# Coalescing Random Walks and Voting on Connected Graphs* 

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#### Abstract

In a coalescing random walk, a set of particles make independent discrete-time random walks on a graph. Whenever one or more particles meet at a vertex, they unite to form a single particle, which then continues a random walk through the graph.

Let $G=(V, E)$, be an undirected and connected graph, with $n$ vertices and $m$ edges. The coalescence time, $C(n)$, is the expected time for all particles to coalesce, when initially one particle is located at each vertex. We study the problem of bounding the coalescence time for general connected graphs, and prove that


$$
C(n)=O\left(\frac{1}{1-\lambda_{2}}\left(\log ^{4} n+\frac{n}{\nu}\right)\right) .
$$

Here $\lambda_{2}$ is the second eigenvalue of the transition matrix of the random walk. To avoid problems arising from e.g. lack of coalescence on bipartite graphs, we assume the random walk can be made lazy if

[^0]required. The value of $\nu$ is given by $\nu=\sum_{v \in V} d^{2}(v) /\left(d^{2} n\right)$, where $d(v)$ is the degree of vertex $v$, and $d=2 m / n$ is the average degree. The parameter $\nu$ is an indicator of the variability of vertex degrees: $1 \leq \nu=O(n)$, with $\nu=1$ for regular graphs.

Our general bound on $C(n)$ holds for all connected graphs. This implies, for example, that $C(n)=O\left(n /\left(1-\lambda_{2}\right)\right)$ for $d$-regular graphs with expansion parameterized by the eigenvalue gap $1-\lambda_{2}$. The bound on $C(n)$ given above is sub-linear for some classes of graphs with skewed degree distributions.

In the voter model, initially each vertex has a distinct opinion, and at each step each vertex changes its opinion to that of a random neighbour. Let $\mathbf{E}(C \boldsymbol{v})$ be the expected time for voting to complete, that is, for a unique opinion to emerge. A system of coalescing particles, where initially one particle is located at each vertex, corresponds to the voter model in that $\mathbf{E}(C \boldsymbol{v})=C(n)$. Thus our result stated above for $C(n)$ also gives general bounds for $\mathbf{E}(C \boldsymbol{v})$.

## 1 Introduction

In a coalescing random walk, a set of particles make independent discretetime random walks in an undirected connected graph. Whenever two or more particles meet at a vertex, then they unite to form a single particle which then continues to make a random walk through the graph.

Let $G=(V, E)$ be an undirected connected graph with $n$ vertices and $m$ edges. The coalescence time is the expected time for all particles to coalesce, when initially one particle is located at each vertex of the graph. We study the problem of bounding the coalescence time for general connected graphs.

For a given graph $G$ we denote the coalescence time of an $n$ particle system by $C(n)=C_{G}(n)$. In order to bound $C(n)$, we study the coalescence time $C(k)=C_{G}(k)$ of a system of $k$ particles for any $2 \leq k \leq n$. The expected time for the $k$ particles to coalesce to a single particle depends on their initial positions. Let $C_{k}\left(i_{1}, \ldots, i_{k}\right)$, be the coalescence time when the particles start from distinct vertices $i_{1}, \ldots, i_{k}$. The worst case expected coalescence time for $k$ particles is

$$
C(k)=\max _{i_{1}, \ldots, i_{k}} \mathbf{E}\left(C_{k}\left(i_{1}, \ldots, i_{k}\right)\right) .
$$

In the special case of two particles, $C(2)$ is more naturally referred to as the (worst case expected) meeting time of two random walks.

A system of $n$ coalescing particles where initially one particle is located at each vertex, corresponds to another classical problem, the voter model, which is defined as follows. Initially each vertex has a distinct opinion, and at each step each vertex changes its opinion to that of a random neighbour.

Let $C \boldsymbol{v}$ be the number of steps for voting to be completed, i.e., for a unique opinion to emerge. The expected completion time of voting, $\mathbf{E}\left(C_{\boldsymbol{v}}\right)$, is called the voting time. The random variable $C \boldsymbol{v}$ has the same distribution, and hence the same expected value, as the coalescence time $C_{n}$ of $n$ coalescing particles, where one particle is initially located at each vertex (see [2]). Thus $C(n) \equiv \mathbf{E}\left(C_{n}\right)=\mathbf{E}(C \boldsymbol{v})$, and any bound on coalescence time $C(n)$ applies equally to the voting time $E(C \boldsymbol{v})$. As the coalescence time is easier to estimate, we focus on this quantity henceforth.

The coalescing random walk is the key ingredient in the self-stabilizing mutual exclusion algorithm of Israeli and Jalfon [8]. Initially each vertex emits a token which makes a random walk on $G$. On meeting at a vertex, tokens coalesce. Provided the graph is connected, and not bipartite, eventually only one token will remain, and the vertex with the token has exclusive access to some resource. The token makes a random walk on $G$, so in the long run it will visit all vertices of $G$ in proportion to their stationary distribution.

## Previous work on coalescing random walks

We summarize some known results for coalescing random walks. There are two distinct models for the transition times of random walks on finite graphs. In the discrete-time model, all walks make transitions synchronously at steps $t=1,2, \ldots$ In the continuous-time model, each walk $W$ waits for a random time $t_{W}$ independently of other walks, and then makes a transition. The wait time $t_{W}$ is an independent exponential random variable with rate 1.

Let $H_{u, v}$ denote the hitting time of vertex $v$ starting from vertex $u$, that is, the random variable which gives the time taken for a random walk starting from vertex $u$ to reach vertex $v$; and let $H_{\max }=\max _{u, v} \mathbf{E}\left(H_{u, v}\right)$. Aldous [1] considers $C(2)$, the meeting time of two random walks, in the continuoustime model, and shows that

$$
C(2)=\Omega(m / \Delta) \quad \text { and } \quad C(2)=O\left(H_{\max }\right)
$$

where $\Delta$ is the maximum degree of $G$. These upper and lower bounds can be far apart, e.g. for a star graph (with loops), $C(2)=\Theta(1)$ whereas $H_{\max }=$ $\Theta(n)$.

