# A Weak Dynamic Programming Principle for Zero-Sum Stochastic Differential Games with Unbounded Controls 

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#### Abstract

We analyze a zero-sum stochastic differential game between two competing players who can choose unbounded controls. The payoffs of the game are defined through backward stochastic differential equations. We prove that each player's priority value satisfies a weak dynamic programming principle and thus solves the associated fully non-linear partial differential equation in the viscosity sense.


Keywords: Zero-sum stochastic differential games, Elliott-Kalton strategies, weak dynamic programming principle, backward stochastic differential equations, viscosity solutions, fully non-linear PDEs.

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## 1 Introduction

In this paper we extend the study of Buckdahn and Li [11] on a zero-sum stochastic differential game (SDG), whose payoffs are generated by backward stochastic differential equations (BSDEs), to the case of super-square-integrable controls (see Remark 2.1).

Since the seminal paper by Fleming and Souganidis [16], the SDG theory has grown rapidly in many aspects (see e.g. the references in [11, [10]). Among these developments, Hamadène et al. [18, 19, 14] introduced a (decoupled) SDE-BSDE system, with controls only in the drift coefficients, to generate the payoffs in their studies of saddle point problems of SDGs. (For the evolution and applications of the BSDE theory, see Pardoux and Peng [27], El Karoui et al. [15] and the references therein.) Later on, [11] as well as its sequels [13, 12, 10] generalized

[^0]the SDE-BSDE framework so that the two competing controllers can also influence the diffusion coefficient of the state dynamics. Unlike [16], [11] used a uniform canonical space $\Omega=\left\{\omega \in \mathbb{C}\left([0, T] ; \mathbb{R}^{d}\right): \omega(0)=0\right\}$ so that admissible control processes can also depend on the information occurring before the start of the game. Such a setting allows the authors of [11] get around a relatively complicated approximation argument of [16] which was due to a measurability issue (see Remark (2.5), and allows them to adopt the notion of stochastic backward semigroups and a BSDE method, developed in [28, 30, to obtain results similar to [16]: the lower and upper values of the SDG satisfy a dynamic programming principle and solve the associated Hamilton-Jacobi-Bellman-Isaacs equations in the viscosity sense. However, [11, [16] as well as some latest advances to the SDG theory (e.g. Bouchard et al. 6] on stochastic target games, Peng and Xu [29] on SDGs in form of a generalized BSDE with random default time) still assume the compactness of control spaces while Pham and Zhang [32] on weak formulation of SDGs assumes the boundedness of coefficients in control variables. We are going to address these particular issues.

In the present paper, since two players take super-square-integrable admissible controls over two separable metric spaces $\mathbb{U}$ and $\mathbb{V}$ not necessarily compact, those approximation methods of [16] and [11] in proving the dynamic programming principle are no longer effective. Instead, we derive a weak form of dynamic programming principle in spirit of Bouchard and Touzi [7] and use it to show that each player's priority value solves the corresponding fully non-linear PDE in the viscosity sense. Vitoria 33 has tried to extend the SDG for unbounded controls by proving a weak dynamic programming principle. However, it still assumed that the control space of the player with priority is compact, see Theorem 75 therein.

Square-integrable controls were initially considered by Krylov [22, Chapter 6], however, for cooperative games (i.e. the so called sup sup case). Browne [9] studied a specific zero-sum investment game between two small investors who control the game via their square-integrable portfolios. Since the PDEs in this case have smooth solutions, the problem can be solved by a verification theorem instead of the dynamic programming principle. Inspired by the "tug-of-war" (a discrete-time random turn game, see e.g. [31] and [25]), Atar and Budhiraja [1] studied a zero-sum stochastic differential game with $\mathbb{U}=\mathbb{V}=\left\{x \in \mathbb{R}^{n}:|x|=1\right\} \times[0, \infty)$ played until the state process exits a given domain. As in Chapter 6 of [22], the authors approximated such a game with unbounded controls by a sequence of games with bounded controls which satisfy a dynamic programming principle. They showed the equicontinuity of the approximating sequence and thus proved that the value function of the game is a unique viscosity solution to the inhomogenous infinity Laplace equation. We do not rely on this approximation scheme but directly prove a weak dynamic programming principle for the game with super-square-integrable controls.

Following the probabilistic setting of [11] (see Remark 2.5), our paper takes the canonical space $\Omega=\{\omega \in$ $\left.\mathbb{C}\left([0, T] ; \mathbb{R}^{d}\right): \omega(0)=0\right\}$, whose coordinator process $B$ is a Brownian motion under the Wiener measure $P$. When the game starts from time $t \in[0, T]$, under the super-square-integrable controls $\mu \in \mathcal{U}_{t}$ and $\nu \in \mathcal{V}_{t}$ selected by player I and II respectively, the state process $X^{t, \xi, \mu, \nu}$ starting from a random initial state $\xi$ will then evolve according to a stochastic differential equation (SDE):

$$
\begin{equation*}
X_{s}=\xi+\int_{t}^{s} b\left(r, X_{r}, \mu_{r}, \nu_{r}\right) d r+\int_{t}^{s} \sigma\left(r, X_{r}, \mu_{r}, \nu_{r}\right) d B_{r}, \quad s \in[t, T] \tag{1.1}
\end{equation*}
$$

where the drift $b$ and the diffusion $\sigma$ are Lipschitz continuous in $x$ and have linear growth in $(u, v)$. The payoff player I will receive from player II is determined by the first component of the unique solution $\left(Y^{t, \xi, \mu, \nu}, Z^{t, \xi, \mu, \nu}\right)$ to the following BSDE:

$$
\begin{equation*}
Y_{s}=g\left(X_{T}^{t, \xi, \mu, \nu}\right)+\int_{s}^{T} f\left(r, X_{r}^{t, \xi, \mu, \nu}, Y_{r}, Z_{r}, \mu_{r}, \nu_{r}\right) d r-\int_{s}^{T} Z_{r} d B_{r}, \quad s \in[t, T] . \tag{1.2}
\end{equation*}
$$

Here the generator $f$ is Lipschitz continuous in $(y, z)$ and also has linear growth in $(u, v)$. When $g$ and $f$ are $2 / p$-Hölder continuous in $x$ for some $p \in(1,2], Y^{t, \xi, \mu, \nu}$ is $p$-integrable. As we see from (1.1) and (1.2) that the controls $\mu, \nu$ influence the game in two aspects: both affect (1.2) via the state process $X^{t, \xi, \mu, \nu}$ and appear directly in the generator $f$ of (1.2) as parameters. In particular, if $f$ is independent of $(y, z), Y$ is in form of the conditional linear expectation of the terminal reward $g\left(X_{T}^{t, \xi, \mu, \nu}\right)$ plus the cumulative reward $\int_{s}^{T} f\left(r, X_{r}^{t, \xi, \mu, \nu}, \mu_{r}, \nu_{r}\right) d r$ (cf. [16]).

When the player (e.g. Player I) with the priority chooses firstly a super-square-integrable control (e.g. $\mu \in \mathcal{U}_{t}$ ), its opponent (e.g. Player II) will select its reacting control via a non-anticipative mapping $\beta: \mathcal{U}_{t} \rightarrow \mathcal{V}_{t}$, called ElliottKalton strategy. In particular, using Elliott-Kalton strategies is essential in proving the dynamic programming
principle. This phenomenon already appears in the controller-stopper games, i.e. when one of the players is endowed with the right of stopping the game instead of using a control; see [2], which shows that if the stopper acts second it is necessary that the stopper uses non-anticipative strategies in order to prove a dynamic programming principle. This type of phenomenon does not appear (or it is implicitly satisfied) if the controllers only control the drift, see e.g. 3] and the references therein, or when there are two stoppers (the so-called Dynkin games), see e.g. [4] and the references therein.

By $w_{1}(t, x) \triangleq \underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}$ we denote Player I's priority value of the game starting from time $t$ and state $x$, where $\mathfrak{B}_{t}$ collects all admissible strategies for Player II. Switching the priority defines Player II's priority value $w_{2}(t, x)$.

Although our setting makes the payoffs $Y_{t}^{t, \xi, \mu, \nu}$ random variables, we can show like [11 that $w_{1}(t, x)$ and $w_{2}(t, x)$ are invariant under Girsanov transformation via functions of the Cameron-Martin space and are thus deterministic, see Lemma 2.2. To assure values $w_{1}(t, x)$ and $w_{2}(t, x)$ are finite, we assume that each player has some control neutralizer for coefficients $(b, \sigma, f)$ (such an assumption holds for additive controls, see Example 2.1), and we impose a growth condition on strategies. These two requirements are also crucial in proving our weak dynamic programming principle. When $\mathbb{U}$ and $\mathbb{V}$ are compact, the control neutralizers become futile and the growth condition holds automatically for strategies. Thus our problem degenerates to [11]'s case, see Remark 2.4.

Although value functions $w_{1}(t, x), w_{2}(t, x)$ are still $2 / p-$ Hölder continuous in $x$ (see Proposition 2.3), they may not be continuous in $t$. Hence we can not follow [11's approach to get a strong form of dynamic programming principle for $w_{1}$ and $w_{2}$. Instead, we prove a weak dynamic programming principle, say for $w_{1}$ :

$$
\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \phi\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right) \leq w_{1}(t, x) \leq \underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \widetilde{\phi}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right),
$$

for any two continuous functions $\phi \leq w_{1} \leq \widetilde{\phi}$. Here $\tau_{\beta, \mu}$ denotes the first existing time of state process $X^{t, x, \mu, \beta(\mu)}$ from the given open ball $O_{\delta}(t, x)$.

To prove the weak dynamic programming principle, we first approximate $w_{1}(t, x)=\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} I(t, x, \beta)$ from above and $I(t, x, \beta) \triangleq \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} Y_{t}^{t, x, \mu, \beta(\mu)}$ from below in a probabilistic sense (see Lemma 4.2) so that we can construct approximately optimal controls/strategies by a pasting technique similar to the one used in [7] and [33]. Then we make a series of estimates and eventually obtain the weak dynamic programming principle by using a stochastic backward semigroup property (2.11), the continuous dependence of payoff process on the initial state (see Lemma 2.3) as well as the control-neutralizer assumption and the growth condition on strategies.

Next, one can deduce from the weak dynamic programming principle and the separability of control space $\mathbb{U}$, $\mathbb{V}$ that the value functions $w_{1}$ and $w_{2}$ are (discontinuous) viscosity solutions of the corresponding fully non-linear PDEs, see Theorem 3.1. Recently, Krylov [24] and [23] studied the regularity of solutions to related fully nonlinear PDEs: The former obtained $C^{1,1} \cap W_{\infty, l o c}^{1,2}$-solutions for the case of bounded measurable coefficients; while the latter showed the existence of $L^{p}$-viscosity solutions in $C^{1+\alpha}$ if the fully non-linear Hamiltonian function is continuous in gradient variable and Lipschitz continuous in Hessian variable.

The rest of the paper is organized as follows: After listing the notations to use, we recall some basic properties of BSDEs in Section 1. In Section 2, we set up the zero-sum stochastic differential games based on BSDEs and present a weak dynamic programming principle for priority values of both players defined via Elliott-Kalton strategies. With help of the weak dynamic programming principle, we show in Section 3 that the priority values are (discontinuous) viscosity solutions of the corresponding fully non-linear PDEs. The proofs of our results are deferred to Section 4 .

### 1.1 Notation and Preliminaries

Let $\left(\mathbb{M}, \rho_{\mathbb{M}}\right)$ be a generic metric space and let $\mathscr{B}(\mathbb{M})$ be the Borel $\sigma$-field of $\mathbb{M}$. For any $x \in \mathbb{M}$ and $\delta>0$, $O_{\delta}(x) \triangleq\left\{x^{\prime} \in \mathbb{M}: \rho_{\mathbb{M}}\left(x, x^{\prime}\right)<\delta\right\}$ and $\bar{O}_{\delta}(x) \triangleq\left\{x^{\prime} \in \mathbb{M}: \rho_{\mathbb{M}}\left(x, x^{\prime}\right) \leq \delta\right\}$ respectively denote the open and closed ball centered at $x$ with radius $\delta$. For any function $\phi: \mathbb{M} \rightarrow \mathbb{R}$, we define the lower/upper semi-continuous envelopes by

$$
\varliminf_{x^{\prime} \rightarrow x} \phi\left(x^{\prime}\right) \triangleq \lim _{n \rightarrow \infty} \uparrow \inf _{x^{\prime} \in O_{\frac{1}{n}}(x)} \phi\left(x^{\prime}\right) \quad \text { and } \quad \varlimsup_{x^{\prime} \rightarrow x} \phi\left(x^{\prime}\right) \triangleq \lim _{n \rightarrow \infty} \downarrow \sup _{x^{\prime} \in O_{\frac{1}{n}}(x)} \phi\left(x^{\prime}\right),
$$

where $\lim _{n \rightarrow \infty} \downarrow$ (resp. $\lim _{n \rightarrow \infty} \uparrow$ ) denotes the limit of a decreasing (resp. increasing) sequence.
Fix $d \in \mathbb{N}$ and a time horizon $T \in(0, \infty)$. We consider the canonical space $\Omega \triangleq\left\{\omega \in \mathbb{C}\left([0, T] ; \mathbb{R}^{d}\right): \omega(0)=0\right\}$ equipped with Wiener measure $P$, under which the canonical process $B$ is a $d$-dimensional Brownian motion. Let $\mathbf{F}=\left\{\mathcal{F}_{t}\right\}_{t \in[0, T]}$ be the filtration generated by $B$ and augmented by all $P$-null sets. We denote by $\mathscr{P}$ the $\mathbf{F}$-progressively measurable $\sigma$-field of $[0, T] \times \Omega$.

Given $t \in[0, T]$, Let $\mathcal{S}_{t, T}$ collect all $\mathbf{F}$-stopping times $\tau$ with $t \leq \tau \leq T, P$-a.s. For any $\tau \in \mathcal{S}_{t, T}$ and $A \in \mathcal{F}_{\tau}$, we define $\llbracket t, \tau \llbracket_{A} \triangleq\{(r, \omega) \in[t, T] \times A: r<\tau(\omega)\}$ and $\llbracket \tau, T \rrbracket_{A} \triangleq\{(r, \omega) \in[t, T] \times A: r \geq \tau(\omega)\}$ for any $A \in \mathcal{F}_{\tau}$. In particular, $\llbracket t, \tau \llbracket \triangleq \llbracket t, \tau \llbracket \Omega$ and $\llbracket \tau, T \rrbracket \triangleq \llbracket \tau, T \rrbracket_{\Omega}$ are the stochastic intervals.

Let $\mathbb{E}$ be a generic Euclidian space. For any $p \in[1, \infty)$ and $t \in[0, T]$, we introduce some spaces of functions:

1) For any sub- $\sigma$-field $\mathcal{G}$ of $\mathcal{F}_{T}$, let $\mathbb{L}^{p}(\mathcal{G}, \mathbb{E})$ be the space of all $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables $\xi$ such that $\|\xi\|_{\mathbb{L}^{p}(\mathcal{G}, \mathbb{E})} \triangleq\left\{E\left[|\xi|^{p}\right]\right\}^{1 / p}<\infty$, and let $\mathbb{L}^{\infty}(\mathcal{G}, \mathbb{E})$ be the space of all $\mathbb{E}$-valued, $\mathcal{G}$-measurable bounded random variables.
2) $\mathbb{C}_{\mathbf{F}}^{p}([t, T], \mathbb{E})$ denotes the space of all $\mathbb{E}$-valued, $\mathbf{F}$-adapted processes $\left\{X_{s}\right\}_{s \in[t, T]}$ with $P$-a.s. continuous paths such that $\|X\|_{\mathbb{C}_{\mathbf{F}}^{p}([t, T], \mathbb{E})} \triangleq\left\{E\left[\sup _{s \in[t, T]}\left|X_{s}\right|^{p}\right]\right\}^{1 / p}<\infty$.
3) $\mathbb{H}_{\mathbf{F}}^{p, l o c}([t, T], \mathbb{E})$ denotes the space of all $\mathbb{E}$-valued, $\mathbf{F}$-progressively measurable processes $\left\{X_{s}\right\}_{s \in[t, T]}$ such that $\int_{t}^{T}\left|X_{s}\right|^{p} d s<\infty, P$-a.s. For any $\widehat{p} \in[1, \infty), \mathbb{H}_{\mathbf{F}}^{p, \widehat{p}}([t, T], \mathbb{E})$ denotes the space of all $\mathbb{E}$-valued, $\mathbf{F}$-progressively measurable processes $\left\{X_{s}\right\}_{s \in[t, T]}$ with $\|X\|_{\mathbb{H}_{\mathbf{F}}^{p, \widehat{p}}([t, T], \mathbb{E})} \triangleq\left\{E\left[\left(\int_{t}^{T}\left|X_{s}\right|^{p} d s\right)^{\widehat{p} / p}\right]\right\}^{1 / \widehat{p}}<\infty$.
4) We also set $\mathbb{G}_{\mathbf{F}}^{p}([t, T]) \triangleq \mathbb{C}_{\mathbf{F}}^{p}([t, T], \mathbb{R}) \times \mathbb{H}_{\mathbf{F}}^{2, p}\left([t, T], \mathbb{R}^{d}\right)$.

If $\mathbb{E}=\mathbb{R}$, we will drop it from the above notations. Moreover, we will use the convention $\inf \emptyset=\infty$.

### 1.2 Backward Stochastic Differential Equations

Given $t \in[0, T]$, a $t$-parameter set $(\eta, f)$ consists of a random variable $\eta \in \mathbb{L}^{0}\left(\mathcal{F}_{T}\right)$ and a function $f:[t, T] \times \Omega \times$ $\mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ that is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}\left(\mathbb{R}^{d}\right) / \mathscr{B}(\mathbb{R})$-measurable. In particular, $(\eta, f)$ is called a $(t, p)$-parameter set for some $p \in[1, \infty)$ if $\eta \in \mathbb{L}^{p}\left(\mathcal{F}_{T}\right)$.

Definition 1.1. Given a $t$-parameter set $(\eta, f)$ for some $t \in[0, T]$, a pair $(Y, Z) \in \mathbb{C}_{\mathbf{F}}^{0}([t, T]) \times \mathbb{H}_{\mathbf{F}}^{2, \text { loc }}\left([t, T], \mathbb{R}^{d}\right)$ is called a solution of the backward stochastic differential equation on the probability space $\left(\Omega, \mathcal{F}_{T}, P\right)$ over period $[t, T]$ with terminal condition $\eta$ and generator $f(\operatorname{BSDE}(t, \eta, f)$ for short $)$ if it holds $P$-a.s. that

$$
\begin{equation*}
Y_{s}=\eta+\int_{s}^{T} f\left(r, Y_{r}, Z_{r}\right) d r-\int_{s}^{T} Z_{r} d B_{r}, \quad s \in[t, T] \tag{1.3}
\end{equation*}
$$

Analogous to Theorem 4.2 of [8], we have the following well-posedness result of BSDE (1.3).
Proposition 1.1. Given $t \in[0, T]$ and $p \in[1, \infty)$, let $(\eta, f)$ be a $(t, p)$-parameter set such that $f$ is Lipschitz continuous in $(y, z)$ : i.e. for some $\gamma>0$, it holds for $d s \times d P-a . s$. $(s, \omega) \in[t, T] \times \Omega$ that

$$
\left|f(s, \omega, y, z)-f\left(s, \omega, y^{\prime}, z^{\prime}\right)\right| \leq \gamma\left(\left|y-y^{\prime}\right|+\left|z-z^{\prime}\right|\right), \quad \forall y, y^{\prime} \in \mathbb{R}, \quad \forall z, z^{\prime} \in \mathbb{R}^{d}
$$

If $E\left[\left(\int_{t}^{T}|f(s, 0,0)| d s\right)^{p}\right]<\infty, B S D E$ (1.3) admits a unique solution $(Y, Z) \in \mathbb{G}_{\mathbf{F}}^{p}([t, T])$ that satisfies

$$
\begin{equation*}
E\left[\sup _{s \in[t, T]}\left|Y_{s}\right|^{\mid} \mid \mathcal{F}_{t}\right] \leq C(T, p, \gamma) E\left[|\eta|^{p}+\left(\int_{t}^{T}|f(s, 0,0)| d s\right)^{p} \mid \mathcal{F}_{t}\right], \quad P-a . s . \tag{1.4}
\end{equation*}
$$

Also, we have the following a priori estimate and comparison results for BSDE (1.3).

Proposition 1.2. Given $t \in[0, T]$ and $p \in[1, \infty)$, let $\left(\eta_{i}, f_{i}\right), i=1,2$ be two $(t, p)$-parameter sets such that $f_{1}$ is Lipschitz continuous in $(y, z)$, and let $\left(Y^{i}, Z^{i}\right) \in \mathbb{G}_{\mathbf{F}}^{p}([t, T]), i=1,2$ be a solution of $\operatorname{BSDE}\left(t, \eta_{i}, f_{i}\right)$.
(1) If $E\left[\left(\int_{t}^{T}\left|f_{1}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f_{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)\right| d s\right)^{\widetilde{p}}\right]<\infty$ for some $\widetilde{p} \in(1, p]$, then it holds $P-a . s$. that

$$
\begin{equation*}
E\left[\sup _{s \in[t, T]}\left|Y_{s}^{1}-Y_{s}^{2}\right|^{\widetilde{p}} \mid \mathcal{F}_{t}\right] \leq C(T, \widetilde{p}, \gamma) E\left[\left|\eta_{1}-\eta_{2}\right|^{\widetilde{p}}+\left(\int_{t}^{T}\left|f_{1}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f_{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)\right| d s\right)^{\widetilde{p}} \mid \mathcal{F}_{t}\right] \tag{1.5}
\end{equation*}
$$

(2) If $\eta_{1} \leq($ resp. $\geq) \eta_{2}, P-a . s$. and if $f_{1}\left(s, Y_{s}^{2}, Z_{s}^{2}\right) \leq($ resp. $\geq) f_{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)$,ds $\times d P-a . s$. on $[t, T] \times \Omega$, then it holds $P$-a.s. that $Y_{s}^{1} \leq($ resp.$\geq) Y_{s}^{2}$ for any $s \in[t, T]$.

## 2 Stochastic Differential Games with Super-square-integrable Controls

Let $\left(\mathbb{U}, \rho_{\mathbb{U}}\right)$ and $\left(\mathbb{V}, \rho_{\mathbb{V}}\right)$ be two separable metric spaces. For some $u_{0} \in \mathbb{U}$ and $v_{0} \in \mathbb{V}$, we define

$$
[u]_{\mathbb{U}} \triangleq \rho_{\mathbb{U}}\left(u, u_{0}\right), \quad \forall u \in \mathbb{U} \quad \text { and } \quad[v]_{\mathbb{V}} \triangleq \rho_{\mathbb{V}}\left(v, v_{0}\right), \quad \forall v \in \mathbb{V}
$$

We shall study a zero-sum stochastic differential game between two players, player I and player II, who choose super-square-integrable $\mathbb{U}$-valued controls and $\mathbb{V}$-valued controls respectively to compete:

Definition 2.1. Given $t \in[0, T]$, an admissible control process $\mu=\left\{\mu_{s}\right\}_{s \in[t, T]}$ for player I over period $[t, T]$ is $a \mathbb{U}$-valued, $\mathbf{F}$-progressively measurable process such that $E \int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s<\infty$ for some $q>2$. Admissible control processes for player II over period $[t, T]$ are defined similarly. We denote by $\mathcal{U}_{t}\left(\right.$ resp. $\left.\mathcal{V}_{t}\right)$ the set of all admissible controls for player I (resp. II) over period $[t, T]$.

Remark 2.1. The reason why we use super-square-integrable controls lies in the fact that in the proof of Proposition 2.2, the set of $\mathbb{U}$-valued (resp. $\mathbb{V}$-valued) square integrable processes is not closed under Girsanov transformation via functions of the Cameron-Martin space (see in particular (4.17)).

Clearly, connecting two $\mathcal{U}_{t}$-controls along some $\tau \in \mathcal{S}_{t, T}$ results in a new $\mathcal{U}_{t}$-control:
Lemma 2.1. Let $t \in[0, T]$ and $\tau \in \mathcal{S}_{t, T}$. For any $\mu^{1}, \mu^{2} \in \mathcal{U}_{t}, \mu_{s} \triangleq \mathbf{1}_{\{s<\tau\}} \mu_{s}^{1}+\mathbf{1}_{\{s \geq \tau\}} \mu_{s}^{2}$, $s \in[t, T]$ defines a $\mathcal{U}_{t}$ - control. Similarly, for any $\nu^{1}, \nu^{2} \in \mathcal{V}_{t}, \nu_{s} \triangleq \mathbf{1}_{\{s<\tau\}} \nu_{s}^{1}+\mathbf{1}_{\{s \geq \tau\}} \nu_{s}^{2}, s \in[t, T]$ defines a $\mathcal{V}_{t}$-control.

### 2.1 Game Setting: A Controlled SDE-BSDE System

Our zero-sum stochastic differential game is formulated via a (decoupled) SDE-BSDE system with the following parameters: Fix $k \in \mathbb{N}, \gamma>0$ and $p \in(1,2]$.

1) Let $b:[0, T] \times \mathbb{R}^{k} \times \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{R}^{k}$ be a $\mathscr{B}([0, T]) \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) \otimes \mathscr{B}(\mathbb{U}) \otimes \mathscr{B}(\mathbb{V}) / \mathscr{B}\left(\mathbb{R}^{k}\right)$-measurable function and let $\sigma:[0, T] \times \mathbb{R}^{k} \times \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{R}^{k \times d}$ be a $\mathscr{B}([0, T]) \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) \otimes \mathscr{B}(\mathbb{U}) \otimes \mathscr{B}(\mathbb{V}) / \mathscr{B}\left(\mathbb{R}^{k \times d}\right)$-measurable function such that for any $(t, u, v) \in[0, T] \times \mathbb{U} \times \mathbb{V}$ and $x, x^{\prime} \in \mathbb{R}^{k}$

$$
\begin{align*}
|b(t, 0, u, v)| & +|\sigma(t, 0, u, v)| \leq \gamma\left(1+[u]_{\mathbb{U}}+[v]_{\mathbb{V}}\right)  \tag{2.1}\\
\text { and } \quad\left|b(t, x, u, v)-b\left(t, x^{\prime}, u, v\right)\right| & +\left|\sigma(t, x, u, v)-\sigma\left(t, x^{\prime}, u, v\right)\right| \leq \gamma\left|x-x^{\prime}\right| . \tag{2.2}
\end{align*}
$$

2) Let $g: \mathbb{R}^{k} \rightarrow \mathbb{R}$ be a $2 / p-$ Hölder continuous function with coefficient $\gamma$.
3) Let $f:[0, T] \times \mathbb{R}^{k} \times \mathbb{R} \times \mathbb{R}^{d} \times \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{R}$ be $\mathscr{B}([0, T]) \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}\left(\mathbb{R}^{d}\right) \otimes \mathscr{B}(\mathbb{U}) \otimes \mathscr{B}(\mathbb{V}) / \mathscr{B}(\mathbb{R})$-measurable function such that for any $(t, u, v) \in[0, T] \times \mathbb{U} \times \mathbb{V}$ and any $(x, y, z),\left(x^{\prime}, y^{\prime}, z^{\prime}\right) \in \mathbb{R}^{k} \times \mathbb{R} \times \mathbb{R}^{d}$

$$
\begin{align*}
|f(t, 0,0,0, u, v)| & \leq \gamma\left(1+[u]_{\mathbb{U}}^{2 / p}+[v]_{\mathbb{V}}^{2 / p}\right)  \tag{2.3}\\
\text { and } \quad\left|f(t, x, y, z, u, v)-f\left(t, x^{\prime}, y^{\prime}, z^{\prime}, u, v\right)\right| & \leq \gamma\left(\left|x-x^{\prime}\right|^{2 / p}+\left|y-y^{\prime}\right|+\left|z-z^{\prime}\right|\right) \tag{2.4}
\end{align*}
$$

For any $\lambda \geq 0$, we let $c_{\lambda}$ denote a generic constant, depending on $\lambda, T, \gamma, p$ and $|g(0)|$, whose form may vary from line to line. (In particular, $c_{0}$ stands for a generic constant depending on $T, \gamma, p$ and $|g(0)|$.)

Also, we would like to introduce two control neutralizers $\psi, \widetilde{\psi}$ for the coefficients: For some $\kappa>0$
(A-u) there exist a function $\psi:[0, T] \times\left(\mathbb{U} \backslash O_{\kappa}\left(u_{0}\right)\right) \rightarrow \mathbb{V}$ that is $\mathscr{B}([0, T]) \times \mathscr{B}\left(\mathbb{U} \backslash O_{\kappa}\left(u_{0}\right)\right) / \mathscr{B}(\mathbb{V})$-measurable and satisfies: for any $(t, x, y, z) \in[0, T] \times \mathbb{R}^{k} \times \mathbb{R} \times \mathbb{R}^{d}$ and $u, u^{\prime} \in \mathbb{U} \backslash O_{\kappa}\left(u_{0}\right)$

$$
\begin{aligned}
& b(t, x, u, \psi(t, u))=b\left(t, x, u^{\prime}, \psi\left(t, u^{\prime}\right)\right), \quad \sigma(t, x, u, \psi(t, u))=\sigma\left(t, x, u^{\prime}, \psi\left(t, u^{\prime}\right)\right) \\
& f(t, x, y, z, u, \psi(t, u))=f\left(t, x, y, z, u^{\prime}, \psi\left(t, u^{\prime}\right)\right) \quad \text { and } \quad[\psi(t, u)]_{\mathbb{V}} \leq \kappa\left(1+[u]_{\mathbb{U}}\right)
\end{aligned}
$$

(A-v) and there exists a function $\tilde{\psi}:[0, T] \times\left(\mathbb{V} \backslash O_{\kappa}\left(v_{0}\right)\right) \rightarrow \mathbb{U}$ that is $\mathscr{B}([0, T]) \times \mathscr{B}\left(\mathbb{V} \backslash O_{\kappa}\left(v_{0}\right)\right) / \mathscr{B}(\mathbb{U})$-measurable and satisfies: for any $(t, x, y, z) \in[0, T] \times \mathbb{R}^{k} \times \mathbb{R} \times \mathbb{R}^{d}$ and $v, v^{\prime} \in \mathbb{V} \backslash O_{\kappa}\left(v_{0}\right)$

$$
\begin{aligned}
& b(t, x, \widetilde{\psi}(t, v), v)=b\left(t, x, \widetilde{\psi}\left(t, v^{\prime}\right), v^{\prime}\right), \quad \sigma(t, x, \widetilde{\psi}(t, v), v)=\sigma\left(t, x, \widetilde{\psi}\left(t, v^{\prime}\right), v^{\prime}\right), \\
& f(t, x, y, z, \widetilde{\psi}(t, v), v)=f\left(t, x, y, z, \widetilde{\psi}\left(t, v^{\prime}\right), v^{\prime}\right) \quad \text { and } \quad[\widetilde{\psi}(t, v)]_{\mathbb{U}} \leq \kappa\left(1+[v]_{\mathbb{V}}\right) .
\end{aligned}
$$

A typical example satisfying both (A-u) and (A-v) is the additive-control case:
Example 2.1. Let $\mathbb{U}=\mathbb{V}=\mathbb{R}^{\ell}$ and consider the following coefficients:

$$
\begin{aligned}
& b(t, x, u, v)=b(t, x, u+v), \quad \sigma(t, x, u, v)=\sigma(t, x, u+v) \quad \text { and } \\
& f(t, x, y, z, u, v)=f(t, x, y, z, u+v), \quad \forall(t, x, y, z, u, v) \in[0, T] \times \mathbb{R}^{k} \times \mathbb{R} \times \mathbb{R}^{d} \times \mathbb{U} \times \mathbb{V}
\end{aligned}
$$

Then $(A-u)$ and $(A-v)$ hold for functions $\psi(u)=-u$ and $\widetilde{\psi}(v)=-v$ respectively.
Here is another example:
Example 2.2. Given $\gamma>0$, let $b_{0}, \sigma_{0}:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be two $\mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{R}) / \mathscr{B}(\mathbb{R})$ - measurable functions and let $f_{0}:[0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be $\mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}) / \mathscr{B}(\mathbb{R})$-measurable function such that for any $t \in[0, T]$ and $(x, y, z),\left(x^{\prime}, y^{\prime}, z^{\prime}\right) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$
\left|b_{0}(t, x)-b_{0}\left(t, x^{\prime}\right)\right|+\left|\sigma_{0}(t, x)-\sigma_{0}\left(t, x^{\prime}\right)\right|+\left|f_{0}(t, x, y, z)-f_{0}\left(t, x^{\prime}, y^{\prime}, z^{\prime}\right)\right| \leq \gamma\left(\left|x-x^{\prime}\right|+\left|y-y^{\prime}\right|+\left|z-z^{\prime}\right|\right)
$$

Also, let $\mathbb{U}=\mathbb{V}=\mathbb{R}, \kappa>0$ and $\varphi:[0, T] \times \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{R}$ be a jointly continuous function such that $\varphi$ is Lipschitz continuous in $(u, v)$ with coefficient $\gamma, \sup _{t \in[0, T]}|\varphi(t, 0,0)| \leq \gamma$, and for any $(t, u, v) \in[0, T] \times \mathbb{U} \times \mathbb{V}$

$$
\begin{equation*}
\inf _{\left|v^{\prime}\right| \leq \kappa|u|} \varphi\left(t, u, v^{\prime}\right) \leq 0 \leq \sup _{\left|v^{\prime}\right| \leq \kappa|u|} \varphi\left(t, u, v^{\prime}\right) \quad \text { and } \inf _{\left|u^{\prime}\right| \leq \kappa|v|} \varphi\left(t, u^{\prime}, v\right) \leq 0 \leq \sup _{\left|u^{\prime}\right| \leq \kappa|v|} \varphi\left(t, u^{\prime}, v\right) \tag{2.5}
\end{equation*}
$$

Then $b(t, x, u, v) \triangleq b_{0}(t, x)+\varphi(t, u, v), \sigma(t, x, u, v) \triangleq \sigma_{0}(t, x)+\varphi(t, u, v)$ and $f(t, x, y, z, u, v) \triangleq f_{0}(t, x, y, z)+$ $\varphi(t, u, v), \forall(t, x, y, z, u, v) \in[0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{V}$ are the measurable functions satisfying (2.1) - (2.4) with $k=d=1$ and $p=2$. We will show at the beginning of Subsection 4.2 that ( $A-u$ ) and (A-v) hold for these coefficients.

When the game begins at time $t \in[0, T]$, player I and player II select admissible controls $\mu \in \mathcal{U}_{t}$ and $\nu \in \mathcal{V}_{t}$ respectively. Then the state process starting from $\xi \in \mathbb{L}^{2}\left(\mathcal{F}_{t}, \mathbb{R}^{k}\right)$ will evolve according to SDE (1.1) on the probability space $\left(\Omega, \mathcal{F}_{T}, P\right)$. The measurability of functions $b, \sigma, \mu$ and $\nu$ implies that

$$
b^{\mu, \nu}(s, \omega, x) \triangleq b\left(s, x, \mu_{s}(\omega), \nu_{s}(\omega)\right), \quad \forall(s, \omega, x) \in[t, T] \times \Omega \times \mathbb{R}^{k}
$$

is $\mathscr{P} \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) / \mathscr{B}\left(\mathbb{R}^{k}\right)$-measurable and that

$$
\sigma^{\mu, \nu}(s, \omega, x) \triangleq \sigma\left(s, x, \mu_{s}(\omega), \nu_{s}(\omega)\right), \quad \forall(s, \omega, x) \in[t, T] \times \Omega \times \mathbb{R}^{k}
$$

is $\mathscr{P} \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) / \mathscr{B}\left(\mathbb{R}^{k \times d}\right)$-measurable. Also, (2.2), (2.1) and Hölder's inequality show that $b^{\mu, \nu}, \sigma^{\mu, \nu}$ are Lipschitz continuous in $x$ and satisfy

$$
E\left[\left(\int_{t}^{T}\left|b^{\mu, \nu}(s, 0)\right| d s\right)^{2}+\left(\int_{t}^{T}\left|\sigma^{\mu, \nu}(s, 0)\right| d s\right)^{2}\right] \leq c_{0}+c_{0} E \int_{t}^{T}\left(\left[\mu_{s}\right]_{\mathbb{U}}^{2}+\left[\nu_{s}\right]_{\mathbb{V}}^{2}\right) d s<\infty
$$

Then it is well-known that the $\operatorname{SDE}(1.1)$ admits a unique solution $\left\{X_{s}^{t, \xi, \mu, \nu}\right\}_{s \in[t, T]} \in \mathbb{C}_{\mathbf{F}}^{2}\left([t, T], \mathbb{R}^{k}\right)$ such that

$$
\begin{align*}
E\left[\sup _{s \in[t, T]}\left|X_{s}^{t, \xi, \mu, \nu}\right|^{2}\right] & \leq c_{0} E\left[|\xi|^{2}\right]+c_{0} E\left[\left(\int_{t}^{T}\left|b^{\mu, \nu}(s, 0)\right| d s\right)^{2}+\left(\int_{t}^{T}\left|\sigma^{\mu, \nu}(s, 0)\right| d s\right)^{2}\right] \\
& \leq c_{0}\left(1+E\left[|\xi|^{2}\right]+E \int_{t}^{T}\left(\left[\mu_{s}\right]_{\mathbb{U}}^{2}+\left[\nu_{s}\right]_{\mathbb{V}}^{2}\right) d s\right)<\infty \tag{2.6}
\end{align*}
$$

Given $s \in[t, T]$, let $[\mu]^{s}$ denote the restriction of $\mu$ over period $[s, T]: i . e .,[\mu]_{r}^{s} \triangleq \mu_{r}, \forall r \in[s, T]$. Clearly, $[\mu]^{s} \in \mathcal{U}_{s}$, similarly, $\left\{[\nu]_{r}^{s} \triangleq \nu_{r}\right\}_{r \in[s, T]} \in \mathcal{V}_{s}$. As

$$
\begin{aligned}
X_{r}^{t, \xi, \mu, \nu} & =X_{s}^{t, \xi, \mu, \nu}+\int_{s}^{r} b\left(r^{\prime}, X_{r^{\prime}}^{t, \xi, \mu, \nu}, \mu_{r^{\prime}}, \nu_{r^{\prime}}\right) d r^{\prime}+\int_{s}^{r} \sigma\left(r^{\prime}, X_{r^{\prime}}^{t, \xi, \mu, \nu}, \mu_{r^{\prime}}, \nu_{r^{\prime}}\right) d B_{r^{\prime}} \\
& =X_{s}^{t, \xi, \mu, \nu}+\int_{s}^{r} b\left(r^{\prime}, X_{r^{\prime}}^{t, \xi, \mu, \nu},[\mu]_{r^{\prime}}^{s},[\nu]_{r^{\prime}}^{s}\right) d r^{\prime}+\int_{s}^{r} \sigma\left(r^{\prime}, X_{r^{\prime}}^{t, \xi, \mu, \nu},[\mu]_{r^{\prime}}^{s},[\nu]_{r^{\prime}}^{s}\right) d B_{r^{\prime}}, \quad r \in[s, T]
\end{aligned}
$$

we see that $\left\{X_{r}^{t, \xi, \mu, \nu}\right\}_{r \in[s, T]} \in \mathbb{C}_{\mathbf{F}}^{2}\left([s, T], \mathbb{R}^{k}\right)$ solves (1.1) with the parameters $\left(s, X_{s}^{t, \xi, \mu, \nu},[\mu]^{s},[\nu]^{s}\right)$. To wit, it holds $P$-a.s. that

$$
\begin{equation*}
X_{r}^{t, \xi, \mu, \nu}=X_{r}^{s, X_{s}^{t, \xi, \mu, \nu},[\mu]^{s},[\nu]^{s}}, \quad \forall r \in[s, T] \tag{2.7}
\end{equation*}
$$

Moreover, the state process depends on controls in the following way:
Lemma 2.2. Given $t \in[0, T]$, let $\xi \in \mathbb{L}^{2}\left(\mathcal{F}_{t}, \mathbb{R}^{k}\right)$ and $(\mu, \nu),(\widetilde{\mu}, \widetilde{\nu}) \in \mathcal{U}_{t} \times \mathcal{V}_{t}$. If $(\mu, \nu)=(\widetilde{\mu}, \widetilde{\nu})$, dr $\times d P-a . s$. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ for some $\tau \in \mathcal{S}_{t, T}$ and $A \in \mathcal{F}_{\tau}$, then it holds $P$-a.s. that

$$
\begin{equation*}
\mathbf{1}_{A} X_{s}^{t, \xi, \mu, \nu}+\mathbf{1}_{A^{c}} X_{\tau \wedge s}^{t, \xi, \mu, \nu}=\mathbf{1}_{A} X_{s}^{t, \xi, \widetilde{\mu}, \widetilde{\nu}}+\mathbf{1}_{A^{c}} X_{\tau \wedge s}^{t, \xi, \widetilde{,}, \widetilde{\nu}}, \quad \forall s \in[t, T] \tag{2.8}
\end{equation*}
$$

Now, let $\Theta$ stand for the quadruplet $(t, \xi, \mu, \nu)$. Given $\tau \in \mathcal{S}_{t, T}$, the measurability of $\left(f, X^{\Theta}, \mu, \nu\right)$ and (2.4) imply that

$$
f_{\tau}^{\Theta}(s, \omega, y, z) \triangleq \mathbf{1}_{\{s<\tau(\omega)\}} f\left(s, X_{s}^{\Theta}(\omega), y, z, \mu_{s}(\omega), \nu_{s}(\omega)\right), \quad \forall(s, \omega, y, z) \in[t, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{d}
$$

is a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}\left(\mathbb{R}^{d}\right) / \mathscr{B}(\mathbb{R})$-measurable function that is Lipschitz continuous in $(y, z)$ with coefficient $\gamma$. And one can deduce from (2.3), (2.4) and Hölder's inequality that

$$
\begin{equation*}
E\left[\left(\int_{t}^{T}\left|f_{\tau}^{\Theta}(s, 0,0)\right| d s\right)^{p}\right] \leq c_{0}+c_{0} E\left[\sup _{s \in[t, T]}\left|X_{s}^{\Theta}\right|^{2}+\int_{t}^{T}\left(\left[\mu_{s}\right]_{\mathbb{U}}^{2}+\left[\nu_{s}\right]_{\mathbb{V}}^{2}\right) d s\right]<\infty \tag{2.9}
\end{equation*}
$$

Thus, for any $\eta \in \mathbb{L}^{p}\left(\mathcal{F}_{\tau}\right)$, Proposition 1.1 shows that the $\operatorname{BSDE}\left(t, \eta, f_{\tau}^{\Theta}\right)$ admits a unique solution $\left(Y^{\Theta}(\tau, \eta), Z^{\Theta}(\tau, \eta)\right)$ $\in \mathbb{G}_{\mathbf{F}}^{p}([t, T])$, which has the following estimate as a consequence of (1.5).

