(\nabla^2 w_*)\partial_\phi f_1(\rho_p,\phi)\partial_t^j\phi, v)| \le C\|\partial_t^j\phi\|_2\|v\nabla^2 w_*\|_2 \le C\|\partial_t^j\phi\|_2\|\nabla w_*\|_{H^m}\|v\|_{H^1},$$

$$|I_6| \le |({}^T(\nabla w_*)\nabla_{\rho_p,\phi}\partial_\phi f_1(\rho_p,\phi)(\nabla\rho_p+\nabla\phi)\partial_t^j\phi, v)| \le C\|\nabla w_*\|_\infty\|\partial_t^j\phi\|_2\|v\|_{H^1}\|\rho_p+\phi\|_{H^m},$$

we obtain by Lemmas 8.2 (ii) and 8.4,

$$|I_4|+|I_5|+|I_6| \le C\{(1+t)^{-\frac{n+1}{2}}M(t)^2 + (1+t)^{-\frac{n+1}{4}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H^1_0}.$$

Therefore, we arrive at

$$|\langle\partial_t^j \boldsymbol{f}_1, v\rangle_{-1}| \le C\{(1+t)^{-\frac{n}{2}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)|||Dw_\infty(t)|||_m\}\|v\|_{H^1_0}.$$

This gives

$$\|\partial_t^j \boldsymbol{f}_1\|_{H^{-1}} \le C\{(1+t)^{-\frac{n}{2}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)|||Dw_\infty(t)|||_m\}.$$

Clearly, one can get the same estimate for $\|\partial_t^j \boldsymbol{f}_2\|_{H^{-1}}$. Concerning $\|\partial_t^j \boldsymbol{f}_3\|_{H^{-1}}$, one can write it as

$$\partial_t^j \boldsymbol{f}_3 = -[\partial_t^j, f_3(x_n,\phi)\phi]\nabla\phi_* - \nabla(f_3(x_n,\phi)\phi\partial_t^j\phi_*) + \nabla(f_3(x_n,\phi)\phi)\partial_t^j\phi_*,$$

and thus the desired estimate is obtained analogously to the one for $\|\partial_t^j \boldsymbol{f}_1\|_{H^{-1}}$. This completes the proof. □

Proof of Proposition 8.1. Since $[\![\widetilde{Q}P_\infty \boldsymbol{F}]\!]_{m-2} \le C[\![\boldsymbol{F}]\!]_{m-2}$, assertion (i) follows from Propositions 8.5 (i)–(iii) and 8.6 (i).

Let us consider $\widetilde{R}(t)$. We know that there holds

$$\widetilde{R}(t) \le C(\sum_{j=1}^{8} R^{(j)}(t) + \sum_{p=1}^{m-1} R_p^{(9)}).$$

Let us first show some basic estimates coming from Propositions 8.5, 8.6 and properties of $P(t)$:

$$[\![Q_0\boldsymbol{F}]\!]_{m-1} \le C(1+t)^{-\frac{2n+1}{4}}M(t)^2, \tag{8.1}$$

$$[\![\widetilde{Q}\boldsymbol{F}]\!]_{m-1} \le C\{(1+t)^{-\frac{2n-1}{4}}M(t)^2 + (1+t)^{-\frac{n-1}{4}}M(t)|||Dw_\infty(t)|||_m\}, \tag{8.2}$$

$$[\![\mathbb{P}\boldsymbol{F}]\!]_{m-1} \le C[\![\boldsymbol{F}]\!]_{m-1}, \tag{8.3}$$

Moreover, there holds

$$[Q_0T_{j,k}\boldsymbol{F}]_\infty = -[\operatorname{div} T_{j,k}(\phi w)]_\infty = -[\operatorname{div}' T_{j,k}(\phi w')]_\infty, \tag{8.4}$$

since $w \in H^1_0$.

Let us begin with $R^{(1)}_{j,k}$. We write

$$R^{(1)}_{j,k} = \frac{\alpha_0}{\gamma^2}([Q_0T_{j,k}\boldsymbol{F}]_\infty, T_{j,k}\sigma_\infty) - \frac{\alpha_0}{\gamma^2}([Q_0T_{j,k}(\mathbb{P}\boldsymbol{F})]_\infty, T_{j,k}\sigma_\infty) + \widetilde{R}^{(1)}_{j,k} - \langle T_{j,k}([Q_0\boldsymbol{F}]_\infty u^{(0)}), T_{j,k}u_\infty\rangle_\Omega$$

$$-\langle T_{j,k}(\mathbb{P}\boldsymbol{F}), T_{j,k}u_\infty\rangle_\Omega + \langle T_{j,k}([Q_0(\mathbb{P}\boldsymbol{F})]_\infty u^{(0)}), T_{j,k}u_\infty\rangle_\Omega = \sum_{l=1}^{6} I_l.$$