The $O\left(H_{\max }\right)$ bound on $C(2)$ implies that $C(n)=O\left(H_{\max } \log n\right)$, since the number of particles halves in $O\left(H_{\max }\right)$ time. Aldous [1] conjectured that $C(n)$ is actually $O\left(H_{\max }\right)$. Earlier results by Cox [5] for the continuous-time model, imply $C(n)=O\left(H_{\max }\right)$ for constant dimension tori and grids.

For regular graphs, in the continuous-time model, Aldous and Fill [2] show that, $C(n) \leq e(\log n+2) H_{\text {max }}, C(n) \leq r n^{2} /(4 s)$ for $r$-regular $s$-edge connected graphs, and $C(n) \sim n$ for complete graphs. Cooper et al. [3] confirmed that the conjecture $C(n)=O\left(H_{\max }\right)$ holds for discrete-time random walks on random regular graphs. This follows from their result that for $r$-regular random graphs $C(n) \sim 2((r-1) /(r-2)) n$, with high probability. We use the notation with high probability (whp), to mean with probability tending to 1 as $n \rightarrow \infty$. The notation $f(n) \sim g(n)$ means that $f(n)=(1 \pm o(1)) g(n)$.

Simple bounds on $H_{\max }$ can be obtained from the commute time between any pair of vertices (see e.g. Corollary 3.3 of Lovasz [10]). For a graph $G$ with $n$ vertices, $m$ edges and minimum degree $\delta(G)$, we have

$$
\begin{equation*}
\frac{m}{2 \delta(G)} \leq H_{\max } \leq \frac{4 m}{\left(1-\lambda_{2}\right) \delta(G)} \tag{1}
\end{equation*}
$$

As $\delta(G) \leq d$ the average degree, it follows that $H_{\max } \geq n / 4$ for any graph. An upper bound, for connected graphs, of $H_{\max } \leq 4 m /\left(1-\lambda_{2}\right)$ follows from $\delta(G) \geq 1$.

## General results for coalescing walks on graphs

In this article, we study the problem of bounding the coalescence time $C(n)$ of any connected graph. We assume that the graphs $G$ we consider are not bipartite, or that if $G$ is bipartite, then the random walks are lazy and pause with probability $1 / 2$ at each step. Equivalently, for the voting process, we assume that vertices may choose their own opinion with this probability.

Our main result, stated formally below, is given in terms of the second eigenvalue of the transition matrix of the random walk, $\lambda_{2}$, and a parameter $\nu$ which measures the variability of the degree sequence. Let $d(v)$ be the degree of vertex $v$, and $d=2 m / n$ the average degree. The parameter $\nu$, the ratio of the average squared degree to the average degree squared, is defined as

$$
\begin{equation*}
\nu=\frac{\sum_{v \in V}(d(v))^{2}}{d^{2} n} \tag{2}
\end{equation*}
$$

This can also be written as $\nu=\left(n \sum_{v \in V} d^{2}(v)\right) /(2 m)^{2}$. The parameter $\nu$ ranges from 1 for regular graphs to $\Theta(n)$ for a star graph. We prove the following general theorem.
Theorem 1 Let $G$ be a connected graph with $n$ vertices, $m$ edges, and let $\nu=\left(n \sum_{v \in V} d^{2}(v)\right) /(2 m)^{2}$. Let $C(n)$ be the expected coalescence time for a system of $n$ particles making a lazy random walk on $G$, where originally one particle starts at each vertex. Then

$$
\begin{equation*}
C(n)=O\left(\frac{1}{1-\lambda_{2}}\left(\log ^{4} n+\frac{n}{\nu}\right)\right) \tag{3}
\end{equation*}
$$

By the equivalence between coalescence and voting, the expected time $\mathbf{E}(C \boldsymbol{v})$ to complete voting on $G$ has the same upper bound as $C(n)$.

Although Theorem 1 is a general statement of our results, the bound (3) can be improved in extremal cases. It is established in (29) of Section 5 that

$$
\begin{equation*}
C(n)=O\left(\frac{1}{1-\lambda_{2}}\left(\frac{m}{\Delta} \log n\right)^{2}\right) \tag{4}
\end{equation*}
$$

This bound is better than (3), if $\Delta=\omega(m / \log n)$. For example, for a star, (3) gives $C(n)=O\left(\log ^{4} n\right)$ and (4) gives $C(n)=O\left(\log ^{2} n\right)$, whereas the correct value is $C(n)=\Theta(\log n)$ (since a star is a bipartite graph, we consider the lazy walk).

Hassin and Peleg [7] showed that voting (hence also coalescence) is completed in expected $O\left(n^{3} \log n\right)$ time on any connected graph. The bound (3) is parameterized by the eigenvalue gap, and can offer a refinement of Hassin and Peleg's bound. As $1-\lambda_{2}=\Omega\left(1 / n^{2}\right)$ for any connected regular graph, coalescence for these graphs is completed in $O\left(n^{3}\right)$ expected time. An example of a (non-regular) graph with coalescence time $\Theta\left(n^{3}\right)$ is given by two cliques of size $n / 4$ joined by a path of length $n / 2$. On the other hand $1-\lambda_{2}=\Theta\left(1 / n^{3}\right)$ for lollypop graphs, indicating that the bound (3) is not tight.

