POLYNOMIAL OPTIMIZATION WITH REAL VARIETIES

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ABSTRACT. We study the optimization problem

min
$$f(x)$$
 s.t. $h(x) = 0, g(x) \ge 0$

with f a polynomial and h, g two tuples of polynomials in $x \in \mathbb{R}^n$. Lasserre's hierarchy is a sequence of sum of squares relaxations for finding the global minimum f_{min} . Let K be the feasible set. We prove the following results: i) If the real variety $V_{\mathbb{R}}(h)$ is finite, then Lasserre's hierarchy has finite convergence, no matter the complex variety $V_{\mathbb{C}}(h)$ is finite or not. This solves an open question in Laurent's survey [6]. ii) If K and $V_{\mathbb{R}}(h)$ have the same vanishing ideal, then the finite convergence of Lasserre's hierarchy is independent of the choice of defining polynomials for the real variety $V_{\mathbb{R}}(h)$. iii) When K is finite, a refined version of Lasserre's hierarchy (using the preordering of g) has finite convergence.

1. INTRODUCTION

Consider the polynomial optimization problem

(1.1)
$$\begin{cases} f_{min} := \min \quad f(x) \\ s.t. \quad h_i(x) = 0 \ (i = 1, \dots, m_1), \\ g_j(x) \ge 0 \ (j = 1, \dots, m_2), \end{cases}$$

where f and all g_i, h_j are real polynomials in $x \in \mathbb{R}^n$. Denote $h := (h_1, \ldots, h_{m_1})$ and $g := (g_1, \ldots, g_{m_2})$. Let K be the feasible set of (1.1). A standard approach for solving (1.1) globally is *Lasserre's hierarchy* of sum of squares (SOS) relaxations [2]. We first give a short review about it. Let $\mathbb{R}[x]$ be the ring of polynomials with real coefficients and in variables $x := (x_1, \ldots, x_n)$. A polynomial p is SOS if there exist $p_1, \ldots, p_k \in \mathbb{R}[x]$ such that $p = p_1^2 + \cdots + p_k^2$. Denote by $\Sigma \mathbb{R}[x]^2$ the set of all SOS polynomials. A subset I of $\mathbb{R}[x]$ is an ideal if $I + I \subseteq I$ and $I \cdot \mathbb{R}[x] \subseteq I$. The tuple h generates the ideal $h_1 \mathbb{R}[x] + \cdots + h_{m_1} \mathbb{R}[x]$, which is denoted as $\langle h \rangle$. The 2k-th truncated ideal generated by h is

$$\langle h \rangle_{2k} := \left\{ \left. \sum_{i=1}^{m_1} \phi_i h_i \right| \begin{array}{c} \operatorname{each} \phi_i \in \mathbb{R}[x] \\ \operatorname{and} \deg(\phi_i h_i) \leq 2k \end{array} \right\},$$

and the k-th truncated quadratic module generated by g is (denote $g_0 = 1$)

$$Q_k(g) := \left\{ \left. \sum_{j=0}^{m_2} \sigma_j g_j \right| \begin{array}{c} \operatorname{each} \sigma_j \in \Sigma \mathbb{R}[x]^2 \\ \operatorname{and} \deg(\sigma_j g_j) \le 2k \end{array} \right\}.$$

¹⁹⁹¹ Mathematics Subject Classification. 65K05, 90C22.

Key words and phrases. polynomials, finite convergence, Lasserre's hierarchy, real variety, semidefinite program, sum of squares.

The research was partially supported by the NSF grant DMS-0844775.

Let \mathbb{N} be the set of nonnegative integers. The union $Q(g) := \bigcup_{k \in \mathbb{N}} Q_k(g)$ is called the *quadratic module* generated by g. Lasserre's hierarchy for (1.1) is the sequence of SOS relaxations $(k \in \mathbb{N})$

(1.2)
$$f_k := \max \quad \gamma \quad s.t. \quad f - \gamma \in \langle h \rangle_{2k} + Q_k(g).$$

The integer k in (1.2) is called a *relaxation order*. The SOS program (1.2) is equivalent to a semidefinite program (SDP) (cf. [3, 6]).

Next, we describe the dual optimization problem of (1.2). Let y be a sequence indexed by $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq 2k$, i.e., y is a *truncated moment sequence* (tms) of degree 2k. Denote by \mathscr{M}_{2k} the space of all tms' whose degrees are 2k. Denote by $\lceil a \rceil$ the smallest integer that is not smaller than a. Denote $d_j := \lceil \deg(g_j)/2 \rceil$, $x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and

$$x]_t := \begin{bmatrix} 1 & x_1 \cdots x_n & x_1^2 & x_1 x_2 & \cdots & x_1^t & \cdots & x_n^t \end{bmatrix}^T.$$

For each $k \ge d_j$, expand the product $g_j[x]_{k-d_j}[x]_{k-d_j}^T$ as

$$g_j[x]_{k-d_j}[x]_{k-d_j}^T = \sum_{|\alpha| \le 2k} A_{\alpha}^{(k,j)} x^{\alpha},$$

where each $A_{\alpha}^{(k,j)}$ is a constant symmetric matrix. The matrix

$$L_{g_j}^{(k)}(y) := \sum_{|\alpha| \le 2k} A_{\alpha}^{(k,j)} y_{\alpha}$$

is called a *localizing matrix*. For $g_0 = 1$, $M_k(y) := L_1^{(k)}(y)$ is called a *moment matrix*. The columns and rows of $L_{g_j}^{(k)}(y)$ are indexed by vectors $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k - d_j$. We refer to Laurent [6, Section 4] for more details about moment and localizing matrices. The dual optimization problem of (1.2) is (cf. [3, 6])

(1.3)
$$\begin{cases} f_k^* := \min_{y \in \mathscr{M}_{2k}} & \langle f, y \rangle \\ \text{s.t.} & L_{h_i}^{(k)}(y) = 0 \ (i = 1, \dots, m_1), \ y_0 = 1, \\ & L_{g_j}^{(k)}(y) \succeq 0 \ (j = 0, 1, \dots, m_2). \end{cases}$$

In the above, $X \succeq 0$ means the matrix X is positive semidefinite.