Here,

$$I_3 = \widetilde{R}^{(1)}_{j,k} = \langle T_{j,k}\boldsymbol{F}, T_{j,k}u_\infty\rangle_\Omega,$$

when $2j+k \le m-1$, and

$$I_3 = \widetilde{R}^{(1)}_{j,k} = -(T_{j,k}(\phi \mathrm{div}\, w), T_{j,k}\phi_\infty \frac{P'(\rho_p)}{\gamma^4\rho_p}) + \frac{1}{2}(\mathrm{div}\,(\frac{P'(\rho_p)}{\gamma^4\rho_p}w), |T_{j,k}\phi_\infty|^2)$$
$$-(w\nabla T_{jk}(\sigma_*\phi^{(0)}+\phi_1), T_{jk}\phi_\infty \frac{P'(\rho_p)}{\gamma^4\rho_p}) - ([T_{j,k}, w]\nabla\phi, T_{j,k}\phi_\infty \frac{P'(\rho_p)}{\gamma^4\rho_p}).$$
$$+(T_{j,k}\boldsymbol{f}_0, T_{j,k}w_\infty\rho_p) + \sum_{l=1}^{3}\langle T_{j,k}\boldsymbol{f}_l, T_{j,k}w_\infty\rho_p\rangle_{-1},$$

when $2j+k=m$.

We first consider I_3. If $2j+k \le m-1$, then by applying (8.1) and (8.2), we have

$$\sum_{2j+k\le m-1} |\langle T_{j,k}\boldsymbol{F}, T_{j,k}u_\infty\rangle_\Omega| \le C[\![\boldsymbol{F}]\!]_{m-1}[\![u_\infty]\!]_{m-1} \le C\{(1+t)^{-\frac{3n}{4}}M(t)^3 + (1+t)^{-\frac{n}{4}}M(t)D_\infty(t)\}.$$

Here, we used relation $a^2b \le \frac{1}{2}(a^3+ab^2)$ and Lemma 8.4 (ii).

In the case $2j+k=m$, we use Lemma 8.4 to calculate

$$|(w\nabla T_{jk}(\sigma_*\phi^{(0)}+\phi_1), T_{jk}\phi_\infty \frac{P'(\rho_p)}{\gamma^4\rho_p})| \le C\|w\|_\infty([\![\sigma_*]\!]_m + [\![\phi_1]\!]_m)[\![\phi_\infty]\!]_m \le C(1+t)^{-n}M(t)^3.$$

From above estimate and Propositions 8.5 (i), (iv), (v) and 8.6 (ii), (iv) we see that

$$\sum_{2j+k\le m} |I_3| \le C\{(1+t)^{-\frac{3n}{4}}M(t)^3 + (1+t)^{\frac{n-1}{4}}M(t)D_\infty(t)\}.$$

We next consider I_5. If $2j+k\le m-1$, then by (8.3) we see that

$$\sum_{2j+k\le m-1} |\langle T_{j,k}(\mathbb{P}\boldsymbol{F}), T_{j,k}u_\infty\rangle_\Omega| \le C[\![\mathbb{P}\boldsymbol{F}]\!]_{m-1}[\![u_\infty]\!]_{m-1} \le C[\![\boldsymbol{F}]\!]_{m-1}[\![u_\infty]\!]_{m-1}.$$

If $2j+k=m$ and $k\ge 1$, then from properties of $P(t)$ we obtain

$$\sum_{2j+k=m|k\ge 1} |\langle T_{j,k}(\mathbb{P}\boldsymbol{F}), T_{j,k}u_\infty\rangle_\Omega| \le \sum_{2j+k=m|k\ge 1} C\|T_{j,k-1}(\mathbb{P}\boldsymbol{F})\|_2\|T_{j,k}u_\infty\|_2 \le C[\![\boldsymbol{F}]\!]_{m-1}[\![u_\infty]\!]_m.$$

In the case $2j=m$, we write

$$|\langle \partial_t^j(\mathbb{P}\boldsymbol{F}), \partial_t^j u_\infty\rangle_\Omega| \le C|\langle[\partial_t^j,\mathbb{P}]\boldsymbol{F}, \partial_t^j u_\infty\rangle_\Omega| + |\langle \mathbb{P}\partial_t^j\boldsymbol{F}, \partial_t^j u_\infty\rangle_\Omega| \le C[\![\boldsymbol{F}]\!]_{m-2}[\![u_\infty]\!]_m + |\langle \mathbb{P}\partial_t^j\boldsymbol{F}, \partial_t^j u_\infty\rangle_\Omega|.$$

To estimate $\langle \mathbb{P}\partial_t^j\boldsymbol{F}, \partial_t^j u_\infty\rangle_\Omega$, we write it as

$$\langle \mathbb{P}\partial_t^j\boldsymbol{F}, \partial_t^j u_\infty\rangle_\Omega = \langle \partial_t^j\boldsymbol{F}, \mathbb{P}^*\partial_t^j u_\infty\rangle_\Omega.$$

Using integration by parts, we have

$$\langle \partial_t^j Q_0\boldsymbol{F}, P^*\partial_t^j u_\infty\rangle_\Omega = (\partial_t^j(\phi w), \nabla\left(Q_0(\mathbb{P}^*\partial_t^j u_\infty)\frac{P'(\rho_p)}{\gamma^4\rho_p}\right)).$$