The parameter $\nu$ is related to the second moment of the degree distribution and measures the variability of the degree sequence. If $\Delta$ is the maximum degree of $G$, then $1 \leq \nu \leq \Delta / d \leq n$. For near regular graphs, when the ratio of the largest to the smallest vertex degree is bounded by a constant, we have $\nu \leq \Delta / d=O(1)$, so the bound (3) becomes

$$
C(n)=O\left(\frac{n}{1-\lambda_{2}}\right)
$$

In particular, if $G$ is an expander in the classic sense that it is regular and its eigenvalue gap $\left(1-\lambda_{2}\right)$ is constant, then $C(n)=O(n)$.

In parallel with our work, Oliveira [11] recently proved the conjecture $C(n)=O\left(H_{\max }\right)$ for continuous-time random walks. The result of Oliveira implies an analogous linear bound $C(n)=O(n)$ for continuous-time random walks on expanders.

We note that the bound (3) is qualitatively different from $O\left(H_{\max }\right)$, as the graph structure is made explicit through the parameter $\nu$. As $H_{\max }=\Omega(n)$ for any graph (see (1)), the bound (3) can improve on $C(n)=O\left(H_{\max }\right)$. This can occur for example if $\nu=\omega(1)$, but also when $\nu=\Theta(1)$, since there are graphs with $H_{\max }=\omega\left(n /\left(1-\lambda_{2}\right)\right)$. Some examples follow.

For graphs with a power law (heavy tailed) degree distribution, Theorem 1 can give sublinear bounds on the coalescence and voting times as the following example shows. Mihail et al. [6] prove that for $2<\alpha<3$, the random $\Theta(n)$-vertex graph with $\left\lceil n / d^{\alpha}\right\rceil$ vertices of degree $d$, for $d=3,4, \ldots, n^{1 / 2}$, has an $\Omega\left(\log ^{-2} n\right)$ eigenvalue gap. For this class of power law graphs, $\nu=$ $\Theta\left(n^{(3-\alpha) / 2}\right)$, so Theorem 1 implies a sublinear $O\left(n^{(\alpha-1) / 2} \log ^{2} n\right)$ voting time, whereas for any graph, $H_{\max }=\Omega(n)$.

There are also examples of graphs with $\nu=\Theta(1)$ for which our bound is asymptotically better than $O\left(H_{\max }\right)$. Consider the graph consisting of $(\log n)$-degree expander $\left(1-\lambda_{2} \leq c<1\right)$ with an additional vertex attached to one of the vertices of the expander. For this graph $\nu=\Theta(1)$ and $1-\lambda_{2}$ is a positive constant, so $C(n)=O(n)$, but $H_{\max }=\Theta(n \log n)$.

It would be interesting to have a general lower bound on $C(n)$ which incorporates the graph structure in a similar way to the upper bounds (3) and (4), but it is not clear what form such a bound might take. A weaker conjecture is $C(n)=\Omega\left(1 /\left(1-\lambda_{2}\right)\right)$. This bound is tight for a path on $n$ vertices, where $1 /\left(1-\lambda_{2}\right)=\Theta\left(n^{2}\right)$ and $C(n)=C(2)=\Theta\left(n^{2}\right)$. Indeed $C(n) \leq n^{2}$, the cover time of the graph by a particle starting from the left most vertex; and $C(2) \geq n^{2} / 4$, the expected hitting time of the central vertex by particles starting from the left most and right most vertices.

## Structure of the paper

The analysis of the coalescence process (that is, the proof of Theorem 1) is divided into two phases. During the first phase the number of particles decreases from the initial $n$ to a threshold value $k^{*}$. This phase is analysed by showing that for a suitably chosen number of steps $t^{*}=t^{*}\left(k^{*}\right)$, the probability
that there exist $k^{*}$ particles which do not have a single meeting between them within the first $t^{*}$ steps is at most $1 / 2$. This implies that with probability at least $1 / 2$, the number of particles at step $t^{*}$ is less than $k^{*}$.

The second phase, when the number of particles decreases from $k^{*}$ to 1 , is analysed by bounding the expected time we have to wait until the first meeting between any of $k$ particles, where $2 \leq k \leq k^{*}$. At the time of this first meeting, the number of particles decreases from $k$ to $k-1$ (with some relatively small probability, the first meeting could involve more than 2 particles, reducing the number of particles to fewer than $k-1$ ). The analysis of the second phase is based on the following theorem bounding the expected time to first meeting between any of $k$ particles.

Theorem 2 Let $k^{*}$ be given by

$$
\begin{equation*}
k^{*}=\max \left\{2, \min \left\{\left(\frac{n}{\nu}\right)^{1 / 2}, \frac{m}{2 \Delta}, \log n\right\}\right\} \tag{5}
\end{equation*}
$$

where $\Delta$ is the maximum degree, and $\nu$ given by (2). For $2 \leq k \leq k^{*}$ particles starting from arbitrary vertices in $G$, let $M_{k}$ be the time to first meeting. Then

$$
\begin{equation*}
\mathbf{E}\left(M_{k}\right)=O\left(\frac{1}{1-\lambda_{2}}\left(k \log n+\frac{n}{\nu k^{2}}\right)\right) . \tag{6}
\end{equation*}
$$

The expression (5) for the threshold value $k^{*}$ is not very transparent, but seems to be necessary to deal with the generality of degree sequences of connected graphs. Provided the maximum degree of the graph satisfies $\Delta \leq 2 m / \log ^{2} n$, then $k^{*}=\log n$. The other terms are there to cover extremal cases such as star graphs. The condition that $k^{*} \geq 2$ ensures there are at least 2 particles to coalesce.

Section 2 gives background material on random walks. Section 3 replaces multiple random walks by a single walk on a suitably defined product graph. Theorem 2 is proven in Section 4 and the proof of Theorem 1 is concluded in Section 5.