Let f_{min}, f_k, f_k^* , respectively, be the optimal values of (1.1), (1.2) and (1.3). It is known that $f_k \leq f_k^* \leq f_{min}$ for all k. The sequences $\{f_k\}$ and $\{f_k^*\}$ are both monotonically increasing. If K has nonempty interior, then (1.3) has an interior point, (1.2) achieves its optimal value and $f_k^* = f_k$, i.e., there is no duality gap between (1.2) and (1.3) (cf. [2]). Under the archimedean condition (there exists R > 0 such that $R - \sum_{i=1}^n x_i^2 \in \langle h \rangle + Q(g)$), Lasserre proved the asymptotic convergence $f_k \to f_{min}$ as $k \to \infty$. The proof uses Putinar's Positivstellensatz [14]. We refer to Lasserre's book [3], Laurent's survey [6] and Marshall's book [9] for the work in this area.

When $f_k = f_{min}$ occurs for some k, we say that Lasserre's hierarchy has finite convergence. An appropriate criterion for checking finite convergence of $\{f_k\}$ is flat truncation, as shown in [13]. For the tuple h, define the complex and real algebraic varieties respectively as

(1.4)
$$V_{\mathbb{C}}(h) = \{ x \in \mathbb{C}^n : h(x) = 0 \}, \quad V_{\mathbb{R}}(h) = V_{\mathbb{C}}(h) \cap \mathbb{R}^n$$

When the complex variety $V_{\mathbb{C}}(h)$ is a finite set, Laurent [5] proved that $\{f_k\}$ has finite convergence to f_{min} . When the real variety $V_{\mathbb{R}}(h)$ is a finite set, Laurent

[6, Theorem 6.15] proved that $\{f_k^*\}$ has finite convergence to f_{min} . In the case that $V_{\mathbb{R}}(h)$ is finite but $V_{\mathbb{C}}(h)$ is infinite, it was unknown whether $\{f_k\}$ has finite convergence to f_{min} or not. Indeed, Laurent [6, Question 6.17] asked:

Does there exist an example with $|V_{\mathbb{C}}(h)| = \infty$, $|V_{\mathbb{R}}(h)| < \infty$ and where $f_k < f_{min}$ for all k?

This question was also asked by Laurent in the workshop *Positive Polynomials* and *Optimization* (Banff, Canada, 2006), and remained open since then, in the author's best knowledge. Semidefinite relaxations are very useful for solving zero-dimensional polynomial systems. We refer to [4, 7].

Our first main result is to give a negative answer to the above question. We prove that if $V_{\mathbb{R}}(h)$ is finite then $f_k = f_{min}$ for all k big enough, no matter $V_{\mathbb{C}}(h)$ is finite or not. This is summarized as follows.

Theorem 1.1. Let f_k , f_{min} be as above. If the real variety $V_{\mathbb{R}}(h)$ is finite, then $f_k = f_{min}$ for all k big enough.

When $V_{\mathbb{R}}(h)$ is finite, Theorem 1.1 implies that there is no duality gap between (1.2) and (1.3), i.e., $f_k - f_k^* = 0$, for k big enough, because $f_k \leq f_k^* \leq f_{min}$. This is a nice property for numerical computations. When primal-dual interior point methods are applied to solve semidefinite programs like (1.2)-(1.3), zero duality gap is often required.

The real variety $V_{\mathbb{R}}(h)$ can be defined by different sets of polynomials, e.g., it can be defined by a single equation like

$$h_1^2(x) + \dots + h_{m_1}^2(x) = 0.$$

Suppose $h' = (h'_1, \ldots, h'_r)$ is a different tuple of polynomials such that $V_{\mathbb{R}}(h') = V_{\mathbb{R}}(h)$. Then, (1.1) is equivalent to

(1.5) min
$$f(x)$$
 s.t. $h'(x) = 0, \quad g(x) \ge 0.$

Like $\langle h \rangle_{2k}$, we similarly define the truncated ideal $\langle h' \rangle_{2k}$. Then, Lasserre's hierarchy for (1.5) is the sequence of SOS relaxations $(k \in \mathbb{N})$

(1.6)
$$f'_k := \max \quad \gamma \quad s.t. \quad f - \gamma \in \langle h' \rangle_{2k} + Q_k(g).$$

Similarly, we have $f'_k \leq f_{min}$ for all k. The following two questions are natural about the two sequences $\{f_k\}$ and $\{f'_k\}$:

- If $\{f_k\}$ has finite convergence to f_{min} , does $\{f'_k\}$ necessarily have finite convergence to f_{min} ?
- If $\{f_k\}$ has no finite convergence to f_{min} , is it possible that $\{f'_k\}$ has finite convergence to f_{min} ?

When the real variety $V_{\mathbb{R}}(h)$ is finite, by Theorem 1.1, the above two questions are solved: the finite convergence of Lasserre's hierarchy is independent of the choice of defining polynomials for $V_{\mathbb{R}}(h)$. When $V_{\mathbb{R}}(h)$ is infinite, do we have a similar result? Indeed, this is true under a general condition on $V_{\mathbb{R}}(h)$ and the feasible set K of (1.1). The vanishing ideal of K is defined as

$$I(K) := \{ p \in \mathbb{R}[x] : p(u) = 0 \,\forall \, u \in K \}.$$

The vanishing ideal of the real variety $V_{\mathbb{R}}(h)$ is

$$I(V_{\mathbb{R}}(h)) := \{ p \in \mathbb{R}[x] : p(u) = 0 \,\forall \, u \in V_{\mathbb{R}}(h) \}.$$

It is also called the *real radical* of $\langle h \rangle$ (cf. [1]).

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Our second main result is the following theorem.