Since

$$\|\nabla\left(Q_0(\mathbb{P}^*\partial_t^j u_\infty)\frac{P'(\rho_p)}{\gamma^4\rho_p}\right)\|_2 \le C\|\partial_t^j u_\infty\|_2,$$

by properties of $\mathbb{P}^*$, we see from Proposition 8.5 (vi) that

$$|\langle \partial_t^j Q_0\boldsymbol{F}, \mathbb{P}^*\partial_t^j u_\infty\rangle_\Omega| \le C(1+t)^{-\frac{3n}{4}}M(t)^3.$$

Since $\widetilde{Q}\mathbb{P}^*\partial_t^j u_\infty \in H_0^1$ one can estimate $|\langle \partial_t^j\widetilde{Q}\boldsymbol{F}, \mathbb{P}^*\partial_t^j u_\infty\rangle_\Omega|$ using Proposition 8.6 (ii), (iv) to obtain

$$\sum_{2j=m} |\langle \partial_t^j\widetilde{Q}\boldsymbol{F}, \mathbb{P}^*\partial_t^j u_\infty\rangle_\Omega| \le C\{(1+t)^{-\frac{3n}{4}}M(t)^3 + (1+t)^{\frac{n-1}{4}}M(t)D_\infty(t)\}.$$

Therefore, we have

$$\sum_{2j+k\le m} |I_5| \le C\{(1+t)^{-\frac{3n}{4}}M(t)^3 + (1+t)^{\frac{n-1}{4}}M(t)D_\infty(t)\}.$$

As for I_4, from (8.4) we compute

$$-\langle T_{j,k}([Q_0\boldsymbol{F}]_\infty u^{(0)}), T_{j,k}u_\infty\rangle_\Omega = \sum_{i=0}^{j}\begin{pmatrix} j \\ i \end{pmatrix}\langle[\mathrm{div}\,'T_{i,k}(\phi w')]_\infty T_{j-i,0}u^{(0)}, T_{j,k}u_\infty\rangle_\Omega.$$

Since

$$\|[\mathrm{div}\,'T_{i,k}(\phi w')]_\infty T_{j-i,0}u^{(0)}\|_2 \le C\|T_{i,k}(\phi w')\|_2,$$

we see using Proposition 8.5 (vi) that

$$\sum_{2j+k\le m} |I_4| \le C(1+t)^{-\frac{3n}{4}}M(t)^3.$$

As for I_1, using (8.4) and integration by parts we obtain

$$\frac{\alpha_0}{\gamma^2}([Q_0T_{j,k}\boldsymbol{F}]_\infty, T_{j,k}\sigma_\infty) = -\frac{\alpha_0}{\gamma^2}([\mathrm{div}\,'T_{j,k}(\phi w')]_\infty, T_{j,k}\sigma_\infty) \le C\|T_{j,k}(\phi w)\|_2\|\nabla' T_{j,k}\sigma_\infty\|_2.$$

and thus by Proposition 8.5 (vi) and Lemma 8.4 (ii) we get

$$\sum_{2j+k\le m} |I_1| \le C(1+t)^{-\frac{3n}{4}}M(t)^3.$$

As for I_2, in the case $1\le 2j+k\le m$ we treat it analogously to I_5 to show

$$|\frac{\alpha_0}{\gamma^2}([Q_0T_{j,k}(\mathbb{P}\boldsymbol{F})]_\infty, T_{j,k}\sigma_\infty)| \le C[\![\boldsymbol{F}]\!]_{m-1}[\![\partial_{x'}\sigma_\infty]\!]_{m-1} + |([Q_0\mathbb{P}\partial_t^{[\frac{m}{2}]}\boldsymbol{F}]_\infty, \partial_t^{[\frac{m}{2}]}\sigma_\infty)|.$$

We further estimate

$$|([Q_0\mathbb{P}\partial_t^{[\frac{m}{2}]}\boldsymbol{F}]_\infty, \partial_t^{[\frac{m}{2}]}\sigma_\infty)| \le \|\mathscr{P}(\partial_t^{[\frac{m}{2}]}\boldsymbol{F})\|_2\|\partial_t^{[\frac{m}{2}]}\sigma_\infty\|_2.$$

Using Plancherel theorem we have

$$\|\mathscr{P}(t)(\partial_t^{[\frac{m}{2}]}\boldsymbol{F}(t))\|_2 = \|\widehat{\chi}_1\langle\partial_t^{[\frac{m}{2}]}\widehat{\boldsymbol{F}}(t), u_{\xi'}^*(t)\rangle\|_2.$$

Therefore, using above relation, analogously to I_5, we estimate

$$\begin{aligned}\|\mathscr{P}(t)(\partial_t^{[\frac{m}{2}]}\boldsymbol{F}(t))\|_2 &\le \|\mathscr{P}(t)(Q_0\partial_t^{[\frac{m}{2}]}\boldsymbol{F}(t))\|_2 + \|\mathscr{P}(t)(\widetilde{Q}\partial_t^{[\frac{m}{2}]}\boldsymbol{F}(t))\|_2 \\ &\le C\{\|\partial_t^{[\frac{m}{2}]}(\phi w)\|_2 + [\![\boldsymbol{f}_0]\!]_m + \sum_{l=1}^{3}\|\partial_t^{[\frac{m}{2}]}\boldsymbol{f}_l\|_{H^{-1}}.\end{aligned}$$