Corollary 2.1. Let $t \in[0, T], \xi \in \mathbb{L}^{2}\left(\mathcal{F}_{t}, \mathbb{R}^{k}\right)$, $(\mu, \nu) \in \mathcal{U}_{t} \times \mathcal{V}_{t}$ and $\tau \in \mathcal{S}_{t, T}$. Given $\eta_{1}, \eta_{2} \in \mathbb{L}^{p}\left(\mathcal{F}_{\tau}\right)$, it holds for any $\widetilde{p} \in(1, p]$ that

$$
\begin{equation*}
E\left[\sup _{s \in[t, T]}\left|Y_{s}^{t, \xi, \mu, \nu}\left(\tau, \eta_{1}\right)-Y_{s}^{t, \xi, \mu, \nu}\left(\tau, \eta_{2}\right)\right|^{\widetilde{p}} \mid \mathcal{F}_{t}\right] \leq c_{\widetilde{p}} E\left[\left|\eta_{1}-\eta_{2}\right|^{\widetilde{p}} \mid \mathcal{F}_{t}\right], \quad P-a . s \tag{2.10}
\end{equation*}
$$

Given another stopping time $\zeta \in \mathcal{S}_{t, T}$ with $\zeta \leq \tau, P$-a.s., one can easily show that $\left\{\left(Y_{\zeta \wedge s}^{\Theta}(\tau, \eta), \mathbf{1}_{\{s<\zeta\}} Z_{s}^{\Theta}(\tau, \eta)\right)\right\}_{s \in[t, T]}$ $\in \mathbb{G}_{\mathbf{F}}^{p}([t, T])$ solves the $\operatorname{BSDE}\left(t, Y_{\zeta}^{\Theta}(\tau, \eta), f_{\zeta}^{\Theta}\right)$. To wit, we have

$$
\begin{equation*}
\left(Y_{s}^{\Theta}\left(\zeta, Y_{\zeta}^{\Theta}(\tau, \eta)\right), Z_{s}^{\Theta}\left(\zeta, Y_{\zeta}^{\Theta}(\tau, \eta)\right)\right)=\left(Y_{\zeta \wedge s}^{\Theta}(\tau, \eta), \mathbf{1}_{\{s<\zeta\}} Z_{s}^{\Theta}(\tau, \eta)\right), \quad s \in[t, T] \tag{2.11}
\end{equation*}
$$

In particular, when $\zeta=\tau$,

$$
\begin{equation*}
\left(Y_{s}^{\Theta}(\tau, \eta), Z_{s}^{\Theta}(\tau, \eta)\right)=\left(Y_{\tau \wedge s}^{\Theta}(\tau, \eta), \mathbf{1}_{\{s<\tau\}} Z_{s}^{\Theta}(\tau, \eta)\right), \quad s \in[t, T] \tag{2.12}
\end{equation*}
$$

On the other hand, if $\tau \in \mathcal{S}_{s, T}$ for some $s \in[t, T]$, letting $\Theta^{s} \triangleq\left(s, X_{s}^{\Theta},[\mu]^{s},[\nu]^{s}\right)$, we can deduce from (2.7) that $\left\{\left(Y_{r}^{\Theta}(\tau, \eta), Z_{r}^{\Theta}(\tau, \eta)\right)\right\}_{r \in[s, T]} \in \mathbb{G}_{\mathbf{F}}^{p}([s, T])$ solves the following $\operatorname{BSDE}\left(s, \eta, f_{\tau}^{\Theta}\right)$ :

$$
\begin{aligned}
Y_{s} & =\eta+\int_{r}^{T} \mathbf{1}_{\left\{r^{\prime}<\tau\right\}} f\left(r^{\prime}, X_{r^{\prime}}^{\Theta}, Y_{r^{\prime}}, Z_{r^{\prime}}, \mu_{r^{\prime}}, \nu_{r^{\prime}}\right) d r^{\prime}-\int_{r}^{T} Z_{r^{\prime}} d B_{r^{\prime}} \\
& =\eta+\int_{r}^{T} \mathbf{1}_{\left\{r^{\prime}<\tau\right\}} f\left(r^{\prime}, X_{r^{\prime}}^{\Theta^{s}}, Y_{r^{\prime}}, Z_{r^{\prime}},[\mu]_{r^{\prime}}^{s},[\nu]_{r^{\prime}}^{s}\right) d r^{\prime}-\int_{r}^{T} Z_{r^{\prime}} d B_{r^{\prime}}, \quad r \in[s, T] .
\end{aligned}
$$

Hence, it holds $P$-a.s. that

$$
\begin{equation*}
Y_{r}^{\Theta}(\tau, \eta)=Y_{r}^{\Theta^{s}}(\tau, \eta), \quad \forall r \in[s, T] \tag{2.13}
\end{equation*}
$$

The $2 / p-$ Hölder continuity of functions $g$ and (2.6) show that $g\left(X_{T}^{\Theta}\right) \in \mathbb{L}^{p}\left(\mathcal{F}_{T}\right)$. Set $J(\Theta) \triangleq Y_{t}^{\Theta}\left(T, g\left(X_{T}^{\Theta}\right)\right)$ From (1.5) and the standard estimate of SDE (1.1), we can deduce the following a priori estimate:

Lemma 2.3. Let $t \in[0, T]$ and $(\mu, \nu) \in \mathcal{U}_{t} \times \mathcal{V}_{t}$. Given $\xi_{1}, \xi_{2} \in \mathbb{L}^{2}\left(\mathcal{F}_{t}, \mathbb{R}^{k}\right)$, it holds for any $\tilde{p} \in(1, p]$ that

$$
\begin{equation*}
E\left[\sup _{s \in[t, T]}\left|Y_{s}^{t, \xi_{1}, \mu, \nu}\left(T, g\left(X_{T}^{t, \xi_{1}, \mu, \nu}\right)\right)-Y_{s}^{t, \xi_{2}, \mu, \nu}\left(T, g\left(X_{T}^{t, \xi_{2}, \mu, \nu}\right)\right)\right|^{\widetilde{p}} \mid \mathcal{F}_{t}\right] \leq c_{\widetilde{p}}\left|\xi_{1}-\xi_{2}\right|^{\frac{2 \tilde{p}}{p}}, \quad P-a . s . \tag{2.14}
\end{equation*}
$$

### 2.2 Definition of the Value Functions and a Weak Dynamic Programming Principle

Now, we are ready to introduce values of the zero-sum stochastic differential games via the following version of Elliott-Kalton strategies (or non-anticipative strategies).

Definition 2.2. Given $t \in[0, T]$, an admissible strategy $\alpha$ for player I over period $[t, T]$ is a mapping $\alpha: \mathcal{V}_{t} \rightarrow \mathcal{U}_{t}$ satisfying: ( $i$ ) There exists a $C_{\alpha}>0$ such that for any $\nu \in \mathcal{V}_{t}\left[(\alpha(\nu))_{s}\right]_{\mathbb{U}} \leq \kappa+C_{\alpha}\left[\nu_{s}\right]_{\mathbb{V}}, d s \times d P-a . s$., where $\kappa$ is the constant that appears in $(A-u)$ and $(A-v)$; (ii) For any $\nu^{1}, \nu^{2} \in \mathcal{V}_{t}, \tau \in \mathcal{S}_{t, T}$ and $A \in \mathcal{F}_{\tau}$, if $\nu^{1}=\nu^{2}$, ds $\times d P-a . s$. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$, then $\alpha\left(\nu^{1}\right)=\alpha\left(\nu^{2}\right)$, $d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$.

Admissible strategies $\beta: \mathcal{U}_{t} \rightarrow \mathcal{V}_{t}$ for player II over period $[t, T]$ are defined similarly. The collection of all admissible strategies for player $I$ (resp. II) over period $[t, T]$ is denoted by $\mathfrak{A}_{t}\left(\right.$ resp. $\left.\mathfrak{B}_{t}\right)$.

Remark 2.2. The condition (ii) of Definition 2.2 is called the nonanticipativity of strategies. It is said in [11, line 4 of page 456$]$ that "From the nonanticipativity of $\beta_{2}$ we have $\beta_{2}\left(u_{2}^{\varepsilon}\right)=\sum_{j \geq 1} \mathbf{1}_{\Delta_{j}} \beta_{2}\left(u_{j}^{2}\right)$, .. ". What actually used in this equality is not the nonanticipativity of $\beta_{2}$ as defined in Definition 3.2 therein, but the requirement:

For any $u, \widetilde{u} \in \mathcal{U}_{t+\delta, T}$ and $A \in \mathcal{F}_{t+\delta}$, if $u=\widetilde{u}$ on $[t+\delta, T] \times A$, then $\beta_{2}(u)=\beta_{2}(\widetilde{u})$ on $[t+\delta, T] \times A$.
Since $\beta_{2}$ is a restriction of strategy $\beta \in \mathcal{B}_{t, T}$ over period $[t+\delta, T]$, (2.15) entails the following condition on $\beta$.
For any $u, \widetilde{u} \in \mathcal{U}_{t, T}$, any $s \in[t, T]$ and any $A \in \mathcal{F}_{s}$, if $u=\widetilde{u}$ on $([t, s) \times \Omega) \cup([s, T] \times A)$, then $\beta(u)=\beta(\widetilde{u})$ on $([t, s) \times \Omega) \cup([s, T] \times A)$.
which is exactly a simple version of our nonanticipativity condition on strategies with $\tau=s$.

For any $(t, x) \in[0, T] \times \mathbb{R}^{k}$, we define

$$
\begin{aligned}
& w_{1}(t, x) \triangleq \underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} J(t, x, \mu, \beta(\mu))=\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(T, g\left(X_{T}^{t, x, \mu, \beta(\mu)}\right)\right) \\
& \text { and } \quad w_{2}(t, x) \triangleq \underset{\alpha \in \mathfrak{A}_{t}}{\operatorname{esssup}} \operatorname{essinf}_{\nu \in \mathcal{V}_{t}} J(t, x, \alpha(\nu), \nu)=\underset{\alpha \in \mathfrak{A}_{t}}{\operatorname{esssup}} \underset{\nu \in \mathcal{V}_{t}}{\operatorname{essinf}} Y_{t}^{t, x, \alpha(\nu), \nu}\left(T, g\left(X_{T}^{t, x, \alpha(\nu), \nu}\right)\right)
\end{aligned}
$$

as player I's and player II's priority values of the zero-sum stochastic differential game that starts from time $t$ with initial state $x$.

Remark 2.3. When $f$ is independent of $(y, z), w_{1}$ and $w_{2}$ are in form of

$$
\begin{aligned}
w_{1}(t, x) & \triangleq \underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} E\left[g\left(X_{T}^{t, x, \mu, \beta(\mu)}\right)+\int_{t}^{T} f\left(s, X_{s}^{t, x, \mu, \beta(\mu)}, \mu_{s},(\beta(\mu))_{s}\right) d s \mid \mathcal{F}_{t}\right] \\
\text { and } w_{2}(t, x) & \triangleq \underset{\alpha \in \mathfrak{A}_{t}}{\operatorname{esssup}} \underset{\nu \in \mathcal{V}_{t}}{\operatorname{essinf}} E\left[g\left(X_{T}^{t, x, \alpha(\nu), \nu}\right)+\int_{t}^{T} f\left(s, X_{s}^{t, x, \alpha(\nu), \nu},(\alpha(\nu))_{s}, \nu_{s}\right) d s \mid \mathcal{F}_{t}\right], \quad \forall(t, x) \in[0, T] \times \mathbb{R}^{k}
\end{aligned}
$$

Remark 2.4. When $U$ and $V$ are compact (say $\mathbb{U}=\bar{O}_{\kappa}\left(u_{0}\right)$ and $\left.\mathbb{V}=\bar{O}_{\kappa}\left(v_{0}\right)\right)$, Assumptions (A-u), (A-v) are no longer needed, and the integrability condition in Definition 2.1 as well as the condition ( $i$ ) in Definition 2.2 hold automatically. Thus our game problem degenerates to the case of [11].

Let us review some basic properties of the essential extrema for the later use (see e.g. [26, Proposition VI-1-1] or [17. Theorem A.32]):
Lemma 2.4. Let $\left\{\xi_{i}\right\}_{i \in \mathcal{I}},\left\{\eta_{i}\right\}_{i \in \mathcal{I}}$ be two classes of $\mathcal{F}_{T}$-measurable random variables with the same index set $\mathcal{I}$.
(1) If $\xi_{i} \leq(=) \eta_{i}, P-$ a.s. holds for all $i \in \mathcal{I}$, then $\underset{i \in \mathcal{I}}{\operatorname{esssup}} \xi_{i} \leq(=) \underset{i \in \mathcal{I}}{\operatorname{esssup}} \eta_{i}, P-a . s$.
(2) For any $A \in \mathcal{F}_{T}$, it holds $P$-a.s. that $\operatorname{esssup}_{i \in \mathcal{I}}\left(\mathbf{1}_{A} \xi_{i}+\mathbf{1}_{A^{c}} \eta_{i}\right)=\mathbf{1}_{A} \underset{i \in \mathcal{I}}{\operatorname{esssup}} \xi_{i}+\mathbf{1}_{A^{c}} \underset{i \in \mathcal{I}}{\operatorname{esssup}} \eta_{i}$. In particular, $\underset{i \in \mathcal{I}}{\operatorname{esssup}}\left(\mathbf{1}_{A} \xi_{i}\right)=\mathbf{1}_{A} \operatorname{esssup}_{i \in \mathcal{I}} \xi_{i}, P-a . s$.
(3) For any $\mathcal{F}_{T}$-measurable random variable $\eta$ and any $\lambda>0$, we have $\operatorname{esssup}_{i \in \mathcal{I}}\left(\lambda \xi_{i}+\eta\right)=\lambda \underset{i \in \mathcal{I}}{\operatorname{esssup}} \xi_{i}+\eta$, $P-a . s$. (1)-(3) also hold when we replace $\underset{i \in \mathcal{I}}{\operatorname{esssup}}$ by $\underset{i \in \mathcal{I}}{\operatorname{essinf}}$.

The values $w_{1}, w_{2}$ are bounded as follows:
Proposition 2.1. For any $(t, x) \in[0, T] \times \mathbb{R}^{k}$, it holds $P-$ a.s. that $\left|w_{1}(t, x)\right|+\left|w_{2}(t, x)\right| \leq c_{\kappa}+c_{0}|x|^{2 / p}$.
Similar to Proposition 3.1 of [11], the following result allows us to regard $w_{1}$ and $w_{2}$ as deterministic functions on $[0, T] \times \mathbb{R}^{k}$ :
Proposition 2.2. Let $i=1,2$. For any $(t, x) \in[0, T] \times \mathbb{R}^{k}$, it holds $P$-a.s. that $w_{i}(t, x)=E\left[w_{i}(t, x)\right]$.
Moreover, as a consequence of (2.14), $w_{1}$ and $w_{2}$ are $2 / p$-Hölder continuous in $x$ :
Proposition 2.3. For any $t \in[0, T]$ and $x_{1}, x_{2} \in \mathbb{R}^{k},\left|w_{1}\left(t, x_{1}\right)-w_{1}\left(t, x_{2}\right)\right|+\left|w_{2}\left(t, x_{1}\right)-w_{2}\left(t, x_{2}\right)\right| \leq c_{0}\left|x_{1}-x_{2}\right|^{2 / p}$.
However, the values $w_{1}, w_{2}$ are generally not continuous in $t$ unless $\mathbb{U}, \mathbb{V}$ are compact.
Remark 2.5. When trying to directly prove the dynamic programming principle, [16] encountered a measurability issue: The pasted strategies for approximation may not be progressively measurable, see page 299 therein. So they first proved that the value functions are unique viscosity solutions to the associated Bellman-Isaacs equations by a time-discretization approach (assuming that the limiting Isaacs equation has a comparison principle), which relies on the following regularity of the approximating values $v_{\pi}$

$$
\left|v_{\pi}(t, x)-v_{\pi}\left(t^{\prime}, x^{\prime}\right)\right| \leq C\left(\left|t-t^{\prime}\right|^{1 / 2}+\left|x-x^{\prime}\right|\right), \quad \forall(t, x),\left(t^{\prime}, x^{\prime}\right) \in[0, T] \times \mathbb{R}^{k}
$$

with a uniform coefficient $C>0$ for all partitions $\pi$ of $[0, T]$. Since our value functions $w_{1}$, $w_{2}$ may not be $1 / 2-$ Hölder continuous in $t$, this method seems not suitable for our problem. Hence, we adopt Buckdahn and Li's probability setting.

The following weak dynamic programming principle for value functions $w_{1}, w_{2}$ is the main result of the paper:
Theorem 2.1. 1) Given $t \in[0, T)$, let $\phi, \widetilde{\phi}:[t, T] \times \mathbb{R}^{k} \rightarrow \mathbb{R}$ be two continuous functions such that $\phi(s, x) \leq$ $w_{1}(s, x) \leq \widetilde{\phi}(s, x),(s, x) \in[t, T] \times \mathbb{R}^{k}$. Then for any $x \in \mathbb{R}^{k}$ and $\delta \in(0, T-t)$, it holds $P-a . s$. that

$$
\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \phi\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right) \leq w_{1}(t, x) \leq \underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \widetilde{\phi}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right),
$$

where $\tau_{\beta, \mu} \triangleq \inf \left\{s \in(t, T]:\left(s, X_{s}^{t, x, \mu, \beta(\mu)}\right) \notin O_{\delta}(t, x)\right\}$.
2) Given $t \in[0, T)$, let $\phi, \widetilde{\phi}:[t, T] \times \mathbb{R}^{k} \rightarrow \mathbb{R}$ be two continuous functions such that $\phi(s, x) \leq w_{2}(s, x) \leq \widetilde{\phi}(s, x)$, $(s, x) \in[t, T] \times \mathbb{R}^{k}$. Then for any $x \in \mathbb{R}^{k}$ and $\delta \in(0, T-t)$, it holds $P$-a.s. that

$$
\underset{\alpha \in \mathfrak{A}_{t}}{\operatorname{esssup}} \underset{\nu \in \mathcal{V}_{t}}{\operatorname{essinf}} Y_{t}^{t, x, \alpha(\nu), \nu}\left(\tau_{\alpha, \nu}, \phi\left(\tau_{\alpha, \nu}, X_{\tau_{\alpha, \nu}}^{t, x, \alpha(\nu), \nu}\right)\right) \leq w_{2}(t, x) \leq \underset{\alpha \in \mathfrak{A}_{t}}{\operatorname{esssup}} \operatorname{essinf}_{\nu \in \mathcal{V}_{t}}^{\operatorname{essinf}} Y_{t}^{t, x, \alpha(\nu), \nu}\left(\tau_{\alpha, \nu}, \widetilde{\phi}\left(\tau_{\alpha, \nu}, X_{\tau_{\alpha, \nu}}^{t, x, \alpha(\nu), \nu}\right)\right),
$$

where $\tau_{\alpha, \nu} \triangleq \inf \left\{s \in(t, T]:\left(s, X_{s}^{t, x, \alpha(\nu), \nu}\right) \notin O_{\delta}(t, x)\right\}$.
The significance of such a weak dynamic programming principle lies in the following fact: Since $w_{i}, i=1,2$ may not be continuous in $t, w_{i}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)$ may not be $\mathcal{F}_{\tau_{\beta, \mu}}$-measurable. Then $Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, w_{i}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right)$ and thus the strong dynamic programming principle may not be well-defined.

## 3 Viscosity Solutions of Related Fully Non-linear PDEs

In this section, we show that the priority values are (discontinuous) viscosity solutions to the following partial differential equation with a fully non-linear Hamiltonian $H$ :

$$
\begin{equation*}
-\frac{\partial}{\partial t} w(t, x)-H\left(t, x, w(t, x), D_{x} w(t, x), D_{x}^{2} w(t, x)\right)=0, \quad \forall(t, x) \in(0, T) \times \mathbb{R}^{k} \tag{3.1}
\end{equation*}
$$

Definition 3.1. Let us denote by $\mathbb{S}_{k}$ the set of all $\mathbb{R}^{k \times k}$ - valued symmetric matrices and let $H:[0, T] \times \mathbb{R}^{k} \times \mathbb{R}^{\prime} \times \mathbb{R}^{k} \times$ $\mathbb{S}_{k} \rightarrow[-\infty, \infty]$. An upper (resp. lower) semicontinuous function $w:[0, T] \times \mathbb{R}^{k} \rightarrow \mathbb{R}$ is called a viscosity subsolution (resp. supersolution) of (3.1) if for any $\left(t_{0}, x_{0}, \varphi\right) \in(0, T) \times \mathbb{R}^{k} \times \mathbb{C}^{1,2}\left([0, T] \times \mathbb{R}^{k}\right)$ such that $w-\varphi$ attains a strict local maximum 0 (resp. strict local minimum 0 ) at $\left(t_{0}, x_{0}\right)$, we have

$$
-\frac{\partial}{\partial t} \varphi\left(t_{0}, x_{0}\right)-H\left(t_{0}, x_{0}, \varphi\left(t_{0}, x_{0}\right), D_{x} \varphi\left(t_{0}, x_{0}\right), D_{x}^{2} \varphi\left(t_{0}, x_{0}\right)\right) \leq(r e s p . \geq) 0
$$

For any $(t, x, y, z, \Gamma, u, v) \in[0, T] \times \mathbb{R}^{k} \times \mathbb{R} \times \mathbb{R}^{d} \times \mathbb{S}_{k} \times \mathbb{U} \times \mathbb{V}$, set

$$
H(t, x, y, z, \Gamma, u, v) \triangleq \frac{1}{2} \operatorname{trace}\left(\sigma \sigma^{T}(t, x, u, v) \Gamma\right)+z \cdot b(t, x, u, v)+f(t, x, y, z \cdot \sigma(t, x, u, v), u, v) .
$$

We consider the following Hamiltonian functions:

$$
\begin{aligned}
\underline{H}_{1}(\Xi) \triangleq \sup _{u \in \mathbb{U}} \lim _{\Xi^{\prime} \rightarrow \Xi} \inf _{v \in \mathscr{O}_{u}} H\left(\Xi^{\prime}, u, v\right), \quad \bar{H}_{1}(\Xi) \triangleq \lim _{n \rightarrow \infty} \downarrow \sup _{u \in \mathbb{U}} \inf _{v \in \mathscr{O}_{u}^{n}} \varlimsup_{u^{\prime} \rightarrow u} \sup _{\Xi^{\prime} \in O_{\frac{1}{n}}(\Xi)} H\left(\Xi^{\prime}, u^{\prime}, v\right), \\
\text { and } \quad \bar{H}_{2}(\Xi) \triangleq \inf _{v \in \mathbb{V}} \varlimsup_{\Xi^{\prime} \rightarrow \Xi} \sup _{u \in \mathscr{O}_{v}} H\left(\Xi^{\prime}, u, v\right), \quad \underline{H}_{2}(\Xi) \triangleq \lim _{n \rightarrow \infty} \uparrow \inf _{v \in \mathbb{V}} \sup _{u \in \mathscr{O}_{v}^{n}} \varliminf_{v^{\prime} \rightarrow v} \inf _{\Xi^{\prime} \in O_{\frac{1}{n}}(\Xi)} H\left(\Xi^{\prime}, u, v^{\prime}\right),
\end{aligned}
$$

where $\Xi=(t, x, y, z, \Gamma), \mathscr{O}_{u}^{n} \triangleq\left\{v \in \mathbb{V}:[v]_{\mathbb{V}} \leq \kappa+n[u]_{\mathbb{U}}\right\}, \mathscr{O}_{v}^{n} \triangleq\left\{u \in \mathbb{U}:[u]_{\mathbb{U}} \leq \kappa+n[v]_{\mathbb{V}}\right\}, \mathscr{O}_{u} \triangleq \cup_{n \in \mathbb{N}} \mathscr{O}_{u}^{n}=$ $\mathbf{1}_{\left\{u=u_{0}\right\}} \bar{O}_{\kappa}\left(v_{0}\right)+\mathbf{1}_{\left\{u \neq u_{0}\right\}} \mathbb{V}$ and $\mathscr{O}_{v} \triangleq \bigcup_{n \in \mathbb{N}} \mathscr{O}_{v}^{n}=\mathbf{1}_{\left\{v=v_{0}\right\}} \bar{O}_{\kappa}\left(u_{0}\right)+\mathbf{1}_{\left\{v \neq v_{0}\right\}} \mathbb{U}$.

Remark 3.1. When $U$ and $V$ are compact (say $\mathbb{U}=\bar{O}_{\kappa}\left(u_{0}\right)$ and $\mathbb{V}=\bar{O}_{\kappa}\left(v_{0}\right)$ ), it holds for any $(u, v) \in \mathbb{U} \times \mathbb{V}$ and $n \in \mathbb{N}$ that $\left(\mathscr{O}_{u}^{n}, \mathscr{O}_{v}^{n}\right)=(\mathbb{V}, \mathbb{U})$. If further assuming as [11] that for any $(x, y, z) \in \mathbb{R}^{k} \times \mathbb{R} \times \mathbb{R}^{d}, b(\cdot, x, \cdot, \cdot)$,
$\sigma(\cdot, x, \cdot, \cdot), f(\cdot, x, y, z, \cdot, \cdot)$ are all continuous in $(t, u, v)$, one can deduce from (2.1)-(2.4) that the continuity of $H(\Xi, u, v)$ in $\Xi$ is uniform in $(u, v)$. It follows that

$$
\underline{H}_{1}(\Xi)=\sup _{u \in \mathbb{U}} \varliminf_{\Xi^{\prime} \rightarrow \Xi} \inf _{v \in \mathbb{V}} H\left(\Xi^{\prime}, u, v\right)=\sup _{u \in \mathbb{U}} \inf _{v \in \mathbb{V}} \underline{\Xi}_{\Xi^{\prime} \rightarrow \Xi} H\left(\Xi^{\prime}, u, v\right)=\sup _{u \in \mathbb{U}} \inf _{v \in \mathbb{V}} H(\Xi, u, v),
$$

and that

$$
\begin{aligned}
\bar{H}_{1}(\Xi) & =\lim _{n \rightarrow \infty} \downarrow \sup _{u \in \mathbb{U}} \inf _{v \in \mathbb{V}} \varlimsup_{u^{\prime} \rightarrow u} \sup _{\Xi^{\prime} \in O_{\frac{1}{n}}(\Xi)} H\left(\Xi^{\prime}, u^{\prime}, v\right)=\sup _{u \in \mathbb{U}} \inf _{v \in \mathbb{V}} \varlimsup_{u^{\prime} \rightarrow u} \lim _{n \rightarrow \infty} \downarrow \sup _{\Xi^{\prime} \in O_{\frac{1}{n}}(\Xi)} H\left(\Xi^{\prime}, u^{\prime}, v\right) \\
& =\sup _{u \in \mathbb{U}} \inf _{v \in \mathbb{V}} \varlimsup_{u^{\prime} \rightarrow u} H\left(\Xi, u^{\prime}, v\right)=\sup _{u \in \mathbb{U}} \inf _{v \in \mathbb{V}} H(\Xi, u, v)=\underline{H}_{1}(\Xi) .
\end{aligned}
$$

Similarly, $\underline{H}_{2}(\Xi)=\bar{H}_{2}(\Xi)=\inf _{v \in \mathbb{V}} \sup _{u \in \mathbb{U}} H(\Xi, u, v)$.
For $i=1,2$, Proposition 2.3 implies that

$$
\underline{w}_{i}(t, x) \triangleq \varliminf_{t^{\prime} \rightarrow t} w_{i}\left(t^{\prime}, x\right)=\varliminf_{\left(t^{\prime}, x^{\prime}\right) \rightarrow(t, x)} w_{i}\left(t^{\prime}, x^{\prime}\right) \text { and } \bar{w}_{i}(t, x) \triangleq \varlimsup_{t^{\prime} \rightarrow t} w_{i}\left(t^{\prime}, x\right)=\varlimsup_{\left(t^{\prime}, x^{\prime}\right) \rightarrow(t, x)} w_{i}\left(t^{\prime}, x^{\prime}\right), \quad \forall(t, x) \in[0, T] \times \mathbb{R}^{k}
$$

In fact, $\underline{w}_{i}$ is the largest lower semicontinuous function below $w_{i}$ (known as the lower semicontinuous envelope of $w_{i}$ ) while $\bar{w}_{i}$ is the smallest upper semicontinuous function above $w_{i}$ (known as the upper semicontinuous envelope of $w_{i}$ ).

Theorem 3.1. For $i=1,2, \underline{w}_{i}\left(\right.$ resp. $\left.\bar{w}_{i}\right)$ is a viscosity supersolution (resp. subsolution) of (3.1) with the fully non-linear Hamiltonian $\underline{H}_{i}\left(\right.$ resp. $\left.\bar{H}_{i}\right)$.

Since there is no regularity, even semi-continuity, in the fully non-linear Hamiltonian functions $\underline{H}_{i}$ and $\bar{H}_{i}$, this existence result of viscosity solutions to the fully non-linear PDEs (3.1) is quite general. In general, a comparison result for the PDEs that we analyze may not hold since it is not clear whether $\underline{H}_{i}=\bar{H}_{i}$ unless the control spaces are compact.

Remark 3.2. Given $i=1,2$ and $x \in \mathbb{R}^{k}$, although $w_{i}(T, x)=g(x)$, it is possible that neither $\underline{w}_{i}(T, x)$ nor $\bar{w}_{i}(T, x)$ equals to $g(x)$ since $w_{i}$ may not be continuous in $t$. This phenomenon already appears in stochastic control problems with unbounded control; see e.g. [5].

## 4 Proofs

### 4.1 Proofs of the results in Section 1

Proof of Proposition 1.1; Set $\mathfrak{f}(s, \omega, y, z) \triangleq 1_{\{s \geq t\}} f(s, \omega, y, z), \forall(s, \omega, y, z) \in[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{d}$. Clearly, $\mathfrak{f}$ is also a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}\left(\mathbb{R}^{d}\right) / \mathscr{B}(\mathbb{R})$-measurable function Lipschitz continuous in $(y, z)$. As $E\left[\left(\int_{0}^{T}|\mathfrak{f}(s, 0,0)| d s\right)^{p}\right]=$ $E\left[\left(\int_{t}^{T}|f(s, 0,0)| d s\right)^{p}\right]<\infty$, Theorem 4.2 of [8] shows that the BSDE

$$
\begin{equation*}
Y_{s}=\eta+\int_{s}^{T} \mathfrak{f}\left(r, Y_{r}, Z_{r}\right) d r-\int_{s}^{T} Z_{r} d B_{r}, \quad s \in[0, T] \tag{4.1}
\end{equation*}
$$

admits a unique solution $(Y, Z) \in \mathbb{G}_{\mathbf{F}}^{p}([0, T])$. In particular, $\left\{\left(Y_{s}, Z_{s}\right)\right\}_{s \in[t, T]} \in \mathbb{G}_{\mathbf{F}}^{p}([t, T])$ solves (1.3).
Suppose that $\left(Y^{\prime}, Z^{\prime}\right)$ is another solution of (1.3) in $\mathbb{G}_{\mathbf{F}}^{p}([t, T])$. Let $\left(\widetilde{Y}^{\prime}, \widetilde{Z}^{\prime}\right) \in \mathbb{G}_{\mathbf{F}}^{p}([0, t])$ be the unique solution of the following BSDE with zero generator:

$$
\tilde{Y}_{s}^{\prime}=Y_{t}^{\prime}-\int_{s}^{t} \widetilde{Z}_{r}^{\prime} d B_{r}, \quad s \in[0, t]
$$

Actually, $\tilde{Y}_{s}^{\prime}=E\left[Y_{t}^{\prime} \mid \mathcal{F}_{s}\right]$. Then $\left(\mathcal{Y}^{\prime}, \mathcal{Z}^{\prime}\right) \triangleq\left\{\left(\mathbf{1}_{\{s<t\}} \widetilde{Y}_{s}^{\prime}+\mathbf{1}_{\{s \geq t\}} Y_{s}^{\prime}, \mathbf{1}_{\{s<t\}} \widetilde{Z}_{s}^{\prime}+\mathbf{1}_{\{s \geq t\}} Z_{s}^{\prime}\right)\right\}_{s \in[0, T]} \in \mathbb{G}_{\mathbf{F}}^{p}([0, T])$ also solves BSDE (4.1). So $\left(\mathcal{Y}^{\prime}, \mathcal{Z}^{\prime}\right)=(Y, Z)$. In particular, $\left(Y_{s}^{\prime}, Z_{s}^{\prime}\right)=\left(Y_{s}, Z_{s}\right), \forall s \in[t, T]$.

Given $A \in \mathcal{F}_{t}$, multiplying $\mathbf{1}_{A}$ to both sides of (1.3) yields that

$$
\mathbf{1}_{A} Y_{s}=\mathbf{1}_{A} \eta+\int_{s}^{T} \mathbf{1}_{A} f\left(r, \mathbf{1}_{A} Y_{r}, \mathbf{1}_{A} Z_{r}\right) d r-\int_{s}^{T} \mathbf{1}_{A} Z_{r} d B_{r}, \quad s \in[t, T]
$$

Let $\left(Y^{A}, Z^{A}\right) \in \mathbb{G}_{\mathbf{F}}^{p}([0, t])$ be the unique solution of the following BSDE with zero generator:

$$
Y_{s}^{A}=\mathbf{1}_{A} Y_{t}-\int_{s}^{t} Z_{r}^{A} d B_{r}, \quad s \in[0, t]
$$

Then $\left(\mathcal{Y}^{A}, \mathcal{Z}^{A}\right) \triangleq\left\{\left(\mathbf{1}_{\{s<t\}} Y_{s}^{A}+\mathbf{1}_{\{s \geq t\}} \mathbf{1}_{A} Y_{s}, \mathbf{1}_{\{s<t\}} Z_{s}^{A}+\mathbf{1}_{\{s \geq t\}} \mathbf{1}_{A} Z_{s}\right)\right\}_{s \in[0, T]} \in \mathbb{G}_{\mathbf{F}}^{p}([0, T])$ solves the BSDE

$$
\mathcal{Y}_{s}^{A}=\mathbf{1}_{A} \eta+\int_{s}^{T} f_{A}\left(r, \mathcal{Y}_{r}^{A}, \mathcal{Z}_{r}^{A}\right) d r-\int_{s}^{T} \mathcal{Z}_{r}^{A} d B_{r}, \quad s \in[0, T]
$$

where $f_{A}(r, \omega, y, z) \triangleq \mathbf{1}_{\{r \geq t\}} \mathbf{1}_{\{\omega \in A\}} f(r, \omega, y, z)$. Since $\left\{\mathbf{1}_{\{r \geq t\} \cap A}\right\}_{r \in[0, T]}$ is a right-continuous $\mathbf{F}$-adapted process, the measurability and Lipschitz continuity of $f$ imply that $f_{A}$ is also a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}\left(\mathbb{R}^{d}\right) / \mathscr{B}(\mathbb{R})$-measurable function Lipschitz continuous in $(y, z)$. Since $E\left[\left(\int_{0}^{T}\left|f_{A}(s, 0,0)\right| d s\right)^{p}\right] \leq E\left[\left(\int_{t}^{T}|f(s, 0,0)| d s\right)^{p}\right]<\infty$, applying Proposition 3.2 of [8] yields that

$$
\begin{aligned}
E\left[\mathbf{1}_{A} \sup _{s \in[t, T]}\left|Y_{s}\right|^{p}\right] & \leq E\left[\sup _{s \in[0, T]}\left|\mathcal{Y}_{s}^{A}\right|^{p}\right] \leq C(T, p, \gamma) E\left[\mathbf{1}_{A}|\eta|^{p}+\left(\int_{0}^{T}\left|f_{A}(s, 0,0)\right| d s\right)^{p}\right] \\
& =C(T, p, \gamma) E\left[\mathbf{1}_{A}|\eta|^{p}+\mathbf{1}_{A}\left(\int_{t}^{T}|f(s, 0,0)| d s\right)^{p}\right]
\end{aligned}
$$

Letting $A$ vary in $\mathcal{F}_{t}$ yields (1.4).
Proof of Proposition 1.2; (1) Set $(\widehat{Y}, \widehat{Z}) \triangleq\left(Y^{1}-Y^{2}, Z^{1}-Z^{2}\right)$, which solves the BSDE

$$
\begin{equation*}
\widehat{Y}_{s}=\eta_{1}-\eta_{2}+\int_{s}^{T} \widehat{f}\left(r, \widehat{Y}_{r}, \widehat{Z}_{r}\right) d r-\int_{s}^{T} \widehat{Z}_{r} d B_{r}, \quad s \in[t, T] \tag{4.2}
\end{equation*}
$$

where $\widehat{f}(r, \omega, y, z) \triangleq f_{1}\left(r, \omega, y+Y_{r}^{2}(\omega), z+Z_{r}^{2}(\omega)\right)-f_{2}\left(r, \omega, Y_{r}^{2}(\omega), Z_{r}^{2}(\omega)\right)$. Clearly, $\widehat{f}$ is a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}\left(\mathbb{R}^{d}\right) / \mathscr{B}(\mathbb{R})-$ measurable function Lipschitz continuous in $(y, z)$. Suppose that $E\left[\left(\int_{t}^{T}|\widehat{f}(s, 0,0)| d s\right)^{\widetilde{p}}\right]=E\left[\left(\int_{t}^{T} \mid f_{1}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-\right.\right.$ $\left.\left.f_{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right) \mid d s\right)^{\widetilde{p}}\right]<\infty$ for some $\widetilde{p} \in(1, p]$. Since $\mathbb{G}_{\mathbf{F}}^{p}([t, T]) \subset \mathbb{G}_{\mathbf{F}}^{\widetilde{p}}([t, T])$ by Hölder's inequality, applying Proposition 1.1 with $p=\widetilde{p}$ shows that $(\widehat{Y}, \widehat{Z})$ is the unique solution of $\operatorname{BSDE}\left(t, \eta_{1}-\eta_{2}, \widehat{f}\right)$ in $\mathbb{G}_{\mathbf{F}}^{\widetilde{p}}([t, T])$ satisfying

$$
E\left[\sup _{s \in[t, T]}\left|\widehat{Y}_{s}\right|^{\widetilde{p}} \mid \mathcal{F}_{t}\right] \leq C(T, \widetilde{p}, \gamma) E\left[\left|\eta_{1}-\eta_{2}\right|^{\widetilde{p}}+\left(\int_{t}^{T}|\widehat{f}(s, 0,0)| d s\right)^{\widetilde{p}} \mid \mathcal{F}_{t}\right], \quad P-\text { a.s. },
$$

which is exactly (1.5).
(2) Next, suppose that $\eta_{1} \leq($ resp. $\geq) \eta_{2}, P-$ a.s. and that $\delta f_{s} \triangleq f_{1}\left(s, Y_{s}^{2}, Z_{s}^{2}\right)-f_{2}\left(s, Y_{s}^{2}, Z_{s}^{2}\right) \leq($ resp. $\geq) 0, d s \times d P-a . s$. on $[t, T] \times \Omega$. By (2.4),

$$
\mathfrak{a}_{s} \triangleq \mathbf{1}_{\left\{\widehat{Y}_{s} \neq 0\right\}} \frac{f_{1}\left(s, Y_{s}^{1}, Z_{s}^{1}\right)-f_{1}\left(s, Y_{s}^{2}, Z_{s}^{1}\right)}{\widehat{Y}_{s}} \in[-\gamma, \gamma], \quad s \in[t, T]
$$

defines an $\mathbf{F}$-progressively measurable bounded process. For $i=1, \cdots, d$, analogous to process $\mathfrak{a}$

$$
\begin{aligned}
\mathfrak{b}_{s}^{i} \triangleq \mathbf{1}_{\left\{Z_{s}^{1, i} \neq Z_{s}^{2, i}\right\}} \frac{1}{Z_{s}^{1, i}-Z_{s}^{2, i}} & \left(f_{1}\left(s, Y_{s}^{2},\left(Z_{s}^{2,1}, \cdots, Z_{s}^{2, i-1}, Z_{s}^{1, i}, \cdots, Z_{s}^{1, n}\right)\right)\right. \\
& \left.-f_{1}\left(s, Y_{s}^{2},\left(Z_{s}^{2,1}, \cdots, Z_{s}^{2, i}, Z_{s}^{1, i+1}, \cdots, Z_{s}^{1, n}\right)\right)\right) \in[-\gamma, \gamma], \quad s \in[t, T]
\end{aligned}
$$

also defines an $\mathbf{F}$-progressively measurable bounded process.
Then we can alternatively express (4.2) as

$$
\widehat{Y}_{s}=\eta_{1}-\eta_{2}+\int_{s}^{T}\left(\mathfrak{a}_{r} \widehat{Y}_{r}+\mathfrak{b}_{r} \cdot \widehat{Z}_{r}+\delta f_{r}\right) d r-\int_{s}^{T} \widehat{Z}_{r} d B_{r}, \quad s \in[t, T]
$$

Define $Q_{s} \triangleq \exp \left\{\int_{t}^{s} \mathfrak{a}_{r} d r-\frac{1}{2} \int_{t}^{s}\left|\mathfrak{b}_{r}\right|^{2} d r+\int_{t}^{s} \mathfrak{b}_{r} d B_{r}\right\}, s \in[t, T]$. Applying integration by parts yields that

$$
\begin{align*}
Q_{s} \widehat{Y}_{s} & =Q_{T} \widehat{Y}_{T}+\int_{s}^{T} Q_{r}\left(\mathfrak{a}_{r} \widehat{Y}_{r}+\mathfrak{b}_{r} \cdot \widehat{Z}_{r}+\delta f_{r}\right) d r-\int_{s}^{T} Q_{r} \widehat{Z}_{r} d B_{r}-\int_{s}^{T} \widehat{Y}_{r} Q_{r} \mathfrak{a}_{r} d r-\int_{s}^{T} \widehat{Y}_{r} Q_{r} \mathfrak{b}_{r} d B_{r}-\int_{s}^{T} Q_{r} \mathfrak{b}_{r} \cdot \widehat{Z}_{r} d r \\
& =Q_{T}\left(\eta_{1}-\eta_{2}\right)+\int_{s}^{T} Q_{r} \delta f_{r} d r-\int_{s}^{T} Q_{r}\left(\widehat{Z}_{r}+\widehat{Y}_{r} \mathfrak{b}_{r}\right) d B_{r}, \quad P-\mathrm{a.s.} \tag{4.3}
\end{align*}
$$

One can deduce from the Burkholder-Davis-Gundy inequality and Hölder's inequality that

$$
\begin{align*}
E\left[\sup _{s \in[t, T]}\left|\int_{t}^{s} Q_{r}\left(\widehat{Z}_{r}+\widehat{Y}_{r} \mathfrak{b}_{r}\right) d B_{r}\right|\right] & \leq c_{0} E\left[\left(\int_{t}^{T} Q_{r}^{2}\left|\widehat{Y}_{r} \mathfrak{b}_{r}+\widehat{Z}_{r}\right|^{2} d r\right)^{\frac{1}{2}}\right] \leq c_{0} E\left[\sup _{s \in[t, T]}\left|Q_{r}\right|\left\{\sup _{s \in[t, T]}\left|\widehat{Y}_{r}\right|+\left(\int_{t}^{T}\left|\widehat{Z}_{r}\right|^{2} d r\right)^{\frac{1}{2}}\right\}\right] \\
& \leq c_{0}\left(E\left[\sup _{s \in[t, T]}\left|Q_{r}\right|^{\widehat{p}}\right]\right)^{1 / \widehat{p}}\left(\left\|\widehat{Y}_{r}\right\|_{\mathbb{C}_{\mathbf{F}}^{p}([t, T])}+\left\|\widehat{Z}_{r}\right\|_{\mathbb{H}_{\mathbf{F}}^{2, p}\left([t, T], \mathbb{R}^{d}\right)}\right) \tag{4.4}
\end{align*}
$$

where $\widehat{p}=\frac{p}{p-1}$. Also, Doob's martingale inequality implies that

$$
\begin{aligned}
E\left[\sup _{s \in[t, T]}\left|Q_{r}\right|^{\widehat{p}}\right] & \leq c_{0} E\left[\left|Q_{T}\right|^{\hat{p}}\right]=c_{0} E\left[\exp \left\{\widehat{p} \int_{t}^{T} \mathfrak{a}_{r} d r+\frac{\widehat{p}^{2}-1}{2} \int_{t}^{T}\left|\mathfrak{b}_{r}\right|^{2} d r-\frac{\widehat{p}^{2}}{2} \int_{t}^{T}\left|\mathfrak{b}_{r}\right|^{2} d r+\widehat{p} \int_{t}^{T} \mathfrak{b}_{r} d B_{r}\right\}\right] \\
& \leq c_{0} \exp \left\{\widehat{p} \gamma T+\frac{\widehat{p}^{2}-1}{2} \gamma^{2} T\right\} E\left[\exp \left\{-\frac{\widehat{p}^{2}}{2} \int_{t}^{T}\left|\mathfrak{b}_{r}\right|^{2} d r+\widehat{p} \int_{t}^{T} \mathfrak{b}_{r} d B_{r}\right\}\right]=c_{0} \exp \left\{\widehat{p} \gamma T+\frac{\widehat{p}^{2}-1}{2} \gamma^{2} T\right\},
\end{aligned}
$$

which together with (4.4) shows that $\left\{\int_{t}^{s} Q_{r}\left(\widehat{Y}_{r} \mathfrak{b}_{r}+\widehat{Z}_{r}\right) d B_{r}\right\}_{s \in[t, T]}$ is a uniformly integrable martingale. Then for any $s \in[t, T]$, taking $E\left[\cdot \mid \mathcal{F}_{s}\right]$ in (4.3) yields that $P$-a.s.

$$
Q_{s} \widehat{Y}_{s}=E\left[Q_{T}\left(\eta_{1}-\eta_{2}\right)+\int_{s}^{T} Q_{r} \delta f_{r} d r \mid \mathcal{F}_{s}\right] \leq(\text { resp. } \geq) 0, \quad \text { thus } \quad \widehat{Y}_{s} \leq(\text { resp. } \geq) 0
$$

By the continuity of process $\widehat{Y}$, it holds $P$-a.s. that $Y_{s}^{1} \leq($ resp. $\geq) Y_{s}^{2}$ for any $s \in[t, T]$.