Theorem 1.2. Let $h' = (h'_1, \ldots, h'_r)$ be a tuple of polynomials in $\mathbb{R}[x]$ such that $V_{\mathbb{R}}(h) = V_{\mathbb{R}}(h')$, and f_k, f'_k, f_{min} be defined as above. Suppose $I(K) = I(V_{\mathbb{R}}(h))$. Then, the sequence $\{f_k\}$ has finite convergence to f_{min} if and only if $\{f'_k\}$ has finite convergence to f_{min} .

In Theorem 1.2, the condition $I(K) = I(V_{\mathbb{R}}(h))$ implies that if a polynomial p identically vanishes on K then it also identically vanishes on $V_{\mathbb{R}}(h)$. It essentially requires that the feasible set K and the real variety $V_{\mathbb{R}}(h)$ have the same Zariski closure. This is often satisfied.

We would like to remark that there does not exist a similar result like Theorem 1.2 for the case of inequalities. That is, the choice of inequality constraining polynomials might affect finite convergence of Lasserre's hierarchy, while the feasible set K is not changed. For instance, consider the problem

$$\min_{x \in \mathbb{R}} \quad 1 - x^2 \quad s.t. \quad 1 - x^2 \ge 0.$$

Clearly, Lasserre's hierarchy for the above converges in one step, and the problem is equivalent to

$$\min_{x \in \mathbb{R}} \quad 1 - x^2 \quad s.t. \quad (1 - x^2)^3 \ge 0.$$

However, Lasserre's sequence $\{f_k\}$ for the above new formulation does not have finite convergence. Indeed, there exists a constant C > 0 such that $f_k \leq -Ck^{-2}$ for all k. This is implied by Stengle [17, Theorem 4].

This paper is organized as follows. Section 2 is mostly to prove Theorem 1.1; Section 3 is mostly to prove Theorem 1.2; Section 4 proves that if only the feasible set K is finite, then a refined version of Lasserre's hierarchy (using the preordering of g) has finite convergence.

2. Optimization with finite real varieties

This section is mostly to prove Theorem 1.1. We begin with a useful lemma.

Lemma 2.1. (i) Let $\ell \geq 1$ be an integer. Then, for all

$$c \ge c_0 := \frac{1}{2\ell} \left(1 - \frac{1}{2\ell} \right)^{2\ell - 1}$$

the univariate polynomial $s_c(t) := 1 + t + ct^{2\ell}$ in t is SOS. (ii) Let $p, q \in \mathbb{R}[x]$ and $\ell \ge 1$ be an integer. Then, for all $\epsilon > 0$ and $c \in \mathbb{R}$,

$$p + \epsilon = \phi_{\epsilon} + \theta_{\epsilon},$$

where

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$$\phi_{\epsilon} = -c\epsilon^{1-2\ell}(p^{2\ell}+q), \quad \theta_{\epsilon} = \epsilon s_c(p/\epsilon) + c\epsilon^{1-2\ell}q$$

(iii) In (ii), assume $c \ge c_0$ as in (i), $p^{2\ell} + q \in \langle h \rangle$ and $q \in Q(g)$ for polynomial tuples h, g. Then, there exists an integer N > 0 such that, for all $\epsilon > 0$,

$$\phi_{\epsilon} \in \langle h \rangle_{2N}, \quad \theta_{\epsilon} \in Q_N(g).$$

Proof. (i) For all c > 0, the univariate polynomial $s_c(t)$ is convex in t over the real line \mathbb{R} and $s'_c(t) = 1 + 2\ell c t^{2\ell-1}$. The polynomial s_c has a unique real critical point $\xi := \left(\frac{-1}{2\ell c}\right)^{\frac{1}{2\ell-1}}$. Note that

$$s_c(\xi) = 1 + \left(\frac{-1}{2\ell c}\right)^{\frac{1}{2\ell-1}} \left(1 - \frac{1}{2\ell}\right).$$

It can be verified that $s_c(\xi) \ge 0$ if and only if $c \ge c_0$. So, when $c \ge c_0$, the univariate polynomial s_c is nonnegative over \mathbb{R} (because $s_c(\xi) \ge 0$, $s'_c(\xi) = 0$ and s_c is convex), and it must be SOS (cf. [15]).

(ii) It can be done by a direct verification.

(iii) By assumption, there exist positive integers N_1, N_2 such that $p^{2\ell} + q \in \langle h \rangle_{2N_1}, q \in Q_{N_2}(g)$. Let $N_0 = \ell \lceil \deg(p)/2 \rceil$. Note that $s_c(p/\epsilon)$ is SOS by (i) and its degree is at most $2N_0$. So, $\epsilon s_c(p/\epsilon) \in Q_{N_0}(g)$ for all $\epsilon > 0$. Then $N := \max(N_0, N_1, N_2)$ works for the proof.

Theorem 1.1 can be proved by using Lemma 2.1.

Proof of Theorem 1.1. When $V_{\mathbb{R}}(h)$ is empty, the feasible set K is also empty, and hence $f_{min} = +\infty$ by convention. By Positivstellensatz (cf. [1, Theorem 4.4.2]), we have $-1 \in \langle h \rangle + \Sigma \mathbb{R}[x]^2$. For all $\gamma > 0$, it holds that

$$f - \gamma = (1 + f/4)^2 + (-1)(\gamma + (1 - f/4)^2) \in \langle h \rangle_{2k} + Q_k(g),$$

for all k big enough. So, for all big k, (1.2) is unbounded from above, and hence $f_k = +\infty$. Hence, Lasserre's hierarchy has finite convergence.