In the case $j=k=0$ we see from Lemma 6.1 (ii) and properties of $\mathbb{P}(t)$,

$$|\frac{\alpha_0}{\gamma^2}([Q_0(\mathbb{P}\boldsymbol{F})]_\infty, \sigma_\infty)| \le C\|\partial_{x'}\sigma_\infty\|_2(\|\phi w'\|_2 + \|\boldsymbol{F}\|_2) \le (1+t)^{-\frac{3n}{4}}M(t)^3.$$

Therefore, using Propositions 8.5 (vi) and 8.6 (ii), (iv) we obtain

$$\sum_{2j+k\le m} |I_2| \le C\{(1+t)^{-\frac{3n}{4}}M(t)^3 + (1+t)^{-\frac{n-1}{4}}M(t)D_\infty(t)\}.$$

As for I_6, it can be treated in a way analogous to I_5 in the case $1\le 2j+k$, thus we get

$$\sum_{2j+k\le m} |\langle T_{j,k}([Q_0(\mathbb{P}\boldsymbol{F})]_\infty u^{(0)}), T_{j,k}u_\infty\rangle_\Omega| \le C\{[\![\boldsymbol{F}]\!]_{m-1}[\![u_\infty]\!]_{m-1} + \|\mathscr{P}(\partial_t^{[\frac{m}{2}]}\boldsymbol{F})\|_2\|\partial_t^{[\frac{m}{2}]}u_\infty\|_2\},$$

and

$$\sum_{2j+k\le m} |I_6| \le C\{(1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{\frac{n-1}{4}} M(t) D_\infty(t)\}.$$

We thus conclude

$$R^{(1)}(t) \le C\{(1+t)^{-\frac{3n}{4}} M(t)^3 + (1+t)^{\frac{n-1}{4}} M(t) D_\infty(t)\}.$$

It is straightforward to show that

$$\begin{aligned}\widetilde{R}(t) \le C\{&R^{(1)}(t) + [\![\boldsymbol{F}]\!]_{m-1}^2 + [\![Q_0(P_\infty^{(0)}\boldsymbol{F})]\!]_{m-1}^2 + [\![P_\infty \boldsymbol{F}]\!]_{m-1}^2 + [\![\phi \operatorname{div} w]\!]_m^2 \\ &+ [\![w\cdot\nabla(\sigma_*\phi^{(0)} + \phi_1)]\!]_m^2 + \sum_{2j+k\le m} \|[\partial_t^j \partial_x^k, w]\cdot\nabla\phi_\infty\|_2^2 \\ &+ \sum_{2j+k+l\le m-1} \left|\left(\operatorname{div}\left(\frac{P'(\rho_p)}{\gamma^4\rho_p} w\right), |T_{j,k}\partial_{x_n}^{l+1}\phi_\infty|^2\right)\right| \\ &+ \sum_{2j+k\le m-1} |(Q_0 T_{j,k}(P_\infty^{(0)}\boldsymbol{F}), T_{j,k}\sigma_\infty)| + [\![\widetilde{Q}(P_\infty \boldsymbol{F})]\!]_{m-1}[\![\partial_{x'}\sigma_\infty]\!]_{m-1}\}.\end{aligned}$$

From definition we have

$$P_\infty^{(0)}(t) = I - \mathbb{P}(t) - P_{\infty,3},$$

which together with

$$|(Q_0 P_{\infty,3} T_{j,k}\boldsymbol{F}, T_{j,k}\sigma_\infty)| \le C\|T_{jk}(\phi w)\|_2 |||D\sigma_\infty|||_{m-1}.$$

gives (analogously to previous computations)

$$\sum_{2j+k\le m-1} |(Q_0 T_{j,k}(P_\infty^{(0)}\boldsymbol{F}), T_{j,k}\sigma_\infty)| \le C(1+t)^{-\frac{3n}{4}} M(t)^3.$$

Since

$$[\![P_\infty^{(0)}\boldsymbol{F}]\!]_{m-1} + [\![P_\infty \boldsymbol{F}]\!]_{m-1} \le C[\![\boldsymbol{F}]\!]_{m-1},$$

using Propositions 8.5, 8.6 and Lemma 8.4 we obtain the desired estimate (ii) in Proposition 8.1. This completes the proof.

□

9 Asymptotic behavior of $\sigma_1(t)$

In this section we show the asymptotic behavior of solutions of (4.1). In the case $n=2$ we prove that it is described by a solution of a 1-dimensional viscous Burgers equation. In the case $n\ge 3$ we show that the asymptotic behavior is described by a linear heat equation, in fact, asymptotic leading term is the same as for the linearized problem.