### 4.2 Proofs of the Results in Section 2

Proof of Example 2.2; For any $(t, u) \in[0, T] \times \mathbb{U}$, the continuity of $\varphi$ and (2.5) show that $\{v \in[-\kappa|u|, \kappa|u|]$ : $\varphi(t, u, v)=0\}$ is a non-empty closed set. So we can define $\mathscr{V}(t, u) \triangleq \min \{v \in[-\kappa|u|, \kappa|u|]: \varphi(t, u, v)=0\}$.

Given $n \in \mathbb{N}$, for any $i=0, \cdots, 2^{n}-1$ and $j \in \mathbb{Z}$, we set $t_{i}^{n}=i 2^{-n} T, u_{j}^{n}=j 2^{-n}$ and $\psi_{i, j}^{n} \triangleq \inf _{(t, u) \in \mathcal{D}_{i, j}^{n}} \mathscr{V}(t, u) \in$ $[-\kappa-\kappa|u|, \kappa+\kappa|u|]$ with

$$
\mathcal{D}_{i, j}^{n} \triangleq \begin{cases}{\left[t_{i}^{n}, t_{i+1}^{n}\right) \times\left[u_{j}^{n}, u_{j+1}^{n}\right),} & \text { if } i<2^{n}-1 \\ {\left[t_{i}^{n}, T\right] \times\left[u_{j}^{n}, u_{j+1}^{n}\right),} & \text { if } i=2^{n}-1\end{cases}
$$

Clearly, $\psi_{n}(t, u) \triangleq \sum_{i=0}^{2^{n}-1} \sum_{j \in \mathbb{Z}} \psi_{i, j}^{n} \mathbf{1}_{\left\{(t, u) \in \mathcal{D}_{i, j}^{n}\right\}} \in[-\kappa-\kappa|u|, \kappa+\kappa|u|], \forall(t, u) \in[0, T] \times \mathbb{U}$ defines a $\mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{U}) / \mathscr{B}(\mathbb{R})$ -measurable function. As $\psi_{n} \leq \psi_{n+1}$, the function $\psi(t, u) \triangleq \lim _{n \rightarrow \infty} \uparrow \psi_{n}(t, u) \in[-\kappa-\kappa|u|, \kappa+\kappa|u|], \forall(t, u) \in[0, T] \times \mathbb{U}$ is also $\mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{U}) / \mathscr{B}(\mathbb{R})$-measurable.

Now, let $(t, u) \in[0, T] \times \mathbb{U}$ and $\varepsilon>0$. By the continuity of $\varphi$ in $t$, there exists a $\delta \in(0, \varepsilon / 3 \gamma)$ such that

$$
\begin{equation*}
|\varphi(s, u, \psi(t, u))-\varphi(t, u, \psi(t, u))|<\varepsilon / 3, \quad \forall s \in[t-\delta, t+\delta] \cap[0, T] \tag{4.5}
\end{equation*}
$$

For any $n>\log _{2}(1 \vee T)-\log _{2}(\delta),(t, u) \in \mathcal{D}_{i, j}^{n}$ for some $(i, j) \in\left\{0, \cdots, 2^{n}-1\right\} \times \mathbb{Z}$, and we can find $\left(t^{\prime}, u^{\prime}\right) \in \mathcal{D}_{i, j}^{n}$ such that $\mathscr{V}\left(t^{\prime}, u^{\prime}\right) \leq \psi_{i, j}^{n}+\delta$. Then (4.5) and the Lipschitz continuity of $\varphi$ in $(u, v)$ show that

$$
\begin{aligned}
& |\varphi(t, u, \psi(t, u))|=\left|\varphi(t, u, \psi(t, u))-\varphi\left(t^{\prime}, u^{\prime}, \mathscr{V}\left(t^{\prime}, u^{\prime}\right)\right)\right| \\
& \quad \leq\left|\varphi(t, u, \psi(t, u))-\varphi\left(t^{\prime}, u, \psi(t, u)\right)\right|+\left|\varphi\left(t^{\prime}, u, \psi(t, u)\right)-\varphi\left(t^{\prime}, u, \psi_{n}(t, u)\right)\right|+\left|\varphi\left(t^{\prime}, u, \psi_{i, j}^{n}\right)-\varphi\left(t^{\prime}, u^{\prime}, \mathscr{V}\left(t^{\prime}, u^{\prime}\right)\right)\right| \\
& \quad \leq \varepsilon / 3+\gamma\left|\psi(t, u)-\psi_{n}(t, u)\right|+\gamma\left(\left|u-u^{\prime}\right|+\left|\psi_{i, j}^{n}-\mathscr{V}\left(t^{\prime}, u^{\prime}\right)\right|\right) \\
& \quad \leq \varepsilon+\gamma\left|\psi(t, u)-\psi_{n}(t, u)\right|+2 \gamma \delta \leq \varepsilon+\gamma\left|\psi(t, u)-\psi_{n}(t, u)\right| .
\end{aligned}
$$

Letting $n \rightarrow \infty$ yields that $|\varphi(t, u, \psi(t, u))| \leq \varepsilon$. Then as $\varepsilon \rightarrow 0$, we obtain that $\varphi(t, u, \psi(t, u))=0$.
Similarly, we can construct a measurable function $\tilde{\psi}$ on $[0, T] \times \mathbb{V}$ such that

$$
\varphi(t, \widetilde{\psi}(t, v), v)=0 \quad \text { and } \quad|\widetilde{\psi}(t, v)| \leq \kappa(1+|v|), \quad \forall(t, v) \in[0, T] \times \mathbb{V} .
$$

Hence (A-u) and (A-v) are satisfied.
Proof of Lemma 2.1; It suffices to prove for $\mathcal{U}_{t}$-controls. Let $s \in[t, T]$ and $U \in \mathscr{B}(\mathbb{U})$. Since $\llbracket t, \tau \llbracket, \llbracket \tau, T \rrbracket \in \mathscr{P}$, we see that both $\mathcal{D}_{1} \triangleq \llbracket t, \tau \llbracket \cap([t, s] \times \Omega)$ and $\mathcal{D}_{2} \triangleq \llbracket \tau, T \rrbracket \cap([t, s] \times \Omega)$ belong to $\mathscr{B}([t, s]) \otimes \mathcal{F}_{s}$. It then follows that

$$
\begin{aligned}
\{(r, \omega) & \left.\in[t, s] \times \Omega: \mu_{r}(\omega) \in U\right\}=\left\{(r, \omega) \in \mathcal{D}_{1}: \mu_{r}^{1}(\omega) \in U\right\} \cup\left\{(r, \omega) \in \mathcal{D}_{2}: \mu_{r}^{2}(\omega) \in U\right\} \\
& =\left(\mathcal{D}_{1} \cap\left\{(r, \omega) \in[t, s] \times \Omega: \mu_{r}^{1}(\omega) \in U\right\}\right) \cup\left(\mathcal{D}_{2} \cap\left\{(r, \omega) \in[t, s] \times \Omega: \mu_{r}^{2}(\omega) \in U\right\}\right) \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s},
\end{aligned}
$$

which shows that the process $\mu$ is $\mathbf{F}-$ progressively measurable.
For $i=1,2$, suppose that $E \int_{t}^{T}\left[\mu_{s}^{i}\right]_{\mathbb{U}}^{q_{i}} d s<\infty$ for some $q_{i}>2$. One can deduce that $E \int_{t}^{T}\left[\mu_{r}\right]_{\mathbb{U}}^{q_{1} \wedge q_{2}} d r \leq$ $E \int_{t}^{T}\left[\mu_{r}^{1}\right]_{\mathbb{U}}^{q_{1} \wedge q_{2}} d r+E \int_{t}^{T}\left[\mu_{r}^{2}\right]_{\mathbb{U}}^{q_{1} \wedge q_{2}} d r<\infty$. Thus $\mu \in \mathcal{U}_{t}$.

Proof of Lemma 2.2; Both $\left\{X_{\tau \wedge s}^{t, \xi, \mu, \nu}\right\}_{s \in[t, T]}$ and $\left\{X_{\tau \wedge s}^{t, \xi, \tilde{\mu}, \tilde{\nu}}\right\}_{s \in[t, T]}$ satisfy the same SDE:

$$
\begin{equation*}
X_{s}=\xi+\int_{t}^{s} b_{\tau}^{\mu, \nu}\left(r, X_{r}\right) d r+\int_{t}^{s} \sigma_{\tau}^{\mu, \nu}\left(r, X_{r}\right) d B_{r}, \quad s \in[t, T], \tag{4.6}
\end{equation*}
$$

where $b_{\tau}^{\mu, \nu}(r, \omega, x) \triangleq \mathbf{1}_{\{r<\tau(\omega)\}} b^{\mu, \nu}(r, \omega, x)$ and $\sigma_{\tau}^{\mu, \nu}(r, \omega, x) \triangleq \mathbf{1}_{\{r<\tau(\omega)\}} \sigma^{\mu, \nu}(r, \omega, x), \forall(r, \omega, x) \in[t, T] \times \Omega \times \mathbb{R}^{k}$. Like $b^{\mu, \nu}$ and $\sigma^{\mu, \nu}, b_{\tau}^{\mu, \nu}$ is a $\mathscr{P} \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) / \mathscr{B}\left(\mathbb{R}^{k}\right)$-measurable function and $\sigma_{\tau}^{\mu, \nu}$ is a $\mathscr{P} \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) / \mathscr{B}\left(\mathbb{R}^{k \times d}\right)$-measurable function that is Lipschitz continuous in $(y, z)$ with coefficient $\gamma$ and satisfies

$$
E\left[\left(\int_{t}^{T}\left|b_{\tau}^{\mu, \nu}(s, 0)\right| d s\right)^{2}+\left(\int_{t}^{T}\left|\sigma_{\tau}^{\mu, \nu}(s, 0)\right| d s\right)^{2}\right]<\infty .
$$

Thus (4.6) has a unique solution. It then holds $P$-a.s. that

$$
\begin{equation*}
X_{\tau \wedge s}^{t, \xi, \mu, \nu}=X_{\tau \wedge s}^{t, \xi, \tilde{\mu}, \tilde{\nu}}, \quad \forall s \in[t, T] . \tag{4.7}
\end{equation*}
$$

One can deduce that

$$
X_{s}^{t, \xi, \mu, \nu}-X_{\tau \wedge s}^{t, \xi, \mu, \nu}=X_{\tau \vee s}^{t, \xi, \mu, \nu}-X_{\tau}^{t, \xi, \mu, \nu}=\int_{\tau}^{\tau \vee s} b\left(r, X_{r}^{t, \xi, \mu, \nu}, \mu_{r}, \nu_{r}\right) d r+\int_{\tau}^{\tau \vee s} \sigma\left(r, X_{r}^{t, \xi, \mu, \nu}, \mu_{r}, \nu_{r}\right) d B_{r}, \quad s \in[t, T] .
$$

Multiplying $\mathbf{1}_{A}$ on both sides yields that

$$
\begin{aligned}
\mathcal{X}_{s} & \triangleq \mathbf{1}_{A}\left(X_{s}^{t, \xi, \mu, \nu}-X_{\tau \wedge s}^{t, \xi, \mu, \nu}\right)=\int_{\tau}^{\tau \vee s} \mathbf{1}_{A} b\left(r, X_{r}^{t, \xi, \mu, \nu}, \mu_{r}, \nu_{r}\right) d r+\int_{\tau}^{\tau \vee s} \mathbf{1}_{A} \sigma\left(r, X_{r}^{t, \xi, \mu, \nu}, \mu_{r}, \nu_{r}\right) d B_{r} \\
& =\int_{t}^{s} \mathbf{1}_{\{r \geq \tau\}} \mathbf{1}_{A} b\left(r, \mathcal{X}_{r}+X_{\tau \wedge r}^{t, \xi, \mu, \nu}, \mu_{r}, \nu_{r}\right) d r+\int_{t}^{s} \mathbf{1}_{\{r \geq \tau\}} \mathbf{1}_{A} \sigma\left(r, \mathcal{X}_{r}+X_{\tau \wedge r}^{t, \xi, \mu, \nu}, \mu_{r}, \nu_{r}\right) d B_{r}, \quad s \in[t, T] .
\end{aligned}
$$

Similarly, we see from (4.7) that

$$
\begin{aligned}
\widetilde{\mathcal{X}}_{s} & \triangleq \mathbf{1}_{A}\left(X_{s}^{t, \xi, \widetilde{\mu}, \widetilde{\nu}}-X_{\tau \wedge s}^{t, \xi, \widetilde{\mu}, \widetilde{\nu}}\right)=\int_{t}^{s} \mathbf{1}_{\{r \geq \tau\}} \mathbf{1}_{A} b\left(r, \widetilde{\mathcal{X}}_{r}+X_{\tau \wedge r}^{t, \xi, \widetilde{\mu}, \widetilde{\nu}}, \widetilde{\mu}_{r}, \widetilde{\nu}_{r}\right) d r+\int_{t}^{s} \mathbf{1}_{\{r \geq \tau\}} \mathbf{1}_{A} \sigma\left(r, \widetilde{\mathcal{X}}_{r}+X_{\tau \wedge r}^{t, \xi, \widetilde{\mu}, \widetilde{\nu}}, \widetilde{\mu}_{r}, \widetilde{\nu}_{r}\right) d B_{r} \\
& =\int_{t}^{s} \mathbf{1}_{\{r \geq \tau\}} \mathbf{1}_{A} b\left(r, \widetilde{\mathcal{X}}_{r}+X_{\tau \wedge r}^{t, \xi, \mu, \nu}, \mu_{r}, \nu_{r}\right) d r+\int_{t}^{s} \mathbf{1}_{\{r \geq \tau\}} \mathbf{1}_{A} \sigma\left(r, \widetilde{\mathcal{X}}_{r}+X_{\tau \wedge r}^{t, \xi, \mu, \nu}, \mu_{r}, \nu_{r}\right) d B_{r}, \quad s \in[t, T] .
\end{aligned}
$$

To wit, $\mathcal{X}, \widetilde{\mathcal{X}} \in \mathbb{C}_{\mathbf{F}}^{2}\left([t, T], \mathbb{R}^{k}\right)$ satisfy the same SDE :

$$
\begin{equation*}
X_{s}=\int_{t}^{s} \widehat{b}\left(r, X_{r}\right) d r+\int_{t}^{s} \widehat{\sigma}\left(r, X_{r}\right) d B_{r}, \quad s \in[t, T] \tag{4.8}
\end{equation*}
$$

where $\widehat{b}(r, \omega, x) \triangleq \mathbf{1}_{\{r \geq \tau(\omega)\}} \mathbf{1}_{\{\omega \in A\}} b\left(r, x+X_{\tau \wedge r}^{t, \xi, \mu, \nu}(\omega), \mu_{r}(\omega), \nu_{r}(\omega)\right)$ and $\widehat{\sigma}(r, \omega, x) \triangleq \mathbf{1}_{\{r \geq \tau(\omega)\}} \mathbf{1}_{\{\omega \in A\}} \sigma^{\mu, \nu}(r, x+$ $\left.X_{\tau \wedge r}^{t, \xi, \mu, \nu}(\omega), \mu_{r}(\omega), \nu_{r}(\omega)\right), \forall(r, \omega, x) \in[t, T] \times \Omega \times \mathbb{R}^{k}$. The measurability of functions $b, X^{t, \xi, \mu, \nu}, \mu$ and $\nu$ implies that the mapping $(r, \omega, x) \rightarrow b\left(r, \omega, x+X_{\tau \wedge r}^{t, \xi, \mu, \nu}(\omega), \mu_{r}(\omega), \nu_{r}(\omega)\right)$ is $\mathscr{P} \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) / \mathscr{B}\left(\mathbb{R}^{k}\right)$-measurable. Clearly, $\left\{\mathbf{1}_{\{r \geq \tau\} \cap A}\right\}_{r \in[t, T]}$ is a right-continuous $\mathbf{F}$-adapted process. Thus $\widehat{b}$ is also $\mathscr{P} \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) / \mathscr{B}\left(\mathbb{R}^{k}\right)$-measurable. Similarly, $\widehat{\sigma}$ is $\mathscr{P} \otimes \mathscr{B}\left(\mathbb{R}^{k}\right) / \mathscr{B}\left(\mathbb{R}^{k \times d}\right)$-measurable. By (2.2), both $\widehat{b}$ and $\widehat{\sigma}$ are Lipschitz continuous in $x$. Since

$$
E\left[\left(\int_{t}^{T}|\widehat{b}(r, 0)| d r\right)^{2}+\left(\int_{t}^{T}|\widehat{\sigma}(r, 0)| d r\right)^{2}\right] \leq c_{0}+c_{0} E\left[\left|X_{\tau}^{t, \xi, \mu, \nu}\right|^{2}\right]+c_{0} E \int_{t}^{T}\left(\left[\mu_{r}\right]_{\mathbb{U}}^{2}+\left[\nu_{r}\right]_{\mathbb{V}}^{2}\right) d r<\infty
$$

by (2.1), (2.2) and Hölder's inequality, the SDE (4.8) admits a unique solution. Hence, $P\left(\mathcal{X}_{s}=\widetilde{\mathcal{X}}_{s}, \forall s \in[t, T]\right)=1$, which together with (4.7) proves (2.8).

Proof of Lemma 2.3: For $i=1,2$, let $\Theta_{i} \triangleq\left(t, \xi_{i}, \mu, \nu\right)$ and set $\left(Y^{i}, Z^{i}\right) \triangleq\left(Y^{\Theta_{i}}\left(T, g\left(X_{T}^{\Theta_{i}}\right)\right), Z^{\Theta_{i}}\left(T, g\left(X_{T}^{\Theta_{i}}\right)\right)\right)$. Given $\widetilde{p} \in(1, p],(2.4)$ and Hölder's inequality show that

$$
E\left[\left(\int_{t}^{T}\left|f_{T}^{\Theta_{1}}\left(r, Y_{r}^{2}, Z_{r}^{2}\right)-f_{T}^{\Theta_{2}}\left(r, Y_{r}^{2}, Z_{r}^{2}\right)\right| d s\right)^{\widetilde{p}}\right] \leq c_{\widetilde{p}} E\left[\sup _{s \in[t, T]}\left|X_{s}^{\Theta_{1}}-X_{s}^{\Theta_{2}}\right|^{\frac{2 \tilde{p}}{p}}\right] \leq c_{\widetilde{p}}\left\{E\left[\sup _{s \in[t, T]}\left|X_{s}^{\Theta_{1}}-X_{s}^{\Theta_{2}}\right|^{2}\right]\right\}^{\frac{\tilde{p}}{p}}<\infty
$$

Then we can deduce from (1.5) that

$$
\begin{aligned}
E\left[\sup _{s \in[t, T]}\left|Y_{s}^{1}-Y_{s}^{2}\right|^{\widetilde{p}} \mid \mathcal{F}_{t}\right] & \leq c_{\widetilde{p}} E\left[\left|g\left(X_{T}^{\Theta_{1}}\right)-g\left(X_{T}^{\Theta_{2}}\right)\right|^{\widetilde{p}}+\int_{t}^{T}\left|f_{T}^{\Theta_{1}}\left(r, Y_{r}^{2}, Z_{r}^{2}\right)-f_{T}^{\Theta_{2}}\left(r, Y_{r}^{2}, Z_{r}^{2}\right)\right|^{\widetilde{p}} d s \mid \mathcal{F}_{t}\right] \\
& \leq c_{\widetilde{p}} E\left[\left.\sup _{s \in[t, T]}\left|X_{s}^{\Theta_{1}}-X_{s}^{\Theta_{2}}\right|^{\frac{2 \tilde{\widetilde{p}}}{p}} \right\rvert\, \mathcal{F}_{t}\right], \quad P \text {-a.s. }
\end{aligned}
$$

Then a standard a priori estimate of SDEs (see e.g. [20, pg. 166-168] and [21, pg. 289-290]) leads to that

$$
E\left[\sup _{s \in[t, T]}\left|Y_{s}^{1}-Y_{s}^{2}\right|^{\widetilde{p}} \mid \mathcal{F}_{t}\right] \leq c_{\widetilde{p}} E\left[\left.\sup _{s \in[t, T]}\left|X_{s}^{\Theta_{1}}-X_{s}^{\Theta_{2}}\right|^{\frac{2 \widetilde{p}}{p}} \right\rvert\, \mathcal{F}_{t}\right] \leq c_{\widetilde{p}}\left|\xi_{1}-\xi_{2}\right|^{\frac{2 \tilde{p}}{p}}, \quad P-\text { a.s. }
$$

Proof of Proposition 2.1: Given $\beta \in \mathfrak{B}_{t}$, (1.4) and Hölder's inequality imply that

$$
\begin{align*}
\left|J\left(t, x, u_{0}, \beta\left(u_{0}\right)\right)\right|^{p} & \leq E\left[\sup _{s \in[t, T]}\left|Y_{s}^{t, x, u_{0}, \beta\left(u_{0}\right)}\left(T, g\left(X_{T}^{t, x, u_{0}, \beta\left(u_{0}\right)}\right)\right)\right|^{p} \mid \mathcal{F}_{t}\right] \\
& \leq c_{0} E\left[\left|g\left(X_{T}^{t, x, u_{0}, \beta\left(u_{0}\right)}\right)\right|^{p}+\int_{t}^{T}\left|f_{T}^{t, x, u_{0}, \beta\left(u_{0}\right)}(s, 0,0)\right|^{p} d s \mid \mathcal{F}_{t}\right], \quad P-\text { a.s. } \tag{4.9}
\end{align*}
$$

Since $\left[\left(\beta\left(u_{0}\right)\right)_{s}\right]_{\mathbb{V}} \leq \kappa, d s \times d P$-a.s., the $2 / p-$ Hölder continuity of $g$, (2.3), (2.4) as well as a conditional-expectation version of (2.6) show that $P$-a.s.

$$
\begin{align*}
& \left|J\left(t, x, u_{0}, \beta\left(u_{0}\right)\right)\right|^{p} \leq c_{0}+c_{0} E\left[\left|X_{T}^{t, x, u_{0}, \beta\left(u_{0}\right)}\right|^{2}+\int_{t}^{T}\left(\left|X_{s}^{t, x, u_{0}, \beta\left(u_{0}\right)}\right|^{2}+\left[\left(\beta\left(u_{0}\right)\right)_{s}\right]_{\mathbb{V}}^{2}\right) d s \mid \mathcal{F}_{t}\right] \\
& \quad \leq c_{\kappa}+c_{0} E\left[\sup _{s \in[t, T]}\left|X_{s}^{t, x, u_{0}, \beta\left(u_{0}\right)}\right|^{2} \mid \mathcal{F}_{t}\right] \leq c_{\kappa}+c_{0}|x|^{2}+c_{0} E\left[\int_{t}^{T}\left[\left(\beta\left(u_{0}\right)\right)_{s}\right]_{\mathbb{V}}^{2} d s \mid \mathcal{F}_{t}\right] \leq c_{\kappa}+c_{0}|x|^{2} \tag{4.10}
\end{align*}
$$

So it follows that

$$
w_{1}(t, x) \geq \underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} J\left(t, x, u_{0}, \beta\left(u_{0}\right)\right) \geq-c_{\kappa}-c_{0}|x|^{2 / p}, \quad P-\text { a.s. }
$$

We extensively set $\psi(t, u) \triangleq v_{0}, \forall(t, u) \in[0, T] \times O_{\kappa}\left(u_{0}\right)$, then it is $\mathscr{B}([0, T]) \times \mathscr{B}(\mathbb{U}) / \mathscr{B}(\mathbb{V})-$ measurable. For any $\mu \in \mathcal{U}_{t}$, the measurability of function $\psi$ and process $\mu$ implies that

$$
\begin{equation*}
\left(\beta_{\psi}(\mu)\right)_{s} \triangleq \psi\left(s, \mu_{s}\right), \quad s \in[t, T] \tag{4.11}
\end{equation*}
$$

defines a $\mathbb{V}$-valued, $\mathbf{F}$-progressively measurable process, and we see from (A-u) that $\left[\left(\beta_{\psi}(\mu)\right)_{s}\right]_{\mathbb{V}} \leq \kappa+\kappa\left[\mu_{s}\right]_{\mathbb{U}}$, $\forall s \in[t, T]$. So $\beta_{\psi}(\mu) \in \mathcal{V}_{t}$. Let $\mu^{1}, \mu^{2} \in \mathcal{U}_{t}$ such that $\mu^{1}=\mu^{2}, d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ for some $\tau \in \mathcal{S}_{t, T}$ and $A \in \mathcal{F}_{\tau}$. It clearly holds $d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ that

$$
\left(\beta_{\psi}\left(\mu^{1}\right)\right)_{s}=\psi\left(s, \mu_{s}^{1}\right)=\psi\left(s, \mu_{s}^{2}\right)=\left(\beta_{\psi}\left(\mu^{2}\right)\right)_{s}
$$

Hence, $\beta_{\psi} \in \mathfrak{B}_{t}$.
Fix a $u_{\sharp} \in \partial O_{\kappa}\left(u_{0}\right)$. For any $\mu \in \mathcal{U}_{t}$, similar to (4.9) and (4.10), we can deduce that $P$-a.s.

$$
\begin{align*}
\left|J\left(t, x, \mu, \beta_{\psi}(\mu)\right)\right|^{p} \leq & c_{0} E\left[\left|g\left(X_{T}^{t, x, \mu, \beta_{\psi}(\mu)}\right)\right|^{p}+\int_{t}^{T}\left|f_{T}^{t, x, \mu, \beta_{\psi}(\mu)}(s, 0,0)\right|^{p} d s \mid \mathcal{F}_{t}\right] \\
\leq & c_{0}+c_{0} E\left[\left|X_{T}^{t, x, \mu, \beta_{\psi}(\mu)}\right|^{2}+\int_{t}^{T}\left(1_{\left\{\mu_{s} \in O_{\kappa}\left(u_{0}\right)\right\}}\left|f\left(s, X_{s}^{t, x, \mu, \beta_{\psi}(\mu)}, 0,0, \mu_{s}, v_{0}\right)\right|^{p}\right.\right. \\
& \left.\left.+\mathbf{1}_{\left\{\mu_{s} \notin O_{\kappa}\left(u_{0}\right)\right\}}\left|f\left(s, X_{s}^{t, x, \mu, \beta_{\psi}(\mu)}, 0,0, u_{\sharp}, \psi\left(s, u_{\sharp}\right)\right)\right|^{p}\right) d s \mid \mathcal{F}_{t}\right]  \tag{4.12}\\
\leq & c_{\kappa}+c_{0} E\left[\sup _{s \in[t, T]}\left|X_{s}^{t, x, \mu, \beta_{\psi}(\mu)}\right|^{2} \mid \mathcal{F}_{t}\right] \\
\leq & c_{\kappa}+c_{0}|x|^{2}+c_{0} E\left[\left(\int_{t}^{T}\left|b\left(s, 0, \mu_{s},\left(\beta_{\psi}(\mu)\right)_{s}\right)\right| d s\right)^{2}+\left(\int_{t}^{T}\left|\sigma\left(s, 0, \mu_{s},\left(\beta_{\psi}(\mu)\right)_{s}\right)\right| d s\right)^{2} \mid \mathcal{F}_{t}\right]
\end{align*}
$$

where we used a conditional-expectation version of (2.6) in the last inequality. Then an analogous decomposition and estimation to (4.12) leads to that $\left|J\left(t, x, \mu, \beta_{\psi}(\mu)\right)\right|^{p} \leq c_{\kappa}+c_{0}|x|^{2}, P$-a.s. It follows that

$$
w_{1}(t, x) \leq \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} J\left(t, x, \mu, \beta_{\psi}(\mu)\right) \leq c_{\kappa}+c_{0}|x|^{2 / p}, \quad P-\text { a.s. }
$$

Similarly, one has $\left|w_{2}(t, x)\right| \leq c_{\kappa}+c_{0}|x|^{2 / p}, P-$ a.s.
Proof of Proposition 2.2; Let $\mathcal{H}$ denote the Cameron-Martin space of all absolutely continuous functions $h \in \Omega$ whose derivative $\dot{h}$ belongs to $\mathbb{L}^{2}\left([0, T], \mathbb{R}^{d}\right)$. For any $h \in \mathcal{H}$, we define $\mathcal{T}_{h}(\omega) \triangleq \omega+h, \forall \omega \in \Omega$. Clearly, $\mathcal{T}_{h}: \Omega \rightarrow \Omega$ is a bijection and its law is given by $P_{h} \triangleq P \circ \mathcal{T}_{h}^{-1}=\exp \left\{\int_{0}^{T} \dot{h}_{s} d B_{s}-\frac{1}{2} \int_{0}^{T}\left|\dot{h}_{s}\right|^{2} d s\right\} P$. Fix $(t, x) \in[0, T] \times \mathbb{R}^{k}$ and set $\mathcal{H}_{t} \triangleq\{h \in \mathcal{H}: h(s)=h(s \wedge t), \forall s \in[0, T]\}$.
a) Let $h \in \mathcal{H}_{t}$. We first show that

$$
\begin{equation*}
\left(\mu\left(\mathcal{T}_{h}\right), \nu\left(\mathcal{T}_{h}\right)\right) \in \mathcal{U}_{t} \times \mathcal{V}_{t}, \quad \forall(\mu, \nu) \in \mathcal{U}_{t} \times \mathcal{V}_{t} \tag{4.13}
\end{equation*}
$$

Fix $\mu \in \mathcal{U}_{t}$. Given $s \in[t, T]$, we set $\Upsilon_{s}^{h}(\mathcal{D}) \triangleq\left\{(r, \omega) \in[t, s] \times \Omega:\left(r, \mathcal{T}_{h}(\omega)\right) \in \mathcal{D}\right\}$ for any $\mathcal{D} \subset[t, s] \times \Omega$. As the mapping

$$
\begin{equation*}
\mathcal{T}_{h}=B+h \text { is } \mathcal{F}_{s} / \mathcal{F}_{s}-\text { measurable } \tag{4.14}
\end{equation*}
$$

it holds for any $\mathcal{E} \in \mathscr{B}([t, s])$ and $A \in \mathcal{F}_{s}$ that

$$
\Upsilon_{s}^{h}(\mathcal{E} \times A)=\left\{(r, \omega) \in[t, s] \times \Omega:\left(r, \mathcal{T}_{h}(\omega)\right) \in \mathcal{E} \times A\right\}=(\mathcal{E} \cap[t, s]) \times \mathcal{T}_{h}^{-1}(A) \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s}
$$

So $\mathcal{E} \times A \in \Lambda_{s}^{h} \triangleq\left\{\mathcal{D} \subset[t, s] \times \Omega: \Upsilon_{s}^{h}(\mathcal{D}) \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s}\right\}$. In particular, $\emptyset \times \emptyset \in \Lambda_{s}^{h}$ and $[t, s] \times \Omega \in \Lambda_{s}^{h}$. For any $\mathcal{D} \in \Lambda_{s}^{h}$ and $\left\{\mathcal{D}_{n}\right\}_{n \in \mathbb{N}} \subset \Lambda_{s}^{h}$, one can deduce that

$$
\begin{aligned}
\Upsilon_{s}^{h}(([t, s] \times \Omega) \backslash \mathcal{D}) & =\left\{(r, \omega) \in[t, s] \times \Omega:\left(r, \mathcal{T}_{h}(\omega)\right) \in([t, s] \times \Omega) \backslash \mathcal{D}\right\} \\
& =([t, s] \times \Omega) \backslash\left\{(r, \omega) \in[t, s] \times \Omega:\left(r, \mathcal{T}_{h}(\omega)\right) \in \mathcal{D}\right\}=([t, s] \times \Omega) \backslash \Upsilon_{s}^{h}(\mathcal{D}) \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s}
\end{aligned}
$$

$$
\text { and } \Upsilon_{s}^{h}\left(\cup_{n \in \mathbb{N}} \mathcal{D}_{n}\right)=\left\{(r, \omega) \in[t, s] \times \Omega:\left(r, \mathcal{T}_{h}(\omega)\right) \in \underset{n \in \mathbb{N}}{\cup} \mathcal{D}_{n}\right\}
$$

$$
=\bigcup_{n \in \mathbb{N}}\left\{(r, \omega) \in[t, s] \times \Omega:\left(r, \mathcal{T}_{h}(\omega)\right) \in \mathcal{D}_{n}\right\}=\bigcup_{n \in \mathbb{N}} \Upsilon_{s}^{h}\left(\mathcal{D}_{n}\right) \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s}
$$

i.e. $([t, s] \times \Omega) \backslash \mathcal{D}, \bigcup_{n \in \mathbb{N}} \mathcal{D}_{n} \in \Lambda_{s}^{h}$. Thus $\Lambda_{s}^{h}$ is a $\sigma$-field of $[t, s] \times \Omega$. It follows that

$$
\begin{equation*}
\mathscr{B}([t, s]) \otimes \mathcal{F}_{s}=\sigma\left\{\mathcal{E} \times A: \mathcal{E} \in \mathscr{B}([t, s]), A \in \mathcal{F}_{s}\right\} \subset \Lambda_{s}^{h} \tag{4.15}
\end{equation*}
$$

Given $U \in \mathscr{B}(\mathbb{U})$, the $\mathbf{F}$-progressive measurability of $\mu$ and (4.15) show that

$$
\mathcal{D}_{U} \triangleq\left\{(r, \omega) \in[t, s] \times \Omega: \mu_{r}(\omega) \in U\right\} \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s} \subset \Lambda_{s}^{h}
$$

That is

$$
\begin{equation*}
\left\{(r, \omega) \in[t, s] \times \Omega: \mu_{r}\left(\mathcal{T}_{h}(\omega)\right) \in U\right\}=\left\{(r, \omega) \in[t, s] \times \Omega:\left(r, \mathcal{T}_{h}(\omega)\right) \in \mathcal{D}_{U}\right\}=\Upsilon_{s}^{h}\left(\mathcal{D}_{U}\right) \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s} \tag{4.16}
\end{equation*}
$$

which shows the $\mathbf{F}$-progressive measurability of process $\mu\left(\mathcal{T}_{h}\right)$.
Suppose that $E \int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s<\infty$ for some $q>2$. Then one can deduce from Hölder's inequality that for any $\widetilde{q} \in(2, q)$

$$
\begin{align*}
& E \int_{t}^{T}\left[\mu_{s}\left(\mathcal{T}_{h}\right)\right]_{\mathbb{U}}^{\widetilde{q}} d s=E_{P_{h}} \int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{\widetilde{q}} d s=E\left[\exp \left\{\int_{0}^{T} \dot{h}_{s} d B_{s}-\frac{1}{2} \int_{0}^{T}\left|\dot{h}_{s}\right|^{2} d s\right\} \int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{\widetilde{q}} d s\right] \\
& \quad \leq T^{\frac{q-\tilde{q}}{q}} \exp \left\{\frac{\widetilde{q}}{2(q-\widetilde{q})} \int_{0}^{T}\left|\dot{h}_{s}\right|^{2} d s\right\} E\left[\exp \left\{\int_{0}^{T} \dot{h}_{s} d B_{s}-\frac{q}{2(q-\widetilde{q})} \int_{0}^{T}\left|\dot{h}_{s}\right|^{2} d s\right\}\left(\int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s\right)^{\frac{\tilde{q}}{q}}\right] \\
& \quad \leq T^{\frac{q-\widetilde{q}}{q}} \exp \left\{\frac{\widetilde{q}}{2(q-\widetilde{q})} \int_{0}^{T}\left|\dot{h}_{s}\right|^{2} d s\right\}\left(E\left[\exp \left\{\frac{q}{q-\widetilde{q}} \int_{0}^{T} \dot{h}_{s} d B_{s}-\frac{q^{2}}{2(q-\widetilde{q})^{2}} \int_{0}^{T}\left|\dot{h}_{s}\right|^{2} d s\right\}\right]\right)^{\frac{q-\widetilde{q}}{q}}\left(E \int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s\right)^{\frac{\widetilde{q}}{q}} \\
& \quad=T^{\frac{q-\widetilde{q}}{q}} \exp \left\{\frac{\widetilde{q}}{2(q-\widetilde{q})} \int_{0}^{T}\left|\dot{h}_{s}\right|^{2} d s\right\}\left(E \int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s\right)^{\frac{\tilde{q}}{q}}<\infty . \tag{4.17}
\end{align*}
$$

Hence, $\mu\left(\mathcal{T}_{h}\right) \in \mathcal{U}_{t}$. Similarly, $\nu\left(\mathcal{T}_{h}\right) \in \mathcal{V}_{t}$ for any $\nu \in \mathcal{V}_{t}$.
b) We next show that

$$
\begin{equation*}
J(t, x, \mu, \nu)\left(\mathcal{T}_{h}\right)=J\left(t, x, \mu\left(\mathcal{T}_{h}\right), \nu\left(\mathcal{T}_{h}\right)\right), \quad P-a . s \tag{4.18}
\end{equation*}
$$

Let $\left\{\Phi_{s}\right\}_{s \in[t, T]}$ be an $\mathbb{R}^{k \times d}$-valued, $\mathbf{F}$-progressively measurable process and set $M_{s} \triangleq \int_{t}^{s} \Phi_{r} d B_{r}, s \in[t, T]$. We know that (see e.g. Problem 3.2.27 of [21], which is proved on page 228 therein) there exists a sequence of $\mathbb{R}^{k \times d}$-valued, $\mathbf{F}$-simple processes $\left\{\Phi_{s}^{n}=\sum_{i=1}^{\ell_{n}} \xi_{i}^{n} \mathbf{1}_{\left\{s \in\left(t_{i}^{n}, t_{i+1}^{n}\right]\right\}}, s \in[t, T]\right\}_{n \in \mathbb{N}}$ (where $t=t_{1}^{n}<\cdots<t_{\ell_{n}+1}^{n}=T$ and $\xi_{i}^{n} \in \mathcal{F}_{t_{i}^{n}}$ for $\left.i=1, \cdots, \ell_{n}\right)$ such that

$$
P-\lim _{n \rightarrow \infty} \int_{t}^{T} \operatorname{trace}\left\{\left(\Phi_{r}^{n}-\Phi_{r}\right)\left(\Phi_{r}^{n}-\Phi_{r}\right)^{T}\right\} d s=0 \quad \text { and } \quad P-\lim _{n \rightarrow \infty} \sup _{s \in[t, T]}\left|M_{s}^{n}-M_{s}\right|=0
$$

where $M_{s}^{n} \triangleq \int_{t}^{s} \Phi_{r}^{n} d B_{s}=\sum_{i=1}^{\ell_{n}} \xi_{i}^{n}\left(B_{s \wedge t_{i+1}^{n}}-B_{s \wedge t_{i}^{n}}\right)$. By the equivalence of $P_{h}$ to $P$, one has

$$
\begin{array}{r}
P_{h}-\lim _{n \rightarrow \infty} \int_{t}^{T} \operatorname{trace}\left\{\left(\Phi_{r}^{n}-\Phi_{r}\right)\left(\Phi_{r}^{n}-\Phi_{r}\right)^{T}\right\} d s=P_{h}-\lim _{n \rightarrow \infty} \sup _{s \in[t, T]}\left|M_{s}^{n}-M_{s}\right|=0 \\
\text { or } P-\lim _{n \rightarrow \infty} \int_{t}^{T} \operatorname{trace}\left\{\left(\Phi_{r}^{n}\left(\mathcal{T}_{h}\right)-\Phi_{r}\left(\mathcal{T}_{h}\right)\right)\left(\Phi_{r}^{n}\left(\mathcal{T}_{h}\right)-\Phi_{r}\left(\mathcal{T}_{h}\right)\right)^{T}\right\} d s=P-\lim _{n \rightarrow \infty} \sup _{s \in[t, T]}\left|M_{s}^{n}\left(\mathcal{T}_{h}\right)-M_{s}\left(\mathcal{T}_{h}\right)\right|=0 \tag{4.19}
\end{array}
$$

Applying Proposition 3.2.26 of [21] yields that

$$
\begin{equation*}
0=P-\lim _{n \rightarrow \infty} \sup _{s \in[t, T]}\left|\int_{t}^{s} \Phi_{r}^{n}\left(\mathcal{T}_{h}\right) d B_{r}-\int_{t}^{s} \Phi_{r}\left(\mathcal{T}_{h}\right) d B_{r}\right| \tag{4.20}
\end{equation*}
$$

As $h \in \mathcal{H}_{t}$, one can deduce that

$$
\begin{aligned}
M_{s}^{n}\left(\mathcal{T}_{h}\right) & =\left(\sum_{i=1}^{\ell_{n}} \xi_{i}^{n}\left(B_{s \wedge t_{i+1}^{n}}-B_{s \wedge t_{i}^{n}}\right)\right)\left(\mathcal{T}_{h}\right)=\sum_{i=1}^{\ell_{n}} \xi_{i}^{n}\left(\mathcal{T}_{h}\right)\left(B_{s \wedge t_{i+1}^{n}}\left(\mathcal{T}_{h}\right)-B_{s \wedge t_{i}^{n}}\left(\mathcal{T}_{h}\right)\right) \\
& =\sum_{i=1}^{\ell_{n}} \xi_{i}^{n}\left(\mathcal{T}_{h}\right)\left(B_{s \wedge t_{i+1}^{n}}-h\left(s \wedge t_{i+1}^{n}\right)-B_{s \wedge t_{i}^{n}}+h\left(s \wedge t_{i}^{n}\right)\right)=\int_{t}^{s} \Phi_{r}^{n}\left(\mathcal{T}_{h}\right) d B_{r}, \quad \forall s \in[t, T]
\end{aligned}
$$

which together with (4.19) and (4.20) leads to that $P$-a.s.