When $V_{\mathbb{R}}(h)$ is nonempty and finite, we can write $V_{\mathbb{R}}(h) = \{u_1, \ldots, u_D\}$ for distinct points $u_1, \ldots, u_D \in \mathbb{R}^n$. Let $\varphi_1, \ldots, \varphi_D \in \mathbb{R}[x]$ be the interpolating polynomials such that $\varphi_i(u_j) = 0$ for $i \neq j$ and $\varphi_i(u_j) = 1$ for i = j. For each u_i , if $f(u_i) - f_{min} \ge 0$, let $a_i := (f(u_i) - f_{min})\varphi_i^2$. If $f(u_i) - f_{min} < 0$, then at least one of $g_1(u_i), \ldots, g_{m_2}(u_i)$ is negative, say, $g_{j_i}(u_i) < 0$, and let

$$u_i := \left(rac{f(u_i) - f_{min}}{g_{j_i}(u_i)}
ight) g_{j_i} \varphi_i^2.$$

Each a_i is a polynomial in Q(g). Let $a := a_1 + \cdots + a_D$. By construction, $a \in Q_{N_1}(g)$ for some integer $N_1 > 0$. The polynomial

$$f := f - f_{min} - a$$

vanishes identically on $V_{\mathbb{R}}(h)$. By Real Nullstellensatz (cf. [1, Corollary 4.1.8]), there exist an integer $\ell > 0$ and $q \in \Sigma \mathbb{R}[x]^2$ such that

$$\hat{f}^{2\ell} + q \in \langle h \rangle$$

Apply Lemma 2.1 to $p := \hat{f}, q$, with the tuples h, g and any $c \ge \frac{1}{2\ell}$. Then, there exists $N \ge N_1$ such that, for all $\epsilon > 0$,

$$\tilde{f} + \epsilon = \phi_{\epsilon} + \theta_{\epsilon},$$

and $\phi_{\epsilon} \in \langle h \rangle_{2N}, \, \theta_{\epsilon} \in Q_N(g)$. Therefore, we get

$$f - (f_{min} - \epsilon) = \phi_{\epsilon} + \sigma_{\epsilon},$$

where $\sigma_{\epsilon} = \theta_{\epsilon} + a \in Q_N(g)$ for all $\epsilon > 0$. This implies that, for all $\epsilon > 0$, $\gamma = f_{min} - \epsilon$ is feasible in (1.2) for the order N. Thus, we get $f_N \ge f_{min}$. Note that $f_k \le f_{min}$

for all k and $\{f_k\}$ is monotonically increasing. So, we must have $f_k = f_{min}$ for all $k \ge N$, i.e., Lasserre's hierarchy has finite convergence.

We present some examples to show the proof of Theorem 1.1.

Example 2.2. Consider the optimization problem

$$\begin{cases} \min & f(x) := x_1 x_2 \\ s.t. & h(x) := (x_1^2 - 1)^2 + (x_2^2 - 1)^2 = 0, \\ g(x) := x_1 + x_2 - 1 \ge 0. \end{cases}$$

Clearly, $V_{\mathbb{R}}(h) = \{(\pm 1, \pm 1)\}, K = \{(1, 1)\}$ and $f_{min} = 1$. Let

$$a = \frac{1}{2}(x_1 + x_2 - 1)(x_1 - x_2)^2 \in Q_2(g),$$

= $f - 1 - a = \frac{1}{2} \left[(x_2^2 - 1)(x_1 - x_2 + 1) - (x_1^2 - 1)(x_1 - x_2 - 1) \right].$

Then, $\hat{f} \equiv 0$ on $V_{\mathbb{R}}(h)$ and

$$\hat{f}^2 + q = \frac{1}{2}((x_1 - x_2)^2 + 1)h \in \langle h \rangle_6,$$

where

$$q = \frac{1}{4} \left((x_1^2 - 1)(x_1 - x_2 + 1) + (x_2^2 - 1)(x_1 - x_2 - 1) \right)^2.$$

For each $\epsilon > 0$, let

 \hat{f}

$$\phi_{\epsilon} = -\frac{1}{4\epsilon}(\hat{f}^2 + q) \in \langle h \rangle_6, \quad \sigma_{\epsilon} = \epsilon \left(1 + \frac{\hat{f}}{2\epsilon}\right)^2 + \frac{1}{4\epsilon}q + a \in Q_3(g).$$

Then, $f - 1 + \epsilon = \phi_{\epsilon} + \sigma_{\epsilon}$ for all $\epsilon > 0$. So, $f_k = 1$ for all $k \ge 3$.

Example 2.3. Let $f \in \mathbb{R}[x]$ be such that f(0) = 0. Consider the problem

$$\begin{cases} \min & f(x) \\ s.t. & h(x) := x_1^{2d} + \dots + x_n^{2d} = 0. \end{cases}$$

Clearly, $f_{min} = 0$. There are no inequality constraints, and we can think that g = 0, as in (1.1). Write f as

$$f = x_1 b_1 + \dots + x_n b_n, \quad b_1, \dots, b_n \in \mathbb{R}[x].$$

Let $\Sigma_{n,2d}$ be the cone of SOS forms in *n* variables and of degree 2*d*. There exists $\lambda > 0$ such that

$$\lambda(t_1^{2d} + \dots + t_n^{2d}) - (t_1 + \dots + t_n)^{2d} \in \Sigma_{n,2d}.$$

This is because $t_1^{2d} + \cdots + t_n^{2d}$ lies in the interior of $\Sigma_{n,2d}$ (cf. [10, Proposition 5.3]). By replacing each t_i by $x_i b_i$ in the above, we know that

$$\psi := \lambda((x_1b_1)^{2d} + \dots + (x_nb_n)^{2d}) - f^{2d} \in \Sigma \mathbb{R}[x]^2.$$

Clearly, it holds that

$$\eta := \lambda \left[\left(\sum_{i=1}^n x_i^{2d} \right) \left(\sum_{i=1}^n b_i^{2d} \right) - \left(\sum_{i=1}^n (x_i b_i)^{2d} \right) \right] \in \Sigma \mathbb{R}[x]^2,$$

$$f^{2d} + \psi + \eta = \lambda (x_1^{2d} + \dots + x_n^{2d}) (b_1^{2d} + \dots + b_n^{2d}) \in \langle h \rangle.$$