## 2 Random walk properties

Let $G=(V, E)$ denote a connected undirected graph, $|V|=n,|E|=m$, and let $d(v)$ be the degree of a vertex $v$. A simple random walk $\mathcal{W}_{u}, u \in V$,
on graph $G$ is a Markov chain modeled by a particle moving from vertex to vertex according to the following rule. The probability of transition from vertex $v$ to vertex $w$ is equal to $1 / d(v)$, if $w$ is a neighbour of $v$, and 0 otherwise. The walk $\mathcal{W}_{u}$ starts from vertex $u$ at $t=0$. Denote by $\mathcal{W}_{u}(t)$ the vertex reached at step $t ; \mathcal{W}_{u}(0)=u$.

We assume $G$ is connected, and the random walk $\mathcal{W}_{u}$ on $G$ is ergodic with stationary distribution $\pi$, where $\pi_{v}=d(v) /(2 m)$. If this is not the case, e.g. $G$ is bipartite, then the walk can be made ergodic, by making it lazy. A random walk is lazy, if it moves from $v$ to one of its neighbours $w$ with probability $1 /(2 d(v))$, and stays at vertex $v$ with probability $1 / 2$.

Let $P=P(G)$ be the matrix of transition probabilities of the walk and let $P_{u}^{t}(v)=\operatorname{Pr}\left(\mathcal{W}_{u}(t)=v\right)$. The eigenvalues of $P(G)$ are real, and can be ordered $\lambda_{1}=1>\lambda_{2} \geq \cdots \geq \lambda_{n}$, where $\lambda_{n}>-1$ as the walk is ergodic. Let $\lambda=\max \left(\lambda_{2},\left|\lambda_{n}\right|\right)<1$. The rate of convergence of the walk is given by

$$
\begin{equation*}
\left|P_{u}^{t}(x)-\pi_{x}\right| \leq\left(\pi_{x} / \pi_{u}\right)^{1 / 2} \lambda^{t} \tag{7}
\end{equation*}
$$

where $|r|$ is the absolute value of the real number $r$. For a proof see for example, Lovasz [10] Theorem 5.1. We assume henceforth that $\lambda=\lambda_{2}$. If not, the standard way to ensure that $\lambda=\lambda_{2}=\lambda_{2}(G)$, is to make the chain lazy.

We use the following definition of mixing time $T_{G}$, for a graph $G$. For all vertices $u$ and $x$ in $G$ and any $t \geq T_{G}$,

$$
\begin{equation*}
\left|P_{u}^{(t)}(x)-\pi_{x}\right| \leq o\left(\frac{1}{n^{2}}\right) \tag{8}
\end{equation*}
$$

For convenience we assume that $T_{G}=\Omega(\log n)$, even if this is not necessary.
Let $\mathbf{E}_{\pi}\left(H_{w}\right)$ denote the expected hitting time of a vertex $w$ from the stationary distribution $\pi$. The quantity $\mathbf{E}_{\pi}\left(H_{w}\right)$ can be expressed as (see e.g. [2], Chapter 2)

$$
\begin{equation*}
\mathbf{E}_{\pi}\left(H_{v}\right)=Z_{v v} / \pi_{v}, \tag{9}
\end{equation*}
$$

where

$$
\begin{equation*}
Z_{v v}=\sum_{t=0}^{\infty}\left(P_{v}^{(t)}(v)-\pi_{v}\right) \tag{10}
\end{equation*}
$$

Let $A_{v}(t ; u)$ denote the event that $\mathcal{W}_{u}$ does not visit vertex $v$ in steps $0, \ldots, t$. The following lemma gives a bound on the probability of this event in terms of $\mathbf{E}_{\pi}\left(H_{v}\right)$ and the mixing time of the walk.

Lemma 1 Let $T=T_{G}$ be a mixing time of a random walk $\mathcal{W}_{u}$ on $G$ satisfying (8). Then

$$
\operatorname{Pr}\left(A_{v}(t ; u)\right) \leq e^{-\left\lfloor t /\left(T+3 \mathbf{E}_{\pi}\left(H_{v}\right)\right)\right\rfloor}
$$

Proof. Let $\rho \equiv P_{u}^{(T)}$ be the distribution of $\mathcal{W}_{u}$ on $G$ after $T$ steps. Then (8) and the fact that $\pi_{x} \geq 1 / n^{2}$ for any connected graph imply

$$
\begin{equation*}
\mathbf{E}_{\rho}\left(H_{v}\right)=(1+o(1)) \mathbf{E}_{\pi}\left(H_{v}\right) . \tag{11}
\end{equation*}
$$

Let $H_{v}(\rho)$ be the time to hit $v$ starting from $\rho$, and let $\tau=T+3 \mathbf{E}_{\pi}\left(H_{v}\right)$. Then, noting that $\mathbf{E}_{\rho}\left(H_{v}\right) \equiv \mathbf{E}\left(H_{v}(\rho)\right)$,

$$
\begin{aligned}
\operatorname{Pr}\left(A_{v}(\tau ; u)\right) & =\operatorname{Pr}\left(A_{v}(T ; u) \text { and } H_{v}(\rho) \geq 3 \mathbf{E}_{\pi}\left(H_{v}\right)\right) \\
& \leq \operatorname{Pr}\left(H_{v}(\rho) \geq 3 \mathbf{E}_{\pi}\left(H_{v}\right)\right) \\
& \leq \operatorname{Pr}\left(H_{v}(\rho) \geq e \cdot \mathbf{E}\left(H_{v}(\rho)\right)\right) \\
& \leq \frac{1}{e}
\end{aligned}
$$

By restarting the process $\mathcal{W}_{u}$ at $\mathcal{W}_{u}(0)=u, \mathcal{W}_{u}(\tau), \mathcal{W}_{u}(2 \tau), \ldots, \mathcal{W}_{u}((\lfloor t / \tau\rfloor-$ 1) $\tau$ ), we obtain

$$
\operatorname{Pr}\left(A_{v}(t ; u)\right) \leq e^{-\lfloor t / \tau\rfloor}
$$

## 3 Multiple random walks

We consider the coalescence of $k \geq 2$ independent random walks on a graph $G=\left(V_{G}, E_{G}\right)$. To do this we replace the $k$ walks by a single walk as follows.