$$
\begin{equation*}
\int_{t}^{s} \Phi_{r}\left(\mathcal{T}_{h}\right) d B_{r}=M_{s}\left(\mathcal{T}_{h}\right)=\left(\int_{t}^{s} \Phi_{r} d B_{r}\right)\left(\mathcal{T}_{h}\right), \quad s \in[t, T] \tag{4.21}
\end{equation*}
$$

Let $(\mu, \nu) \in \mathcal{U}_{t} \times \mathcal{V}_{t}$ and set $\Theta=(t, x, \mu, \nu)$. By (4.14), the process $X^{\Theta}\left(\mathcal{T}_{h}\right)$ is $\mathbf{F}$-adapted, and the equivalence of $P_{h}$ to $P$ implies that $X^{\Theta}\left(\mathcal{T}_{h}\right)$ has $P$-a.s. continuous paths. Suppose that $E \int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s+E \int_{t}^{T}\left[\nu_{s}\right]_{\mathbb{V}}^{q} d s<\infty$ for some $q>2$. A standard estimate of SDEs (see e.g. [20, pg. 166-168] and [21, pg. 289-290]) shows that

$$
\begin{align*}
E\left[\sup _{s \in[t, T]}\left|X_{s}^{\Theta}\right|^{q}\right] & \leq c_{q}|x|^{q}+c_{q} E\left[\left(\int_{t}^{T}\left|b^{\mu, \nu}(s, 0)\right| d s\right)^{q}+\left(\int_{t}^{T}\left|\sigma^{\mu, \nu}(s, 0)\right| d s\right)^{q}\right] \\
& \leq c_{q}\left(1+|x|^{q}+E \int_{t}^{T}\left(\left[\mu_{s}\right]_{\mathbb{U}}^{q}+\left[\nu_{s}\right]_{\mathbb{V}}^{q}\right) d s\right)<\infty \tag{4.22}
\end{align*}
$$

Similar to (4.17), one can deduce that $E\left[\sup _{s \in[t, T]}\left|X_{s}^{\Theta}\left(\mathcal{T}_{h}\right)\right|^{\widetilde{q}}\right]<\infty$ for any $\widetilde{q} \in[2, q)$. In particular, $X^{\Theta}\left(\mathcal{T}_{h}\right) \in$ $\mathbb{C}_{\mathbf{F}}^{2}\left([t, T], \mathbb{R}^{k}\right)$. It follows from (4.21) that

$$
\begin{aligned}
X_{s}^{\Theta}\left(\mathcal{T}_{h}\right) & =x+\int_{t}^{s} b\left(r, X_{r}^{\Theta}\left(\mathcal{T}_{h}\right), \mu_{r}\left(\mathcal{T}_{h}\right), \nu_{r}\left(\mathcal{T}_{h}\right)\right) d r+\left(\int_{t}^{s} \sigma\left(r, X_{r}^{\Theta}, \mu_{r}, \nu_{r}\right) d B_{r}\right)\left(\mathcal{T}_{h}\right) \\
& =x+\int_{t}^{s} b\left(r, X_{r}^{\Theta}\left(\mathcal{T}_{h}\right), \mu_{r}\left(\mathcal{T}_{h}\right), \nu_{r}\left(\mathcal{T}_{h}\right)\right) d r+\int_{t}^{s} \sigma\left(r, X_{r}^{\Theta}\left(\mathcal{T}_{h}\right), \mu_{r}\left(\mathcal{T}_{h}\right), \nu_{r}\left(\mathcal{T}_{h}\right)\right) d B_{r}, \quad s \in[t, T]
\end{aligned}
$$

Thus the uniqueness of $\operatorname{SDE}$ (1.1) with parameters $\Theta_{h}=\left(t, x, \mu\left(\mathcal{T}_{h}\right), \nu\left(\mathcal{T}_{h}\right)\right)$ shows that

$$
\begin{equation*}
X_{s}^{\Theta_{h}}=X_{s}^{\Theta}\left(\mathcal{T}_{h}\right), \quad \forall s \in[t, T] \tag{4.23}
\end{equation*}
$$

Let $(\widehat{Y}, \widehat{Z})=\left(Y^{\Theta}\left(T, g\left(X_{T}^{\Theta}\right)\right), Z^{\Theta}\left(T, g\left(X_{T}^{\Theta}\right)\right)\right)$. Analogous to $X^{\Theta}\left(\mathcal{T}_{h}\right), \widehat{Y}\left(\mathcal{T}_{h}\right)$ is an $\mathbf{F}$-adapted continuous process. And using the similar arguments that leads to (4.16), we see that the process $\widehat{Z}\left(\mathcal{T}_{h}\right)$ is $\mathbf{F}$-progressively measurable. By (4.22), $g\left(X_{T}^{\Theta}\right) \in \mathbb{L}^{\frac{p q}{2}}\left(\mathcal{F}_{T}\right)$, and a similar argument to (2.9) yields that

$$
E\left[\left(\int_{t}^{T}\left|f_{T}^{\Theta}(s, 0,0)\right| d s\right)^{\frac{p q}{2}}\right] \leq c_{q}+c_{q} E\left[\sup _{s \in[t, T]}\left|X_{s}^{\Theta}\right|^{q}+\int_{t}^{T}\left(\left[\mu_{s}\right]_{\mathbb{U}}^{q}+\left[\nu_{s}\right]_{\mathbb{V}}^{q}\right) d s\right]<\infty
$$

Then we know from Proposition 1.1 that the unique solution $(\widehat{Y}, \widehat{Z})$ of $\operatorname{BSDE}\left(t, g\left(X_{T}^{\Theta}\right), f_{T}^{\Theta}\right)$ in $\mathbb{G}_{\mathbf{F}}^{p}([t, T])$ actually belongs to $\mathbb{G}_{\mathbf{F}}^{\frac{p q}{2}}([t, T])$. Similar to (4.17), one can deduce that $E\left[\sup _{s \in[t, T]}\left|\widehat{Y}_{s}\left(\mathcal{T}_{h}\right)\right|^{\widetilde{q}}+\left(\int_{t}^{T}\left|\widehat{Z}_{s}\left(\mathcal{T}_{h}\right)\right|^{2} d s\right)^{\widetilde{q} / 2}\right]<\infty$ for any $\widetilde{q} \in\left[p, \frac{p q}{2}\right)$. In particular, $\left(\widehat{Y}\left(\mathcal{T}_{h}\right), \widehat{Z}\left(\mathcal{T}_{h}\right)\right) \in \mathbb{G}_{\mathbf{F}}^{p}([t, T])$.

Applying (4.21) again, we can deduce from (4.23) that

$$
\begin{aligned}
\widehat{Y}_{s}\left(\mathcal{T}_{h}\right) & =g\left(X_{T}^{\Theta}\left(\mathcal{T}_{h}\right)\right)+\int_{s}^{T} f\left(r, X_{r}^{\Theta}\left(\mathcal{T}_{h}\right), \widehat{Y}_{r}\left(\mathcal{T}_{h}\right), \widehat{Z}_{r}\left(\mathcal{T}_{h}\right), \mu_{r}\left(\mathcal{T}_{h}\right), \nu_{r}\left(\mathcal{T}_{h}\right)\right) d r-\left(\int_{s}^{T} \widehat{Z}_{r} d B_{r}\right)\left(\mathcal{T}_{h}\right) \\
& =g\left(X_{T}^{\Theta_{h}}\right)+\int_{s}^{T} f\left(r, X_{r}^{\Theta_{h}}, \widehat{Y}_{r}\left(\mathcal{T}_{h}\right), \widehat{Z}_{r}\left(\mathcal{T}_{h}\right), \mu_{r}\left(\mathcal{T}_{h}\right), \nu_{r}\left(\mathcal{T}_{h}\right)\right) d r-\int_{t}^{s} \widehat{Z}_{r}\left(\mathcal{T}_{h}\right) d B_{r}, \quad s \in[t, T]
\end{aligned}
$$

Thus the uniqueness of $\operatorname{BSDE}\left(t, g\left(X_{T}^{\Theta_{h}}\right), f_{T}^{\Theta_{h}}\right)$ implies that $P$-a.s.

$$
Y_{s}^{\Theta_{h}}\left(T, g\left(X_{T}^{\Theta_{h}}\right)\right)=\widehat{Y}_{s}\left(\mathcal{T}_{h}\right), \quad s \in[t, T]
$$

In particular,

$$
J(t, x, \mu, \nu)\left(\mathcal{T}_{h}\right)=\widehat{Y}_{t}\left(\mathcal{T}_{h}\right)=Y_{t}^{\Theta_{h}}\left(T, g\left(X_{T}^{\Theta_{h}}\right)\right)=J\left(t, x, \mu\left(\mathcal{T}_{h}\right), \nu\left(\mathcal{T}_{h}\right)\right), \quad P \text {-a.s. }
$$

c) Now, we show that $w_{1}(t, x)\left(\mathcal{T}_{h}\right)=w_{1}(t, x), P-a . s$.

Let $\beta \in \mathfrak{B}_{t}$ and define

$$
\beta_{h}(\mu) \triangleq \beta\left(\mu\left(\mathcal{T}_{-h}\right)\right)\left(\mathcal{T}_{h}\right), \quad \forall \mu \in \mathcal{U}_{t}
$$

Similar to (4.13), $\mu\left(\mathcal{T}_{-h}\right) \in \mathcal{U}_{t}$ as $-h$ also belongs to $\mathcal{H}$. It follows that $\beta\left(\mu\left(\mathcal{T}_{-h}\right)\right) \in \mathcal{V}_{t}$. Using (4.13) again shows that $\beta_{h}(\mu)=\beta\left(\mu\left(\mathcal{T}_{-h}\right)\right)\left(\mathcal{T}_{h}\right) \in \mathcal{V}_{t}$. Since $\left[\left(\beta\left(\mu\left(\mathcal{T}_{-h}\right)\right)\right)_{s}\right]_{\mathbb{V}} \leq \kappa+C_{\beta}\left[\mu_{s}\left(\mathcal{T}_{-h}\right)\right]_{\mathbb{U}}, d s \times d P$-a.s., the equivalence of $P_{h}$ to $P$ shows that $\left[\left(\beta\left(\mu\left(\mathcal{T}_{-h}\right)\right)\right)_{s}\right]_{\mathbb{V}} \leq \kappa+C_{\beta}\left[\mu_{s}\left(\mathcal{T}_{-h}\right)\right]_{\mathbb{U}}, d s \times d P_{h}-$ a.s., or

$$
\left[\left(\beta_{h}(\mu)\right)_{s}\right]_{\mathbb{V}}=\left[\left(\beta\left(\mu\left(\mathcal{T}_{-h}\right)\right)\left(\mathcal{T}_{h}\right)\right)_{s}\right]_{\mathbb{V}} \leq \kappa+C_{\beta}\left[\mu_{s}\right]_{\mathbb{U}}, \quad d s \times d P-a . s
$$

Let $\mu^{1}, \mu^{2} \in \mathcal{U}_{t}$ such that $\mu^{1}=\mu^{2}, d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ for some $\tau \in \mathcal{S}_{t, T}$ and $A \in \mathcal{F}_{\tau}$. By the equivalence of $P_{-h}$ to $P, \mu^{1}=\mu^{2}, d s \times d P_{-h}-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$, or $\mu^{1}\left(\mathcal{T}_{-h}\right)=\mu^{2}\left(\mathcal{T}_{-h}\right), d s \times d P-$ a.s. on $\llbracket t, \tau\left(\mathcal{T}_{-h}\right) \llbracket \cup \llbracket \tau\left(\mathcal{T}_{-h}\right), T \rrbracket_{\mathcal{T}_{h}(A)}$. Given $s \in[t, T]$, similar to (4.14), $\mathcal{T}_{-h}$ is also $\mathcal{F}_{s} / \mathcal{F}_{s}-$ measurable. It follows that

$$
\begin{aligned}
\left\{\tau\left(\mathcal{T}_{-h}\right) \leq s\right\} & =\left\{\omega: \mathcal{T}_{-h}(\omega) \in\{\tau \leq s\}\right\}=\mathcal{T}_{-h}^{-1}(\{\tau \leq s\}) \in \mathcal{F}_{s} \\
\text { and } \mathcal{T}_{h}(A) \cap\left\{\tau\left(\mathcal{T}_{-h}\right) \leq s\right\} & =\mathcal{T}_{-h}^{-1}(A) \cap \mathcal{T}_{-h}^{-1}(\{\tau \leq s\})=\mathcal{T}_{-h}^{-1}(A \cap\{\tau \leq s\}) \in \mathcal{F}_{s}
\end{aligned}
$$

which shows that $\tau\left(\mathcal{T}_{-h}\right)$ is an $\mathbf{F}$-stopping time and $\mathcal{T}_{h}(A) \in \mathcal{F}_{\tau\left(\mathcal{T}_{-h}\right)}$. As $t \leq \tau \leq T, P$-a.s., the equivalence of $P_{-h}$ to $P$ shows that $t \leq \tau \leq T, P_{-h}$ a.s., or $t \leq \tau\left(\mathcal{T}_{-h}\right) \leq T, P-$ a.s. So $\tau\left(\mathcal{T}_{-h}\right) \in \mathcal{S}_{t, T}$, and we see from Definition 2.2 that $\beta\left(\mu^{1}\left(\mathcal{T}_{-h}\right)\right)=\beta\left(\mu^{2}\left(\mathcal{T}_{-h}\right)\right)$, $d s \times d P-$ a.s. on $\llbracket t, \tau\left(\mathcal{T}_{-h}\right) \llbracket \cup \llbracket \tau\left(\mathcal{T}_{-h}\right), T \rrbracket \mathcal{T}_{h}(A)$. The equivalence of $P_{h}$ to $P$ then shows that $\beta\left(\mu^{1}\left(\mathcal{T}_{-h}\right)\right)=\beta\left(\mu^{2}\left(\mathcal{T}_{-h}\right)\right)$, $d s \times d P_{h}$-a.s. on $\llbracket t, \tau\left(\mathcal{T}_{-h}\right) \llbracket \cup \llbracket \tau\left(\mathcal{T}_{-h}\right), T \rrbracket_{\mathcal{T}_{h}(A)}$, or $\beta_{h}\left(\mu^{1}\right)=$ $\beta\left(\mu^{1}\left(\mathcal{T}_{-h}\right)\right)\left(\mathcal{T}_{h}\right)=\beta\left(\mu^{2}\left(\mathcal{T}_{-h}\right)\right)\left(\mathcal{T}_{h}\right)=\beta_{h}\left(\mu^{2}\right), d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$. Hence, $\beta_{h} \in \mathfrak{B}_{t}$.

Set $I(t, x, \beta) \triangleq \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} J(t, x, \mu, \beta(\mu))$. For any $\mu \in \mathcal{U}_{t}$, as $I(t, x, \beta) \geq J(t, x, \mu, \beta(\mu)), P-$ a.s., the equivalence of $P_{h}$ to $P$ shows that $I(t, x, \beta) \geq J(t, x, \mu, \beta(\mu)), P_{h}-$ a.s., or

$$
\begin{equation*}
I(t, x, \beta)\left(\mathcal{T}_{h}\right) \geq J(t, x, \mu, \beta(\mu))\left(\mathcal{T}_{h}\right), \quad P-\text { a.s. } \tag{4.24}
\end{equation*}
$$

Let $\xi$ be another random variable such that $\xi \geq J(t, x, \mu, \beta(\mu))\left(\mathcal{T}_{h}\right), P-$ a.s., or $\xi\left(\mathcal{T}_{-h}\right) \geq J(t, x, \mu, \beta(\mu))$, $P_{h}-$ a.s. for any $\mu \in \mathcal{U}_{t}$. By the equivalence of $P_{h}$ to $P$, it holds for any $\mu \in \mathcal{U}_{t}$ that $\xi\left(\mathcal{T}_{-h}\right) \geq J(t, x, \mu, \beta(\mu)), P-$ a.s. Taking essential supremum over $\mu \in \mathcal{U}_{t}$ yields that $\xi\left(\mathcal{T}_{-h}\right) \geq I(t, x, \beta), P-$ a.s. or $\xi \geq I(t, x, \beta)\left(\mathcal{T}_{h}\right), P_{-h}-$ a.s. Then it follows from the equivalence of $P_{-h}$ to $P$ that $\xi \geq I(t, x, \beta)\left(\mathcal{T}_{h}\right)$, $P$-a.s., which together with (4.24) implies that

$$
\begin{equation*}
\operatorname{esssup}_{\mu \in \mathcal{U}_{t}}\left(J(t, x, \mu, \beta(\mu))\left(\mathcal{T}_{h}\right)\right)=I(t, x, \beta)\left(\mathcal{T}_{h}\right)=\left(\operatorname{esssup}_{\mu \in \mathcal{U}_{t}} J(t, x, \mu, \beta(\mu))\right)\left(\mathcal{T}_{h}\right), \quad P \text {-a.s. } \tag{4.25}
\end{equation*}
$$

Similarly, $\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}}\left(I(t, x, \beta)\left(\mathcal{T}_{h}\right)\right)=\left(\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} I(t, x, \beta)\right)\left(\mathcal{T}_{h}\right), P$-a.s., which together (4.18) and (4.25) yields that

$$
\begin{align*}
w_{1}(t, x)\left(\mathcal{T}_{h}\right) & =\left(\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} I(t, x, \beta)\right)\left(\mathcal{T}_{h}\right)=\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}}\left(I(t, x, \beta)\left(\mathcal{T}_{h}\right)\right)=\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}}\left(J(t, x, \mu, \beta(\mu))\left(\mathcal{T}_{h}\right)\right) \\
& =\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} J\left(t, x, \mu\left(\mathcal{T}_{h}\right), \beta_{h}\left(\mu\left(\mathcal{T}_{h}\right)\right)\right)=\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} J\left(t, x, \mu, \beta_{h}(\mu)\right) \\
& =\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} J(t, x, \mu, \beta(\mu))=w_{1}(t, x), \quad P-\text { a.s., } \tag{4.26}
\end{align*}
$$

where we used the facts that $\left\{\mu\left(\mathcal{T}_{h}\right): \mu \in \mathcal{U}_{t}\right\}=\mathcal{U}_{t}$ and $\left\{\beta_{h}: \beta \in \mathfrak{B}_{t}\right\}=\mathfrak{B}_{t}$.
d) As an $\mathcal{F}_{t}-$ measurable random variable, $w_{1}(t, x)$ only depends on the restriction of $\omega \in \Omega$ to the time interval $[0, t]$. So (4.26) holds even for any $h \in \mathcal{H}$. Then an application of Lemma 3.4 of [11 yields that $w_{1}(t, x)=E\left[w_{1}(t, x)\right]$, $P$-a.s. Similarly, one can deduce that $w_{2}(t, x)=E\left[w_{2}(t, x)\right], P-$ a.s.

Proof of Proposition [2.3: Let $t \in[0, T]$ and $x_{1}, x_{2} \in \mathbb{R}^{k}$. For any $(\beta, \mu) \in \mathfrak{B}_{t} \times \mathcal{U}_{t}$, (2.14) implies that

$$
\left|J\left(t, x_{1}, \mu, \beta(\mu)\right)-J\left(t, x_{2}, \mu, \beta(\mu)\right)\right|^{p} \leq c_{0}\left|x_{1}-x_{2}\right|^{2}, \quad P-\text { a.s. }
$$

which leads to that

$$
J\left(t, x_{2}, \mu, \beta(\mu)\right)-c_{0}\left|x_{1}-x_{2}\right|^{2 / p} \leq J\left(t, x_{1}, \mu, \beta(\mu)\right) \leq J\left(t, x_{2}, \mu, \beta(\mu)\right)+c_{0}\left|x_{1}-x_{2}\right|^{2 / p}, \quad P-\text { a.s. }
$$

Taking essential supremum over $\mu \in \mathcal{U}_{t}$ and then taking essential infimum over $\beta \in \mathfrak{B}_{t}$ yield that

$$
w_{1}\left(t, x_{2}\right)-c_{0}\left|x_{1}-x_{2}\right|^{2 / p} \leq w_{1}\left(t, x_{1}\right) \leq w_{1}\left(t, x_{2}\right)+c_{0}\left|x_{1}-x_{2}\right|^{2 / p} .
$$

So $\left|w_{1}\left(t, x_{1}\right)-w_{1}\left(t, x_{2}\right)\right| \leq c_{0}\left|x_{1}-x_{2}\right|^{2 / p}$. Similarly, one has $\left|w_{2}\left(t, x_{1}\right)-w_{2}\left(t, x_{2}\right)\right| \leq c_{0}\left|x_{1}-x_{2}\right|^{2 / p}$.

### 4.3 Proof of the Weak Dynamic Programming Principle

To prove the weak dynamic programming principle (Theorem 2.1), we begin with two auxiliary result. The first one shows that the pasting of state processes (resp. payoff processes) is exactly the state process (resp. payoff process) with the pasted controls.

Lemma 4.1. Given $t \in[0, T]$, let $\left\{A_{i}\right\}_{i=1}^{n} \subset \mathcal{F}_{t}$ be a partition of $\Omega$. For any $\left\{\left(\xi_{i}, \mu^{i}, \nu^{i}\right)\right\}_{i=0}^{n} \subset \mathbb{L}^{2}\left(\mathcal{F}_{t}, \mathbb{R}^{k}\right) \times \mathcal{U}_{t} \times \mathcal{V}_{t}$, if $\xi_{0}=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} \xi_{i}, P-a . s$. and if $\left(\mu^{0}, \nu^{0}\right)=\left(\sum_{i=1}^{n} \mathbf{1}_{A_{i}} \mu^{i}, \sum_{i=1}^{n} \mathbf{1}_{A_{i}} \nu^{i}\right)$,ds $\times d P-a . s$., then it holds $P-a . s$. that

$$
\begin{equation*}
X_{s}^{t, \xi_{0}, \mu^{0}, \nu^{0}}=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} X_{s}^{t, \xi_{i}, \mu^{i}, \nu^{i}}, \quad \forall s \in[t, T] . \tag{4.27}
\end{equation*}
$$

Moreover, for any $\left\{\left(\tau_{i}, \eta_{i}\right)\right\}_{i=0}^{n} \subset \mathcal{S}_{t, T} \times \mathbb{L}^{p}\left(\mathcal{F}_{T}\right)$ such that each $\eta_{i}$ is $\mathcal{F}_{\tau_{i}}-$ measurable, if $\tau_{0}=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} \tau_{i}, P-$ a.s. and if $\eta_{0}=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} \eta_{i}, P-a . s$., then it holds $P-$ a.s. that

$$
\begin{equation*}
Y_{s}^{t, \xi_{0}, \mu^{0}, \nu^{0}}\left(\tau_{0}, \eta_{0}\right)=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} Y_{s}^{t, \xi_{i}, \mu^{i}, \nu^{i}}\left(\tau_{i}, \eta_{i}\right), \quad \forall s \in[t, T] . \tag{4.28}
\end{equation*}
$$

In particular, one has

$$
\begin{equation*}
J\left(t, \xi_{0}, \mu^{0}, \nu^{0}\right)=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} J\left(t, \xi_{i}, \mu^{i}, \nu^{i}\right), \quad P-a . s . \tag{4.29}
\end{equation*}
$$

Proof: Let $\left(X^{i}, Y^{i}, Z^{i}\right)=\left(X^{t, \xi_{i}, \mu^{i}, \nu^{i}}, Y^{t, \xi_{i}, \mu^{i}, \nu^{i}}\left(\tau_{i}, \eta_{i}\right), Z^{t, \xi_{i}, \mu^{i}, \nu^{i}}\left(\tau_{i}, \eta_{i}\right)\right)$ for $i=0, \cdots, n$. We define

$$
(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \triangleq \sum_{i=1}^{n} \mathbf{1}_{A_{i}}\left(X^{i}, Y^{i}, Z^{i}\right) \in \mathbb{C}_{\mathbf{F}}^{2}\left([t, T], \mathbb{R}^{k}\right) \times \mathbb{G}_{\mathbf{F}}^{p}([t, T])
$$

For any $s \in[t, T]$ and $i=1, \cdots, n$, multiplying $\mathbf{1}_{A_{i}}$ to SDE (1.1) with parameters $\left(t, \xi_{i}, \mu^{i}, \nu^{i}\right)$, we can deduce that

$$
\begin{align*}
\mathbf{1}_{A_{i}} X_{s}^{i} & =\mathbf{1}_{A_{i}} \xi_{i}+\mathbf{1}_{A_{i}} \int_{t}^{s} b\left(r, X_{r}^{i}, \mu_{r}^{i}, \nu_{r}^{i}\right) d r+\mathbf{1}_{A_{i}} \int_{t}^{s} \sigma\left(r, X_{r}^{i}, \mu_{r}^{i}, \nu_{r}^{i}\right) d B_{r} \\
& =\mathbf{1}_{A_{i}} \xi_{i}+\int_{t}^{s} \mathbf{1}_{A_{i}} b\left(r, X_{r}^{i}, \mu_{r}^{i}, \nu_{r}^{i}\right) d r+\int_{t}^{s} \mathbf{1}_{A_{i}} \sigma\left(r, X_{r}^{i}, \mu_{r}^{i}, \nu_{r}^{i}\right) d B_{r} \\
& =\mathbf{1}_{A_{i}} \xi_{i}+\int_{t}^{s} \mathbf{1}_{A_{i}} b\left(r, \mathcal{X}_{r}, \mu_{r}^{0}, \nu_{r}^{0}\right) d r+\int_{t}^{s} \mathbf{1}_{A_{i}} \sigma\left(r, \mathcal{X}_{r}, \mu_{r}^{0}, \nu_{r}^{0}\right) d B_{r} \\
& =\mathbf{1}_{A_{i}} \xi_{i}+\mathbf{1}_{A_{i}} \int_{t}^{s} b\left(r, \mathcal{X}_{r}, \mu_{r}^{0}, \nu_{r}^{0}\right) d r+\mathbf{1}_{A_{i}} \int_{t}^{s} \sigma\left(r, \mathcal{X}_{r}, \mu_{r}^{0}, \nu_{r}^{0}\right) d B_{r}, \quad P-\text { a.s. } \tag{4.30}
\end{align*}
$$

Adding them up over $i \in\{1, \cdots, n\}$ and using the continuity of process $\mathcal{X}$ show that $P$-a.s.

$$
\mathcal{X}_{s}=\xi_{0}+\int_{t}^{s} b\left(r, \mathcal{X}_{r}, \mu_{r}^{0}, \nu_{r}^{0}\right) d r+\int_{t}^{s} \sigma\left(r, \mathcal{X}_{r}, \mu_{r}^{0}, \nu_{r}^{0}\right) d B_{r}, \quad s \in[t, T]
$$

So $\mathcal{X}=X^{t, \xi_{0}, \mu^{0}, \nu^{0}}$, i.e. 4.27).
Next, for any $s \in[t, T]$ and $i=1, \cdots, n$, similar to (4.30), multiplying $\mathbf{1}_{A_{i}}$ to $\operatorname{BSDE}\left(t, \eta_{i}, f_{\tau_{i}}^{t, \xi_{i}, \mu^{i}, \nu^{i}}\right)$ yields that

$$
\begin{aligned}
\mathbf{1}_{A_{i}} Y_{s}^{i} & =\mathbf{1}_{A_{i}} \eta_{i}+\mathbf{1}_{A_{i}} \int_{s}^{T} \mathbf{1}_{\left\{r<\tau_{i}\right\}} f\left(r, X_{r}^{i}, Y_{r}^{i}, Z_{r}^{i}, \mu_{r}^{i}, \nu_{r}^{i}\right) d r-\mathbf{1}_{A_{i}} \int_{s}^{T} Z_{r}^{i} d B_{r} \\
& =\mathbf{1}_{A_{i}} \eta_{i}+\mathbf{1}_{A_{i}} \int_{s}^{T} \mathbf{1}_{\left\{r<\tau_{0}\right\}} f\left(r, \mathcal{X}_{r}, \mathcal{Y}_{r}, \mathcal{Z}_{r}, \mu_{r}^{0}, \nu_{r}^{0}\right) d r-\mathbf{1}_{A_{i}} \int_{s}^{T} \mathcal{Z}_{r} d B_{r}, \quad P-\text { a.s. }
\end{aligned}
$$

Adding them up and using the continuity of process $\mathcal{Y}$, we obtain that $P$-a.s.

$$
\mathcal{Y}_{s}=\eta_{0}+\int_{s}^{T} \mathbf{1}_{\left\{r<\tau_{0}\right\}} f\left(r, X_{r}^{t, \xi_{0}, \mu^{0}, \nu^{0}}, \mathcal{Y}_{r}, \mathcal{Z}_{r}, \mu_{r}^{0}, \nu_{r}^{0}\right) d r-\int_{s}^{T} \mathcal{Z}_{r} d B_{r}, \quad s \in[t, T]
$$

Thus $(\mathcal{Y}, \mathcal{Z})=\left(Y^{t, \xi_{0}, \mu^{0}, \nu^{0}}\left(\tau_{0}, \eta_{0}\right), Z^{t, \xi_{0}, \mu^{0}, \nu^{0}}\left(\tau_{0}, \eta_{0}\right)\right)$, proving (4.28).
Taking $\tau_{i}=T$ and $\eta_{i}=g\left(X_{T}^{t, \xi_{i}, \mu^{i}, \nu^{i}}\right) \in \mathbb{L}^{p}\left(\mathcal{F}_{T}\right)$ for $i=0, \cdots, n$, we see from (4.27) that

$$
\sum_{i=1}^{n} \mathbf{1}_{A_{i}} \eta_{i}=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} g\left(X_{T}^{t, \xi_{i}, \mu^{i}, \nu^{i}}\right)=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} g\left(X_{T}^{t, \xi_{0}, \mu^{0}, \nu^{0}}\right)=g\left(X_{T}^{t, \xi_{0}, \mu^{0}, \nu^{0}}\right)=\eta_{0}, \quad P-\text { a.s. }
$$

Then (4.28) shows that $P$-a.s.

$$
J\left(t, \xi_{0}, \mu^{0}, \nu^{0}\right)=Y_{t}^{t, \xi_{0}, \mu^{0}, \nu^{0}}\left(T, \eta_{0}\right)=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} Y_{t}^{t, \xi_{i}, \mu^{i}, \nu^{i}}\left(T, \eta_{i}\right)=\sum_{i=1}^{n} \mathbf{1}_{A_{i}} J\left(t, \xi_{i}, \mu^{i}, \nu^{i}\right)
$$

In the next Lemma, we approach $I(t, x, \beta) \triangleq \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} Y_{t}^{t, x, \mu, \beta(\mu)}$ from above and $w_{1}(t, x)=\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} I(t, x, \beta)$ from below:

Lemma 4.2. Let $(t, x) \in[0, T] \times \mathbb{R}^{k}$ and $\varepsilon>0$. For any $\beta \in \mathfrak{B}_{t}$, there exist $\left\{\left(A_{n}, \mu^{n}\right)\right\}_{n \in \mathbb{N}} \subset \mathcal{F}_{t} \times \mathcal{U}_{t}$ with $\lim _{n \rightarrow \infty} \uparrow \mathbf{1}_{A_{n}}=1, P-$ a.s. such that for any $n \in \mathbb{N}$

$$
\begin{equation*}
J\left(t, x, \mu^{n}, \beta\left(\mu^{n}\right)\right) \geq(I(t, x, \beta)-\varepsilon) \wedge \varepsilon^{-1}, \quad P-a . s . \text { on } A_{n} \tag{4.31}
\end{equation*}
$$

Similarly, there exist $\left\{\left(\mathcal{A}_{n}, \beta_{n}\right)\right\}_{n \in \mathbb{N}} \subset \mathcal{F}_{t} \times \mathfrak{B}_{t}$ with $\lim _{n \rightarrow \infty} \uparrow \mathbf{1}_{\mathcal{A}_{n}}=1$, P-a.s. such that for any $n \in \mathbb{N}$

$$
\begin{equation*}
w_{1}(t, x) \geq I\left(t, x, \beta_{n}\right)-\varepsilon, \quad P-\text { a.s. on } \mathcal{A}_{n} \tag{4.32}
\end{equation*}
$$

Proof: (i) Let $\beta \in \mathfrak{B}_{t}$. Given $\mu^{1}, \mu^{2} \in \mathcal{U}_{t}$, we set $A \triangleq\left\{J\left(t, x, \mu^{1}, \beta\left(\mu^{1}\right)\right) \geq J\left(t, x, \mu^{2}, \beta\left(\mu^{2}\right)\right)\right\} \in \mathcal{F}_{t}$ and define $\widehat{\mu}_{s} \triangleq \mathbf{1}_{A} \mu_{s}^{1}+\mathbf{1}_{A^{c}} \mu_{s}^{2}, s \in[t, T]$. Clearly, $\widehat{\mu}$ is an $\mathbf{F}$-progressively measurable process. For $i=1,2$, suppose that $E \int_{t}^{T}\left[\mu_{s}^{i}\right]_{\mathbb{U}}^{q_{i}^{s}} d s<\infty$ for some $q_{i}>2$. It follows that $E \int_{t}^{T}\left[\widehat{\mu}_{s}\right]_{\mathbb{U}}^{q_{1} \wedge q_{2}} d s \leq E \int_{t}^{T}\left[\mu_{s}^{1}\right]_{\mathbb{U}}^{q_{1} \wedge q_{2}} d s+E \int_{t}^{T}\left[\mu_{s}^{2}\right]_{\mathbb{U}}^{q_{1} \wedge q_{2}} d s<\infty$. Thus, $\widehat{\mu} \in \mathcal{U}_{t}$. As $\widehat{\mu}=\mu^{1}$ on $[t, T] \times A$, taking $(\tau, A)=(t, A)$ in Definition 2.2 yields that $\beta(\widehat{\mu})=\beta\left(\mu^{1}\right), d s \times d P-$ a.s. on $[t, T] \times A$. Similarly, $\beta(\widehat{\mu})=\beta\left(\mu^{2}\right), d s \times d P-$ a.s. on $[t, T] \times A^{c}$. So $\beta(\widehat{\mu})=\mathbf{1}_{A} \beta\left(\mu^{1}\right)+\mathbf{1}_{A^{c}} \beta\left(\mu^{2}\right), d s \times d P-$ a.s. Then (4.29) shows that

$$
J(t, x, \widehat{\mu}, \beta(\widehat{\mu}))=\mathbf{1}_{A} J\left(t, x, \mu^{1}, \beta\left(\mu^{1}\right)\right)+\mathbf{1}_{A^{c}} J\left(t, x, \mu^{2}, \beta\left(\mu^{2}\right)\right)=J\left(t, x, \mu^{1}, \beta\left(\mu^{1}\right)\right) \vee J\left(t, x, \mu^{2}, \beta\left(\mu^{2}\right)\right), \quad P-\text { a.s. }
$$

which shows that the collection $\{J(t, x, \mu, \beta(\mu))\}_{\mu \in \mathcal{U}_{t}}$ is directed upwards (see Theorem A. 32 of [17]). By Proposition VI-1-1 of [26] or Theorem A. 32 of [17], there exists a sequence $\left\{\widetilde{\mu}^{i}\right\}_{i \in \mathbb{N}} \subset \mathcal{U}_{t}$ such that

$$
\begin{equation*}
I(t, x, \beta)=\underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} J(t, x, \mu, \beta(\mu))=\lim _{i \rightarrow \infty} \uparrow J\left(t, x, \widetilde{\mu}^{i}, \beta\left(\widetilde{\mu}^{i}\right)\right), \quad P-\text { a.s. } \tag{4.33}
\end{equation*}
$$

So $I(t, x, \beta)$ is $\mathcal{F}_{t}-$ measurable.
For any $i \in \mathbb{N}$, we set $\widetilde{A}_{i} \triangleq\left\{J\left(t, x, \widetilde{\mu}^{i}, \beta\left(\widetilde{\mu}^{i}\right)\right) \geq(I(t, x, \beta)-\varepsilon) \wedge \varepsilon^{-1}\right\} \in \mathcal{F}_{t}$ and $\widehat{A}_{i} \triangleq \widetilde{A}_{i} \backslash \cup_{j<i} \widetilde{A}_{j} \in \mathcal{F}_{t}$. Fix $n \in \mathbb{N}$ and set $A_{n} \triangleq \bigcup_{i=1}^{n} \widehat{A}_{i} \in \mathcal{F}_{t}$. Similar to $\widehat{\mu}, \mu^{n} \triangleq \sum_{i=1}^{n} \mathbf{1}_{\widehat{A}_{i}} \widetilde{\mu}^{i}+\mathbf{1}_{A_{n}^{c}} \widetilde{\mu}^{1}$ also defines a $\mathcal{U}_{t}-$ process. For $i=1, \cdots, n$, as $\mu^{n}=\widetilde{\mu}^{i}$ on $[t, T] \times \widehat{A}_{i}$, taking $(\tau, A)=\left(t, \widehat{A}_{i}\right)$ in Definition 2.2 shows that $\beta\left(\mu^{n}\right)=\beta\left(\widetilde{\mu}^{i}\right), d s \times d P-$ a.s. on $[t, T] \times \widehat{A}_{i}$. Then (4.29) implies that $\mathbf{1}_{\widehat{A}_{i}} J\left(t, x, \mu^{n}, \beta\left(\mu^{n}\right)\right)=\mathbf{1}_{\widehat{A}_{i}} J\left(t, x, \widetilde{\mu}^{i}, \beta\left(\widetilde{\mu}^{i}\right)\right), P$ a.s. Adding them up over $i \in\{1, \cdots, n\}$ gives

$$
\mathbf{1}_{A_{n}} J\left(t, x, \mu^{n}, \beta\left(\mu^{n}\right)\right)=\sum_{i=1}^{n} \mathbf{1}_{\widehat{A}_{i}} J\left(t, x, \widetilde{\mu}^{i}, \beta\left(\widetilde{\mu}^{i}\right)\right) \geq \mathbf{1}_{A_{n}}\left((I(t, x, \beta)-\varepsilon) \wedge \varepsilon^{-1}\right), \quad P-\text { a.s. }
$$

Let $\mathcal{N}$ be the $P$-null set such that (4.33) holds on $\mathcal{N}^{c}$. Clearly, $\{I(t, x, \beta)<\infty\} \cap \mathcal{N}^{c} \subset \underset{i \in \mathbb{N}}{\cup}\left\{J\left(t, x, \widetilde{\mu}^{i}, \beta\left(\widetilde{\mu}^{i}\right)\right) \geq\right.$ $I(t, x, \beta)-\varepsilon\}$ and $\{I(t, x, \beta)=\infty\} \cap \mathcal{N}^{c} \subset \underset{i \in \mathbb{N}}{\bigcup}\left\{J\left(t, x, \widetilde{\mu}^{i}, \beta\left(\widetilde{\mu}^{i}\right)\right) \geq \varepsilon^{-1}\right\}$. It follows that

$$
\mathcal{N}^{c} \subset \cup_{i \in \mathbb{N}}\left(\left\{J\left(t, x, \widetilde{\mu}^{i}, \beta\left(\widetilde{\mu}^{i}\right)\right) \geq I(t, x, \beta)-\varepsilon\right\} \cup\left\{J\left(t, x, \widetilde{\mu}^{i}, \beta\left(\widetilde{\mu}^{i}\right)\right) \geq \varepsilon^{-1}\right\}\right)=\cup_{i \in \mathbb{N}} \widetilde{A}_{i}=\bigcup_{i \in \mathbb{N}} \widehat{A}_{i}=\cup_{n \in \mathbb{N}} A_{n}
$$

So $\lim _{n \rightarrow \infty} \uparrow \mathbf{1}_{A_{n}}=1, P$-a.s.
(ii) Let $\beta_{1}, \beta_{2} \in \mathfrak{B}_{t}$. We just showed that $I\left(t, x, \beta_{1}\right)$ and $I\left(t, x, \beta_{2}\right)$ are $\mathcal{F}_{t}-$ measurable, so $\mathcal{A}_{o} \triangleq\left\{I\left(t, x, \beta_{1}\right) \leq\right.$ $\left.I\left(t, x, \beta_{2}\right)\right\}$ belongs to $\mathcal{F}_{t}$. For any $\mu \in \mathcal{U}_{t}$, similar to $\widehat{\mu}$ above, $\beta_{o}(\mu) \triangleq \mathbf{1}_{\mathcal{A}_{o}} \beta_{1}(\mu)+\mathbf{1}_{\mathcal{A}_{o}^{c}} \beta_{2}(\mu)$ defines a $\mathcal{V}_{t}-$ process. For $i=1,2$, letting $C_{i}>0$ be the constant associated to $\beta_{i}$ in Definition 2.2(i), we see that

$$
\left[\left(\beta_{o}(\mu)\right)_{s}\right]_{\mathbb{V}}=\mathbf{1}_{\mathcal{A}_{o}}\left[\left(\beta_{1}(\mu)\right)_{s}\right]_{\mathbb{V}}+\mathbf{1}_{\mathcal{A}_{o}^{c}}\left[\left(\beta_{2}(\mu)\right)_{s}\right]_{\mathbb{V}} \leq \kappa+\left(C_{1} \vee C_{2}\right)\left[\mu_{s}\right]_{\mathbb{U}}, \quad d s \times d P-\text { a.s. }
$$

Let $\mu^{1}, \mu^{2} \in \mathcal{U}_{t}$ such that $\mu^{1}=\mu^{2}, d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ for some $\tau \in \mathcal{S}_{t, T}$ and $A \in \mathcal{F}_{\tau}$. By Definition 2.2. $\beta_{1}\left(\mu^{1}\right)=\beta_{1}\left(\mu^{2}\right)$ and $\beta_{2}\left(\mu^{1}\right)=\beta_{2}\left(\mu^{2}\right), d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$. Then it follows that for $d s \times d P-$ a.s. $(s, \omega) \in \llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$

$$
\begin{equation*}
\left(\beta_{o}\left(\mu^{1}\right)\right)_{s}(\omega)=\mathbf{1}_{\mathcal{A}_{o}}\left(\beta_{1}\left(\mu^{1}\right)\right)_{s}(\omega)+\mathbf{1}_{\mathcal{A}_{o}^{c}}\left(\beta_{2}\left(\mu^{1}\right)\right)_{s}(\omega)=\mathbf{1}_{\mathcal{A}_{o}}\left(\beta_{1}\left(\mu^{2}\right)\right)_{s}(\omega)+\mathbf{1}_{\mathcal{A}_{o}^{c}}\left(\beta_{2}\left(\mu^{2}\right)\right)_{s}(\omega)=\left(\beta_{o}\left(\mu^{2}\right)\right)_{s}(\omega) \tag{4.34}
\end{equation*}
$$

Hence, $\beta_{o} \in \mathfrak{B}_{t}$.
For any $\mu \in \mathcal{U}_{t}$, (4.29) shows that $J\left(t, x, \mu, \beta_{o}(\mu)\right)=\mathbf{1}_{\mathcal{A}_{o}} J\left(t, x, \mu, \beta_{1}(\mu)\right)+\mathbf{1}_{\mathcal{A}_{o}^{c}} J\left(t, x, \mu, \beta_{2}(\mu)\right)$, $P$-a.s. Then taking essential supremum over $\mu \in \mathcal{U}_{t}$ and using Lemma 2.4 (2) yield that

$$
I\left(t, x, \beta_{o}\right)=\mathbf{1}_{\mathcal{A}_{o}} I\left(t, x, \beta_{1}\right)+\mathbf{1}_{\mathcal{A}_{o}^{c}} I\left(t, x, \beta_{2}\right)=I\left(t, x, \beta_{1}\right) \wedge I\left(t, x, \beta_{2}\right), \quad P-\text { a.s. }
$$

Thus the collection $\{I(t, x, \beta)\}_{\beta \in \mathfrak{B}_{t}}$ is directed downwards (see Theorem A. 32 of [17]). By Proposition VI-1-1 of [26] or Theorem A. 32 of [17], one can find a sequence $\left\{\widetilde{\beta}_{i}\right\}_{i \in \mathbb{N}} \subset \mathfrak{B}_{t}$ such that

$$
\begin{equation*}
w_{1}(t, x)=\underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} I(t, x, \beta)=\lim _{i \rightarrow \infty} \downarrow I\left(t, x, \widetilde{\beta}_{i}\right), \quad P-\text { a.s. } \tag{4.35}
\end{equation*}
$$

For any $i \in \mathbb{N}$, we set $\widetilde{\mathcal{A}}_{i} \triangleq\left\{I\left(t, x, \widetilde{\beta}_{i}\right) \leq w_{1}(t, x)+\varepsilon\right\} \in \mathcal{F}_{t}$ and $\widehat{\mathcal{A}}_{i} \triangleq \widetilde{\mathcal{A}}_{i} \backslash \underset{j<i}{\cup} \widetilde{\mathcal{A}}_{j} \in \mathcal{F}_{t}$. Fix $n \in \mathbb{N}$ and set $\mathcal{A}_{n} \triangleq \bigcup_{i=1}^{n} \widehat{\mathcal{A}}_{i} \in \mathcal{F}_{t}$. For any $\mu \in \mathcal{U}_{t}$, similar to $\widehat{\mu}$ above, $\beta_{n}(\mu) \triangleq \sum_{i=1}^{n} \mathbf{1}_{\widehat{\mathcal{A}}_{i}} \widetilde{\beta}_{i}(\mu)+\mathbf{1}_{\mathcal{A}_{n}^{c}} \widetilde{\beta}_{1}(\mu)$ defines a $\mathcal{V}_{t}-$ process. For $i=1, \cdots, n$, let $\widetilde{C}_{i}>0$ be the constant associated to $\widetilde{\beta}_{i}$ in Definition 2.2 (i). Setting $C_{n} \triangleq \max \left\{\widetilde{C}_{i}: i=1, \cdots, n\right\}$, we can deduce that

$$
\left[\left(\beta_{n}(\mu)\right)_{s}\right]_{\mathbb{V}}=\sum_{i=1}^{n} \mathbf{1}_{\widehat{\mathcal{A}}_{i}}\left[\left(\widetilde{\beta}_{i}(\mu)\right)_{s}\right]_{\mathbb{V}}+\mathbf{1}_{\mathcal{A}_{n}^{c}}\left[\left(\widetilde{\beta}_{1}(\mu)\right)_{s}\right]_{\mathbb{V}} \leq \kappa+C_{n}\left[\mu_{s}\right]_{\mathbb{U}}, \quad d s \times d P-a . s
$$

Let $\mu^{1}, \mu^{2} \in \mathcal{U}_{t}$ such that $\mu^{1}=\mu^{2}, d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ for some $\tau \in \mathcal{S}_{t, T}$ and $A \in \mathcal{F}_{\tau}$. Similar to (4.34), it holds for $d s \times d P-$ a.s. $(s, \omega) \in \llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ that

$$
\left(\beta_{n}\left(\mu^{1}\right)\right)_{s}(\omega)=\sum_{i=1}^{n} \mathbf{1}_{\widehat{\mathcal{A}}_{i}}\left(\widetilde{\beta}_{i}\left(\mu^{1}\right)\right)_{s}(\omega)+\mathbf{1}_{\mathcal{A}_{n}^{c}}\left(\widetilde{\beta}_{1}\left(\mu^{1}\right)\right)_{s}(\omega)=\sum_{i=1}^{n} \mathbf{1}_{\widehat{\mathcal{A}}_{i}}\left(\widetilde{\beta}_{i}\left(\mu^{2}\right)\right)_{s}(\omega)+\mathbf{1}_{\mathcal{A}_{n}^{c}}\left(\widetilde{\beta}_{1}\left(\mu^{2}\right)\right)_{s}(\omega)=\left(\beta_{n}\left(\mu^{2}\right)\right)_{s}(\omega)
$$

So $\beta_{n} \in \mathfrak{B}_{t}$. For any $\mu \in \mathcal{U}_{t}$, applying (4.29) again yields that $\mathbf{1}_{\mathcal{A}_{n}} J\left(t, x, \mu, \beta_{n}(\mu)\right)=\sum_{i=1}^{n} \mathbf{1}_{\widehat{\mathcal{A}}_{i}} J\left(t, x, \mu, \widetilde{\beta}_{i}(\mu)\right)$, $P$-a.s. Taking essential supremum over $\mu \in \mathcal{U}_{t}$ and using Lemma 2.4 (2) again yield that

$$
\mathbf{1}_{\mathcal{A}_{n}} I\left(t, x, \beta_{n}\right)=\sum_{i=1}^{n} \mathbf{1}_{\widehat{\mathcal{A}}_{i}} I\left(t, x, \widetilde{\beta}_{i}\right) \leq \mathbf{1}_{\mathcal{A}_{n}}\left(w_{1}(t, x)+\varepsilon\right), \quad P-\text { a.s. }
$$

Let $\tilde{\mathcal{N}}$ be the $P$-null set such that (4.35) holds on $\tilde{\mathcal{N}}^{c}$. As $\left|w_{1}(t, x)\right|<\infty$ by Proposition 2.1 and Proposition 2.2 we see that $\bigcup_{n \in \mathbb{N}} \mathcal{A}_{n}=\bigcup_{i \in \mathbb{N}} \widehat{\mathcal{A}}_{i}=\bigcup_{i \in \mathbb{N}} \widetilde{\mathcal{A}}_{i}=\widetilde{\mathcal{N}}^{c}$.