In this section we assume that $\nu \ge \nu_0$ and $\gamma^2/(\nu+\widetilde{\nu}) \ge \gamma_0^2$. Let us note that $\sigma_1(t)$ is given by

$$\sigma_1(t) = \mathscr{P}(t)u(t),\ t\ge 0,$$

where $u(t)$ is a global in time solution of (4.1). Existence of $u(t)$ was proved in Sections 3–8.

First let us treat the case $n=2$.

Lemma 9.1 *Let $n=2$ and $\sigma(t)$ is a solution of*

$$\begin{gathered}\partial_t\sigma - \kappa_1\partial_{x_1}^2\sigma + \kappa_0\partial_{x_1}\sigma + \omega_0\partial_{x_1}(\sigma^2) = 0, \\ \sigma|_{t=0} = \sigma_0,\end{gathered} \tag{9.1}$$

where $\kappa_0\in\mathbb{R}$, $\kappa_1>0$ are the numbers given in (4.6) *and $\omega_0 = \frac{1}{T}\int_0^T [\phi^{(0)} w^{(0),1}(z)] - \langle \boldsymbol{F}_1(z), u^{*(1)}(z)\rangle dz$ and $\sigma_0 = [Q_0 u_0] = [\phi_0]$. Then we can write*

$$\sigma(t) = \mathscr{H}(t)\sigma_0 - \omega_0\int_0^t \mathscr{H}(t-z)\partial_{x_1}(\sigma^2(z))dz. \tag{9.2}$$

Theorem 9.2 *Let $n = 2$. For any $\delta > 0$ there exists $\varepsilon_7 > 0$ such that if $\|u_0\|_{H^m \cap L^1} \le \varepsilon_7$, then*

$$\|\sigma_1(t) - \sigma(t)\|_2 \le C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^m\cap L^1},$$

for $t \ge 0$.

(3.7) now follows from (5.13) and Theorem 9.2. To prove Theorem 9.2, we employ the following well-known decay properties of $\sigma(t)$.

Lemma 9.3 *Let $n = 2$ and $\sigma(t)$ is a solution of (9.1) with $\|\sigma_0\|_{H^1\cap L^1} \ll 1$. Then*

$$\|\partial_{x_1}^k \sigma(t)\|_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|\sigma_0\|_{H^1\cap L^1} \quad (k = 0, 1),$$

$$\|\sigma(t)\|_\infty \le C(1+t)^{-\frac{1}{2}}\|\sigma_0\|_{H^1\cap L^1}.$$

We introduce a quantity. Let $\sigma_1(t)$ and $\sigma(t)$ be solutions of (5.1) for $s = 0$ and (9.1), respectively. We define $N(t)$ by

$$N(t) = \sup_{0\le z\le t}(1+z)^{\frac{3}{4}-\delta}\|\sigma_1(z) - \sigma(z)\|_{H^1}.$$

Theorem 9.2 would then follow if we could show that $N(t) \le C\|u_0\|_{H^m\cap L^1}$.

Proof of Theorem 9.2. It is obvious that estimate holds for $0 \le t < 1$. Let us show that it holds for $t \ge 1$. Assume $t \ge 1$. From (5.1) we have that for $s = 0$

$$\sigma_1(t) = e^{t\Lambda}\mathscr{P}(0)u_0 + \int_0^t e^{(t-z)\Lambda}\mathscr{P}(z)\boldsymbol{F}(z)dz. \tag{9.3}$$

We next rewrite $e^{(t-z)\Lambda}\mathscr{P}(z)\boldsymbol{F}(z)$. By Lemma 6.1 (ii), we have

$$\mathscr{P}(z)\boldsymbol{F}(z) = -\partial_{x_1}[\phi w^1]_1 + \partial_{x_1}\mathscr{P}^{(1)}(z)\boldsymbol{F}(z) + \partial_{x_1}^2\mathscr{P}^{(2)}(z)\boldsymbol{F}(z)$$

$$= -a_{11}(z)\partial_{x_1}(\sigma_1^2) - \partial_{x_1}([\phi w^1]_1 - [\phi^{(0)}w^{(0),1}\sigma_1^2]_1)$$

$$+\partial_{x_1}\mathscr{P}^{(1)}(z)(\sigma_1^2\boldsymbol{F}_1(z) + \boldsymbol{F}_2(z)) + \partial_{x_1}^2\mathscr{P}^{(2)}(z)\boldsymbol{F}(z).$$

Here $a_{11}(z) = [\phi^{(0)}w^{(0),1}(z)]$. Since

$$\mathscr{F}\{\mathscr{P}^{(1)}(z)(\sigma_1^2\boldsymbol{F}_1(z))\} = \widehat{\chi}_1\langle\widehat{(\sigma_1^2)}\boldsymbol{F}_1(z), u^{*(1)}(z)\rangle = \widehat{\chi}_1\langle\boldsymbol{F}_1(z), u^{*(1)}(z)\rangle\widehat{(\sigma_1^2)} = -a_{12}(z)\sigma_1^2,$$

where $a_{12}(z) = -\langle\boldsymbol{F}_1(z), u^{*(1)}(z)\rangle$.