Let graph $Q=Q_{k}=\left(V_{Q}, E_{Q}\right)$ have vertex set $V_{Q}=V^{k}$. Thus a vertex $\boldsymbol{v}$ of $Q_{k}$ is a $k$-tuple $\boldsymbol{v}=\left(v_{1}, v_{2}, \ldots, v_{k}\right)$ of vertices $v_{i} \in V_{G}, i=1, \ldots, k$, with repeats allowed. Two vertices $\boldsymbol{v}, \boldsymbol{w} \in V_{Q}$ are adjacent if $\left\{v_{1}, w_{1}\right\}, \ldots,\left\{v_{k}, w_{k}\right\}$ are edges of $G$. There is a direct equivalence between $k$ random walks $\mathcal{W}_{u_{i}}(t)$ on $G$ with starting positions $u_{i}$ and a single random walk $\mathcal{W}_{\boldsymbol{u}}(t)$ on $Q_{k}$ with starting position $\boldsymbol{u}=\left(u_{1}, u_{2}, \ldots, u_{k}\right)$.

For any starting positions $\boldsymbol{u}=\left(u_{1}, \ldots, u_{k}\right)$ of the walks, let $M_{k}(\boldsymbol{u})$ be the time until the first meeting in $G$. Let $S_{k} \subseteq V\left(Q_{k}\right)$, the diagonal set of vertices, be defined by

$$
S=S_{k}=\left\{\left(v_{1}, \ldots, v_{k}\right): v_{i}=v_{j} \text { some } 1 \leq i<j \leq k\right\}
$$

If the random walk on $Q_{k}$ visits this set, two particles occupy the same vertex in the underlying graph $G$ and a (coalescing) meeting occurs.

The number of visits to the set $S_{k}$ by a random walk is not a readily manipulated quantity. An easier approach is to contract $S_{k}$ to a single vertex $\gamma=\gamma_{k}=\gamma\left(S_{k}\right)$, thus replacing $Q_{k}$ by a graph $\Gamma=\Gamma_{k}$. On contraction, all edges, including loops, are retained. Thus $d_{\Gamma}(\gamma)=d_{Q}(S)$, where $d_{F}$ denotes vertex degree in graph $F$, and the degree $d_{F}(X)$ of a set $X$ is the sum of the degrees of the vertices in $X$. Moreover $\Gamma$ and $Q$ have the same total degree, and the degree of any vertex of $\Gamma$ other than $\gamma$ is the same as in graph $Q$. Let $\pi$ and $\hat{\pi}$ be the stationary distributions of a random walk on $Q$ and $\Gamma$, respectively. If $\boldsymbol{v} \notin S$ then $\hat{\pi}_{\boldsymbol{v}}=\pi_{\boldsymbol{v}}$, and $\hat{\pi}_{\gamma}=\pi_{S} \equiv \sum_{\boldsymbol{x} \in S} \pi_{\boldsymbol{x}}$.

It follows that, if $T_{\Gamma}$ is a mixing time satisfying (8) in $\Gamma$, then

$$
\begin{equation*}
\mathbf{E}\left(M_{k}(\boldsymbol{u})\right) \leq T_{\Gamma}+(1+o(1)) \mathbf{E}_{\widehat{\pi}}\left(H_{\gamma_{k}}\right) \tag{12}
\end{equation*}
$$

where $\mathbf{E}_{\widehat{\pi}}\left(H_{\gamma_{k}}\right)$ is the hitting time of $\gamma_{k}$ in $\Gamma$ from stationarity.
Since we have replaced $k$ individual walks on $G$ by a single walk on $Q_{k}$, and then on $\Gamma$, we need to relate mixing times on $T_{Q}$ and $T_{\Gamma}$ directly to a given mixing time $T_{G}$ of a single random walk on the underlying graph $G$. (We will need $T_{\Gamma}$ in two places: in the bound (12) and when applying Lemma 1 to graph $\Gamma$.)

Lemma 2 For random walks in graphs $G, Q$ and $\Gamma$, there are mixing times

$$
\begin{equation*}
T_{G}=O\left(\frac{\log n}{1-\lambda_{2}(G)}\right), T_{Q}=O\left(k T_{G}\right), T_{\Gamma}=O\left(k T_{G}\right) \tag{13}
\end{equation*}
$$

such that

$$
\max _{u, x \in V_{F}}\left|P_{u}^{t}(x)-\pi_{x}\right|=o\left(1 / n_{F}^{2}\right), \quad \text { for any } t \geq T_{F}
$$

where $F$ is any of the graphs $G, Q$ or $\Gamma$, and $n_{F}=\left|V_{F}\right|$.
Proof. The bound on $T_{G}$ is well known (see for example, Sinclair [12]): use (7), observing that $\pi_{x} / \pi_{u}=O(n)$ and $\lambda_{2}^{1 /\left(1-\lambda_{2}\right)}$ has a constant $c<1$ upper bound. To use (7) also to derive bounds on $T_{Q}$ and $T_{\Gamma}$, we need to know the eigenvalues of $Q_{k}$ and $\Gamma$ in terms of the eigenvalues of $G$. We have $\lambda_{2}(\Gamma) \leq \lambda_{2}\left(Q_{k}\right)$ and $\lambda_{2}\left(Q_{k}\right)=\lambda_{2}(G)$. This follows from established results, as we next explain.