In the proof of the weak dynamic programming principle below, we first use Lemma 4.2 to construct approximately optimal controls/strategies by pasting locally approximately optimal ones according to a finite partition of $\bar{O}_{\delta}(t, x)$ determined by the continuity of test functions $\phi$ and $\widetilde{\phi}$. After a series of estimates on state processes and payoff processes, we obtain the weak dynamic programming principle by using the stochastic backward semigroup property (2.11), the continuous dependence of payoff process on the initial state (see Lemma [2.3) as well as the control-neutralizer assumption and the growth condition on strategies.
Proof of Theorem 2.1; 1) For any $m \in \mathbb{N}$ and $(s, \mathfrak{x}) \in[t, T] \times \mathbb{R}^{k}$, the continuity of $\phi, \widetilde{\phi}$ shows that there exists a $\delta_{s, \mathfrak{x}}^{m} \in(0,1 / m)$ such that

$$
\begin{equation*}
\left|\phi\left(s^{\prime}, \mathfrak{x}^{\prime}\right)-\phi(s, \mathfrak{x})\right|+\left|\widetilde{\phi}\left(s^{\prime}, \mathfrak{x}^{\prime}\right)-\widetilde{\phi}(s, \mathfrak{x})\right| \leq 1 / m, \quad \forall\left(s^{\prime}, \mathfrak{x}^{\prime}\right) \in\left[\left(s-\delta_{s, \mathfrak{x}}^{m}\right) \vee t,\left(s+\delta_{s, \mathfrak{x}}^{m}\right) \wedge T\right] \times \bar{O}_{\delta_{s, \mathfrak{x}}^{m}}(\mathfrak{x}) \tag{4.36}
\end{equation*}
$$

By classical covering theory, $\left\{\mathfrak{D}_{m}(s, \mathfrak{x}) \triangleq\left(s-\delta_{s, \mathfrak{x}}^{m}, s+\delta_{s, \mathfrak{x}}^{m}\right) \times O_{\delta_{s, \mathfrak{x}}^{m}}(\mathfrak{x})\right\}_{(s, \mathfrak{x}) \in[t, T] \times \mathbb{R}^{k}}$ has a finite subcollection $\left\{\mathfrak{D}_{m}\left(s_{i}, x_{i}\right)\right\}_{i=1}^{N_{m}}$ to cover $\bar{O}_{\delta}(t, x)$. For $i=1, \cdots, N_{m}$, we set $t_{i} \triangleq\left(s_{i}+\delta_{s_{i}, x_{i}}^{m}\right) \wedge T$.

Fix $(\beta, \mu) \in \mathfrak{B}_{t} \times \mathcal{U}_{t}$ and simply denote $\tau_{\beta, \mu}$ by $\tau$. By Lemma 2.1, $\widehat{\mu}_{s} \triangleq \mathbf{1}_{\{s<\tau\}} \mu_{s}+\mathbf{1}_{\{s \geq \tau\}} u_{0}, s \in[t, T]$ defines a $\mathcal{U}_{t}$-control. We set $\Theta \triangleq(t, x, \mu, \beta(\mu))$ and $\widehat{\Theta} \triangleq(t, x, \widehat{\mu}, \beta(\widehat{\mu}))$.
1a) Given $s \in[t, T)$, we first show that along $\left.\widehat{\mu}\right|_{[t, s]}$, the restriction of $\beta$ over $[s, T]$ is still an admissible strategy, which will be used in the next step to choose the locally approximately optimal controls, see (4.38).

Let $\widetilde{\mu} \in \mathcal{U}_{s}$. The process $\left(\widehat{\mu} \oplus_{s} \widetilde{\mu}\right)_{r} \triangleq \mathbf{1}_{\{r<s\}} \widehat{\mu}_{r}+\mathbf{1}_{\{r \geq s\}} \widetilde{\mu}_{r}, r \in[t, T]$ is clearly $\mathbf{F}$-progressively measurable. Suppose that $E \int_{t}^{T}\left[\mu_{r}\right]_{\mathbb{U}}^{q} d r+E \int_{s}^{T}\left[\widetilde{\mu}_{r}\right]_{\mathbb{U}}^{\widetilde{q}} d r<\infty$ for some $q>2$ and $\widetilde{q}>2$. It follows that

$$
E \int_{t}^{T}\left[\left(\widehat{\mu} \oplus_{s} \widetilde{\mu}\right)_{r}\right]_{\mathbb{U}}^{q \wedge \widetilde{q}} d r \leq E \int_{t}^{T}\left[\mu_{r}\right]_{\mathbb{U}}^{q \wedge \widetilde{q}} d r+E \int_{s}^{T}\left[\widetilde{\mu}_{r}\right]_{\mathbb{U}}^{q \wedge \widetilde{q}} d r<\infty
$$

Thus, $\widehat{\mu} \oplus_{s} \widetilde{\mu} \in \mathcal{U}_{t}$. Then we can define

$$
\begin{equation*}
\beta^{s}(\widetilde{\mu}) \triangleq\left[\beta\left(\widehat{\mu} \oplus_{s} \widetilde{\mu}\right)\right]^{s} \in \mathcal{V}_{s} \tag{4.37}
\end{equation*}
$$

For $d r \times d P-$ a.s. $(r, \omega) \in[s, T] \times \Omega$,

$$
\left[\left(\beta^{s}(\widetilde{\mu})\right)_{r}(\omega)\right]_{\mathbb{V}}=\left[\left(\beta\left(\widehat{\mu} \oplus_{s} \widetilde{\mu}\right)\right)_{r}(\omega)\right]_{\mathbb{V}} \leq \kappa+C_{\beta}\left[\left(\widehat{\mu} \oplus_{s} \widetilde{\mu}\right)_{r}(\omega)\right]_{\mathbb{U}}=\kappa+C_{\beta}\left[\widetilde{\mu}_{r}(\omega)\right]_{\mathbb{U}}
$$

Let $\widetilde{\mu}^{1}, \widetilde{\mu}^{2} \in \mathcal{U}_{s}$ such that $\widetilde{\mu}^{1}=\widetilde{\mu}^{2}, d r \times d P-$ a.s. on $\llbracket s, \zeta \llbracket \cup \llbracket \zeta, T \rrbracket_{A}$ for some $\zeta \in \mathcal{S}_{s, T}$ and $A \in \mathcal{F}_{\zeta}$. Then $\widehat{\mu} \oplus_{s} \widetilde{\mu}^{1}=$ $\widehat{\mu} \oplus_{s} \widetilde{\mu}^{2}, d r \times d P$-a.s. on $\llbracket t, \zeta \llbracket \cup \llbracket \zeta, T \rrbracket_{A}$. By Definition 2.2, $\beta\left(\widehat{\mu} \oplus_{s} \widetilde{\mu}^{1}\right)=\beta\left(\widehat{\mu} \oplus_{s} \widetilde{\mu}^{2}\right), d r \times d P-$ a.s. on $\llbracket t, \zeta \llbracket \cup \llbracket \zeta, T \rrbracket_{A}$. It follows that for $d r \times d P-$ a.s. $(r, \omega) \in \llbracket s, \zeta \llbracket \cup \llbracket \zeta, T \rrbracket_{A}$

$$
\left(\beta^{s}\left(\widetilde{\mu}^{1}\right)\right)_{r}(\omega)=\left(\beta\left(\widehat{\mu} \oplus_{s} \widetilde{\mu}^{1}\right)\right)_{r}(\omega)=\left(\beta\left(\widehat{\mu} \oplus_{s} \widetilde{\mu}^{2}\right)\right)_{r}(\omega)=\left(\beta^{s}\left(\widetilde{\mu}^{2}\right)\right)_{r}(\omega)
$$

Hence, $\beta^{s} \in \mathfrak{B}_{s}$.
1b) Fix $m \in \mathbb{N}$ with $m \geq C_{x, \delta}^{\phi} \triangleq \sup \left\{|\phi(s, \mathfrak{x})|:(s, \mathfrak{x}) \in \bar{O}_{\delta+3}(t, x) \cap\left([t, T] \times \mathbb{R}^{k}\right)\right\}$. According to the finite cover $\left\{\mathfrak{D}_{m}\left(s_{i}, x_{i}\right)\right\}_{i=1}^{N_{m}}$ of $\bar{O}_{\delta}(t, x)$, we use (4.31) to construct the $1 / m$-optimal control $\mu^{m}$ for player I under strategy $\beta$ by pasting together local $1 / m$-optimal controls.

Given $i=1, \cdots, N_{m}$, (4.31) shows that there exists $\left\{\left(A_{n}^{m, i}, \mu_{n}^{m, i}\right)\right\}_{n \in \mathbb{N}} \subset \mathcal{F}_{t_{i}} \times \mathcal{U}_{t_{i}}$ with $\lim _{n \rightarrow \infty} \uparrow \mathbf{1}_{A_{n}^{m, i}}=1$, $P$-a.s. such that for any $n \in \mathbb{N}$

$$
\begin{equation*}
J\left(t_{i}, x_{i}, \mu_{n}^{m, i}, \beta^{t_{i}}\left(\mu_{n}^{m, i}\right)\right) \geq\left(I\left(t_{i}, x_{i}, \beta^{t_{i}}\right)-1 / m\right) \wedge m, \quad P-\text { a.s. on } A_{n}^{m, i} \tag{4.38}
\end{equation*}
$$

As $Y^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right) \in \mathbb{C}_{\mathbf{F}}^{p}([t, T])$, the Monotone Convergence Theorem shows that

$$
\lim _{n \rightarrow \infty} \downarrow E\left[\mathbf{1}_{\left(A_{n}^{m, i}\right)^{c}}\left(\sup _{s \in[t, T]}\left|Y_{s}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right)\right|^{p}+\left(C_{x, \delta}^{\phi}\right)^{p}\right)\right]=0
$$

So there exists an $n(m, i) \in \mathbb{N}$ such that $E\left[\mathbb{1}_{\left(A_{n(m, i)}^{m, i}\right)}{ }^{c}\left(\sup _{s \in[t, T]}\left|Y_{s}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right)\right|^{p}+\left(C_{x, \delta}^{\phi}\right)^{p}\right)\right] \leq m^{-(1+p)} N_{m}^{-1}$. Set $\left(A_{i}^{m}, \mu_{i}^{m}\right) \triangleq\left(A_{n(m, i)}^{m, i}, \mu_{n(m, i)}^{m, i}\right)$ and $\widetilde{A}_{i}^{m} \triangleq\left\{\left(\tau, X_{\tau}^{\Theta}\right) \in \mathfrak{D}_{m}\left(s_{i}, x_{i}\right) \backslash \cup_{j<i} \mathfrak{D}_{m}\left(s_{j}, x_{j}\right)\right\} \in \mathcal{F}_{\tau}$. As $\widetilde{A}_{i}^{m} \subset\left\{\left(\tau, X_{\tau}^{\Theta}\right) \in\right.$ $\left.\mathfrak{D}_{m}\left(s_{i}, x_{i}\right)\right\} \subset\left\{\tau \leq t_{i}\right\}$, we see that $\widetilde{A}_{i}^{m}=\widetilde{A}_{i}^{m} \cap\left\{\tau \leq t_{i}\right\} \in \mathcal{F}_{t_{i}}$.

By the continuity of process $X^{\Theta},\left(\tau, X_{\tau}^{\Theta}\right) \in \partial O_{\delta}(t, x), P-$ a.s. So $\left\{\widetilde{A}_{i}^{m}\right\}_{i=1}^{N_{m}}$ forms a partition of $\mathcal{N}^{c}$ for some $P-$ null set $\mathcal{N}$. Then we can define an $\mathbf{F}$-stopping time $\tau_{m} \triangleq \sum_{i=1}^{N_{m}} \mathbf{1}_{\widetilde{A}_{i}^{m}} t_{i}+\mathbf{1}_{\mathcal{N}} T \geq \tau$ as well as a process

$$
\begin{aligned}
\mu_{s}^{m} & \triangleq \mathbf{1}_{\left\{s<\tau_{m}\right\}} \widehat{\mu}_{s}+\mathbf{1}_{\left\{s \geq \tau_{m}\right\}}\left(\sum_{i=1}^{N_{m}} \mathbf{1}_{\widetilde{A}_{i}^{m} \cap A_{i}^{m}}\left(\mu_{i}^{m}\right)_{s}+\mathbf{1}_{A_{m}} \widehat{\mu}_{s}\right) \\
& =\mathbf{1}_{A_{m}} \widehat{\mu}_{s}+\sum_{i=1}^{N_{m}} \mathbf{1}_{\widetilde{A}_{i}^{m} \cap A_{i}^{m}}\left(\mathbf{1}_{\left\{s<t_{i}\right\}} \widehat{\mu}_{s}+\mathbf{1}_{\left\{s \geq t_{i}\right\}}\left(\mu_{i}^{m}\right)_{s}\right), \quad \forall s \in[t, T],
\end{aligned}
$$

where $A_{m} \triangleq\left({ }_{i=1}^{N_{m}}\left(\widetilde{A}_{i}^{m} \backslash A_{i}^{m}\right)\right) \cup \mathcal{N}$.
Let $s \in[t, T]$ and $U \in \mathscr{B}(\mathbb{U})$. As $\llbracket t, \tau \llbracket \in \mathscr{P}$, we see that $\mathcal{D} \triangleq \llbracket t, \tau \llbracket \cap([t, s] \times \Omega) \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s}$. The $\mathbf{F}$-progressive measurability of $\widehat{\mu}$ then implies that

$$
\begin{equation*}
\left\{(r, \omega) \in \mathcal{D}: \mu_{r}^{m}(\omega) \in U\right\}=\left\{(r, \omega) \in \mathcal{D}: \widehat{\mu}_{r}(\omega) \in U\right\}=\mathcal{D} \cap\left\{(r, \omega) \in[t, s] \times \Omega: \widehat{\mu}_{r}(\omega) \in U\right\} \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s} \tag{4.39}
\end{equation*}
$$

Given $i=1, \cdots, N_{m}$, we set $\bar{A}_{i}^{m} \triangleq\left(\widetilde{A}_{i}^{m} \backslash A_{i}^{m}\right) \cup \mathcal{N} \in \mathcal{F}_{t_{i}}$. If $s<t_{i}$, both $\mathcal{D}_{i}^{m} \triangleq \llbracket \tau, T \rrbracket_{\tilde{A}_{i}^{m} \cap A_{i}^{m}} \cap([t, s] \times \Omega)=$ $\left(\left[t_{i}, T\right] \cap[t, s]\right) \times\left(\widetilde{A}_{i}^{m} \cap A_{i}^{m}\right)$ and $\widehat{\mathcal{D}}_{i}^{m} \triangleq \llbracket \tau, T \rrbracket_{\bar{A}_{i}^{m}} \cap([t, s] \times \Omega)=\left(\left[t_{i}, T\right] \cap[t, s]\right) \times \bar{A}_{i}^{m}$ are empty. Otherwise, if $s \geq t_{i}$, both $\mathcal{D}_{i}^{m}=\left[t_{i}, s\right] \times\left(\widetilde{A}_{i}^{m} \cap A_{i}^{m}\right)$ and $\widehat{\mathcal{D}}_{i}^{m}=\left[t_{i}, s\right] \times \bar{A}_{i}^{m}$ belong to $\mathscr{B}\left(\left[t_{i}, s\right]\right) \otimes \mathcal{F}_{s}$. Using a similar argument to (4.39) on the $\mathbf{F}$-progressive measurability of process $\mu_{i}^{m}$ yields that

$$
\begin{aligned}
\left\{(r, \omega) \in \mathcal{D}_{i}^{m}: \mu_{r}^{m}(\omega) \in U\right\} & =\left\{(r, \omega) \in \mathcal{D}_{i}^{m}:\left(\mu_{i}^{m}\right)_{r}(\omega) \in U\right\} \in \mathscr{B}\left(\left[t_{i}, s\right]\right) \otimes \mathcal{F}_{s} \subset \mathscr{B}([t, s]) \otimes \mathcal{F}_{s} \\
\text { and }\left\{(r, \omega) \in \widehat{\mathcal{D}}_{i}^{m}: \mu_{r}^{m}(\omega) \in U\right\} & =\left\{(r, \omega) \in \widehat{\mathcal{D}}_{i}^{m}: \widehat{\mu}_{r}(\omega) \in U\right\} \in \mathscr{B}([t, s]) \otimes \mathcal{F}_{s},
\end{aligned}
$$

both of which together with (4.39) shows the $\mathbf{F}$-progressive measurability of $\mu^{m}$. For $i=1, \cdots, N_{m}$, suppose that $E \int_{t_{i}}^{T}\left[\left(\mu_{i}^{m}\right)_{r}\right]_{\mathbb{U}}^{q_{i}} d r<\infty$ for some $q_{i}>2$. Setting $q_{*} \triangleq q \wedge \min \left\{q_{i}: i=1, \cdots, N_{m}\right\}$, we can deduce that

$$
E \int_{t}^{T}\left[\mu_{r}^{m}\right]_{\mathbb{U}}^{q_{*}} d r \leq E \int_{t}^{T}\left[\mu_{r}\right]_{\mathbb{U}}^{q_{*}} d r+\sum_{i=1}^{N_{m}} E \int_{t_{i}}^{T}\left[\left(\mu_{i}^{m}\right)_{r}\right]_{\mathbb{U}}^{q_{*}} d r<\infty
$$

Hence, $\mu^{m} \in \mathcal{U}_{t}$.
1c) Next, set $\Theta_{m} \triangleq\left(t, x, \mu^{m}, \beta\left(\mu^{m}\right)\right)$. We shall use a series of estimates on state processes $X^{t, \xi, \mu, \nu} /$ payoff processes $Y^{t, \xi, \mu, \nu}$, a stochastic backward semigroup property (2.11) as well as the continuous dependence of $Y^{t, \xi, \mu, \nu}$ on $\xi$ to demonstrate how $J\left(t, x, \mu^{m}, \beta\left(\mu^{m}\right)\right)$ deviates from $Y_{t}^{\Theta}\left(\tau, \phi\left(\tau, X_{\tau}^{\Theta}\right)\right)$, which will eventually lead to

$$
\begin{equation*}
w_{1}(t, x) \geq \underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \phi\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right), \quad P-a . s . \tag{4.40}
\end{equation*}
$$

As $\mu^{m}=\widehat{\mu}=\mu$ on $\llbracket t, \tau \llbracket$, taking $(\tau, A)=(\tau, \emptyset)$ in Definition 2.2 shows that $\beta\left(\mu^{m}\right)=\beta(\mu), d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket$, and then applying (2.8) with $(\tau, A)=(\tau, \emptyset)$ yields that $P$-a.s.

$$
\begin{equation*}
X_{s}^{\Theta_{m}}=X_{s}^{\Theta} \in \bar{O}_{\delta}(x), \quad \forall s \in[t, \tau] \tag{4.41}
\end{equation*}
$$

Thus, for any $\eta \in \mathbb{L}^{p}\left(\mathcal{F}_{\tau}\right)$, the $\operatorname{BSDE}\left(t, \eta, f_{\tau}^{\Theta}\right)$ and the $\operatorname{BSDE}\left(t, \eta, f_{\tau}^{\Theta}\right)$ are essentially the same. To wit,

$$
\begin{equation*}
\left(Y^{\Theta_{m}}(\tau, \eta), Z^{\Theta_{m}}(\tau, \eta)\right)=\left(Y^{\Theta}(\tau, \eta), Z^{\Theta}(\tau, \eta)\right) \tag{4.42}
\end{equation*}
$$

Given $A \in \mathcal{F}_{t}$, we see from (4.41) that

$$
\begin{aligned}
\mathbf{1}_{A} X_{\tau_{m} \wedge s}^{\Theta_{m}} & =\mathbf{1}_{A} X_{\tau \wedge s}^{\Theta_{m}}+\mathbf{1}_{A} \int_{\tau \wedge s}^{\tau_{m} \wedge s} b\left(r, X_{r}^{\Theta_{m}}, \mu_{r}^{m},\left(\beta\left(\mu^{m}\right)\right)_{r}\right) d r+\mathbf{1}_{A} \int_{\tau \wedge s}^{\tau_{m} \wedge s} \sigma\left(r, X_{r}^{\Theta_{m}}, \mu_{r}^{m},\left(\beta\left(\mu^{m}\right)\right)_{r}\right) d B_{r}, \\
& =\mathbf{1}_{A} X_{\tau \wedge s}^{\Theta}+\int_{\tau \wedge s}^{\tau_{m} \wedge s} \mathbf{1}_{A} b\left(r, X_{\tau_{m} \wedge r}^{\Theta_{m}}, u_{0},\left(\beta\left(\mu^{m}\right)\right)_{r}\right) d r+\int_{\tau \wedge s}^{\tau_{m} \wedge s} \mathbf{1}_{A} \sigma\left(r, X_{\tau_{m} \wedge r}^{\Theta_{m}}, u_{0},\left(\beta\left(\mu^{m}\right)\right)_{r}\right) d B_{r}, \quad s \in[t, T]
\end{aligned}
$$

It follows that

$$
\begin{align*}
\mathbf{1}_{A} \sup _{r \in[t, s]}\left|X_{\tau_{m} \wedge \wedge}^{\Theta_{m}}-X_{\tau \wedge r}^{\Theta}\right| \leq & \int_{\tau \wedge s}^{\tau_{m} \wedge s} \mathbf{1}_{A}\left|b\left(r, X_{\tau_{m} \wedge r}^{\Theta_{m}}, u_{0},\left(\beta\left(\mu^{m}\right)\right)_{r}\right)\right| d r \\
& +\sup _{r \in[t, s]}\left|\int_{\tau \wedge r}^{\tau_{m} \wedge r} \mathbf{1}_{A} \sigma\left(r^{\prime}, X_{\tau_{m} \wedge r^{\prime}}^{\Theta_{m}}, u_{0},\left(\beta\left(\mu^{m}\right)\right)_{r^{\prime}}\right) d B_{r^{\prime}}\right|, \quad s \in[t, T] \tag{4.43}
\end{align*}
$$

Let $C(\kappa, x, \delta)$ denote a generic constant, depending on $\kappa+|x|+\delta, C_{x, \delta}^{\phi}, T, \gamma, p$ and $|g(0)|$, whose form may vary from line to line. Squaring both sides of (4.43) and taking expectation, we can deduce from Hölder's inequality, Doob's martingale inequality, (2.1), (2.2), (4.41) and Fubini's Theorem that

$$
\begin{align*}
& E\left[\mathbf{1}_{A} \sup _{r \in[t, s]}\left|X_{\tau_{m} \wedge r}^{\Theta_{m}}-X_{\tau \wedge r}^{\Theta}\right|^{2}\right] \\
& \quad \leq 4 E \int_{\tau \wedge s}^{\tau_{m} \wedge s} \mathbf{1}_{A}\left|b\left(r, X_{\tau_{m} \wedge r}^{\Theta_{m}}, u_{0},\left(\beta\left(\mu^{m}\right)\right)_{r}\right)\right|^{2} d r+8 E \int_{\tau \wedge s}^{\tau_{m} \wedge s} \mathbf{1}_{A}\left|\sigma\left(r, X_{\tau_{m} \wedge r}^{\Theta_{m}}, u_{0},\left(\beta\left(\mu^{m}\right)\right)_{r}\right)\right|^{2} d r \\
& \quad \leq 12 \gamma^{2} E \int_{\tau \wedge s}^{\tau_{m} \wedge s} \mathbf{1}_{A}\left(\left|X_{\tau_{m} \wedge r}^{\Theta_{m}}-X_{\tau \wedge r}^{\Theta}\right|+\left|X_{\tau \wedge r}^{\Theta}\right|+1+\left[\left(\beta\left(\mu^{m}\right)\right)_{r}\right]_{\mathbb{V}}\right)^{2} d r \\
& \quad \leq 24 \gamma^{2} \int_{t}^{s} E\left[\mathbf{1}_{A} \sup _{r^{\prime} \in[t, r]}\left|X_{\tau_{m} \wedge r^{\prime}}^{\Theta_{m}}-X_{\tau \wedge r^{\prime}}^{\Theta}\right|^{2}\right] d r+\frac{C(\kappa, x, \delta)}{m} P(A), \quad \forall s \in[t, T] \tag{4.44}
\end{align*}
$$

where we used the facts that

$$
\begin{equation*}
\tau_{m}-\tau \leq \sum_{i=1}^{N_{m}} \mathbf{1}_{\widetilde{A}_{i}^{m}} 2 \delta_{s_{i}, x_{i}}^{m}<\frac{2}{m}, P-\text { a.s. } \quad \text { and } \quad\left[\left(\beta\left(\mu^{m}\right)\right)_{r}\right]_{\mathbb{V}} \leq \kappa, d r \times d P-a . s . \text { on } \llbracket \tau, \tau_{m} \llbracket . \tag{4.45}
\end{equation*}
$$

Then an application of Gronwall's inequality yields that

$$
E\left[\mathbf{1}_{A} \sup _{r \in[t, s]}\left|X_{\tau_{m} \wedge r}^{\Theta_{m}}-X_{\tau \wedge r}^{\Theta}\right|^{2}\right] \leq \frac{C(\kappa, x, \delta)}{m} P(A) e^{24 \gamma^{2}(s-t)}, \quad \forall s \in[t, T]
$$

In particular, $E\left[\mathbf{1}_{A} \sup _{r \in[t, T]}\left|X_{\tau_{m} \wedge r}^{\Theta_{m}}-X_{\tau \wedge r}^{\Theta}\right|^{2}\right] \leq \frac{C(\kappa, x, \delta)}{m} P(A)$. Letting $A$ vary in $\mathcal{F}_{t}$ yields that

$$
\begin{equation*}
E\left[\sup _{r \in[t, T]}\left|X_{\tau_{m} \wedge r}^{\Theta_{m}}-X_{\tau \wedge r}^{\Theta}\right|^{2} \mid \mathcal{F}_{t}\right] \leq \frac{C(\kappa, x, \delta)}{m}, \quad P-\text { a.s. } \tag{4.46}
\end{equation*}
$$

Let $i=1, \cdots, N_{m}$ and set $\Theta_{m}^{t_{i}} \triangleq\left(t_{i}, X_{t_{i}}^{\Theta_{m}},\left[\mu^{m}\right]^{t_{i}},\left[\beta\left(\mu^{m}\right)\right]^{t_{i}}\right)$. We see from (2.7) that $X_{T}^{\Theta_{m}}=X_{T}^{\Theta_{m}^{t_{i}}}, P-$ a.s. It then follows from (2.13) that

$$
\begin{equation*}
Y_{t_{i}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)=Y_{t_{i}}^{\Theta_{m}^{t_{i}}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)=Y_{t_{i}}^{\Theta_{m}^{t_{i}}}\left(T, g\left(X_{T}^{\Theta_{m}^{t_{i}}}\right)\right)=J\left(\Theta_{m}^{t_{i}}\right), \quad P-\text { a.s. } \tag{4.47}
\end{equation*}
$$

Similar to $\mu^{m}$,

$$
\begin{aligned}
\left(\widehat{\mu}_{i}^{m}\right)_{s} & \triangleq \mathbf{1}_{\left\{s<\tau_{m}\right\}} \widehat{\mu}_{s}+\mathbf{1}_{\left\{s \geq \tau_{m}\right\}}\left(\mathbf{1}_{\widetilde{A}_{i}^{m} \cap A_{i}^{m}}\left(\mu_{i}^{m}\right)_{s}+\mathbf{1}_{\left(\widetilde{A}_{i}^{m} \cap A_{i}^{m}\right)^{c}} \widehat{\mu}_{s}\right) \\
& =\mathbf{1}_{\widetilde{A}_{i}^{m} \cap A_{i}^{m}}\left(\mathbf{1}_{\left\{s<t_{i}\right\}} \widehat{\mu}_{s}+\mathbf{1}_{\left\{s \geq t_{i}\right\}}\left(\mu_{i}^{m}\right)_{s}\right)+\mathbf{1}_{\left(\widetilde{A}_{i}^{m} \cap A_{i}^{m}\right)^{c}} \widehat{\mu}_{s}, \quad s \in[t, T]
\end{aligned}
$$

also defines a $\mathcal{U}_{t}$-process. As $\mu^{m}=\widehat{\mu}_{i}^{m}$ on $\llbracket t, \tau_{m} \llbracket \cup \llbracket \tau_{m}, T \rrbracket_{\widetilde{A}_{i}^{m} \cap A_{i}^{m}}$ and $\widehat{\mu}_{i}^{m}=\widehat{\mu} \oplus_{t_{i}} \mu_{i}^{m}$ on $\left(\left[t, t_{i}\right) \times \Omega\right) \cup\left(\left[t_{i}, T\right] \times\left(\widetilde{A}_{i}^{m} \cap\right.\right.$ $\left.A_{i}^{m}\right)$ ), Definition 2.2 shows that $\beta\left(\mu^{m}\right)=\beta\left(\widehat{\mu}_{i}^{m}\right), d s \times d P-$ a.s. on $\llbracket t, \tau_{m} \llbracket \cup \llbracket \tau_{m}, T \rrbracket_{\widetilde{A}_{i}^{m} \cap A_{i}^{m}}$ and $\beta\left(\widehat{\mu}_{i}^{m}\right)=\beta\left(\widehat{\mu} \oplus_{t_{i}} \mu_{i}^{m}\right)$, $d s \times d P-$ a.s. on $\left(\left[t, t_{i}\right) \times \Omega\right) \cup\left(\left[t_{i}, T\right] \times\left(\widetilde{A}_{i}^{m} \cap A_{i}^{m}\right)\right)$. Thus $\left(\mu^{m}, \beta\left(\mu^{m}\right)\right)=\left(\widehat{\mu} \oplus_{t_{i}} \mu_{i}^{m}, \beta\left(\widehat{\mu} \oplus_{t_{i}} \mu_{i}^{m}\right)\right)$, ds $\times d P-$ a.s. on $\llbracket \tau_{m}, T \rrbracket \widetilde{A}_{i}^{m} \cap A_{i}^{m}=\left[t_{i}, T\right] \times\left(\widetilde{A}_{i}^{m} \cap A_{i}^{m}\right)$. From (4.37), one has $\left(\left[\mu^{m}\right]^{t_{i}},\left[\beta\left(\mu^{m}\right)\right]^{t_{i}}\right)=\left(\mu_{i}^{m}, \beta^{t_{i}}\left(\mu_{i}^{m}\right)\right), d s \times d P-$ a.s. on $\left[t_{i}, T\right] \times\left(\widetilde{A}_{i}^{m} \cap A_{i}^{m}\right)$. Then by (4.47), (4.29) and (2.14), it holds $P-$ a.s. on $\widetilde{A}_{i}^{m} \cap A_{i}^{m} \in \mathcal{F}_{t_{i}}$ that

$$
Y_{\tau_{m}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)=Y_{t_{i}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)=J\left(t_{i}, X_{\tau_{m}}^{\Theta_{m}}, \mu_{i}^{m}, \beta^{t_{i}}\left(\mu_{i}^{m}\right)\right) \geq J\left(t_{i}, X_{\tau}^{\Theta}, \mu_{i}^{m}, \beta^{t_{i}}\left(\mu_{i}^{m}\right)\right)-c_{0}\left|X_{\tau_{m}}^{\Theta_{m}}-X_{\tau}^{\Theta}\right|^{2 / p}
$$

Since $\mathfrak{D}_{m}\left(s_{i}, x_{i}\right) \cap \bar{O}_{\delta}(t, x) \neq \emptyset$, it is easy to see that

$$
\overline{\mathfrak{D}}_{m}\left(s_{i}, x_{i}\right)=\left[s_{i}-\delta_{s_{i}, x_{i}}^{m}, s_{i}+\delta_{s_{i}, x_{i}}^{m}\right] \times \bar{O}_{\delta_{s_{i}, x_{i}}^{m}}\left(x_{i}\right) \subset \bar{O}_{\delta+2 \sqrt{2} \delta_{s_{i}, x_{i}}^{m}}(t, x) \subset \bar{O}_{\delta+\frac{2 \sqrt{2}}{m}}(t, x) \subset \bar{O}_{\delta+3}(t, x)
$$

So $\phi\left(t_{i}, x_{i}\right) \leq C_{x, \delta}^{\phi}<m+1 / m$. On the other hand, one has $\phi\left(t_{i}, x_{i}\right) \leq w_{1}\left(t_{i}, x_{i}\right) \leq I\left(t_{i}, x_{i}, \beta^{t_{i}}\right)$, $P$-a.s. Then it follows from (4.38) that

$$
\phi\left(t_{i}, x_{i}\right) \leq I\left(t_{i}, x_{i}, \beta^{t_{i}}\right) \wedge(m+1 / m) \leq J\left(t_{i}, x_{i}, \mu_{i}^{m}, \beta^{t_{i}}\left(\mu_{i}^{m}\right)\right)+1 / m, \quad P-\text { a.s. on } A_{i}^{m}
$$

As $\left|X_{\tau}^{\Theta}-x_{i}\right|^{2 / p}<\left(\delta_{s_{i}, x_{i}}^{m}\right)^{2 / p}<m^{-2 / p} \leq 1 / m$ on $\widetilde{A}_{i}^{m}$, we can also deduce from (2.14), (4.36) and the continuity of $\phi$ that it holds $P-$ a.s. on $\widetilde{A}_{i}^{m} \cap A_{i}^{m}$ that
$J\left(t_{i}, X_{\tau}^{\Theta}, \mu_{i}^{m}, \beta^{t_{i}}\left(\mu_{i}^{m}\right)\right) \geq J\left(t_{i}, x_{i}, \mu_{i}^{m}, \beta^{t_{i}}\left(\mu_{i}^{m}\right)\right)-\frac{c_{0}}{m} \geq \phi\left(t_{i}, x_{i}\right)-\frac{c_{0}}{m} \geq \phi\left(s_{i}, x_{i}\right)-\frac{c_{0}}{m} \geq \phi\left(\tau, X_{\tau}^{\Theta}\right)-\frac{c_{0}}{m} \triangleq \eta_{m} \in \mathbb{L}^{\infty}\left(\mathcal{F}_{\tau}\right)$.
Thus it holds $P$-a.s. on $\cup_{i=1}^{N_{m}}\left(\widetilde{A}_{i}^{m} \cap A_{i}^{m}\right)$ that

$$
\begin{equation*}
Y_{\tau_{m}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right) \geq \eta_{m}-c_{0}\left|X_{\tau_{m}}^{\Theta_{m}}-X_{\tau}^{\Theta}\right|^{2 / p} \triangleq \widetilde{\eta}_{m} \in \mathbb{L}^{p}\left(\mathcal{F}_{\tau_{m}}\right) \tag{4.48}
\end{equation*}
$$

By (2.10), it holds $P$-a.s. that

$$
\begin{equation*}
\left|Y_{t}^{\Theta}\left(\tau, \eta_{m}\right)-Y_{t}^{\Theta}\left(\tau, \phi\left(\tau, X_{\tau}^{\Theta}\right)\right)\right|^{p} \leq c_{0} E\left[\left|\eta_{m}-\phi\left(\tau, X_{\tau}^{\Theta}\right)\right|^{p} \mid \mathcal{F}_{t}\right] \leq \frac{c_{0}}{m^{p}} \tag{4.49}
\end{equation*}
$$

Let $\left(Y^{m}, Z^{m}\right) \in \mathbb{G}_{\mathbf{F}}^{p}([t, T])$ be the unique solution of the following BSDE with zero generator:

$$
Y_{s}^{m}=Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)-\int_{s}^{T} Z_{r}^{m} d B_{r}, \quad s \in[t, T]
$$

For any $s \in[t, T]$, one can deduce that

$$
Y_{\tau \wedge s}^{m}=E\left[Y_{\tau \wedge s}^{m} \mid \mathcal{F}_{\tau}\right]=E\left[Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)-\int_{\tau \wedge s}^{T} Z_{r}^{m} d B_{r} \mid \mathcal{F}_{\tau}\right]=Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)-\int_{\tau \wedge s}^{\tau} Z_{r}^{m} d B_{r}, \quad P-\text { a.s. }
$$

By the continuity of process $Y^{m}$, it holds $P$-a.s. that

$$
\begin{equation*}
Y_{\tau \wedge s}^{m}=Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)-\int_{\tau \wedge s}^{\tau} Z_{r}^{m} d B_{r}=Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)-\int_{s}^{T} \mathbf{1}_{\{r<\tau\}} Z_{r}^{m} d B_{r}, \quad s \in[t, T] \tag{4.50}
\end{equation*}
$$

Thus, we see that $\left(Y_{s}^{m}, Z_{s}^{m}\right)=\left(Y_{\tau \wedge s}^{m}, \mathbf{1}_{\{s<\tau\}} Z_{s}^{m}\right), s \in[t, T]$. Also, taking $\left[\cdot \mid \mathcal{F}_{\tau \wedge s}\right]$ in (4.50) shows that $P$-a.s.

$$
Y_{s}^{m}=Y_{\tau \wedge s}^{m}=E\left[Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right) \mid \mathcal{F}_{\tau \wedge s}\right], \quad \forall s \in[t, T]
$$