Using properties of $e^{(t-z)\Lambda}$, we thus arrive at

$$e^{(t-z)\Lambda}\mathscr{P}(z)\boldsymbol{F}(z) = -a_1(z)e^{(t-z)\Lambda}\partial_{x_1}(\sigma_1^2(z))$$

$$-e^{(t-z)\Lambda}\partial_{x_1}\{[\phi w^1]_1(z) - [\phi^{(0)}w^{(0),1}(z)]\sigma_1^2(z)\}$$

$$+e^{(t-z)\Lambda}h_5(z) + e^{(t-z)\Lambda}h_6(z),$$

where $a_1(z) = a_{11}(z) + a_{12}(z)$, $\sup_{z\in J_T}|a_1(z)| \le C$ and

$$h_5(z) = \partial_{x_1}\mathscr{P}^{(1)}(z)\boldsymbol{F}_2(z) + \partial_{x_1}^2\mathscr{P}^{(2)}(z)\boldsymbol{F}_2(z),$$

$$h_6(z) = \partial_{x_1}^2\mathscr{P}^{(2)}(z)(\sigma_1^2\boldsymbol{F}_1(z)).$$

It then follows from (9.2) and (9.3) that

$$\sigma_1(t) - \sigma(t) = \sum_{j=0}^6 I_j(t),$$

where

$$I_0(t) = e^{t\Lambda}\mathscr{P}(0)u_0 - \mathscr{H}(t)\sigma_0,$$

$$I_1(t) = -\int_0^t \omega_0 \mathscr{H}(t-z)\partial_{x_1}(\sigma_1^2(z) - \sigma^2(z))dz,$$

$$I_2(t) = -\int_0^t \omega_0 (e^{(t-z)\Lambda} - \mathscr{H}(t-z))\partial_{x_1}(\sigma_1^2)dz,$$

$$I_3(t) = -\int_0^t (a_1(z) - \omega_0)e^{(t-z)\Lambda}\partial_{x_1}(\sigma_1^2)dz,$$

$$I_4(t) = -\int_0^t \partial_{x_1} e^{(t-z)\Lambda}([\phi w^1]_1(z) - [\phi^{(0)}w^{(0),1}(z)]\sigma_1^2(z))dz,$$

$$I_j(t) = \int_0^t e^{(t-z)\Lambda}h_j(z)dz, \quad j = 5, 6.$$

Let us show estimates on I_j, $j = 0, \dots, 6$.

As for I_0, from (4.10) we see

$$\|I_0(t)\|_{H^1} \le Ct^{-\frac{3}{4}}\|u_0\|_{L^1}.$$

Let us consider $I_1(t)$. By Lemma 9.3, (5.12) and the definition of $M(t)$ and $N(t)$, we have

$$\|(\sigma_1^2 - \sigma^2)(z)\|_1 \le \|(\sigma_1 + \sigma)(z)\|_2 \|(\sigma_1 - \sigma)(z)\|_2 \le C(1+z)^{-1+\delta}N(t)\|u_0\|_{H^m \cap L^1},$$

for $\|u_0\|_{H^m\cap L^1} \le \varepsilon_3$. Furthermore, by Lemma 8.2 (iii) we have $\|(\sigma_1 - \sigma)(z)\|_\infty \le C(1+z)^{-\frac{3}{4}+\delta}N(t)$, and hence,

$$\|\partial_{x_1}(\sigma_1^2 - \sigma^2)(z)\|_2 \le C\{\|(\sigma_1 + \sigma)(z)\|_\infty \|\partial_{x_1}(\sigma_1 - \sigma)(z)\|_2 + \|(\sigma_1 - \sigma)(z)\|_\infty \|\partial_{x_1}(\sigma_1 + \sigma)(z)\|_2\}$$

$$\le C(1+z)^{-\frac{5}{4}+\delta}\|u_0\|_{H^m\cap L^1}N(t).$$

It then follows from (4.8) that for $k = 0, 1$,

$$\|\partial_{x_1}^k I_1(t)\|_2 \le C\{\int_0^{\frac{t-1}{2}} (t-z)^{-\frac{3}{4}-\frac{k}{2}}(1+z)^{-1+\delta}dz + \int_{\frac{t-1}{2}}^{t-\frac{1}{2}} (t-z)^{-\frac{3}{4}-\frac{k}{2}}(1+z)^{-1+\delta}dz$$

$$+ \int_{t-\frac{1}{2}}^t (t-z)^{-\frac{k}{2}}(1+z)^{-\frac{5}{4}+\delta}dz\}\|u_0\|_{H^m\cap L^1}N(t) \le C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^m\cap L^1}N(t).$$

As for $I_2(t)$, we see from (4.11) that for $k = 0, 1$,

$$\|\partial_{x_1}^k I_2(t)\|_2 \le C\{\int_0^{\frac{t-1}{2}} (t-z)^{-\frac{5}{4}-\frac{k}{2}}\|\sigma_1^2(z)\|_1 dz + \int_{\frac{t-1}{2}}^{t-\frac{1}{2}} (t-z)^{-\frac{3}{4}-\frac{k}{2}}\|\partial_{x_1}(\sigma_1^2(z))\|_1 dz + \int_{t-\frac{1}{2}}^t (t-z)^{-\frac{k}{2}}\|\partial_{x_1}(\sigma_1^2(z))\|_2 dz\}.$$