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Let $q := \psi + \eta \in \Sigma \mathbb{R}[x]^2$. Clearly, $f \equiv 0$ on $V_{\mathbb{R}}(h)$, and $f^{2d} + q \in \langle h \rangle$. Suppose $\deg(f) = r$. Apply Lemma 2.1 with $c = \frac{1}{2d}$, $\ell = d$ and p = f. For each $\epsilon > 0$, let

$$\phi_{\epsilon} = -\frac{1}{2d} \epsilon^{1-2d} (f^{2d} + q) \in \langle h \rangle_{2dr},$$

$$\sigma_{\epsilon} = \epsilon \left(1 + f/\epsilon + \frac{1}{2d} (f/\epsilon)^{2d} \right) + \frac{1}{2d} \epsilon^{1-2d} q \in Q_{dr}(0).$$

$$= \sigma_{\epsilon} + \phi_{\epsilon} \text{ for all } \epsilon > 0. \text{ So, } f_{k} = 0 \text{ for all } k > dr.$$

Then, $f + \epsilon = \sigma_{\epsilon} + \phi_{\epsilon}$

We would like to remark that the SOS relaxation (1.2) might not achieve its optimal value f_k for any order k, even if $\{f_k\}$ has finite convergence to f_{min} . For instance, consider the problem

min
$$x_1$$
 s.t. $x_1^2 + x_2^2 + \dots + x_n^2 = 0.$

By Example 2.3, we know $f_k = 0$ for all $k \ge 1$. However, for any $\phi \in \mathbb{R}[x]$, the polynomial $\varphi = x_1 - (x_1^2 + x_2^2 + \dots + x_n^2)\phi$ cannot be SOS (because $\varphi(0) =$ $0, \nabla \varphi(0) \neq 0$, and 0 can not be a minimizer of φ). For this problem, (1.2) does not have a maximizer, for any order $k \geq 1$.

However, in Theorem 1.1, if the ideal $\langle h \rangle$ is real, i.e., $\langle h \rangle = I(V_{\mathbb{R}}(h))$ (cf. [1, Section 4.1]), then (1.2) achieves its optimum for all big k.

Proposition 2.4. In Theorem 1.1, if, in addition, the ideal $\langle h \rangle$ is real, then (1.2) achieves its optimum for all k big enough.

Proof. Let a be from the proof of Theorem 1.1. We know that $\hat{f} = f - f_{min} - a$ identically vanishes on $V_{\mathbb{R}}(h)$. So, $\hat{f} \in I(V_{\mathbb{R}}(h))$. Since $\langle h \rangle$ is real, $I(V_{\mathbb{R}}(h)) = \langle h \rangle$ and $\hat{f} \in \langle h \rangle$. The identity $f - f_{min} = a + \hat{f}$ implies that $\gamma = f_{min}$ is feasible in (1.2) if k is big enough. Thus, (1.2) achieves its optimum f_{min} for all big k. \Box

When $V_{\mathbb{R}}(h)$ is not finite, the conclusion of Proposition 2.4 also holds under some other conditions.

Proposition 2.5. Let h and K be as in (1.1). Suppose that f_{min} is finite and Lasserre's hierarchy has finite convergence. If $\langle h \rangle = I(K)$, then (1.2) achieves its optimum for all k big enough.

Proof. There exists N_1 such that $f_k = f_{min}$ for all $k \ge N_1$. By the condition that $I(K) = \langle h \rangle$, we know the quotient set $Q_k(g)/\langle h \rangle$ is closed for all k (cf. Laurent [6, Theorem 3.35] or Marshall [8, Theorem 3.1]). Let $\{\gamma_i\}_{i=1}^{\infty}$ be a sequence such that each γ_i is feasible for (1.2) with $k = N_1$ and $\gamma_i \to f_{min}$ as $i \to \infty$. Clearly, each $f - \gamma_i \in Q_{N_1}(g)/\langle h \rangle$ and $f - \gamma_i \to f - f_{min}$. Hence, $f - f_{min} \in Q_{N_1}(g)/\langle h \rangle$, i.e., there exists $\phi^* \in \langle h \rangle$ and $\sigma^* \in Q_{N_1}(g)$ such that

$$f - f_{min} = \phi^* + \sigma^*.$$

Let $N_2 \ge N_1$ be such that $\phi^* \in \langle h \rangle_{2N_2}$. Then, $\gamma = f_{min}, \phi^*, \sigma^*$) is feasible for (1.2) with order $k \ge N_2$. Hence, (1.2) achieves its optimum for all $k \ge N_2$.

3. Optimization with general real varieties

This section is mostly to prove Theorem 1.2. We first prove a result that similar to Theorem 1.2 by using generators of the real radical $I(V_{\mathbb{R}}(h))$.

Let $h_1^{rad}, \ldots, h_t^{rad}$ be a set of generators for $I(V_{\mathbb{R}}(h))$, i.e.,

$$I(V_{\mathbb{R}}(h)) = \langle h_1^{rad}, \dots, h_t^{rad} \rangle$$

Denote $h^{rad} := (h_1^{rad}, \ldots, h_t^{rad})$. Define $\langle h^{rad} \rangle_{2k}$ similarly as for $\langle h \rangle_{2k}$. Clearly, (1.1) is equivalent to

(3.1)
$$\begin{cases} \min f(x) \\ s.t. \quad h_i^{rad}(x) = 0 \ (i = 1, \dots, t), \\ g_j(x) \ge 0 \ (j = 1, \dots, m_2). \end{cases}$$

Lasserre's hierarchy for (3.1) is the sequence of SOS relaxations $(k \in \mathbb{N})$

(3.2)
$$f_k^{rad} := \max \quad \gamma \quad s.t. \quad f - \gamma \in \langle h^{rad} \rangle_{2k} + Q_k(g)$$

We also have $f_k^{rad} \leq f_{min}$ for all k.

Theorem 3.1. Let h, f_{min} and K be as in (1.1). Suppose that f_{min} is finite and $I(K) = I(V_{\mathbb{R}}(h)) = \langle h^{rad} \rangle$. Let f_k (resp., f_k^{rad}) be the optimal value of (1.2) (resp., (3.2)). Then, the sequence $\{f_k\}$ has finite convergence to f_{min} if and only if $\{f_k^{rad}\}$ has finite convergence to f_{min} .