In the notation of Markov processes, the random walk on $Q_{k}$ is known as the tensor product chain, and its eigenvalues are the $k$-wise products of the

| Graph | vertices | Stationary distribution $\pi$ | Mixing time |
| :--- | :--- | :--- | :--- |
| $G$ | $n_{G}=n$ | $\pi_{v}=d(v) / 2 m$ | $T_{G}=O\left(\log n /\left(1-\lambda_{2}\right)\right)$ |
| $Q_{k}$ | $n_{Q}=n^{k}$ | $\pi_{\boldsymbol{v}}=d\left(v_{1}\right) \cdots d\left(v_{k}\right) /(2 m)^{k}$ | $T_{Q}=O\left(k T_{G}\right)$ |
| $\Gamma_{k}$ | $n_{\Gamma} \leq n^{k}$ | $\pi_{\gamma} \geq k^{2} \nu /(8 n)$ | $T_{\Gamma}=O\left(T_{Q}\right)$ |

Table 1: The main parameters of the random walks on graphs $G, Q_{k}$ and $\Gamma_{k}$.
eigenvalues of $G$. Thus, assuming $\lambda_{2}(G) \geq \lambda_{n}(G)$, it follows that $\lambda_{2}\left(Q_{k}\right)=$ $\lambda_{2}(G)$. See [9] page 168 for more details.

In the notation of [2, Ch. 3], the random walk on $\Gamma$ is the random walk on $Q_{k}$ with $S$ collapsed to $\gamma(S)$. It is proved in [2, Ch. 3], Corollary 27, that if a subset $A$ of vertices is collapsed to a single vertex, then the second eigenvalue of the transition matrix cannot increase (in that corollary the variable $\left.\tau_{2}=1 /\left(1-\lambda_{2}\right)\right)$. Thus $\lambda_{2}(Q) \geq \lambda_{2}(\Gamma)$.

We get the factor $k$ in the bounds (13) on the mixing times $T_{Q}$ and $T_{\Gamma}$, because $\pi_{x} / \pi_{u}=O\left(n^{2 k}\right)$ and we need $\left|P_{u}^{T}(x)-\pi_{x}\right|=o\left(1 / n^{2 k}\right)$, as the number of vertices in graphs $Q$ and $\Gamma$ is $O\left(n^{k}\right)$.

For reference, we record the salient facts for the graphs $G, Q, \Gamma$ in Table 1. The bound on $\pi_{\gamma}$ will be established in Lemma 4.

## 4 Hitting time from stationarity - Proof of Theorem 2

The proof of Theorem 2 is based on Inequality (12) and on a good upper bound on the expected hitting time of vertex $\gamma$ by a random walk in $\Gamma$ which starts from the stationary distribution. We obtain such a bound using (9) by deriving an upper bound on $Z_{\gamma \gamma}$ (Lemma 3) and a lower bound on the stationary probability $\pi_{\gamma}=\widehat{\pi}_{\gamma}$ (Lemma 4$)$.

Lemma 3 Let $F$ be a graph with the eigenvalue gap $1-\lambda_{2}$, then

$$
\begin{equation*}
Z_{v v} \leq \frac{1}{1-\lambda_{2}} \tag{14}
\end{equation*}
$$

In particular, for any vertex $v$ of $G, Q$ or $\Gamma, Z_{v v} \leq 1 /\left(1-\lambda_{2}(G)\right)$.

Proof. Let $\lambda_{2}=\lambda_{2}(F)$. Using (7) with $x=u=v$ gives

$$
\left|P_{v}^{t}(v)-\pi_{v}\right| \leq \lambda_{2}^{t}
$$

and thus

$$
Z_{v v}=\sum_{t \geq 0}\left(P_{v}^{t}(v)-\pi_{v}\right) \leq \sum_{t \geq 0} \lambda_{2}^{t}=\frac{1}{1-\lambda_{2}} .
$$

The proof of Lemma 2 establishes that $\left(1-\lambda_{2}(\Gamma)\right) \geq 1-\lambda_{2}(Q)=1-\lambda_{2}(G)$.

Lemma 4 Let $G$ be a connected graph with $n$ vertices and $m$ edges. Let

$$
\begin{equation*}
k^{*}=\max \left\{2, \min \left\{\left(\frac{n}{\nu}\right)^{1 / 2}, \frac{m}{2 \Delta}, \log n\right\}\right\}, \tag{15}
\end{equation*}
$$

where $\Delta$ is the maximum degree of $G$ and $\nu=\left(n /(2 m)^{2}\right) \sum_{v \in V} d^{2}(v)$. Let $k$ be integer, $2 \leq k \leq k^{*}$. Let $\gamma=\gamma_{k}$ in $\Gamma$ be the contraction of $S=S_{k}$ in $Q$. Then

$$
\begin{equation*}
\pi_{\gamma}=\frac{d(\gamma)}{(2 m)^{k}} \geq \frac{k^{2} \nu}{8 n} \tag{16}
\end{equation*}
$$

Proof. By definition, $d(\gamma)=d(S)$. If $k=2$, then,

$$
d(S)=\sum_{v \in V} d^{2}(v)=(2 m)^{2} \frac{\nu}{n}
$$

If $3 \leq k \leq k^{*}$, for $1 \leq x<y \leq k$, define the following subsets of $S$ :

$$
S_{(x, y)}=\left\{\left(v_{1}, \ldots, v_{k}\right): v_{x}=v_{y}\right\}
$$

We have

$$
S=\bigcup_{1 \leq x<y \leq k} S_{(x, y)}
$$

and

$$
d\left(S_{(x, y)}\right)=(2 m)^{k-2} \sum_{v \in V} d^{2}(v)=(2 m)^{k} \frac{\nu}{n}
$$