From Lemma 6.3 we have

$$\le C\{\int_0^{\frac{t-1}{2}} (t-z)^{-\frac{5}{4}-\frac{k}{2}}(1+z)^{-\frac{1}{2}}dz + \int_{\frac{t-1}{2}}^{t-\frac{1}{2}} (t-z)^{-\frac{3}{4}-\frac{k}{2}}(1+z)^{-1}dz + \int_{t-\frac{1}{2}}^t (t-z)^{-\frac{k}{2}}(1+z)^{-\frac{5}{4}}dz\}M(t)^2$$

$$\le C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^m\cap L^1}^2.$$

As for $I_3(t)$, let us define $b(t) = \int_0^t a_1(z) - \omega_0 dz$. Then $\partial_t b(t) = a_1(t) - \omega_0$ and $b(0) = b(T) = 0$. Since $a_1(t+T) = a_1(t)$ we have $\partial_t b(t+T) = \partial_t b(t)$ and thus $b(t+T) = b(t)$. We arrive at $\sup_{z\in J_T}|b(z)| \le C$. We write

$$I_3(t) = -\int_0^t \partial_z b(z)e^{(t-z)\Lambda}\partial_{x_1}(\sigma_1^2)dz = -\left[b(z)e^{(t-z)\Lambda}\partial_{x_1}(\sigma_1^2(z))\right]_0^t + \int_0^t b(z)\partial_z\left(e^{(t-z)\Lambda}\partial_{x_1}(\sigma_1^2(z))\right)dz$$

$$= -b(t)\partial_{x_1}(\sigma_1^2(t)) - \int_0^t b(z)e^{(t-z)\Lambda}\Lambda\partial_{x_1}(\sigma_1^2(z))dz + \int_0^t b(z)\partial_{x_1}e^{(t-z)\Lambda}\partial_z(\sigma_1^2(z))dz \equiv J_1(t) + J_2(t) + J_3(t).$$

From Lemma 6.3 (vii) we have for $k = 0, 1$,

$$\|\partial_{x_1}^k J_1(t)\|_2 \le C(1+t)^{-\frac{5}{4}}M(t)^2.$$

We see from (4.9) and Lemma 6.3 that

$$\|J_2(t)\|_2 \le C\{\int_0^{\frac{t}{2}}(1+t-z)^{-\frac{5}{4}}\|\sigma_1^2(z)\|_1 dz + \int_{\frac{t}{2}}^t (1+t-z)^{-\frac{3}{4}}\|\partial_{x_1}(\sigma_1^2(z))\|_1 dz\}$$

$$\le C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^m\cap L^1}^2.$$

As for J_3, using (6.3) we calculate

$$J_3(t) = 2\int_0^t b(z)\partial_{x_1}e^{(t-z)\Lambda}\sigma_1(z)\Lambda\sigma_1(z)dz + 2\int_0^t b(z)\partial_{x_1}e^{(t-z)\Lambda}\sigma_1(z)\mathscr{P}(z)\boldsymbol{F}(z)dz \equiv J_{31} + J_{32}.$$

Using (4.9) and Lemma 6.3 we calculate

$$\|J_{31}\|_2 \le C\int_0^t (1+t-z)^{-\frac{3}{4}}\|\sigma_1(z)\Lambda\sigma_1(z)\|_1 dz \le C\int_0^t (1+t-z)^{-\frac{3}{4}}\|\sigma_1(z)\|_2\|\Lambda\sigma_1(z)\|_2 dz$$

$$\le CM(t)^2\int_0^t (1+t-z)^{-\frac{3}{4}}(1+z)^{-1}dz \le C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m\cap L^1}^2.$$

Analogously we obtain for J_{32} that

$$\|J_{32}\|_2 \le C\int_0^t (1+t-z)^{-\frac{3}{4}}\|\sigma_1(z)\|_2\|\boldsymbol{F}(z)\|_2 dz \le C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m\cap L^1}^3.$$

As for $I_4(t)$, we have

$$\|[\phi w^1]_1(z) - [\phi^{(0)}w^{(0),1}(z)]\sigma_1^2(z)\|_1 \le C\{\|\sigma_1(z)\|_2\|u(z) - \sigma_1(z)u^{(0)}(z)\|_2 + \|u(z) - \sigma_1(z)u^{(0)}(z)\|_2^2\}$$

$$\le C(1+z)^{-1}M(z)^2.$$

Thus, (4.9) gives us

$$\|\partial_{x_1}^k I_4(t)\|_2 \le CM(t)^2\int_0^t (1+t-z)^{-\frac{3}{4}-\frac{k}{2}}(1+z)^{-1}dz \le C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m\cap L^1}^2.$$

To estimate $I_5(t)$, we write $h_5(z)$ as

$$h_5(z) = \partial_{x_1}\left(\mathscr{P}^{(1)}(z)\boldsymbol{F}_2(z) + \partial_{x_1}\mathscr{P}^{(2)}(z)\boldsymbol{F}_2(z)\right).$$