Proof. First, assume that $\{f_k^{rad}\}$ has finite convergence to f_{min} . The feasible set of (3.1) is K and $\langle h^{rad} \rangle = I(K)$. Apply Proposition 2.5 to Lasserre's sequence $\{f_k^{rad}\}$ for (3.1) with the tuple h^{rad} . We know that (3.2) achieves its optimum f_{min} for all big k, say, for all $k \geq N_1$. Let $p \in \langle h^{rad} \rangle_{2N_1}$ and $\sigma_1 \in Q_{N_1}(g)$ be such that

$$f - f_{min} = p + \sigma_1.$$

Since $\langle h^{rad} \rangle = I(V_{\mathbb{R}}(h)), \ p \equiv 0$ on $V_{\mathbb{R}}(h)$. By Real Nullstellensatz (cf. [1, Corollary 4.1.8]), there exist an integer $\ell > 0$ and $q \in \Sigma \mathbb{R}[x]^2$ such that

$$p^{2\ell} + q \in \langle h \rangle.$$

By Lemma 2.1, there exists $N_2 > 0$ such that, for all $\epsilon > 0$,

$$p + \epsilon = \phi_{\epsilon} + \theta_{\epsilon}$$

with $\phi_{\epsilon} \in \langle h \rangle_{2N_2}$, $\theta_{\epsilon} \in Q_{N_2}(g)$. Let $\sigma_{\epsilon} = \theta_{\epsilon} + \sigma_1$ and $N_3 = \max(N_1, N_2)$. Then,

$$f - (f_{min} - \epsilon) = \sigma_{\epsilon} + \phi_{\epsilon}, \quad \sigma_{\epsilon} \in Q_{N_3}(g), \quad \phi_{\epsilon} \in \langle h \rangle_{2N_3}.$$

Hence, $f_k = f_{min}$ for all $k \ge N_3$, i.e., $\{f_k\}$ has finite convergence to f_{min} .

Second, assume that $\{f_k\}$ has finite convergence to f_{min} , say, $f_k = f_{min}$ for all $k \ge M_1$. Thus, for every $\epsilon > 0$, there exist $\phi_{\epsilon} \in \langle h \rangle_{2M_1}$, $\sigma_{\epsilon} \in Q_{M_1}(g)$ such that

$$f - (f_{min} - \epsilon) = \phi_{\epsilon} + \sigma_{\epsilon}$$

Note that each $h_i \in I(V_{\mathbb{R}}(h)) = \langle h^{rad} \rangle$. So, there exists $M_2 \geq M_1$ such that $\langle h \rangle_{2M_1} \subseteq \langle h^{rad} \rangle_{2M_2}$ and $Q_{M_1}(g) \subseteq Q_{M_2}(g)$. This implies that $f_k^{rad} \geq f_{min} - \epsilon$ for all $k \geq M_2$ and for all $\epsilon > 0$. Hence, $f_k^{rad} \geq f_{min}$ for all $k \geq M_2$. Since $f_k^{rad} \leq f_{min}$ for all k, we know that $\{f_k^{rad}\}$ has finite convergence to f_{min} .

Theorem 1.2 can be proved by using Theorem 3.1.

Proof of Theorem 1.2. If $f_{min} = -\infty$, then $f_k, f'_k \leq f_{min} = -\infty$ for all k, and the conclusion of Theorem 1.2 is clearly true. If $f_{min} = +\infty$, then $K = \emptyset$ and $I(K) = \mathbb{R}[x]$; so, $I(V_{\mathbb{R}}(h)) = I(K) = \mathbb{R}[x]$ and $V_{\mathbb{R}}(h) = \emptyset$. The conclusion of Theorem 1.2 is also true, as shown at the beginning of the proof of Theorem 1.1.

Now we prove Theorem 1.2 when f_{min} is finite. Let h^{rad} and f_k^{rad} be as in Theorem 3.1. By Theorem 3.1, $\{f_k\}$ has finite convergence to f_{min} if and only if $\{f_k^{rad}\}$ has finite convergence to f_{min} . For the same reason, since $V_{\mathbb{R}}(h) = V_{\mathbb{R}}(h')$ and (1.1) is equivalent to (1.5), $\{f'_k\}$ has finite convergence to f_{min} if and only if

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 $\{f_k^{rad}\}$ has finite convergence to f_{min} . This shows that $\{f_k\}$ has finite convergence to f_{min} if and only if $\{f'_k\}$ has finite convergence to f_{min} .

A direct consequence of of Theorem 1.2 is that we can reduce the number of equality constraints in polynomial optimization, while finite convergence of Lasserre's hierarchy is not lost. As is well known, every real variety can be defined by a single equation. Let

$$h^{sq}(x) := h_1^2(x) + \dots + h_{m_1}^2(x).$$

Then, (1.1) is equivalent to

(3.3) min
$$f(x)$$
 s.t. $h^{sq}(x) = 0, g(x) \ge 0$

Lasserre's hierarchy for (3.3) is the sequence of SOS relaxations $(k \in \mathbb{N})$

(3.4)
$$f_k^{sq} := \max \quad \gamma \quad s.t. \quad f - \gamma \in \langle h^{sq} \rangle_{2k} + Q_k(g).$$

If $I(K) = I(V_{\mathbb{R}}(h))$, by Theorem 1.2, $\{f_k\}$ has finite convergence to f_{min} if and only if $\{f_k^{sq}\}$ has finite convergence to f_{min} . We show an example of this.