For $\{x, y\} \neq\{p, q\}, d\left(S_{(x, y)} \cap S_{(p, q)}\right)$ equals to

$$
\begin{array}{ll}
(2 m)^{k-4} \sum_{v, u \in V} d^{2}(v) d^{2}(u), & \text { if }\{x, y\} \cap\{p, q\}=\emptyset, \text { or } \\
(2 m)^{k-3} \sum_{v \in V} d^{3}(v), & \text { if }|\{x, y\} \cap\{p, q\}|=1 .
\end{array}
$$

Therefore, from the inclusion-exclusion principle,

$$
\begin{align*}
d(S) & \geq \sum_{\{x, y\}} d\left(S_{(x, y)}\right)-\sum_{\{x, y\} \neq\{p, q\}} d\left(S_{(x, y)} \cap S_{(p, q)}\right) \\
& \geq\binom{ k}{2}(2 m)^{k} \frac{\nu}{n}-3\binom{k}{4}(2 m)^{k} \frac{\nu^{2}}{n^{2}}-3\binom{k}{3}(2 m)^{k} \frac{\Delta \nu}{2 m n}  \tag{17}\\
& \geq\binom{ k}{2}(2 m)^{k} \frac{\nu}{n}\left(1-\frac{k^{2} \nu}{4 n}-\frac{k \Delta}{2 m}\right)  \tag{18}\\
& \geq\binom{ k}{2}(2 m)^{k} \frac{\nu}{2 n} \tag{19}
\end{align*}
$$

The factor 3 in (17) occurs as the number of ways to partition 4 objects into disjoint sets of size 2 , and partition 3 objects into sets of size 2 with single intersection, respectively. The bound (19) follows from (18), by noting the upper bound on $k$ in (15).

Proof of Theorem 2. Let $M_{k}$ be the time of the first meeting among $k \leq k^{*}$ particles in $G$, and let $\gamma=\gamma_{k}$ be the contraction of the diagonal set $S=S_{k}$. Using (9) for graph $\Gamma$ and with $v=\gamma$, and Lemmas 3 and 4 we have, that the hitting time $H_{\gamma}$ of $\gamma$ from stationarity has expected value

$$
\begin{align*}
\mathbf{E}_{\pi}\left(H_{\gamma}\right) & \leq \frac{1}{\pi(\gamma)} \frac{1}{1-\lambda_{2}}  \tag{20}\\
& \leq \frac{8}{k^{2}} \frac{n}{\nu} \frac{1}{1-\lambda_{2}} . \tag{21}
\end{align*}
$$

Since $T_{\Gamma}=O\left(k T_{G}\right)$, and referring to (12) and Table 1,

$$
\begin{align*}
\mathbf{E}\left(M_{k}\right) & \leq O\left(k T_{G}\right)+(1+o(1)) \mathbf{E}_{\pi}\left(H_{\gamma}\right)  \tag{22}\\
& =O\left(\frac{1}{1-\lambda_{2}}\left(k \log n+\frac{n}{\nu k^{2}}\right)\right) . \tag{23}
\end{align*}
$$

Let $C_{k}$ be the time for $k \leq k^{*}$ particles to coalesce. For use in the proof of Theorem 1 in the next section, we state an upper bound on $\mathbf{E}\left(C_{k}\right)$ which follows directly from Theorem 2. Using (23) and noting that $\sum_{s}\left(1 / s^{2}\right) \leq$ $\pi^{2} / 6$ is constant, we have

$$
\begin{equation*}
\mathbf{E}\left(C_{k}\right) \leq \sum_{s=2}^{k} \mathbf{E}\left(M_{s}\right)=O\left(\frac{1}{1-\lambda_{2}}\left(k^{2} \log n+\frac{n}{\nu}\right)\right) \tag{24}
\end{equation*}
$$

## 5 Coalescence time: Proof of Theorem 1

We consider the case of $n$ coalescing particles, where each particle is initially located at a distinct vertex of the graph. The purpose of this section is to conclude the proof that for any connected graph

$$
\begin{equation*}
C(n)=O\left(\frac{1}{1-\lambda_{2}}\left(\log ^{4} n+\frac{n}{\nu}\right)\right) \tag{25}
\end{equation*}
$$

To establish this result, we first prove that the probability that there exist $k^{*}$ particles which do not have a single meeting between them within the first $t^{*}$ steps is at most $1 / 2$, if

$$
t^{*}=k^{*} \log n\left(T_{\Gamma}+3 \mathbf{E}_{\pi}\left(H_{\gamma}\right)\right),
$$

where $\Gamma=\Gamma_{k^{*}}, \gamma=\gamma_{k^{*}}$ and the value of $k^{*}$ is given in (15). An upper bound on the expected time $\mathbf{E}\left(C_{k}\right)$ for $k \leq k^{*}$ particles to coalesce is given in (24) above, and we can deal with that part separately.

Let $\mathcal{P}=\mathcal{P}(\boldsymbol{v})$ be the set of $k^{*}$ particles starting from vertices $\boldsymbol{v}=$ $\left(v_{1}, \ldots, v_{k^{*}}\right)$. The probability that the particles in $\mathcal{P}$ do not meet by time $t$ is the same as the probability that the random walk in $\Gamma$ starting from $\boldsymbol{v}$ does not visit $\gamma$ by time $t$. We apply Lemma 1 to graph $\Gamma$, vertex $\gamma$ and $t=t^{*}$, and obtained that

$$
\begin{aligned}
\operatorname{Pr}(\text { no meeting among particles in } & \left.\mathcal{P} \text { before } t^{*}\right) \\
& \leq e^{-k^{*} \log n}=n^{-k^{*}}
\end{aligned}
$$