On the other hand, let $\left(\widetilde{Y}^{m}, \widetilde{Z}^{m}\right) \in \mathbb{G}_{\mathbf{F}}^{p}([t, T])$ be the unique solution of the following BSDE with zero generator:

$$
\begin{equation*}
\widetilde{Y}_{s}^{m}=\eta_{m}-\int_{s}^{T} \widetilde{Z}_{r}^{m} d B_{r}, \quad s \in[t, T] \tag{4.51}
\end{equation*}
$$

Similar to $\left(Y^{m}, Z^{m}\right)$, it holds $P$-a.s. that

$$
\begin{equation*}
\left(\widetilde{Y}_{s}^{m}, \widetilde{Z}_{s}^{m}\right)=\left(\widetilde{Y}_{\tau \wedge s}^{m}, \mathbf{1}_{\{s<\tau\}} \widetilde{Z}_{s}^{m}\right) \quad \text { and } \quad \widetilde{Y}_{s}^{m}=E\left[\eta_{m} \mid \mathcal{F}_{\tau \wedge s}\right], \quad \forall s \in[t, T] \tag{4.52}
\end{equation*}
$$

We can deduce that $\left(\mathcal{Y}^{m}, \mathcal{Z}^{m}\right) \triangleq\left\{\left(\mathbf{1}_{\{s<\tau\}} Y_{s}^{m}+\mathbf{1}_{\{s \geq \tau\}} Y_{s}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right), \mathbf{1}_{\{s<\tau\}} Z_{s}^{m}+\mathbf{1}_{\{s \geq \tau\}} Z_{s}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)\right)\right\}_{s \in[t, T]} \in$ $\mathbb{G}_{\mathbf{F}}^{p}([t, T])$ solves the following BSDE

$$
\begin{align*}
\mathcal{Y}_{s}^{m} & =\mathbf{1}_{\{s \geq \tau\}} Y_{s}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)+\mathbf{1}_{\{s<\tau\}} Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)-\mathbf{1}_{\{s<\tau\}} \int_{s}^{T} Z_{r}^{m} d B_{r}=Y_{\tau \vee s}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)-\mathbf{1}_{\{s<\tau\}} \int_{s}^{T} \mathbf{1}_{\{r<\tau\}} Z_{r}^{m} d B_{r} \\
& =\eta_{m}+\int_{\tau \vee s}^{T} f_{\tau_{m}}^{\Theta_{m}}\left(r, Y_{r}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right), Z_{r}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)\right) d r-\int_{\tau \vee s}^{T} Z_{r}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right) d B_{r}-\int_{s}^{T} \mathbf{1}_{\{r<\tau\}} Z_{r}^{m} d B_{r} \\
& =\eta_{m}+\int_{s}^{T} \mathbf{1}_{\{r \geq \tau\}} f_{\tau_{m}}^{\Theta_{m}}\left(r, \mathcal{Y}_{r}^{m}, \mathcal{Z}_{r}^{m}\right) d r-\int_{s}^{T} \mathcal{Z}_{r}^{m} d B_{r}, \quad s \in[t, T] \tag{4.53}
\end{align*}
$$

Since (2.4), Hölder's inequality and (2.9) imply that

$$
E\left[\int_{t}^{T} \mathbf{1}_{\{s \geq \tau\}}\left|f_{\tau_{m}}^{\Theta_{m}}\left(s, \widetilde{Y}_{s}^{m}, \widetilde{Z}_{s}^{m}\right)\right|^{p} d s\right] \leq c_{p} E\left[\int_{t}^{T}\left|f_{\tau_{m}}^{\Theta_{m}}(s, 0,0)\right|^{p} d s+\sup _{s \in[t, T]}\left|\widetilde{Y}_{s}^{m}\right|^{p}+\left(\int_{t}^{T}\left|\widetilde{Z}_{s}^{m}\right|^{2} d s\right)^{p / 2}\right]<\infty
$$

applying (1.5) to $\mathcal{Y}^{m}-\tilde{Y}^{m}$ and using (4.52) yield that

$$
\begin{align*}
E\left[\left|Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)-\eta_{m}\right|^{p} \mid \mathcal{F}_{t}\right] & =E\left[\left|\mathcal{Y}_{\tau}^{m}-\widetilde{Y}_{\tau}^{m}\right|^{p} \mid \mathcal{F}_{t}\right] \leq E\left[\sup _{s \in[t, T]}\left|\mathcal{Y}_{s}^{m}-\widetilde{Y}_{s}^{m}\right|^{p} \mid \mathcal{F}_{t}\right] \leq c_{0} E\left[\int_{\tau}^{T}\left|f_{\tau_{m}}^{\Theta_{m}}\left(s, \widetilde{Y}_{s}^{m}, \widetilde{Z}_{s}^{m}\right)\right|^{p} d s \mid \mathcal{F}_{t}\right] \\
& =c_{0} E\left[\int_{\tau}^{\tau_{m}}\left|f\left(s, X_{\tau_{m} \wedge s}^{\Theta_{m}}, \eta_{m}, 0, u_{0},\left(\beta\left(\mu^{m}\right)\right)_{s}\right)\right|^{p} d s \mid \mathcal{F}_{t}\right], \quad P-\text { a.s. } \tag{4.54}
\end{align*}
$$

Then one can deduce from (2.10), (2.3), (2.4), (4.41), (4.45) and (4.46) that

$$
\begin{align*}
& \left|Y_{t}^{\Theta_{m}}\left(\tau, Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)\right)-Y_{t}^{\Theta_{m}}\left(\tau, \eta_{m}\right)\right|^{p} \leq c_{0} E\left[\left|Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)-\eta_{m}\right|^{p} \mid \mathcal{F}_{t}\right] \\
& \quad \leq c_{0} E\left[\int_{\tau}^{\tau_{m}}\left(1+\left|X_{\tau_{m} \wedge s}^{\Theta_{m}}-X_{\tau \wedge s}^{\Theta}\right|^{2}+\left|X_{\tau \wedge s}^{\Theta}\right|^{2}+\left|\eta_{m}\right|^{p}+\left[\left(\beta\left(\mu^{m}\right)\right)_{s}\right]_{\mathbb{V}}^{2}\right) d s \mid \mathcal{F}_{t}\right] \\
& \quad \leq c_{0} E\left[\left(\tau_{m}-\tau\right) \cdot \sup _{s \in[t, T]}\left|X_{\tau_{m} \wedge s}^{\Theta_{m}}-X_{\tau \wedge s}^{\Theta}\right|^{2} \mid \mathcal{F}_{t}\right]+\frac{c_{0}}{m}\left\{1+(|x|+\delta)^{2}+\left(C_{x, \delta}^{\phi}+\frac{c_{0}}{m}\right)^{p}+\kappa^{2}\right\} \\
& \quad \leq \frac{C(\kappa, x, \delta)}{m^{2}}+\frac{C(\kappa, x, \delta)}{m}+\frac{c_{0}}{m^{p+1}} \leq \frac{C(\kappa, x, \delta)}{m}, \quad P-\text { a.s. } \tag{4.55}
\end{align*}
$$

Applying (2.11) with $(\zeta, \tau, \eta)=\left(\tau, \tau_{m}, \eta_{m}\right)$, applying (4.42) with $\eta=\eta_{m}$ and using (4.49) yield that $P$-a.s.

$$
\begin{align*}
Y_{t}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right) & =Y_{t}^{\Theta_{m}}\left(\tau, Y_{\tau}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)\right) \geq Y_{t}^{\Theta_{m}}\left(\tau, \eta_{m}\right)-\frac{C(\kappa, x, \delta)}{m^{1 / p}} \\
& =Y_{t}^{\Theta}\left(\tau, \eta_{m}\right)-\frac{C(\kappa, x, \delta)}{m^{1 / p}} \geq Y_{t}^{\Theta}\left(\tau, \phi\left(\tau, X_{\tau}^{\Theta}\right)\right)-\frac{C(\kappa, x, \delta)}{m^{1 / p}} \tag{4.56}
\end{align*}
$$

As $\mu^{m}=\widehat{\mu}$ on $\llbracket t, \tau_{m} \llbracket$, taking $(\tau, A)=\left(\tau_{m}, \emptyset\right)$ in Definition 2.2 shows that $\beta\left(\mu^{m}\right)=\beta(\widehat{\mu}), d s \times d P-$ a.s. on $\llbracket t, \tau_{m} \llbracket$, and then applying (2.8) with $(\tau, A)=\left(\tau_{m}, \emptyset\right)$ yields that $P$-a.s.

$$
\begin{equation*}
X_{s}^{\Theta_{m}}=X_{s}^{\widehat{\Theta}}, \quad \forall s \in\left[t, \tau_{m}\right] \tag{4.57}
\end{equation*}
$$

Given $i=1, \cdots, N_{m}$, (4.57) shows that $X_{t_{i}}^{\Theta_{m}}=X_{t_{i}}^{\widehat{\Theta}}, P-$ a.s. on $\widetilde{A}_{i}^{m} \backslash A_{i}^{m}$. As $\mu^{m}=\widehat{\mu}$ on $\llbracket t, \tau_{m} \llbracket \cup \llbracket \tau_{m}, T \rrbracket_{\tilde{A}_{i}^{m} \backslash A_{i}^{m}}$, Definition 2.2 shows that $\beta\left(\mu^{m}\right)=\beta(\widehat{\mu}), d s \times d P-$ a.s. on $\llbracket t, \tau_{m} \llbracket \cup \llbracket \tau_{m}, T \rrbracket_{\tilde{A}_{i}^{m} \backslash A_{i}^{m}}$. So $\left(\left[\mu^{m}\right]^{t_{i}},\left[\beta\left(\mu^{m}\right)\right]^{t_{i}}\right)=\left([\widehat{\mu}]^{t_{i}},[\beta(\widehat{\mu})]^{t_{i}}\right)$
holds $d s \times d P-$ a.s. on $\llbracket \tau_{m}, T \rrbracket_{\widetilde{A}_{i}^{m} \backslash A_{i}^{m}}=\left[t_{i}, T\right] \times\left(\widetilde{A}_{i}^{m} \backslash A_{i}^{m}\right)$. Then by (4.29) and a similar argument to (4.47), it holds $P$-a.s. on $\widetilde{A}_{i}^{m} \backslash A_{i}^{m}$ that

$$
\begin{equation*}
Y_{\tau_{m}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)=Y_{t_{i}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)=J\left(\Theta_{m}^{t_{i}}\right)=J\left(\widehat{\Theta}^{t_{i}}\right)=Y_{t_{i}}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right)=Y_{\tau_{m}}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right) \tag{4.58}
\end{equation*}
$$

where $\widehat{\Theta}^{t_{i}} \triangleq\left(t_{i}, X_{t_{i}}^{\widehat{\Theta}},[\widehat{\mu}]^{t_{i}},[\beta(\widehat{\mu})]^{t_{i}}\right)$.
Let $\widehat{\eta}_{m} \triangleq Y_{\tau_{m}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right) \wedge \widetilde{\eta}_{m} \in \mathbb{L}^{p}\left(\mathcal{F}_{\tau_{m}}\right)$ and set $\widetilde{A}_{m} \triangleq\left\{Y_{\tau_{m}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)<\widetilde{\eta}_{m}\right\} \in \mathcal{F}_{\tau_{m}}$. Clearly, $\mathbf{1}_{\widetilde{A}_{m}} \leq \mathbf{1}_{A_{m}}, P$-a.s. Applying (2.10) again, we can deduce from (4.46) and (4.58) that $P$-a.s.

$$
\begin{align*}
\left|Y_{t}^{\Theta_{m}}\left(\tau_{m}, \widehat{\eta}_{m}\right)-Y_{t}^{\Theta_{m}}\left(\tau_{m}, \eta_{m}\right)\right|^{p} & \leq c_{0} E\left[\left|\widehat{\eta}_{m}-\eta_{m}\right|^{p} \mid \mathcal{F}_{t}\right]=c_{0} E\left[\mathbf{1}_{\widetilde{A}_{m}^{c}}\left|\widetilde{\eta}_{m}-\eta_{m}\right|^{p}+\mathbf{1}_{\widetilde{A}_{m}}\left|Y_{\tau_{m}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)-\eta_{m}\right|^{p} \mid \mathcal{F}_{t}\right] \\
& \leq c_{0} E\left[\left|X_{\tau_{m}}^{\Theta_{m}}-X_{\tau}^{\Theta}\right|^{2}+\mathbf{1}_{A_{m}}\left|Y_{\tau_{m}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta}\right)\right)-\eta_{m}\right|^{p} \mid \mathcal{F}_{t}\right] \\
& \leq \frac{C(\kappa, x, \delta)}{m}+c_{0} E\left[\mathbf{1}_{A_{m}}\left|Y_{\tau_{m}}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right)-\phi\left(\tau, X_{\tau}^{\Theta}\right)\right|^{p} \mid \mathcal{F}_{t}\right]+\frac{c_{0}}{m^{p}} \\
& \leq \frac{C(\kappa, x, \delta)}{m}+c_{0} E\left[\mathbf{1}_{A_{m}}\left(\sup _{s \in[t, T]}\left|Y_{s}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right)\right|^{p}+\left(C_{x, \delta}^{\phi}\right)^{p}\right) \mid \mathcal{F}_{t}\right] \tag{4.59}
\end{align*}
$$

Applying (2.11) with $(\zeta, \tau, \eta)=\left(\tau_{m}, T, g\left(X_{T}^{\Theta_{m}}\right)\right)$, we see from Proposition 1.2 (2), (4.59) and (4.56) that $P$-a.s.

$$
\begin{align*}
& Y_{t}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)=Y_{t}^{\Theta_{m}}\left(\tau_{m}, Y_{\tau_{m}}^{\Theta_{m}}\left(T, g\left(X_{T}^{\Theta_{m}}\right)\right)\right) \geq Y_{t}^{\Theta_{m}}\left(\tau_{m}, \widehat{\eta}_{m}\right) \\
& \quad \geq Y_{t}^{\Theta}\left(\tau, \phi\left(\tau, X_{\tau}^{\Theta}\right)\right)-\frac{C(\kappa, x, \delta)}{m^{1 / p}}-c_{0}\left\{E\left[\mathbf{1}_{A_{m}}\left(\sup _{s \in[t, T]}\left|Y_{s}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right)\right|^{p}+\left(C_{x, \delta}^{\phi}\right)^{p}\right) \mid \mathcal{F}_{t}\right]\right\}^{\frac{1}{p}} \tag{4.60}
\end{align*}
$$

Letting $\widehat{A}_{m} \triangleq\left\{E\left[\mathbf{1}_{A_{m}}\left(\sup _{s \in[t, T]}\left|Y_{s}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right)\right|^{p}+\left(C_{x, \delta}^{\phi}\right)^{p}\right) \mid \mathcal{F}_{t}\right]>1 / m\right\}$, one can deduce that

$$
\begin{aligned}
P\left(\widehat{A}_{m}\right) & \leq m E\left[E\left[\mathbf{1}_{A_{m}}\left(\sup _{s \in[t, T]}\left|Y_{s}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right)\right|^{p}+\left(C_{x, \delta}^{\phi}\right)^{p}\right) \mid \mathcal{F}_{t}\right]\right] \\
& \leq \sum_{i=1}^{N_{m}} m E\left[\mathbf{1}_{\left(A_{i}^{m}\right)^{c}}\left(\sup _{s \in[t, T]}\left|Y_{s}^{\widehat{\Theta}}\left(T, g\left(X_{T}^{\widehat{\Theta}}\right)\right)\right|^{p}+\left(C_{x, \delta}^{\phi}\right)^{p}\right)\right] \leq m^{-p}
\end{aligned}
$$

Multiplying $\mathbf{1}_{\widehat{A}_{m}^{c}}$ to both sides of (4.60) yields that

$$
\begin{equation*}
\mathbf{1}_{\widehat{A}_{m}^{c}} I(t, x, \beta) \geq \mathbf{1}_{\widehat{A}_{m}^{c}} J\left(t, x, \mu^{m}, \beta\left(\mu^{m}\right)\right) \geq \mathbf{1}_{\widehat{A}_{m}^{c}} Y_{t}^{\Theta}\left(\tau, \phi\left(\tau, X_{\tau}^{\Theta}\right)\right)-\frac{C(\kappa, x, \delta)}{m^{1 / p}}, \quad P-\text { a.s. } \tag{4.61}
\end{equation*}
$$

As $\sum_{m \in \mathbb{N}} P\left(\widehat{A}_{m}\right) \leq \sum_{m \in \mathbb{N}} m^{-p}<\infty$, Borel-Cantelli theorem shows that $P\left(\varlimsup_{m \rightarrow \infty} \mathbf{1}_{\widehat{A}_{m}}=1\right)=0$. It follows that $P\left(\varlimsup_{m \rightarrow \infty} \mathbf{1}_{\widehat{A}_{m}}=0\right)=1$ and thus

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbf{1}_{\widehat{A}_{m}}=0, \quad P \text {-a.s. } \tag{4.62}
\end{equation*}
$$

So letting $m \rightarrow \infty$ in (4.61) yields that $I(t, x, \beta) \geq Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \phi\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right), P$-a.s. Taking essential supremum over $\mu \in \mathcal{U}_{t}$ and then taking essential infimum over $\beta \in \mathfrak{B}_{t}$, we obtain (4.40).
1d) Now let us show the other inequality of Theorem 2.1 (1). Similar to $\mu^{m}$, we shall first use (4.32) to construct the $1 / m$-optimal strategy $\beta_{m}$ by pasting together local $1 / m$-optimal strategies with respect to the finite cover $\left\{\mathfrak{D}_{m}\left(s_{i}, x_{i}\right)\right\}_{i=1}^{N_{m}}$ of $\bar{O}_{\delta}(t, x)$.

Fix $m \in \mathbb{N}$. For $i=1, \cdots, N_{m}$, (4.32) shows that there exists $\left(\mathcal{A}_{i}^{m}, \beta_{i}^{m}\right) \in \mathcal{F}_{t_{i}} \times \mathfrak{B}_{t_{i}}$ with $P\left(\mathcal{A}_{i}^{m}\right) \geq 1-m^{\frac{1+p^{2}}{1-p}} N_{m}^{-1}$ such that

$$
\begin{equation*}
\widetilde{\phi}\left(t_{i}, x_{i}\right) \geq w_{1}\left(t_{i}, x_{i}\right) \geq I\left(t_{i}, x_{i}, \beta_{i}^{m}\right)-1 / m, \quad P-\text { a.s. on } \mathcal{A}_{i}^{m} \tag{4.63}
\end{equation*}
$$

Let $\beta_{\psi}$ be the $\mathfrak{B}_{t}-$ strategy considered in (4.11) and fix $\beta \in \mathfrak{B}_{t}$. For any $\mu \in \mathcal{U}_{t}$, we simply denote $\tau_{\beta, \mu}$ by $\tau_{\mu}$ and define

$$
(\widehat{\beta}(\mu))_{s} \triangleq \mathbf{1}_{\left\{s<\tau_{\mu}\right\}}(\beta(\mu))_{s}+\mathbf{1}_{\left\{s \geq \tau_{\mu}\right\}}\left(\beta_{\psi}(\mu)\right)_{s}, \quad \forall s \in[t, T]
$$

which is a $\mathcal{V}_{t}-$ control by Lemma 2.1. By (A-u), it holds $d s \times d P-$ a.s. that

$$
\begin{equation*}
\left[(\widehat{\beta}(\mu))_{s}\right]_{\mathbb{V}}=\mathbf{1}_{\left\{s<\tau_{\mu}\right\}}\left[(\beta(\mu))_{s}\right]_{\mathbb{V}}+\mathbf{1}_{\left\{s \geq \tau_{\mu}\right\}}\left[\left(\beta_{\psi}(\mu)\right)_{s}\right]_{\mathbb{V}} \leq \kappa+\left(C_{\beta} \vee \kappa\right)\left[\mu_{s}\right]_{\mathbb{U}} \tag{4.64}
\end{equation*}
$$

To see $\widehat{\beta} \in \mathfrak{B}_{t}$, we let $\mu^{1}, \mu^{2} \in \mathcal{U}_{t}$ such that $\mu^{1}=\mu^{2}, d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ for some $\tau \in \mathcal{S}_{t, T}$ and $A \in \mathcal{F}_{\tau}$. Since $\beta\left(\mu^{1}\right)=\beta\left(\mu^{2}\right), d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ by Definition $\sqrt{2.2}$, it holds $d s \times d P-$ a.s. on $\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket t, \tau_{\mu^{1}} \wedge \tau_{\mu^{2}} \llbracket$ that

$$
\begin{equation*}
\left(\widehat{\beta}\left(\mu^{1}\right)\right)_{s}=\left(\beta\left(\mu^{1}\right)\right)_{s}=\left(\beta\left(\mu^{2}\right)\right)_{s}=\left(\widehat{\beta}\left(\mu^{2}\right)\right)_{s} \tag{4.65}
\end{equation*}
$$

And (2.8) shows that except on a $P$-null set $\mathcal{N}$

$$
\begin{equation*}
\mathbf{1}_{A} X_{s}^{\Theta} \Theta_{\mu^{1}}+\mathbf{1}_{A^{c}} X_{\tau \wedge s}^{\Theta_{\mu^{1}}}=\mathbf{1}_{A} X_{s}^{\Theta_{\mu^{2}}}+\mathbf{1}_{A^{c}} X_{\tau \wedge s}^{\Theta_{\mu^{2}}}, \quad \forall s \in[t, T] \tag{4.66}
\end{equation*}
$$

Then it holds for any $\omega \in A \cap \mathcal{N}^{c}$ that

$$
\tau_{\mu^{1}}(\omega)=\inf \left\{s \in(t, T]:\left(s, X_{s}^{\Theta} \mu^{1}(\omega)\right) \notin O_{\delta}(t, x)\right\}=\inf \left\{s \in(t, T]:\left(s, X_{s}^{\Theta_{\mu^{2}}}(\omega)\right) \notin O_{\delta}(t, x)\right\}=\tau_{\mu^{2}}(\omega)
$$

Let $A_{o} \triangleq\left\{\tau \geq \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}\right\}$. We can deduce from (4.66) that for any $\omega \in A_{o} \cap\left\{\tau_{\mu^{1}} \leq \tau_{\mu^{2}}\right\} \cap \mathcal{N}^{c}$

$$
\begin{aligned}
\tau_{\mu^{1}}(\omega) & =\inf \left\{s \in(t, T]:\left(s, X_{s}^{\Theta} \Theta_{\mu^{1}}(\omega)\right) \notin O_{\delta}(t, x)\right\}=\inf \left\{s \in(t, \tau(\omega)]:\left(s, X_{s}^{\Theta}(\omega)\right) \notin O_{\delta}(t, x)\right\} \\
& =\inf \left\{s \in(t, \tau(\omega)]:\left(s, X_{s}^{\Theta} \Theta_{\mu^{2}}(\omega)\right) \notin O_{\delta}(t, x)\right\} \geq \inf \left\{s \in(t, T]:\left(s, X_{s}^{\Theta}(\omega)\right) \notin O_{\delta}(t, x)\right\}=\tau_{\mu^{2}}(\omega) \geq \tau_{\mu^{1}}(\omega)
\end{aligned}
$$

Similarly, it holds on $A_{o} \cap\left\{\tau_{\mu^{2}} \leq \tau_{\mu^{1}}\right\} \cap \mathcal{N}^{c}$ that $\tau_{\mu^{1}}=\tau_{\mu^{2}}$. So

$$
\begin{equation*}
\tau_{\mu^{1}}=\tau_{\mu^{2}} \text { on } \widetilde{A} \triangleq\left(A \cup A_{o}\right) \cap \mathcal{N}^{c} \tag{4.67}
\end{equation*}
$$

Since $\llbracket t, \tau \llbracket \cap \llbracket \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}, T \rrbracket=\llbracket \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}, \tau \rrbracket_{A_{o}}$ and $\llbracket t, T \rrbracket_{A} \cap \llbracket \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}, T \rrbracket=\llbracket \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}, T \rrbracket_{A}$, (4.67) leads to that

$$
\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}, T \rrbracket_{\mathcal{N}^{c}} \subset \llbracket \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}, T \rrbracket_{\widetilde{A}}=\llbracket \tau_{\mu^{1}}, T \rrbracket_{\widetilde{A}} \cap \llbracket \tau_{\mu^{2}}, T \rrbracket_{\widetilde{A}}
$$

Thus it holds $d s \times d P-$ a.s. on $\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}, T \rrbracket$ that $\left(\widehat{\beta}\left(\mu^{1}\right)\right)_{s}=\psi\left(s, \mu_{s}^{1}\right)=\psi\left(s, \mu_{s}^{2}\right)=\left(\widehat{\beta}\left(\mu^{2}\right)\right)_{s}$, which together with (4.65) shows that $\widehat{\beta} \in \mathfrak{B}_{t}$.

Given $\mu \in \mathcal{U}_{t}$, we set $\Theta_{\mu} \triangleq(t, x, \mu, \beta(\mu))$ and $\widehat{\Theta}_{\mu} \triangleq(t, x, \mu, \widehat{\beta}(\mu))$. For $i=1, \cdots, N_{m}$, analogous to $\widetilde{A}_{i}^{m}$ of part (1b), $\mathcal{A}_{i}^{\mu, m} \triangleq\left\{\left(\tau_{\mu}, X_{\tau_{\mu}}^{\Theta_{\mu}}\right) \in \mathfrak{D}_{m}\left(s_{i}, x_{i}\right) \backslash \bigcup_{j<i}^{\cup} \mathfrak{D}_{m}\left(s_{j}, x_{j}\right)\right\}$ belongs to $\mathcal{F}_{\tau_{\mu}} \cap \mathcal{F}_{t_{i}}$. By the continuity of process $X^{\Theta_{\mu}}$, $\left(\tau_{\mu}, X_{\tau_{\mu}}^{\Theta_{\mu}}\right) \in \partial O_{\delta}(t, x), P$-a.s. So $\left\{\mathcal{A}_{i}^{\mu, m}\right\}_{i=1}^{N_{m}}$ forms a partition of $\mathcal{N}_{\mu}^{c}$ for some $P$-null set $\mathcal{N}_{\mu}$. Then we can define an $\mathbf{F}$-stopping time $\tau_{\mu}^{m} \triangleq \sum_{i=1}^{N_{m}} \mathbf{1}_{\mathcal{A}_{i}^{\mu, m}} t_{i}+\mathbf{1}_{\mathcal{N}_{\mu}} T \geq \tau_{\mu}$ as well as a process

$$
\begin{align*}
\left(\beta_{m}(\mu)\right)_{s} & \triangleq \mathbf{1}_{\left\{s<\tau_{\mu}^{m}\right\}}(\widehat{\beta}(\mu))_{s}+\mathbf{1}_{\left\{s \geq \tau_{\mu}^{m}\right\}}\left(\sum_{i=1}^{N_{m}} \mathbf{1}_{\mathcal{A}_{i}^{\mu, m} \cap \mathcal{A}_{i}^{m}}\left(\beta_{i}^{m}\left([\mu]^{t_{i}}\right)\right)_{s}+\mathbf{1}_{\mathcal{A}_{\mu}^{m}}(\widehat{\beta}(\mu))_{s}\right) \\
& =\mathbf{1}_{\mathcal{A}_{\mu}^{m}}(\widehat{\beta}(\mu))_{s}+\sum_{i=1}^{N_{m}} \mathbf{1}_{\mathcal{A}_{i}^{\mu, m} \cap \mathcal{A}_{i}^{m}}\left(\mathbf{1}_{\left\{s<t_{i}\right\}}(\widehat{\beta}(\mu))_{s}+\mathbf{1}_{\left\{s \geq t_{i}\right\}}\left(\beta_{i}^{m}\left([\mu]^{t_{i}}\right)\right)_{s}\right), \quad \forall s \in[t, T], \tag{4.68}
\end{align*}
$$

where $\mathcal{A}_{\mu}^{m}=(\underbrace{\mathcal{N}_{m}}_{i=1}\left(\mathcal{A}_{i}^{\mu, m} \backslash \mathcal{A}_{i}^{m}\right)) \cup \mathcal{N}_{\mu}$.
We claim that $\beta_{m}$ is a $\mathfrak{B}_{t}$-strategy. Using a similar argument to that in part (1b) for the measurability of the pasted control $\mu^{m}$, one can deduce that the process $\beta_{m}(\mu)$ is $\mathbf{F}$-progressively measurable. For $i=1, \cdots, N_{m}$, let
$C_{i}^{m}>0$ be the constant associated to $\beta_{i}^{m}$ in Definition 2.2 (i). Setting $C_{m}=C_{\beta} \vee \kappa \vee \max \left\{C_{i}^{m}: i=1, \cdots, N_{m}\right\}$, we can deduce from (4.64) and (A-u) that $d s \times d P-$ a.s.

$$
\begin{align*}
& {\left[\left(\beta_{m}(\mu)\right)_{s}\right]_{\mathbb{V}}=\mathbf{1}_{\left\{s<\tau_{\mu}^{m}\right\}}\left[(\widehat{\beta}(\mu))_{s}\right]_{\mathbb{V}}+\mathbf{1}_{\left\{s \geq \tau_{\mu}^{m}\right\}}\left(\sum_{i=1}^{N_{m}} \mathbf{1}_{\mathcal{A}_{i}^{\mu, m} \cap \mathcal{A}_{i}^{m}}\left[\left(\beta_{i}^{m}\left([\mu]^{t_{i}}\right)\right)_{s}\right]_{\mathbb{V}}+\mathbf{1}_{\mathcal{A}_{\mu}^{m}}\left[(\widehat{\beta}(\mu))_{s}\right]_{\mathbb{V}}\right)} \\
& \quad \leq\left(\mathbf{1}_{\left\{s<\tau_{\mu}^{m}\right\}}+\mathbf{1}_{\left\{s \geq \tau_{\mu}^{m}\right\}} \mathbf{1}_{\mathcal{A}_{\mu}^{m}}\right)\left(\kappa+\left(C_{\beta} \vee \kappa\right)\left[\mu_{s}\right]_{\mathbb{U}}\right)+\mathbf{1}_{\left\{s \geq \tau_{\mu}^{m}\right\}} \sum_{i=1}^{N_{m}} \mathbf{1}_{\mathcal{A}_{i}^{\mu, m} \cap \mathcal{A}_{i}^{m}}\left(\kappa+C_{i}^{m}\left[[\mu]_{s}^{t_{i}}\right]_{\mathbb{U}}\right) \leq \kappa+C_{m}\left[\mu_{s}\right]_{\mathbb{U}} . \tag{4.69}
\end{align*}
$$

Let $E \int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s<\infty$ for some $q>2$. It follows from (4.69) that

$$
E \int_{t}^{T}\left[\left(\beta_{m}(\mu)\right)_{s}\right]_{\mathbb{V}}^{q} d s \leq 2^{q-1} \kappa^{q} T+2^{q-1} C_{m}^{q} E \int_{t}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s<\infty
$$

Hence $\beta_{m}(\mu) \in \mathcal{V}_{t}$.
Let $\mu^{1}, \mu^{2} \in \mathcal{U}_{t}$ such that $\mu^{1}=\mu^{2}, d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ for some $\tau \in \mathcal{S}_{t, T}$ and $A \in \mathcal{F}_{\tau}$. As $\widehat{\beta}\left(\mu^{1}\right)=\widehat{\beta}\left(\mu^{2}\right)$, $d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ by Definition 2.2, it holds $d s \times d P-$ a.s. on $\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket t, \tau_{\mu^{1}}^{m} \wedge \tau_{\mu^{2}}^{m} \llbracket$ that

$$
\begin{equation*}
\left(\beta_{m}\left(\mu^{1}\right)\right)_{s}=\left(\widehat{\beta}\left(\mu^{1}\right)\right)_{s}=\left(\widehat{\beta}\left(\mu^{2}\right)\right)_{s}=\left(\beta_{m}\left(\mu^{2}\right)\right)_{s} \tag{4.70}
\end{equation*}
$$

Definition 2.2 also shows that $\left(\mu^{1}, \beta\left(\mu^{1}\right)\right)=\left(\mu^{2}, \beta\left(\mu^{2}\right)\right), d s \times d P-$ a.s. on $\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$. So we again have (4.66) except on a $P$-null set $\mathcal{N}$, and (4.67) still holds on $\widetilde{A} \triangleq\left(A \cup A_{o}\right) \cap \mathcal{N}^{c}$ with $A_{o}=\left\{\tau \geq \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}\right\}$. Plugging (4.67) into (4.66) yields that

$$
\begin{equation*}
X_{\tau_{\mu^{1}}}^{\Theta}=X_{\tau_{\mu^{2}}}^{\Theta_{\mu^{2}}} \text { holds on } \widetilde{A} \tag{4.71}
\end{equation*}
$$

Given $i=1, \cdots, N_{m}$. since it holds $d s \times d P-$ a.s. on $\left(\llbracket t, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}\right) \cap\left(\left[t_{i}, T\right] \times \Omega\right)=\llbracket t_{i}, \tau \vee t_{i} \llbracket \cup \llbracket \tau \vee t_{i}, T \rrbracket_{A}$ that $\left(\left[\mu^{1}\right]^{t_{i}}\right)_{s}=\mu_{s}^{1}=\mu_{s}^{2}=\left(\left[\mu^{2}\right]^{t_{i}}\right)_{s}$, taking $(\tau, A)=\left(\tau \vee t_{i}, A\right)$ in Definition [2.2 with respect to $\beta_{i}^{m}$ yields that for $d s \times d P-$ a.s. $(s, \omega) \in \llbracket t_{i}, \tau \vee t_{i} \llbracket \cup \llbracket \tau \vee t_{i}, T \rrbracket_{A}=\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap\left(\left[t_{i}, T\right] \times \Omega\right)$

$$
\begin{equation*}
\left(\beta_{i}^{m}\left(\left[\mu^{1}\right]^{t_{i}}\right)\right)_{s}(\omega)=\left(\beta_{i}^{m}\left(\left[\mu^{2}\right]^{t_{i}}\right)\right)_{s}(\omega) \tag{4.72}
\end{equation*}
$$

Given $\omega \in \mathcal{A}_{i} \triangleq \widetilde{A} \cap \mathcal{A}_{i}^{\mu^{1}, m}$, (4.67) and (4.71) imply that

$$
\left(\tau_{\mu^{2}}(\omega), X_{\tau_{\mu^{2}}(\omega)}^{\Theta_{\mu^{2}}}(\omega)\right)=\left(\tau_{\mu^{1}}(\omega), X_{\tau_{\mu^{1}}(\omega)}^{\Theta}(\omega)\right) \in \mathfrak{D}_{m}\left(s_{i}, x_{i}\right) \backslash \cup_{j<i} \mathfrak{D}_{m}\left(s_{j}, x_{j}\right), \quad \text { i.e., } \omega \in \mathcal{A}_{i}^{\mu^{2}, m}
$$

So $\mathcal{A}_{i} \subset \mathcal{A}_{i}^{\mu^{1}, m} \cap \mathcal{A}_{i}^{\mu^{2}, m}$, and it follows that $\mathbf{1}_{\mathcal{A}_{i}} \tau_{\mu^{1}}^{m}=\mathbf{1}_{\mathcal{A}_{i}} t_{i}=\mathbf{1}_{\mathcal{A}_{i}} \tau_{\mu^{2}}^{m}$. Then one can deduce that

$$
\begin{align*}
\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket \tau_{\mu^{1}}^{m} \wedge \tau_{\mu^{2}}^{m}, T \rrbracket_{\mathcal{A}_{i} \cap \mathcal{A}_{i}^{m}} & =\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap\left(\left[t_{i}, T\right] \times\left(\mathcal{A}_{i} \cap \mathcal{A}_{i}^{m}\right)\right) \\
& \subset\left[t_{i}, T\right] \times\left(\mathcal{A}_{i}^{\mu^{1}, m} \cap \mathcal{A}_{i}^{\mu^{2}, m} \cap \mathcal{A}_{i}^{m}\right) \tag{4.73}
\end{align*}
$$

which together with (4.72) shows that for $d s \times d P-$ a.s. $(s, \omega) \in\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket \tau_{\mu^{1}}^{m} \wedge \tau_{\mu^{2}}^{m}, T \rrbracket \rrbracket_{\mathcal{A}_{i} \cap \mathcal{A}_{i}^{m}}$

$$
\begin{equation*}
\left(\beta_{m}\left(\mu^{1}\right)\right)_{s}(\omega)=\left(\beta_{i}^{m}\left(\left[\mu^{1}\right]^{t_{i}}\right)\right)_{s}(\omega)=\left(\beta_{i}^{m}\left(\left[\mu^{2}\right]^{t_{i}}\right)\right)_{s}(\omega)=\left(\beta_{m}\left(\mu^{2}\right)\right)_{s}(\omega) . \tag{4.74}
\end{equation*}
$$

Analogous to (4.73), $\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket \tau_{\mu^{1}}^{m} \wedge \tau_{\mu^{2}}^{m}, T \rrbracket_{\mathcal{A}_{i} \backslash \mathcal{A}_{i}^{m}} \subset\left[t_{i}, T\right] \times\left(\left(\mathcal{A}_{i}^{\mu^{1}, m} \backslash \mathcal{A}_{i}^{m}\right) \cap\left(\mathcal{A}_{i}^{\mu^{2}, m} \backslash \mathcal{A}_{i}^{m}\right)\right)$. So (4.70) also holds $d s \times d P-$ a.s. on $\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket \tau_{\mu^{1}}^{m} \wedge \tau_{\mu^{2}}^{m}, T \rrbracket_{\mathcal{A}_{i} \backslash \mathcal{A}_{i}^{m}}$. Combining this with (4.74) and then letting $i$ run over $\left\{1, \cdots, N_{m}\right\}$ yield that

$$
\begin{equation*}
\left(\beta_{m}\left(\mu^{1}\right)\right)_{s}=\left(\beta_{m}\left(\mu^{2}\right)\right)_{s}, \quad d s \times d P-a . s . \text { on }\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket \tau_{\mu^{1}}^{m} \wedge \tau_{\mu^{2}}^{m}, T \rrbracket_{A \cup A_{o}} \tag{4.75}
\end{equation*}
$$

As $\llbracket \tau_{\mu^{1}}^{m} \wedge \tau_{\mu^{2}}^{m}, T \rrbracket_{A^{c} \cap A_{o}^{c}} \subset \llbracket \tau_{\mu^{1}} \wedge \tau_{\mu^{2}}, T \rrbracket_{A^{c} \cap A_{o}^{c}} \subset \llbracket \tau, T \rrbracket_{A^{c} \cap A_{o}^{c}} \subset \llbracket \tau, T \rrbracket_{A^{c}}$, one can deduce that $\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap$ $\llbracket \tau_{\mu^{1}}^{m} \wedge \tau_{\mu^{2}}^{m}, T \rrbracket_{A \cup A_{o}}=\left(\llbracket t, \tau \llbracket \cup \llbracket t, T \rrbracket_{A}\right) \cap \llbracket \tau_{\mu^{1}}^{m} \wedge \tau_{\mu^{2}}^{m}, T \rrbracket$. Therefore, (4.75) together with (4.70) implies that $\beta_{m} \in \mathfrak{B}_{t}$.