Example 3.2. Consider the optimization problem:

(3.5)
$$\begin{cases} \min \quad f(x) := x_1 x_2 x_3 - 2x_3 \\ s.t. \quad h^{sq}(x) := (x_1^2 - x_2)^2 + (x_1^3 - x_3)^2 = 0. \end{cases}$$

It has no inequality constraints, and we can think that g = 0. Its feasible set is the curve parameterized as (x_1, x_1^2, x_1^3) . The minimum $f_{min} = -1$. We show that the sequence $\{f_k^{sq}\}$ for (3.5) has finite convergence. Let $\sigma_1 = (x_1^3 - 1)^2$ and

$$p = (x_1^3 - 2)(x_3 - x_1^3) + x_1 x_3 (x_2 - x_1^2).$$

Then, $f + 1 = p + \sigma_1$. Clearly, $p \equiv 0$ on $V_{\mathbb{R}}(h^{sq})$ and

$$p^2 + q = h^{sq}\psi$$

where

$$q = \left(x_1 x_3 (x_3 - x_1^3) - (x_1^3 - 2)(x_2 - x_1^2)\right)^2, \quad \psi = x_1^2 x_3^2 + (x_1^3 - 2)^2.$$

For all $\epsilon > 0$, we have $f + 1 + \epsilon = \phi_{\epsilon} + \sigma_{\epsilon}$ where

$$\phi_{\epsilon} = \frac{-1}{4\epsilon} \psi h^{sq} \in \langle h^{sq} \rangle_{12}, \quad \sigma_{\epsilon} = \epsilon \left(1 + \frac{p}{2\epsilon} \right)^2 + \frac{1}{4\epsilon} q + \sigma_1 \in Q_6(0).$$

= -1 for all $k \ge 6$.

So, $f_k^{sq} = -1$ for all $k \ge 6$.

We show an application of Theorem 1.2 in gradient SOS relaxations for minimizing polynomials [11]. Consider the unconstrained optimization problem

(3.6)
$$\min_{x \in \mathbb{P}^n} \quad f(x).$$

If (3.6) has a minimizer, then it is equivalent to

(3.7)
$$\min_{x \in \mathbb{R}^n} \quad f(x) \quad s.t. \quad \nabla f(x) = 0.$$

When $\langle \nabla f \rangle$ is radical, Lasserre's hierarchy for (3.7) has finite convergence [11]. Indeed, the finite convergence also occurs even if $\langle \nabla f \rangle$ is not radical, as shown in [12]. Clearly, (3.7) is equivalent to

(3.8)
$$\min_{x \in \mathbb{R}^n} \quad f(x) \quad s.t. \quad \|\nabla f(x)\|_2^2 = 0.$$

An advantage of (3.8) over (3.7) is that (3.8) has a single equality constraint. By Theorem 1.2, Lasserre's hierarchy of (3.8) also has finite convergence.

Example 3.3. ([2, 11]) Consider the polynomial optimization problem

$$\min_{x \in \mathbb{R}^2} \quad f(x) := x_1^2 x_2^2 (x_1^2 + x_2^2 - 1)$$

The minimum $f_{min} = -1/27$ is achieved at $(\pm 1, \pm 1)/\sqrt{3}$. We have

$$\partial f/\partial x_1 = 2x_1x_2^2(2x_1^2 + x_2^2 - 1), \quad \partial f/\partial x_2 = 2x_1^2x_2(x_1^2 + 2x_2^2 - 1).$$

This optimization problem is equivalent to

$$\min_{x \in \mathbb{R}^2} \quad f(x) \quad s.t. \quad \|\nabla f(x)\|_2^2 = 0$$

where $\|\nabla f(x)\|_2^2$ has the representation

$$4x_1^2x_2^2\Big(x_2^2(2x_1^2+x_2^2-1)^2+x_1^2(x_1^2+2x_2^2-1)^2\Big).$$

Let $\sigma_1 = 3(x_1^2 x_2^2 - 1/9)^2$ and $\hat{f} = f + 1/27 - \sigma_1$. Then, $\hat{f} \equiv 0$ on $V_{\mathbb{R}}(\nabla f)$ and $\hat{f}^2 + q = \|\nabla f\|_2^2 \psi$,

where q,ψ are SOS polynomials given as

$$a = x_1^2 - 1/3, \quad b = x_2^2 - 1/3,$$

$$s_1 = x_1^2 x_2^2 \Big(x_2^4 (2a+b)^2 + x_1^4 (a+2b)^2 \Big), \quad s_2 = 4(a+b)^2 (a^2+b^2) + a^4 + b^4,$$

$$q = \frac{9}{2} (s_1 (a^2+b^2) + x_1^4 x_2^4 s_2), \quad \psi = \frac{9}{8} (x_1^2 + x_2^2) (a^2+b^2).$$

For each $\epsilon > 0$, let

$$\phi_{\epsilon} = -\frac{1}{4\epsilon} \|\nabla f\|_2^2 \psi \in \langle \|\nabla f\|_2^2 \rangle_{16}, \quad \sigma_{\epsilon} = \epsilon \left(1 + \hat{f}/2\epsilon\right)^2 + \frac{q}{4\epsilon} + \sigma_1 \in Q_8(0).$$

Then, $f + 1/27 + \epsilon = \sigma_{\epsilon} + \phi_{\epsilon}$ for all $\epsilon > 0$. So, $f_k^{grad} = -1/27$ for all $k \ge 8$. \Box

4. Optimization over finite semialgebraic sets

In this section, we consider the case that the feasible set K of (1.1) is a finite set while the real variety $V_{\mathbb{R}}(h)$ is not necessarily. To apply Theorem 1.1, a natural idea is to introduce new variables z_1, \ldots, z_{m_2} . Then, K can be equivalently defined by the equations

$$h(x) = 0, \quad g_1(x) - z_1^2 = \dots = g_{m_2}(x) - z_{m_2}^2 = 0.$$

Clearly, K is a finite set if and only if the above equations have finitely many real solutions. If K is finite, by Theorem 1.1, Lasserre's hierarchy has finite convergence if we use the above equivalent polynomial equalities in both x_1, \ldots, x_n and z_1, \ldots, z_{m_2} . However, this approach introduces new variables z_1, \ldots, z_{m_2} , which typically make the resulting SOS relaxations very difficult to solve. To get a finitely convergent hierarchy of SOS relaxations that only uses the original polynomials in x, we need stronger relaxations than (1.2).