Using (4.14) and Lemma 6.3 (v), we have

$$\|\partial_{x_1}^k I_5(t)\|_2 \le CM(t)^2\int_0^t (1+t-z)^{-\frac{3}{4}-\frac{k}{2}}(1+z)^{-1}dz \le C(1+t)^{-\frac{3}{4}}\log(1+t)\|u_0\|_{H^m\cap L^1}^2.$$

As for $I_6(t)$, we write $h_6(z)$ as

$$h_6(z) = \begin{cases} \partial_{x_1}^2\mathscr{P}^{(2)}(z)(\sigma_1^2\boldsymbol{F}_1)(z) & \text{for } z \in [0, \frac{t}{2}], \\ \partial_{x_1}\mathscr{P}^{(2)}(z)(\partial_{x_1}(\sigma_1^2)\boldsymbol{F}_1)(z) & \text{for } z \in [\frac{t}{2}, t]. \end{cases}$$

We see from (4.14) and Lemma 6.3 that

$$\|\partial_{x_1}^k I_6(t)\|_2 \le C\{\int_0^{\frac{t}{2}}(1+t-z)^{-\frac{5}{4}-\frac{k}{2}}\|\sigma_1^2(z)\|_1 dz + \int_{\frac{t}{2}}^t (1+t-z)^{-\frac{3}{4}-\frac{k}{2}}\|\partial_{x_1}(\sigma_1)^2\|_1 dz\}M(t)^2$$

$$\leq C\{\int_0^{\frac{t}{2}}(1+t-z)^{-\frac{5}{4}-\frac{k}{2}}(1+z)^{-\frac{1}{2}}dz+\int_{\frac{t}{2}}^t(1+t-z)^{-\frac{3}{4}-\frac{k}{2}}(1+z)^{-1}dz\}M(t)^2\leq C(1+t)^{-\frac{3}{4}}\|u_0\|_{H^m\cap L^1}^2.$$

We thus obtain

$$\|(\sigma_1-\sigma)(t)\|_{H^1}\leq C(1+t)^{-\frac{3}{4}+\delta}\|u_0\|_{H^m\cap L^1}\{1+\|u_0\|_{H^m\cap L^1}+\|u_0\|_{H^m\cap L^1}^2+N(t)\},$$

which yields

$$N(t)\leq\|u_0\|_{H^m\cap L^1}\{1+\|u_0\|_{H^m\cap L^1}+\|u_0\|_{H^m\cap L^1}^2+N(t)\}.$$

The desired result now follows by taking $\|u_0\|_{H^m\cap L^1}$ suitably small. This completes the proof. □

Now let us show the asymptotic behavior in cases $n\geq 3$.

Theorem 9.4 *Let $n\geq 3$. There exists $\varepsilon_8>0$ such that if $\|u_0\|_{H^m\cap L^1}\leq\varepsilon_8$, then*

$$\|\sigma_1(t)-\mathscr{H}(t)\sigma_0\|_2\leq C(1+t)^{-\frac{n+1}{4}}\eta_n(t)\|u_0\|_{H^m\cap L^1},$$

where $\eta_n(t)=\log(1+t)$ when $n=3$ and $\eta_n(t)=1$ when $n\geq 4$ and $t\geq 0$.

Proof. From (9.3) we see that

$$\sigma_1(t)-\mathscr{H}(t)\sigma_0=e^{t\Lambda}\mathscr{P}(0)u_0-\mathscr{H}(t)\sigma_0+\int_0^t e^{(t-z)\Lambda}\mathscr{P}(z)\boldsymbol{F}(z)dz.$$

Estimate (4.10) then implies

$$\|e^{t\Lambda}\mathscr{P}(0)u_0-\mathscr{H}(t)\sigma_0\|_2\leq Ct^{-\frac{n-1}{4}-\frac{1}{2}}\|u_0\|_{L^1(\Omega)}.$$

By Lemma 6.1 (ii) we have

$$\mathscr{P}(z)\boldsymbol{F}(z)=-\mathrm{div}'[\phi(z)w'(z)]_1+\mathrm{div}'\mathscr{P}^{(1)}(z)\boldsymbol{F}(z)+\Delta'\mathscr{P}^{(2)}(z)\boldsymbol{F}(z),$$

and thus by using (4.14) and Lemma 6.3 we obtain

$$\|\int_0^t e^{(t-z)\Lambda}\mathscr{P}(z)\boldsymbol{F}(z)dz\|_2\leq C\int_0^t(1+t-z)^{-\frac{n-1}{4}-\frac{1}{2}}(\|[\phi w'](z)\|_1+\|\boldsymbol{F}(z)\|_1)dz$$

$$\leq CM(t)^2\int_0^t(1+t-z)^{-\frac{n-1}{4}-\frac{1}{2}}(1+z)^{-\frac{n-1}{2}}dz\leq C(1+t)^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t)\|u_0\|_{H^m\cap L^1}.$$

This concludes the proof. □

(3.9) now follows from (5.13) and Theorem 9.4.

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