1e) Next, let $\mu \in \mathcal{U}_{t}$ and $\Theta_{\mu}^{m} \triangleq\left(t, x, \mu, \beta_{m}(\mu)\right)$. We shall do similar estimates to those in part (c) to conclude

$$
\begin{equation*}
w_{1}(t, x) \leq \underset{\beta \in \mathfrak{B}_{t}}{\operatorname{essinf}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \widetilde{\phi}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right), \quad P-a . s \tag{4.76}
\end{equation*}
$$

As $\beta_{m}(\mu)=\widehat{\beta}(\mu)=\beta(\mu)$ on $\llbracket t, \tau_{\mu} \llbracket$, taking $(\tau, A)=\left(\tau_{\mu}, \emptyset\right)$ in (2.8) yields that $P$-a.s.

$$
\begin{equation*}
X_{s}^{\Theta_{\mu}^{m}}=X_{s}^{\widehat{\Theta}_{\mu}}=X_{s}^{\Theta_{\mu}} \in \bar{O}_{\delta}(x), \quad \forall s \in\left[t, \tau_{\mu}\right] \tag{4.77}
\end{equation*}
$$

Thus, for any $\eta \in \mathbb{L}^{p}\left(\mathcal{F}_{\tau_{\mu}}\right)$, the $\operatorname{BSDE}\left(t, \eta, f_{\tau_{\mu}}^{\Theta_{\mu}^{m}}\right)$ and the $\operatorname{BSDE}\left(t, \eta, f_{\tau_{\mu}}^{\Theta_{\mu}}\right)$ are essentially the same. To wit,

$$
\begin{equation*}
\left(Y^{\Theta_{\mu}^{m}}\left(\tau_{\mu}, \eta\right), Z^{\Theta_{\mu}^{m}}\left(\tau_{\mu}, \eta\right)\right)=\left(Y^{\Theta_{\mu}}\left(\tau_{\mu}, \eta\right), Z^{\Theta_{\mu}}\left(\tau_{\mu}, \eta\right)\right) \tag{4.78}
\end{equation*}
$$

Given $A \in \mathcal{F}_{t}$, similar to (4.43), we can deduce from (4.77) that

$$
\begin{aligned}
\mathbf{1}_{A} \sup _{r \in[t, s]}\left|X_{\tau_{\mu}^{m} \wedge r}^{\Theta_{\mu}^{m}}-X_{\tau_{\mu} \wedge r}^{\Theta_{\mu}}\right| \leq & \int_{\tau_{\mu} \wedge s}^{\tau_{\mu}^{m} \wedge s} \mathbf{1}_{A}\left|b\left(r, X_{\tau_{\mu}^{m} \wedge r}^{\Theta_{\mu}^{m}}, \mu_{r}, \psi\left(r, \mu_{r}\right)\right)\right| d r \\
& +\sup _{r \in[t, s]}\left|\int_{\tau_{\mu} \wedge r}^{\tau_{\mu}^{m} \wedge r} \mathbf{1}_{A} \sigma\left(r^{\prime}, X_{\tau_{\mu}^{m} \wedge r^{\prime}}^{\Theta_{\mu}^{m}} \mu_{r^{\prime}}, \psi\left(r^{\prime}, \mu_{r^{\prime}}\right)\right) d B_{r^{\prime}}\right|, \quad s \in[t, T] .
\end{aligned}
$$

where we used the fact that $\beta_{m}(\mu)=\widehat{\beta}(\mu)=\beta_{\psi}(\mu)$ on $\llbracket \tau_{\mu}, \tau_{\mu}^{m} \llbracket$. Let $\widetilde{C}(\kappa, x, \delta)$ denote a generic constant, depending on $\kappa+|x|+\delta, C_{x, \delta}^{\widetilde{\phi}} \triangleq \sup \left\{|\widetilde{\phi}(s, \mathfrak{x})|:(s, \mathfrak{x}) \in \bar{O}_{\delta+3}(t, x) \cap\left([t, T] \times \mathbb{R}^{k}\right)\right\}, T, \gamma, p$ and $|g(0)|$, whose form may vary from line to line. Since $\tau_{\mu}^{m}-\tau_{\mu} \leq \sum_{i=1}^{N_{m}} \mathbf{1}_{\mathcal{A}_{i}^{\mu, m}} 2 \delta_{s_{i}, x_{i}}^{m}<\frac{2}{m}, P$ a.s., using similar arguments to those that lead to (4.44) and using an analogous decomposition and estimation to (4.12), we can deduce that

$$
\begin{aligned}
& E\left[\mathbf{1}_{A} \sup _{r \in[t, s]}\left|X_{\tau_{\mu}^{m} \wedge r}^{\Theta_{m}^{m}}-X_{\tau_{\mu} \wedge r}^{\Theta_{\mu}}\right|^{2}\right] \\
& \quad \leq 4 E \int_{\tau_{\mu} \wedge s}^{\tau_{\mu}^{m} \wedge s} \mathbf{1}_{A}\left|b\left(r, X_{\tau_{\mu}^{m} \wedge r}^{\Theta_{\mu}^{m}}, \mu_{r}, \psi\left(r, \mu_{r}\right)\right)\right|^{2} d r+8 E \int_{\tau_{\mu} \wedge s}^{\tau_{\mu}^{m} \wedge s} \mathbf{1}_{A}\left|\sigma\left(r, X_{\tau_{\mu}^{m} \wedge r}^{\Theta_{\mu}^{m}}, \mu_{r}, \psi\left(r, \mu_{r}\right)\right)\right|^{2} d r \\
& \quad \leq 24 \gamma^{2} \int_{t}^{s} E\left[\mathbf{1}_{A} \sup _{r^{\prime} \in[t, r]}\left|X_{\tau_{\mu}^{m} \wedge r^{\prime}}^{\Theta_{\mu}^{m}}-X_{\tau_{\mu} \wedge r^{\prime}}^{\Theta_{\mu}}\right|^{2}\right] d r+\frac{\widetilde{C}(\kappa, x, \delta)}{m} P(A), \quad \forall s \in[t, T] .
\end{aligned}
$$

Then similar to (4.46), an application of Gronwall's inequality leads to that

$$
\begin{equation*}
E\left[\sup _{r \in[t, T]}\left|X_{\tau_{\mu}^{m} \wedge r}^{\Theta_{m}^{m}}-X_{\tau_{\mu} \wedge r}^{\Theta_{\mu}}\right|^{2} \mid \mathcal{F}_{t}\right] \leq \frac{\widetilde{C}(\kappa, x, \delta)}{m}, \quad P-\text { a.s. } \tag{4.79}
\end{equation*}
$$

Let $i=1, \cdots, N_{m}$ and set $\Theta_{\mu}^{m, t_{i}} \triangleq\left(t_{i}, X_{t_{i}}^{\Theta_{\mu}^{m}},[\mu]^{t_{i}},\left[\beta_{m}(\mu)\right]^{t_{i}}\right)$. Similar to (4.47), it holds $P$-a.s. that

$$
\begin{equation*}
Y_{t_{i}}^{\Theta_{\mu}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)=J\left(\Theta_{\mu}^{m, t_{i}}\right) \tag{4.80}
\end{equation*}
$$

Since $\left[\beta_{m}(\mu)\right]_{r}^{t_{i}}(\omega)=\left(\beta_{m}(\mu)\right)_{r}(\omega)=\left(\beta_{i}^{m}\left([\mu]^{t_{i}}\right)\right)_{r}(\omega)$ for any $(r, \omega) \in\left[t_{i}, T\right] \times\left(\mathcal{A}_{i}^{\mu, m} \cap \mathcal{A}_{i}^{m}\right)$, one can deduce from (4.80), (4.29) and (2.14) that it holds $P$-a.s. on $\mathcal{A}_{i}^{\mu, m} \cap \mathcal{A}_{i}^{m} \in \mathcal{F}_{t_{i}}$ that
$Y_{\tau_{\mu}^{m}}^{\Theta_{\mu}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)=Y_{t_{i}}^{\Theta_{\mu}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)=J\left(t_{i}, X_{\tau_{\mu}^{m}}^{\Theta_{\mu}^{m}},[\mu]^{t_{i}}, \beta_{i}^{m}\left([\mu]^{t_{i}}\right)\right) \leq J\left(t_{i}, X_{\tau_{\mu}}^{\Theta_{\mu}},[\mu]^{t_{i}}, \beta_{i}^{m}\left([\mu]^{t_{i}}\right)\right)+c_{0}\left|X_{\tau_{\mu}^{m}}^{\Theta_{\mu}^{m}}-X_{\tau_{\mu}}^{\Theta_{\mu}}\right|^{2 / p}$.
As $\left|X_{\tau_{\mu}}^{\Theta_{\mu}}-x_{i}\right|^{2 / p}<\left(\delta_{s_{i}, x_{i}}^{m}\right)^{2 / p}<m^{-2 / p} \leq 1 / m$ on $\mathcal{A}_{i}^{\mu, m}$, we can also deduce from (2.14), (4.63), (4.36) and the continuity of $\widetilde{\phi}$ that it holds $P$-a.s. on $\mathcal{A}_{i}^{\mu, m} \cap \mathcal{A}_{i}^{m}$ that

$$
\begin{aligned}
J\left(t_{i}, X_{\tau_{\mu}}^{\Theta_{\mu}},[\mu]^{t_{i}}, \beta_{i}^{m}\left([\mu]^{t_{i}}\right)\right) & \leq J\left(t_{i}, x_{i},[\mu]^{t_{i}}, \beta_{i}^{m}\left([\mu]^{t_{i}}\right)\right)+\frac{c_{0}}{m} \leq I\left(t_{i}, x_{i}, \beta_{i}^{m}\right)+\frac{c_{0}}{m} \leq \widetilde{\phi}\left(t_{i}, x_{i}\right)+\frac{c_{0}}{m} \\
& \leq \widetilde{\phi}\left(s_{i}, x_{i}\right)+\frac{c_{0}}{m} \leq \widetilde{\phi}\left(\tau_{\mu}, X_{\tau_{\mu}}^{\Theta_{\mu}}\right)+\frac{c_{0}}{m} \triangleq \eta_{\mu}^{m} \in \mathbb{L}^{\infty}\left(\mathcal{F}_{\tau_{\mu}}\right)
\end{aligned}
$$

Thus it holds $P-$ a.s. on $\cup_{i=1}^{N_{m}}\left(\mathcal{A}_{i}^{\mu, m} \cap \mathcal{A}_{i}^{m}\right)$ that

$$
Y_{\tau_{\mu}^{m}}^{\Theta_{m}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right) \leq \eta_{\mu}^{m}+c_{0}\left|X_{\tau_{\mu}^{m}}^{\Theta_{m}^{m}}-X_{\tau_{\mu}}^{\Theta_{\mu}}\right|^{2 / p} \triangleq \widetilde{\eta}_{\mu}^{m} \in \mathbb{L}^{p}\left(\mathcal{F}_{\tau_{\mu}^{m}}\right) .
$$

By (2.10), it holds $P$-a.s. that

$$
\begin{equation*}
\left|Y_{t}^{\Theta_{\mu}}\left(\tau_{\mu}, \eta_{\mu}^{m}\right)-Y_{t}^{\Theta_{\mu}}\left(\tau_{\mu}, \widetilde{\phi}\left(\tau_{\mu}, X_{\tau_{\mu}}^{\Theta_{\mu}}\right)\right)\right|^{p} \leq c_{0} E\left[\left|\eta_{\mu}^{m}-\widetilde{\phi}\left(\tau_{\mu}, X_{\tau_{\mu}}^{\Theta_{\mu}}\right)\right|^{p} \mid \mathcal{F}_{t}\right] \leq \frac{c_{0}}{m^{p}} . \tag{4.81}
\end{equation*}
$$

Similar to (4.54), one can deduce that

$$
\begin{aligned}
E\left[\left|Y_{\tau_{\mu}}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}^{m}, \eta_{\mu}^{m}\right)-\eta_{\mu}^{m}\right|^{p} \mid \mathcal{F}_{t}\right] & \leq c_{0} E\left[\int_{\tau_{\mu}}^{T}\left|f_{\tau_{\mu}^{m}}^{\Theta_{m}^{m}}\left(s, \widetilde{Y}_{s}^{m, \mu}, \widetilde{Z}_{s}^{m, \mu}\right)\right|^{p} d s \mid \mathcal{F}_{t}\right] \\
& =c_{0} E\left[\int_{\tau_{\mu}}^{\tau_{\mu}^{m}}\left|f\left(s, X_{\tau_{\mu}^{m} \wedge s}^{\Theta_{m}^{m}}, \eta_{\mu}^{m}, 0, \mu_{s}, \psi\left(s, \mu_{s}\right)\right)\right|^{p} d s \mid \mathcal{F}_{t}\right], \quad P-\text { a.s. }
\end{aligned}
$$

Using an analogous decomposition and estimation to (4.12), similar to (4.55), we can deduce from (4.79) that

$$
\begin{aligned}
& \left|Y_{t}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}, Y_{\tau_{\mu}}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}^{m}, \eta_{\mu}^{m}\right)\right)-Y_{t}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}, \eta_{\mu}^{m}\right)\right|^{p} \leq E\left[\left|Y_{\tau_{\mu}}^{\Theta^{m}}\left(\tau_{\mu}^{m}, \eta_{\mu}^{m}\right)-\eta_{\mu}^{m}\right|^{p} \mid \mathcal{F}_{t}\right] \\
& \quad \leq c_{0} E\left[\int_{\tau_{\mu}}^{\tau_{\mu}^{m}}\left(\left|X_{\tau_{\mu}^{m} \wedge s}^{\Theta_{m}^{m}}-X_{\tau_{\mu} \wedge s}^{\Theta_{\mu}}\right|^{2}+\left|X_{\tau_{\mu} \wedge s}^{\Theta_{\mu}}\right|^{2}+\left|\eta_{\mu}^{m}\right|^{p}+c_{\kappa}\right) d s \mid \mathcal{F}_{t}\right] \leq \frac{\widetilde{C}(\kappa, x, \delta)}{m}, \quad P-\text { a.s. }
\end{aligned}
$$

Applying (2.11) with $(\zeta, \tau, \eta)=\left(\tau_{\mu}, \tau_{\mu}^{m}, \eta_{\mu}^{m}\right)$, applying (4.78) with $\eta=\eta_{\mu}^{m}$ and using (4.81) yield that $P$-a.s.

$$
\begin{align*}
& Y_{t}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}^{m}, \eta_{\mu}^{m}\right)=Y_{t}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}, Y_{\tau_{\mu}}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}^{m}, \eta_{\mu}^{m}\right)\right) \leq Y_{t}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}, \eta_{\mu}^{m}\right)+\frac{\widetilde{C}(\kappa, x, \delta)}{m^{1 / p}}=Y_{t}^{\Theta_{\mu}}\left(\tau_{\mu}, \eta_{\mu}^{m}\right)+\frac{\widetilde{C}(\kappa, x, \delta)}{m^{1 / p}} \\
& \quad \leq Y_{t}^{\Theta_{\mu}}\left(\tau_{\mu}, \widetilde{\phi}\left(\tau_{\mu}, X_{\tau_{\mu}}^{\Theta_{\mu}}\right)\right)+\frac{\widetilde{C}(\kappa, x, \delta)}{m^{1 / p}} \leq \underset{\mu \in \mathcal{U}_{t}}{\operatorname{esssup}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \widetilde{\phi}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right)+\frac{\widetilde{C}(\kappa, x, \delta)}{m^{1 / p}} . \tag{4.82}
\end{align*}
$$

As $\beta_{m}(\mu)=\widehat{\beta}(\mu), d s \times d P-$ a.s. on $\llbracket t, \tau_{\mu}^{m} \llbracket$, applying (2.8) with $(\tau, A)=\left(\tau_{\mu}^{m}, \emptyset\right)$ yields that $P$-a.s.

$$
\begin{equation*}
X_{s}^{\Theta_{\mu}^{m}}=X_{s}^{\widehat{\Theta}_{\mu}}, \quad \forall s \in\left[t, \tau_{\mu}^{m}\right] . \tag{4.83}
\end{equation*}
$$

Given $i=1, \cdots, N_{m}$, (4.83) shows that $X_{t_{i}}^{\Theta_{\mu}^{m}}=X_{t_{i}}^{\widehat{\Theta}_{\mu}}, P-$ a.s. on $\mathcal{A}_{i}^{\mu, m} \backslash \mathcal{A}_{i}^{m}$. Since $\left[\beta_{m}(\mu)\right]_{r}^{t_{i}}(\omega)=\left(\beta_{m}(\mu)\right)_{r}(\omega)=$ $(\widehat{\beta}(\mu))_{r}(\omega)=[\widehat{\beta}(\mu)]_{r}^{t_{i}}(\omega)$ holds $d s \times d P$-a.s. on $\llbracket \tau_{\mu}^{m}, T \rrbracket_{\mathcal{A}_{i}^{\mu, m} \backslash \mathcal{A}_{i}^{m}}=\left[t_{i}, T\right] \times\left(\mathcal{A}_{i}^{\mu, m} \backslash \mathcal{A}_{i}^{m}\right)$. Then by (4.29) and a similar argument to (4.80), it holds $P$-a.s. on $\mathcal{A}_{i}^{\mu, m} \backslash \mathcal{A}_{i}^{m}$ that

$$
\begin{equation*}
Y_{\tau_{\mu}^{m}}^{\Theta_{m}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)=Y_{t_{i}}^{\Theta_{\mu}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)=J\left(\Theta_{\mu}^{m, t_{i}}\right)=J\left(\widehat{\Theta}_{\mu}^{t_{i}}\right)=Y_{t_{i}}^{\widehat{\Theta}_{\mu}}\left(T, g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right)=Y_{\tau_{\mu}^{m}}^{\widehat{\Theta}_{\mu}}\left(T, g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right), \tag{4.84}
\end{equation*}
$$

where $\widehat{\Theta}_{\mu}^{t_{\mu}} \triangleq\left(t_{i}, X_{t_{i}}^{\widehat{\Theta}_{\mu}},[\mu]^{t_{i}},[\widehat{\beta}(\mu)]^{t_{i}}\right)$.
Given $A \in \mathcal{F}_{t}$, one can deduce that

$$
\begin{aligned}
\mathbf{1}_{A} X_{\tau_{\mu} \vee s}^{\widehat{\Theta}_{\mu}} & =\mathbf{1}_{A} X_{\tau_{\mu}}^{\widehat{\Theta}_{\mu}}+\mathbf{1}_{A} \int_{\tau_{\mu}}^{\tau_{\mu} \vee s} b\left(r, X_{r}^{\widehat{\Theta}_{\mu}}, \mu_{r},(\widehat{\beta}(\mu))_{r}\right) d r+\mathbf{1}_{A} \int_{\tau_{\mu}}^{\tau_{\mu} \vee s} \sigma\left(r, X_{r}^{\widehat{\Theta}_{\mu}}, \mu_{r},(\widehat{\beta}(\mu))_{r}\right) d B_{r} \\
& =\mathbf{1}_{A} X_{\tau_{\mu}}^{\widehat{\Theta}_{\mu}}+\int_{t}^{s} \mathbf{1}_{\left\{r \geq \tau_{\mu}\right\}} \mathbf{1}_{A} b\left(r, X_{r}^{\widehat{\Theta}_{\mu}}, \mu_{r}, \psi\left(r, \mu_{r}\right)\right) d r+\int_{t}^{s} \mathbf{1}_{\left\{r \geq \tau_{\mu}\right\}} \mathbf{1}_{A} \sigma\left(r, X_{r}^{\widehat{\Theta}_{\mu}}, \mu_{r}, \psi\left(r, \mu_{r}\right)\right) d B_{r}, \quad s \in[t, T] .
\end{aligned}
$$

It then follows from (4.77) that

$$
\begin{aligned}
\mathbf{1}_{A} \sup _{r \in[t, s]}\left|X_{\tau_{\mu} \vee r}^{\widehat{\Theta}_{\mu}}\right| \leq & \mathbf{1}_{A}(|x|+\delta)+\int_{t}^{s} \mathbf{1}_{\left\{r \geq \tau_{\mu}\right\}} \mathbf{1}_{A}\left|b\left(r, X_{r}^{\widehat{\Theta}_{\mu}}, \mu_{r}, \psi\left(r, \mu_{r}\right)\right)\right| d r \\
& +\sup _{r \in[t, s]}\left|\int_{t}^{r} \mathbf{1}_{\left\{r^{\prime} \geq \tau_{\mu}\right\}} \mathbf{1}_{A} \sigma\left(r^{\prime}, X_{r^{\prime}}^{\widehat{\Theta}_{\mu}}, \mu_{r^{\prime}}, \psi\left(r^{\prime}, \mu_{r^{\prime}}\right)\right) d B_{r^{\prime}}\right|, \quad s \in[t, T] .
\end{aligned}
$$

Using an analogous decomposition and estimation to (4.12), one can deduce from Hölder's inequality, Doob's martingale inequality, (2.1), (2.2), (4.77) and Fubini's Theorem that

$$
\begin{aligned}
& E\left[\mathbf{1}_{A} \sup _{r \in[t, s]}\left|X_{\tau_{\mu} \vee r}^{\widehat{\Theta}_{\mu}}\right|^{2}\right] \leq \widetilde{C}(\kappa, x, \delta) P(A)+c_{0} E \int_{t}^{s} \mathbf{1}_{\left\{r \geq \tau_{\mu}\right\}} \mathbf{1}_{A}\left(\left|b\left(r, X_{r}^{\widehat{\Theta}_{\mu}}, \mu_{r}, \psi\left(r, \mu_{r}\right)\right)\right|^{2}+\left|\sigma\left(r, X_{r}^{\widehat{\Theta}_{\mu}}, \mu_{r}, \psi\left(r, \mu_{r}\right)\right)\right|^{2}\right) d r \\
& \quad \leq \widetilde{C}(\kappa, x, \delta) P(A)+c_{0} E \int_{t}^{s} \mathbf{1}_{\left\{r \geq \tau_{\mu}\right\}} \mathbf{1}_{A}\left|X_{\tau_{\mu} \vee r}^{\widehat{\Theta}_{\mu}}\right|^{2} d r \leq \widetilde{C}(\kappa, x, \delta) P(A)+c_{0} \int_{t}^{s} E\left[\mathbf{1}_{A} \sup _{r^{\prime} \in[t, r]} \mid X_{\tau_{\mu} \vee r^{\prime}} \widehat{\Theta}^{2}\right] d r, \quad \forall s \in[t, T]
\end{aligned}
$$

Then an application of Gronwall's inequality shows that $E\left[\mathbf{1}_{A} \sup _{r \in[t, s]}\left|X_{\tau_{\mu} \vee r}^{\widehat{\Theta}_{\mu}}\right|^{2}\right] \leq \widetilde{C}(\kappa, x, \delta) P(A) e^{c_{0}(s-t)}, s \in[t, T]$.
In particular, $E\left[\mathbf{1}_{A} \sup _{r \in\left[\tau_{\mu}, T\right]}\left|X_{r}^{\widehat{\Theta}_{\mu}}\right|^{2}\right]=E\left[\mathbf{1}_{A} \sup _{r \in[t, T]}\left|X_{\tau_{\mu} \vee r}^{\widehat{\Theta}_{\mu}}\right|^{2}\right] \leq \widetilde{C}(\kappa, x, \delta) P(A)$. Letting $A$ vary in $\mathcal{F}_{t}$ yields that

$$
\begin{equation*}
E\left[\sup _{r \in\left[\tau_{\mu}, T\right]}\left|X_{r}^{\widehat{\Theta}_{\mu}}\right|^{2} \mid \mathcal{F}_{t}\right] \leq \widetilde{C}(\kappa, x, \delta), \quad P-\text { a.s. } \tag{4.85}
\end{equation*}
$$

Let $\left(\widehat{Y}^{\mu}, \widehat{Z}^{\mu}\right) \in \mathbb{G}_{\mathbf{F}}^{p}([t, T])$ be the unique solution of the following BSDE with zero generator:

$$
\widehat{Y}_{s}^{\mu}=Y_{\tau_{\mu}}^{\widehat{\Theta}_{\mu}}\left(T, g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right)-\int_{s}^{T} \widehat{Z}_{r}^{\mu} d B_{r}, \quad s \in[t, T]
$$

Analogous to (4.53), $\left(\widehat{\mathcal{Y}}^{\mu}, \widehat{\mathcal{Z}}^{\mu}\right) \triangleq\left\{\left(\mathbf{1}_{\left\{s<\tau_{\mu}\right\}} \widehat{Y}_{s}^{\mu}+\mathbf{1}_{\left\{s \geq \tau_{\mu}\right\}} Y_{s}^{\widehat{\Theta}_{\mu}}\left(T, g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right), \mathbf{1}_{\left\{s<\tau_{\mu}\right\}} \widehat{Z}_{s}^{\mu}+\mathbf{1}_{\left\{s \geq \tau_{\mu}\right\}} Z_{s}^{\widehat{\Theta}_{\mu}}\left(T, g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right)\right)\right\}_{s \in[t, T]}$ $\in \mathbb{G}_{\mathbf{F}}^{p}([t, T])$ solves the following BSDE

$$
\widehat{\mathcal{Y}}_{s}^{\mu}=g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)+\int_{s}^{T} \mathbf{1}_{\left\{r \geq \tau_{\mu}\right\}} f_{T}^{\widehat{\Theta}_{\mu}}\left(r, \widehat{\mathcal{Y}}_{r}^{\mu}, \widehat{\mathcal{Z}}_{r}^{\mu}\right) d r-\int_{s}^{T} \widehat{\mathcal{Z}}_{r}^{\mu} d B_{r}, \quad s \in[0, T]
$$

Then (2.9), (1.4) and Hölder's inequality imply that $P$-a.s.

$$
\begin{aligned}
E\left[\sup _{s \in\left[\tau_{\mu}, T\right]}\left|Y_{s}^{\widehat{\Theta}_{\mu}}\left(T, g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right)\right|^{p} \mid \mathcal{F}_{t}\right] & \leq E\left[\sup _{s \in[t, T]}\left|\widehat{\mathcal{Y}}_{s}^{\mu}\right|^{p} \mid \mathcal{F}_{t}\right] \leq c_{0} E\left[\left|g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right|^{p}+\int_{\tau_{\mu}}^{T}\left|f_{T}^{\widehat{\Theta}_{\mu}}(s, 0,0)\right|^{p} d s \mid \mathcal{F}_{t}\right] \\
& =c_{0} E\left[\left|g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right|^{p}+\int_{\tau_{\mu}}^{T}\left|f\left(s, X_{s}^{\widehat{\Theta}_{\mu}}, 0,0, \mu_{s}, \psi\left(s, \mu_{s}\right)\right)\right|^{p} d s \mid \mathcal{F}_{t}\right]
\end{aligned}
$$

Using an analogous decomposition and estimation to (4.12), we can then deduce from (2.3), (2.4) and (4.85) that

$$
E\left[\sup _{s \in\left[\tau_{\mu}, T\right]}\left|Y_{s}^{\widehat{\Theta}_{\mu}}\left(T, g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right)\right|^{p} \mid \mathcal{F}_{t}\right] \leq c_{\kappa}+c_{0} E\left[\sup _{s \in\left[\tau_{\mu}, T\right]}\left|X_{s}^{\widehat{\Theta}_{\mu}}\right|^{2} \mid \mathcal{F}_{t}\right] \leq \widetilde{C}(\kappa, x, \delta), \quad P \text {-a.s. }
$$

Let $\widehat{\eta}_{\mu}^{m} \triangleq Y_{\tau_{\mu}^{m}}^{\Theta_{\mu}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right) \vee \widetilde{\eta}_{\mu}^{m} \in \mathbb{L}^{p}\left(\mathcal{F}_{\tau_{\mu}^{m}}\right)$ and set $\widetilde{\mathcal{A}}_{\mu}^{m} \triangleq\left\{Y_{\tau_{\mu}^{m}}^{\Theta_{\mu}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)>\widetilde{\eta}_{\mu}^{m}\right\} \in \mathcal{F}_{\tau_{\mu}^{m}}$. Clearly, $\mathbf{1}_{\widetilde{\mathcal{A}}_{\mu}^{m}} \leq \mathbf{1}_{\mathcal{A}_{\mu}^{m}}, P$-a.s. Applying (2.10) with $\widetilde{p}=\frac{1+p}{2}$, we can deduce from Hölder's inequality, (4.79) and (4.84) that

$$
\begin{align*}
\mid Y_{t}^{\Theta_{\mu}^{m}} & \left(\tau_{\mu}^{m}, \widehat{\eta}_{\mu}^{m}\right)-\left.Y_{t}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}^{m}, \eta_{\mu}^{m}\right)\right|^{\widetilde{p}} \leq c_{0} E\left[\left|\widehat{\eta}_{\mu}^{m}-\eta_{\mu}^{m}\right|^{\widetilde{p}} \mid \mathcal{F}_{t}\right]=c_{0} E\left[\mathbf{1}_{\left(\widetilde{\mathcal{A}}_{\mu}^{m}\right)^{c}}\left|\widetilde{\eta}_{\mu}^{m}-\eta_{\mu}^{m}\right|^{\widetilde{p}}+\mathbf{1}_{\widetilde{\mathcal{A}}_{\mu}^{m}}\left|Y_{\tau_{\mu}^{m}}^{\Theta_{\mu}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)-\eta_{\mu}^{m}\right|^{\widetilde{p}} \mid \mathcal{F}_{t}\right] \\
& \leq c_{0} E\left[\left|X_{\tau_{\mu}^{m}}^{\Theta_{\mu}^{m}}-X_{\tau_{\mu}}^{\Theta_{\mu}}\right|^{\frac{2 \tilde{p}}{p}}\right]+c_{0}\left\{E\left[\mathbf{1}_{\widetilde{\mathcal{A}}_{\mu}^{m}} \mid \mathcal{F}_{t}\right]\right\}^{\frac{p-\widetilde{\widetilde{p}}}{p}}\left\{E\left[\left|Y_{\tau_{\mu}^{m}}^{\widehat{\Theta}_{\mu}}\left(T, g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right)-\eta_{\mu}^{m}\right|^{p} \mid \mathcal{F}_{t}\right]\right\}^{\frac{\tilde{p}}{p}} \\
& \leq c_{0}\left\{E\left[\left|X_{\tau_{\mu}^{m}}^{\Theta_{m}^{m}}-X_{\tau_{\mu}}^{\Theta_{\mu}}\right|^{2}\right]\right\}^{\frac{\tilde{p}}{p}}+c_{0}\left\{E\left[\mathbf{1}_{\mathcal{A}_{\mu}^{m}} \mid \mathcal{F}_{t}\right]\right\}^{\frac{p-\widetilde{p}}{p}}\left\{E\left[\left.\sup _{s \in\left[\tau_{\mu}, T\right]}\left|Y_{s}^{\widehat{\Theta}_{\mu}}\left(T, g\left(X_{T}^{\widehat{\Theta}_{\mu}}\right)\right)\right|^{p}+\left(C_{x, \delta}^{\widetilde{\phi}}+\frac{c_{0}}{m}\right)^{p} \right\rvert\, \mathcal{F}_{t}\right]\right\}^{\frac{\tilde{p}}{p}} \\
& \leq \frac{\widetilde{C}(\kappa, x, \delta)}{m^{\widetilde{p} / p}+\widetilde{C}(\kappa, x, \delta)\left\{E\left[\mathbf{1}_{\cup_{i=1}^{N_{m}}\left(\mathcal{A}_{i}^{m}\right)^{c}} \mid \mathcal{F}_{t}\right]\right\}^{\frac{p-\tilde{p}}{p}}, \quad P-\text { a.s. }} \tag{4.86}
\end{align*}
$$

Applying (2.11) with $(\zeta, \tau, \eta)=\left(\tau_{\mu}^{m}, T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)$, we see from Proposition 1.2 (2), (4.86) and (4.82) that $P$-a.s.

$$
\begin{align*}
& Y_{t}^{\Theta_{\mu}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)=Y_{t}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}^{m}, Y_{\tau_{\mu}^{m}}^{\Theta_{m}^{m}}\left(T, g\left(X_{T}^{\Theta_{\mu}^{m}}\right)\right)\right) \leq Y_{t}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}^{m}, \widehat{\eta}_{\mu}^{m}\right) \\
& \quad \leq Y_{t}^{\Theta_{\mu}^{m}}\left(\tau_{\mu}^{m}, \eta_{\mu}^{m}\right)+\frac{\widetilde{C}(\kappa, x, \delta)}{m^{1 / p}}+\widetilde{C}(\kappa, x, \delta)\left\{E\left[\mathbf{1}_{\cup_{i=1}^{N_{m}}\left(\mathcal{A}_{i}^{m}\right)^{c}} \mid \mathcal{F}_{t}\right]\right\}^{\frac{p-\tilde{\tilde{p}}}{p}} \\
& \quad \leq \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \widetilde{\phi}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right)+\frac{\widetilde{C}(\kappa, x, \delta)}{m^{1 / p}}+\widetilde{C}(\kappa, x, \delta)\left\{E\left[\mathbf{1}_{\cup_{i=1}^{N m}\left(\mathcal{A}_{i}^{m}\right)^{c}} \mid \mathcal{F}_{t}\right]\right\}^{\frac{p-\widetilde{p}}{p p}} \tag{4.87}
\end{align*}
$$

Letting $\widehat{\mathcal{A}}_{m} \triangleq\left\{E\left[\mathbf{1}_{\cup_{i=1}^{N_{m}}\left(\mathcal{A}_{i}^{m}\right)^{c}} \mid \mathcal{F}_{t}\right]>m^{\frac{1+p}{1-p}}\right\}$, one can deduce that

$$
P\left(\widehat{\mathcal{A}}_{m}\right) \leq m^{\frac{1+p}{p-1}} E\left[E\left[\mathbf{1}_{\cup_{i=1}^{N_{m}}\left(\mathcal{A}_{i}^{m}\right)^{c}} \mid \mathcal{F}_{t}\right]\right]=m^{\frac{1+p}{p-1}} P\left(\cup_{i=1}^{N_{m}}\left(\mathcal{A}_{i}^{m}\right)^{c}\right) \leq m^{\frac{1+p}{p-1}} \sum_{i=1}^{N_{m}} P\left(\left(\mathcal{A}_{i}^{m}\right)^{c}\right) \leq m^{-p}
$$

Multiplying $\mathbf{1}_{\widehat{\mathcal{A}}_{m}^{c}}$ to both sides of (4.87) yields that

$$
\mathbf{1}_{\widehat{\mathcal{A}}_{m}^{c}} J\left(t, x, \mu, \beta_{m}(\mu)\right) \leq \mathbf{1}_{\widehat{\mathcal{A}}_{m}^{c}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \widetilde{\phi}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right)+\frac{\widetilde{C}(\kappa, x, \delta)}{m^{1 / p}}, \quad P-\text { a.s. }
$$

Since $\widehat{\mathcal{A}}_{m}$ does not depend on $\mu$ nor on $\beta$, taking essential supremum over $\mu \in \mathcal{U}_{t}$ and applying Lemma 2.4 (2) yield that

$$
\mathbf{1}_{\widehat{\mathcal{A}}_{m}^{c}} w_{1}(t, x) \leq \mathbf{1}_{\widehat{\mathcal{A}}_{m}^{c}} I\left(t, x, \beta_{m}\right) \leq \mathbf{1}_{\widehat{\mathcal{A}}_{m}^{c}} \operatorname{esssup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \widetilde{\phi}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right)+\frac{\widetilde{C}(\kappa, x, \delta)}{m^{1 / p}}, \quad P-\text { a.s. }
$$

Then taking essential infimum over $\beta \in \mathfrak{B}_{t}$ and using Lemma 2.4 (2) again, we obtain

$$
\begin{equation*}
\mathbf{1}_{\widehat{\mathcal{A}}_{m}^{c}} w_{1}(t, x) \leq \mathbf{1}_{\widehat{\mathcal{A}}_{m}^{c}} \operatorname{exsinf}_{\beta \in \mathfrak{B}_{t}}^{\operatorname{essinsup}} \operatorname{essup}_{\mu \in \mathcal{U}_{t}} Y_{t}^{t, x, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \widetilde{\phi}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t, x, \mu, \beta(\mu)}\right)\right)+\frac{\widetilde{C}(\kappa, x, \delta)}{m^{1 / p}}, \quad P-\text { a.s. } \tag{4.88}
\end{equation*}
$$

As $\sum_{m \in \mathbb{N}} P\left(\widehat{\mathcal{A}}_{m}\right) \leq \sum_{m \in \mathbb{N}} m^{-p}<\infty$, similar to (4.62), Borel-Cantelli theorem implies that $\lim _{m \rightarrow \infty} \mathbf{1}_{\widehat{\mathcal{A}}_{m}}=0, P$-a.s. Thus, letting $m \rightarrow \infty$ in (4.88) yields (4.76).
2) For any $(t, x, y, z, u, v) \in[0, T] \times \mathbb{R}^{k} \times \mathbb{R} \times \mathbb{R}^{d} \times \mathbb{U} \times \mathbb{V}$, we define

$$
\mathfrak{g}(x) \triangleq-g(x) \quad \text { and } \quad \mathfrak{f}(t, x, y, z, u, v) \triangleq-f(t, x,-y,-z, u, v)
$$

Given $(\mu, \nu) \in \mathcal{U}_{t} \times \mathcal{V}_{t}$, we let $\Theta$ stand for $(t, x, \mu, \nu)$. For any $\tau \in \mathcal{S}_{t, T}$ and any $\eta \in \mathbb{L}^{p}\left(\mathcal{F}_{\tau}\right)$, let $\left(\mathcal{Y}^{\Theta}(\tau, \eta), \mathcal{Z}^{\Theta}(\tau, \eta)\right)$ denote the unique solution of the $\operatorname{BSDE}\left(t, \eta, \mathfrak{f}_{\tau}^{\Theta}\right)$ in $\mathbb{G}_{\mathbf{F}}^{q}([t, T])$, where

$$
\mathfrak{f}_{\tau}^{\Theta}(s, \omega, y, z) \triangleq \mathbf{1}_{\{s<\tau(\omega)\}} \mathfrak{f}\left(s, X_{s}^{\Theta}(\omega), y, z, \mu_{s}(\omega), \nu_{s}(\omega)\right), \quad \forall(s, \omega, y, z) \in[t, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{d}
$$

Multiplying -1 in the $\operatorname{BSDE}\left(t, \eta, \mathfrak{f}_{\tau}^{\Theta}\right)$ shows that $\left(-\mathcal{Y}^{\Theta}(\tau, \eta),-\mathcal{Z}^{\Theta}(\tau, \eta)\right) \in \mathbb{G}_{\mathbf{F}}^{q}([t, T])$ solves the $\operatorname{BSDE}\left(t,-\eta, f_{\tau}^{\Theta}\right)$. To wit

$$
\begin{equation*}
\left(-\mathcal{Y}^{\Theta}(\tau, \eta),-\mathcal{Z}^{\Theta}(\tau, \eta)\right)=\left(Y^{\Theta}(\tau,-\eta), Z^{\Theta}(\tau,-\eta)\right) \tag{4.89}
\end{equation*}
$$

Given $(t, x) \in[0, T] \times \mathbb{R}^{k}$, let us consider the situation where player II acts first by choosing a $\mathcal{V}_{t}-$ control to maximize $\mathcal{Y}_{t}^{t, x, \alpha(\nu), \nu}\left(T, \mathfrak{g}\left(X_{T}^{t, x, \alpha(\nu), \nu}\right)\right)$, where $\alpha \in \mathfrak{A}_{t}$ is player I's strategic response. The corresponding priority value of player II is $\mathfrak{w}_{2}(t, x) \triangleq \underset{\alpha \in \mathfrak{R}_{t}}{\operatorname{essinf}} \operatorname{esssup}_{\nu \in \mathcal{V}_{t}} \mathcal{Y}_{t}^{t, x, \alpha(\nu), \nu}\left(T, \mathfrak{g}\left(X_{T}^{t, x, \alpha(\nu), \nu}\right)\right)$. We see from (4.89) that

$$
-\mathfrak{w}_{2}(t, x)=\underset{\alpha \in \mathfrak{A}_{t}}{\operatorname{esssup}} \operatorname{essinf}_{\nu \in \mathcal{V}_{t}}-\mathcal{Y}_{t}^{t, x, \alpha(\nu), \nu}\left(T, \mathfrak{g}\left(X_{T}^{t, x, \alpha(\nu), \nu}\right)\right)=\underset{\alpha \in \mathfrak{A}_{t}}{\operatorname{esssup}} \operatorname{essinf}_{\nu \in \mathcal{V}_{t}} Y_{t}^{t, x, \alpha(\nu), \nu}\left(T, g\left(X_{T}^{t, x, \alpha(\nu), \nu}\right)\right)=w_{2}(t, x)
$$

Let $t \in(0, T]$ and let $\phi, \widetilde{\phi}:[t, T] \times \mathbb{R}^{k} \rightarrow \mathbb{R}$ be two continuous functions satisfying $\phi(s, x) \leq w_{2}(s, x) \leq \widetilde{\phi}(s, x)$, $(s, x) \in[t, T] \times \mathbb{R}^{k}$. As $-\widetilde{\phi}(s, x) \leq \mathfrak{w}_{2}(s, x) \leq-\phi(s, x),(s, x) \in[t, T] \times \mathbb{R}^{k}$, applying the weak dynamic programming principle of part (1) yields that for any $x \in \mathbb{R}^{k}$ and $\delta \in(0, T-t]$

$$
\begin{aligned}
& \underset{\alpha \in \mathfrak{A}_{t}}{\operatorname{essinf}} \underset{\nu \in \mathcal{V}_{t}}{\operatorname{esssup}} \mathcal{Y}_{t}^{t, x, \alpha(\nu), \nu}\left(\tau_{\alpha, \nu},-\widetilde{\phi}\left(\tau_{\alpha, \nu}, X_{\tau_{\alpha, \nu}}^{t, x, \alpha(\nu), \nu}\right)\right) \\
& \quad \leq \mathfrak{w}_{2}(t, x) \leq \underset{\alpha \in \mathfrak{A}_{t}}{\operatorname{essinf}} \underset{\nu \in \mathcal{V}_{t}}{\operatorname{esssup}} \mathcal{Y}_{t}^{t, x, \alpha(\nu), \nu}\left(\tau_{\alpha, \nu},-\phi\left(\tau_{\alpha, \nu}, X_{\tau_{\alpha, \nu}}^{t, x, \alpha(\nu), \nu}\right)\right), \quad P-\text { a.s. }
\end{aligned}
$$

Multiplying -1 above and using (4.89), we obtain the weak dynamic programming principle for $w_{2}$.

### 4.4 Proofs of Section 3

We will prove that $\underline{w}_{i}$ and $\bar{w}_{i}, i=1,2$ are viscosity solutions of (3.1) in a standard way: Assume oppositely that the corresponding inequality of (3.1) does not hold for some test function $\varphi$. We decompose $\underline{H}_{i}$ or $\bar{H}_{i}$ with $\varphi$ in the reverse inequality until we reach a similar reverse inequality satisfied by a control $\widehat{\mu}$ or a strategy $\widehat{\beta}$. Then applying the comparison result of BSDE, i.e. Proposition 1.2 (2), to such an inequality leads to a contradiction to the weak dynamic programming principle.
Proof of Theorem 3.1; We only need to prove for $\underline{w}_{1}$ and $\bar{w}_{1}$, then the results of $\bar{w}_{2}$ and $\underline{w}_{2}$ follow by a similar transformation to that used in the proof of Theorem 2.1, part (2).
a) We first show that $\underline{w}_{1}$ is a viscosity supersolution of (3.1) with Hamiltonian $\underline{H}_{1}$. Let $\left(t_{0}, x_{0}, \varphi\right) \in(0, T) \times \mathbb{R}^{k} \times$ $\mathbb{C}^{1,2}\left([0, T] \times \mathbb{R}^{k}\right)$ be such that $\underline{w}_{1}\left(t_{0}, x_{0}\right)=\varphi\left(t_{0}, x_{0}\right)$ and that $\underline{w}_{1}-\varphi$ attains a strict local minimum at $\left(t_{0}, x_{0}\right)$, i.e., for some $\delta_{0} \in\left(0, t_{0} \wedge\left(T-t_{0}\right)\right)$

$$
\begin{equation*}
\left(\underline{w}_{1}-\varphi\right)(t, x)>\left(\underline{w}_{1}-\varphi\right)\left(t_{0}, x_{0}\right)=0, \quad \forall(t, x) \in O_{\delta_{0}}\left(t_{0}, x_{0}\right) \backslash\left\{\left(t_{0}, x_{0}\right)\right\} . \tag{4.90}
\end{equation*}
$$

We simply denote $\left(\varphi\left(t_{0}, x_{0}\right), D_{x} \varphi\left(t_{0}, x_{0}\right), D_{x}^{2} \varphi\left(t_{0}, x_{0}\right)\right)$ by $\left(y_{0}, z_{0}, \Gamma_{0}\right)$. If $\underline{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right)=-\infty$, then

$$
-\frac{\partial}{\partial t} \varphi\left(t_{0}, x_{0}\right)-\underline{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right) \geq 0
$$

holds automatically. To make a contradiction, we assume that when $\underline{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right)>-\infty$,

$$
\begin{equation*}
\varrho \triangleq \frac{\partial}{\partial t} \varphi\left(t_{0}, x_{0}\right)+\underline{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right)>0 \tag{4.91}
\end{equation*}
$$

For any $(t, x, y, z, \Gamma, u, v) \in[0, T] \times \mathbb{R}^{k} \times \mathbb{R} \times \mathbb{R}^{d} \times \mathbb{S}_{k} \times \mathbb{U} \times \mathbb{V}$, one can deduce from (2.1) - (2.4) that

$$
\begin{align*}
& |H(t, x, y, z, \Gamma, u, v)| \leq \frac{1}{4}\left|\sigma \sigma^{T}(t, x, u, v)\right|^{2}+\frac{1}{4}|\Gamma|^{2}+\gamma|z||b(t, x, u, v)|+\gamma\left(1+|x|^{2 / p}+|y|+\gamma|z||\sigma(t, x, u, v)|+[u]_{\mathbb{U}}^{2 / p}+[v]_{\mathbb{V}}^{2 / p}\right) \\
& \quad \leq \frac{1}{4} \gamma^{2}\left(1+|x|+[u]_{\mathbb{U}}+[v]_{\mathbb{V}}\right)^{2}+\frac{1}{4}|\Gamma|^{2}+\left(\gamma+\gamma^{2}\right)|z|\left(1+|x|+[u]_{\mathbb{U}}+[v]_{\mathbb{V}}\right)+\gamma\left(1+|x|^{2 / p}+|y|+[u]_{\mathbb{U}}^{2 / p}+[v]_{\mathbb{V}}^{2 / p}\right) \tag{4.92}
\end{align*}
$$

Set $C_{\varphi}^{0} \triangleq\left|y_{0}\right|+\left|z_{0}\right|+\left|\Gamma_{0}\right|=\left|\varphi\left(t_{0}, x_{0}\right)\right|+\left|D_{x} \varphi\left(t_{0}, x_{0}\right)\right|+\left|D_{x}^{2} \varphi\left(t_{0}, x_{0}\right)\right|$, and fix a $u_{\sharp} \in \partial O_{\kappa}\left(u_{0}\right)$. For any $u \notin O_{\kappa}\left(u_{0}\right)$, we see from (A-u) that $\psi\left(t_{0}, u\right) \in \mathscr{O}_{u}$, and it follows from (4.92) that

$$
\begin{align*}
& \inf _{v \in \mathscr{O}_{u}} H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u, v\right) \leq\left|H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u, \psi\left(t_{0}, u\right)\right)\right| \\
& \quad=\left|H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u_{\sharp}, \psi\left(t_{0}, u_{\sharp}\right)\right)\right| \leq \frac{1}{4}\left(C_{\varphi}^{0}\right)^{2}+C_{\varphi}^{0} C\left(\kappa, x_{0}\right)+C\left(\kappa, x_{0}\right) . \tag{4.93}
\end{align*}
$$

Here $C\left(\kappa, x_{0}\right)$ denotes a generic constant, depending on $\kappa,\left|x_{0}\right|, T, \gamma, p$ and $|g(0)|$, whose form may vary from line to line.