Let $Pr_k(g)$ be the k-th truncated quadratic module generated by the set of all possible cross products:

 $g_1, \ldots, g_{m_2}, g_1g_2, \ldots, g_{m_1-1}g_{m_1}, \ldots, g_1g_2\cdots g_{m_2}.$

The set $Pr_k(g)$ is also called the k-th truncated preordering generated by $g = (g_1, \ldots, g_{m_2})$ (cf. [3, 6, 9]). Consider the sequence of SOS relaxations $(k \in \mathbb{N})$

(4.1) $f_k^{pre} := \max \quad \gamma \quad s.t. \quad f - \gamma \in \langle h \rangle_{2k} + Pr_k(g).$

If K is compact, then $\{f_k^{pre}\}$ asymptotically converges to f_{min} (cf. [2, 16]). When K is finite, the sequence of optimal values of the dual problem of (4.1) has finite convergence, as shown by Lasserre, Laurent, and Rostalski [4, Remark 4.9]. Here, we show that the same result holds for the sequence $\{f_k^{pre}\}$.

Theorem 4.1. Let f_k^{pre} , f_{min} be as above. If the feasible set K of (1.1) is finite, then the sequence $\{f_k^{pre}\}$ has finite convergence to f_{min} .

Proof. The set K consists of finitely many points, say, $u_1, \ldots, u_D \in \mathbb{R}^n$. Let $\varphi_1, \ldots, \varphi_D \in \mathbb{R}[x]$ be the interpolating polynomials such that $\varphi_i(u_j) = 0$ for $i \neq j$ and $\varphi_i(u_j) = 1$ for i = j. Then, $f(u_i) - f_{min} \geq 0$ for all i. Let

$$a := \sum_{i=1}^{D} (f(u_i) - f_{min})\varphi_i^2 \in \Sigma \mathbb{R}[x]^2$$

The polynomial $\hat{f} := f - f_{min} - a$ vanishes identically on K. By Positivstellensatz (cf. [1, Corollary 4.4.3]), there exist integers $\ell > 0$ and $N_1 > 0$ such that

$$q \in Pr_{N_1}(g), \qquad \hat{f}^{2\ell} + q \in \langle h \rangle_{2N_1}$$

Applying Lemma 2.1 with $c \geq \frac{1}{2\ell}$ and $p = \hat{f}$, we get that, for all $\epsilon > 0$,

$$f - (f_{min} - \epsilon) = p + \epsilon + a = \sigma_{\epsilon} + \phi_{\epsilon},$$

$$\phi_{\epsilon} = -c\epsilon^{1-2\ell}(\hat{f}^{2\ell} + q) \in \langle h \rangle_{2N_1},$$

$$\sigma_{\epsilon} = \epsilon \left(1 + \hat{f}/\epsilon + c(\hat{f}/\epsilon)^{2\ell}\right) + c\epsilon^{1-2\ell}q + a.$$

Let $N \geq N_1$ be such that $\sigma_{\epsilon} \in Pr_N(g)$ for all $\epsilon > 0$. Like before, we have $f_k^{pre} = f_{min}$ for all $k \geq N$.

We illustrate the proof of Theorem 4.1 with the following example.

Example 4.2. Consider the optimization problem

$$\begin{array}{ll} \min & -x_1^2 - x_2^2 \\ s.t. & x_1^3 \ge 0, x_2^3 \ge 0, -x_1 - x_2 - x_1 x_2 \ge 0. \end{array}$$

Let f, g_1, g_2, g_3 be the objective, the first, second and third constraining polynomials respectively. Clearly, $K = \{(0,0)\}$ and $f_{min} = 0$. We have $f \equiv 0$ on K and

$$f^4 + q = 0$$

where

 $q = \sigma_0 + g_1 \sigma_1 + g_2 \sigma_2 + g_1 g_2 \sigma_{12} + g_3 \sigma_3.$

In the above, the SOS polynomials $\sigma_0, \sigma_1, \sigma_2, \sigma_{12}, \sigma_3$ are given as:

$$\begin{aligned} \sigma_0 &= (x_1^2 - x_2^2)^4 + 6(x_1^4 - x_2^4)^2, \ \sigma_{12} &= 32(x_1^2 + x_2^2 + x_1^4 + x_2^4 + x_1^6 + x_2^6), \\ \sigma_1 &= 8\left(x_1^6(x_2 + 1/2 + 4x_2^2) + x_1^4(x_1^2/2 + 2x_1x_2 + 2x_2^2) + x_1^4(2x_2 + 1/2 + 2x_2^2) + x_1^2(x_1^2/2 + x_1x_2 + 4x_2^2) + 4(x_2^4 + x_2^6 + x_2^8)\right), \\ \sigma_2 &= 8\left(x_2^6(x_1 + 1/2 + 4x_1^2) + x_2^4(x_2^2/2 + 2x_2x_1 + 2x_1^2) + x_2^4(2x_1 + 1/2 + 2x_1^2) + x_2^2(x_2^2/2 + x_2x_1 + 4x_1^2) + 4(x_1^4 + x_1^6 + x_1^8)\right), \\ \sigma_3 &= 8\left((x_1^8 + x_2^8 + x_1^7 + x_2^7 + x_1^6 + x_2^6) + 4x_1^2x_2^2(x_1^2 + x_2^2 + x_1^4 + x_2^4 + x_1^6 + x_2^6)\right). \end{aligned}$$

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Apply Lemma 2.1 with c = 1/4 and p = f. For each $\epsilon > 0$, let

$$\phi_{\epsilon} = 0, \quad \sigma_{\epsilon} := \epsilon \left(1 + \frac{f}{\epsilon} + \frac{f^4}{4\epsilon^4} \right) + \frac{1}{4\epsilon^3} q \in Pr_6(g).$$

Then, $f + \epsilon = \phi_{\epsilon} + \sigma_{\epsilon}$ for all $\epsilon > 0$. So, $f_k^{pre} = 0$ for all $k \ge 6$.

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