In the coalescence process, we can assume that if two or more particles meet at the same vertex, then the lowest index particle survives (and continues its random walk) while the other particles die. Thus if there are $k^{*}$ or more particles after $t^{*}$ steps, then there is a set $\mathcal{P}$ of $k^{*}$ particles which do not meet within $t^{*}$ steps. Therefore,

$$
\begin{align*}
& \operatorname{Pr}\left(\text { at least } k^{*} \text { particles after } t^{*} \text { steps }\right) \\
& \quad \leq \operatorname{Pr}\left(\text { exists a set } \mathcal{P} \text { of } k^{*} \text { particles with no meeting before } t^{*}\right) \\
& \quad \leq\binom{ n}{k^{*}} n^{-k^{*}} \leq \frac{1}{2} \tag{26}
\end{align*}
$$

The last inequality holds because $\binom{n}{k} \leq n^{k} / k$ ! and $k^{*} \geq 2$. The bound (26) implies that the expected number of steps until fewer than $k^{*}$ particles remain is at most $t^{*}+\frac{1}{2}\left(2 t^{*}\right)+\frac{1}{4}\left(3 t^{*}\right)+\cdots=4 t^{*}$. Therefore, using $T_{\Gamma}=$
$O\left(k^{*} \log n /\left(1-\lambda_{2}\right)\right)$ from Lemma 2, the bound on $\mathbf{E}_{\pi}\left(H_{\gamma}\right)$ given in (21), and the bound on $\mathbf{E}\left(C_{k^{*}}\right)$ given in (24), we obtain the bound (25):

$$
\begin{align*}
C(n) & \leq 4 t^{*}+\mathbf{E}\left(C_{k^{*}}\right) \\
& =O\left(\frac{\left(k^{*} \log n\right)^{2}}{1-\lambda_{2}}+\frac{1}{1-\lambda_{2}} \frac{\log n}{k^{*}} \frac{n}{\nu}+\frac{1}{1-\lambda_{2}}\left(\left(k^{*}\right)^{2} \log n+\frac{n}{\nu}\right)\right) \\
& =O\left(\frac{1}{1-\lambda_{2}}\left(\left(k^{*} \log n\right)^{2}+\frac{\log n}{k^{*}} \frac{n}{\nu}\right)\right)  \tag{27}\\
& =O\left(\frac{1}{1-\lambda_{2}}\left(\log ^{4} n+\frac{n}{\nu}\right)\right) .
\end{align*}
$$

The last bound above is obvious if $k^{*}=\log n$. If $k^{*}<\log n$, then the last bound holds because the second term in the sum in (27), that is $(n / \nu) \log n / k^{*}$, is $O\left(\log ^{3} n\right)$. Indeed, if $k^{*}<\log n$, then from the definition of $k^{*}$, either $\left(\frac{n}{\nu}\right)^{1 / 2}<\log n$ or $\frac{m}{2 \Delta}<\log n$. If the former, then the second term in the sum in (27) is clearly $O\left(\log ^{3} n\right)$. Observe that

$$
\begin{equation*}
\frac{n}{\nu} \leq \frac{n^{2} d^{2}}{\Delta^{2}}=\left(\frac{2 m}{\Delta}\right)^{2} \tag{28}
\end{equation*}
$$

Thus if $\frac{m}{2 \Delta}<\log n$, then $n / \nu=O\left(\log ^{2} n\right)$, and the second term in the sum in (27) is again $O\left(\log ^{3} n\right)$.

We conclude by noting that since $k^{*} \leq \frac{m}{2 \Delta}$ and (28), then (27) implies

$$
\begin{equation*}
C(n)=O\left(\frac{1}{1-\lambda_{2}}\left(\frac{m}{\Delta} \log n\right)^{2}\right) \tag{29}
\end{equation*}
$$

The above bound is better than (25), if $\Delta=\omega(m / \log n)$.

## References

[1] D. Aldous. Meeting times for independent Markov chains. Stochastic Processes and their Applications 38(2):185-193, August 1991.
[2] D. Aldous and J. Fill. Reversible Markov Chains and Random Walks on Graphs, http://stat-www.berkeley.edu/pub/users/aldous/RWG/book.html.
[3] C. Cooper, A. M. Frieze, and T. Radzik. Multiple Random Walks in Random Regular Graphs. SIAM J. Discrete Math. 23(4):1738-1761, 2009.
[4] C. Cooper, R. Elsässer, H. Ono, T. Radzik. Coalescing random walks and voting on graphs. In PODC 2012: Proceedings of the 2012 ACM Symposium on Principles of Distributed Computing, pages 47-56, July 2012.
[5] J. T. Cox. Coalescing random walks and voter model consensus times on the torus in $\mathbb{Z}^{d}$. The Annals of Probability 17(4):1333-1366, October 1989.
[6] C. Gkantsidis, M. Mihail, and A. Saberi. Conductance and congestion in power law graphs. In SIGMETRICS 2003: Proceedings of 2003 ACM SIGMETRICS Intl. Conf. on Measurement and Modeling of Computer Systems, New York, NY, USA, pages 148-159, 2003.
[7] Y. Hassin and D. Peleg. Distributed probabilistic polling and applications to proportionate agreement. Information \& Computation 171(2):248-268, December 2001.
[8] A. Israeli and M. Jalfon. Token management schemes and random walks yeild self stabilizing mutual exclusion. In PODC 1990: Proceedings of the 9th Annual ACM Symposium on Principles of Distributed Computing, Quebec City, Quebec, Canada, pages 119-131, August 1990.
[9] D. Levin, Y. Peres, and E. Wilmer. Markov Chains and Mixing Times. American Mathematical Society, 2009.
[10] L. Lovász. Random walks on graphs: a survey. Bolyai Society Mathematical Studies. Combinatorics, Paul Erdős is Eighty 2:1-46, Keszthely, Hungary, 1993.
[11] R. Oliveira. On the coalescence time of reversible random walks. Trans. Amer. Math. Soc. 364(4): 2109-2128, 2012.
[12] A. Sinclair. Improved bounds for mixing rates of Markov chains and multicommodity flow. Combinatorics, Probability and Computing 1(4):351370, December 1992.


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