Similarly, it holds for any $u \in O_{\kappa}\left(u_{0}\right)$ that

$$
\inf _{v \in \mathscr{O}_{u}} H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u, v\right) \leq\left|H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u, v_{0}\right)\right| \leq \frac{1}{4}\left(C_{\varphi}^{0}\right)^{2}+C_{\varphi}^{0} C\left(\kappa, x_{0}\right)+C\left(\kappa, x_{0}\right)
$$

which together with (4.93) implies that

$$
\underline{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right) \leq \sup _{u \in \mathbb{U}} \inf _{v \in \mathscr{O}_{u}} H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u, v\right) \leq \frac{1}{4}\left(C_{\varphi}^{0}\right)^{2}+C_{\varphi}^{0} C\left(\kappa, x_{0}\right)+C\left(\kappa, x_{0}\right)<\infty .
$$

Thus $\varrho<\infty$.
As $\varphi \in \mathbb{C}^{1,2}\left([0, T] \times \mathbb{R}^{k}\right)$, we see from (4.91) that for some $\widehat{u} \in \mathbb{U}$

$$
\underset{(t, x) \rightarrow\left(t_{0}, x_{0}\right)}{ } \inf _{v \in \mathscr{O}_{\widehat{u}}}^{\lim } H\left(t, x, \varphi(t, x), D_{x} \varphi(t, x), D_{x}^{2} \varphi(t, x), \widehat{u}, v\right) \geq \frac{3}{4} \varrho-\frac{\partial}{\partial t} \varphi\left(t_{0}, x_{0}\right)
$$

Moreover, there exists a $\delta \in\left(0, \delta_{0}\right)$ such that

$$
\begin{equation*}
\inf _{v \in \mathscr{O}_{\widehat{u}}} H\left(t, x, \varphi(t, x), D_{x} \varphi(t, x), D_{x}^{2} \varphi(t, x), \widehat{u}, v\right) \geq \frac{1}{2} \varrho-\frac{\partial}{\partial t} \varphi(t, x), \quad \forall(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right) \tag{4.94}
\end{equation*}
$$

Let $\wp \triangleq \inf \left\{\left(\underline{w}_{1}-\varphi\right)(t, x):(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right) \backslash O_{\frac{\delta}{3}}\left(t_{0}, x_{0}\right)\right\}$. Since the set $\bar{O}_{\delta}\left(t_{0}, x_{0}\right) \backslash O_{\frac{\delta}{3}}\left(t_{0}, x_{0}\right)$ is compact, there exists a sequence $\left\{\left(t_{n}, x_{n}\right)\right\}_{n \in \mathbb{N}}$ on $\bar{O}_{\delta}\left(t_{0}, x_{0}\right) \backslash O_{\frac{\delta}{3}}\left(t_{0}, x_{0}\right)$ that converges to some $\left(t_{*}, x_{*}\right) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right) \backslash O_{\frac{\delta}{3}}\left(t_{0}, x_{0}\right)$ and satisfies $\wp=\lim _{n \rightarrow \infty} \downarrow\left(\underline{w}_{1}-\varphi\right)\left(t_{n}, x_{n}\right)$. The lower semicontinuity of $\underline{w}_{1}$ and the continuity of $\varphi$ imply that $\underline{w}_{1}-\varphi$ is also lower semicontinuous. It follows that $\wp \leq\left(\underline{w}_{1}-\varphi\right)\left(t_{*}, x_{*}\right) \leq \lim _{n \rightarrow \infty} \downarrow\left(\underline{w}_{1}-\varphi\right)\left(t_{n}, x_{n}\right)=\wp$, which together with (4.90) shows that

$$
\begin{equation*}
\wp=\min \left\{\left(\underline{w}_{1}-\varphi\right)(t, x):(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right) \backslash O_{\frac{\delta}{3}}\left(t_{0}, x_{0}\right)\right\}=\left(\underline{w}_{1}-\varphi\right)\left(t_{*}, x_{*}\right)>0 . \tag{4.95}
\end{equation*}
$$

Then we set $\widetilde{\wp} \triangleq \frac{\wp \wedge \varrho}{2(1 \vee \gamma) T}>0$ and let $\left\{\left(t_{j}, x_{j}\right)\right\}_{j \in \mathbb{N}}$ be a sequence of $O_{\frac{\delta}{6}}\left(t_{0}, x_{0}\right)$ such that

$$
\lim _{j \rightarrow \infty}\left(t_{j}, x_{j}\right)=\left(t_{0}, x_{0}\right) \quad \text { and } \quad \lim _{j \rightarrow \infty} w_{1}\left(t_{j}, x_{j}\right)=\underline{w}_{1}\left(t_{0}, x_{0}\right)=\varphi\left(t_{0}, x_{0}\right)=\lim _{j \rightarrow \infty} \varphi\left(t_{j}, x_{j}\right)
$$

So one can find a $j \in \mathbb{N}$ such that

$$
\begin{equation*}
\left|w_{1}\left(t_{j}, x_{j}\right)-\varphi\left(t_{j}, x_{j}\right)\right|<\frac{5}{6} \widetilde{\wp} t_{0} . \tag{4.96}
\end{equation*}
$$

Clearly, $\widehat{\mu}_{s} \triangleq \widehat{u}, s \in\left[t_{j}, T\right]$ is a constant $\mathcal{U}_{t_{j}}$-process. Fix $\beta \in \mathfrak{B}_{t_{j}}$. We set $\Theta \triangleq\left(t_{j}, x_{j}, \widehat{\mu}, \beta(\widehat{\mu})\right)$ and define

$$
\tau=\tau_{\beta, \widehat{\mu}} \triangleq \inf \left\{s \in\left(t_{j}, T\right]:\left(s, X_{s}^{\Theta}\right) \notin O_{\frac{2}{3} \delta}\left(t_{j}, x_{j}\right)\right\} \in \mathcal{S}_{t_{j}, T}
$$

Since $\left|\left(T, X_{T}^{\Theta}\right)-\left(t_{j}, x_{j}\right)\right| \geq T-t_{j} \geq T-t_{0}-\left|t_{j}-t_{0}\right|>\delta_{0}-\frac{\delta}{6}>\frac{5}{6} \delta>\frac{2}{3} \delta$, the continuity of $X^{\Theta}$ implies that $P-a . s$.

$$
\begin{equation*}
\tau<T \text { and }\left(\tau \wedge s, X_{\tau \wedge s}^{\Theta}\right) \in \bar{O}_{\frac{2}{3} \delta}\left(t_{j}, x_{j}\right) \subset \bar{O}_{\frac{5}{6} \delta}\left(t_{0}, x_{0}\right), \quad \forall s \in\left[t_{j}, T\right] \tag{4.97}
\end{equation*}
$$

$$
\begin{equation*}
\text { in particular, } \quad\left(\tau, X_{\tau}^{\Theta}\right) \in \partial O_{\frac{2}{3} \delta}\left(t_{j}, x_{j}\right) \subset \bar{O}_{\frac{5}{6} \delta}\left(t_{0}, x_{0}\right) \backslash O_{\frac{\delta}{2}}\left(t_{0}, x_{0}\right) \tag{4.98}
\end{equation*}
$$

The continuity of $\varphi, X^{\Theta}$ and (4.97) show that $\mathcal{Y}_{s} \triangleq \varphi\left(\tau \wedge s, X_{\tau \wedge s}^{\Theta}\right)+\widetilde{\wp}(\tau \wedge s), s \in\left[t_{j}, T\right]$ defines a bounded $\mathbf{F}$-adapted continuous process. By Itô's formula,

$$
\begin{equation*}
\mathcal{Y}_{s}=\mathcal{Y}_{T}+\int_{s}^{T} \mathfrak{f}_{r} d r-\int_{s}^{T} \mathcal{Z}_{r} d B_{r}, \quad s \in\left[t_{j}, T\right] \tag{4.99}
\end{equation*}
$$

where $\mathcal{Z}_{r}=\mathbf{1}_{\{r<\tau\}} D_{x} \varphi\left(r, X_{r}^{\Theta}\right) \cdot \sigma\left(r, X_{r}^{\Theta}, \widehat{u},(\beta(\widehat{\mu}))_{r}\right)$ and
$\mathfrak{f}_{r}=-\mathbf{1}_{\{r<\tau\}}\left\{\widetilde{\wp}+\frac{\partial \varphi}{\partial t}\left(r, X_{r}^{\Theta}\right)+D_{x} \varphi\left(r, X_{r}^{\Theta}\right) \cdot b\left(r, X_{r}^{\Theta}, \widehat{u},(\beta(\widehat{\mu}))_{r}\right)+\frac{1}{2} \operatorname{trace}\left(\sigma \sigma^{T}\left(r, X_{r}^{\Theta}, \widehat{u},(\beta(\widehat{\mu}))_{r}\right) \cdot D_{x}^{2} \varphi\left(r, X_{r}^{\Theta}\right)\right)\right\}$.

As $\varphi \in \mathbb{C}^{1,2}\left([t, T] \times \mathbb{R}^{k}\right)$, the measurability of $b, \sigma, X^{\Theta}, \widehat{u}$ and $\beta(\widehat{\mu})$ implies that both $\mathcal{Z}$ and $\mathfrak{f}$ are $\mathbf{F}$-progressively measurable. And one can deduce from (2.1), (2.2), (4.97) and Hölder's inequality that

$$
\begin{align*}
& E\left[\left(\int_{t_{j}}^{T}\left|\mathcal{Z}_{s}\right|^{2} d s\right)^{p / 2}\right] \leq\left(\gamma \widetilde{C}_{\varphi}\right)^{p} E\left[\left(\int_{t_{j}}^{\tau}\left(1+\left|X_{s}^{\Theta}\right|+[\widehat{u}]_{\mathbb{U}}+\left[(\beta(\widehat{\mu}))_{s}\right]_{\mathbb{V}}\right)^{2} d s\right)^{p / 2}\right] \\
& \quad \leq c_{0} \widetilde{C}_{\varphi}^{p}\left(\left(1+\left|x_{0}\right|+\delta+[\widehat{u}]_{\mathbb{U}}\right)^{p}+\left\{E \int_{t_{j}}^{T}\left[(\beta(\widehat{\mu}))_{s}\right]_{\mathbb{V}}^{2} d s\right\}^{p / 2}\right)<\infty, \quad \text { i.e. } \mathcal{Z} \in \mathbb{H}_{\mathbf{F}}^{2, p}\left(\left[t_{j}, T\right], \mathbb{R}^{d}\right), \tag{4.100}
\end{align*}
$$

where $\widetilde{C}_{\varphi} \triangleq \sup _{(t, x) \in \bar{O}_{\frac{5}{6} \delta}\left(t_{0}, x_{0}\right)}\left|D_{x} \varphi(t, x)\right|<\infty$. Hence, $\left\{\left(\mathcal{Y}_{s}, \mathcal{Z}_{s}\right)\right\}_{s \in\left[t_{j}, T\right]}$ solves the $\operatorname{BSDE}\left(t_{j}, \mathcal{Y}_{T}, \mathfrak{f}\right)$.
Let $\ell(x)=c_{\kappa}+c_{0}|x|^{2 / p}, x \in \mathbb{R}^{k}$ be the function appeared in Proposition 2.1. Let $\theta_{1}:[0, T] \times \mathbb{R}^{k} \rightarrow[0,1]$ be a continuous function such that $\theta_{1} \equiv 0$ on $\bar{O}_{\frac{5}{6} \delta}\left(t_{0}, x_{0}\right)$ and $\theta_{1} \equiv 1$ on $\left([0, T] \times \mathbb{R}^{k}\right) \backslash O_{\delta}\left(t_{0}, x_{0}\right)$. Also, let $\theta_{2}:[0, T] \times \mathbb{R}^{k} \rightarrow$ $[0,1]$ be another continuous function such that $\theta_{2} \equiv 0$ on $\bar{O}_{\frac{\delta}{3}}\left(t_{0}, x_{0}\right)$ and $\theta_{2} \equiv 1$ on $\left([0, T] \times \mathbb{R}^{k}\right) \backslash O_{\frac{\delta}{2}}\left(t_{0}, x_{0}\right)$. Define

$$
\begin{equation*}
\phi(t, x) \triangleq-\theta_{1}(t, x) \ell(x)+\left(1-\theta_{1}(t, x)\right)\left(\varphi(t, x)+\wp \theta_{2}(t, x)\right), \quad \forall(t, x) \in\left[t_{j}, T\right] \times \mathbb{R}^{k} \tag{4.101}
\end{equation*}
$$

which is a continuous function satisfying $\phi \leq w_{1}$ : given $(t, x) \in\left[t_{j}, T\right] \times \mathbb{R}^{k}$,

- if $(t, x) \in \bar{O}_{\frac{\delta}{3}}\left(t_{0}, x_{0}\right)$, (4.90) shows that $\phi(t, x)=\varphi(t, x) \leq \underline{w}_{1}(t, x) \leq w_{1}(t, x)$;
- if $(t, x) \in O_{\delta}\left(t_{0}, x_{0}\right) \backslash \bar{O}_{\frac{\delta}{3}}\left(t_{0}, x_{0}\right)$, since $\varphi(t, x)+\wp \theta_{2}(t, x) \leq \varphi(t, x)+\wp \leq \underline{w}_{1}(t, x) \leq w_{1}(t, x)$ by (4.95), one can deduce from Proposition 2.1 that $\phi(t, x) \leq w_{1}(t, x)$;
- if $(t, x) \notin O_{\delta}\left(t_{0}, x_{0}\right), \phi(t, x)=-\ell(x) \leq w_{1}(t, x)$.

Then we can deduce from (4.98) that

$$
\begin{equation*}
\mathcal{Y}_{T}=\varphi\left(\tau, X_{\tau}^{\Theta}\right)+\widetilde{\wp} T<\varphi\left(\tau, X_{\tau}^{\Theta}\right)+\wp=\phi\left(\tau, X_{\tau}^{\Theta}\right), \quad P-\mathrm{a.s.} \tag{4.102}
\end{equation*}
$$

Since it holds $d s \times d P$-a.s. on $\left[t_{j}, T\right] \times \Omega$ that $\left[(\beta(\widehat{\mu}))_{s}\right]_{\mathbb{V}} \leq \kappa+C_{\beta}\left[\widehat{\mu}_{s}\right]_{\mathbb{U}}=\kappa+C_{\beta}[\widehat{u}]_{\mathbb{U}} \in \mathscr{O}_{\widehat{u}}$, (4.97), (4.94) and (2.4) imply that for $d s \times d P-$ a.s. $(s, \omega) \in\left[t_{j}, T\right] \times \Omega$

$$
\begin{align*}
\mathfrak{f}_{s}(\omega) & \leq \mathbf{1}_{\{s<\tau(\omega)\}}\left\{-\widetilde{\wp}-\frac{1}{2} \varrho+f\left(s, \omega, X_{s}^{\Theta}(\omega), \mathcal{Y}_{s}(\omega)-\widetilde{\wp} s, \mathcal{Z}_{s}(\omega), \widehat{u},(\beta(\widehat{\mu}))_{s}(\omega)\right)\right\} \\
& \leq \mathbf{1}_{\{s<\tau(\omega)\}}\left\{-\widetilde{\wp}-\frac{1}{2} \varrho+\gamma \widetilde{\wp} T+f\left(s, \omega, X_{s}^{\Theta}(\omega), \mathcal{Y}_{s}(\omega), \mathcal{Z}_{s}(\omega), \widehat{u},(\beta(\widehat{\mu}))_{s}(\omega)\right)\right\} \leq f_{\tau}^{\Theta}\left(s, \omega, \mathcal{Y}_{s}(\omega), \mathcal{Z}_{s}(\omega)\right) . \tag{4.103}
\end{align*}
$$

As $f_{\tau}^{\Theta}$ is Lipschitz continuous in $(y, z)$, Proposition 1.2 (2) implies that $P$-a.s.

$$
\mathcal{Y}_{s} \leq Y_{s}^{\Theta}\left(\tau, \phi\left(\tau, X_{\tau}^{\Theta}\right)\right), \quad \forall s \in\left[t_{j}, T\right]
$$

Letting $s=t_{j}$ and using the fact that $t_{j}>t_{0}-\frac{1}{6} \delta>t_{0}-\frac{1}{6} \delta_{0}>\frac{5}{6} t_{0}$, we obtain

$$
\begin{aligned}
\varphi\left(t_{j}, x_{j}\right)+\frac{5}{6} \widetilde{\wp} t_{0} & <\varphi\left(t_{j}, x_{j}\right)+\widetilde{\wp} t_{j}=\mathcal{Y}_{t_{j}} \leq Y_{t_{j}}^{t_{j}, x_{j}, \widehat{\mu}, \beta(\widehat{\mu})}\left(\tau, \phi\left(\tau, X_{\tau}^{t_{j}, x_{j}, \widehat{\mu}, \beta(\widehat{\mu})}\right)\right) \\
& \leq \operatorname{esssup}_{\mu \in \mathcal{U}_{j}} Y_{t_{j}}^{t_{j}, x_{j}, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \phi\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t_{j}, x_{j}, \mu, \beta(\mu)}\right)\right), \quad P-\text { a.s. },
\end{aligned}
$$

where $\tau_{\beta, \mu} \triangleq \inf \left\{s \in\left(t_{j}, T\right]:\left(s, X_{s}^{t_{j}, x_{j}, \mu, \beta(\mu)}\right) \notin O_{\frac{2}{3} \delta}\left(t_{j}, x_{j}\right)\right\}, \forall \mu \in \mathcal{U}_{t_{j}}$. Taking essential infimum over $\beta \in \mathfrak{B}_{t_{j}}$ and applying Theorem 2.1] with $(t, x, \delta)=\left(t_{j}, x_{j}, \frac{2}{3} \delta\right)$, we see from (4.96) that $P$-a.s.

$$
\varphi\left(t_{j}, x_{j}\right)+\frac{5}{6} \widetilde{\wp} t_{0} \leq \operatorname{essinf}_{\beta \in \mathfrak{B}_{t_{j}}}^{\operatorname{essssu}} \exp _{t_{j}} Y_{t_{j}}^{t_{j}, x_{j}, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \phi\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t_{j}, x_{j}, \mu, \beta(\mu)}\right)\right) \leq w_{1}\left(t_{j}, x_{j}\right)<\varphi\left(t_{j}, x_{j}\right)+\frac{5}{6} \widetilde{\wp} t_{0} .
$$

A contradiction appears. Therefore, $\underline{w}_{1}$ is a viscosity supersolution of (3.1) with Hamiltonian $\underline{H}_{1}$.
b) Next, we show that $\bar{w}_{1}$ is a viscosity subsolution of (3.1) with Hamiltonian $\bar{H}_{1}$. Let $\left(t_{0}, x_{0}, \varphi\right) \in(0, T) \times \mathbb{R}^{k} \times$ $\mathbb{C}^{1,2}\left([0, T] \times \mathbb{R}^{k}\right)$ be such that $\bar{w}_{1}\left(t_{0}, x_{0}\right)=\varphi\left(t_{0}, x_{0}\right)$ and that $\bar{w}_{1}-\varphi$ attains a strict local maximum at $\left(t_{0}, x_{0}\right)$, i.e., for some $\delta_{0} \in\left(0, t_{0} \wedge\left(T-t_{0}\right)\right)$

$$
\left(\bar{w}_{1}-\varphi\right)(t, x)<\left(\bar{w}_{1}-\varphi\right)\left(t_{0}, x_{0}\right)=0, \quad \forall(t, x) \in O_{\delta_{0}}\left(t_{0}, x_{0}\right) \backslash\left\{\left(t_{0}, x_{0}\right)\right\}
$$

We still denote $\left(\varphi\left(t_{0}, x_{0}\right), D_{x} \varphi\left(t_{0}, x_{0}\right), D_{x}^{2} \varphi\left(t_{0}, x_{0}\right)\right)$ by $\left(y_{0}, z_{0}, \Gamma_{0}\right)$. If $\bar{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right)=\infty$, then

$$
-\frac{\partial}{\partial t} \varphi\left(t_{0}, x_{0}\right)-\bar{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right) \leq 0
$$

holds automatically. To make a contradiction, we assume that when $\bar{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right)<\infty$,

$$
\begin{equation*}
\varrho \triangleq-\frac{\partial}{\partial t} \varphi\left(t_{0}, x_{0}\right)-\bar{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right)>0 \tag{4.104}
\end{equation*}
$$

It is easy to see that

$$
\begin{align*}
\bar{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right) & \geq \lim _{n \rightarrow \infty} \downarrow \sup _{u \in \mathbb{U}} \inf _{v \in \mathscr{O}_{u}^{n}} H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u, v\right) \geq \lim _{n \rightarrow \infty} \downarrow \inf _{v \in \mathscr{O}_{u_{0}}^{n}} H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u_{0}, v\right) \\
& =\inf _{v \in \bar{O}_{k}\left(v_{0}\right)} H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u_{0}, v\right) . \tag{4.105}
\end{align*}
$$

For any $v \in \bar{O}_{\kappa}\left(v_{0}\right)$, one can deduce from (4.92) that $\left|H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u_{0}, v\right)\right| \leq \frac{1}{4}\left(C_{\varphi}^{0}\right)^{2}+C_{\varphi}^{0} C\left(\kappa, x_{0}\right)+C\left(\kappa, x_{0}\right)$, where $C_{\varphi}^{0}=\left|\varphi\left(t_{0}, x_{0}\right)\right|+\left|D_{x} \varphi\left(t_{0}, x_{0}\right)\right|+\left|D_{x}^{2} \varphi\left(t_{0}, x_{0}\right)\right|$ as set in part (a). It then follows from (4.105) that

$$
\bar{H}_{1}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right) \geq \inf _{v \in \bar{O}_{\kappa}\left(v_{0}\right)} H\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}, u_{0}, v\right) \geq-\frac{1}{4}\left(C_{\varphi}^{0}\right)^{2}-C_{\varphi}^{0} C\left(\kappa, x_{0}\right)-C\left(\kappa, x_{0}\right)>-\infty
$$

Thus $\varrho<\infty$.
Then one can find an $m \in \mathbb{N}$ such that

$$
\begin{equation*}
-\frac{\partial \varphi}{\partial t}\left(t_{0}, x_{0}\right)-\frac{7}{8} \varrho \geq \sup _{u \in \mathbb{U}} \inf _{v \in \mathscr{O}_{u}^{m}} \varlimsup_{u^{\prime} \rightarrow u} \sup _{(t, x, y, z, \Gamma) \in O_{1 / m}\left(t_{0}, x_{0}, y_{0}, z_{0}, \Gamma_{0}\right)} H\left(t, x, y, z, \Gamma, u^{\prime}, v\right) \tag{4.106}
\end{equation*}
$$

As $\varphi \in \mathbb{C}^{1,2}\left([0, T] \times \mathbb{R}^{k}\right)$, there exists a $\delta<\frac{1}{2 m} \wedge \delta_{0}$ such that for any $(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right)$

$$
\begin{equation*}
\left|\frac{\partial \varphi}{\partial t}(t, x)-\frac{\partial \varphi}{\partial t}\left(t_{0}, x_{0}\right)\right| \leq \frac{1}{8} \varrho \tag{4.107}
\end{equation*}
$$

$$
\text { and } \quad\left|\varphi(t, x)-\varphi\left(t_{0}, x_{0}\right)\right| \vee\left|D_{x} \varphi(t, x)-D_{x} \varphi\left(t_{0}, x_{0}\right)\right| \vee\left|D_{x}^{2} \varphi(t, x)-D_{x}^{2} \varphi\left(t_{0}, x_{0}\right)\right| \leq \frac{1}{2 m}
$$

the latter of which together with (4.106) implies that

$$
-\frac{\partial \varphi}{\partial t}\left(t_{0}, x_{0}\right)-\frac{7}{8} \varrho \geq \sup _{u \in \mathbb{U}} \inf _{v \in \mathscr{O}_{u}^{m}} \varlimsup_{u^{\prime} \rightarrow u} \sup _{(t, x) \in \bar{\sigma}_{\delta}\left(t_{0}, x_{0}\right)} H\left(t, x, \varphi(t, x), D_{x} \varphi(t, x), D_{x}^{2} \varphi(t, x), u^{\prime}, v\right)
$$

Then for any $u \in \mathbb{U}$, there exists a $\mathfrak{P}_{o}(u) \in \mathscr{O}_{u}^{m}$ such that

$$
-\frac{\partial \varphi}{\partial t}\left(t_{0}, x_{0}\right)-\frac{3}{4} \varrho \geq \varlimsup_{u^{\prime} \rightarrow u} \sup _{(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right)} H\left(t, x, \varphi(t, x), D_{x} \varphi(t, x), D_{x}^{2} \varphi(t, x), u^{\prime}, \mathfrak{P}_{o}(u)\right)
$$

and we can find a $\lambda(u) \in(0,1)$ such that for any $u^{\prime} \in O_{\lambda(u)}(u)$

$$
\begin{equation*}
-\frac{\partial \varphi}{\partial t}\left(t_{0}, x_{0}\right)-\frac{5}{8} \varrho \geq \sup _{(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right)} H\left(t, x, \varphi(t, x), D_{x} \varphi(t, x), D_{x}^{2} \varphi(t, x), u^{\prime}, \mathfrak{P}_{o}(u)\right) \tag{4.108}
\end{equation*}
$$

Set $\widetilde{\lambda}\left(u_{0}\right)=\lambda\left(u_{0}\right)$ and $\widetilde{\lambda}(u)=\lambda(u) \wedge\left(\frac{1}{2}[u]_{\mathbb{U}}\right)$ for any $u \in \mathbb{U} \backslash\left\{u_{0}\right\}$. Since the separable metric space $\mathbb{U}$ is Lindelöf, $\left\{\mathfrak{O}(u) \triangleq O_{\widetilde{\lambda}(u)}(u)\right\}_{u \in \mathbb{U}}$ has a countable subcollection $\left\{\mathfrak{O}\left(u_{i}\right)\right\}_{i \in \mathbb{N}}$ to cover $\mathbb{U}$. It is clear that

$$
\mathfrak{P}(u) \triangleq \sum_{i \in \mathbb{N}} \mathbf{1}_{\left\{u \in \mathfrak{O}\left(u_{i}\right) \backslash \cup_{j<i} \mathfrak{O}\left(u_{j}\right)\right\}} \mathfrak{P}_{o}\left(u_{i}\right) \in \mathbb{V}, \quad \forall u \in \mathbb{U}
$$

defines a $\mathscr{B}(\mathbb{U}) / \mathscr{B}(\mathbb{V})$-measurable function.
Given $u \in \mathbb{U}$, there exists an $i \in \mathbb{N}$ such that $u \in \mathfrak{O}\left(u_{i}\right) \backslash \underset{j<i}{\cup} \mathfrak{O}\left(u_{j}\right)$. If $u_{i}=u_{0}$,

$$
\begin{equation*}
[\mathfrak{P}(u)]_{\mathbb{V}}=\left[\mathfrak{P}_{0}\left(u_{i}\right)\right]_{\mathbb{V}} \leq \kappa+m\left[u_{i}\right]_{\mathbb{U}}=\kappa \leq \kappa+m[u]_{\mathbb{U}} . \tag{4.109}
\end{equation*}
$$

On the other hand, if $u_{i} \neq u_{0}$, then $\left[u_{i}\right]_{\mathbb{U}} \leq[u]_{\mathbb{U}}+\rho_{\mathbb{U}}\left(u, u_{i}\right) \leq[u]_{\mathbb{U}}+\widetilde{\lambda}\left(u_{i}\right) \leq[u]_{\mathbb{U}}+\frac{1}{2}\left[u_{i}\right]_{\mathbb{U}}$, and it follows that

$$
\begin{equation*}
[\mathfrak{P}(u)]_{\mathbb{V}}=\left[\mathfrak{P}_{0}\left(u_{i}\right)\right]_{\mathbb{V}} \leq \kappa+m\left[u_{i}\right]_{\mathbb{U}} \leq \kappa+2 m[u]_{\mathbb{U}} \tag{4.110}
\end{equation*}
$$

Also, we see from (4.108) that

$$
\begin{aligned}
-\frac{\partial \varphi}{\partial t}\left(t_{0}, x_{0}\right)-\frac{5}{8} \varrho & \geq \sup _{(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right)} H\left(t, x, \varphi(t, x), D_{x} \varphi(t, x), D_{x}^{2} \varphi(t, x), u, \mathfrak{P}_{0}\left(u_{i}\right)\right) \\
& =\sup _{(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right)} H\left(t, x, \varphi(t, x), D_{x} \varphi(t, x), D_{x}^{2} \varphi(t, x), u, \mathfrak{P}(u)\right)
\end{aligned}
$$

which together with (4.107) implies that

$$
\begin{equation*}
-\frac{\partial \varphi}{\partial t}(t, x)-\frac{1}{2} \varrho \geq H\left(t, x, \varphi(t, x), D_{x} \varphi(t, x), D_{x}^{2} \varphi(t, x), u, \mathfrak{P}(u)\right), \quad \forall(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right), \quad \forall u \in \mathbb{U} \tag{4.111}
\end{equation*}
$$

Similar to (4.95), we set $\wp \triangleq \min \left\{\left(\varphi-\bar{w}_{1}\right)(t, x):(t, x) \in \bar{O}_{\delta}\left(t_{0}, x_{0}\right) \backslash O_{\frac{\delta}{3}}\left(t_{0}, x_{0}\right)\right\}>0$ and $\widetilde{\wp} \triangleq \frac{\wp \wedge \varrho}{2(1 \vee \gamma) T}>0$. Let $\left\{\left(t_{j}, x_{j}\right)\right\}_{j \in \mathbb{N}}$ be a sequence of $O_{\frac{\delta}{6}}\left(t_{0}, x_{0}\right)$ such that

$$
\lim _{j \rightarrow \infty}\left(t_{j}, x_{j}\right)=\left(t_{0}, x_{0}\right) \quad \text { and } \quad \lim _{j \rightarrow \infty} w_{1}\left(t_{j}, x_{j}\right)=\bar{w}_{1}\left(t_{0}, x_{0}\right)=\varphi\left(t_{0}, x_{0}\right)=\lim _{j \rightarrow \infty} \varphi\left(t_{j}, x_{j}\right)
$$

So one can find a $j \in \mathbb{N}$ that

$$
\begin{equation*}
\left|w_{1}\left(t_{j}, x_{j}\right)-\varphi\left(t_{j}, x_{j}\right)\right|<\frac{5}{6} \widetilde{\wp} t_{0} \tag{4.112}
\end{equation*}
$$

For any $\mu \in \mathcal{U}_{t_{j}}$, the measurability of function $\mathfrak{P}$ shows that $(\widehat{\beta}(\mu))_{s} \triangleq \mathfrak{P}\left(\mu_{s}\right), s \in\left[t_{j}, T\right]$ is a $\mathbb{V}$-valued, $\mathbf{F}$-progressively measurable process. By (4.109) and (4.110),

$$
\left[(\widehat{\beta}(\mu))_{s}\right]_{\mathbb{V}}=\left[\mathfrak{P}\left(\mu_{s}\right)\right]_{\mathbb{V}} \leq \kappa+2 m\left[\mu_{s}\right]_{\mathbb{U}}, \quad \forall s \in\left[t_{j}, T\right]
$$

Let $E \int_{t_{j}}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s<\infty$ for some $q>2$. It then follows that

$$
E \int_{t_{j}}^{T}\left[(\widehat{\beta}(\mu))_{s}\right]_{\mathbb{V}}^{q} d s \leq 2^{q-1} \kappa^{q} T+2^{2 q-1} m^{q} E \int_{t_{j}}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{q} d s<\infty
$$

So $\widehat{\beta}(\mu) \in \mathcal{V}_{t_{j}}$. Let $\mu^{1}, \mu^{2} \in \mathcal{U}_{t_{j}}$ such that $\mu^{1}=\mu^{2}, d s \times d P-$ a.s. on $\llbracket t_{j}, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$ for some $\tau \in \mathcal{S}_{t_{j}, T}$ and $A \in \mathcal{F}_{\tau}$. Then it directly follows that $\left(\widehat{\beta}\left(\mu^{1}\right)\right)_{s}=\mathfrak{P}\left(\mu_{s}^{1}\right)=\mathfrak{P}\left(\mu_{s}^{2}\right)=\left(\widehat{\beta}\left(\mu^{2}\right)\right)_{s}$, ds $\times d P-$ a.s. on $\llbracket t_{j}, \tau \llbracket \cup \llbracket \tau, T \rrbracket_{A}$. Hence, $\widehat{\beta} \in \mathfrak{B}_{t_{j}}$.

Let $\mu \in \mathcal{U}_{t_{j}}$. We set $\Theta_{\mu} \triangleq\left(t_{j}, x_{j}, \mu, \widehat{\beta}(\mu)\right)$ and define

$$
\tau_{\mu}=\tau_{\widehat{\beta}, \mu} \triangleq \inf \left\{s \in\left(t_{j}, T\right]:\left(s, X_{s}^{\Theta_{\mu}}\right) \notin O_{\frac{2}{3} \delta}\left(t_{j}, x_{j}\right)\right\} \in \mathcal{S}_{t_{j}, T}
$$

As $\left|\left(T, X_{T}^{\Theta_{\mu}}\right)-\left(t_{j}, x_{j}\right)\right| \geq T-t_{j} \geq T-t_{0}-\left|t_{j}-t_{0}\right|>\delta_{0}-\frac{\delta}{6}>\frac{2}{3} \delta$, the continuity of $X^{\Theta_{\mu}}$ implies that $P$-a.s.

$$
\begin{equation*}
\tau_{\mu}<T \quad \text { and } \quad\left(\tau_{\mu} \wedge s, X_{\tau_{\mu} \wedge s}^{\Theta}\right) \in \bar{O}_{\frac{2}{3} \delta}\left(t_{j}, x_{j}\right) \subset \bar{O}_{\frac{5}{6} \delta}\left(t_{0}, x_{0}\right), \quad \forall s \in\left[t_{j}, T\right] \tag{4.113}
\end{equation*}
$$

In particular, $\quad\left(\tau_{\mu}, X_{\tau_{\mu}}^{\Theta_{\mu}}\right) \in \partial O_{\frac{2}{3} \delta}\left(t_{j}, x_{j}\right) \subset \bar{O}_{\frac{5}{6} \delta}\left(t_{0}, x_{0}\right) \backslash O_{\frac{\delta}{2}}\left(t_{0}, x_{0}\right)$.
The continuity of $\varphi, X^{\Theta_{\mu}}$ and (4.113) show that $\mathcal{Y}_{s}^{\mu} \triangleq \varphi\left(\tau_{\mu} \wedge s, X_{\tau_{\mu} \wedge s}^{\Theta_{\mu}}\right)-\widetilde{\wp}\left(\tau_{\mu} \wedge s\right), s \in\left[t_{j}, T\right]$ defines a bounded $\mathbf{F}$-adapted continuous process. Applying Itô's formula yields that

$$
\begin{equation*}
\mathcal{Y}_{s}^{\mu}=\mathcal{Y}_{T}^{\mu}+\int_{s}^{T} \mathfrak{f}_{r}^{\mu} d r-\int_{s}^{T} \mathcal{Z}_{r}^{\mu} d B_{r}, \quad s \in\left[t_{j}, T\right] \tag{4.115}
\end{equation*}
$$

where $\mathcal{Z}_{r}^{\mu} \triangleq \mathbf{1}_{\left\{r<\tau_{\mu}\right\}} D_{x} \varphi\left(r, X_{r}^{\Theta_{\mu}}\right) \cdot \sigma\left(r, X_{r}^{\Theta_{\mu}}, \mu_{r},(\widehat{\beta}(\mu))_{r}\right)$ and
$\mathfrak{f}_{r}^{\mu} \triangleq \mathbf{1}_{\left\{r<\tau_{\mu}\right\}}\left\{\widetilde{\emptyset}-\frac{\partial \varphi}{\partial t}\left(r, X_{r}^{\Theta_{\mu}}\right)-D_{x} \varphi\left(r, X_{r}^{\Theta_{\mu}}\right) \cdot b\left(r, X_{r}^{\Theta_{\mu}}, \mu_{r},(\widehat{\beta}(\mu))_{r}\right)-\frac{1}{2} \operatorname{trace}\left(\sigma \sigma^{T}\left(r, X_{r}^{\Theta_{\mu}}, \mu_{r},(\widehat{\beta}(\mu))_{r}\right) \cdot D_{x}^{2} \varphi\left(r, X_{r}^{\Theta_{\mu}}\right)\right)\right\}$.
As $\varphi \in \mathbb{C}^{1,2}\left([t, T] \times \mathbb{R}^{k}\right)$, the measurability of $b, \sigma, X^{\Theta_{\mu}}, \mu$ and $\widehat{\beta}(\mu)$ implies that both $\mathcal{Z}^{\mu}$ and $\mathfrak{f}^{\mu}$ are $\mathbf{F}$-progressively measurable. Let $\widetilde{C}_{\varphi} \triangleq \sup _{(t, x) \in \overline{\bar{D}}_{\frac{5}{5} \delta} \delta\left(t_{0}, x_{0}\right)}\left|D_{x} \varphi(t, x)\right|<\infty$. Similar to (4.100), we see from (2.1), (2.2) and (4.113) that

$$
\begin{equation*}
E\left[\left(\int_{t_{j}}^{T}\left|\mathcal{Z}_{s}^{\mu}\right|^{2} d s\right)^{p / 2}\right] \leq c_{0} \widetilde{C}_{\varphi}^{p}\left(\left(1+\left|x_{0}\right|+\delta\right)^{p}+\left\{E \int_{t_{j}}^{T}\left[\mu_{s}\right]_{\mathbb{U}}^{2} d s\right\}^{p / 2}+\left\{E \int_{t_{j}}^{T}\left[(\widehat{\beta}(\mu))_{s}\right]_{\mathbb{V}}^{2} d s\right\}^{p / 2}\right)<\infty \tag{4.116}
\end{equation*}
$$

i.e. $\mathcal{Z}^{\mu} \in \mathbb{H}_{\mathbf{F}}^{2, p}\left(\left[t_{j}, T\right], \mathbb{R}^{d}\right)$. Hence, $\left\{\left(\mathcal{Y}_{s}^{\mu}, \mathcal{Z}_{s}^{\mu}\right)\right\}_{s \in\left[t_{j}, T\right]}$ solves the $\operatorname{BSDE}\left(t_{j}, \mathcal{Y}_{T}^{\mu}, \mathfrak{f}^{\mu}\right)$.

Let $\ell, \theta_{1}$ and $\theta_{2}$ still be the continuous functions considered in part (a). Like $\phi$ in (4.101),

$$
\widetilde{\phi}(t, x) \triangleq \theta_{1}(t, x) \ell(x)+\left(1-\theta_{1}(t, x)\right)\left(\varphi(t, x)-\wp \theta_{2}(t, x)\right), \quad \forall(t, x) \in\left[t_{j}, T\right] \times \mathbb{R}^{k}
$$

define a continuous function with $\tilde{\phi} \geq w_{1}$. Similar to (4.102) and (4.103), we can deduce from (4.114), (4.113), (4.111) and (2.4) that $\mathcal{Y}_{T}^{\mu} \geq \widetilde{\phi}\left(\tau_{\mu}, X_{\tau_{\mu}}^{\Theta_{\mu}}\right), P-$ a.s. and that $\mathfrak{f}_{s}^{\mu}(\omega) \geq f_{\tau_{\mu}}^{\Theta_{\mu}}\left(s, \omega, \mathcal{Y}_{s}^{\mu}(\omega), \mathcal{Z}_{s}^{\mu}(\omega)\right)$ for $d s \times d P$-a.s. $(s, \omega) \in$ $\left[t_{j}, T\right] \times \Omega$. As $f_{\tau_{\mu}}^{\Theta_{\mu}}$ is Lipschitz continuous in (y,z), we know from Proposition 1.2 (2) that $P$-a.s.

$$
\mathcal{Y}_{s}^{\mu} \geq Y_{s}^{\Theta_{\mu}}\left(\tau_{\mu}, \widetilde{\phi}\left(\tau_{\mu}, X_{\tau_{\mu}}^{\Theta_{\mu}}\right)\right), \quad \forall s \in\left[t_{j}, T\right] .
$$

Letting $s=t_{j}$ and using the fact that $t_{j}>t_{0}-\frac{1}{6} \delta>t_{0}-\frac{1}{6} \delta_{0}>\frac{5}{6} t_{0}$, we obtain

$$
\varphi\left(t_{j}, x_{j}\right)-\frac{5}{6} \widetilde{\wp} t_{0}>\varphi\left(t_{j}, x_{j}\right)-\widetilde{\wp} t_{j}=\mathcal{Y}_{t_{j}}^{\mu} \geq Y_{t_{j}}^{t_{j}, x_{j}, \mu, \widehat{\beta}(\mu)}\left(\tau_{\mu}, \widetilde{\phi}\left(\tau_{\mu}, X_{\tau_{\mu}}^{t_{j}, x_{j}, \mu, \widehat{\beta}(\mu)}\right)\right), \quad P-\text { a.s. }
$$

Taking essential supremum over $\mu \in \mathcal{U}_{t_{j}}$ and applying Theorem [2.1] with $(t, x, \delta)=\left(t_{j}, x_{j}, \frac{2}{3} \delta\right)$, we see from (4.112) that $P$-a.s.

$$
\begin{aligned}
\varphi\left(t_{j}, x_{j}\right)-\frac{5}{6} \widetilde{\wp} t_{0} & \geq \underset{\mu \in \mathcal{U}_{t_{j}}}{\operatorname{esssup}} Y_{t_{j}}^{t_{j}, x_{j}, \mu, \widehat{\beta}(\mu)}\left(\tau_{\mu}, \widetilde{\phi}\left(\tau_{\mu}, X_{\tau_{\mu}}^{t_{j}, x_{j}, \mu, \widehat{\beta}(\mu)}\right)\right) \geq \underset{\beta \in \mathfrak{B}_{t_{j}}}{\operatorname{essinf}} \underset{\mu \in \mathcal{U}_{t_{j}}}{\operatorname{esssup}} Y_{t_{j}}^{t_{j}, x_{j}, \mu, \beta(\mu)}\left(\tau_{\beta, \mu}, \widetilde{\phi}\left(\tau_{\beta, \mu}, X_{\tau_{\beta, \mu}}^{t_{j}, x_{j}, \mu, \beta(\mu)}\right)\right) \\
& \geq w_{1}\left(t_{j}, x_{j}\right)>\varphi\left(t_{j}, x_{j}\right)-\frac{5}{6} \widetilde{\wp} t_{0},
\end{aligned}
$$

where $\tau_{\beta, \mu} \triangleq \inf \left\{s \in\left(t_{j}, T\right]:\left(s, X_{s}^{t_{j}, x_{j}, \mu, \beta(\mu)}\right) \notin O_{\frac{2}{3} \delta} \delta\left(t_{j}, x_{j}\right)\right\}$. A contradiction appears. Therefore, $\bar{w}_{1}$ is a viscosity supersolution of (3.1) with Hamiltonian $\bar{H}_{1}$.

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